ONE-DIMENSIONAL MAGNETOGASDYNAMICS EQUATIONS: LIE GROUP CLASSIFICATION, CONSERVATION LAWS

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A Thesis Submitted in Partial Fulfillment of the Requirements for the

Degree of Master of Science in Applied Mathematics

Suranaree University of Technology

Academic Year 2022

สมการแก๊สพลศาสตร์เชิงแม่เหล็กในหนึ่งมิติ:
การจำแนกกรุปของลี, กฎการอนุรักษ์

นายพชรพล มุกดาสนิท

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์
มหาวิทยาลัยเทคโนโลยีสุรนารี
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คำสำคัญ : ลากรานจ์เจียนของมวล/สมการออยเลอร์–ลากรานจ์/กฎการอนุรักษ์/ทฤษฎีของนออีเธอร์

ด้วยการสร้างแบบจำลองทางคณิตศาสตร์ของปรากฏการณ์ทางฟิสิกส์เป็นหนึ่งในการศึกษาหลักของกลศาสตร์ต่อเนื่อง วิทยานิพนธ์นี้จึงได้ศึกษาการวิเคราะห์สมมาตรของสมการพลศาสตร์เชิงแม่เหล็กสำหรับก๊าซในอุดมคติสมบูรณ์หนึ่งมิติภายใต้สภาพนำไฟฟ้าเป็นอนันต์ ทั้งนี้สมการดังกล่าวถูกพิจารณาในระบบพิกัดลากรานจ์ของมวล และในการหาผลเฉลยของปัญหาเฮล์มโฮลซ์ได้ค้นพบลากรานจ์เจียนที่สมนัยซึ่งเป็นสมการออยเลอร์–ลากรานจ์ของปัญหาดังกล่าว สำหรับการใช้ทฤษฎีบทของนออีเธอร์มีการใช้การจำแนกกรุปสำหรับเอนโทรปีและสนามแม่เหล็ก ซึ่งการจำแนกกรุปดังกล่าวด้วยแนวคิดทางพีชคณิตทำให้การวิเคราะห์ง่ายขึ้นเป็นอย่างมากลากรานจ์เจียนและการจำแนกกรุปที่ได้ทำให้สามารถใช้ทฤษฎีบทของนออีเธอร์ในการสร้างกฎการอนุรักษ์ต่าง ๆ นอกจากนี้ในวิทยานิพนธ์ได้แสดงกฎการอนุรักษ์ต่างๆในรูปแบบของตัวแปรทางฟิสิกส์อีกด้วย

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา พชรพล มุกดาสนิท

ปีการศึกษา 2565 ลายมือชื่ออาจารย์ที่ปรึกษา

POTCHARAPOL MUKDASANIT : ONE-DIMENSIONAL MAGNETOGASDYNAMICS EQUATIONS: LIE GROUP CLASSIFICATION, CONSERVATION LAWS. THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D. 38 PP.

Keyword: MASS LAGRANGIAN/ EULER-LAGRANGE EQUATIONS/ CONSERVATION LAWS/ NOETHER'S THEOREM/ GROUP CLASSIFICATION

Mathematical modeling of physical phenomena is one of the main studies in continuum mechanics. This thesis is devoted to the symmetry analysis of the one-dimensional magnetogasdynamics equations of an ideal perfect gas with infinite electrical conductivity. The equations are considered in mass Lagrangian coordinates. Solving the Helmholtz problem, a corresponding Lagrangian such that the equations in the study are Euler-Lagrange equations is found. For using Noether's theorem a group classification with respect to the entropy and magnetic field is performed. The group classification is made by an algebraic approach, which essentially simplifies the analysis. The found Lagrangian and the group classification allow us to use Noether's theorem for constructing conservation laws. Physical interpretation of the found conservation laws is given.

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ACKNOWLEDGEMENTS

First and foremost, I wish to express my profound gratitude to my research advisor, Prof. Sergey Meleshko, for the opportunity to conduct research under his guidance. It has been both an honor and a privilege to study and work under his mentorship.

In addition, I would like to extend my special thanks to the Institute for the Promotion of Technology (IPST) and the Development and Promotion of Science and Technology Talents Project (DPST), for granting me a three-year scholarship at Suranaree University of Technology.

Lastly, my appreciation goes to my family and my friends. Their constant encouragement and support have been invaluable, and their respect for my decisions.

Potcharapol Mukdasanit

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CHAPTER I

INTRODUCTION

Fluid mechanics is a field that seeks to describe the behavior of fluid waves, as understanding these behaviors can aid in predicting fluid motion phenomena. One specific area of fluid dynamics is magnetogasdynamics, which investigates the motion of electrically conducting fluids in a magnetic field. The magnetogasdynamics equations (MGD), also referred to as magnetohydrodynamics equations (MHD), describe various phenomena related to plasma flows, such as plasma confinement, solar, and heliospheric plasma.

The MGD equations in Eulerian coordinates are presented in many books, for example, Kulikovskii and Lyubimov (1965), Landau and Lifshitz (1987), Webb (2018) and many others. However, the study of MGD equations in Lagrangian coordinates has been less successful, with limited publications available in the literature. Thus, there is a need to investigate the MGD equations in Lagrangian coordinates to better understand the principles and develop more models of fluid behavior.

In this thesis, we consider the plane one-dimensional MGD equations of a polytropic gas, which can represent the motion of an MGD fluid. We assume that the fluids are inviscid with infinite electrical conductivity. Additionally, for the one-dimensional case, the magnetic field H $= (H_1, H_2, H_3)$ has the property of a constant H_1. We examine the equations in mass Lagrangian coordinates, aiming to identify a Lagrangian and perform group classification. Our research's objective is to construct conservation laws by applying Noether's theorem (Noether, 1918).

1.1 Research Objective

The research objective is to find the general form of a Lagrangian for plane one-dimensional flows of magnetogasdynamics equations, to obtain their group classification and use them for constructing conservation laws by Noether's theorem.

1.2 Scope and Limitations

This project focuses on one-dimensional equations that model the behavior of MGD. We specifically consider cases where $H_1 = 0$ and investigate the case of a polytropic gas.

1.3 Research Procedure

In this study, we first examine the magnetogasdynamics equations and investigate the relations between Eulerian and Lagrangian coordinates. Following this, we aim to find a Lagrangian and make group classification. Lastly, the conservation laws are constructed by using Noether's theorem.

CHAPTER II

LITERATURE REVIEW

This section introduces the magnetogasdynamics equations, focusing on their application to one-dimensional flows. It then proceeds to explain the group analysis method, the derivation of prolongation formulas, the process of group classification, and the application of Noether's theorem.

2.1 Magnetogasdynamics equations

The equations of magnetogasdynamics (MGD) describe motion of electrically conducting fluids under the action of the internal forces related with a magnetic field.

2.1.1 Three-dimensional MGD

The magnetogasdynamics equations (MGD) in Eulerian coordinates can be written in many forms. In this work we take the dimensionless form of the MGD equations with infinite conductivity

$$\rho_t + \mathrm{div}(\rho \mathrm{u}) = 0, \tag{2.1a}$$

$$\mathrm{u}_t + (\mathrm{u} \cdot \nabla)\mathrm{u} = -\frac{1}{\rho}\nabla p + (\nabla \times \mathrm{H}) \times \mathrm{H}, \tag{2.1b}$$

$$\epsilon_t + (\mathrm{u} \cdot \nabla)\epsilon = -\frac{p}{\rho}\mathrm{div}(\mathrm{u}), \tag{2.1c}$$

$$\mathrm{H}_t = \nabla \times [\mathrm{u} \times \mathrm{H}], \quad \mathrm{div}(\mathrm{H}) = 0. \tag{2.1d}$$

Here ρ is the density, p is the pressure, ϵ is the internal energy per unit volume, $\mathbf{x} = (x, y, z)$ is the coordinates, $\mathrm{u} = (u, v, w)$ is the velocity, and $\mathrm{H} = (H_1, H_2, H_3)$ is the magnetic field (the dependent variables $\rho, \mathrm{u}, \epsilon, \mathrm{H}$ are functions of the independent variables (t, x, y, z));

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

is the gradient operator, and

$$\text{div} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

is the divergence operator. Equations (2.1a)-(2.1c) are equations of conservation laws of mass, momentum, and energy, respectively, and equations (2.1d) are Faraday's law and Gauss's law. Here, we denote the patial derivative of the dependent variable with respect to the independent variable by using subscript by the dependent variable. For example,

$$\rho_t = \frac{\partial \rho}{\partial t}, \qquad \rho_x = \frac{\partial \rho}{\partial x}, \qquad \rho_y = \frac{\partial \rho}{\partial y}, \qquad \rho_z = \frac{\partial \rho}{\partial z}.$$

2.1.2 First thermodynamics law

The first law of thermodynamics is a formulation of the law of conservation of energy, adapted for thermodynamic processes. For a reversible process, the relation between the change in the internal energy of a closed system, the quantity of heat, and the quantity of work performed by the system on the surrounding, the 1-st thermodynamics law can be written in the form

$$TdS = d\epsilon + pd\frac{1}{\rho}, \tag{2.2}$$

where T is the temperature, and S is entropy.

