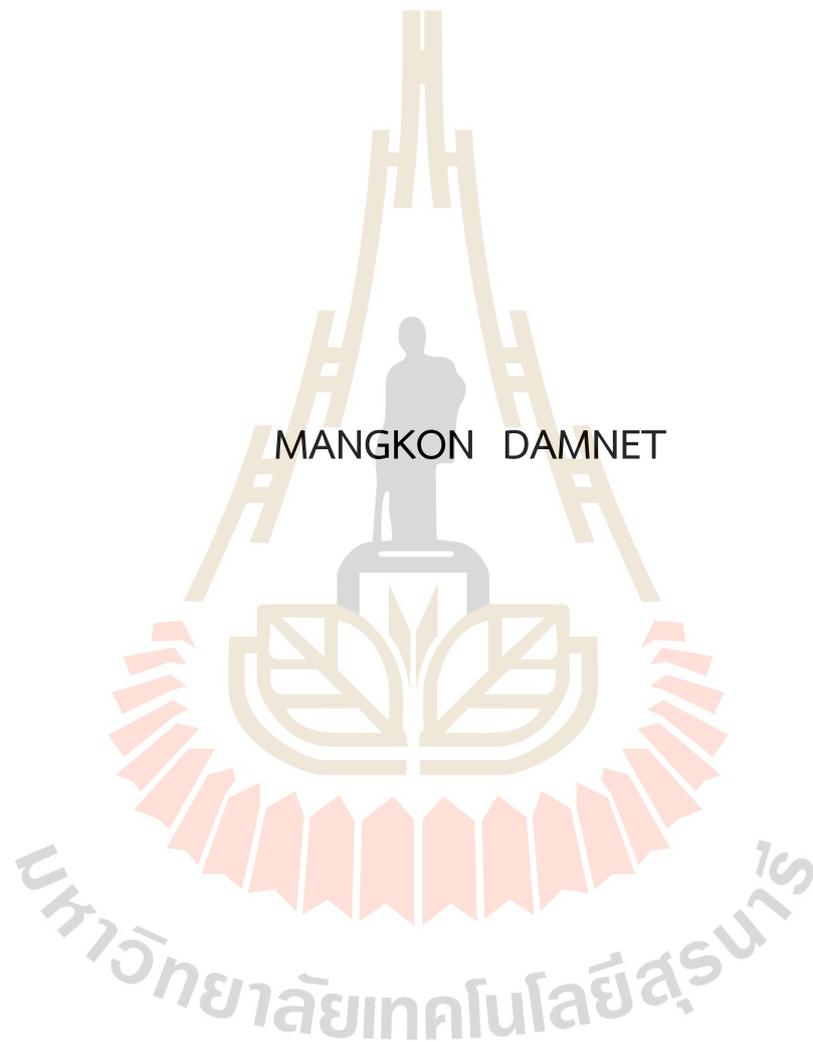# BAYESIAN ANALYSIS OF WRAPPED DISTRIBUTIONS
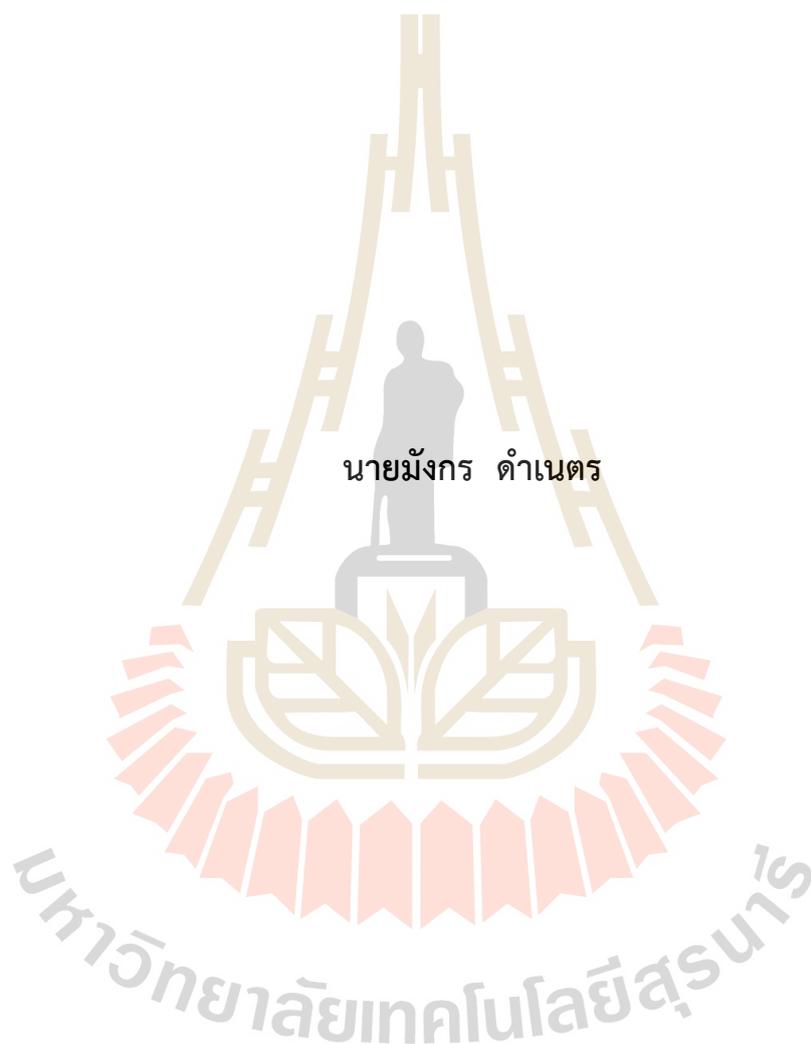
MANGKON DAMNET

A Thesis Submitted in Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy in Applied Mathematics

Suranaree University of Technology

Academic Year 2024

การวิเคราะห์แบบเบส์ของการแจกแจงห่อหุ้ม

นายมังกร ดำเนตร

BAYESIAN ANALYSIS OF WRAPPED DISTRIBUTIONS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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มังกร ดำเนตร : การวิเคราะห์แบบเบส์ของการแจกแจงห่อหุ้ม (BAYESIAN ANALYSIS OF WRAPPED DISTRIBUTIONS) อาจารย์ที่ปรึกษา : ผู้ช่วยศาสตราจารย์ ดร.เจษฎา ตัณฑนุช, 75 หน้า.

คำสำคัญ: การแจกแจงห่อหุ้ม/การแจกแจงเลขชี้กำลังห่อหุ้มภายหลัง/ตัวประมาณค่าเบส์/ฟังก์ชัน การหาค่าความเสี่ยงต่ำสุด

วิทยานิพนธ์ฉบับนี้ศึกษาการขยายการวิเคราะห์แบบเบส์โดยใช้การแจกแจงห่อหุ้ม โดย นำเสนอแนวคิดของการแจกแจงห่อหุ้มภายหลังภายใต้ความน่าจะเป็นก่อนทั้งแบบไม่มีข้อมูลและมี ข้อมูล สำหรับกรณีความน่าจะเป็นก่อนแบบมีข้อมูลในที่นี้ได้นำเสนอตัวแบบที่ใช้ข้อมูลก่อนเป็นการ แจกแจงทางสถิติแกมมาเพื่อนำไปใช้กับการแจกแจงเลขชี้กำลังห่อหุ้มภายหลัง นอกจากนี้ได้นำเสนอ เครื่องมือวัดทางสถิติที่สำคัญของการแจกแจงทางสถิติเหล่านั้นได้แก่ ค่าเฉลี่ย ความแปรปรวน ความ เบ้ ความโด่ง และฟังก์ชันก่อกำเนิดโมเมนต์ ทั้งนี้ได้แสดงการหาตัวประมาณค่าเบส์และฟังก์ชันการ หาค่าความเสี่ยงต่ำสุด และวิเคราะห์กราฟฟังก์ชันความหนาแน่นของความน่าจะเป็น รวมถึงสมบัติ ทางสถิติอื่น ๆ ในตอนท้ายมีการนำเสนอ 2 ตัวอย่างในชีวิตจริงที่สามารถประยุกต์ใช้การแจกแจงเลข ชี้กำลังห่อหุ้มภายหลังในการอธิบายปรากฏการณ์นั้นได้

สาขาวิชาคณิตศาสตร์และภูมิสารสนเทศ	ลายมือชื่อนักศึกษา _____
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	ลายมือชื่ออาจารย์ที่ปรึกษาร่วม Amornrat. S

MANGKON DAMNET : BAYESIAN ANALYSIS OF WRAPPED DISTRIBUTIONS. THESIS
ADVISOR : ASST. PROF. JESSADA TANTHANUCH, Ph.D. 75 PP.

Keyword: WRAPPED DISTRIBUTION, POSTERIOR WRAPPED EXPONENTIAL DISTRIBUTION,
BAYESIAN ESTIMATOR, RISK MINIMIZATION FUNCTION.

This thesis investigates the extension of Bayesian analysis using wrapped distributions. It introduces the concept of posterior wrapped distributions, derived under both uninformative and informative priors, with the latter modeled using a gamma distribution, to formulate posterior wrapped exponential distributions. Key statistical measures, including mean, variance, skewness, kurtosis, and moment generating functions, are presented for these distributions. Bayesian estimators and risk minimization functions are derived, and the probability density function curves along with their statistical properties are thoroughly examined. Furthermore, the practical application of posterior wrapped exponential distributions is illustrated through two real-life examples, demonstrating their utility in empirical contexts.
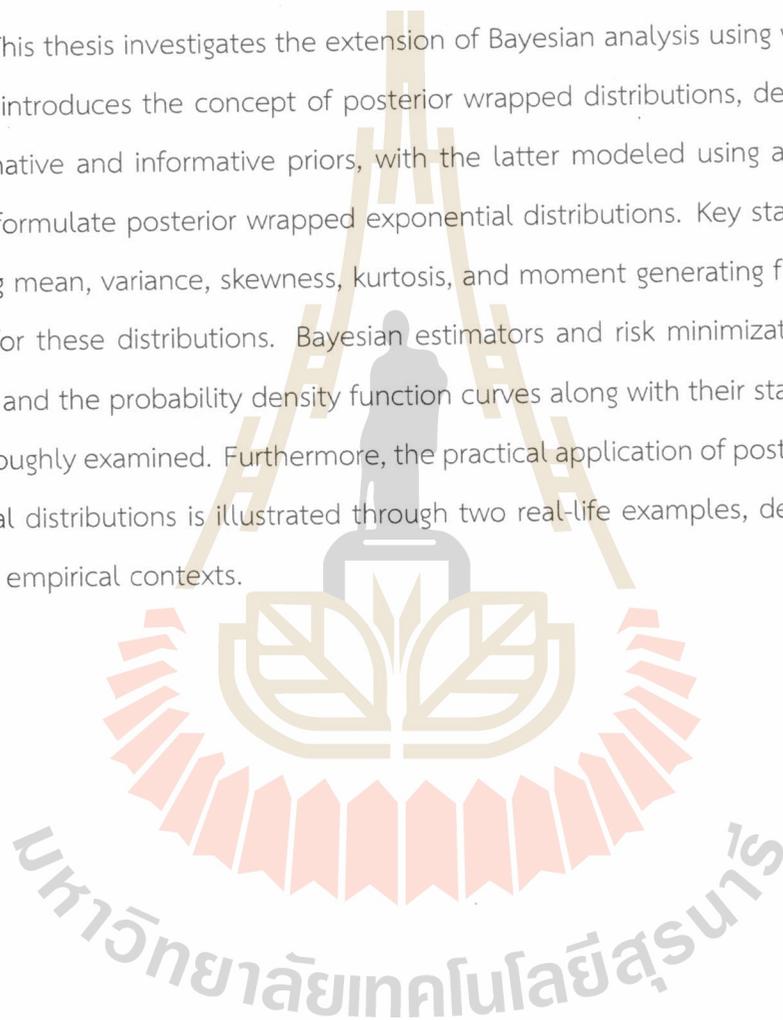
School of Mathematical Sciences	Student's Signature _Mangkon_
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ACKNOWLEDGEMENTS

Mangkon Damnet

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CHAPTER I

INTRODUCTION

By the concept of innovation waves, first introduced by economist and historian Carlota Perez, the "Sixth Innovation Wave" refers to the current phase of technological development, building on previous waves of innovation (Perez, 1983). The sixth innovation wave represents a new era of technological and social development, with significant opportunities and challenges for individuals, organizations, and society as a whole. The sixth wave is characterized by the convergence of several technologies, including artificial intelligence, big data, the internet of things, robotics, blockchain, and quantum computing (Bangkok Post, 2020). Many of these technologies are associated with mathematical and statistical frameworks. Specifically, mathematics and statistics are critical for:

- Data analysis: These technologies generate vast amounts of data that can be analyzed to extract insights and make predictions. Statistics provides the tools and techniques for analyzing and interpreting this data.

- Machine learning: Machine learning is a subfield of artificial intelligence that involves using statistical algorithms to identify patterns and make predictions. Statistics provides the mathematical framework for these algorithms.

- Experimental design: In order to make accurate predictions and draw conclusions from data, experiments must be designed in a statistically sound manner. This involves determining sample sizes, controlling for confounding variables, and randomizing treatment assignments.

- Quality control: Many of these technologies are used in manufacturing and other industries where quality control is essential. Statistical process control methods are used to monitor and maintain the quality of products and processes.

- Risk management: Many of these technologies introduce new risks and challenges, such as cybersecurity threats, ethical concerns, and potential job displacement. Statistics provides the tools for assessing and managing these risks.

Probability distributions play a crucial role in statistical inference and modeling, particularly in describing the uncertainty associated with random variables. Classical probability distributions are suited for variables assumed to be linear and can take on values along an unbounded straight line. However, many variables in real life are constrained to a circular range and have a periodic nature. For example, angles and times of day are circular variables that can take on values between 0 and 2π or 0 and 24 hours, respectively. While classical probability distributions such as the normal distribution and the exponential distribution are well suited for linear variables, they do not account for the circular nature of these circular variables. In probability theory, a wrapped distribution is a circular version of a classical probability distribution that accounts for the circular nature of the variable being measured (Mardia, 1972; Mardia and Jupp, 2001).

However, traditional statistical methods are often limited by restrictive assumptions and the necessity of large sample sizes to achieve reliable results, which can be challenging when working with wrapped distributions. These limitations hinder their effective application in real-world data analysis. In contrast, Bayesian statistics provides a more flexible and effective approach, particularly in situations where data is scarce or conventional assumptions may not be entirely applicable. Bayesian statistics provides a powerful framework for estimating parameters within wrapped distributions, offering refinement beyond traditional approaches. Unlike conventional methods, which may rely on restrictive assumptions or require substantial sample sizes to achieve accuracy, the Bayesian approach incorporates prior knowledge into the estimation process, enabling more nuanced and robust parameter estimation even with limited data (Ravindran and Ghosh, 2004).

The thesis focuses on analyzing wrapped distributions. Wrapped distributions belong to the subclass of circular distributions, which describe the distribution of circular variables. The use of probability distributions is central to Bayesian analysis as it provides

a way to represent the uncertainty associated with the parameters of interest. Probability distributions are used to model the prior information or beliefs about the parameters, as well as the likelihood of the observed data. The choice of prior distribution is crucial in Bayesian analysis as it impacts posterior distribution and resulting inference. The Bayesian analysis of wrapped distributions will be investigated, particularly the conjugate prior distributions associated with them.

1.1   Research Objective

The aim of this thesis is to advance the study of wrapped distributions by exploring an extended Bayesian analysis within this context. The focus is placed on illustrating examples of posterior wrapped distributions, with particular emphasis on their probability density functions (PDFs), cumulative distribution functions (CDFs), and graphical representations.

1.2   Scope and Limitations

This thesis primarily investigates continuous wrapped distributions within the framework of circular data, where the period is defined as 2π. The study aims to enhance the understanding of these distributions through an extended Bayesian analysis to derive posterior wrapped distributions. Key statistical properties, including the mean, variance, skewness, kurtosis, and moments, are examined, with particular emphasis on their PDFs, CDFs, and graphical representations.

In addressing the theoretical development of wrapped distributions, the study primarily focuses on their structural properties and estimation methods rather than empirical applications. Expressions for fundamental statistical measures are systematically derived, highlighting the role of Bayesian inference in improving parameter estimation. By emphasizing theoretical formulations, this work establishes a foundation for further exploration of wrapped distributions. The scope is explicitly confined to circular data with a fixed period of 2π, with non-circular and higher-dimensional extensions beyond the present analysis.

1.3 Research Procedure

The research work will proceed as follows:

1. Study Wrapped Distributions, with an emphasis on continuous forms within the context of circular data with period 2π.

2. Study Bayesian Statistics Theory with a focus on its application to wrapped distributions.

3. Analyze the wrapped form of probability distributions, particularly considering their probability density functions (PDFs), cumulative distribution functions (CDFs), and graphical representations.

4. Apply Bayesian Statistics Theory to derive key statistical measures, including mean, variance, skewness, kurtosis, and higher moments of wrapped distributions.

5. Investigate theoretical applications of the developed theory, focusing on examples that illustrate the methods and concepts.

1.4 Expected Result

The expected result of this research is the development of a deeper understanding of wrapped distributions, particularly through the application of Bayesian Statistics Theory. This includes a comprehensive analysis of continuous wrapped distributions within the context of circular data with period 2π, the derivation of important statistical measures such as mean, variance, skewness, kurtosis, and higher moments, and the clear representation of their probability density functions (PDFs), cumulative distribution functions (CDFs), and graphical depictions. Additionally, the research is expected to provide insights into the role of Bayesian methods in parameter estimation for wrapped distributions, focusing on theoretical development and statistical inference. The study will culminate in illustrative examples that demonstrate the methods and applications of the developed theory, contributing to a more thorough theoretical foundation for wrapped distributions and their potential applications in various contexts.

CHAPTER II

LITERATURE REVIEW

In this chapter, fundamental concepts of mathematics and statistics relevant to Bayesian inference and the analysis of circular data are reviewed. The majority of the concepts on probability and random variables, including Probability Space, Conditional Probability, Independent Events, Classes of Probability Spaces, and Random Variables, are based on the works of Grimmett and Stirzaker (2020), Grimmett and Welsh (2017), Destrempes and Cloutier (2010), and Park, Song, and Yoon (2022).

2.1 Probability Space

The foundation of probability theory lies in the concept of a random experiment (or trial), which is an action whose outcome is uncertain. This concept is subsequently formalized as a mathematical construct called a *probability space*.

2.1.1 Sample Space

In probability theory, a sample space represents the set of all possible outcomes of an experiment. It provides a comprehensive view of the experiment's structure, forming the basis for defining events and assigning probabilities. The sample space is often denoted as Ω, and its nature depends on whether the experiment is discrete or continuous. Understanding the sample space is essential for developing accurate probabilistic models and drawing meaningful inferences.

