CHAPTER III

RESEARCH METHODOLOGY

3.1 The Wrapped Exponential Distribution with an Uninformative Prior Distribution

A scenario is considered in this study where the observed data are assumed to exhibit a specific pattern, prompting the adoption of an improper prior distribution for the parameter, characterized by the prior probability density function $h(\lambda) = 1$. This uninformative improper prior is selected to assign a constant density across all positive values of the parameter λ, reflecting a non-normalizable distribution that does not integrate to a finite value over an unbounded domain. Within the framework of the Wrapped Exponential Distribution, frequently applied to model circular or angular data such as directions or orientations, this improper prior is employed to represent a state of complete ignorance regarding the parameter λ. Consequently, the inference process is driven primarily by the observed data, ensuring that conclusions are not swayed by restrictive prior assumptions about the parameter's behavior.

Through Bayesian analysis of the wrapped exponential distribution with this prior distribution, we obtain the following posterior PDF:

$$
\begin{aligned}
k\left(\lambda|\theta\right) &= \frac{\left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)\cdot 1}{\int_0^\infty \left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)\cdot 1 d\lambda} \\
&= \frac{\left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)}{\int_0^\infty \frac{\left(\frac{y}{2\pi}\right)^{2-1} e^{-\left(\frac{y}{2\pi}\right)\theta}}{1-e^{-y}} \frac{dy}{2\pi}} \qquad (\text{let } y = 2\pi\lambda) \\
&= \frac{\left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)}{\frac{1}{(2\pi)^2}\int_0^\infty \frac{y^{2-1} e^{-\left(\frac{\theta}{2\pi}\right)y}}{1-e^{-y}} dy}.
\end{aligned}
$$

Since the integral term can be represented by the Hurwitz Zeta function, therefore

$$k\left(\lambda|\theta\right)=\frac{4\pi^2\lambda e^{-\lambda\theta}}{\left(1-e^{-2\pi\lambda}\right)\zeta\left(2,\frac{\theta}{2\pi}\right)},$$

where $\Gamma(\cdot)$is the gamma function and $\zeta(\cdot,\cdot)$ is the Hurwitz Zeta function. We say that the distribution characterized by this posterior PDF is *the posterior wrapped exponential distribution with an uninformative prior distribution* (PWEU distribution). Here $\theta\in[0,2\pi)$ is the random variable of the wrapped exponential distribution representing the time between events and $\lambda>0$ is the rate parameter of the exponential distribution determining the average number of events per unit time.

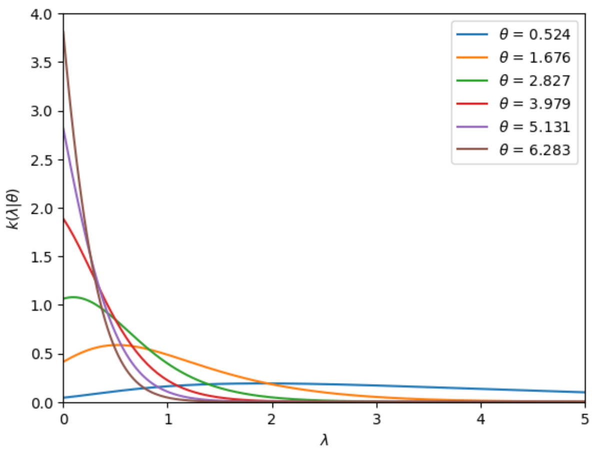


Figure 3.1 The curves of the posterior PDF of the PWEU distribution.

3.2 The Wrapped Exponential Distribution with the Gamma Prior Distribution

A case is examined where the observed data are assumed to follow a wrapped exponential distribution, and a gamma distribution is adopted as the prior for the parameter λ. The prior probability density function is given by

$$f\left(x\mid\alpha,\beta\right)=\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}x^{\alpha-1}e^{-\beta x},$$

where $\alpha>0$ is the shape parameter and $\beta>0$ is the scale parameter. This informative prior is selected to reflect prior knowledge about the rate parameter λ, with beliefs about its magnitude and variability incorporated through the parameters α and β. Within the context of the Wrapped Exponential Distribution, commonly employed to model circular or angular data such as time intervals or orientations, the gamma prior is utilized to provide a flexible framework that balances prior assumptions with empirical evidence. Consequently, the Bayesian inference process is shaped by combining the influence of the observed data with the prior distribution, resulting in a posterior that reflects both sources of information.

The following presents the Bayesian analysis of the wrapped exponential distribution with the gamma prior distribution:

$$\begin{aligned}
k(\lambda \mid \theta, \alpha, \beta) &= \frac{\left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\right)}{\int_0^\infty \left(\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\right)d\lambda} \\
&= \frac{\frac{\lambda^\alpha e^{-\lambda(\theta+\beta)}}{1-e^{-2\pi\lambda}}}{\int_0^\infty \frac{\lambda^{(\alpha+1)-1}e^{-\lambda(\theta+\beta)}}{1-e^{-2\pi\lambda}}d\lambda} \\
&= \frac{\frac{\lambda^\alpha e^{-\lambda(\theta+\beta)}}{1-e^{-2\pi\lambda}}}{\int_0^\infty \frac{\left(\frac{y}{2\pi}\right)^{(\alpha+1)-1}e^{-\left(\frac{y}{2\pi}\right)(\theta+\beta)}}{1-e^{-y}}\frac{dy}{2\pi}} \qquad (\text{let } y = 2\pi\lambda) \\
&= \frac{\frac{\lambda^\alpha e^{-\lambda(\theta+\beta)}}{1-e^{-2\pi\lambda}}}{\frac{1}{(2\pi)^{\alpha+1}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\int_0^\infty \frac{y^{(\alpha+1)-1}e^{-\left(\frac{\theta+\beta}{2\pi}\right)y}}{1-e^{-y}}dy}.
\end{aligned}$$

