Pergamon

PII: S0021-8928(97)00019-1

J. Appl. Maths Mechs, Vol. 61, No. 1, pp. 157–163, 1997

0021-8928/97 $24.00+0.00

STEADY SPATIAL NON-ISENTROPIC DOUBLE-WAVE TYPE GAS FLOWS†

S. V. MELESHKO

Novosibirsk

(*Received* 20 *January* 1995)

The non-isentropic steady spatial double-wave equations of an ideal gas with an equation of state of the form $\tau = g(p)A^2(S)$ are investigated in two cases, omitted previously [1]: when $H \neq 0, F_2'' = cF_3''$ and when $H = 0$ with straight level lines.

The analysis given here completes the classification of spatial steady non-isentropic double waves with an arbitrary equation of state $\tau = \tau(p, S)$ when there is a functional arbitrariness in the general solution of the Cauchy problem. Partial solutions of this kind for a polytropic gas can be found in [2–5].‡

We will consider double waves

$$\mathbf{v} = \mathbf{v}(p, S)$$

which are irreducible to invariant solutions of the equation describing the flow of an ideal gas in the spatial steady non-isobaric and non-isentropic case

$$\frac{d\mathbf{v}}{dt} + \tau\nabla p = 0, \quad \frac{d\tau}{dt} - \tau \operatorname{div} \mathbf{v} = 0, \quad \frac{dS}{dt} = 0 \tag{1}$$

with equation of state $\tau = \tau(p, S)$, $\tau_p \neq 0$, $\tau_S \neq 0$. Here $\mathbf{v} = (u_1, u_2, u_3)$ is the velocity, p is the pressure, S is the entropy, τ is the specific volume, $d/dt = u_\alpha d/dx_\alpha$ (summation is carried out over repeated Greek subscripts from 1 to 3, unless otherwise stated), and the following notation is also used below

$$H = \tau_p + \mathbf{v}_p\mathbf{v}_p, \quad \zeta = \tau_S + \mathbf{v}_p\mathbf{v}_S, \quad \xi = 2\zeta - \tau_S, \quad \mathbf{b} = \mathbf{v} \times \mathbf{v}_S, \quad \lambda = \mathbf{b}\mathbf{v}_p$$

It follows from the investigation carried out below and results obtained previously in [1] that there are only the following forms of spatial non-isentropic, non-isobaric steady double-wave type flows of an ideal gas, irreducible to invariant solutions with functional arbitrariness.

1. Double waves with an arbitrariness in one function of one argument and an equation of state $\tau = g(p)A^2(S)$, in which $u_1 = h_1(p)A(S)$, while the other coordinates of the velocity u_2 and u_3 either have the form $u_2 = h_2(p)A(S)$, $u_3 = u_3(S)$ or the form $u_2 = u_2(S)$, $u_3 = u_3(S)$. In the first case the functions $h_1(p)$, $h_2(p)$ and $g(p)$ satisfy a system of two ordinary differential equations (21) [1] ($F_2 = h'_2/h'_1, h_1h''_1 + h_2h''_2 \neq 0$). In the second case the functions $u_2 = u_2(S)$, $u_3 = u_3(S)$ are arbitrary while $h_1(p)$ and $g(p)$ are related by the equation $g + h_1h'_1 = 0$ ($h_1h''_1 \neq 0$).

These solutions for a polytropic gas were considered previously in [1], where the functions $u_2(S)$ and $u_3(S)$ are linearly related to $A(S)$, i.e. only a special class of solutions of the double-wave form was indicated.

2. Double waves with straight level lines with an arbitrariness in two functions of one argument, which are arbitrary functions of the solution of Eq. (31) [1]. For $\tau(p, S)$, $\mathbf{v}(p, S)$ there is an overdetermined system consisting of five differential equations: (3), (30) and (32). An analysis of this overdetermined system is difficult in the general case of the equations of state. But in a special case, this system is only compatible for a polytropic gas with polytropic index $\gamma = 2$ and has a solution with an arbitrariness in one function of one argument.