A two-parametric model is a model that specifies two parameters as arbitrary, and the others are functions of them. From these two parameters, the other parameter can be found by using the first law of thermodynamics. The MGD system (2.1) should be complemented by an equation of state, which takes the form:

$$\epsilon = \epsilon(\rho, S).$$

We examine a medium characterized by an ideal gas, which obeys the Boyle-Mariotte law

$$p = \rho RT; \tag{2.3}$$

here R is the specific gas constant.

2.1.3 Polytropic gas

The polytropic ideal gas is defined by a linear relation between the internal energy function ϵ and the temperature (Chorin et al., 1990), (Ovsiannikov, 2003),

$$\epsilon = c_v T, \tag{2.4}$$

where c_v is the specific heat of the gas at constant volume, which is constant.

The 1-st thermodynamics form (2.2) changes to

$$TdS = c_v dT + pd\frac{1}{\rho}. \tag{2.5}$$

Next, apply the two-parameter model, where $\epsilon = \epsilon(\rho, S)$ and $T = T(\rho, S)$ to equation (2.4), and then apply the result to equation (2.5)

$$TdS = \epsilon_\rho d\rho + \epsilon_S dS - \frac{p}{\rho^2}d\rho,$$

thus, we get

$$\epsilon_S = T \quad \text{and} \quad \epsilon_\rho = \frac{RT}{\rho}, \tag{2.6}$$

where the subindex means derivative, for example $\epsilon_S = \frac{\partial \epsilon}{\partial S}$, $\epsilon_\rho = \frac{\partial \epsilon}{\partial \rho}$.
Substituting ϵ from equation (2.4) into equation (2.6), we have

$$T = c_v T_S, \qquad \frac{RT}{\rho} = c_v T_\rho. \tag{2.7}$$

Integrating equation (2.7), we get

$$T = Ce^{\frac{1}{c_v}S},$$

where $C = C(\rho)$ is an arbitrary function, and

$$\frac{RT}{\rho} = c_v T_\rho.$$

Substituting the function T that we get into the equation above, we get

$$\frac{RCe^{\frac{S}{c_v}}}{\rho} = c_v C' e^{\frac{S}{c_v}},$$

$$C' = \frac{R}{c_v}\frac{C}{\rho},$$

then

$$C = K\rho^{\frac{R}{cv}},$$

where K is constant.

Substituting C back into the function T we obtain the result

$$T = K\rho^{\frac{R}{cv}} e^{\frac{S}{cv}}, \quad \text{and} \quad p = RK\rho^{\frac{R}{cv}+1} e^{\frac{S}{cv}}.$$

Let

$$\gamma = \frac{R}{c_v} + 1 \quad \text{and} \quad K = e^{-\frac{S_0}{cv}},$$

where S_0 is constant.

Following that, the pressure is

$$p = A(S)\rho^{\gamma},$$

where $A(S) = Re^{\frac{S-S_0}{cv}}$ and $c_v = \frac{R}{\gamma-1}$.

Then, the internal energy becomes

$$\epsilon = \frac{RT}{\gamma - 1}. \tag{2.8}$$

We consider the system of equations described by the polytropic gas. From equation (2.3) and equation (2.8), we obtain the equation of state

$$\epsilon = \frac{1}{\gamma - 1}\frac{p}{\rho}. \tag{2.9}$$

The pressure, the density and the entropy are related by the equation

$$p = \tilde{S}\rho^{\gamma}, \tag{2.10}$$

where $\tilde{S} = A(S)$ and $\gamma > 1$ is the polytropic exponent (further the sign " $\sim$ " is omitted).

The equation of conservation of energy (2.1c) can be written in different forms. It can be the equation for the pressure p

$$p_t + (\mathrm{u} \cdot \nabla)p + \gamma p \mathrm{div}(\mathrm{u}) = 0, \tag{2.11}$$

or the equation for the entropy S

$$S_t + (\mathrm{u} \cdot \nabla)S = 0. \tag{2.12}$$

Rewrite system (2.1)

$$\frac{d}{dt}\rho + \rho\mathrm{div}(\mathrm{u}) = 0, \tag{2.13a}$$

$$\frac{d}{dt}\mathrm{u} + \frac{1}{\rho}\nabla(p + \frac{1}{2}\mathrm{H}^2) - \frac{1}{\rho}(\mathrm{H} \cdot \nabla)\mathrm{H} = 0, \tag{2.13b}$$

$$\frac{d}{dt}\epsilon + \frac{p}{\rho}\mathrm{div}(\mathrm{u}) = 0, \tag{2.13c}$$

$$\frac{d}{dt}\mathrm{H} = (\mathrm{H} \cdot \nabla)\mathrm{u} - \mathrm{H}\mathrm{div}(\mathrm{u}), \qquad \mathrm{div}(\mathrm{H}) = 0, \tag{2.13d}$$

here $\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathrm{u} \cdot \nabla)$ is the material derivative.

Equation (2.11) and equation (2.12) get rewritten as

$$\frac{d}{dt}p + \gamma p\mathrm{div}(\mathrm{u}) = 0, \tag{2.14}$$

and

$$\frac{d}{dt}S = 0. \tag{2.15}$$

2.1.4 Plane one-dimensional equations

The plane one-dimensional MGD equations represent one-dimensional MGD flows (Kulikovskii et al., 1965). The system of equations contains all dependent variables, which are functions of only two independent scalar variables: t and x. Note that in this project, one-dimensional means that the variables depend on single-direction variable and one time-variable. In this case, the equations

$$(H_1)_t = 0, \quad \text{and} \quad \mathrm{div}(\mathrm{H}) = (H_1)_x = 0,$$

yield that H_1 is constant.

System of equations (2.13), equation (2.14), and equation (2.15) are reduced to

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{2.16a}$$

$$\rho(u_t + uu_x) + \left(\frac{1}{2}(H_2^2 + H_3^2) + p\right)_x = 0, \tag{2.16b}$$

$$\rho(v_t + uv_x) = H_1(H_2)_x, \tag{2.16c}$$

$$\rho(w_t + uw_x) = H_1(H_3)_x, \tag{2.16d}$$

$$(H_2)_t + (uH_2)_x = H_1 v_x, \tag{2.16e}$$

$$(H_3)_t + (uH_3)_x = H_1 w_x, \tag{2.16f}$$

$$S_t + uS_x = 0. \tag{2.16g}$$

An equation of the form

$$(\phi)_t + (\psi)_x = G$$

is called a conservation law, where ϕ, ψ, and G are arbitrary functions.

In the case that G is zero, we can apply Noether's theorem to identify the conservation law.