By the representing the integral term with the Hurwitz Zeta function, we obtain

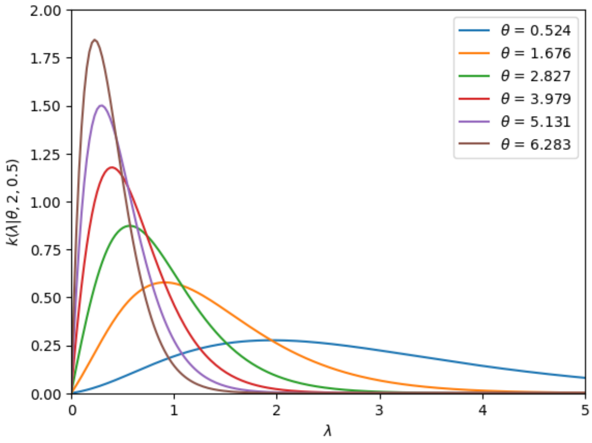
$$k(\lambda \mid \theta, \alpha, \beta) = \frac{(2\pi)^{\alpha+1}\lambda^\alpha e^{-\lambda(\theta+\beta)}}{\Gamma(\alpha+1)\left(1-e^{-2\pi\lambda}\right)\zeta\left(\alpha+1, \frac{\theta+\beta}{2\pi}\right)},$$

where $\Gamma(\cdot)$ is the gamma function and $\zeta(\cdot,\cdot)$ is the Hurwitz Zeta function.

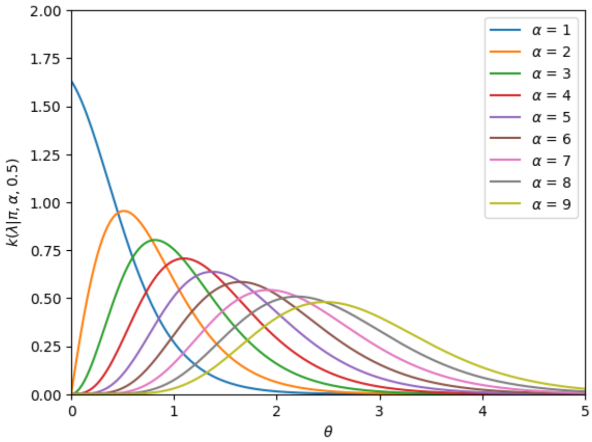
The posterior PDF for the wrapped exponential distribution with the gamma prior distribution as the PDF given by the following:

$$k_{\text{PWEG}}(\lambda|\theta, \alpha, \beta) = \frac{(2\pi)^{\alpha+1}\lambda^\alpha e^{-\lambda(\theta+\beta)}}{\Gamma(\alpha+1)\left(1-e^{-2\pi\lambda}\right)\zeta\left(\alpha+1, \frac{\theta+\beta}{2\pi}\right)}, \qquad \lambda, \alpha, \beta > 0, \theta \in [0, 2\pi),$$

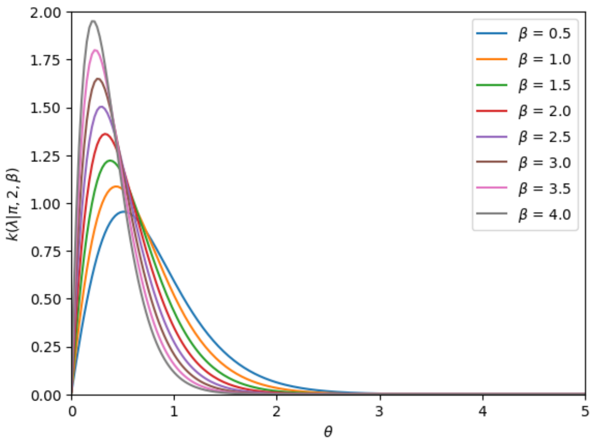
where θ, α and β are the parameters defined in the previous section. To keep it simple, we refer to the posterior wrapped exponential distribution with the gamma prior distribution as PWEG distribution and its probability density function as PWEG PDF. For better clarity, graphs of the posterior PDF for the wrapped exponential distribution with various parameters are presented in Figure 3.2.



(a) The curves of the PWEG PDF with fixed values of $\alpha = 2$ and $\beta = 0.5$, and varying θ.



(b) The curves of the PWEG PDF with fixed values of $\theta = \pi$ and $\beta = 0.5$, and varying α.



(c) The curves of the PWEG PDF with fixed values of $\theta = \pi$ and $\alpha = 2$, and varying β.

Figure 3.2 The curves of the PWEU PDF.