3. Double waves with straight level lines with an arbitrariness in two functions of one argument and with an equation of state $\tau = g(p)A^2(S)$. In this case $u_2 = u_2(S)$, $u_3 = u_3(S)$ are arbitrary ($(u_i/A)_S \neq 0, i = 2, 3$), $u_1 = h_1(p)A(S)$. Here $h_1(p) = k_1p + k$ and $g(p) = -k_1h_1$.

4. Double waves with an arbitrariness in one function of two arguments, which is an arbitrary function of the solution of Eq. (1.25) from [1]

$$-gQ_p + h'_1(\chi' - Q(h'_2/h'_1)' - x_3(h'_3/h'_1)')(h_3Q_{x3} - h_2) = 0$$

†*Prikl. Mat. Mekh.* Vol. 61, No. 1, pp. 159–167, 1997.

‡See also: Zubov, Ye. N., Double waves for the spatial steady equations of gas dynamics. Candidate dissertation, Sverdlovsk, 1978.

The functions $g(p)$ and $h_i(p)$ $(i = 1, 2, 3)$ are related to Eqs (1.24) from [1]

$$h_\alpha \ddot{h}_\alpha = 0, \quad h_\alpha h'_\alpha = -g$$

Here $\chi = \chi(p)$, $Q = Q(p, S, x_3)$, $u_i = h_i(p)A(S)$ $(i = 1, 2, 3)$ and the equation of state is $\tau = g(p)A^2(S)$.

Theorem. Only the four forms of spatial non-isentropic, non-isobaric steady double-wave type flows of an ideal gas, irreducible to invariant solutions with functional arbitrariness, indicated above exist.

Proof. After introducing the new relationship $\varphi = (\operatorname{div} \mathbf{v})/\tau_p$, we obtain from (1)

$$\mathbf{v}\nabla p - \tau\varphi = 0, \quad \mathbf{v}\nabla S = 0, \quad \mathbf{v}_S \nabla S = H\varphi \tag{2}$$

$$\boldsymbol{\Phi} \equiv (\Phi_1, \Phi_2, \Phi_3) = \nabla p + \varphi \mathbf{v}_p = 0$$

Since the flow is isobaric we have $\varphi \neq 0$, $\mathbf{v}_p \neq 0$ (to fix our ideas we will assume that $u_{1p} \neq 0$). It follows from (2) that

$$\tau + \mathbf{v}\mathbf{v}_p = 0 \tag{3}$$

(the Bernoulli integral).

Differentiating D_i totally with respect to the spatial variable x_i $(i = 1, 2, 3)$ and setting up the following combinations from Eqs (2), we obtain

$$\mathbf{D} \times \boldsymbol{\Phi} = -\mathbf{v}_p \times \nabla\varphi - \mathbf{v}_{pS}\nabla S - \varphi^2 \mathbf{v}_p \times \mathbf{v}_{pp} = 0 \tag{4}$$

$$D(H\varphi - \mathbf{v}_S \nabla S)/Dt = H d\varphi/dt - \varphi(\tau \mathbf{v}_{pS} + \mathbf{v}_p(\mathbf{v}_p \mathbf{v}_S))\nabla S + \varphi^2(H^2 + \tau H_p) = 0 \tag{5}$$

$$D\boldsymbol{\Phi}/Dt = \mathbf{v}_p d\varphi/dt + \tau\nabla\varphi + \varphi\zeta\nabla S - \varphi^2(\mathbf{v}_p H - \tau \mathbf{v}_{pp}) = 0 \tag{6}$$

where $\mathbf{D} = (D_1, D_2, D_3)$, $D/Dt = \mathbf{v}\mathbf{D}$.

Eliminating the derivatives ∇_φ from (4) using (6) we obtain

$$(\zeta \mathbf{v}_p - \tau \mathbf{v}_{pS}) \times \nabla S = 0 \tag{7}$$

Henceforth we will need to distinguish two cases: $H \neq 0$ and $H = 0$.

