

CHAPTER IV

RESULTS AND DISCUSSION

This chapter presents the findings obtained through the methodology outlined in Chapter III, Research Methodology. The primary objective of this section is to interpret and evaluate the results generated by the Python software developed as part of this study. The complete code implementation is provided in the Appendix for reference.

4.1 Characterization and Estimation of the PWEU Distribution

This section presents the moment generating function (MGF), mean, variance, skewness, kurtosis, Bayesian estimators, and risk minimization function for the PWEU distribution. The discussion covers key statistical measures that describe the distribution's characteristics and provides essential tools for estimation in practical applications.

4.1.1 Moment Generating Function and Statistical Measures

Specifically, let $k_{PWEU}(\lambda|\theta)$ represent the posterior PDF of the PWEU distribution. Also, let M_{PWEU} , μ_{PWEU} , σ_{PWEU}^2 , γ_{PWEU} , and κ_{PWEU} represent the MGF, mean, variance, skewness, and kurtosis of the same distribution, respectively. For the sake of simplicity, the new function is introduced, $\Phi(a) = \frac{\zeta\left(a, \frac{\theta}{2\pi}\right)}{\zeta\left(2, \frac{\theta}{2\pi}\right)}$. The MGF and statistical measures of the PWEU distribution can be found as follows:

- MGF

$$\begin{aligned}
M_{\text{PWEU}}(t) &= E[e^{t\Lambda}|\theta] \\
&= \frac{4\pi^2}{\zeta\left(2, \frac{\theta}{2\pi}\right)} \int_0^\infty e^{t\lambda} \left(\frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}} \right) d\lambda \\
&= \frac{4\pi^2}{\zeta\left(2, \frac{\theta}{2\pi}\right)} \int_0^\infty \frac{\lambda^{2-1} e^{-(\theta-t)\lambda}}{1 - e^{-2\pi\lambda}} d\lambda \\
&= \frac{4\pi^2}{\zeta\left(2, \frac{\theta}{2\pi}\right)} \int_0^\infty \frac{\left(\frac{u}{2\pi}\right)^{2-1} e^{-\left(\frac{\theta-t}{2\pi}\right)u}}{1 - e^{-u}} \frac{du}{2\pi} \\
&= \frac{\Gamma(2)}{\zeta\left(2, \frac{\theta}{2\pi}\right)} \left(\frac{1}{\Gamma(2)} \int_0^\infty \frac{u^{2-1} e^{-\left(\frac{\theta-t}{2\pi}\right)u}}{1 - e^{-u}} du \right) \\
&= \frac{\zeta\left(2, \frac{\theta-t}{2\pi}\right)}{\zeta\left(2, \frac{\theta}{2\pi}\right)}
\end{aligned}$$

- Mean

$$\mu_{\text{PWEU}} = \mathbb{E}[\Lambda|\theta] = M'_{\text{PWEU}}(0) = \frac{2\zeta\left(3, \frac{\theta}{2\pi}\right)}{2\pi\zeta\left(2, \frac{\theta}{2\pi}\right)} = \frac{1}{\pi}\Phi(3),$$

where $\Lambda|\theta$ is the wrapped random variable with parameter θ .

- Variance

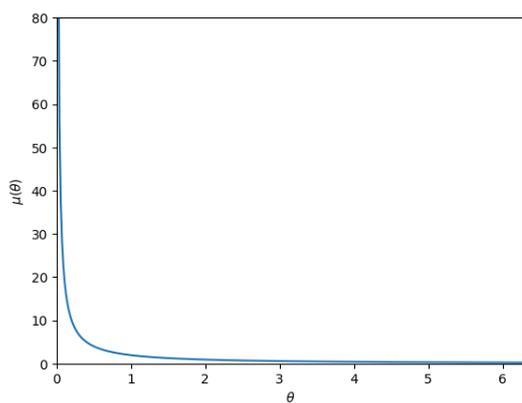
$$\begin{aligned}
\sigma_{\text{PWEU}}^2 &= \mathbb{E}[(\Lambda|\theta - \mu_{\text{PWEU}})^2] \\
&= \mathbb{E}[(\Lambda|\theta)^2] - \mu_{\text{PWEU}}^2 \\
&= M''_{\text{PWEU}}(0) - \mu_{\text{PWEU}}^2 \\
&= \frac{6\zeta\left(4, \frac{\theta}{2\pi}\right)}{(2\pi)^2 \zeta\left(2, \frac{\theta}{2\pi}\right)} - \left(\frac{1}{\pi}\Phi(3)\right)^2 \\
&= \frac{1}{\pi^2} \left[\frac{3}{2}\Phi(4) - (\Phi(3))^2 \right]
\end{aligned}$$

- Skewness

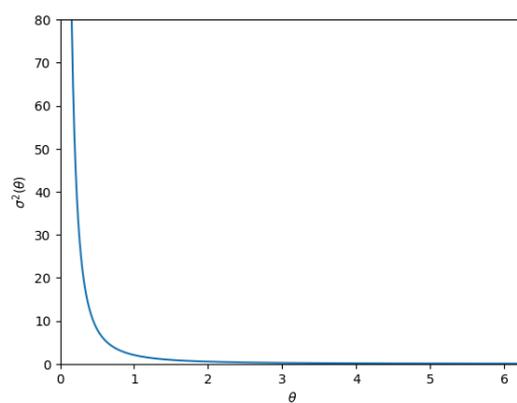
$$\begin{aligned}
\gamma_{\text{PWEU}} &= \frac{\mathbb{E}[(\Lambda|\theta - \mu_{\text{PWEU}})^3]}{\sigma_{\text{PWEU}}^3} \\
&= \frac{\mathbb{E}[(\Lambda|\theta)^3] - 3\mu_{\text{PWEU}}\sigma_{\text{PWEU}}^2 - \mu_{\text{PWEU}}^3}{\sigma_{\text{PWEU}}^3} \\
&= \frac{M'''_{\text{PWEU}}(0) - 3\mu_{\text{PWEU}}\sigma_{\text{PWEU}}^2 - \mu_{\text{PWEU}}^3}{\sigma_{\text{PWEU}}^3} \\
&= \frac{\frac{3}{\pi^3}\Phi(5) - 3\left(\frac{1}{\pi}\Phi(3)\right)\left(\frac{1}{\pi^2}\left[\frac{3}{2}\Phi(4) - (\Phi(3))^2\right]\right) - \left(\frac{1}{\pi}\Phi(3)\right)^3}{\left[\frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)\right]^{3/2}} \\
&= \frac{\frac{3}{\pi^3}\Phi(5) - \frac{3}{\pi^3}\left[\frac{3}{2}\Phi(3)\Phi(4) - (\Phi(3))^3\right] - \frac{1}{\pi^3}(\Phi(3))^3}{\left[\frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)\right]^{3/2}} \\
&= \frac{3\Phi(5) - \frac{9}{2}\Phi(3)\Phi(4) + 2(\Phi(3))^3}{\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)^{3/2}}
\end{aligned}$$