Definition 2.1. An experiment that can be consistently repeated under controlled conditions, but whose outcome remains uncertain, is referred to as a random experiment.

Definition 2.2. The collection of all possible outcomes of an experiment is called the sample space of the experiment. This sample space, often represented by Ω.

Definition 2.3. An element of the sample space is referred to as a sample point or an elementary outcome.

Definition 2.4. For combined random experiments involving two discrete sample spaces, Ω_1 of size m and Ω_2 of size n, the sample space $\Omega = \Omega_1 \times \Omega_2$, which has a size of mn, is called a discrete combined space.

2.1.2 Event Space

The event space is the collection of all events that can occur in a given experiment, with each event being a subset of the sample space. It represents specific outcomes or sets of outcomes that satisfy certain conditions. The event space allows for the formal definition of events in probabilistic models, enabling probability calculations for each event. Its structure is crucial for understanding the relationships between events and their probabilities.

Definition 2.5. A collection $\mathcal{F}$ of subsets of a sample space Ω is called a *σ-field* if it satisfies the following conditions:

1. $\emptyset \in \mathcal{F}$;

2. If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$;

3. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, where A^c is the complement of A.

A σ-field derived from a sample space is called an event space, and each element within this event space is referred to as an event.

Definition 2.6. An event that consists of a singleton set is referred to as an elementary event.

Definition 2.7. Let the sample space be the real line $\mathbb{R}$, the σ-algebra

$$\mathcal{B}(\mathbb{R}) = \sigma(\text{all open intervals})$$

generated by all open intervals (a, b) in $\mathbb{R}$ is known as the Borel σ-field, Borel algebra, or Borel field of $\mathbb{R}$.

2.1.3   Probability Measure

A probability measure provides the formal framework to quantify the likelihood of events within a sample space. This begins with the definition of a measurable space, followed by the introduction of a probability measure function, which assigns probabilities to events. These definitions collectively form a probability space, the essential structure for calculating and analyzing probabilities.

Definition 2.8. The pair $(\Omega, \mathcal{F})$ consisting of a sample space Ω and an event space $\mathcal{F}$ is referred to as a measurable space.

Definition 2.9. A mapping $P : \mathcal{F} \to \mathbb{R}$ is called a probability measure on the measurable space $(\Omega, \mathcal{F})$ if it satisfies the following axioms:

1. $P(A) \geq 0$ for all $A \in \mathcal{F}$.

2. $P(\Omega) = 1$ and $P(\emptyset) = 0$.

3. If a finite number of events $\{B_i\}_{i=1}^n$ are mutually exclusive (meaning $B_i \cap B_j = \emptyset$ whenever $i \neq j$), then

$$P \left(\bigcup_{i=1}^n B_i \right) = \sum_{i=1}^n P(B_i).$$

4. If a countable number of events $\{B_i\}_{i=1}^\infty$ are mutually exclusive, then

$$P \left(\bigcup_{i=1}^\infty B_i \right) = \sum_{i=1}^\infty P(B_i).$$

The probability measure P assigns a real number $P(G)$, called the probability of an event G, and it is also denoted by $P\{G\}$ or $\Pr\{G\}$. A probability measure is sometimes referred to as a probability function, probability distribution, or simply distribution.

Definition 2.10. The triplet $(\Omega, \mathcal{F}, P)$ consisting of a sample space Ω, an event space $\mathcal{F}$ of Ω, and a probability measure P is referred to as a *probability space*.

2.2 Conditional Probability

Conditional probability is a fundamental concept and a powerful analytical tool in probability theory. It quantifies the likelihood of an event occurring given the occurrence of another event. This concept is particularly essential when addressing problems with incomplete or partial information, as it often simplifies complex analyses and provides an efficient means of deriving probabilistic results.

Definition 2.11. The probability of an event given that another event has occurred is referred to as conditional probability. Specifically, the conditional probability of event A, given that event B has occurred, is denoted by $\mathbb{P}(A|B)$ and is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)},$$

where $\mathbb{P}(B) > 0$.

Theorem 2.1. *For any events A and B such that $0 < \mathbb{P}(B) < 1$,*

$$\mathbb{P}(A) = \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\mathbb{P}(B^c).$$

More generally, let $B_1, B_2, \ldots, B_n$ be a partition of Ω such that $\mathbb{P}(B_i) > 0$ for all i. The following theorem, called the total probability theorem, is then obtained:

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i).$$

2.3 Independent Events

The occurrence of an event B can influence the likelihood of another event A, replacing the original probability $P(A)$ with the conditional probability $P(A \mid B)$. If $P(A \mid B) = P(A)$, the events A and B are defined as independent. More formally, independence requires that, for $P(A) > 0$ and $P(B) > 0$, the conditions $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$ are satisfied.

Definition 2.12. If the probability of the intersection of two events A and B, denoted $P(A \cap B)$, is equal to the product of their individual probabilities, i.e.,

$$P(A \cap B) = P(A)P(B),$$

then the events A and B are defined as independent, or mutually independent. Otherwise, the events are dependent.

2.4   Classes of Probability Spaces

The concepts of probability mass functions (PMFs) and probability density functions (PDFs) are introduced here. As discussed by Kim (2010), these functions provide an equivalent representation of the probability measure for describing a probability space. Furthermore, they serve as more practical and efficient tools for performing mathematical operations, particularly differentiation and integration, which are essential in the analysis of probability distributions.

2.4.1   Discrete Probability Spaces

In a discrete probability space, where the sample space Ω is a countable set, it is common to assume $\Omega = \{0, 1, \dots\}$ or $\Omega = \{1, 2, \dots\}$, with the event space $\mathcal{F}$ defined as the power set of Ω, denoted as $\mathcal{F} = 2^{\Omega}$.

Definition 2.13. In a discrete probability space, a function $p(\omega)$ that assigns a real number to each sample point $\omega \in \Omega$ is called a PMF if it satisfies the following conditions:

$$p(\omega) \geq 0, \quad \forall \omega \in \Omega$$

and

$$\sum_{\omega \in \Omega} p(\omega) = 1.$$

2.4.2   Continuous Probability Spaces

Let us now consider a continuous probability space with the measurable space $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Here, the sample space Ω is the set of real numbers $\mathbb{R}$, and the event space $\mathcal{F}$ is the Borel sigma-algebra $\mathcal{B}(\mathbb{R})$.

Definition 2.14. Let $(\Omega, \mathcal{B})$ be a measurable space. A real-valued Borel function f is called a probability density function, density function, or simply a density, if it satisfies the following two conditions:

$$f(r) \geq 0, \quad \forall r \in \Omega$$

and

$$\int_\Omega f(r)\, dr = 1.$$

2.5 Random Variables

Random phenomena are represented by functions on a probability space, known as *random variables*. The values a random variable assumes are determined by chance, and their likelihoods are captured by a function known as the *distribution function*.

Definition 2.15. A *random variable* is a function $X : \Omega \to \mathbb{R}$ such that for each $x \in \mathbb{R}$, the set

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$$

is an element of the sigma-algebra $\mathcal{F}$. This property is referred to as $\mathcal{F}$-measurability.

The distribution of a random variable is a fundamental statistical concept. It describes all the possible values that a random variable can take and the frequency with which each value occurs. Every random variable is associated with a *distribution function*, which is a key tool in statistical analysis.

Definition 2.16. The *distribution function* of a random variable X is defined as the function

$$F : \mathbb{R} \to [0, 1]$$

where $F(x) = \mathbb{P}(X \leq x)$.

The distribution function satisfies the following properties:

Theorem 2.2. *Let F be a distribution function. Then, the following conditions hold:*

1. $\lim\limits_{x\to-\infty} F(x) = 0$,

2. $\lim\limits_{x\to\infty} F(x) = 1$,

3. F is non-decreasing: if $x < y$, then $F(x) \leq F(y)$,

4. F is right-continuous: as $h \to 0^+$, $F(x + h) \to F(x)$.

Additionally, the following relationships hold between the distribution function of a random variable X and the probability measure $\mathbb{P}$.

Theorem 2.3. *Let F denote the distribution function of X. Then*

1. $\mathbb{P}(X > x) = 1 - F(x)$,

2. $\mathbb{P}(x < X \leq y) = F(y) - F(x)$,

3. $\mathbb{P}(X = x) = F(x) - \lim_{y\to x^-} F(y)$.

Random variables can be divided into two primary types: *discrete* and *continuous*.

Definition 2.17. A random variable X is called *discrete* if it takes values from a countable subset of $\mathbb{R}$, denoted $\{x_1, x_2, \dots\}$. For a discrete random variable, the *probability mass function* $f : \mathbb{R} \to [0, 1]$ is given by $f(x) = \mathbb{P}(X = x)$.

Definition 2.18. A random variable X is said to be *continuous* if its distribution function is expressible as

$$F(x) = \int_{-\infty}^{x} f(u)\, du \quad \text{for } x \in \mathbb{R},$$

where $f : \mathbb{R} \to [0, \infty)$ is an integrable function, known as the *probability density function* of X.

2.6   Expected Value and Variance

Let $x_1, x_2, \dots, x_N$ denote the outcomes from N repetitions of an experiment. The arithmetic mean of these outcomes is given by

$$m = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

Now consider a set of N discrete random variables, each having the same probability mass function f. For any given value x, approximately $Nf(x)$ of the variables X_i, where $i = 1, \ldots, N$, will take the value x. Hence, the average value can be written as

$$m = \frac{1}{N} \sum_x xNf(x) = \sum_x xf(x),$$

where the summation runs over all possible values of X_i. This average is referred to as the *expectation* or *mean value* of the distribution with mass function f.

Definition 2.19. The *expectation* or *mean* of a discrete random variable X with mass function f is defined by

$$\mathbb{E}(X) = \sum_{x:f(x)>0} xf(x), \tag{2.1}$$

provided that the sum converges absolutely.

The expectation, as given in equation (2.1), represents the average of the possible values of X, which can also be expressed as

$$\mathbb{E}(X) = \sum_x x\mathbb{P}(X = x),$$

indicating that each value is weighted by its corresponding probability.

Lemma 2.4. *Let X be a random variable with mass function f, and let $g : \mathbb{R} \to \mathbb{R}$ be a function. Then, the expected value of $g(X)$ is given by*

$$\mathbb{E}(g(X)) = \sum_x g(x)f(x),$$

assuming that the sum converges absolutely.

Definition 2.20. For any positive integer k, the kth *moment* m_k of X is defined as

$$m_k = \mathbb{E}(X^k) = \sum_x x^k\mathbb{P}(X = x),$$

provided the sum converges.

The kth *central moment* σ_k is defined as

$$\sigma_k = \mathbb{E}\left((X - m_1)^k\right),$$

where $m_1 = \mathbb{E}(X)$.

Definition 2.21. Two commonly used moments are:

- $m_1 = \mathbb{E}(X)$, which is the *mean* (or *expectation*) of X, and

- $\sigma_2 = \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$, which is the *variance* of X.

These two quantities describe the central tendency and spread of X; that is, m_1 gives the average value of X, while σ_2 quantifies the extent to which X deviates from this average. The mean is often denoted as μ, and the variance is denoted as $\text{var}(X)$. The standard deviation is the positive square root of the variance, $\sigma = \sqrt{\text{var}(X)}$.

In addition, higher moments provide further insight into the shape of the distribution. The *skewness* measures the asymmetry of the distribution, while the *kurtosis* describes the thickness of the tails and the peakedness of the distribution relative to a normal distribution.

Definition 2.22. The *skewness* of a random variable X, denoted γ_1, is defined as

$$\gamma_1 = \mathbb{E}\left[\left(\frac{X - \mathbb{E}(X)}{\sqrt{\text{var}(X)}}\right)^3\right],$$

provided the expectation exists. This measures the degree of asymmetry around the mean, with positive values indicating a longer right tail and negative values indicating a longer left tail.

Definition 2.23. The *kurtosis* of a random variable X, denoted γ_2, is defined as

$$\gamma_2 = \mathbb{E}\left[\left(\frac{X - \mathbb{E}(X)}{\sqrt{\text{var}(X)}}\right)^4\right],$$

provided the expectation exists. This quantifies the concentration of data in the tails and the peakedness relative to a Gaussian distribution, where a value greater than 3 indicates heavier tails or a sharper peak (leptokurtic), and a value less than 3 suggests lighter tails or a flatter peak (platykurtic).

Definition 2.24. The *moment generating function* (MGF) of a random variable X, denoted $M_X(t)$, is defined as

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx,$$

for a continuous random variable with density function f, provided the integral exists for some t in a neighborhood of 0. The MGF is a powerful tool that encapsulates all moments of X. Specifically, the kth moment of X can be obtained by taking the kth derivative of $M_X(t)$ with respect to t and evaluating at $t = 0$:

$$m_k = \mathbb{E}(X^k) = \left. \frac{d^k}{dt^k} M_X(t) \right|_{t=0},$$

assuming the derivatives exist. Thus, the mean (m_1), variance (via $\mathbb{E}(X^2) - [\mathbb{E}(X)]^2$), skewness, and kurtosis can all be derived from the MGF, linking it directly to the previously defined quantities.

Definition 2.25. The *expectation* of a continuous random variable X with density function f is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) \, dx,$$

provided the integral exists.

Thus, the *mean, variance, skewness,* and *kurtosis* for a continuous random variable are defined as $\mu = \mathbb{E}(X)$, $\sigma_2 = \mathbb{E}\left((X - \mathbb{E}(X))^2 \right)$, $\gamma_1 = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma} \right)^3 \right]$, and $\gamma_2 = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma} \right)^4 \right]$, respectively, provided all expectations exist.

2.7 Bayesian Statistics

Bayesian statistics is a branch of statistical theory grounded in the Bayesian interpretation of probability, where probability quantifies the degree of belief or certainty regarding a specific event or outcome. This framework leverages Bayes' theorem to update probabilities as new data are observed, seamlessly integrating prior knowledge with newly acquired information. Such an approach provides enhanced flexibility and adaptability in modeling and inference, particularly in scenarios where data are limited or noisy. By employing Bayes' theorem, parameters of a statistical model or probability distribution can be directly estimated through the assignment of a probability distribution to these parameters, reflecting beliefs about their plausible values. Furthermore, Bayesian statistics establishes a structured methodology for incorporating prior knowledge into the analysis,

utilizing concepts such as prior distributions, Bayesian estimators, loss functions, and risk functions to refine inferences and decision-making processes (Lee, 2012).

Theorem 2.5. *Bayes' Theorem serves as a cornerstone for computing the conditional probability of an event based on prior knowledge of related events. It is expressed as*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)},$$

where $\mathbb{P}(A|B)$ denotes the posterior probability, representing the updated likelihood of event A given the occurrence of event B, $\mathbb{P}(B|A)$ is the likelihood, indicating the probability of observing B given that A has occurred, $\mathbb{P}(A)$ is the prior probability of event A, and $\mathbb{P}(B)$ is the marginal probability of event B, acting as a normalizing constant.

2.7.1 Specification of the Prior

In Bayesian analysis, the prior distribution, denoted $\mathbb{P}(A)$ in Bayes' theorem, encapsulates initial beliefs about the parameter of interest before observing the data. Priors can be classified as *informative*, based on existing knowledge or expert opinion, or *uninformative*, designed to exert minimal influence on the posterior, such as a uniform or improper prior (e.g., $h(\theta) = 1$ for $\theta > 0$). The choice of prior is critical, as it shapes the posterior distribution and, consequently, the resulting inferences. For a parameter θ, the prior distribution $p(\theta)$ is specified to reflect the analyst's belief about θ, and when combined with the likelihood $p(x|\theta)$ via Bayes' theorem, it yields the posterior $p(\theta|x) \propto p(x|\theta)p(\theta)$.

2.7.2 Bayesian Estimator

A Bayesian estimator is derived from the posterior distribution to provide a point estimate of the parameter θ. Common estimators include the posterior mean, $\mathbb{E}(\theta|x) = \int \theta p(\theta|x)\, d\theta$, which minimizes the expected squared error, and the posterior median, which is robust to outliers. The choice of estimator often depends on the decision-theoretic framework, where an estimator is selected to optimize a specified criterion, such as minimizing a loss function. The process of obtaining a Bayesian estimator involves the following steps:

1. **Specify the Prior**: Define the prior distribution $p(\theta)$ based on prior knowledge or an uninformative assumption.

2. **Determine the Likelihood**: Model the likelihood $p(x|\theta)$ based on the observed data x and the assumed distribution of the random variable.

3. **Compute the Posterior**: Apply Bayes' theorem to obtain the posterior distribution, $p(\theta|x) = \frac{p(x|\theta)p(\theta)}{\int p(x|\theta')p(\theta')\,d\theta'}$, where the denominator is the marginal likelihood, ensuring normalization.

4. **Select a Loss Function**: Choose a loss function $L(\theta, \hat{\theta})$ (e.g., squared error loss, $L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$) to quantify the penalty of estimation errors.

5. **Derive the Estimator**: Minimize the expected posterior loss, $\mathbb{E}[L(\theta, \hat{\theta})|x] = \int L(\theta, \hat{\theta})p(\theta|x)\,d\theta$, with respect to $\hat{\theta}$. For squared error loss, this yields the posterior mean; for absolute error loss, it yields the posterior median.

This systematic approach underscores the Bayesian paradigm's ability to synthesize prior beliefs and observed data into a coherent estimate, guided by the chosen loss function.

2.7.3   Loss Function and Risk Function

The evaluation of a Bayesian estimator involves the use of a *loss function*, denoted $L(\theta, \hat{\theta})$, which quantifies the cost of estimating the true parameter θ with an estimator $\hat{\theta}$. Common loss functions include squared error loss, $L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$, and absolute error loss, $L(\theta, \hat{\theta}) = |\theta - \hat{\theta}|$. The *risk function*, or expected loss, is defined as

$$R(\theta, \hat{\theta}) = \mathbb{E}[L(\theta, \hat{\theta})] = \int L(\theta, \hat{\theta})p(x|\theta)\,dx,$$

representing the average loss over all possible data x given θ. In Bayesian decision theory, the optimal estimator minimizes the expected posterior risk, $\mathbb{E}[L(\theta, \hat{\theta})|x] = \int L(\theta, \hat{\theta})p(\theta|x)\,d\theta$, integrating the loss over the posterior distribution. This framework links the prior, likelihood, and posterior to decision-making under uncertainty.

Using Bayes' theorem, initial assumptions about the parameter θ are revised based on the evidence provided by the data x. The posterior distribution $p(\theta|x)$ offers a refined

estimate of θ, informed by both the prior $p(\theta)$ and the likelihood $p(x|\theta)$, while Bayesian estimators, guided by loss and risk functions, facilitate optimal inference and prediction.

2.8 Wrapped Distributions

Wrapped distributions are frequently used in directional statistics to describe the distribution of circular data. This is because circular data repeats at regular intervals, such as every 360 degrees (or 2π radians), and this property is not captured by traditional probability distributions (Fisher, 1993).