1. Suppose $H \neq 0$. If φ is expressed from the third equation of (2), and we substitute this quantity into the remaining equations of this system, then, instead of (7), we obtain a homogeneous system of quasilinear differential equations in p and S. From the fact that it is forbidden to reduce the double waves to invariant solutions [6] and from Eqs (2) and (6) we have the equation

$$\tau \mathbf{v}_{pS} - \zeta \mathbf{v}_p = 0 \tag{8}$$

whence it follows that vector functions $\mathbf{F} = (F_1(p), F_2(p), F_3(p))$ exist such that $(F_1 \equiv 1)$

(9)

Since $H \neq 0$, it follows from the second and third equations of (2) that $\mathbf{b} \neq 0$. To fix our ideas we will assume that $b_1 \equiv u_2 u_{3S} - u_3 u_{2S} \neq 0$.

The system of equations which the functions $\mathbf{U} = (p, S, \varphi)'$ must satisfy is written in the form of the following overdetermined system of quasilinear differential equations

$$\mathbf{U}_{x_3} + G_3 \mathbf{U}_{x_1} = \mathbf{f}_1, \quad \mathbf{U}_{x_2} + G_2 \mathbf{U}_{x_1} = \mathbf{f}_2$$

$$p_{x_1} + \varphi u_{1p} = 0, \quad \varphi e_1 S_{x_1} + \varphi_{x_1} = f \tag{10}$$

$$b_1 G_i = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & -b_i & 0 \\ 0 & \varphi e_i & 0 \end{Vmatrix}, \quad \mathbf{e} \equiv (e_1, e_2, e_3) = \frac{\xi\lambda}{\tau H}\mathbf{v}_p + \frac{\xi}{\tau}\mathbf{b}$$

with functions $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}$ which are independent of the derivatives $\mathbf{U}_{x_i}$ $(i = 1–3)$.

For a solution of system (1) having functional arbitrariness to exist we must have

$$\xi\lambda(b_1 \mathbf{v}_p - u_{1p}\mathbf{b}) = 0 \tag{11}$$

This follows from a consideration of the matrix compiled from the coefficients of the leading derivatives in the extended system (10). The results of an analysis of the case when $\lambda(b_1\mathbf{v}_p - u_{1p}\mathbf{b}) = 0$ or $\xi \neq 0$, given previously in [1], show that either contradictory equations are obtained or reduction to plane flow occurs. Below we will investigate the case when $\xi = 0$, which was not considered previously.

When the condition $\xi = 0$ is satisfied, some of the equations of system (10) take the form

$$\boldsymbol{\Psi} \equiv (\Psi_1, \Psi_2, \Psi_3) = \tau H \nabla\varphi + \varphi H \zeta \nabla S - \varphi^2 \mathbf{c} = 0$$

$$\mathbf{c} \equiv (2H^2 + \tau H_p)\mathbf{v}_p - \tau H \mathbf{v}_{pp}$$

From the equations $\mathbf{D} \times \boldsymbol{\Psi} = 0$ we obtain another three first-order (scalar) equations

$$\mathbf{a} \times \nabla S + \varphi \mathbf{d} \times \mathbf{v}_p = 0 \tag{12}$$

Here

$$\mathbf{a} = \tau H[H_p / H - 2u_{1pp} / u_{1p} + 2H / \tau]_S \mathbf{F}$$

$$\mathbf{d} = (2H^2 + \tau H_p)\mathbf{v}_{pp} - \tau H \mathbf{v}_{ppp}$$

From the fact that it is forbidden to reduce the double wave to an invariant solution in Eqs (10) and (12) it follows that $a = 0$ and $d_i = F_i D_i$, $(i = 2, 3)$, i.e.

$$[H_p / H - 2u_{1pp} / u_{1p} + 2H / \tau]_S = 0 \tag{13}$$

$$F_i'(u_{1p}(2H^2 + \tau H_p) - 2\tau H u_{1pp}) - \tau H u_{1p} F_i'' = 0, \quad (i = 2,3) \tag{14}$$

When (3), (9) and (14) are satisfied and $\xi = 0$, system (10) is in involution and has a solution with an arbitrariness in one function of one argument x_i. Hence, when $H \neq 0$ it remains to investigate the compatibility of only the system consisting of Eqs (3), (9), (14) and $\xi = 0$. This can be split into two cases: $F_2' = F_3' = 0$ and $(F_2')^2 + (F_3') \neq 0$.