Next, we aim to express the equations in conservation form, as it makes them simpler to analyze and identify the conservation law. Hence, system (2.16) becomes

$$\rho_t + (\rho u)_x = 0, \tag{2.17a}$$

$$(\rho u)_t + \left(\rho u^2 + \frac{1}{2}(H_2^2 + H_3^2) + p\right)_x = 0, \tag{2.17b}$$

$$(\rho v)_t + (\rho uv - H_1 H_2)_x = 0, \tag{2.17c}$$

$$(\rho w)_t + (\rho uw - H_1 H_3)_x = 0, \tag{2.17d}$$

$$(H_2)_t + (uH_2 - H_1 v)_x = 0, \tag{2.17e}$$

$$(H_3)_t + (uH_3 - H_1 w)_x = 0, \tag{2.17f}$$

$$\left(\rho(\epsilon + \frac{1}{2}u^2)\right)_t + \left(\rho u(\epsilon + \frac{1}{2}u^2) + u(p + \frac{1}{2}(H_2^2 + H_3^2))\right)_x = 0. \tag{2.17g}$$

2.2 Lagrangian coordinates

In this part, we scope on one-dimensional space. In continuum mechanics, there are two ways for analyzing a field $\mathcal{F}$, when fluid is moving. The first one is called the Eulerian way, where the field $\mathcal{F}$ is expressed in (x, t). Here t is the time and x is the position in the fixed coordinate system at the time t_0.

The second way is called the Lagrangian way. In this case, the function $\mathcal{F}$ depends on (ξ, t), where the position ξ is considered in the coordinate, which is moving with a particle. The coordinate $x = \xi$ at the time t_0.

Relations between the Eulerian and Lagrangian coordinates are defined by the function $\varphi(\xi, t)$

$$x = \varphi(\xi, t). \tag{2.18}$$

The velocity of a particle labeled by position ξ is $\tilde{u}(\xi, t) = \varphi_t(\xi, t)$. The velocities in Lagrangian coordinate system $\tilde{u}(\xi, t)$ and in Eulerian coordinate system $u(x, t)$ are related by the formula

$$\tilde{u}(\xi, t) = u(\varphi(\xi, t), t).$$

A similar relation between the density of fluid, presented in Eulerian coordinates $\rho(x, t)$, and in Lagrangian coordinates $\tilde{\rho}(\xi, t)$ is

$$\rho(\varphi(\xi, t), t) = \tilde{\rho}(\xi, t).$$

The advantage of using Lagrangian coordinates is that the equations of conservation law of mass and conservation law of energy can be integrated

$$\tilde{\rho}(\xi, t) = \frac{\rho_0(\xi)}{\varphi_\xi(\xi, t)}, \quad \text{and} \quad S = S(\xi),$$

where $\rho_0(\xi)$ is a function of integration.

Introducing the variable s such that $\xi = \alpha(s)$, where

$$\alpha'(s) = \rho_0(\alpha(s)).$$

The first equation is simplified to coordinate of (s, t)

$$\hat{\rho}(s, t) = \frac{1}{\hat{\varphi}_s(s, t)}, \tag{2.19}$$

where $\hat{\varphi}(s,t) = \varphi(\alpha(s),t)$ and $\hat{\rho}(s,t) = \tilde{\rho}(\alpha(s),t)$.

Note also that

$$\hat{u}(s,t) = \tilde{u}(\alpha(s),t) = \hat{\varphi}_t(s,t).$$

Summarizing, the following independent variables are used:
(x,t) are Eulerian coordinates, (ξ,t) are Lagrangian coordinates, (s,t) are mass Lagrangian coordinates. Further the sign "^" is omitted for mass Lagrangian coordinates.

By using these relations, system (2.17) becomes

$$\varphi_{tt} + \left(\frac{1}{2}(H_2^2 + H_3^2) + p\right)_s = 0, \tag{2.20a}$$

$$(H_2\varphi_s)_t = 0, \tag{2.20b}$$

$$(H_3\varphi_s)_t = 0, \tag{2.20c}$$

$$S_t = 0. \tag{2.20d}$$

2.3 Group analysis method

The group analysis method is one of the methods for finding exact solutions and properties of differential equations. For ordinary differential equations, it can reduce the order of the equations. For partial differential equations, it provides a representation of an invariant solution with a reduced number of the independent variables. We review here the basic concepts of group analysis. Details can be found in the books Ovsiannikov (1982), Ibragimov (1985), and Olver (1986). For the simplicity, we consider s, t, and φ are variables.

Definition 2.1. Let T_a be a set of invertible transformations depending on a real valued parameter $a \in \triangle \subset R, (s,t,\varphi) \in D \subset R^3$,

$$\bar{s} = f(s,t,\varphi,a),$$

$$\bar{t} = g(s,t,\varphi,a),$$

$$\bar{\varphi} = q(s,t,\varphi,a),$$

where $\triangle$ is a symmetric interval around 0, $(\bar{s}, \bar{t}, \bar{\varphi}) \in D$, and $D \subset R^3$ is an open set.

A local one-parameter Lie group of transformations requires to have the following four properties:

1. For $a = 0$ the transformation T_0 is the identity transformation:

$$f(s,t,\varphi,0) = s,$$
$$g(s,t,\varphi,0) = t,$$
$$q(s,t,\varphi,0) = \varphi.$$

2. A composition of two transformations satisfies the property:

$$T_b T_a = T_{a+b},$$

for any $a, b, a+b \in \triangle$ and $(s,t,\varphi) \in D$. That is for all $(s,t,\varphi) \in D$ and $a, b, a+b \in \triangle$ one has

$$f(f(s,t,\varphi,a), g(s,t,\varphi,a), q(s,t,\varphi,a), b) = f(s,t,\varphi,a+b),$$
$$g(f(s,t,\varphi,a), g(s,t,\varphi,a), q(s,t,\varphi,a), b) = g(s,t,\varphi,a+b),$$
$$q(f(s,t,\varphi,a), g(s,t,\varphi,a), q(s,t,\varphi,a), b) = q(s,t,\varphi,a+b).$$

3. If for $\forall (s,t,\varphi) \in D$ one has

$$f(s,t,\varphi,a_0) = s,$$
$$g(s,t,\varphi,a_0) = t,$$
$$q(s,t,\varphi,a_0) = \varphi,$$

then

$$a_0 = 0.$$

4. The functions f, g, q are sufficiently smooth.

A local one-parameter Lie group is simply called a Lie group.

Definition 2.2. Let (f, g, q) compose a Lie group of transformations. The infinitesimal generator

$$X = \xi(s,t,\varphi)\frac{\partial}{\partial s} + \eta(s,t,\varphi)\frac{\partial}{\partial t} + \zeta(s,t,\varphi)\frac{\partial}{\partial \varphi},$$

where

$$\xi(s,t,\varphi) = \left(\frac{\partial}{\partial a} f(s,t,\varphi,a)\right)_{|a=0},$$
$$\eta(s,t,\varphi) = \left(\frac{\partial}{\partial a} g(s,t,\varphi,a)\right)_{|a=0},$$
$$\zeta(s,t,\varphi) = \left(\frac{\partial}{\partial a} q(s,t,\varphi,a)\right)_{|a=0},$$

is called the generator of the group of transformations.

On the other hand, a Lie group can be reconstructed from its generator by solving the Lie equations

$$\frac{\partial f}{\partial a} = \xi(f,g,q),$$
$$\frac{\partial g}{\partial a} = \eta(f,g,q),$$
$$\frac{\partial q}{\partial a} = \zeta(f,g,q),$$

with the initial conditions

$$f_{|a=0} = s,$$
$$g_{|a=0} = t,$$
$$q_{|a=0} = \varphi.$$

2.4 Prolongations formulas

Consider that φ is a dependent variable and (t,s) are independent variables.

Definition 2.3. Let the generator be

$$X = \xi\frac{\partial}{\partial s} + \eta\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial \varphi},$$

then the generator

$$X^p = X + \zeta^{\varphi_s}\frac{\partial}{\partial \varphi_s} + \zeta^{\varphi_t}\frac{\partial}{\partial \varphi_t} + \zeta^{\varphi_{ss}}\frac{\partial}{\partial \varphi_{ss}} + \zeta^{\varphi_{ts}}\frac{\partial}{\partial \varphi_{ts}} + \zeta^{\varphi_{tt}}\frac{\partial}{\partial \varphi_{tt}},$$

with the coefficients

$$\xi(s,t,\varphi), \quad \eta(s,t,\varphi), \quad \zeta(s,t,\varphi),$$

$$\begin{aligned}
\zeta^{\varphi_s} &= D_s\zeta - \varphi_s D_s\xi - \varphi_t D_s\eta, \\
\zeta^{\varphi_t} &= D_t\zeta - \varphi_s D_t\xi - \varphi_t D_t\eta, \\
\zeta^{\varphi_{ss}} &= D_s\zeta^s - \varphi_{ss} D_s\xi - \varphi_{ts} D_s\eta, \\
\zeta^{\varphi_{ts}} &= D_s\zeta^t - \varphi_{ts} D_s\xi - \varphi_{tt} D_s\eta, \\
\zeta^{\varphi_{tt}} &= D_t\zeta^t - \varphi_{ts} D_t\xi - \varphi_{tt} D_t\eta,
\end{aligned}$$

is called the second-order prolongation of the generator X.
Here D_t and D_s are total differentiation operators with respect to t and s, respectively.