- Kurtosis

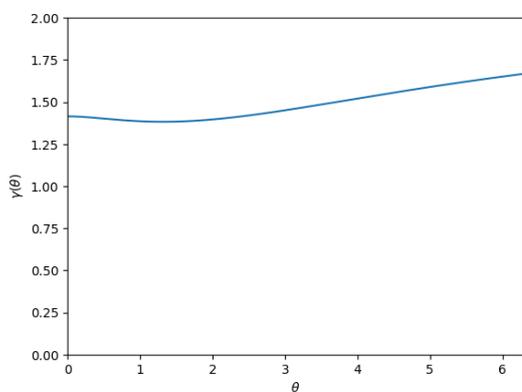
$$\begin{aligned}
\kappa_{\text{PWEU}} &= \frac{\mathbb{E}[(\Lambda|\theta - \mu_{\text{PWEU}})^4]}{\sigma_{\text{PWEU}}^4} \\
&= \frac{\mathbb{E}[(\Lambda|\theta)^4] - 4\mu_{\text{PWEU}}\mathbb{E}[(\Lambda|\theta)^3] + 6\mu_{\text{PWEU}}^2\sigma_{\text{PWEU}}^2 + 3\mu_{\text{PWEU}}^4}{(\sigma_{\text{PWEU}}^2)^2} \\
&= \frac{M^{(4)}_{\text{PWEU}}(0) - 4\mu_{\text{PWEU}}M'''_{\text{PWEU}}(0) + 6\mu_{\text{PWEU}}^2\sigma_{\text{PWEU}}^2 + 3\mu_{\text{PWEU}}^4}{(\sigma_{\text{PWEU}}^2)^2} \\
&= \frac{\frac{15}{2\pi^4}\Phi(6) - 4\left(\frac{1}{\pi}\Phi(3)\right)\left(\frac{3}{\pi^3}\Phi(5)\right) + 6\left(\frac{1}{\pi}\Phi(3)\right)^2\left(\frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)\right) + 3\left(\frac{1}{\pi}\Phi(3)\right)^4}{\left[\frac{1}{\pi^2}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)\right]^2} \\
&= \frac{\frac{15}{2\pi^4}\Phi(6) - \frac{12}{\pi^4}\Phi(3)\Phi(5) + \frac{6}{\pi^4}\left((\Phi(3))^2\Phi(4) - (\Phi(3))^4\right) + \frac{3}{\pi^4}(\Phi(3))^4}{\frac{1}{\pi^4}\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)^2} \\
&= \frac{\frac{15}{2}\Phi(6) - 12\Phi(3)\Phi(5) + 6(\Phi(3))^2\Phi(4) - 3(\Phi(3))^4}{\left(\frac{3}{2}\Phi(4) - (\Phi(3))^2\right)^2}.
\end{aligned}$$



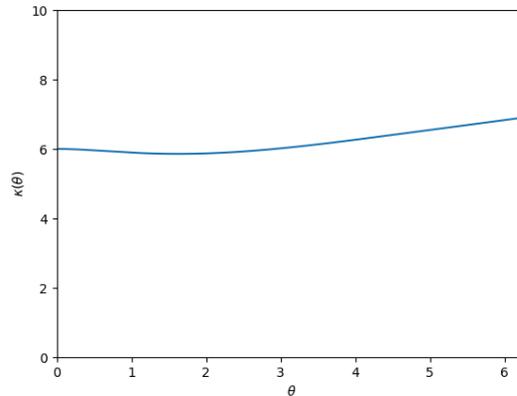
(a) The curve of mean for the PWEU distribution.



(b) The curve of variance for the PWEU distribution.



(c) The curve of skewness for the PWEU distribution.



(d) The curve of kurtosis for the PWEU distribution.

Figure 4.1 The curves of statistical measures for the PWEU distribution.

4.1.2 Bayesian Estimators and Risk Functions for the PWEU distribution

In this section, we present Bayesian estimators and risk minimization functions under loss functions of squared error type and precautionary type for the PWEU distribution. These functions are crucial for estimating parameter values based on empirical data, which has a wrapped random component.