Suppose initially that $F_2' = F_3' = 0$, i.e. $F_i =$ const. Without loss of generality we can assume that $F_2 = F_3 = 0$ or $u_2 = u_2(S)$, $u_3 = u_3(S)$. Here $u_2 = u_2(S)$, $u_3 = u_3(S)$ are assumed to be arbitrary functions of the entropy. From the condition $\xi = 0$ and (3) we obtain the arbitrary relations

$$u_{1p} = -\tau / u_1, \quad u_{1S} = \tau_S u_1 / (2\tau) \tag{15}$$

After cross-differentiating u_1 with respect to p and S in the last equations and equating the mixed derivatives we have $(\tau_S / \tau)_p = 0$. Hence we obtain $\tau = A^2(S)g(p)$, and after integrating (15) we have $u_1^2 = -2A^2(S)\int g(p)dp$. One can verify here by direct substitution that $\mathbf{a} = 0$. Hence, in this case the equation of state must have the form $\tau = A^2(S)g(p)$, and the components of the velocity are

$$u_1 = h_1(p)A(S), \quad u_2 = u_2(S), \quad u_3 = u_3(S)$$

where $h_1(p)$ is found from the equation $h_1h_1' + g = 0$, while the functions $u_2(S)$, $u_3 = u_3(S)$ are arbitrary.

Note that for a polytropic gas the same condition was obtained in [1] for the function $h_1(p)$ but the functions $u_2 = u_2(S)$, $u_3 = u_3(S)$ were assumed to be related to $A(S)$, i.e. a narrower class of solutions is indicated (because of the additional assumption).

Suppose now that $(F_2')^2 + (F_3')^2 \neq 0$. Since $\tau H u_{1p} \neq 0$, by virtue of the linearity of Eqs (14) with respect to F_i and F'' $(i = 2, 3)$, we can assume without loss of generality that $F_2' \neq 0$ and $F_3 = 0$.

From (14) we obtain $a_1 = 0$ and

$$H_p / H = 2u_{1pp} / u_{1p} - 2H / \tau + F_2'' / F_2'$$

After substituting $H = \tau_p + u_{1p}^2(1 + F_2^2)$ here we obtain the equation

$$\frac{2u_{1pp}\tau_p}{u_{1p}H} + \frac{F_2''}{F_2'} - \frac{2H}{\tau} - \frac{\tau_{pp} + 2F_2F_2'u_{1p}^2}{H} = 0 \tag{16}$$

From (3) we can determine the component of the velocity $u_2 = -(u_1 + \tau/u_{1p})/F_2$. Substituting its expression into (9) $(i = 3)$ and into $\xi = 0$ (taking (3) into account), we obtain

$$u_{1pp} = u_{1p}\left[\frac{H}{\tau} - \frac{F_2'(u_1u_{1p} + \tau)}{\tau F_2}\right], \quad u_{1pS} = \frac{\tau_S u_{1p}}{2\tau} \tag{17}$$

After cross-differentiation of Eqs (17) we obtain

$$u_1(2\tau u_{1S}-(BF_2)/(u_{1p}F'_2))/\tau_S \quad (B=-\tau\tau_{ps}-\tau_p\tau_s) \tag{18}$$

and after differentiating (18) with respect to p and substituting it into (17) we obtain

$$4Bu_{1S}=\frac{\tau_S}{u_{1p}}\left[-2B+\frac{BF_2}{F_2'}\left(\frac{2B}{\tau\tau_S}+\frac{2\tau_p}{\tau}+u_{1p}^2(1+F_2^2)+\frac{F_2''}{F_2'}\right)-\frac{B_pF_2}{F_2'}\right] \tag{19}$$