2.5 Group classification

A Lie group and the equations in this study are related by the notion of an admitted Lie group.

2.5.1 Admitted Lie group

The first step of the group analysis method consists of finding the admitted Lie group. One of the definitions of the Lie group admitted by a system of differential equations is defined by the property that a solution of the differential equation is mapped by a transformation from a Lie group into a solution of the same equation.

Consider a system of differential equations defined by the equation

$$F(t,s,\varphi,\varphi_t,\varphi_s,\varphi_{tt},\varphi_{ts},\varphi_{ss}) = 0,$$

where F is a vector-function.

A Lie group of transformations satisfying the determining equation

$$X^p F_{|_{F=0}} = 0,$$

is called a Lie group admitted by equation $F = 0$, or simply an admitted Lie group.

Most differential equations of fluid dynamics include arbitrary elements denoted here by f, which can be functions and constants. A Lie group admitted by differential equations including these elements depends on concrete choice of these elements. Further the concept of the group classification is discussed.

2.5.2 Equivalence transformation

Let the studied equation be

$$F(t, s, \varphi, \varphi_t, \varphi_s, \varphi_{tt}, \varphi_{ts}, \varphi_{ss}, f) = 0.$$

One of the problems of group analysis is to find transformations which change arbitrary element f, but do not change the structure of the equations

$$F(\bar{t}, \bar{s}, \bar{\varphi}, \bar{\varphi}_{\bar{t}}, \bar{\varphi}_{\bar{s}}, \bar{\varphi}_{\bar{t}\bar{t}}, \bar{\varphi}_{\bar{t}\bar{s}}, \bar{\varphi}_{\bar{s}\bar{s}}, \bar{f}) = 0.$$

Such transformations are called equivalence transformations. A local Lie group of equivalence transformations is called an equivalence group.

Two systems of equations are called equivalent if there exists an equivalence transformation from the equivalence group mapping one into another. This property also defines an equivalence between admitted Lie groups. Separation of the systems of equations into classes of equivalent systems is called the group classification.

2.5.3 Equivalence group

A set of equivalence transformations composes a Lie group, called an equivalence group. A generator of an equivalence group is denoted as X^e and takes the form

$$X^e = \xi^s(s, t, \varphi, f)\partial_s + \xi^t(s, t, \varphi, f)\partial_t + \zeta^\varphi(s, t, \varphi, f)\partial_\varphi + \zeta^f(s, t, \varphi, f)\partial_f,$$

where $f = f(s, t, \varphi)$ is an arbitrary element contained in the equations studied.

Prolongation formulas of a generator of an equivalence group

$$\tilde{X}^e = X^e + \zeta^{\varphi_s}\partial_{\varphi_s} + \zeta^{\varphi_t}\partial_{\varphi_t} + \zeta^{f_s}\partial_{f_s} + \zeta^{f_t}\partial_{f_t} + \zeta^{f_\varphi}\partial_{f_\varphi}$$

differ from prolongation formulas of an admitted generator. The coordinates related to the dependent functions are

$$\zeta^{\varphi_\mu} = D^e_\mu \zeta^\varphi - \varphi_s D^e_\mu \xi^s - \varphi_t D^e_\mu \xi^t, \;\; D^e_\mu = \partial_\mu + \varphi_\mu \partial_\varphi + (f_\varphi \varphi_\mu + f_\mu)\partial_f,$$

where $\mu = s$ and $\mu = t$. The formulas define the coordinates of the prolonged generator $\tilde{X}^e$, that are related with the arbitrary elements

$$\zeta^{f_\lambda} = \tilde{D}^e_\lambda \zeta^f - f_s \tilde{D}^e_\lambda \xi^s - f_t \tilde{D}^e_\lambda \xi^t - f_\varphi \tilde{D}^e_\lambda \zeta^\varphi, \;\; \tilde{D}^e_\lambda = \partial_\lambda + f_\lambda \partial_f,$$

where $\lambda = s$, $\lambda = t$ and $\lambda = \varphi$.

The determining equation is constructed by applying the prolonged generator $\tilde{X}^e$ to the equations studied

$$\tilde{X}^e F_{|_{F=0}} = 0.$$

Remark 2.1. For finding an equivalence group, the equations studied can be extended by additional equations corresponding to the conditions related with arbitrary elements. For example, if the function $S = S(s)$ only depends on s, where the dependent variable $\varphi(s,t)$ depends on s and t, then the additional equations are

$$S_t = 0 \text{ and } S_\varphi = 0.$$

2.5.4 Lie algebra

Let $z = (z_1, z_2, ..., z_n) \in V$, where V be an open set in R^n, and

$$X_i = \zeta^\alpha_i(z)\partial_{z_\alpha}, \qquad (i = 1, ..., m),$$

be a set of generators. We use the standard notation: summation with respect to a repeated index for all values of the index.

Definition 2.4. The generator

$$[X_i, X_j] = \zeta^\alpha_{ij}(z)\partial_{z_\alpha}$$

with the coefficients

$$\zeta^\alpha_{ij} = X_i(\zeta^\alpha_j) - X_j(\zeta^\alpha_i)$$

is called the commutator of the generators X_i and X_j.

The operation of commutation has the following properties:

1. $[X_i, X_i] = 0$.
2. $[X_i, X_j] = -[X_j, X_i]$.

Definition 2.5. A vector space of generators which is closed with respect to commutation is called a Lie algebra. The set of admitted generators of a system of differential equations composes a Lie algebra (Ovsiannikov, 1982).

2.5.5 Inner automorphisms

Two Lie algebras $\tilde{L}$ and L are called isomorphic if there exists an isomorphism κ of vector spaces of L and $\tilde{L}$

$$\kappa : \tilde{L} \to L,$$

conserving the commutators

$$[\kappa(X), \kappa(Y)]_L = \kappa([X, Y]_{\tilde{L}}).$$

In the case when $\tilde{L}$ equals to L, κ is called an automorphism. Choose a generator Y and observe changes of other generators along the trajectory of Y. This can be described by the following formulas

$$\frac{d}{dt}\hat{X} = [\hat{X}, Y], \qquad \hat{X}_{|t=0} = X. \tag{2.21}$$

In the particular case, where L has a finite dimension, we aim to express elements of L through a basis. Let L_n be an n-th dimensional Lie algebra of generators with a basis $X_1, X_2, ..., X_n$. Any generator X can be written in the following form

$$X = x_\alpha X_\alpha.$$

As L_n is closed under the commutation, the commutator of any two generators can be expressed as a linear combination of the basis generators

$$[X_i, X_j] = C_{ij}^\gamma X_\gamma,$$

where C_{ij}^γ are called structure constants.

For the Lie algebra L_n it is possible to construct an isomorphic Lie algebra defined on the space R^n:

$$\kappa(X_i) = e_i, \quad (i = 1, 2, ..., n),$$

where $\{e_i\}_{i=1}^n$ is a basis in R^n.

The commutator on the space R^n is defined by the formula

$$[e_i, e_j] = C_{ij}^{\alpha} e_{\alpha}, \quad (i, j = 1, 2, ..., n).$$

Formula (2.21) becomes

$$\frac{d}{dt}\hat{x}_{\gamma} = [\hat{x}, y]_{\gamma} = \hat{x}_{\alpha} C_{\alpha k}^{\gamma}, \ \hat{x}_{\gamma|_{t=0}} = x_{\gamma}, \quad (\gamma = 1, 2, ..., r), \tag{2.22}$$

where $\hat{X} = \hat{x}_{\gamma} X_{\gamma}$.