Definition 2.26. If X is a random variable with values on the line and distribution function $F(x)$, the random variable X_W of the wrapped distribution is given by

$$X_W = x \pmod{2\pi}$$

and the distribution function $F_W(\theta)$ of X_W is given by

$$F_W(\theta) = \sum_{k=-\infty}^{\infty} \left[F(\theta + 2\pi k) - F(2\pi k) \right], \quad 0 < \theta \leq 2\pi.$$

Specifically, if X is a discrete random variable with integer range, the for a fixed $m \in \mathbb{N}$, X_W is defined by

$$X_W = 2\pi x \pmod{2\pi m},$$

which is a random variable with r aligns in the lattice $\left[\frac{2\pi r}{m}, r = 0, 1, \ldots, m-1 \right]$ on the circle. The probability mass function $\mathbb{P}_W \left(X_W = \frac{2\pi r}{m} \right)$ of X_W is given by

$$\mathbb{P}_W \left(X_W = \frac{2\pi r}{m} \right) = \sum_{k=-\infty}^{\infty} f(r + km), \quad r = 0, 1, \ldots, m-1.$$

If X is a continuous random variable, the probability density function $f_W(\theta)$ of X_W is

$$f_W(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2\pi k).$$

2.9 Distributions Related Wrapped Data

Distributions related to wrapped data are used to model circular or periodic data, and are important in a variety of fields including meteorology, biology, and engineering.

These distributions treat data as if it were on a circle, with a periodicity of 2π, allowing for more accurate modeling of data that exhibits circular or periodic patterns.

2.9.1   Wrapped Binomial Distribution

A wrapped binomial distribution is the wrapped version of a binomial distribution. It is often used to model circular data that have a binary outcome. The PMF of a wrapped binomial distribution is given by (Girija, Dattatreya Rao, and Srihari, 2014a, 2014b)

$$f_W\left(\theta = \frac{2\pi r}{m}\Big|n,p\right) = \sum_{k=0}^{\left\lfloor \frac{n-r}{m} \right\rfloor} \binom{n}{r+km}p^{r+km}(1-p)^{n-(r+km)}, \quad r = 0,1,\ldots,m-1,$$

where θ is the observed angle, n is the number of trials, p is the probability of success, and $n \geq m$.

2.9.2   Wrapped Geometric Distribution

The wrapped geometric distribution still has the same concept with the classical geometric distribution. It is used to model circular data that have a binary outcome and exhibit clustering or periodicity. The PMF of a wrapped geometric distribution is given by (Jacob, and Jayakumar, 2013)

$$f_W\left(\theta = \frac{2\pi r}{m}\Big|p\right) = \sum_{k=0}^{\infty}(1-p)^{r+km}p, \quad r = 0,1,\ldots,m-1,$$

where θ is the observed angle, and p is the probability of success.

2.9.3   Wrapped Poisson Distribution

The wrapped Poisson distribution is a circular distribution obtained by wrapping a Poisson distribution. Circular data that have a countable outcome can be modeled by this distribution. Its PMF is given by (Mardia, and Jupp, 2001)

$$f_W\left(\theta = \frac{2\pi r}{m}\Big|\lambda\right) = e^{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{r+km}}{(r+km)!} \quad r = 0,1,\ldots,m-1,$$

where θ is the observed angle, and λ is the rate parameter.

2.9.4 Wrapped Normal Distribution

The other well-known wrapped distribution is the wrapped normal distribution. Circular data, such as angles or directions, is often modeled by this distribution. Its PDF is given by (Mardia, and Jupp, 2001)

$$f_W(\theta|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \exp\left\{ -\frac{(2k\pi + \theta - \mu)^2}{2\sigma^2} \right\},$$

where θ is the observed angle, μ is the mean angle, and σ is the standard deviation.

2.9.5 Wrapped Cauchy Distribution

For the circular data that exhibit heavy-tailed behavior, the wrapped Cauchy distribution is often used to model. It has a PDF given by (Jammalamadaka, and Gupta, 2014)

$$f_W(\theta|\mu,\gamma) = \frac{1}{2\pi}\frac{1 - \gamma^2}{1 + \gamma^2 - 2\gamma\cos(\theta - \mu)},$$

where θ is the observed angle, μ is the mean angle, and γ is the concentration parameter.

2.9.6 Wrapped Exponential Distribution

The wrapped exponential distribution is a probability distribution commonly used to model cyclic or periodic phenomena. It finds applications in various fields, including the analysis of animal migration patterns, seasonal cycle modeling of waves in oceanography, and diagnosing daily cycles of pill intake in the human body.

This distribution is derived by mapping the exponential distribution onto a circular or periodic domain. To recall, the distribution of a random variable X is exponential if its PDF is given by:

$$f(x \mid \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

where X is a continuous random variable and $\lambda > 0$ is the rate parameter, representing the inverse of the average time between events.

Hence, the distribution of a random variable X follows a wrapped exponential distribution if its PDF is given by:

$$
\begin{aligned}
f_\Theta(\theta \mid \lambda) &= \sum_{k=-\infty}^{\infty} f_X(\theta + 2\pi k \mid \lambda), \theta \in [0, 2\pi) \\
&= \sum_{k=0}^{\infty} \lambda e^{-\lambda(\theta + 2\pi k)} \\
&= \lambda e^{-\lambda\theta} \sum_{k=0}^{\infty} e^{(-2\pi\lambda)k} \\
&= \frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}},
\end{aligned}
$$

where Θ is a wrapped random variable and λ is a parameter. The PDF of the wrapped exponential distribution is a decreasing function. Consequently, as the angle increases, the likelihood of observing a larger angle diminishes.

2.9.7   Hurwitz Zeta Function

The Hurwitz Zeta function can be seen as an extension of the Riemann Zeta function. It is named after the Austrian mathematician Adolf Hurwitz, who introduced it in the late 19th century. Its definition is as follows

$$
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},
$$

where

- a is a real number, $a \neq 0, -1, -2, \ldots$, and

- s is a complex number with real part greater than 1 ($\mathrm{Re}(s) > 1$).

However, the Hurwitz Zeta function can be also represented in the integral form as

$$
\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} dx.
$$

2.10   Related Research

Ravindran and Ghosh (2011) discusses the use of Bayesian methodology for analyzing circular data. They propose using wrapped distributions, which are probability distributions that are defined on a circle rather than a line, to model circular data. The context shows how the wrapped Cauchy distribution and the wrapped normal distribution can be used to model circular data. The performance of these models to traditional methods such as circular-linear regression and circular kernel density estimation was compared. Through a simulation study and the analysis of real data on bird migration patterns, the authors provide evidence for the effectiveness of their Bayesian methodology using wrapped distributions for the analysis of circular data. The article also establishes that this approach offers a flexible and efficient means of analyzing circular data, and is capable of producing more accurate results when compared to traditional methods.

Jacob and Jayakumar (2013) propose a new distribution called the wrapped geometric (WG) distribution for modeling circular data that takes values in the set of positive integers. The authors derive the probability mass function and cumulative distribution function of the WG and compare it to other circular distributions, such as the von Mises and wrapped Cauchy distributions. They demonstrate the usefulness of the WG through simulation studies and by applying it to real data sets related to animal movement and wind direction. The authors conclude that the WG is a useful tool for modeling circular count data that takes values in the set of positive integers, particularly in cases where other circular count distributions may not be appropriate.

Umehara, Okada, and Naruse (2018) propose a method for estimating time series of angles that are wrapped or unwrapped using sequential Bayesian filters. The authors consider both circular data, such as wind direction, and linear data, such as stock prices, and develop a framework that can handle both types of data. The authors also propose a method for estimating hyperparameters, such as the observation noise variance and the transition noise variance, using the maximum a posteriori estimator. They demonstrate the effectiveness of their approach through simulation studies and by applying it to real data sets related to wind direction and stock prices. The authors conclude that their method is

a powerful tool for estimating time series of angles that are wrapped or unwrapped, and it can be used in a wide range of applications, including weather forecasting and financial analysis.

Srihari, Girija, and Rao (2018a) introduce a new probability distribution called the discrete wrapped exponential (DWE) distribution for modeling circular data that is discrete in nature. The authors derive the probability mass function and cumulative distribution function of the DWE and compare it to other circular distributions, such as the wrapped Cauchy and von Mises distributions. They demonstrate the usefulness of the DWE through simulation studies and by applying it to real data sets related to wind direction and animal movement. The authors conclude that the DWE is a useful tool for modeling discrete circular data, particularly in cases where other circular distributions may not be appropriate. They also discuss some characteristics of the DWE, such as its moments and order statistics.

Srihari, Girija, and Rao (2018b) propose a new distribution called the wrapped negative binomial (WNB) distribution for modeling circular count data. The authors derive the probability mass function and cumulative distribution function of the WNB and compare it to other circular count distributions, such as the wrapped Poisson and the wrapped binomial. They demonstrate the usefulness of the WNB through simulation studies and by applying it to real data sets related to bird migration and crime patterns. The authors conclude that the WNB is a flexible and useful tool for modeling circular count data, particularly in cases where other circular count distributions may not be appropriate.

Malay, Xiaolong, Ashis, and Ruoyang (2019) propose a method for selecting non-subjective prior distributions for Bayesian analysis of wrapped Cauchy distributions. The authors demonstrate that traditional prior distributions, such as the Jeffreys prior, can be inappropriate for wrapped Cauchy distributions, and they propose a new class of priors based on the confluent hypergeometric function. The authors demonstrate the effectiveness of their approach through simulation studies and by applying it to real data sets related to animal movement and wind direction. The authors conclude that their method provides a useful tool for Bayesian analysis of wrapped Cauchy distributions, and it can be used in a wide range of applications, including ecology and meteorology.

Ignacio, Concha, and Pedro (2019) propose a method for classification of circular data using wrapped Cauchy distributions within a Bayesian framework. The authors demonstrate that traditional classifiers, such as linear discriminant analysis, are not appropriate for circular data, and they introduce the use of wrapped distributions to model circular data. They develop a Bayesian framework that uses the wrapped Cauchy distribution to model circular data and demonstrate the effectiveness of their approach through simulation studies and by applying it to real data sets related to wind direction and animal movement. The authors conclude that their method is a powerful tool for classification of circular data, and it can be used in a wide range of applications, including meteorology and animal behavior analysis.

Girija, Srihari, and Srinivas (2019) introduces a new probability distribution called the discrete wrapped Cauchy distribution (DWCD) for modeling circular data that is discrete in nature. The authors derive the probability density function and cumulative distribution function of the DWCD and compare it to other circular distributions, such as the wrapped Cauchy and von Mises distributions. They demonstrate the utility of the DWCD through simulation studies and by applying it to real data sets related to wind direction and animal movement. The authors conclude that the DWCD is a useful tool for modeling discrete circular data, particularly in cases where other circular distributions may not be appropriate.

Coelho-Barros, Achcar, Martinez, Davarzani, and Grabsch (2019) propose a Bayesian approach for analyzing data using the segmented Weibull distribution. The authors derive the probability density function and cumulative distribution function of the segmented Weibull distribution and develop a Bayesian framework for estimating the parameters of the distribution. They propose a non-informative prior distribution for the parameters and demonstrate how to use Markov chain Monte Carlo methods to estimate the posterior distribution of the parameters. The authors demonstrate the effectiveness of their approach through simulation studies and by applying it to real data sets related to cancer survival times and wind speed. The authors conclude that their approach provides a useful tool for modeling data with changing hazard rates, and it can be used in a wide range of applications, including engineering and health sciences.

Yu and González (2020) propose a Bayesian approach for estimating fault and volcano surface ground deformation models from wrapped satellite interferometric phase data. The authors use the wrapped Cauchy distribution to model the phase data and develop a Bayesian inversion algorithm to estimate the surface ground deformation models. The authors demonstrate the effectiveness of their approach through simulation studies and by applying it to real data sets related to the 2010 Chile earthquake and the 2011 eruption of the Puyehue-Cordón Caulle volcano in Chile. The authors conclude that their method is a powerful tool for estimating surface ground deformation models from wrapped satellite interferometric phase data, and it can be used in a wide range of applications, including earthquake and volcanic activity monitoring.

Bailey and Codling (2020) discusses the emergence of the wrapped Cauchy distribution in mixed directional data, which is data that contains a mixture of directional and non-directional components. The authors demonstrate that when directional data is mixed with non-directional data, the resulting distribution often exhibits characteristics of the wrapped Cauchy distribution. They develop a statistical framework for modeling mixed directional data using the wrapped Cauchy distribution and demonstrate its effectiveness through simulation studies and by applying it to real data sets related to animal movement and ocean currents. The authors conclude that their approach provides a useful tool for analyzing mixed directional data, and it can be used in a wide range of applications, including ecology and oceanography.

Okhli and Nooghabi (2020) propose a Bayesian approach for modeling positive-valued insurance claim data with outliers using the contaminated exponential distribution. The authors demonstrate that traditional distributions, such as the exponential and gamma distributions, can be inadequate for modeling insurance claim data with outliers, and they propose the use of the contaminated exponential distribution to model such data. They develop a Bayesian framework for estimating the parameters of the contaminated exponential distribution and demonstrate its effectiveness through simulation studies and by applying it to real insurance claim data. The authors conclude that this approach provides a useful tool for modeling insurance claim data with outliers and can be used in a wide range of applications in the insurance industry.

Chaudhary (2020) propose a Bayesian approach for analyzing data using the two-parameter exponentiated log-logistic (TELL) distribution. The author derives the probability density function, cumulative distribution function, and hazard function of the TELL distribution and discusses its various characteristics, such as its moments, skewness, and kurtosis. The author then develops a Bayesian framework for estimating the parameters of the TELL distribution, including the shape parameter, scale parameter, and exponentiation parameter, and demonstrates the effectiveness of their approach through simulation studies and by applying it to real data sets related to income and survival times. The author concludes that their approach provides a useful tool for modeling data with heavy tails and can be applied in a wide range of applications, including economics and health sciences.

Ghaderinezhad, Ley, and Loperfido (2020) proposes a Bayesian approach for analyzing data using skew-symmetric distributions, which are distributions that exhibit skewness in both positive and negative directions. The authors develop a general framework for modeling skew-symmetric distributions and demonstrate its usefulness by applying it to several specific distributions, including the skew normal, skew Student's t, and skew Laplace distributions. They discuss how to select appropriate prior distributions for the parameters of skew-symmetric distributions and develop a Bayesian inference algorithm for estimating the parameters. The authors demonstrate the effectiveness of their approach through simulation studies and by applying it to real data sets related to finance and economics. The authors conclude that their approach provides a powerful tool for analyzing data with skewness in both positive and negative directions, and it can be used in a wide range of applications.

Aslam and Feroze (2020) propose a Bayesian approach for analyzing data using a two-component mixture of Weibull distributions. The authors develop a likelihood function for the mixture distribution and propose an approximation technique based on the Laplace approximation and the saddlepoint approximation to estimate the posterior distribution of the parameters. They also develop a Bayesian model selection criterion based on the integrated completed likelihood to determine the number of components in the mixture distribution. The authors demonstrate the effectiveness of their approach

through simulation studies and by applying it to real data sets related to survival times and wind speed. The authors conclude that their approach provides a powerful tool for analyzing data using mixture models, and it can be used in a wide range of applications, including engineering and environmental sciences.
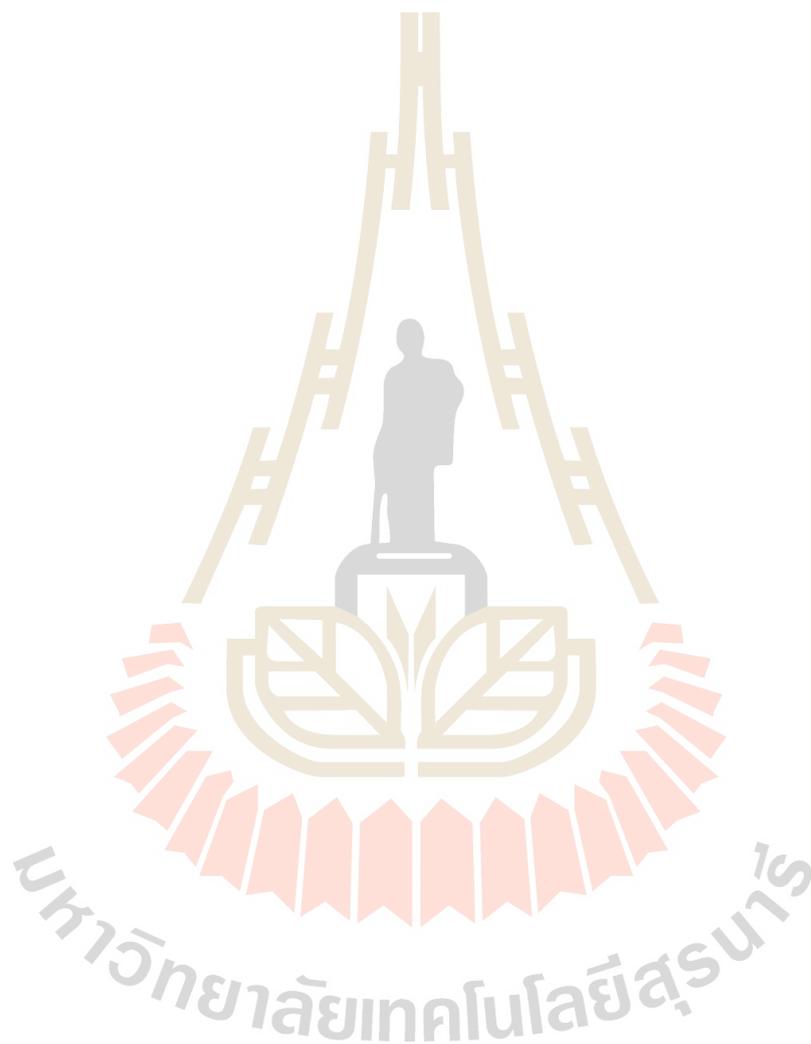
CHAPTER III

RESEARCH METHODOLOGY

3.1 The Wrapped Exponential Distribution with an Uninformative Prior Distribution

A scenario is considered in this study where the observed data are assumed to exhibit a specific pattern, prompting the adoption of an improper prior distribution for the parameter, characterized by the prior probability density function $h(\lambda) = 1$. This uninformative improper prior is selected to assign a constant density across all positive values of the parameter λ, reflecting a non-normalizable distribution that does not integrate to a finite value over an unbounded domain. Within the framework of the Wrapped Exponential Distribution, frequently applied to model circular or angular data such as directions or orientations, this improper prior is employed to represent a state of complete ignorance regarding the parameter λ. Consequently, the inference process is driven primarily by the observed data, ensuring that conclusions are not swayed by restrictive prior assumptions about the parameter's behavior.