If $B\neq 0$, we can determine u_1 and u_{1S} in terms of u_{1p} from (18) and (19). Then, after differentiating (18) with respect to S we have

$$(B_p/B)_S=B/\tau^2 \tag{20}$$

and after substituting the first expression of (17) into (16) we obtain an equation, differentiation of which with respect to S taking (20) into account gives the relation $\tau_p + u_{1p}^2(1+F_2^2)=0$, which contradicts the condition $H\neq 0$. Hence, we need to assume that $B=0$, which corresponds to the equation of state $\tau=g(p)A^2(S)$. For this equation of state $u_1h_1(p)A(S)$, and we have the following system of two ordinary differential equations for the functions $F_2(p)$ and $H_1(p)$

$$\begin{aligned}&F_2F_2''((h_1')^2(1+F_2^2)+g')=2(F_2')^2h_1'(g'h_1-h_1'g)+F_2F_2'gg''+\\&+2((F_2')^2g+(h_1')^2F_2F_2'(1+F_2^2))((h_1')^2(1+F_2^2)+g')\\&F_2gh_1''=F_2(1+F_2^2)(h_1')^3-F_2'h_1(h_1')^2+h_1'(F_2g'-gF_2')\end{aligned} \tag{21}$$

The function $u_2=u_2(S)$ remains arbitrary.

As in the previous case, a similar solution for a polytropic gas was obtained previously by Zubov (see the earlier footnote), but the function $u_2(S)$ was related linearly to the function $A(S)$.

Hence, if $H\neq 0$, steady double-wave type non-isentropic non-isobaric flows, which have a functional arbitrariness and are not reducible to invariant solutions, only exist for the equations of state $\tau=A^2(S)g(p)$.

2. Suppose now that $H=0$. It follows from the fact that education of system (2), (5), (7) to invariant solutions is forbidden, that the following equations are satisfied

$$\mathbf{vg}=0,\ \mathbf{v}_S\mathbf{g}=0,\ \mathbf{g}^2+\xi\mathbf{v}_p\mathbf{g}=0 \quad (\mathbf{g}=\tau\mathbf{v}_{pS}-\zeta\mathbf{v}_p) \tag{22}$$

For the further investigation we will change to new independent variables p, S, and x_3 (we can assume without loss of generality that the inequality $p_{x_1}S_{x_2}-p_{x2}S_{x_1}\neq 0$ holds), i.e. $x_1=P(P,S,x_3)$, $x_2=Q(p,S,x_3)$.

After making this change Eqs (1) can be written as

$$\begin{aligned}&BP_p-AQ_p=0,\ u_{1p}BP_S-(\tau+u_{1p}A)Q_S=0\\&(\tau+u_{2p}B)P_S-u_{2p}AQ_S=0\\&(u_{3p}B-\tau Q_{x_3})P_s-(u_{3p}A-\tau P_{x_3})Q_S=0\\&(u_{2S}-u_{3S}Q_{x_3})P_p-(u_{1S}-u_{3S}P_{x_3})Q_p=0\end{aligned} \tag{23}$$

where

$$(A\equiv u_1-u_3P_{x_3},\ B\equiv u_2-u_3Q_{x_3})$$

$$P_pQ_S-P_SQ_p\neq 0 \tag{24}$$

The investigation of system (23) can be split into two cases: (a) $\mathbf{v}_S\times\mathbf{v}_{pS}=0$ (this case was investigated previously in [1]), and (b) $\mathbf{v}_S\times\mathbf{v}_{pS}\neq 0$ (this case was eliminated from consideration in [1] and in Zubov's dissertation).

We will consider case (b) below. To fix our ideas we will assume that $R=u_{2pS}u_{1p}-u_{1pS}u_{2p}\neq 0$.

Since we must assume that $P_S^2+Q_S^2\neq 0$, we obtain from the second and third equations of (23) that $A^2+B^2\neq 0$ and

$$\begin{aligned}&A^2+B^2\neq 0 \text{ and}\\&\tau+u_{1p}A+u_{2p}B=0\end{aligned} \tag{25}$$

or, by virtue of (3)

$$u_{3p}=U_{1p}P_{x_3}+u_{2p}Q_{x_3}=0$$

Integrating this equation with respect to x_3 we obtain

$$x_3 u_{3p} + u_{1p} P + u_{2p} Q = \chi \tag{26}$$

with arbitrary function $\chi = \chi(p, S)$. It follows from (25) that the second and third equations of system (23) reduce to

$$u_{1p} P_S + u_{2p} Q_S = 0 \tag{27}$$

Differentiating (26) with respect to S and then substituting (27) we obtain the relation