- **Squared Error Loss Function**

For the squared error loss function (SEL function), represented as $\ell(\vartheta, \hat{\vartheta}) = (\vartheta - \hat{\vartheta})^2$, where ϑ denotes the estimate and $\hat{\vartheta}$ represents the true value, the corresponding Bayesian estimator and risk minimization function are outlined below:

- Bayesian Estimator (BE)

The derivation of the Bayesian Estimator under the Squared Error Loss Function begins by minimizing the expected loss:

$$\mathbb{E} \left[(\Lambda|_{\theta} - \hat{\lambda})^2 \right].$$

To find the estimator $\hat{\lambda}$ that minimizes this expression, we take the derivative with respect to $\hat{\lambda}$ and set it equal to zero:

$$\frac{d}{d\hat{\lambda}} \mathbb{E} \left[(\Lambda|_{\theta} - \hat{\lambda})^2 \right] = -2\mathbb{E} [\Lambda|_{\theta}] + 2\hat{\lambda} = 0.$$

Solving this equation:

$$2\hat{\lambda} = 2\mathbb{E} [\Lambda|_{\theta}],$$

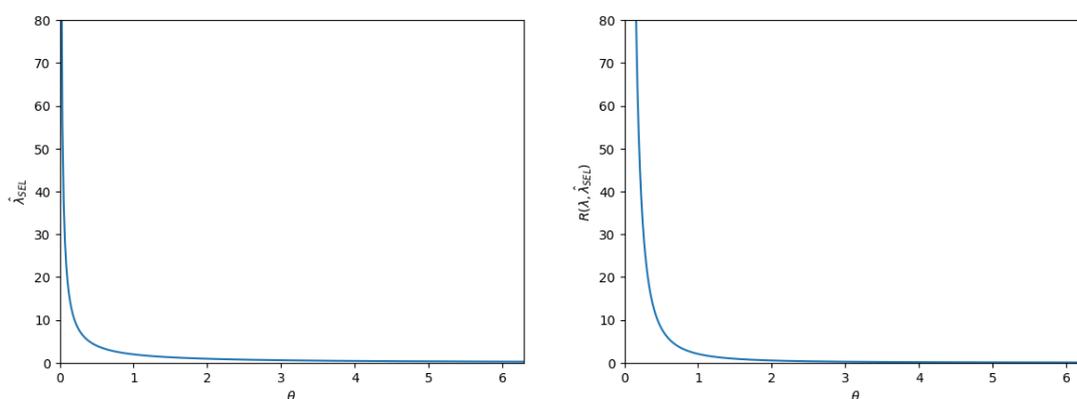
$$\hat{\lambda} = \mathbb{E} [\Lambda|_{\theta}].$$

Thus, the Bayesian Estimator under SEL is the expected value of $\Lambda|_{\theta}$. For the PWEU distribution, this expectation is given by:

$$\hat{\lambda}_{\text{SEL}} = \mathbb{E} [\Lambda|_{\theta}] = \mu_{\text{PWEU}} = \frac{1}{\pi} \Phi(3).$$

– Risk Minimization Function (RMF)

$$\begin{aligned}
 R(\lambda, \hat{\lambda}_{\text{SEL}}) &= \mathbb{E} \left[\ell(\Lambda|_{\theta}, \hat{\lambda}_{\text{SEL}}) \right] \\
 &= \mathbb{E} \left[(\Lambda|_{\theta} - \hat{\lambda}_{\text{SEL}})^2 \right] \\
 &= \sigma_{\text{PWEU}}^2 \\
 &= \frac{1}{\pi^2} \left(\frac{3}{2} \Phi(4) - (\Phi(3))^2 \right).
 \end{aligned}$$



(a) The curve of BE under the SEL function. (b) The curve of RMF under the SEL function.

Figure 4.2 The curves of Bayesian estimator and risk minimization function for the PWEU distribution.

- **Precautionary Loss Function**

For the precautionary loss function (PL function) $\ell(\vartheta, \hat{\vartheta}) = \frac{(\vartheta - \hat{\vartheta})^2}{\hat{\vartheta}}$, its Bayesian estimator and risk minimization function are as follows:

– Bayesian Estimator

The derivation of the Bayesian Estimator under the Precautionary Loss Function begins by minimizing the expected loss:

$$\mathbb{E} \left[\ell(\Lambda|_{\theta}, \hat{\lambda}) \right] = \mathbb{E} \left[\frac{(\Lambda|_{\theta} - \hat{\lambda})^2}{\hat{\lambda}} \right].$$

To find the estimator $\hat{\lambda}$ that minimizes this expression, we take the derivative

with respect to $\hat{\lambda}$ and set it equal to zero:

$$\frac{d}{d\hat{\lambda}} \mathbb{E} \left[\frac{(\Lambda|_{\theta} - \hat{\lambda})^2}{\hat{\lambda}} \right] = 0.$$

Expanding the expected loss:

$$\mathbb{E} \left[\frac{(\Lambda|_{\theta} - \hat{\lambda})^2}{\hat{\lambda}} \right] = \mathbb{E} \left[\frac{\Lambda|_{\theta}^2 - 2\hat{\lambda}\Lambda|_{\theta} + \hat{\lambda}^2}{\hat{\lambda}} \right] = \frac{\mathbb{E}[\Lambda|_{\theta}^2]}{\hat{\lambda}} - 2\mathbb{E}[\Lambda|_{\theta}] + \hat{\lambda}.$$

Taking the derivative:

$$\frac{d}{d\hat{\lambda}} \left(\frac{\mathbb{E}[\Lambda|_{\theta}^2]}{\hat{\lambda}} - 2\mathbb{E}[\Lambda|_{\theta}] + \hat{\lambda} \right) = -\frac{\mathbb{E}[\Lambda|_{\theta}^2]}{\hat{\lambda}^2} + 1 = 0.$$

Solving this equation:

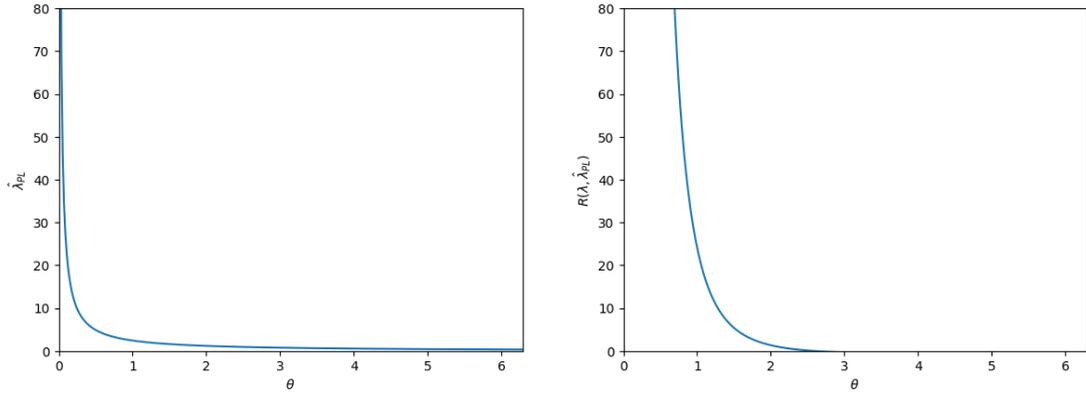
$$\begin{aligned} -\frac{\mathbb{E}[\Lambda|_{\theta}^2]}{\hat{\lambda}^2} + 1 &= 0, \\ \frac{\mathbb{E}[\Lambda|_{\theta}^2]}{\hat{\lambda}^2} &= 1, \\ \hat{\lambda}^2 &= \mathbb{E}[\Lambda|_{\theta}^2], \\ \hat{\lambda} &= \sqrt{\mathbb{E}[\Lambda|_{\theta}^2]}. \end{aligned}$$

Thus, the Bayesian Estimator under PL is the square root of the second moment of $\Lambda|_{\theta}$. For the PWEU distribution, this is given by:

$$\hat{\lambda}_{\text{PL}} = \sqrt{\mathbb{E}[(\Lambda|_{\theta})^2]} = \sqrt{\frac{3}{2\pi^2}\Phi(4)} = \frac{1}{2\pi}\sqrt{6\Phi(4)}.$$

– Risk Minimization Function

$$\begin{aligned} R(\lambda, \hat{\lambda}_{\text{PL}}) &= \mathbb{E} \left[\ell(\Lambda|_{\theta}, \hat{\lambda}_{\text{PL}}) \right] \\ &= \mathbb{E} \left[\frac{(\Lambda|_{\theta} - \hat{\lambda}_{\text{PL}})^2}{\hat{\lambda}_{\text{PL}}} \right] \\ &= 2 \left(\left(\frac{3}{2\pi^2}\Phi(4) \right)^{3/2} - \frac{1}{\pi}\Phi(3) \right). \end{aligned}$$



(a) The curve of BE under the PL function. (b) The curve of RMF under the PL function.

Figure 4.3 The curves of Bayesian estimator and risk minimization function for the PWEU distribution.

4.2 Characterization and Estimation of the PWEU Distribution

The following section delves into the characterization and estimation of the PWEU distribution, focusing on its moment generating function (MGF) and key statistical measures such as mean, variance, skewness, and kurtosis. Mathematical formulations are derived using special functions like the Hurwitz zeta function. Additionally, Bayesian estimators for parameter estimation, using squared error loss and precautionary loss functions, are examined. Graphical representations of these estimators are provided to illustrate their sensitivity to changes in parameters α and β .

4.2.1 Moment Generating Function and Statistical Measures

This section presents MGF, mean, variance, skewness, and kurtosis of the PWEU distribution, denoted as M_{PWEU} , μ_{PWEU} , σ_{PWEU}^2 , γ_{PWEU} , and κ_{PWEU} , respectively. Let function

$$\Psi(b) = \frac{\zeta\left(\alpha + b, \frac{\theta + \beta}{2\pi}\right)}{\zeta\left(\alpha + 1, \frac{\theta + \beta}{2\pi}\right)},$$

be introduced, therefore

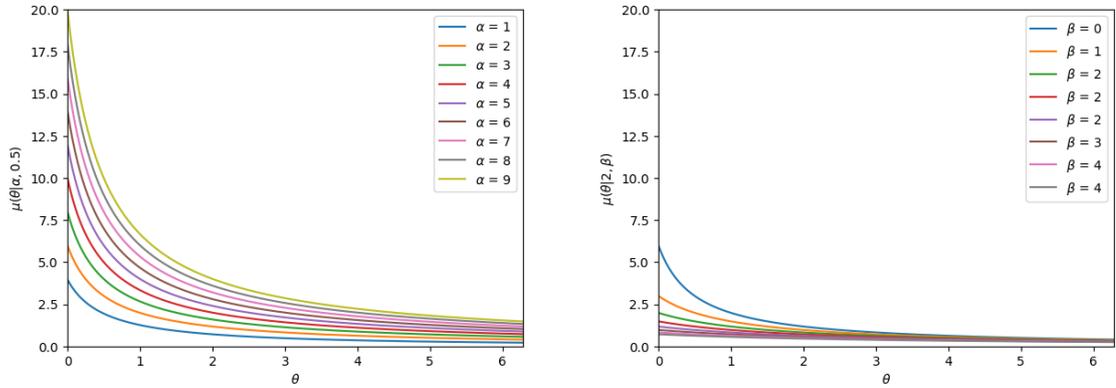
- MGF

$$\begin{aligned}
M_{\text{PWEg}}(t) &= E[e^{t\Lambda} | \theta, \alpha, \beta] \\
&= \frac{(2\pi)^{\alpha+1}}{\Gamma(\alpha+1) \zeta(\alpha+1, \frac{\theta+\beta}{2\pi})} \int_0^\infty e^{t\lambda} \left(\frac{\lambda^\alpha e^{-\lambda(\theta+\beta)}}{(1-e^{-2\pi\lambda})} \right) d\lambda \\
&= \frac{(2\pi)^{\alpha+1}}{\Gamma(\alpha+1) \zeta(\alpha+1, \frac{\theta+\beta}{2\pi})} \int_0^\infty \frac{\lambda^{(\alpha+1)-1} e^{-(\theta+\beta-t)\lambda}}{(1-e^{-2\pi\lambda})} d\lambda \\
&= \frac{(2\pi)^{\alpha+1}}{\Gamma(\alpha+1) \zeta(\alpha+1, \frac{\theta+\beta}{2\pi})} \int_0^\infty \frac{\left(\frac{u}{2\pi}\right)^{(\alpha+1)-1} e^{-\left(\frac{\theta+\beta-t}{2\pi}\right)u}}{1-e^{-u}} \frac{du}{2\pi} \\
&= \frac{1}{\zeta(\alpha+1, \frac{\theta+\beta}{2\pi})} \left(\frac{1}{\Gamma(\alpha+1)} \int_0^\infty \frac{u^{(\alpha+1)-1} e^{-\left(\frac{\theta+\beta-t}{2\pi}\right)u}}{1-e^{-u}} du \right) \\
&= \frac{\zeta(\alpha+1, \frac{\theta+\beta-t}{2\pi})}{\zeta(\alpha+1, \frac{\theta+\beta}{2\pi})}
\end{aligned}$$