Due to the linearity of the Cauchy problem (2.22), its solution can be represented in the matrix form

$$\hat{x} = A_y(t)x.$$

The latter is an automorphism, which is called an inner automorphism. Equation (2.22) can be rewritten in form

$$x_{\alpha} C_{\alpha k}^{\gamma} \partial_{x_{\gamma}}. \tag{2.23}$$

2.5.6 Classification of the Lie algebra

Definition 2.6. A vector subspace $L' \subset L$ of a Lie algebra L is called a subalgebra if it is a Lie algebra.

Similar subalgebras of the same dimension constitute a class. This means that if one considers two elements from the same class L_1 and L_2 then there exists an isomorphism A such that $AL_1 = L_2$.

Definition 2.7. A set of all representatives (one representative from each class) is called an optimal system of subalgebras.

2.6 Noether's theorem

Noether's theorem is used for finding conservation laws of variational equations with symmetries. Here, we present a simplified version of this theorem restricted to second-order PDEs with the two independent variables (t, s). Consider a Lagrangian depending on first-order derivatives

$$\mathcal{L} = \mathcal{L}(t, s, \varphi, \varphi_t, \varphi_s), \qquad \varphi = (\varphi^1, ..., \varphi^m).$$

The Lagrangian provides the second-order Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta \varphi^i} = 0, \quad (i = 1, ..., m). \tag{2.24}$$

The operator

$$\frac{\delta}{\delta \varphi^i} = \frac{\partial}{\partial \varphi^i} - D_t \frac{\partial}{\partial \varphi^i_t} - D_s \frac{\partial}{\partial \varphi^i_s}$$

is called the variational operator.

Let a Lie group of equation (2.24) be given by the generator

$$X = \xi^t(t, s, \varphi)\frac{\partial}{\partial t} + \xi^s(t, s, \varphi)\frac{\partial}{\partial s} + \eta^i(t, s, \varphi)\frac{\partial}{\partial \varphi^i}.$$

The generator X is prolonged to the second-order derivatives, contained in the Euler-Lagrange equation, according to the standard prolongation formulas.

Noether's theorem (Noether, 1918) is based on the following identities. The first identity relates the invariance of the elementary action with invariance of the Lagrangian

$$X\mathcal{L} + \mathcal{L}(D_t\xi^t + D_s\xi^s) = (\eta^i - \xi^t\varphi^i_t - \xi^s\varphi^i_s)\frac{\delta \mathcal{L}}{\delta \varphi^i} + D_t(N^t\mathcal{L}) + D_s(N^s\mathcal{L}),$$

where

$$N^t = \xi^t + (\eta - \xi^t\varphi^i_t - \xi^s\varphi^i_s)\frac{\partial}{\partial \varphi^i_t}, \qquad N^s = \xi^s + (\eta - \xi^t\varphi^i_t - \xi^s\varphi^i_s)\frac{\partial}{\partial \varphi^i_s},$$

which are called Noether's operators (the details on symmetries and conservation laws can found in Ibragimov (2011)).

If the generator X and the Lagrangian $\mathcal{L}$ satisfy the condition

$$X\mathcal{L} + \mathcal{L}(D_t\xi^t + D_s\xi^s) = 0,$$

then the symmetry X is called a variational symmetry.

If X and $\mathcal{L}$ satisfy the condition

$$X\mathcal{L} + \mathcal{L}(D_t\xi^t + D_s\xi^s) = D_tB_1 + D_sB_2$$

with nontrivial $B_1(t, s, \varphi)$ and $B_2(t, s, \varphi)$, we say that X is a divergence symmetry (Bassel-Hagen, 1921).

Another identity relates the invariance of the Lagrangian with the invariance of the Euler-Lagrange equations:

$$\begin{aligned}&\frac{\delta}{\delta\varphi^j}\left(X\mathcal{L} + \mathcal{L}(D_t\xi^t + D_s\xi^s)\right)\\&= X\left(\frac{\delta\mathcal{L}}{\delta\varphi^j}\right) + \left(\frac{\partial\eta^k}{\partial\varphi^j} - \frac{\partial\xi^t}{\partial\varphi^j}\varphi_t^k - \frac{\partial\xi^s}{\partial\varphi^j}\varphi_s^k + \delta_{jk}(D_t\xi^t + D_s\xi^s)\right)\frac{\delta\mathcal{L}}{\delta\varphi^k}, \quad j = 1, 2, ..., m,\end{aligned}$$

where δ_{jk} is the Kronecker symbol.

Noether's theorem is formulated as follows:

Theorem 2.1. Let the Lagrangian function $\mathcal{L}$ satisfies the equation

$$X\mathcal{L} + \mathcal{L}(D_t\xi^t + D_s\xi^s) = D_tB_1 + D_sB_2, \tag{2.25}$$

where X is a generator and $B_i = B_i(t, s, \varphi)$, $i = 1, 2$.

Then the generator X is a symmetry of the Euler-Lagrange equations, and the Euler-Lagrange equations possess the conservation law

$$D_t(N^t\mathcal{L} - B_1) + D_s(N^s\mathcal{L} - B_2) = 0. \tag{2.26}$$

CHAPTER III

RESEARCH METHODOLOGY

This chapter outlines the process employed in this research. Firstly, the equations are transformed from their Eulerian forms to Lagrangian forms. Secondly, we identify the Lagrangian. Lastly, we explain Noether's condition.

3.1 Tools

In this research the experiments have been performed on the Reduce computer algebra system (Hearn, 1987) programming on a computer with Windows 10 64 bit OS with an Intel (R) Core (TM) i5-10400F CPU @ 2.90GHz with Memory 16 GB and equipped with NVIDIA GeForce RTX 3060 Ti.

3.2 Transform the equations from Eulerian form into Lagrangian form

The magnetogasdynamics equations (2.17a)-(2.17g) are presented in the Eulerian form. In order to apply Noether's theorem, we need to rewrite the magnetogasdynamics equations in mass Lagrangian coordinates. We use relations (2.18) and (2.19) to transform system (2.17). After integration equations (2.20a)-(2.20d), they reduce to the equation

$$\varphi_{tt} + (B\varphi_s^{-2} + S\varphi_s^{-\gamma})_s = 0, \tag{3.1}$$

where $B = \frac{1}{2}(H_{20}^2 + H_{30}^2)$, $H_{20} = H_{20}(s)$, $H_{30} = H_{30}(s)$ and $S = S(s)$.

3.3 Find a Lagrangian

For finding a Lagrangian for which the studied equation is the Euler-Lagrange equation we have to solve the following problem. Let $\mathcal{L}(t, s, \varphi, \varphi_t, \varphi_s)$ be a corresponding

Lagrangian. Substituting $\mathcal{L}$ and φ_{tt} found from equation (3.1), into the equation $\frac{\delta\mathcal{L}}{\delta\varphi} = 0$, and splitting it with respect to the parametric derivatives we obtain an overdetermined system of equations for the function $\mathcal{L}$.

During solving this overdetermined system we have to solve equations, which have the following forms.

1. Consider an equation of the form

$$P(x, y) = 0,$$

where

$$P(x, y) = \sum_{j=0}^{n} a_j(x) y^j,$$

and $y \in R^1$. The general solution of this equation is

$$a_j(x) = 0, \qquad (j = 0, 1, ..., n).$$

This operation is called splitting.

2. If $\mathcal{K}_x = 0$, where $\mathcal{K} = \mathcal{K}(x, y)$, then $\mathcal{K}$ does not depend on x.

3. The general solution of an equation of the form

$$\mathcal{K}_{xx}(x, y) = 0,$$

is

$$\mathcal{K}(x, y) = a(y)x + b(y),$$

where $a(y)$ and $b(y)$ are arbitrary functions.

4. If the equation is

$$\mathcal{K}_{xy}(x, y) = 0,$$

then its general solution is

$$\mathcal{K}(x, y) = a(x) + b(y),$$

where $a(x)$ and $b(y)$ are arbitrary functions.

5. If the equation is

$$\mathcal{K}_t - \mathcal{P}_x = 0,$$

then its general solution is

$$\mathcal{K} = h_x,$$

and

$$\mathcal{P} = h_t,$$

where $h = h(x,t)$ is an arbitrary function

After solving all the equations, the found Lagrangian is

$$\mathcal{L} = \frac{1}{2}\varphi_t^2 - \frac{S_0}{\gamma - 1}\varphi_s^{1-\gamma} - \frac{B}{\varphi_s}. \tag{3.2}$$

Further steps of the research consist of finding the equivalence group and group classification. Results of the latter study are applied for constructing conservation laws by using Noether's theorem.