$$u_{1pS} P + u_{2pS} Q = \chi_S - x_3 u_{3pS} \tag{28}$$

We then obtain from (26) and (28)

$$P = \frac{1}{R}\left(-x_3 \begin{vmatrix} u_{3p} & u_{2p} \\ u_{3pS} & u_{2pS} \end{vmatrix} + \begin{vmatrix} \chi & u_{2p} \\ \chi_S & u_{2pS} \end{vmatrix}\right) \tag{29}$$

$$Q = \frac{1}{R}\left(x_3 \begin{vmatrix} u_{3p} & u_{1p} \\ u_{3pS} & u_{1pS} \end{vmatrix} - \begin{vmatrix} \chi & u_{1p} \\ \chi_S & u_{1pS} \end{vmatrix}\right)$$

If $\mathbf{b} = 0$, we have $\mathbf{v} = u_1\mathbf{F}$, where $\mathbf{F} = \mathbf{F}(p)$, $F_1' \equiv 1$, $F_2' \neq 0$. After substituting (29) into the first equation of (23), we obtain $(F_3'/F_2') = 0$ if $F_3 = k_1F_2 + k_2$ with constants k_1 and k_2, and this implies that $u_3 = k_1u_2 + k_2u_1$, i.e. reduction to an invariant solution. Hence we must consider $\mathbf{b} \neq 0$, i.e. the vectors $\mathbf{v}$, $\mathbf{v}_S$ and $\mathbf{b}$ are not coplanar. From (22) we obtain

$$\tau \mathbf{v}_{pS} = \zeta \mathbf{v}_p - \xi\lambda \mathbf{b} / \mathbf{b}^2 \tag{30}$$

Here, since $R \neq 0$, we have $\xi\lambda \neq 0$.

After substituting (29) into (23) and taking (30) into account there remains only a single second-order linear hyperbolic equation with respect to the function $\chi = \chi(p, S)$ in system (23)

$$\chi_{pS} + A\chi_p + B\chi_S + C\chi = 0 \tag{31}$$

Here

$$A = -\frac{\zeta}{\tau},\quad C = A_p + AB + \frac{\xi\tau_p}{2\tau^2}$$

$$B = -\frac{\tau\mathbf{b}^2}{\zeta\lambda}\left[\frac{\xi\lambda}{\tau\mathbf{b}^2}\begin{vmatrix} u_{1p} & u_{2p} \\ b_{1p} & b_{2p} \end{vmatrix}\begin{vmatrix} u_{1p} & u_{2p} \\ b_1 & b_2 \end{vmatrix}^{-1} + \left(\frac{\xi\lambda}{\tau\mathbf{b}^2}\right)_p\right]$$

Hence, for $H = 0$ in case (b) flows with straight level lines exist if the functions τ and $u_i(p, S)$ $(i = 1, 2, 3)$ satisfy the overdetermined system consisting of five differential equations: (3), (30) and

$$H \equiv \tau_p + \mathbf{v}_p\mathbf{v}_p = 0 \tag{32}$$

It is difficult to analyse this overdetermined system in the general case of the equation of state. We will do this for an equation of state of the form $\tau = g(p)A^2(S)$.

Differentiating (32) with respect to S and substituting (30) we obtain a relation from which it follows from the form of the equation of state and from the fact that $\xi \neq 0$, that

$$\tau_p + 2\lambda^2 / \mathbf{b}^2 = 0 \tag{33}$$

Differentiating (3), (32) and (33) with respect to p we obtain

$$\begin{gathered} \mathbf{v}\mathbf{v}_{pp} = 0,\quad \mathbf{v}_p\mathbf{v}_{pp} = -\tau_p / 2 \\ \mathbf{b}\mathbf{v}_{pp} = \lambda g''/(2g') - \mathbf{b}_p(g' A^2\mathbf{b} + \mathbf{v}_p) \end{gathered} \tag{34}$$

The determinant for the derivative $\mathbf{v}_{pp}$ in (34) will be $d \equiv \tau(\mathbf{v}\mathbf{v}_s) + \mathbf{v}^2(\mathbf{v}_p\mathbf{v}_S)$.