- Mean

$$\mu_{\text{PWEg}} = \mathbb{E}[\Lambda | \theta, \alpha, \beta] = M'_{\text{PWEg}}(0) = \frac{(\alpha+1) \zeta(\alpha+2, \frac{\theta+\beta}{2\pi})}{2\pi \zeta(\alpha+1, \frac{\theta+\beta}{2\pi})} = \frac{\alpha+1}{2\pi} \Psi(2),$$

where $\Lambda |_{\theta, \alpha, \beta}$ is the wrapped random variable with parameters θ , α and β .



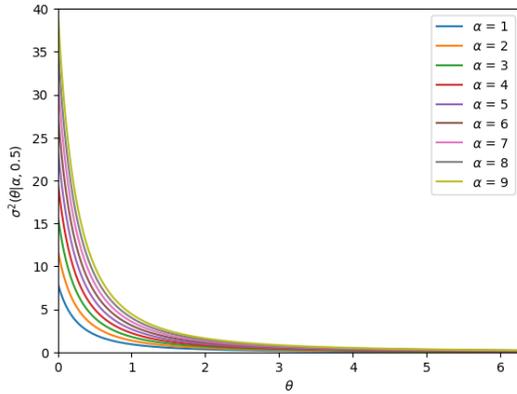
(a) The curves of mean with fixed values of $\beta = 0.5$ and and varying α .

(b) The curves of mean with fixed values of $\alpha = 2$ and and varying β .

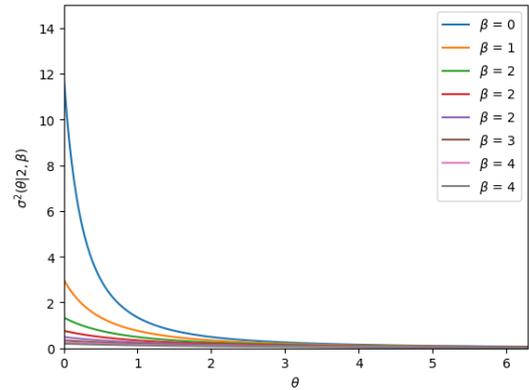
Figure 4.4 The curves of mean for the PWEg distribution.

- Variance

$$\begin{aligned}
 \sigma_{\text{PWEg}}^2 &= \mathbb{E} [(\Lambda|_{\theta, \alpha, \beta} - \mu_{\text{PWEg}})^2] \\
 &= \mathbb{E} [(\Lambda|_{\theta, \alpha, \beta})^2] - \mu_{\text{PWEg}}^2 \\
 &= M''_{\text{PWEg}}(0) - \mu_{\text{PWEg}}^2 \\
 &= \frac{(\alpha + 1)(\alpha + 2) \zeta\left(\alpha + 3, \frac{\theta + \beta}{2\pi}\right)}{(2\pi)^2 \zeta\left(\alpha + 1, \frac{\theta + \beta}{2\pi}\right)} - \left(\frac{\alpha + 1}{2\pi} \Psi(2)\right)^2 \\
 &= \frac{(\alpha + 1)(\alpha + 2)}{4\pi^2} \Psi(3) - \left(\frac{\alpha + 1}{2\pi} \Psi(2)\right)^2 \\
 &= \frac{\alpha + 1}{4\pi^2} ((\alpha + 2)\Psi(3) - (\alpha + 1)(\Psi(2))^2)
 \end{aligned}$$



(a) The curves of variance with fixed values of $\beta = 0.5$ and varying α .

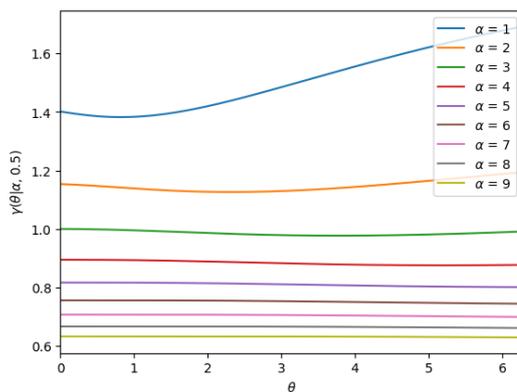


(b) The curves of mean with fixed values of $\alpha = 2$ and varying β .

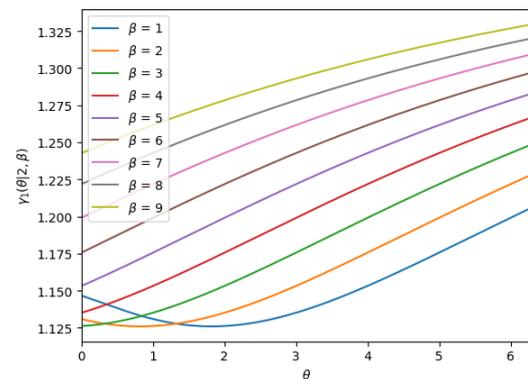
Figure 4.5 The curves of variance for the PWEg distribution.