3.4 Noether's condition

Let $X_i, (i = 1, 2, ..., n)$ be admitted generators, and a linear combination of these generators be

$$X = k_i X_i,$$

where k_i are constant.

In order to check Noether's condition we apply the variational derivative to the expression $X\mathcal{L} + \mathcal{L}(D_t\xi^t + D_s\xi^s)$:

$$\frac{\delta}{\delta\varphi}\left(X\mathcal{L} + \mathcal{L}(D_t\xi^t + D_s\xi^s)\right) = 0. \tag{3.3}$$

This gives conditions for the constants k_i.

CHAPTER IV

RESULTS AND DISCUSSION

This chapter is devoted to the main results obtained and discussions. Among the main results we focus on the group classification of equations (3.1).

4.1 Group Classification

Group classification consists of 2 steps.

1. Find the equivalence group, which is used for separation of equation (3.2) into equivalence classes with respect to the group. For equivalence group, equation $\frac{\delta\mathcal{L}}{\delta\varphi}=0$ should be extended by the equations

$$S_t=0,\quad S_\varphi=0,\quad B_t=0,\quad B_\varphi=0. \tag{4.1}$$

The equivalence group is found by solving the determining equation

$$\tilde{X}^e(F)_{|F=0}=0,$$

where F consists of the equations $\frac{\delta\mathcal{L}}{\delta\varphi}=0$, and equations (4.1).

2. Find the admitted Lie group for each class

$$X^p\left(\frac{\delta\mathcal{L}}{\delta\varphi}\right)_{|\frac{\delta\mathcal{L}}{\delta\varphi}=0}=0. \tag{4.2}$$

4.1.1 Equivalence group

The equivalence transformations for equation (4.2) have generators of the form

$$X^e=\xi^t\frac{\partial}{\partial t}+\xi^s\frac{\partial}{\partial s}+\eta^\varphi\frac{\partial}{\partial\varphi}+\eta^S\frac{\partial}{\partial S}+\eta^B\frac{\partial}{\partial B},$$

where all coefficients depend on (t,s,φ,S,B).

Computations provide the generators

$$X_1^e = \frac{\partial}{\partial t}, \quad X_2^e = \frac{\partial}{\partial s},$$
$$X_3^e = \frac{\partial}{\partial \varphi}, \quad X_4^e = t\frac{\partial}{\partial \varphi},$$
$$X_5^e = t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s} + \varphi\frac{\partial}{\partial \varphi}, \quad X_6^e = t\frac{\partial}{\partial t} - 2s\frac{\partial}{\partial s} + 2(\gamma-2)S\frac{\partial}{\partial S},$$
$$X_7^e = (1-\gamma)t\frac{\partial}{\partial t} - 2s\frac{\partial}{\partial s} + 2(\gamma-2)B\frac{\partial}{\partial B},$$

where X_1^e, X_2^e, and X_3^e are shift of t, s, and φ, respectively, X_4^e is Gallilean transformation, X_5^e, X_6^e, and X_7^e are scaling.

There are also two involutions

$$E_1 : \tilde{s} = -s, \quad \tilde{\varphi} = -\varphi.$$
$$E_2 : \tilde{t} = -t. \tag{4.3}$$

4.1.2 Admitted Lie group

Partially solving the determining equation we obtain that an admitted generator has the following form

$$X = \sum_{i=1}^{i=7} x_i X_i,$$

where $X_1 = \varphi\partial_\varphi,\ \ X_2 = \partial_t,\ \ X_3 = t\partial_t,\ \ X_4 = \partial_\varphi,\ \ X_5 = t\partial_\varphi,\ \ X_6 = \partial_s,\ \ X_7 = s\partial_s,$ x_i are constants. The generators X_2, X_4, X_5 are admitted for all arbitrary elements, they are basis generators of the kernel. The generators X_i, $(i = 1, ..., 7)$ compose a Lie algebra L_7. The remaining equations, which are not solved, are called classifying equations. The classifying equations are

$$-3Bx_1 + 2Bx_3 + B_s x_6 + (sB_s + B)x_7 = 0, \tag{4.4a}$$

$$S(\gamma+1)x_1 - 2Sx_3 - S_s x_6 + (S - \gamma S - sS_s)x_7 = 0. \tag{4.4b}$$

Latter equations contain constants x_i and functions B and S. We have to solve equation (4.4a) and equation (4.4b) with respect to the constants and the functions together.

If $x_6 = 0$ and $x_7 = 0$, then this leads to the contradiction that $X = 0$. Indeed, let $x_6 = 0$, $x_7 = 0$: $X = x_1X_1 + x_3X_3$, $(x_1^2 + x_3^2 \neq 0)$. Equations (4.4a) and (4.4b) become

$$B(-3x_1 + 2x_3) = 0, \qquad S((\gamma + 1)x_1 - 2x_3) = 0.$$

As $BS \neq 0$, then for* $\gamma \neq 2$ the latter equations lead to the contradiction

$$x_1 = 0, \qquad x_3 = 0.$$

For solving the classifying equations we use the algebraic approach.

For the algebraic approach we note that action of equivalence transformations is equivalent to the inner automorphisms. The set of inner automorphisms is constructed on the base of commutator table.

Table 4.1 The commutator table.

	X_1	X_3	X_6	X_7	X_2	X_4	X_5
X_1	0	0	0	0	0	$-X_4$	$-X_5$
X_3	0	0	0	0	$-X_2$	0	X_5
X_6	0	0	0	X_6	0	0	0
X_7	0	0	$-X_6$	0	0	0	0
X_2	0	X_2	0	0	0	0	X_4
X_4	X_4	0	0	0	0	0	0
X_5	X_5	$-X_5$	0	0	$-X_4$	0	0

*The case $\gamma = 2$ corresponds to the classical gas dynamics equations, which are excluded from our consideration.

The inner automorphisms are

$A_1 : x_4\partial_{x_4} + x_5\partial_{x_5} \Rightarrow \tilde{x}_4 = x_4 e^{a_1},\ \ \tilde{x}_5 = x_5 e^{a_1},$

$A_2 : x_3\partial_{x_2} + x_5\partial_{x_4} \quad \Rightarrow \tilde{x}_2 = x_2 + a_2 x_3,\ \ \tilde{x}_4 = x_4 + a_2 x_5,$

$A_3 : -x_2\partial_{x_2} + x_5\partial_{x_5} \Rightarrow \tilde{x}_2 = x_2 e^{-a_3},\ \ \tilde{x}_5 = x_5 e^{a_3},$

$A_4 : x_1\partial_{x_4} \quad \Rightarrow \tilde{x}_4 = x_4 + a_4 x_1,$

$A_5 : (x_1 - x_3)\partial_{x_5} - x_2\partial_{x_4} \Rightarrow \tilde{x}_5 = x_5 + a_5(x_1 - x_3),\ \ \tilde{x}_4 = x_4 + a_5 x_2,$

$A_6 : x_7\partial_{x_6} \quad \Rightarrow \tilde{x}_6 = x_6 + a_6 x_7,$

$A_7 : -x_6\partial_{x_6} \Rightarrow \tilde{x}_6 = x_6 e^{a_7}.$

To find the four-dimensional and higher-dimensional optimal system of subalgebras of the Lie algebra L_7, we aim to simplify all the coefficients of the generator

$$X = x_1X_1 + x_3X_3 + x_6X_6 + x_7X_7.$$

Other three generators are basis generators of the kernel of the admitted Lie algebras. Hereinafter, the generators from the kernel are omitted. Considering the inner automorphisms, one notes that x_7 is invariant of the inner automorphisms.

Case $x_7 \neq 0$

In the first case, we assume that $x_7 \neq 0$. Using the inner automorphism A_6, we can reduce $x_6 = 0$. Dividing each term in the generator X by x_7, we obtain

$$X = \alpha_1X_1 + \beta_1X_3 + X_7,$$

where α_1 and β_1 are constant.