If $d = 0$, after differentiating d with respect to p we obtain an equation, using which together with the first two equations of (34), we obtain the second derivatives

$$\mathbf{v}_{pp} = -\mathbf{k}\xi(2\tau^2 + \tau_p\mathbf{v}^2)/(2\tau\mathbf{v}^2) - \mathbf{b}\tau_{pp}/(2\lambda)\quad (\mathbf{k} = \mathbf{v}\times\mathbf{v}_p) \tag{35}$$

After differentiating (33) with respect to p, taking (35) and the relation $d = 0$ into account, we find $\mathbf{v}^2 = -2\tau\tau_p/\tau_{pp}$. Hence we obtain $\mathbf{v}\mathbf{v}_S = -(2\tau\tau_p/\tau_{pp})_S$, and from the relation $d = 0$ it follows that $\mathbf{v}_p\mathbf{v}_S = -\tau_S/2$. But we then obtain $\xi = 0$, which is impossible in the case in question. Hence we must have $d \neq 0$.

Since $d \neq 0$, from Eqs (34) we obtain the second derivatives

$$d\mathbf{v}_{pp} = b\mathbf{k} + \tau_{pp}(\mathbf{v}(\mathbf{v}\mathbf{v}_S) - \mathbf{v}_S\mathbf{v}^2)/2 \tag{36}$$

$$(b = \tau_{pp}\lambda/(2\tau_p) + \tau_p(\tau\mathbf{v}_S + (\mathbf{v}_p\mathbf{v}_S)\mathbf{v}^2)/(2\tau\lambda))$$

We can verify by direct calculations that the equation $(\mathbf{v}_{pp})_S - (\mathbf{v}_{pS})_p = 0$ is satisfied. Hence, the new relations, containing derivatives of v no higher than the second order, can only be obtained after differentiating (33) with respect to S

$$\mathbf{a}\mathbf{v}_{SS} - \tau_p\xi\mathbf{b}^2/(2\tau) = 0 \quad (\mathbf{a} \equiv 2\lambda\mathbf{k} - \tau_p(\mathbf{v}^2\mathbf{v}_S - (\mathbf{v}\mathbf{v}_S)\mathbf{v})) \tag{37}$$

Corollaries of Eqs (3), (30), (32) and (33) will be useful later. Since the vectors $\mathbf{v}$, $\mathbf{v}_S$ and $\mathbf{b}$ are not coplanar, the vectors $\mathbf{v}_p$ and $\mathbf{k}$ can be expressed linearly in terms of them

$$\mathbf{v}_p = [-\mathbf{v}(\tau\mathbf{v}_S^2 + (\mathbf{v}\mathbf{v}_S)(\mathbf{v}_p\mathbf{v}_S)) + \mathbf{v}_S(\tau(\mathbf{v}\mathbf{v}_S) + \mathbf{v}^2(\mathbf{v}_p\mathbf{v}_S)) + \lambda\mathbf{b}]/\mathbf{b}^2 \tag{38}$$

$$\mathbf{k} = [\mathbf{v}\lambda(\mathbf{v}\mathbf{v}_S) - \mathbf{v}_S\lambda\mathbf{v}^2 + \mathbf{b}d]/\mathbf{b}^2 \tag{39}$$

Hence we have the relation

$$2(\tau\mathbf{v}_S + (\mathbf{v}_p\mathbf{v}_s)\mathbf{v})^2 + \tau_p\mathbf{b}^2 = 0 \tag{40}$$

Then, taking (39) into account, we can write (37) in the form

$$4\lambda\tau d\mathbf{b}\mathbf{v}_{SS} = \tau_p\xi(\mathbf{b}^2)^2 \tag{41}$$

New equations containing derivatives of u_i no higher than the second order can now be obtained only after differentiating Eq. (41) with respect to p. It turns out that after this differentiation and using (41) the equation $F = 0$ is obtained, containing derivatives of u_i no higher than the first order (it will not be given here in view of its length). This equation must then be differentiated with respect to p and S. In view of the length of the further calculations, which were carried out on a computer in the REDUCE system, we will only describe the results.