- Skewness

$$\begin{aligned}
 \gamma_{\text{PWEg}} &= \frac{\mathbb{E} [(\Lambda|_{\theta, \alpha, \beta} - \mu_{\text{PWEg}})^3]}{\sigma_{\text{PWEg}}^3} \\
 &= \frac{\mathbb{E} [(\Lambda|_{\theta, \alpha, \beta})^3] - 3\mu_{\text{PWEg}}\sigma_{\text{PWEg}}^2 - \mu_{\text{PWEg}}^3}{\sigma_{\text{PWEg}}^3} \\
 &= \frac{M'''_{\text{PWEg}}(0) - 3\mu_{\text{PWEg}}\sigma_{\text{PWEg}}^2 - \mu_{\text{PWEg}}^3}{\sigma_{\text{PWEg}}^3} \\
 &= \frac{\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{8\pi^3} \Psi(4) - 3\left(\frac{\alpha+1}{2\pi} \Psi(2)\right)\left(\frac{\alpha+1}{4\pi^2} ((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2)\right) - \left(\frac{\alpha+1}{2\pi} \Psi(2)\right)^3}{\left(\frac{\alpha+1}{4\pi^2} ((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2)\right)^{3/2}} \\
 &= \frac{\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{8\pi^3} \Psi(4) - \frac{3(\alpha+1)^2}{8\pi^3} ((\alpha+2)\Psi(3)\Psi(2) - (\alpha+1)(\Psi(2))^3) - \frac{(\alpha+1)^3}{8\pi^3} (\Psi(2))^3}{\left(\frac{\alpha+1}{4\pi^2} ((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2)\right)^{3/2}} \\
 &= \frac{(\alpha+1)(\alpha+2)(\alpha+3)\Psi(4) - 3(\alpha+1)^2((\alpha+2)\Psi(3)\Psi(2) - (\alpha+1)(\Psi(2))^3) - (\alpha+1)^3(\Psi(2))^3}{((\alpha+1)(\alpha+2)\Psi(3) - (\alpha+1)^2(\Psi(2))^2)^{3/2}}
 \end{aligned}$$



(a) The curves of skewness with fixed values of $\beta = 0.5$ and and varying α .

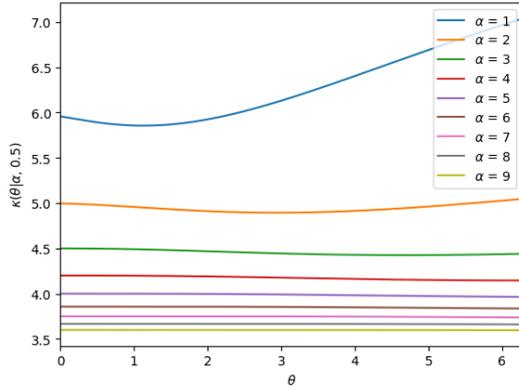


(b) The curves skewness with fixed values of $\alpha = 2$ and and varying β .

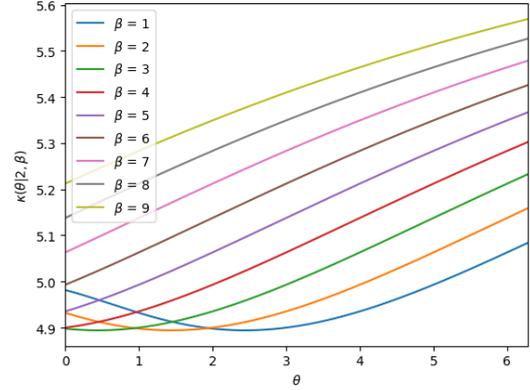
Figure 4.6 The curves of skewness for the PWEg distribution.

• Kurtosis

$$\begin{aligned}
\kappa_{\text{PWEg}} &= \frac{\mathbb{E} \left[(\Lambda|_{\theta, \alpha, \beta} - \mu_{\text{PWEg}})^4 \right]}{\sigma_{\text{PWEg}}^4} \\
&= \frac{\mathbb{E} \left[(\Lambda|_{\theta, \alpha, \beta})^4 \right] - 4\mu_{\text{PWEg}} \mathbb{E} \left[(\Lambda|_{\theta, \alpha, \beta})^3 \right] + 6\mu_{\text{PWEg}}^2 \sigma_{\text{PWEg}}^2 + 3\mu_{\text{PWEg}}^4}{(\sigma_{\text{PWEg}}^2)^2} \\
&= \frac{M_{\text{PWEg}}^{(4)}(0) - 4\mu_{\text{PWEg}} M_{\text{PWEg}}'''(0) + 6\mu_{\text{PWEg}}^2 \sigma_{\text{PWEg}}^2 + 3\mu_{\text{PWEg}}^4}{(\sigma_{\text{PWEg}}^2)^2} \\
&= \frac{\frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{(2\pi)^4} \Psi(5) - 4 \left(\frac{\alpha+1}{2\pi} \Psi(2) \right) \left(\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{8\pi^3} \Psi(4) \right) + 6 \left(\frac{\alpha+1}{2\pi} \Psi(2) \right)^2 \left(\frac{\alpha+1}{4\pi^2} ((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2) \right) + 3 \left(\frac{\alpha+1}{2\pi} \Psi(2) \right)^4}{\left(\frac{\alpha+1}{4\pi^2} ((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2) \right)^2} \\
&= \frac{\frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{16\pi^4} \Psi(5) - \frac{4(\alpha+1)^2(\alpha+2)(\alpha+3)}{16\pi^4} \Psi(2)\Psi(4) + \frac{6(\alpha+1)^3}{16\pi^4} ((\alpha+2)(\Psi(2))^2\Psi(3) - (\alpha+1)(\Psi(2))^4) + \frac{3(\alpha+1)^4}{16\pi^4} (\Psi(2))^4}{\frac{(\alpha+1)^4}{16\pi^4} ((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2)^2} \\
&= \frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\Psi(5) - 4(\alpha+1)^2(\alpha+2)(\alpha+3)\Psi(2)\Psi(4) + 6(\alpha+1)^3((\alpha+2)(\Psi(2))^2\Psi(3) - (\alpha+1)(\Psi(2))^4) + 3(\alpha+1)^4(\Psi(2))^4}{((\alpha+1)^2(\alpha+2)\Psi(3) - (\alpha+1)^3(\Psi(2))^2)^2}
\end{aligned}$$



(a) The curves of kurtosis with fixed values of $\beta = 0.5$ and varying α .