Case $x_7 = 0$

In the second case, assuming $x_7 = 0$, we get

$$X = x_1X_1 + x_3X_3 + x_6X_6. \tag{4.5}$$

Let $x_6 \neq 0$. In this case, we can assume that $x_6 = 1$. Considering the condition $x_1 \neq 0$ and using involution E_1 of (4.3) and the inner automorphism A_7, we can reduce $x_1 = 1$. Generator (4.5) becomes

$$X = X_1 + \beta_2X_3 + X_6,$$

where β_2 is constant.

For another case, where $x_1 = 0$, generator (4.5) becomes

$$X = \alpha_3 X_3 + X_6,$$

where α_3 is constant. Similar to the previous case, using E_1 and A_7, the latter generator can be reduced to either

$$X = X_3 + X_6 \quad \text{or} \quad X = X_6.$$

As noted earlier, the basis elements of an admitted subalgebras of L_7 should contain either the generator X_6 or X_7, thus it is not necessary to analyze the case

$$x_1 X_1 + x_3 X_3.$$

For higher dimensional subalgebras of the Lie algebra L_7, which are admitted, we have to analyze the only case

$$\{X, Y\},$$

where

$$X = X_7 + \alpha X_1 + \beta X_3,$$
$$Y = X_6 + x_1 X_1 + x_3 X_3.$$

Taking the commutator

$$[X, Y] = -X_6,$$

and using the condition for the generators to compose a Lie algebra, we have

$$-X_6 = hX + \mu Y,$$

where h and μ are constant. The latter leads to $x_1 = 0$, $x_3 = 0$. This gives the subalgebra

$$\{X_6, X_7 + \alpha X_1 + \beta X_3\}.$$

Therefore, for solving equation (4.4a) and equation (4.4b), we only use the subalgebras with the basis generators

$$\begin{aligned} &\{\alpha_1 X_1 + \beta_1 X_3 + X_7\}, \quad \{X_1 + \beta_2 X_3 + X_6\}, \\ &\{X_3 + X_6\}, \quad \{X_6\}, \quad \{X_6, X_7 + \alpha X_1 + \beta X_3\}. \end{aligned} \tag{4.6}$$

All of these subalgebras also include the generators X_2, X_4, X_5. From the optimal system of subalgebras (4.6) the coefficients of the classifying equations (4.4a) and (4.4b) can be found. This is explained further.

1. At first, we consider the subalgebra from the optimal system

$$\{\alpha_1 X_1 + \beta_1 X_3 + X_7, X_2, X_4, X_5\}. \tag{4.7}$$

From this subalgebra, we get $x_1 = \alpha_1$, $x_3 = \beta_1$, $x_7 = 1$, and $x_2 = x_4 = x_5 = 1$, then the classifying equations (4.4a) and (4.4b) become

$$sB_s + (2\beta_1 - 3\alpha_1 + 1)B = 0, \tag{4.8}$$

$$sS_s - ((\gamma + 1)\alpha_1 - 2\beta_1 + (1 - \gamma))S = 0. \tag{4.9}$$

Integrating equations (4.8) and (4.9), we obtain

$$B = C_1 s^{(3\alpha_1 - 2\beta_1 - 1)}, \;\; S = C_2 s^{((\gamma+1)\alpha_1 - 2\beta_1 + (1-\gamma))},$$

where C_1 and C_2 are constant.

2. Consider the case in which the subalgebra is

$$\{X_1 + \beta_2 X_3 + X_6, X_2, X_4, X_5\}. \tag{4.10}$$

From subalgebra (4.10), we have $x_1 = x_6 = 1$, $x_3 = \beta_2$, and $x_2 = x_4 = x_5 = 1$, Then the classifying equations (4.4a) and (4.4b) become

$$B_s + (2\beta_2 - 3)B = 0, \tag{4.11}$$

$$S_s - ((\gamma + 1) - 2\beta_2)S = 0. \tag{4.12}$$

Integrating equations (4.11) and (4.12), we get

$$B = C_3 e^{(3 - 2\beta_2)}, \;\; S = C_4 e^{((\gamma+1) - 2\beta_2)},$$

where C_3 and C_4 are constant.

3. Choosing the subalgebra

$$\{X_3 + X_6, X_2, X_4, X_5\} \tag{4.13}$$

from the optimal system, the coefficients are $x_3 = x_6 = 1$, $x_1 = x_7 = 0$, and $x_2 = x_4 = x_5 = 1$. Then the classifying equations (4.4a) and (4.4b) become

$$B_s + 2B = 0, \tag{4.14}$$

$$S_s + 2S = 0. \tag{4.15}$$

Integrating equations (4.14) and (4.15), we have

$$B = C_5 s^{-2}, \;\; S = C_6 s^{-2},$$

where C_5 and C_6 are constant.

4. One more subalgebra from the optimal system is

$$\{X_6, X_2, X_4, X_5\}. \tag{4.16}$$

From subalgebra (4.16), we get $x_6 = 1$, $x_1 = x_3 = x_7 = 0$, and $x_2 = x_4 = x_5 = 1$. The classifying equations (4.4a) and (4.4b) lead to

$$B_s = 0, \tag{4.17}$$

$$S_s = 0. \tag{4.18}$$

The general solution of equations (4.17) and (4.18) is

$$B = C_7, \;\; S = C_8,$$

where C_7 and C_8 are constant.

5. Lastly, the subalgebra of the optimal system is

$$\{X_6, X_7 + \alpha X_1 + \beta X_3, X_2, X_4, X_5\}. \tag{4.19}$$

From subalgebra (4.19), we get two cases for the coefficients. The first case is $x_6 = 1$, $x_1 = x_3 = x_7 = 0$, and $x_2 = x_4 = x_5 = 1$, then the classifying equations (4.4a) and (4.4b) become

$$B_s = 0, \tag{4.20}$$

$$S_s = 0. \tag{4.21}$$

Hence, as in the previous case

$$B = C_7, \;\; S = C_8,$$

The second case gives $x_1 = \alpha$, $x_3 = \beta$, $x_6 = 0$, $x_7 = 1$, and $x_2 = x_4 = x_5 = 1$. As from the first case, we get that B and S are constant, then by virtue of the condition $BS \neq 0$, the classifying equations (4.4a) and (4.4b) reduce to the equations

$$-3x_1 + 2x_3 + 1 = 0, \tag{4.22}$$

$$(\gamma + 1)x_1 - 2x_3 + (1 - \gamma) = 0. \tag{4.23}$$

Summing equation (4.22) and equation (4.23), we obtain

$$(\gamma - 2)(x_1 - 1) = 0.$$

From the condition that $\gamma \neq 2$, we get

$$x_1 = 1 \quad \text{and} \quad x_3 = 1.$$

A summary of the calculations above is presented in Table 4.2.

Table 4.2 Representations of the functions $S(s)$ and $B(s)$.

Case	$S(s)$	$B(s)$	Symmetry Extensioin	Conditions
1	S_0	B_0	$\partial_s,\ t\partial_t + s\partial_s + \varphi\partial_\varphi$	
2	$S_0 s^\alpha$	$B_0 s^\beta$	$(2(\gamma-2) + 3\alpha - \beta(\gamma+1))t\partial_t$ $+2(\gamma-2)s\partial_s + 2(\gamma - 2 + \alpha - \beta)\varphi\partial_\varphi$	$\alpha^2 + \beta^2 \neq 0$
3	$S_0 e^{ps}$	$B_0 e^{qs}$	$(-q(\gamma+1) + 3p)t\partial_t$ $+2(\gamma-2)\partial_s + 2(p-q)\varphi\partial_\varphi$	$p^2 + q^2 \neq 0$

4.2 Application of Noether's theorem

4.2.1 Noether's condition

Applying the Lagrangian (3.2) and the data from Table 4.2, the conservation laws are found. However, Noether's conditions must be satisfied. A summary of solving of these conditions is presented in Table 4.3.

Table 4.3 Additional variational symmetries of $S(s)$ and $B(s)$.