Through Bayesian analysis of the wrapped exponential distribution with this prior distribution, we obtain the following posterior PDF:

$$
\begin{aligned}
k\left(\lambda|\theta\right) &= \frac{\left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)\cdot 1}{\int_0^\infty \left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)\cdot 1\, d\lambda} \\
&= \frac{\left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)}{\int_0^\infty \frac{\left(\frac{y}{2\pi}\right)^{2-1}e^{-\left(\frac{y}{2\pi}\right)\theta}}{1-e^{-y}}\frac{dy}{2\pi}} \quad (\text{let } y = 2\pi\lambda) \\
&= \frac{\left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)}{\frac{1}{(2\pi)^2}\int_0^\infty \frac{y^{2-1}e^{-\left(\frac{\theta}{2\pi}\right)y}}{1-e^{-y}}dy}.
\end{aligned}
$$

Since the integral term can be represented by the Hurwitz Zeta function, therefore

$$k\left(\lambda|\theta\right) = \frac{4\pi^2\lambda e^{-\lambda\theta}}{\left(1 - e^{-2\pi\lambda}\right)\zeta\left(2, \frac{\theta}{2\pi}\right)},$$

where $\Gamma(\cdot)$ is the gamma function and $\zeta(\cdot, \cdot)$ is the Hurwitz Zeta function. We say that the distribution characterized by this posterior PDF is *the posterior wrapped exponential distribution with an uninformative prior distribution* (PWEU distribution). Here $\theta \in [0, 2\pi)$ is the random variable of the wrapped exponential distribution representing the time between events and $\lambda > 0$ is the rate parameter of the exponential distribution determining the average number of events per unit time.
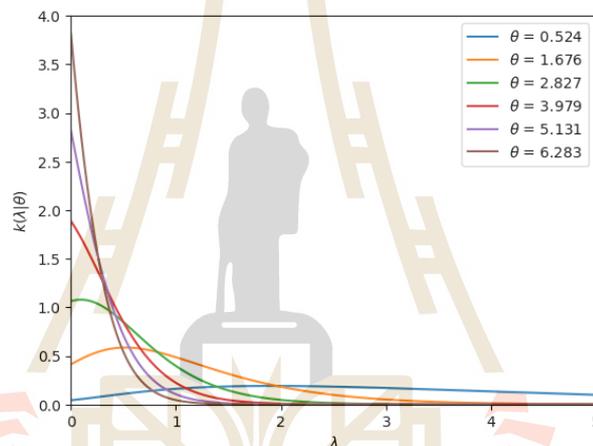
Figure 3.1 The curves of the posterior PDF of the PWEU distribution.

3.2 The Wrapped Exponential Distribution with the Gamma Prior Distribution

A case is examined where the observed data are assumed to follow a wrapped exponential distribution, and a gamma distribution is adopted as the prior for the parameter λ. The prior probability density function is given by

$$f\left(x \mid \alpha, \beta\right) = \frac{\beta^\alpha}{\Gamma\left(\alpha\right)}x^{\alpha-1}e^{-\beta x},$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter. This informative prior is selected to reflect prior knowledge about the rate parameter λ, with beliefs about

its magnitude and variability incorporated through the parameters α and β. Within the context of the Wrapped Exponential Distribution, commonly employed to model circular or angular data such as time intervals or orientations, the gamma prior is utilized to provide a flexible framework that balances prior assumptions with empirical evidence. Consequently, the Bayesian inference process is shaped by combining the influence of the observed data with the prior distribution, resulting in a posterior that reflects both sources of information.

The following presents the Bayesian analysis of the wrapped exponential distribution with the gamma prior distribution:

$$k(\lambda \mid \theta, \alpha, \beta) = \frac{\left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\right)}{\int_0^\infty \left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\right)d\lambda}$$

$$= \frac{\frac{\lambda^\alpha e^{-\lambda(\theta+\beta)}}{1-e^{-2\pi\lambda}}}{\int_0^\infty \frac{\lambda^{(\alpha+1)-1}e^{-\lambda(\theta+\beta)}}{1-e^{-2\pi\lambda}}d\lambda}$$

$$= \frac{\frac{\lambda^\alpha e^{-\lambda(\theta+\beta)}}{1-e^{-2\pi\lambda}}}{\int_0^\infty \frac{\left(\frac{y}{2\pi}\right)^{(\alpha+1)-1}e^{-\left(\frac{y}{2\pi}\right)(\theta+\beta)}}{1-e^{-y}}\frac{dy}{2\pi}} \qquad (\text{let } y = 2\pi\lambda)$$

$$= \frac{\frac{\lambda^\alpha e^{-\lambda(\theta+\beta)}}{1-e^{-2\pi\lambda}}}{\frac{1}{(2\pi)^{\alpha+1}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\int_0^\infty \frac{y^{(\alpha+1)-1}e^{-\left(\frac{\theta+\beta}{2\pi}\right)y}}{1-e^{-y}}dy}.$$

By the representing the integral term with the Hurwitz Zeta function, we obtain

$$k(\lambda \mid \theta, \alpha, \beta) = \frac{(2\pi)^{\alpha+1}\lambda^\alpha e^{-\lambda(\theta+\beta)}}{\Gamma(\alpha+1)\left(1-e^{-2\pi\lambda}\right)\zeta\left(\alpha+1, \frac{\theta+\beta}{2\pi}\right)},$$

where $\Gamma(\cdot)$ is the gamma function and $\zeta(\cdot,\cdot)$ is the Hurwitz Zeta function.

The posterior PDF for the wrapped exponential distribution with the gamma prior distribution as the PDF given by the following:

$$k_{\text{PWEG}}\left(\lambda|\theta,\alpha,\beta\right) = \frac{(2\pi)^{\alpha+1}\lambda^\alpha e^{-\lambda(\theta+\beta)}}{\Gamma(\alpha+1)\left(1-e^{-2\pi\lambda}\right)\zeta\left(\alpha+1, \frac{\theta+\beta}{2\pi}\right)}, \qquad \lambda, \alpha, \beta > 0, \theta \in [0, 2\pi),$$

where θ, α and β are the parameters defined in the previous section. To keep it simple, we refer to the posterior wrapped exponential distribution with the gamma prior distribution as PWEG distribution and its probability density function as PWEG PDF. For better clarity, graphs of the posterior PDF for the wrapped exponential distribution with various parameters are presented in Figure 3.2.
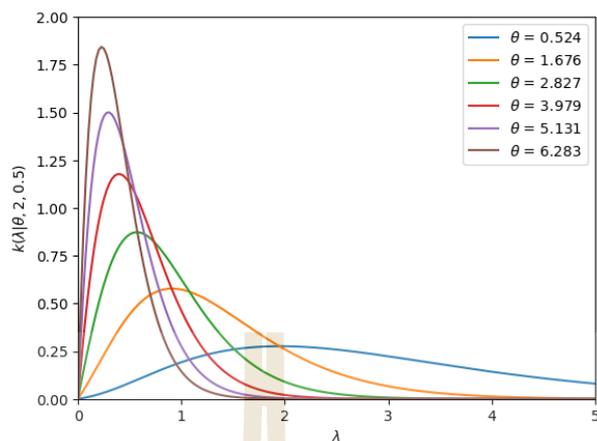
(a) The curves of the PWEG PDF with fixed values of $\alpha = 2$ and $\beta = 0.5$, and varying θ.
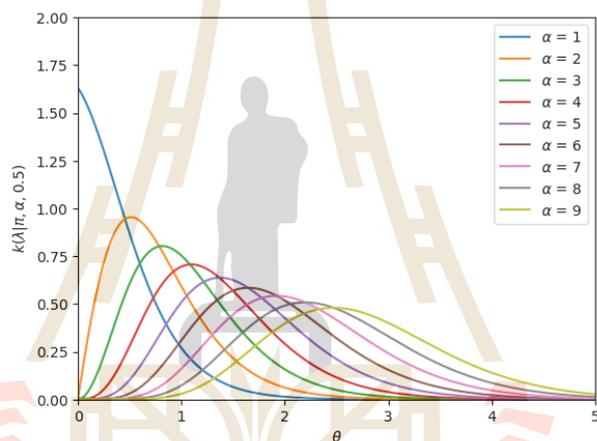
(b) The curves of the PWEG PDF with fixed values of $\theta = \pi$ and $\beta = 0.5$, and varying α.
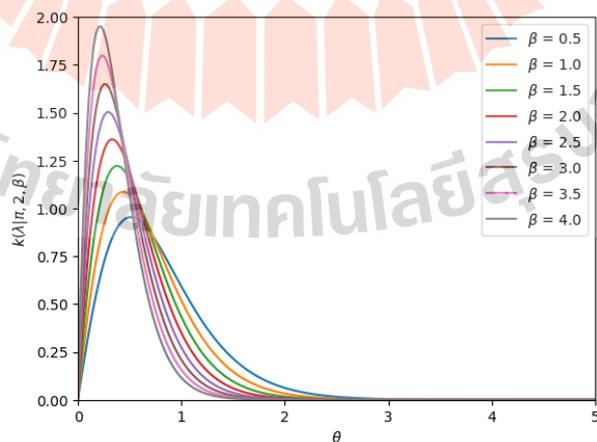
(c) The curves of the PWEG PDF with fixed values of $\theta = \pi$ and $\alpha = 2$, and varying β.

Figure 3.2 The curves of the PWEU PDF.

CHAPTER IV

RESULTS AND DISCUSSION

This chapter presents the findings obtained through the methodology outlined in Chapter III, Research Methodology. The primary objective of this section is to interpret and evaluate the results generated by the Python software developed as part of this study. The complete code implementation is provided in the Appendix for reference.

4.1 Characterization and Estimation of the PWEU Distribution

This section presents the moment generating function (MGF), mean, variance, skewness, kurtosis, Bayesian estimators, and risk minimization function for the PWEU distribution. The discussion covers key statistical measures that describe the distribution's characteristics and provides essential tools for estimation in practical applications.

4.1.1 Moment Generating Function and Statistical Measures

Specifically, let $k_{\mathsf{PWEU}}\left(\lambda|\theta\right)$ represent the posterior PDF of the PWEU distribution. Also, let $M_{\mathsf{PWEU}}, \mu_{\mathsf{PWEU}}, \sigma^2_{\mathsf{PWEU}}, \gamma_{\mathsf{PWEU}}$, and κ_{PWEU} represent the MGF, mean, variance, skewness, and kurtosis of the same distribution, respectively. For the sake of simplicity, the new function is introduced, $\Phi(a) = \dfrac{\zeta\left(a, \frac{\theta}{2\pi}\right)}{\zeta\left(2, \frac{\theta}{2\pi}\right)}$. The MGF and statistical measures of the PWEU distribution can be found as follows:

- MGF

$$
\begin{aligned}
M_{\mathsf{PWEU}}(t) &= E\left[e^{t\Lambda}|\theta\right] \\
&= \frac{4\pi^2}{\zeta\left(2,\frac{\theta}{2\pi}\right)} \int_0^\infty e^{t\lambda} \left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right) d\lambda \\
&= \frac{4\pi^2}{\zeta\left(2,\frac{\theta}{2\pi}\right)} \int_0^\infty \frac{\lambda^{2-1} e^{-(\theta-t)\lambda}}{1-e^{-2\pi\lambda}} \, d\lambda \\
&= \frac{4\pi^2}{\zeta\left(2,\frac{\theta}{2\pi}\right)} \int_0^\infty \frac{\left(\frac{u}{2\pi}\right)^{2-1} e^{-\left(\frac{\theta-t}{2\pi}\right)u}}{1-e^{-u}} \, \frac{du}{2\pi} \\
&= \frac{\Gamma(2)}{\zeta\left(2,\frac{\theta}{2\pi}\right)} \left(\frac{1}{\Gamma(2)} \int_0^\infty \frac{u^{2-1} e^{-\left(\frac{\theta-t}{2\pi}\right)u}}{1-e^{-u}} \, du\right) \\
&= \frac{\zeta\left(2,\frac{\theta-t}{2\pi}\right)}{\zeta\left(2,\frac{\theta}{2\pi}\right)}
\end{aligned}
$$

- Mean

$$
\mu_{\mathsf{PWEU}} = \mathbb{E}\left[\Lambda|\theta\right] = M'_{\mathsf{PWEU}}(0) = \frac{2\zeta\left(3,\frac{\theta}{2\pi}\right)}{2\pi\zeta\left(2,\frac{\theta}{2\pi}\right)} = \frac{1}{\pi}\Phi(3),
$$

where $\Lambda|_\theta$ is the wrapped random variable with parameter θ.

- Variance

$$
\begin{aligned}
\sigma_{\mathsf{PWEU}}^2 &= \mathbb{E}\left[(\Lambda|_\theta - \mu_{\mathsf{PWEU}})^2\right] \\
&= \mathbb{E}\left[(\Lambda|_\theta)^2\right] - \mu_{\mathsf{PWEU}}^2 \\
&= M''_{\mathsf{PWEU}}(0) - \mu_{\mathsf{PWEU}}^2 \\
&= \frac{6\zeta\left(4,\frac{\theta}{2\pi}\right)}{(2\pi)^2 \zeta\left(2,\frac{\theta}{2\pi}\right)} - \left(\frac{1}{\pi}\Phi(3)\right)^2 \\
&= \frac{1}{\pi^2}\left[\frac{3}{2}\Phi(4) - (\Phi(3))^2\right]
\end{aligned}
$$

• Skewness

$$
\begin{aligned}
\gamma_{\mathsf{PWEU}} &= \frac{\mathbb{E}\left[(\Lambda|_\theta - \mu_{\mathsf{PWEU}})^3\right]}{\sigma_{\mathsf{PWEU}}^3} \\
&= \frac{\mathbb{E}\left[(\Lambda|_\theta)^3\right] - 3\mu_{\mathsf{PWEU}}\sigma_{\mathsf{PWEU}}^2 - \mu_{\mathsf{PWEU}}^2}{\sigma_{\mathsf{PWEU}}^3} \\
&= \frac{M_{\mathsf{PWEU}}'''(0) - 3\mu_{\mathsf{PWEU}}\sigma_{\mathsf{PWEU}}^2 - \mu_{\mathsf{PWEU}}^2}{\sigma_{\mathsf{PWEU}}^3} \\
&= \frac{\frac{3}{\pi^3}\Phi(5) - 3\left(\frac{1}{\pi}\Phi(3)\right)\left(\frac{1}{\pi^2}\left[\frac{3}{2}\Phi(4) - (\Phi(3))^2\right]\right) - \left(\frac{1}{\pi}\Phi(3)\right)^3}{\left[\frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)\right]^{3/2}} \\
&= \frac{\frac{3}{\pi^3}\Phi(5) - \frac{3}{\pi^3}\left[\frac{3}{2}\Phi(3)\Phi(4) - (\Phi(3))^3\right] - \frac{1}{\pi^3}(\Phi(3))^3}{\left[\frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)\right]^{3/2}} \\
&= \frac{3\Phi(5) - \frac{9}{2}\Phi(3)\Phi(4) + 2(\Phi(3))^3}{\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)^{3/2}}
\end{aligned}
$$

• Kurtosis

$$
\begin{aligned}
\kappa_{\mathsf{PWEU}} &= \frac{\mathbb{E}\left[(\Lambda|_\theta - \mu_{\mathsf{PWEU}})^4\right]}{\sigma_{\mathsf{PWEU}}^4} \\
&= \frac{\mathbb{E}\left[(\Lambda|_\theta)^4\right] - 4\mu_{\mathsf{PWEU}}\mathbb{E}\left[(\Lambda|_\theta)^3\right] + 6\mu_{\mathsf{PWEU}}^2\sigma_{\mathsf{PWEU}}^2 + 3\mu_{\mathsf{PWEU}}^4}{(\sigma_{\mathsf{PWEU}}^2)^2} \\
&= \frac{M_{\mathsf{PWEU}}^{(4)}(0) - 4\mu_{\mathsf{PWEU}}M_{\mathsf{PWEU}}'''(0) + 6\mu_{\mathsf{PWEU}}^2\sigma_{\mathsf{PWEU}}^2 + 3\mu_{\mathsf{PWEU}}^4}{(\sigma_{\mathsf{PWEU}}^2)^2} \\
&= \frac{\frac{15}{2\pi^4}\Phi(6) - 4\left(\frac{1}{\pi}\Phi(3)\right)\left(\frac{3}{\pi^3}\Phi(5)\right) + 6\left(\frac{1}{\pi}\Phi(3)\right)^2\left(\frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)\right) + 3\left(\frac{1}{\pi}\Phi(3)\right)^4}{\left[\frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)\right]^2} \\
&= \frac{\frac{15}{2\pi^4}\Phi(6) - \frac{12}{\pi^4}\Phi(3)\Phi(5) + \frac{6}{\pi^4}\left((\Phi(3))^2\Phi(4) - (\Phi(3))^4\right) + \frac{3}{\pi^4}(\Phi(3))^4}{\frac{1}{\pi^4}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)^2} \\
&= \frac{\frac{15}{2}\Phi(6) - 12\Phi(3)\Phi(5) + 6(\Phi(3))^2\Phi(4) - 3(\Phi(3))^4}{\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)^2}.
\end{aligned}
$$
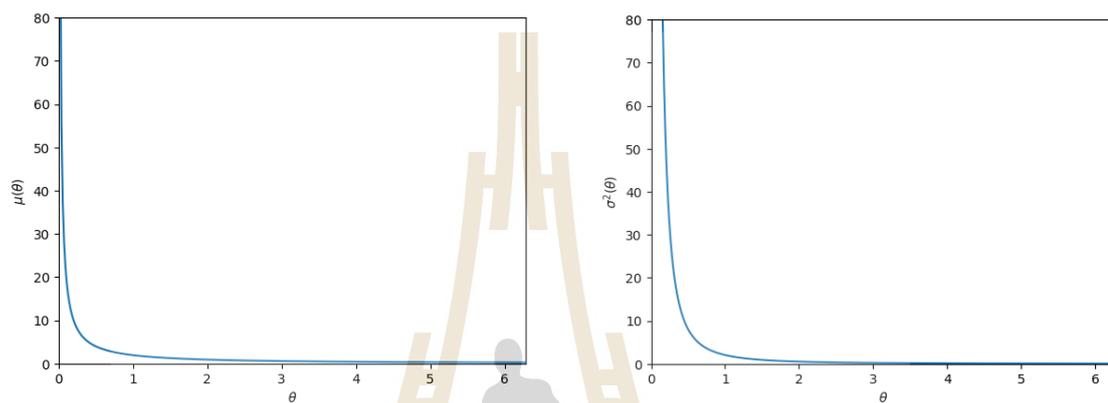
(a) The curve of mean for the PWEU distribution.