(b) The curves kurtosis with fixed values of $\alpha = 2$ and varying β .

Figure 4.7 The curves of kurtosis for the PWEg distribution.

4.2.2 Bayesian Estimators and Risk Functions for the PWEГ Distribution

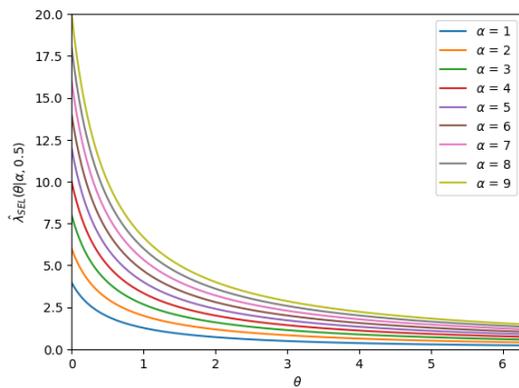
In this section, Bayesian estimators and risk minimization functions under loss functions of squared error type and precautionary type for the PWEГ distribution are presented.

- **Squared Error Loss Function**

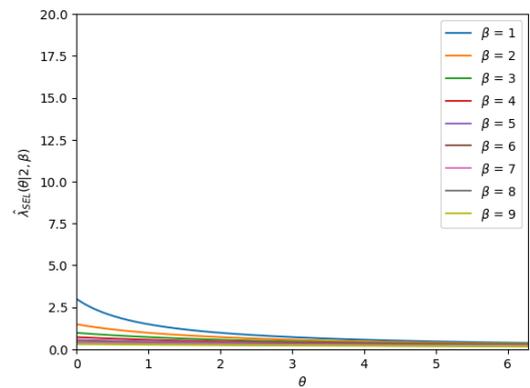
For the SEL function, the corresponding Bayesian estimator and risk minimization function are as follows:

- Bayesian Estimator

$$\hat{\lambda}_{\text{SEL}} = \mathbb{E}[\Lambda|\theta, \alpha, \beta] = \frac{\alpha + 1}{2\pi} \Psi(2),$$



(a) The curves of BE with fixed values of $\beta = 0.5$ and and varying α under the SEL function.

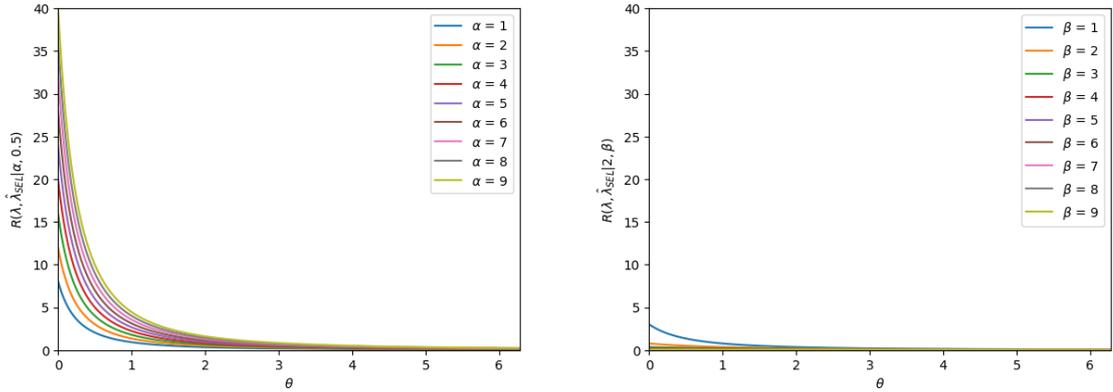


(b) The curves BE with fixed values of $\alpha = 2$ and and varying β under the SEL function.

Figure 4.8 The curves of Bayesian estimator (under the SEL function) for the PWEГ distribution.

- Minimizes Risk Function

$$\begin{aligned} R(\lambda, \hat{\lambda}_{\text{SEL}}) &= \mathbb{E} \left[\left(\Lambda|\theta, \alpha, \beta - \hat{\lambda}_{\text{SEL}} \right)^2 \right] \\ &= \frac{\alpha + 1}{4\pi^2} \left((\alpha + 2)\Psi(3) - (\alpha + 1)(\Psi(2))^2 \right). \end{aligned}$$



(a) The curves of RMF with fixed values of $\beta = 0.5$ and and varying α under the SEL function.

(b) The curves RMF with fixed values of $\alpha = 2$ and and varying β under the SEL function.

Figure 4.9 The curves of risk minimization function (under the SEL function) for the PWEG distribution.

- **Precautionary Loss Function**

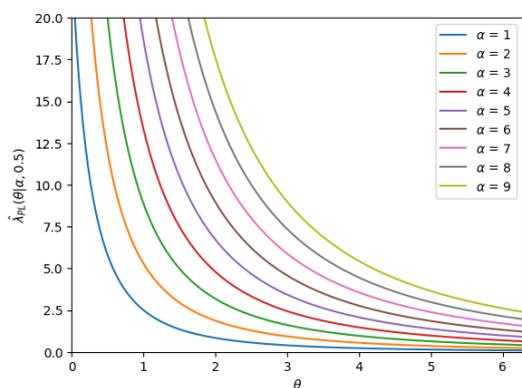
The corresponding Bayesian estimator and risk minimization function for PL function are as follows:

- Bayesian Estimator

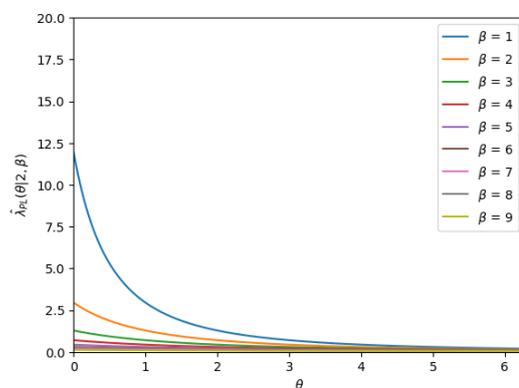
$$\hat{\lambda}_{\text{PL}} = \sqrt{\mathbb{E} [(\Lambda|_{\theta, \alpha, \beta})^2]} = \sqrt{\frac{(\alpha + 1)(\alpha + 2)}{4\pi^2} \Psi(3)},$$

- Minimizes Risk Function

$$\begin{aligned} R(\lambda, \hat{\lambda}_{\text{PL}}) &= \mathbb{E} \left[\frac{(\Lambda|_{\theta, \alpha, \beta} - \hat{\lambda}_{\text{PL}})^2}{\hat{\lambda}_{\text{PL}}} \right] \\ &= 2 \left(\left(\frac{(\alpha + 1)(\alpha + 2)}{4\pi^2} \Psi(3) \right)^{3/2} - \frac{\alpha + 1}{2\pi} \Psi(2) \right). \end{aligned}$$

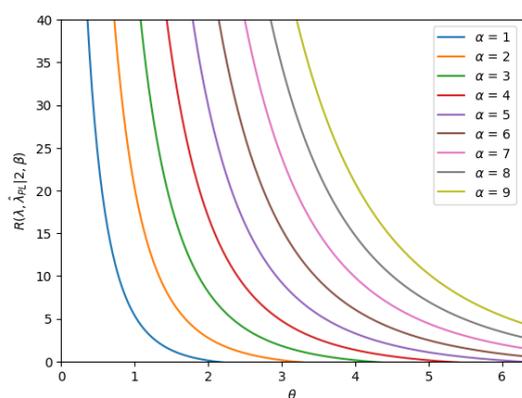


(a) The curves of BE with fixed values of $\beta = 0.5$ and and varying α under the PL function.

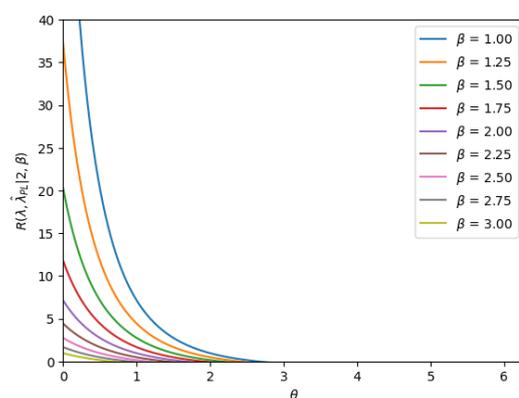


(b) The curves BE with fixed values of $\alpha = 2$ and and varying β under the PL function.

Figure 4.10 The curves of Bayesian estimator (under the PL function) for the PWEg distribution.



(a) The curves of RMF with fixed values of $\beta = 0.5$ and and varying α under the PL function.



(b) The curves RMF with fixed values of $\alpha = 2$ and and varying β under the PL function.

Figure 4.11 The curves of risk minimization function (under the PL function) for the PWEg distribution.

Table 4.1 Table of Properties of PWEU and PWEU Distributions.

Property	PWEU Distribution	PWEU Distribution
Support	$\lambda > 0$	$\lambda > 0$
Parameter	$\theta \in [0, 2\pi)$	$\theta \in [0, 2\pi), \alpha, \beta > 0$
PDF	$\frac{4\pi^2 \lambda e^{-\lambda\theta}}{(1 - e^{-2\pi\lambda}) \zeta(2, \frac{\theta}{2\pi})}$	$\frac{(2\pi)^{\alpha+1} \lambda e^{-\lambda(\theta+\beta)}}{\Gamma(\alpha+1)(1 - e^{-2\pi\lambda}) \zeta(\alpha+1, \frac{\theta+\beta}{2\pi})}$
MGF	$\frac{\zeta(2, \frac{\theta-t}{2\pi})}{\zeta(2, \frac{\theta}{2\pi})}$	$\frac{\zeta(\alpha+1, \frac{\theta+\beta-t}{2\pi})}{\zeta(\alpha+1, \frac{\theta+\beta}{2\pi})}$
Mean	$\frac{1}{\pi} \Phi(3)$	$\frac{\alpha+1}{2\pi} \Psi(2)$
Variance	$\frac{1}{\pi^2} [\frac{3}{2} \Phi(4) - (\Phi(3))^2]$	$\frac{\alpha+1}{4\pi^2} ((\alpha+2)\Psi(3) - (\alpha+1)(\Psi(2))^2)$
skewness	$\frac{3\Phi(5) - \frac{9}{2}\Phi(3)\Phi(4) + 2(\Phi(3))^3}{(\frac{3}{2}\Phi(4) - (\Phi(3))^2)^{3/2}}$	$\frac{(\alpha+1)(\alpha+2)\Psi(3) - (\alpha+1)^2(\Psi(2))^2}{(\alpha+1)(\alpha+2)(\alpha+3)\Psi(4)}$
kurtosis	$\frac{\frac{15}{2}\Phi(6) - 12\Phi(3)\Phi(5) + 6(\Phi(3))^2\Phi(4) - 3(\Phi(3))^4}{(\frac{3}{2}\Phi(4) - (\Phi(3))^2)^2}$	$\frac{(\alpha+1)(\alpha+2)(\alpha+3)\Psi(5) - 4(\alpha+1)^2(\alpha+2)(\alpha+3)\Psi(2)\Psi(4) + 6(\alpha+1)^3((\alpha+2)\Psi(2))^2\Psi(3) - (\alpha+1)(\Psi(2))^4 + 3(\alpha+1)^4(\Psi(2))^4}{((\alpha+1)^2(\alpha+2)\Psi(3) - (\alpha+1)^3(\Psi(2))^2)^2}$