Case	$S(s)$	$B(s)$	Symmetry Extensioin	Conditions
1	S_0	B_0	∂_s	
2	$S_0 s^{\alpha}$	$B_0 s^{\beta}$	$(2\beta+5)t\partial_t - s\partial_s + (\beta+3)\varphi\partial_\varphi$	$\alpha^2+\beta^2 \neq 0$, $\alpha+\beta(\gamma-3) = -4(\gamma-2)$ (Note that $\gamma \neq 3$ if $\alpha = 0$)
3	$S_0 e^{ps}$	$B_0 e^{qs}$	$2qt\partial_t - \partial_s + q\varphi\partial_\varphi$	$p^2+q^2 \neq 0$ $p+q(\gamma-3) = 0$ (Note that $\gamma = 3$ if $p = 0$)

4.2.2 Conservation laws

Here the conservation laws of equation (3.1) are discussed.

The symmetries of the basis generators $X_2 = \partial_t$, $X_4 = \partial_\varphi$, and $X_5 = t\partial_\varphi$ provide conservation laws of energy, momentum and motion of the center of mass, respectively. They represent

- conservation laws of energy

$$D_t^L\left(\frac{u^2}{2} + \frac{1}{\gamma-1}\frac{p}{\rho} + \frac{H_{20}^2+H_{30}^2}{2\rho}\right) + D_s\left(u\left(p + \frac{H_{20}^2+H_{30}^2}{2}\right)\right) = 0; \tag{4.24}$$

- conservation laws of momentum:

$$D_t^L(u) + D_s\left(p + \frac{H_{20}^2+H_{30}^2}{2}\right) = 0; \tag{4.25}$$

- conservation laws of motion of the center of mass:

$$D_t^L(tu-x)+D_s\left(t\left(p+\frac{H_{20}^2+H_{30}^2}{2}\right)\right)=0. \tag{4.26}$$

The additional symmetries provide the following conservation laws:

1. Case $(S(s),B(s))=(S_0,B_0)$. In this case the extension of the kernel of admitted Lie algebras is defined by the generator ∂_s.

   The conservation law in Lagrangian coordinates is

$$D_t^L(\varphi_s\varphi_t)+D_s\left(-\frac{\varphi_t^2}{2}+\frac{\gamma S}{\gamma-1}\varphi_s^{1-\gamma}+\frac{2B}{\varphi_s}\right)=0, \tag{4.27}$$

   and using the physical variables, we get

$$D_t^L\left(\frac{u}{\rho}\right)+D_s\left(-\frac{u^2}{2}+\frac{\gamma S}{\gamma-1}\rho^{\gamma-1}+\frac{(H_{20}^2+H_{30}^2)}{\rho}\right)=0. \tag{4.28}$$

2. Case $(S(s),B(s))=(S_0s^\alpha,B_0s^\beta)$, where the extension is given by the generator $(2\beta+5)t\partial_t-s\partial_s+(\beta+3)\varphi\partial_\varphi$.

   The conservation law in Lagrangian coordinates is

$$\begin{aligned}D_t^L&\left((2\beta+5)t\left(\frac{\varphi_t^2}{2}+\frac{S}{\gamma-1}\varphi_s^{1-\gamma}+\frac{B}{\varphi_s}\right)-s\varphi_s\varphi_t-(\beta+3)\varphi\varphi_t\right)\\&+D_s\left(((2\beta+5)t\varphi_t-(\beta+3)\varphi)\left(S\varphi_s^{-\gamma}+\frac{B}{\varphi_s^2}\right)\right.\\&\left.+s\left(\frac{\varphi_t^2}{2}-\frac{\gamma S}{\gamma-1}\varphi_s^{1-\gamma}-\frac{2B}{\varphi_s}\right)\right)=0,\end{aligned} \tag{4.29}$$

   and using the physical variables, we have

$$\begin{aligned}D_t^L&\left((2\beta+5)t\left(\frac{u^2}{2}+\frac{S}{\gamma-1}\rho^{\gamma-1}+\frac{H_{20}^2+H_{30}^2}{2\rho}\right)-s\frac{u}{\rho}-(\beta+3)xu\right)\\&+D_s\left\{((2\beta+5)tu-(\beta+3)x)\left(S\rho^\gamma+\frac{H_{20}^2+H_{30}^2}{2}\right)\right.\\&\left.+s\left(\frac{u^2}{2}-\frac{\gamma S}{\gamma-1}\rho^{\gamma-1}-\frac{H_{20}^2+H_{30}^2}{\rho}\right)\right\}=0.\end{aligned} \tag{4.30}$$

3. Case $(S(s),B(S))=(S_0e^{ps},B_0e^{qs})$. The generator used for this case is $2qt\partial_t-\partial_s+q\varphi\partial_\varphi$.

The conservation law in Lagrangian coordinates is

$$D_t^L\left(2qt\left(\frac{\varphi_t^2}{2}+\frac{S}{\gamma-1}\varphi_s^{1-\gamma}+\frac{B}{\varphi_s}\right)-\varphi_s\varphi_t-q\varphi\varphi_t\right)$$
$$+D_s\left(q(2t\varphi_t-\varphi)\left(S\varphi_s^{-\gamma}+\frac{B}{\varphi_s^2}\right)+\frac{\varphi_t^2}{2}-\frac{\gamma S}{\gamma-1}\varphi_s^{1-\gamma}-\frac{2B}{\varphi_s}\right)=0, \quad (4.31)$$

and using the physical variables, we obtain

$$\begin{aligned}D_t^L\left(2qt\left(\frac{u^2}{2}+\frac{S}{\gamma-1}\rho^{\gamma-1}+\frac{H_{20}^2+H_{30}^2}{2\rho}\right)-\frac{u}{\rho}-qxu\right)&\\ +D_s\Bigg\{q(2tu-x)\left(S\rho^{\gamma}+\frac{H_{20}^2+H_{30}^2}{2}\right)&\\ +\frac{u^2}{2}-\frac{\gamma S}{\gamma-1}\rho^{\gamma-1}-\frac{H_{20}^2+H_{30}^2}{\rho}\Bigg\}=0.&\end{aligned} \quad (4.32)$$

CHAPTER V

CONCLUSION AND RECOMMENDATION

The focus of this research is on Lie group analysis and conservation laws of one-dimensional magnetogasdynamics equations, which are described in the mass Lagrangian coordinates, and case of infinite conductivity. We found a Lagrangian and applied the group analysis method. The conservation laws were found using Noether's theorem in Lagrangian coordinates. Finally, the additional conservation laws are written in Lagrangian coordinates using the physical variables.

In the future, we plan to expand our study to consider the three-dimensional case, and the case of finite conductivity.

REFERENCES

REFERENCES

Bessel-Hagen, E. (1921), Über die Erhaltungssätze der Elektrodynamik, *Math. Ann., 84,* 258–276.

Chorin, A. J. and Marsden, J. E. (1990). *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag.

Hearn, A. C. (1987). *REDUCE Users Manual, ver. 3.3*, Santa Monica: The Rand Corporation CP 78.

Ibragimov, N. H. (1985). *Transformation Groups Applied to Mathematical Physics*, Reidel, Boston.

Ibragimov, N. H. (1994,1995,1996). *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 1,2,3, CRC Press, Boca Raton.

Ibragimov, N. H. (2011). Nonlinear Self-adjointness and Conservation Laws, *J. Phys. A., 44*(10), 432002.

Kulikovskii, A. G. and Lyubimov, G. A. (1965). *Magnetohydrodynamics*, Addison-Wesley, Berlin/Boston.

Landau, L. D. and Lifshitz, E. M. (1987). *Fluid Mechanics*, second ed., Pergamon Press.

Noether, E. (1918). Invariante Variationsprobleme. Nachrichten von der königliche Gesellschaft der Wissenschaften zu Göttingen, *Mathematisch-physikalische Klasse, Heft 2,* 235-257. English translation, *Transport Theory Statist. Phys. 1,* 183-207.

Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*, Springer, New York.

Ovsiannikov, L. V. (1982). *Group Analysis of Differential Equations*, Academic Press, New York.

Ovsiannikov, L. V. (2003). *Lectures on the Gas Dynamics Equations*, Institute of Computer Studies, Moscow.

Webb, G. (2018). *Magnetohydrodynamics and Fluid Dynamics: Action Principles and Conservation Laws*, in: Lecture Notes in Physics, vol. 946, Springer, Heidelberg.

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