(b) The curve of variance for the PWEU distribution.
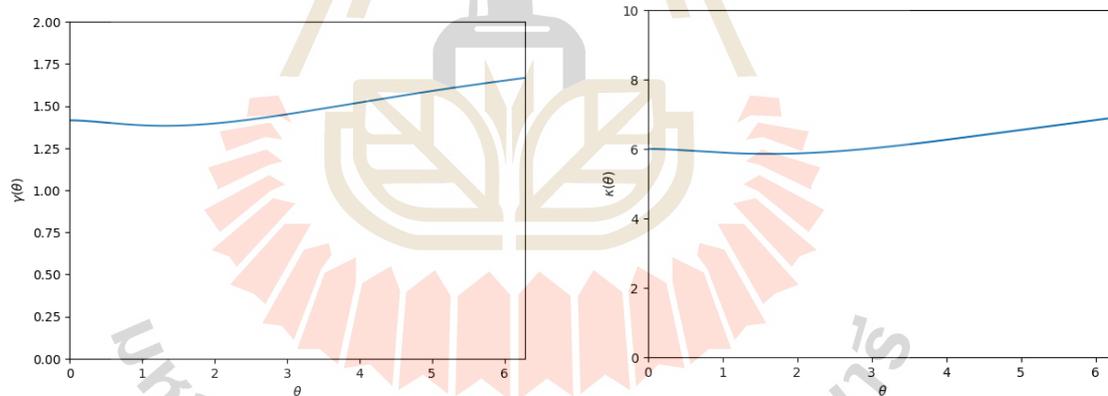
(c) The curve of skewness for the PWEU distribution.

(d) The curve of kurtosis for the PWEU distribution.

Figure 4.1 The curves of statistical measures for the PWEU distribution.

4.1.2 Bayesian Estimators and Risk Functions for the PWEU distribution

In this section, we present Bayesian estimators and risk minimization functions under loss functions of squared error type and precautionary type for the PWEU distribution. These functions are crucial for estimating parameter values based on empirical data, which has a wrapped random component.

- **Squared Error Loss Function**

  For the squared error loss function (SEL function), represented as $\ell\left(\vartheta, \hat{\vartheta}\right) = \left(\vartheta - \hat{\vartheta}\right)^2$, where ϑ denotes the estimate and $\hat{\vartheta}$ represents the true value, the corresponding Bayesian estimator and risk minimization function are outlined below:

  - Bayesian Estimator (BE)

    The derivation of the Bayesian Estimator under the Squared Error Loss Function begins by minimizing the expected loss:

    $$\mathbb{E}\left[\left(\Lambda|_\theta - \hat{\lambda}\right)^2\right].$$

    To find the estimator $\hat{\lambda}$ that minimizes this expression, we take the derivative with respect to $\hat{\lambda}$ and set it equal to zero:

    $$\frac{d}{d\hat{\lambda}}\mathbb{E}\left[\left(\Lambda|_\theta - \hat{\lambda}\right)^2\right] = -2\mathbb{E}\left[\Lambda|_\theta\right] + 2\hat{\lambda} = 0.$$

    Solving this equation:
    $$2\hat{\lambda} = 2\mathbb{E}\left[\Lambda|_\theta\right],$$
    $$\hat{\lambda} = \mathbb{E}\left[\Lambda|_\theta\right].$$

    Thus, the Bayesian Estimator under SEL is the expected value of $\Lambda|_\theta$. For the PWEU distribution, this expectation is given by:

    $$\hat{\lambda}_{\text{SEL}} = \mathbb{E}\left[\Lambda|_\theta\right] = \mu_{\text{PWEU}} = \frac{1}{\pi}\Phi(3).$$

– Risk Minimization Function (RMF)

$$R\left(\lambda, \hat{\lambda}_{\text{SEL}}\right) = \mathbb{E}\left[\ell\left(\Lambda|_\theta, \hat{\lambda}_{\text{SEL}}\right)\right]$$

$$= \mathbb{E}\left[\left(\Lambda|_\theta - \hat{\lambda}_{\text{SEL}}\right)^2\right]$$

$$= \sigma^2_{\text{PWEU}}$$

$$= \frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right).$$
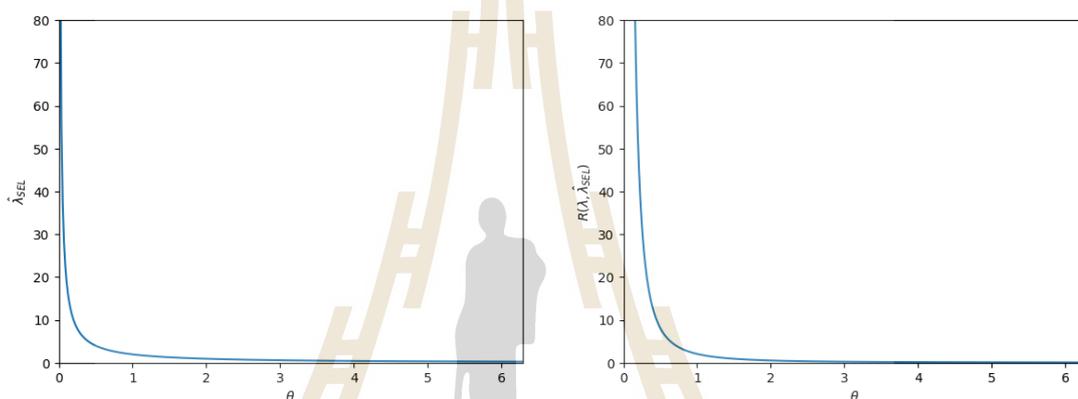
(a) The curve of BE under the SEL function.　　(b) The curve of RMF under the SEL function.

Figure 4.2 The curves of Bayesian estimator and risk minimization function for the PWEU distribution.

• **Precautionary Loss Function**

For the precautionary loss function (PL function) $\ell\left(\vartheta, \hat{\vartheta}\right) = \frac{\left(\vartheta - \hat{\vartheta}\right)^2}{\hat{\vartheta}}$, its Bayesian estimator and risk minimization function are as follows:

– Bayesian Estimator

The derivation of the Bayesian Estimator under the Precautionary Loss Function begins by minimizing the expected loss:

$$\mathbb{E}\left[\ell\left(\Lambda|_\theta, \hat{\lambda}\right)\right] = \mathbb{E}\left[\frac{\left(\Lambda|_\theta - \hat{\lambda}\right)^2}{\hat{\lambda}}\right].$$

To find the estimator $\hat{\lambda}$ that minimizes this expression, we take the derivative

with respect to $\hat{\lambda}$ and set it equal to zero:

$$\frac{d}{d\hat{\lambda}}\mathbb{E}\left[\frac{\left(\Lambda|_\theta - \hat{\lambda}\right)^2}{\hat{\lambda}}\right] = 0.$$

Expanding the expected loss:

$$\mathbb{E}\left[\frac{\left(\Lambda|_\theta - \hat{\lambda}\right)^2}{\hat{\lambda}}\right] = \mathbb{E}\left[\frac{\Lambda|_\theta^2 - 2\hat{\lambda}\Lambda|_\theta + \hat{\lambda}^2}{\hat{\lambda}}\right] = \frac{\mathbb{E}\left[\Lambda|_\theta^2\right]}{\hat{\lambda}} - 2\mathbb{E}\left[\Lambda|_\theta\right] + \hat{\lambda}.$$

Taking the derivative:

$$\frac{d}{d\hat{\lambda}}\left(\frac{\mathbb{E}\left[\Lambda|_\theta^2\right]}{\hat{\lambda}} - 2\mathbb{E}\left[\Lambda|_\theta\right] + \hat{\lambda}\right) = -\frac{\mathbb{E}\left[\Lambda|_\theta^2\right]}{\hat{\lambda}^2} + 1 = 0.$$

Solving this equation:

$$-\frac{\mathbb{E}\left[\Lambda|_\theta^2\right]}{\hat{\lambda}^2} + 1 = 0,$$

$$\frac{\mathbb{E}\left[\Lambda|_\theta^2\right]}{\hat{\lambda}^2} = 1,$$

$$\hat{\lambda}^2 = \mathbb{E}\left[\Lambda|_\theta^2\right],$$

$$\hat{\lambda} = \sqrt{\mathbb{E}\left[\Lambda|_\theta^2\right]}.$$

Thus, the Bayesian Estimator under PL is the square root of the second moment of $\Lambda|_\theta$. For the PWEU distribution, this is given by:

$$\hat{\lambda}_{\text{PL}} = \sqrt{\mathbb{E}\left[(\Lambda|_\theta)^2\right]} = \sqrt{\frac{3}{2\pi^2}\Phi(4)} = \frac{1}{2\pi}\sqrt{6\Phi(4)}.$$

– Risk Minimization Function

$$R\left(\lambda, \hat{\lambda}_{\text{PL}}\right) = \mathbb{E}\left[\ell\left(\Lambda|_\theta, \hat{\lambda}_{\text{PL}}\right)\right]$$

$$= \mathbb{E}\left[\frac{\left(\Lambda|_\theta - \hat{\lambda}_{\text{PL}}\right)^2}{\hat{\lambda}_{\text{PL}}}\right]$$

$$= 2\left(\left(\frac{3}{2\pi^2}\Phi(4)\right)^{3/2} - \frac{1}{\pi}\Phi(3)\right).$$
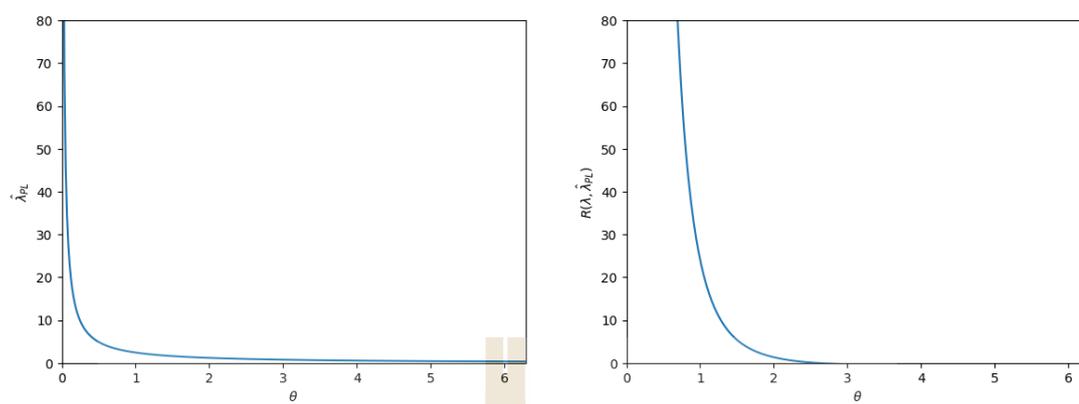
(a) The curve of BE under the PL function.　　(b) The curve of RMF under the PL function.

Figure 4.3 The curves of Bayesian estimator and risk minimization function for the PWEU distribution.

4.2　Characterization and Estimation of the PWEG Distribution

The following section delves into the characterization and estimation of the PWEG distribution, focusing on its moment generating function (MGF) and key statistical measures such as mean, variance, skewness, and kurtosis. Mathematical formulations are derived using special functions like the Hurwitz zeta function. Additionally, Bayesian estimators for parameter estimation, using squared error loss and precautionary loss functions, are examined. Graphical representations of these estimators are provided to illustrate their sensitivity to changes in parameters α and β.

4.2.1　Moment Generating Function and Statistical Measures

This section presents MGF, mean, variance, skewness, and kurtosis of the PWEG distribution, denoted as $M_{\text{PWEG}}, \mu_{\text{PWEG}}, \sigma^2_{\text{PWEG}}, \gamma_{\text{PWEG}}$, and κ_{PWEG}, respectively. Let function

$$\Psi(b) = \frac{\zeta\left(\alpha + b, \frac{\theta+\beta}{2\pi}\right)}{\zeta\left(\alpha + 1, \frac{\theta+\beta}{2\pi}\right)},$$

be introduced, therefore

- MGF

$$M_{\text{PWEG}}(t) = E\left[e^{t\Lambda}|_{\theta,\alpha,\beta}\right]$$

$$= \frac{(2\pi)^{\alpha+1}}{\Gamma(\alpha+1)\zeta\left(\alpha+1,\frac{\theta+\beta}{2\pi}\right)} \int_0^\infty e^{t\lambda}\left(\frac{\lambda^\alpha e^{-\lambda(\theta+\beta)}}{(1-e^{-2\pi\lambda})}\right) d\lambda$$

$$= \frac{(2\pi)^{\alpha+1}}{\Gamma(\alpha+1)\zeta\left(\alpha+1,\frac{\theta+\beta}{2\pi}\right)} \int_0^\infty \frac{\lambda^{(\alpha+1)-1}e^{-(\theta+\beta-t)\lambda}}{(1-e^{-2\pi\lambda})} d\lambda$$

$$= \frac{(2\pi)^{\alpha+1}}{\Gamma(\alpha+1)\zeta\left(\alpha+1,\frac{\theta+\beta}{2\pi}\right)} \int_0^\infty \frac{\left(\frac{u}{2\pi}\right)^{(\alpha+1)-1}e^{-\left(\frac{\theta+\beta-t}{2\pi}\right)u}}{1-e^{-u}} \frac{du}{2\pi}$$

$$= \frac{1}{\zeta\left(\alpha+1,\frac{\theta+\beta}{2\pi}\right)}\left(\frac{1}{\Gamma(\alpha+1)} \int_0^\infty \frac{u^{(\alpha+1)-1}e^{-\left(\frac{\theta+\beta-t}{2\pi}\right)u}}{1-e^{-u}} du\right)$$

$$= \frac{\zeta\left(\alpha+1,\frac{\theta+\beta-t}{2\pi}\right)}{\zeta\left(\alpha+1,\frac{\theta+\beta}{2\pi}\right)}$$

- Mean

$$\mu_{\text{PWEG}} = \mathbb{E}[\Lambda|_{\theta,\alpha,\beta}] = M'_{\text{PWEG}}(0) = \frac{(\alpha+1)\zeta\left(\alpha+2,\frac{\theta+\beta}{2\pi}\right)}{2\pi\zeta\left(\alpha+1,\frac{\theta+\beta}{2\pi}\right)} = \frac{\alpha+1}{2\pi}\Psi(2),$$

where $\Lambda|_{\theta,\alpha,\beta}$ is the wrapped random variable with parameters θ, α and β.
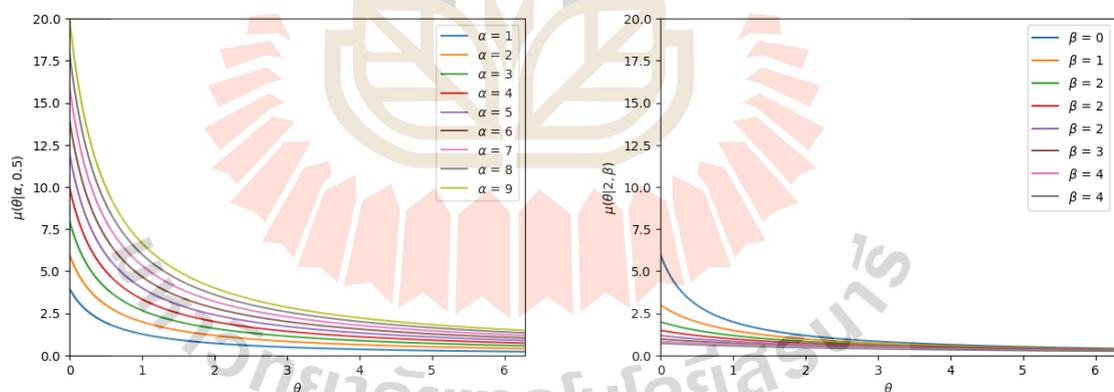
(a) The curves of mean with fixed values of $\beta = 0.5$ and and varying α.

(b) The curves of mean with fixed values of $\alpha = 2$ and and varying β.

Figure 4.4 The curves of mean for the PWEG distribution.

- Variance

$$\sigma^2_{\text{PWEG}} = \mathbb{E}\left[(\Lambda|_{\theta,\alpha,\beta} - \mu_{\text{PWEG}})^2\right]$$

$$= \mathbb{E}\left[(\Lambda|_{\theta,\alpha,\beta})^2\right] - \mu^2_{\text{PWEG}}$$

$$= M''_{\text{PWEG}}(0) - \mu^2_{\text{PWEG}}$$

$$= \frac{(\alpha+1)(\alpha+2)\zeta\left(\alpha+3,\frac{\theta+\beta}{2\pi}\right)}{(2\pi)^2\zeta\left(\alpha+1,\frac{\theta+\beta}{2\pi}\right)} - \left(\frac{\alpha+1}{2\pi}\Psi(2)\right)^2$$

$$= \frac{(\alpha+1)(\alpha+2)}{4\pi^2}\Psi(3) - \left(\frac{\alpha+1}{2\pi}\Psi(2)\right)^2$$

$$= \frac{\alpha+1}{4\pi^2}\left((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2\right)$$

(a) The curves of variance with fixed values of $\beta = 0.5$ and and varying α.

(b) The curves of mean with fixed values of $\alpha = 2$ and and varying β.

Figure 4.5 The curves of variance for the PWEG distribution.