We have $(\mathbf{v}^2 + 2\tau^2/\tau_p)(\tau\tau_{pp} - \tau_p^2) \neq 0$ (otherwise we obtain a contradiction of the condition $\xi \neq 0$). Then, from the equation $F = 0$ we find an expression for $\mathbf{v}\mathbf{v}_S$, after substituting which into the equation

$$\partial(\mathbf{v}\mathbf{v}_S)/\partial p + \tau_S = 0 \tag{42}$$

we obtain

$$2\tau_p^3 b_0\mathbf{v}^2 + \tau^2(4\tau_p^2 b_0 + 6\tau_p^2 a^2 - 3a^3) = 0$$

$$(a \equiv \tau\tau_{pp} - \tau_p^2,\ b_0 = (\tau\tau_p a_p - 2a(a + \tau_p^2))) \tag{43}$$

Suppose $b_0 \neq 0$. Since Eq. (42) holds by virtue of (30), after substituting into it the product $\mathbf{v}\mathbf{v}_S$, obtained from $F = 0$, we determine $\mathbf{v}^2$, and after substituting $\mathbf{v}^2$ into (3) we obtain

$$(\tau\tau_{pp} - \tau_p^2)(\tau\tau_{pp} - 3\tau_p^2) = 0$$

which contradicts the condition $b_0 \neq 0$.

Suppose now that $b_0 = 0$. From (43) we then obtain $\tau_{pp} - 3\tau_p^2 = 0$, which corresponds to the equation of state of a polytropic gas with polytropy index $\gamma = 2$. Then $\mathbf{v}\mathbf{v}_S = \mathbf{v}^2A'/A$ and from the relation $\partial(\mathbf{v}\mathbf{v}_S)/\partial S - \mathbf{v}_{SS} - \mathbf{v}_S^2 = 0$ we obtain the equation

$$\mathbf{v}\mathbf{v}_{SS} - (\mathbf{v}^2/A^2)(AA'' + 2(AA'g + \mathbf{v}_p\mathbf{v}_s)^2/(\mathbf{v}^2 g' + 2g^2A^2) = 0 \tag{44}$$

differentiation of which with respect to p leads to an identity. This implies that the overdetermined system consisting of Eqs (3), (30), (32) and (33) is in involution.

Note that in this case, Eq. (3) and $\mathbf{v}\mathbf{v}_S = \mathbf{v}^2A'/A$ has the integral

$$\mathbf{v}^2 = (c_1 - 4p^{1/2})A^2 \quad (c_1 = \text{const})$$

Hence, for the equation of state of the form $\tau = g(p)A^2(S)$, the system of equations (3), (30), (32), (33) is only compatible for equations of state of a polytropic gas with polytropy index $\gamma = 0$ and has a solution with an arbitrariness in a single function of a single argument.

This completes the proof of the theorem.

This research was carried out with financial support from the Russian Foundation for Basic Research (93-013-17361).

REFERENCES

1. Meleshko, S. V., Non-isentropic steady spatial and plane unsteady double waves. *Prikl. Mat. Mekh.*, 1989, **53**(2), 255–260.
2. Zubov, Ye. N., Plane steady gas flows with straight level lines when there is a Bernoulli integral. *Dokl. Akad. Nauk SSSR*, 1976, **227**(1), 57–59.
3. Zubov, Ye. N., A class of spatial steady flows of a polytropic gas with an adiabatic index of two. In *Numerical Methods of the Mechanics of a Continuous Medium*. Vychisl. Tsentr Sib. Otd. Akad. Nauk SSSR, 1978, **9**(3), 92–100.
4. Zubov, Ye. N., *Spatial steady flows of an ideal gas in a degenerate hodograph when there is a Bernoulli integral*. Trudy Inst. Mat. Mekh. Ural. Nauch. Tsentra Akad. Nauk SSSR, 1978, **25**, 31–42.
5. Zubov, Ye. N., Spatial steady flows of an ideal gas with straight level lines. In *Numerical Methods of the Mechanics of a Continuous Medium*. Vychisl. Tsentr Sib. Otd. Akad. Nauk SSSR, 1978, **9**(1), 47–64.
6. Ovsyannikov, L. V., *Group Analysis of Differential Equations*. Nauka, Moscow, 1978.

Translated by R.C.G.