- Skewness

γ_{PWEG}

$$= \frac{\mathbb{E}\left[\left(\Lambda|_{\theta,\alpha,\beta} - \mu_{\text{PWEG}}\right)^3\right]}{\sigma_{\text{PWEG}}^3}$$

$$= \frac{\mathbb{E}\left[\left(\Lambda|_{\theta,\alpha,\beta}\right)^3\right] - 3\mu_{\text{PWEG}}\sigma_{\text{PWEG}}^2 - \mu_{\text{PWEG}}^2}{\sigma_{\text{PWEG}}^3}$$

$$= \frac{M_{\text{PWEG}}'''(0) - 3\mu_{\text{PWEG}}\sigma_{\text{PWEG}}^2 - \mu_{\text{PWEG}}^2}{\sigma_{\text{PWEG}}^3}$$

$$= \frac{\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{8\pi^3}\Psi(4) - 3\left(\frac{\alpha+1}{2\pi}\Psi(2)\right)\left(\frac{\alpha+1}{4\pi^2}\left((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2\right)\right) - \left(\frac{\alpha+1}{2\pi}\Psi(2)\right)^3}{\left(\frac{\alpha+1}{4\pi^2}\left((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2\right)\right)^{3/2}}$$

$$= \frac{\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{8\pi^3}\Psi(4) - \frac{3(\alpha+1)^2}{8\pi^3}\left((\alpha+2)\Psi(3)\Psi(2) - (\alpha+1)(\Psi(2))^3\right) - \frac{(\alpha+1)^3}{8\pi^3}(\Psi(2))^3}{\left(\frac{\alpha+1}{4\pi^2}\left((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2\right)\right)^{3/2}}$$

$$= \frac{(\alpha+1)(\alpha+2)(\alpha+3)\Psi(4) - 3(\alpha+1)^2\left((\alpha+2)\Psi(3)\Psi(2) - (\alpha+1)(\Psi(2))^3\right) - (\alpha+1)^3(\Psi(2))^3}{\left((\alpha+1)(\alpha+2)\Psi(3) - (\alpha+1)^2(\Psi(2))^2\right)^{3/2}}$$
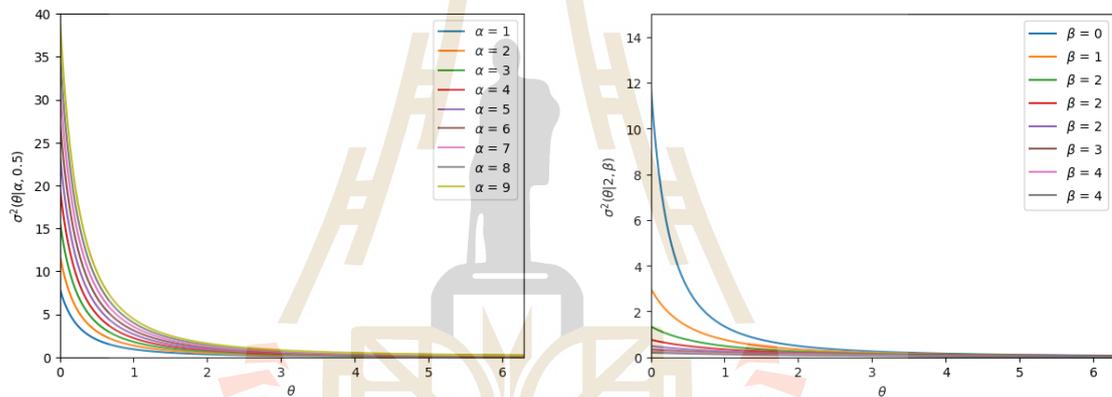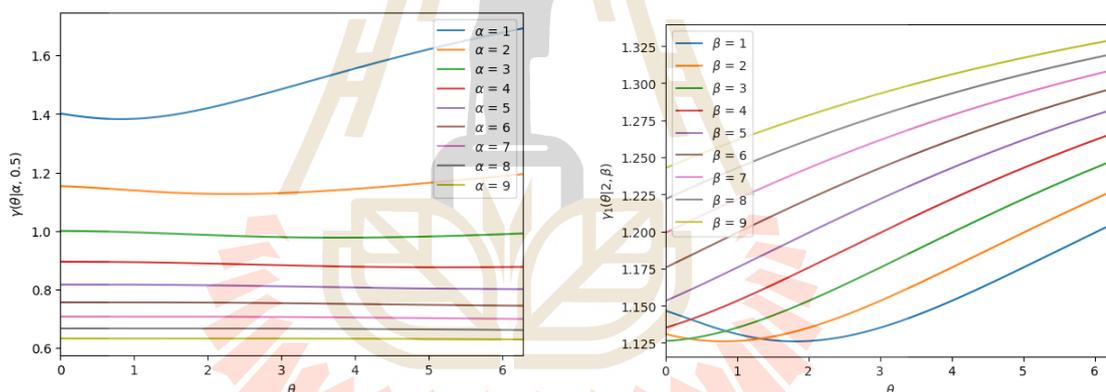

(a) The curves of skewness with fixed values of $\beta = 0.5$ and and varying α.

(b) The curves skewness with fixed values of $\alpha = 2$ and and varying β.

Figure 4.6 The curves of skewness for the PWEG distribution.

- Kurtosis

κ_{PWEG}

$$= \frac{\mathbb{E}\left[(\Lambda|_{\theta,\alpha,\beta} - \mu_{\mathsf{PWEG}})^4\right]}{\sigma_{\mathsf{PWEG}}^4}$$

$$= \frac{\mathbb{E}\left[(\Lambda|_{\theta,\alpha,\beta})^4\right] - 4\mu_{\mathsf{PWEG}}\mathbb{E}\left[(\Lambda|_{\theta,\alpha,\beta})^3\right] + 6\mu_{\mathsf{PWEG}}^2\sigma_{\mathsf{PWEG}}^2 + 3\mu_{\mathsf{PWEG}}^4}{(\sigma_{\mathsf{PWEG}}^2)^2}$$

$$= \frac{M_{\mathsf{PWEG}}^{(4)}(0) - 4\mu_{\mathsf{PWEG}}M_{\mathsf{PWEG}}'''(0) + 6\mu_{\mathsf{PWEG}}^2\sigma_{\mathsf{PWEG}}^2 + 3\mu_{\mathsf{PWEG}}^4}{(\sigma_{\mathsf{PWEG}}^2)^2}$$

$$= \frac{\frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{(2\pi)^4}\Psi(5) - 4\left(\frac{\alpha+1}{2\pi}\Psi(2)\right)\left(\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{8\pi^3}\Psi(4)\right) + 6\left(\frac{\alpha+1}{2\pi}\Psi(2)\right)^2\left(\frac{\alpha+1}{4\pi^2}\left((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2\right)\right) + 3\left(\frac{\alpha+1}{2\pi}\Psi(2)\right)^4}{\left(\frac{\alpha+1}{4\pi^2}((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2)\right)^2}$$

$$= \frac{\frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{16\pi^4}\Psi(5) - \frac{4(\alpha+1)^2(\alpha+2)(\alpha+3)}{16\pi^4}\Psi(2)\Psi(4) + \frac{6(\alpha+1)^3}{16\pi^4}\left((\alpha+2)(\Psi(2))^2\Psi(3) - (\alpha+1)(\Psi(2))^4\right) + \frac{3(\alpha+1)^4}{16\pi^4}(\Psi(2))^4}{\frac{(\alpha+1)^4}{16\pi^4}\left((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2\right)^2}$$

$$= \frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\Psi(5) - 4(\alpha+1)^2(\alpha+2)(\alpha+3)\Psi(2)\Psi(4) + 6(\alpha+1)^3\left((\alpha+2)(\Psi(2))^2\Psi(3) - (\alpha+1)(\Psi(2))^4\right) + 3(\alpha+1)^4(\Psi(2))^4}{\left((\alpha+1)^2(\alpha+2)\Psi(3) - (\alpha+1)^3(\Psi(2))^2\right)^2}$$
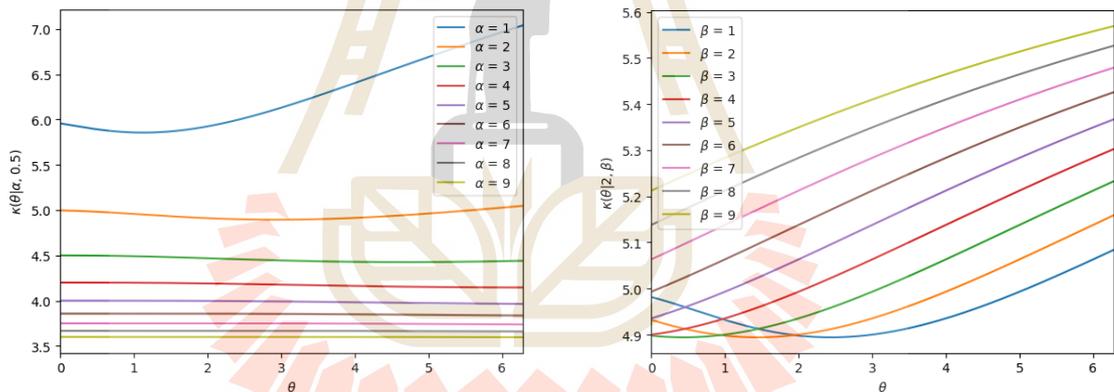

(a) The curves of kurtosis with fixed values of $\beta = 0.5$ and and varying α.

(b) The curves kurtosis with fixed values of $\alpha = 2$ and and varying β.

Figure 4.7 The curves of kurtosis for the PWEG distribution.

4.2.2 Bayesian Estimators and Risk Functions for the PWEG Distribution

In this section, Bayesian estimators and risk minimization functions under loss functions of squared error type and precautionary type for the PWEG distribution are presented.

- **Squared Error Loss Function**

  For the SEL function, the corresponding Bayesian estimator and risk minimization function are as follows:

  – Bayesian Estimator

  $$\hat{\lambda}_{\mathsf{SEL}} = \mathbb{E}\left[\Lambda|_{\theta,\alpha,\beta}\right] = \frac{\alpha+1}{2\pi}\Psi(2),$$
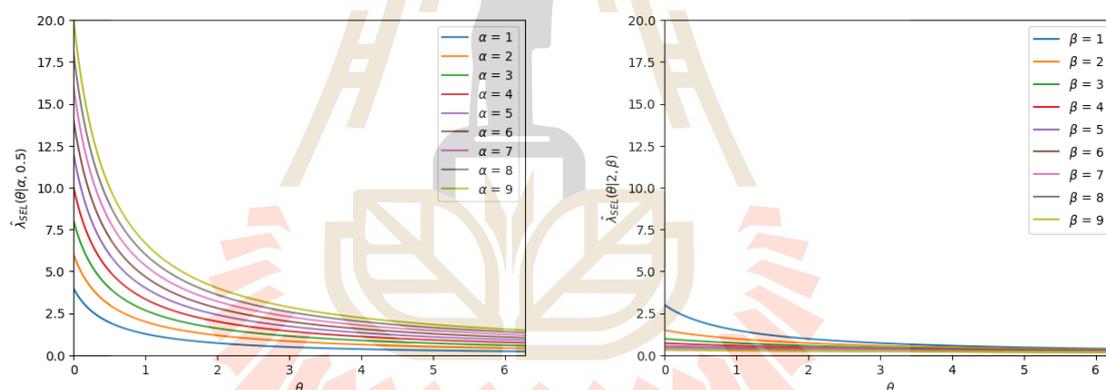
(a) The curves of BE with fixed values of $\beta = 0.5$ and and varying α under the SEL function.

(b) The curves BE with fixed values of $\alpha = 2$ and and varying β under the SEL function.

Figure 4.8 The curves of Bayesian estimator (under the SEL function) for the PWEG distribution.

  – Minimizes Risk Function

  $$R\left(\lambda, \hat{\lambda}_{\mathsf{SEL}}\right) = \mathbb{E}\left[\left(\Lambda|_{\theta,\alpha,\beta} - \hat{\lambda}_{\mathsf{SEL}}\right)^2\right]$$
  $$= \frac{\alpha+1}{4\pi^2}\left((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2\right).$$

(a) The curves of RMF with fixed values of $\beta = 0.5$ and and varying α under the SEL function.

(b) The curves RMF with fixed values of $\alpha = 2$ and and varying β under the SEL function.

Figure 4.9 The curves of risk minimization function (under the SEL function) for the PWEG distribution.

- **Precautionary Loss Function**

  The corresponding Bayesian estimator and risk minimization function for PL function are as follows:

  – Bayesian Estimator

  $$\hat{\lambda}_{\text{PL}} = \sqrt{\mathbb{E}\left[(\Lambda|_{\theta,\alpha,\beta})^2\right]} = \sqrt{\frac{(\alpha+1)(\alpha+2)}{4\pi^2}\Psi(3)},$$

  – Minimizes Risk Function

  $$R\left(\lambda, \hat{\lambda}_{\text{PL}}\right) = \mathbb{E}\left[\frac{\left(\Lambda|_{\theta,\alpha,\beta} - \hat{\lambda}_{\text{PL}}\right)^2}{\hat{\lambda}_{\text{PL}}}\right]$$

  $$= 2\left(\left(\frac{(\alpha+1)(\alpha+2)}{4\pi^2}\Psi(3)\right)^{3/2} - \frac{\alpha+1}{2\pi}\Psi(2)\right).$$
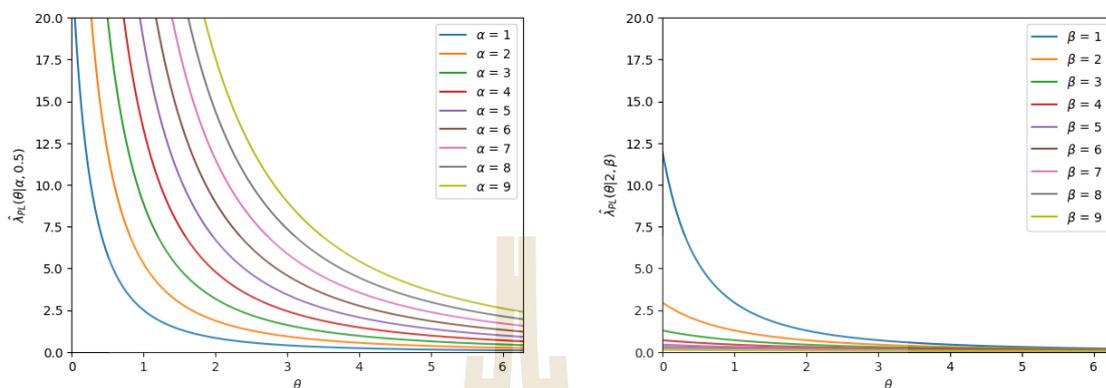
(a) The curves of BE with fixed values of $\beta = 0.5$ and and varying α under the PL function.

(b) The curves BE with fixed values of $\alpha = 2$ and and varying β under the PL function.

Figure 4.10 The curves of Bayesian estimator (under the PL function) for the PWEG distribution.
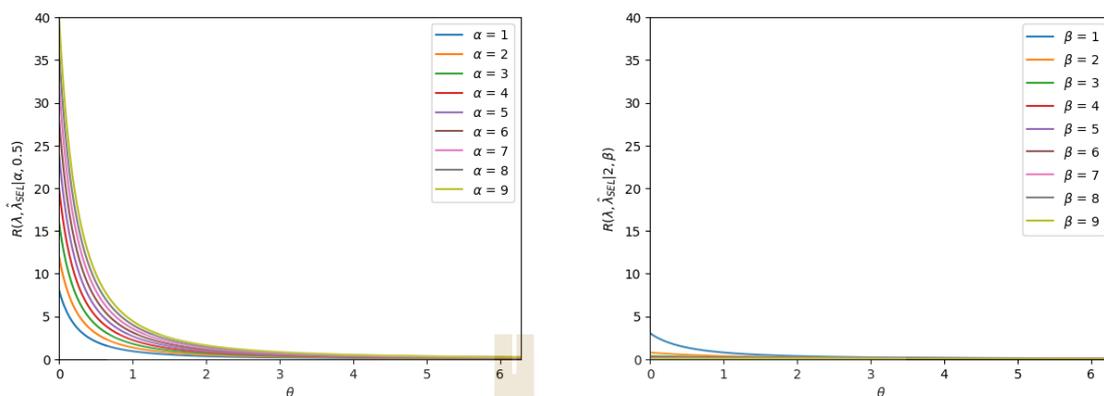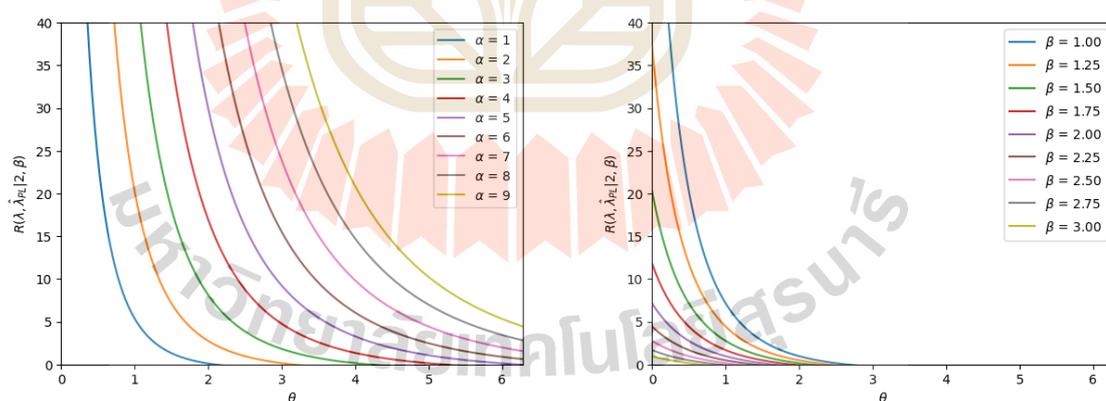

(a) The curves of RMF with fixed values of $\beta = 0.5$ and and varying α under the PL function.

(b) The curves RMF with fixed values of $\alpha = 2$ and and varying β under the PL function.

Figure 4.11 The curves of risk minimization function (under the PL function) for the PWEG distribution.

Table 4.1 Table of Properties of PWEU and PWEG Distributions.

Property	PWEU Distribution	PWEG Distribution
Support	$\lambda > 0$	$\lambda > 0$
Parameter	$\theta \in [0, 2\pi)$	$\theta \in [0, 2\pi),\, \alpha, \beta > 0$
PDF	$\dfrac{4\pi^2\lambda e^{-\lambda\theta}}{(1-e^{-2\pi\lambda})}\zeta\left(2, \frac{\theta}{2\pi}\right)$	$\dfrac{(2\pi)^{\alpha+1}\lambda^\alpha e^{-\lambda(\theta+\beta)}}{\Gamma(\alpha+1)(1-e^{-2\pi\lambda})}\zeta\left(\alpha+1, \frac{\theta+\beta}{2\pi}\right)$
MGF	$\dfrac{\zeta\left(2, \frac{\theta-t}{2\pi}\right)}{\zeta\left(2, \frac{\theta}{2\pi}\right)}$	$\dfrac{\zeta\left(\alpha+1, \frac{\theta+\beta-t}{2\pi}\right)}{\zeta\left(\alpha+1, \frac{\theta+\beta}{2\pi}\right)}$
Mean	$\dfrac{1}{\pi}\Phi(3)$	$\dfrac{\alpha+1}{2\pi}\Psi(2)$
Variance	$\dfrac{1}{\pi^2}\left[\frac{3}{2}\Phi(4) - (\Phi(3))^2\right]$	$\dfrac{\alpha+1}{4\pi^2}\left((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2\right)$
skewness	$\dfrac{3\Phi(5)-\frac{9}{2}\Phi(3)\Phi(4)+2(\Phi(3))^3}{\left(\frac{3}{2}\Phi(4)-(\Phi(3))^2\right)^{3/2}}$	$\dfrac{(\alpha+1)(\alpha+2)(\alpha+3)\Psi(4) -3(\alpha+1)^2\left((\alpha+2)\Psi(3)\Psi(2) - (\alpha+1)(\Psi(2))^3\right) - (\alpha+1)^3(\Psi(2))^3}{\left((\alpha+1)(\alpha+2)\Psi(3)-(\alpha+1)^2(\Psi(2))^2\right)^{3/2}}$
kurtosis	$\dfrac{\frac{15}{2}\Phi(6)-12\Phi(3)\Phi(5)+6(\Phi(3))^2\Phi(4)-3(\Phi(3))^4}{\left(\frac{3}{2}\Phi(4)-(\Phi(3))^2\right)^2}$	$\dfrac{\begin{array}{l}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\Psi(5)-4(\alpha+1)^2(\alpha+2)(\alpha+3)\Psi(2)\Psi(4)\\ +6(\alpha+1)^3\left((\alpha+2)(\Psi(2))^2\Psi(3)-(\alpha+1)(\Psi(2))^4\right)+3(\alpha+1)^4(\Psi(2))^4\end{array}}{\left((\alpha+1)^2(\alpha+2)\Psi(3)-(\alpha+1)^3(\Psi(2))^2\right)^2}$

CHAPTER V

CONCLUSION

This study extends Bayesian analysis to wrapped distributions, specifically illustrating the posterior wrapped exponential distribution. By examining both uninformative and gamma prior distributions, closed-form expressions for key statistical measures including the moment generating function (MGF), mean, variance, skewness, and kurtosis are derived in terms of the Hurwitz Zeta function. Graphical representations further elucidate these properties, enhancing practical interpretability. Additionally, the study investigates essential loss functions such as the Squared Error Loss Function (SEL function) and the Precautionary Loss Function (PL function), alongside Bayesian estimators and risk minimization techniques for parameter estimation.

The applicability of the posterior wrapped distribution is demonstrated through real-world scenarios where periodicity and non-constant failure rates are prominent. The wrapped exponential distribution with an uninformative prior effectively models failure patterns in hard disk drives (HDDs), while the gamma prior variant captures cyclical service time distributions in queueing networks within interactive computing systems. These applications underscore the model's flexibility in representing periodic stochastic processes.

Future research may extend this framework by exploring alternative prior distributions tailored to specific applications. Additionally, other loss functions such as Absolute Error Loss, Linex Loss, and Hinge Loss could be investigated to refine Bayesian estimation techniques and improve model robustness in diverse settings.
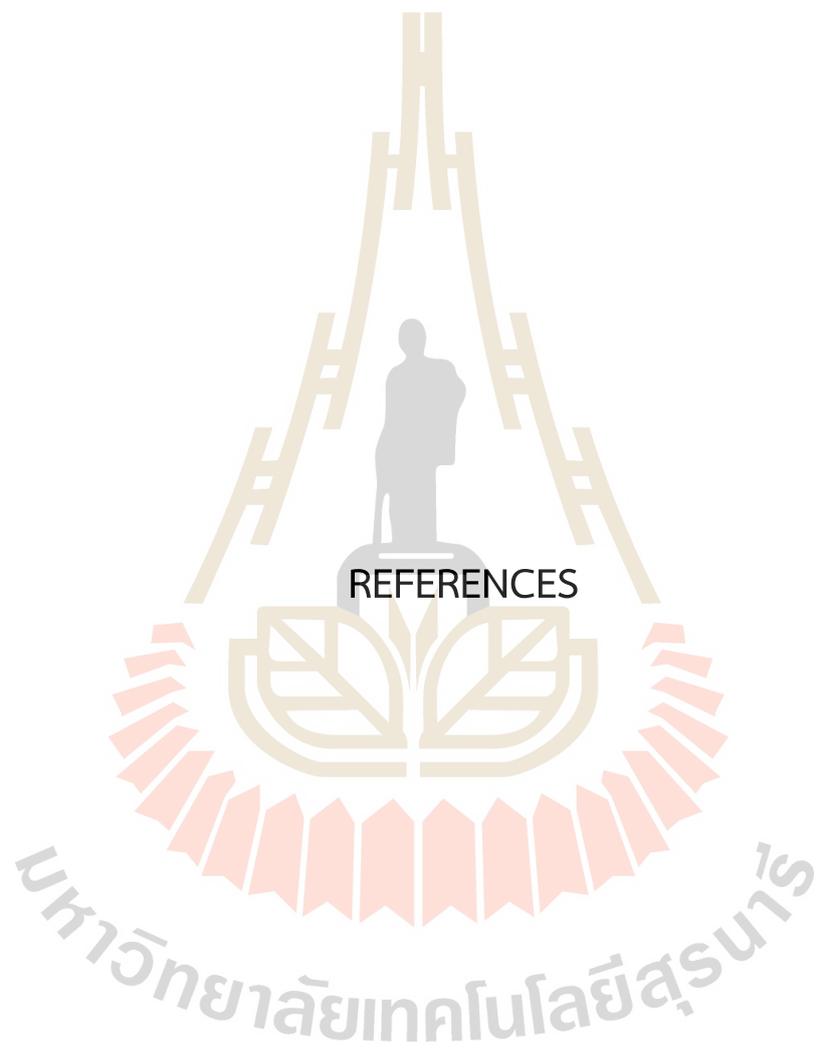
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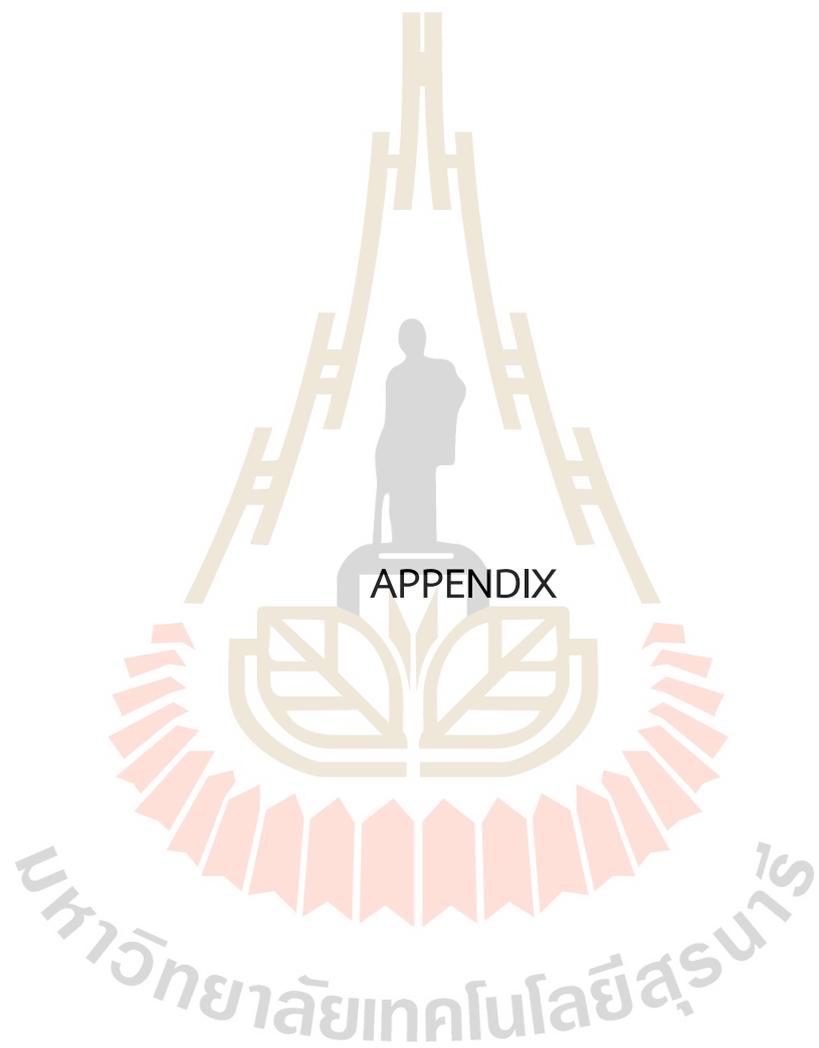
APPENDIX
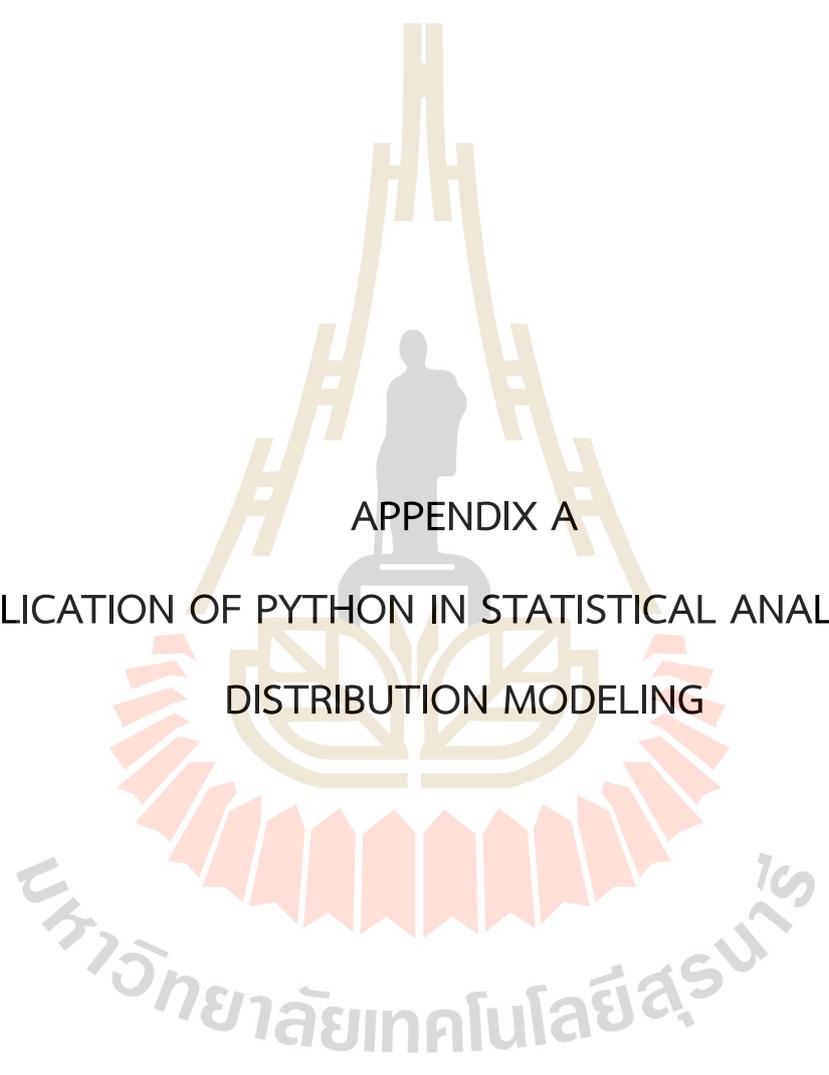
APPENDIX A

APPLICATION OF PYTHON IN STATISTICAL ANALYSIS AND

DISTRIBUTION MODELING

This chapter presents some Python code using in this thesis.

A.1 The Curves of the Posterior Probability Density Function

This section illustrates the visualization of the posterior probability density function (PDF) curves for different parameters, using Python code to generate plots for various distributions. The curves demonstrate how the posterior PDF varies with respect to different values of the parameters and help visualize the model's behavior for different data points.

A.1.1 The Curve of the Posterior PDF for the PWEU Distribution

This plot shows the curve of the posterior PDF for the PWEU model, where the value of θ is varied across a specific range. The graph is generated by plotting the value of $k(\lambda|\theta)$ for different values of λ and θ, allowing for a deeper understanding of the behavior of this distribution.

```python
1  import numpy as np
2  import matplotlib.pyplot as plt
3  import scipy.special as ss
4
5  lambda_value = np.linspace(0.001, 5, 200)
6  theta_value = np.linspace(np.pi/6, 2 * np.pi, 6)
7  len_theta = len(theta_value)
8
9  def g(lambda0, theta):
10     numerator = 4 * np.pi ** 2 * lambda0 * np.exp(-lambda0 * theta)
11     denominator = (1 - np.exp(-2 * np.pi * lambda0)) * ss.zeta(2, theta/(2 * np.pi))
12     return numerator / denominator
13
14 for i in range(len_theta):
15     y = g(lambda_value, theta_value[i])
16     plt.plot(lambda_value, y, label=f'$\\theta$ = {theta_value[i]:.3f}')
17
18 plt.xlabel('$\\theta$')
```

```
19  plt.ylabel('$k(\lambda|\\theta)$')
20  #plt.title('PWEU$(\lambda|\\theta)$')
21  plt.legend()
22  plt.ylim(0,4)
23  plt.xlim(0,5)
24  plt.show()
```

A.1.2   The Curve of the Posterior PDF for the PWEG Distribution

The graph presents the posterior PDF curve for the PWEG model with given parameters, highlighting the relationship between λ and θ when $\alpha = 2$ and $\beta = 0.5$. By varying θ, the plot visually represents how the PWEG distribution behaves under different conditions, offering insights into the characteristics of this specific model.

```python
1  import numpy as np
2  import matplotlib.pyplot as plt
3  import scipy.special as ss
4
5  lambda_value = np.linspace(0.001, 5, 200)
6  theta_value = np.linspace(np.pi/6, 2 * np.pi, 6)
7  len_theta = len(theta_value)
8
9  def gg(lambda0, theta, alpha, beta):
10     numerator = ((2 * np.pi) ** (alpha + 1)) * (lambda0 ** alpha) * np.exp(-lambda0 * (
           theta + beta))
11     denominator = ss.gamma(alpha + 1) * (1 - np.exp(-2 * np.pi * lambda0)) * ss.zeta(
           alpha + 1, (theta + beta) / (2 * np.pi))
12     return numerator / denominator
13
14 for i in range(len_theta):
15     y = gg(lambda_value, theta_value[i], 2, 0.5)
16     plt.plot(lambda_value, y, label=f'$\\theta$ = {theta_value[i]:.3f}')
17
18 plt.xlabel('$\lambda$')
19 plt.ylabel('$k(\lambda|\\theta, 2, 0.5)$')
20 #plt.title('PWEG$(\lambda|\\theta, \\alpha=2, \\beta=0.5)$')
21 plt.ylim(0,2)
22 plt.xlim(0,5)
23 plt.legend()
24 plt.show()
```

A.2   Moment Generating Function, Statistical Measures

This section introduces several fundamental statistical measures for the PWEU and PWEG distributions, including the moment-generating function (MGF), mean, variance, skewness, and kurtosis. These measures are critical for understanding the distribution's properties and are essential for analyzing the behavior of the data modeled by these distributions.

A.2.1   The Statistical Measures for the PWEU Distribution

This section describes the calculation of the statistical measures for the PWEU distribution. It includes the computation of the moment-generating function (MGF), mean, variance, skewness, and kurtosis for this model. The corresponding Python code allows the user to calculate and visualize these statistics, helping to better understand the underlying distribution.

- MGF

```
1  import numpy as np
2  import scipy.special as ss
3
4  def mgf(t, theta):
5      numerator = ss.zeta(2, (theta - t) / (2 * np.pi))
6      denominator = ss.zeta(2, theta / (2 * np.pi))
7      return numerator / denominator
```

- Mean

```
1  import numpy as np
2  import matplotlib.pyplot as plt
3  import scipy.special as ss
4
5  theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6
7  def zeta(k, theta):
```

```
8      return ss.gamma(2 + k) * ss.zeta(2 + k, theta / (2 * np.pi)) / np.power(2 *
           np.pi, k)
9  def mu(theta):
10     numerator = zeta(1, theta)
11     denominator = zeta(0, theta)
12     return numerator / denominator
13
14 plt.plot(theta_value2, mu(theta_value2))
15 plt.xlabel('$\\theta$')
16 plt.ylabel('$\mu(\\theta)$')
17 #plt.title('Graph of $\mu(\\theta)$')
18 plt.ylim(0, 80)
19 plt.xlim(0, 2 * np.pi)
20 plt.show()
```

- Variance

```
1  import numpy as np
2  import matplotlib.pyplot as plt
3  import scipy.special as ss
4
5  theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6
7
8  def zeta(k, theta):
9      return ss.gamma(2 + k) * ss.zeta(2 + k, theta / (2 * np.pi)) / np.power(2 *
             np.pi, k)
10 def mu(theta):
11     numerator = zeta(1, theta)
12     denominator = zeta(0, theta)
13     return numerator / denominator
14 def raw_2m(theta):
15     numerator = zeta(2, theta)
16     denominator = zeta(0, theta)
17     return numerator / denominator
18 def sigma_s(theta):
```

```
19      return raw_2m(theta) - np.power(mu(theta), 2)
20
21  plt.plot(theta_value2, sigma_s(theta_value2))
22  plt.xlabel('$\\theta$')
23  plt.ylabel('$\sigma^{2}(\\theta)$')
24  #plt.title('Graph of $\sigma^{2}(\\theta)$')
25  plt.ylim(0, 80)
26  plt.xlim(0, 2 * np.pi)
27  plt.show()
```

- Skewness

```
1   import numpy as np
2   import matplotlib.pyplot as plt
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6
7   def zeta(k, theta):
8       return ss.gamma(2 + k) * ss.zeta(2 + k, theta / (2 * np.pi)) / np.power(2 *
           np.pi, k)
9   def mu(theta):
10      numerator = zeta(1, theta)
11      denominator = zeta(0, theta)
12      return numerator / denominator
13  def raw_2m(theta):
14      numerator = zeta(2, theta)
15      denominator = zeta(0, theta)
16      return numerator / denominator
17  def sigma_s(theta):
18      return raw_2m(theta) - np.power(mu(theta), 2)
19  def raw_3m(theta):
20      numerator = zeta(3, theta)
21      denominator = zeta(0, theta)
22      return numerator / denominator
23  def skew(theta):
```

```
24    numerator = raw_3m(theta) - 3 * mu(theta) * sigma_s(theta) - np.power(mu(
          theta), 3)
25    denominator = np.power(np.sqrt(sigma_s(theta)), 3)
26    return numerator / denominator
27
28  plt.plot(theta_value2, skew(theta_value2))
29  plt.xlabel('$\\theta$')
30  plt.ylabel('$\gamma(\\theta)$')
31  #plt.title('Graph of $\gamma_{1}(\\theta)$')
32  plt.ylim(0, 2)
33  plt.xlim(0, 2 * np.pi)
34  plt.show()
```

- Kurtosis

```
1   import numpy as np
2   import matplotlib.pyplot as plt
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6
7   def zeta(k, theta):
8       return ss.gamma(2 + k) * ss.zeta(2 + k, theta / (2 * np.pi)) / np.power(2 *
            np.pi, k)
9   def mu(theta):
10      numerator = zeta(1, theta)
11      denominator = zeta(0, theta)
12      return numerator / denominator
13  def raw_2m(theta):
14      numerator = zeta(2, theta)
15      denominator = zeta(0, theta)
16      return numerator / denominator
17  def sigma_s(theta):
18      return raw_2m(theta) - np.power(mu(theta), 2)
19  def raw_3m(theta):
20      numerator = zeta(3, theta)
```

```
21      denominator = zeta(0, theta)
22      return numerator / denominator
23  def raw_4m(theta):
24      numerator = zeta(4, theta)
25      denominator = zeta(0, theta)
26      return numerator / denominator
27  def Kurt(theta):
28      numerator = raw_4m(theta) - (4 * mu(theta) * raw_3m(theta)) + (6 * np.power(
            mu(theta), 2) * sigma_s(theta)) + (3 * np.power(mu(theta), 4))
29      denominator = np.power(sigma_s(theta), 2)
30      return numerator / denominator
31
32  plt.plot(theta_value2, Kurt(theta_value2))
33  plt.xlabel('$\\theta$')
34  plt.ylabel('$\kappa(\\theta)$')
35  #plt.title('Graph of $\\beta_{2}(\\theta)$')
36  plt.ylim(0, 10)
37  plt.xlim(0, 2 * np.pi)
38  plt.show()
```

A.2.2   The Statistical Measures for the PWEG Distribution

In this subsection, the statistical measures for the PWEG distribution are computed and visualized. The code provided allows for the calculation of the MGF, mean, variance, skewness, and kurtosis, with visualizations to help interpret the effects of varying parameters such as α and β. These measures offer critical insights into the shape and characteristics of the distribution.

- MGF

```
1  import numpy as np
2  import scipy.special as ss
3
4  def mgf_g(t, theta, alpha, beta):
5      numerator = ss.zeta(alpha + 1, (theta + beta - t) / (2 * np.pi))
```

```
6       denominator = ss.zeta(alpha + 1, (theta + beta) / (2 * np.pi))
7       return numerator / denominator
```

- Mean

```
1   import numpy as np
2   import matplotlib.pyplot as plt
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6   alpha_value = np.linspace(1, 9, 9)
7   len_alpha = len(alpha_value)
8
9   def zeta_g(k, theta, alpha, beta):
10      return ss.gamma(alpha + 1 + k) * ss.zeta(alpha + 1 + k, (theta + beta) / (2
            * np.pi)) / np.power(2 * np.pi, k)
11  def mu_g(theta, alpha, beta):
12      numerator = zeta_g(1, theta, alpha, beta)
13      denominator = zeta_g(0, theta, alpha, beta)
14      return numerator / denominator
15
16  for i in range(len_alpha):
17      y = mu_g(theta_value2, alpha_value[i], 0.5)
18      plt.plot(theta_value2, y, label=f'$\\alpha$ = {alpha_value[i]:.0f}')
19
20  plt.xlabel('$\\theta$')
21  plt.ylabel('$\mu(\\theta|\\alpha, 0.5)$')
22  #plt.title('Graph of $\mu(\\theta|\\alpha, \\beta = 0.5)$')
23  plt.ylim(0, 20)
24  plt.xlim(0, 2 * np.pi)
25  plt.legend()
26  plt.show()
```

- Variance

```
1   import numpy as np
2   import matplotlib.pyplot as plt
```

```
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6   alpha_value = np.linspace(1, 9, 9)
7   len_alpha = len(alpha_value)
8
9   def zeta_g(k, theta, alpha, beta):
10      return ss.gamma(alpha + 1 + k) * ss.zeta(alpha + 1 + k, (theta + beta) / (2
            * np.pi)) / np.power(2 * np.pi, k)
11  def raw_2m_g(theta, alpha, beta):
12      numerator = zeta_g(2, theta, alpha, beta)
13      denominator = zeta_g(0, theta, alpha, beta)
14      return numerator / denominator
15  def mu_g(theta, alpha, beta):
16      numerator = zeta_g(1, theta, alpha, beta)
17      denominator = zeta_g(0, theta, alpha, beta)
18      return numerator / denominator
19  def sigma_s_g(theta, alpha, beta):
20      return raw_2m_g(theta, alpha, beta) - np.power(mu_g(theta, alpha, beta), 2)
21
22  for i in range(len_alpha):
23      y = sigma_s_g(theta_value2, alpha_value[i], 0.5)
24      plt.plot(theta_value2, y, label=f'$\\alpha$ = {alpha_value[i]:.0f}')
25
26  plt.xlabel('$\\theta$')
27  plt.ylabel('$\sigma^{2}(\\theta| \\alpha, 0.5)$')
28  #plt.title('Graph of $\sigma^{2}(\\theta| \\alpha, \\beta = 0.5)$')
29  plt.ylim(0, 40)
30  plt.xlim(0, 2 * np.pi)
31  plt.legend()
32  plt.show()
```

- Skewness

```
1   import numpy as np
2   import matplotlib.pyplot as plt
```

```
3   import scipy.special as ss

4

5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)

6   alpha_value = np.linspace(1, 9, 9)

7   len_alpha = len(alpha_value)

8

9   def zeta_g(k, theta, alpha, beta):

10      return ss.gamma(alpha + 1 + k) * ss.zeta(alpha + 1 + k, (theta + beta) / (2
            * np.pi)) / np.power(2 * np.pi, k)

11  def raw_2m_g(theta, alpha, beta):

12      numerator = zeta_g(2, theta, alpha, beta)

13      denominator = zeta_g(0, theta, alpha, beta)

14      return numerator / denominator

15  def mu_g(theta, alpha, beta):

16      numerator = zeta_g(1, theta, alpha, beta)

17      denominator = zeta_g(0, theta, alpha, beta)

18      return numerator / denominator

19  def sigma_s_g(theta, alpha, beta):

20      return raw_2m_g(theta, alpha, beta) - np.power(mu_g(theta, alpha, beta), 2)

21  def raw_3m_g(theta, alpha, beta):

22      numerator = zeta_g(3, theta, alpha, beta)

23      denominator = zeta_g(0, theta, alpha, beta)

24      return numerator / denominator

25  def skew_g(theta, alpha, beta):

26      numerator = raw_3m_g(theta, alpha, beta) - (3 * mu_g(theta, alpha, beta) *
            sigma_s_g(theta, alpha, beta)) - np.power(mu_g(theta, alpha, beta), 3)

27      denominator = np.power(np.sqrt(sigma_s_g(theta, alpha, beta)), 3)

28      return numerator / denominator

29

30  for i in range(len_alpha):

31      y = skew_g(theta_value2, alpha_value[i], 0.5)

32      plt.plot(theta_value2, y, label=f'$\\alpha$ = {alpha_value[i]:.0f}')

33

34  plt.xlabel('$\\theta$')

35  plt.ylabel('$\gamma(\\theta | \\alpha, 0.5)$')

36  #plt.title('Graph of $\gamma{1}(\\theta, \\alpha, \\beta = 0.5)$')
```

```
37  plt.ylim()
38  plt.xlim(0, 2 * np.pi)
39  plt.legend()
40  plt.show()
```

- Kurtosis

```
1   import numpy as np
2   import matplotlib.pyplot as plt
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6
7   def zeta_g(k, theta, alpha, beta):
8       return ss.gamma(alpha + 1 + k) * ss.zeta(alpha + 1 + k, (theta + beta) / (2
            * np.pi)) / np.power(2 * np.pi, k)
9   def raw_2m_g(theta, alpha, beta):
10      numerator = zeta_g(2, theta, alpha, beta)
11      denominator = zeta_g(0, theta, alpha, beta)
12      return numerator / denominator
13  def mu_g(theta, alpha, beta):
14      numerator = zeta_g(1, theta, alpha, beta)
15      denominator = zeta_g(0, theta, alpha, beta)
16      return numerator / denominator
17  def sigma_s_g(theta, alpha, beta):
18      return raw_2m_g(theta, alpha, beta) - np.power(mu_g(theta, alpha, beta), 2)
19  def raw_3m_g(theta, alpha, beta):
20      numerator = zeta_g(3, theta, alpha, beta)
21      denominator = zeta_g(0, theta, alpha, beta)
22      return numerator / denominator
23  def raw_4m_g(theta, alpha, beta):
24      numerator = zeta_g(4, theta, alpha, beta)
25      denominator = zeta_g(0, theta, alpha, beta)
26      return numerator / denominator
27  def Kurt_g(theta, alpha, beta):
28      numerator = raw_4m_g(theta, alpha, beta) - (4 * mu_g(theta, alpha, beta) *
```

```
          raw_3m_g(theta, alpha, beta)) + (6 * np.power(mu_g(theta, alpha, beta),
          2) * sigma_s_g(theta, alpha, beta)) + (3 * np.power(mu_g(theta, alpha,
          beta), 4))
29    denominator = np.power(sigma_s_g(theta, alpha, beta), 2)
30    return numerator / denominator
31 def Kurt_ex_g(theta, alpha, beta):
32    return Kurt_g(theta, alpha, beta) - 3
33
34 for i in range(len_alpha):
35    y = Kurt_ex_g(theta_value2, alpha_value[i], 0.5)
36    plt.plot(theta_value2, y)
37
38 plt.xlabel('$\\theta$')
39 plt.ylabel('$\gamma_{2}(\\theta, \\alpha, \\beta = 0.5)$')
40 plt.title('Graph of $\gamma_{2}(\\theta, \\alpha, \\beta = 0.5)$')
41 plt.ylim()
42 plt.xlim(0, 2 * np.pi)
43 plt.show
```

A.3   Bayesian Estimators and Risk Functions

This section delves into the concept of Bayesian estimators and their associated risk functions. Using the PWEU distribution, the Bayesian Estimators (BE) and Risk Minimization Functions (RMF) are derived and visualized. The plots generated here help to understand the efficiency and effectiveness of the Bayesian approach in estimating model parameters under different loss functions.

A.3.1   Bayesian Estimators for the PWEU Distribution

This part explains the process of deriving and visualizing Bayesian Estimators for the PWEU model using a squared error loss function (SEL). The corresponding Python code demonstrates how to compute the Bayesian estimator and visualize the results, helping to assess the performance of the estimator under varying values of θ.

- Squared Error Loss Function

    - Bayesian Estimator (BE)

```python
1   import numpy as np
2   import matplotlib.pyplot as plt
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6
7   def zeta(k, theta):
8       return ss.gamma(2 + k) * ss.zeta(2 + k, theta / (2 * np.pi)) / np.power
            (2 * np.pi, k)
9   def mu(theta):
10      numerator = zeta(1, theta)
11      denominator = zeta(0, theta)
12      return numerator / denominator
13
14  plt.plot(theta_value2, mu(theta_value2))
15  plt.xlabel('$\\theta$')
16  plt.ylabel('$\hat\lambda_{SEL}$')
17  #plt.title('Graph of $\hat{\lambda}(\\theta)$')
18  plt.ylim(0, 80)
19  plt.xlim(0, 2 * np.pi)
20  plt.show()
```

    - Risk Minimization Function (RMF)

```python
1   import numpy as np
2   import matplotlib.pyplot as plt
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6
```

```
7
8  def zeta(k, theta):
9      return ss.gamma(2 + k) * ss.zeta(2 + k, theta / (2 * np.pi)) / np.power
            (2 * np.pi, k)
10  def mu(theta):
11      numerator = zeta(1, theta)
12      denominator = zeta(0, theta)
13      return numerator / denominator
14  def raw_2m(theta):
15      numerator = zeta(2, theta)
16      denominator = zeta(0, theta)
17      return numerator / denominator
18  def sigma_s(theta):
19      return raw_2m(theta) - np.power(mu(theta), 2)
20
21  plt.plot(theta_value2, sigma_s(theta_value2))
22  #plt.title('Graph of $R(\lambda,\hat{\lambda})$')
23  plt.xlabel('$\\theta$')
24  plt.ylabel('$R(\lambda,\hat{\lambda}_{SEL})$')
25  plt.ylim(0, 80)
26  plt.xlim(0, 2 * np.pi)
27  plt.show()
```

- Precautionary Loss Function

  – Bayesian Estimator (BE)

```
1  import numpy as np
2  import matplotlib.pyplot as plt
3  import scipy.special as ss
4
5  theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6
7  def zeta(k, theta):
```

```
 8      return ss.gamma(2 + k) * ss.zeta(2 + k, theta / (2 * np.pi)) / np.power
            (2 * np.pi, k)
 9  def raw_2m(theta):
10      numerator = zeta(2, theta)
11      denominator = zeta(0, theta)
12      return numerator / denominator
13
14  plt.plot(theta_value2, np.power(raw_2m(theta_value2),0.5))
15  plt.xlabel('$\\theta$')
16  plt.ylabel('$\hat\lambda_{PL}$')
17  plt.ylim(0, 80)
18  plt.xlim(0, 2 * np.pi)
19  plt.show()
```

– Risk Minimization Function (RMF)

```
 1  import numpy as np
 2  import matplotlib.pyplot as plt
 3  import scipy.special as ss
 4
 5  theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
 6
 7  def zeta(k, theta):
 8      return ss.gamma(2 + k) * ss.zeta(2 + k, theta / (2 * np.pi)) / np.power
            (2 * np.pi, k)
 9  def mu(theta):
10      numerator = zeta(1, theta)
11      denominator = zeta(0, theta)
12      return numerator / denominator
13  def raw_2m(theta):
14      numerator = zeta(2, theta)
15      denominator = zeta(0, theta)
16      return numerator / denominator
17  def sigma_s(theta):
18      return 2 * (np.power(raw_2m(theta),3/2) - mu(theta))
```

```
19
20  plt.plot(theta_value2, sigma_s(theta_value2))
21  plt.xlabel('$\\theta$')
22  plt.ylabel('$R(\lambda,\hat\lambda_{PL})$')
23  plt.ylim(0, 80)
24  plt.xlim(0, 2 * np.pi)
25  plt.show()
```

A.3.2   Bayesian Estimators for the PWEG Distribution

In this section, Bayesian Estimators for the PWEG distribution are computed and visualized, using a squared error loss function (SEL). The provided Python code demonstrates the calculation of the Bayesian Estimator and shows the resulting plots. These visualizations help in understanding how the estimator behaves as the parameters α and β change and the model's sensitivity to different loss functions.

- Squared Error Loss Function

  – Bayesian Estimator (BE)

```
 1  import numpy as np
 2  import matplotlib.pyplot as plt
 3  import scipy.special as ss
 4
 5  theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
 6  alpha_value = np.linspace(1, 9, 9)
 7  len_alpha = len(alpha_value)
 8
 9  def zeta_g(k, theta, alpha, beta):
10      return ss.gamma(alpha + 1 + k) * ss.zeta(alpha + 1 + k, (theta + beta)
            / (2 * np.pi)) / np.power(2 * np.pi, k)
11  def mu_g(theta, alpha, beta):
12      numerator = zeta_g(1, theta, alpha, beta)
```

```
13      denominator = zeta_g(0, theta, alpha, beta)

14      return numerator / denominator

15

16  for i in range(len_alpha):

17      y = mu_g(theta_value2, alpha_value[i], 0.5)

18      plt.plot(theta_value2, y, label=f'$\\alpha$ = {alpha_value[i]:.0f}')

19

20  plt.xlabel('$\\theta$')

21  plt.ylabel('$\hat\lambda_{SEL}(\\theta | \\alpha, 0.5)$')

22  plt.ylim(0, 20)

23  plt.xlim(0, 2 * np.pi)

24  plt.legend()

25  plt.show()
```

– Risk Minimization Function (RMF)

```
1   import numpy as np

2   import matplotlib.pyplot as plt

3   import scipy.special as ss

4

5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)

6   alpha_value = np.linspace(1, 9, 9)

7   len_alpha = len(alpha_value)

8

9   def zeta_g(k, theta, alpha, beta):

10      return ss.gamma(alpha + 1 + k) * ss.zeta(alpha + 1 + k, (theta + beta)
        / (2 * np.pi)) / np.power(2 * np.pi, k)

11  def raw_2m_g(theta, alpha, beta):

12      numerator = zeta_g(2, theta, alpha, beta)

13      denominator = zeta_g(0, theta, alpha, beta)

14      return numerator / denominator

15  def mu_g(theta, alpha, beta):

16      numerator = zeta_g(1, theta, alpha, beta)

17      denominator = zeta_g(0, theta, alpha, beta)

18      return numerator / denominator
```

```python
19  def sigma_s_g(theta, alpha, beta):
20      return raw_2m_g(theta, alpha, beta) - np.power(mu_g(theta, alpha, beta)
            , 2)
21
22  for i in range(len_alpha):
23      y = sigma_s_g(theta_value2, alpha_value[i], 0.5)
24      plt.plot(theta_value2, y, label=f'$\\alpha$ = {alpha_value[i]:.0f}')
25
26  plt.xlabel('$\\theta$')
27  plt.ylabel('$R(\lambda,\hat\lambda_{SEL} | \\alpha, 0.5)$')
28  plt.ylim(0, 40)
29  plt.xlim(0, 2 * np.pi)
30  plt.legend()
31  plt.show()
```

- Precautionary Loss Function

  – Bayesian Estimator (BE)

```python
1   import numpy as np
2   import matplotlib.pyplot as plt
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6   alpha_value = np.linspace(1, 9, 9)
7   len_alpha = len(alpha_value)
8
9   def zeta_g(k, theta, alpha, beta):
10      return ss.gamma(alpha + 1 + k) * ss.zeta(alpha + 1 + k, (theta + beta)
            / (2 * np.pi)) / np.power(2 * np.pi, k)
11  def raw_2m_g(theta, alpha, beta):
12      numerator = zeta_g(2, theta, alpha, beta)
13      denominator = zeta_g(0, theta, alpha, beta)
14      return numerator / denominator
```

```
15
16  for i in range(len_alpha):
17      y = raw_2m_g(theta_value2, alpha_value[i], 0.5)
18      plt.plot(theta_value2, y, label=f'$\\alpha$ = {alpha_value[i]:.0f}')
19
20  plt.xlabel('$\\theta$')
21  plt.ylabel('$\hat\lambda_{PL}(\\theta | \\alpha, 0.5)$')
22  plt.ylim(0, 20)
23  plt.xlim(0, 2 * np.pi)
24  plt.legend()
25  plt.show()
```

– Risk Minimization Function (RMF)

```
1   import numpy as np
2   import matplotlib.pyplot as plt
3   import scipy.special as ss
4
5   theta_value2 = np.linspace(0.001, 2 * np.pi, 360)
6   alpha_value = np.linspace(1, 9, 9)
7   len_alpha = len(alpha_value)
8
9   def zeta_g(k, theta, alpha, beta):
10      return ss.gamma(alpha + 1 + k) * ss.zeta(alpha + 1 + k, (theta + beta)
            / (2 * np.pi)) / np.power(2 * np.pi, k)
11  def raw_2m_g(theta, alpha, beta):
12      numerator = zeta_g(2, theta, alpha, beta)
13      denominator = zeta_g(0, theta, alpha, beta)
14      return numerator / denominator
15  def mu_g(theta, alpha, beta):
16      numerator = zeta_g(1, theta, alpha, beta)
17      denominator = zeta_g(0, theta, alpha, beta)
18      return numerator / denominator
19  def sigma_s_g(theta, alpha, beta):
20      return 2*(np.power(raw_2m_g(theta, alpha, beta),3/2) - mu_g(theta,
```

```
        alpha, beta))
21
22  for i in range(len_alpha):
23      y = sigma_s_g(theta_value2, alpha_value[i], 0.5)
24      plt.plot(theta_value2, y, label=f'$\\alpha$ = {alpha_value[i]:.0f}')
25
26  plt.xlabel('$\\theta$')
27  plt.ylabel('$R(\lambda,\hat\lambda_{PL} | 2, \\beta)$')
28  plt.ylim(0, 40)
29  plt.xlim(0, 2 * np.pi)
30  plt.legend()
31  plt.show()
```

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- Damnet, M., Suriyawichitseranee, A., and Tanthanuch, J. (2023). Utilizing Bayesian Analysis of Wrapped Distributions in Computer Technology., Proceedings of *the 20th International and National Conference on Applied Computer Technology and Information System (ACTIS).* 25 August 2023, 97 - 101.

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