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On periodic solutions of nonlinear evolution equations in Banach spaces ☆

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Abstract

We prove an existence result for T-periodic solutions to nonlinear evolution equations of the form

$$\dot{x}(t) + A\big(t, x(t)\big) = f\big(t, x(t)\big), \quad 0 < t < T.$$

Here $V \hookrightarrow H \hookrightarrow V^*$ is an evolution triple, $A : I \times V \to V^*$ is a uniformly monotone operator, and $f : I \times H \to V^*$ is a Caratheodory mapping which is Hölder continuous with respect to x in H and exponent $0 < \alpha \leqslant 1$. For illustration, an example of a quasi-linear parabolic differential equation is worked out in detail.

Keywords: Evolution equation; Time-periodic solution; Quasi-linear parabolic differential equation

1. Introduction

In this paper, we establish an existence result of periodic solutions for a class of nonlinear evolution equations in Banach spaces. Our approach will be based on techniques and results of the theory of monotone operators and the Leray–Schauder fixed point theorem.

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The problem of existence of periodic solutions for nonlinear evolution equations has been studied by several authors. However, most of the works concentrated on semilinear systems. We refer to the works of Browder [2], Prüss [7], and Becker [1]. The first fully nonlinear existence results for the periodic problem, were obtained by Vrabie [9] and Hirano [4]. Vrabie assumes that the nonlinear time invariant operator A monitoring the evolution equation is such that $A - \lambda I$ is m-accretive for some $\lambda > 0$, while Hirano [4] considers time invariant nonlinear operator A which is a subdifferential of a convex function defined in a Hilbert space. Moreover, Hu and Papageorgiou [5] consider evolution inclusions defined in an evolution triple of Hilbert spaces. They proved the existence of a periodic solution for a problem with a Caratheodory multivalued perturbation $F(t, x)$ defined on $I \times H$ into H.

The time-dependent systems for the periodic problem were investigated recently by Kandilakis and Papageorgiou [6] and Shioji [8]. Kandilakis and Papageorgiou consider multivalued perturbations while Shioji consider single-valued perturbation. However, the assumptions on the monitoring operator A and on the perturbation $F(t, x)$ of both papers imply that the operator $A + F$ is a pseudo-monotone operator.

In this paper, we also consider a time-dependent systems with a single-valued perturbation. Here, the perturbation $f(t, x)$ is assumed to be continuous and defined on $I \times H$ with values in V^*. Our assumptions on the monitoring operator A and on the perturbation $f(t, x)$ do not imply that $A + f$ is pseudo-monotone.

2. System description

The mathematical setting of our problem is the following. Let H be a real separable Hilbert space, V be a dense subspace of H having structure of a reflexive Banach space, with the continuous embedding $V \hookrightarrow H \hookrightarrow V^*$, where V^* is the topological dual space of V. The system model considered here is based on this evolution triple. Let the embedding $V \hookrightarrow H$ be compact.

Let $\langle x, y \rangle$ denote the pairing of an element $x \in V^*$ and an element $y \in V$. If $x, y \in H$, then $\langle x, y \rangle = (x, y)$, where (x, y) is the scalar product on H. The norm in any Banach space X will be denoted by $\|\cdot\|_X$. Let T be a fixed positive constant and $I = (0, T)$. Let $p, q \geqslant 1$ be such that $p^{-1} + q^{-1} = 1$ and $2 \leqslant p < +\infty$.

We denote $L_p(I, V)$ by X. Then the dual space of X is $L_q(I, V^*)$ and is denoted by X^*. For p, q satisfy the above conditions, it follows from reflexivity of V that both X and X^* are reflexive Banach spaces (see Zeidler [10, p. 411]). Define

$$W_{pq} = \{x\colon\ x \in X,\ \dot{x} \in X^*\},$$

where the derivative in this definition should be understood in the sense of distribution. Furnished with the norm $\|x\|_{W_{pq}} = \|x\|_X + \|\dot{x}\|_{X^*}$, the space $(W_{pq}, \|\cdot\|)$ becomes a Banach space which is clearly reflexive and separable. Moreover, W_{pq} embeds into $C(\bar{I}, H)$ continuously (see Proposition 23.23 of [10]). So every element in W_{pq} has a representative in $C(\bar{I}, H)$. Because of the embedding $V \hookrightarrow H$ is compact, the embedding $W_{pq} \hookrightarrow L_p(I, H)$ is also compact (see Problem 23.13 of [10]). The pairing between X and X^* is denoted by $\langle\langle\, , \rangle\rangle$.

Consider the following equation

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t)), \quad t \in I, \\ x(0) = x(T), \end{cases} \tag{2.1}$$

where the operators $A : I \times V \to V^*$ and $f : I \times H \to V^*$. By a solution x of problem (2.1), we mean a function $x \in \{x \in W_{pq}\colon x(0) = x(T)\}$ such that

$$\langle \dot{x}(t), v \rangle + \langle A(t, x), v \rangle = \langle f(t, x), v \rangle$$

for all $v \in V$ and almost all $t \in I$.

We need the following hypotheses on the data problem (2.1).

(A1) $A : I \times V \to V^*$ is an operator such that

(1) $t \mapsto A(t, x)$ is measurable;

(2) For each $t \in I$, the operator $A(t) : V \to V^*$ is uniformly monotone and hemicontinuous, that is, there exists a constant $C_1 \geqslant 0$ such that

$$\langle A(t, x_1) - A(t, x_2), x_1 - x_2 \rangle \geqslant C_1 \|x_1 - x_2\|_V^p, \quad \text{for all } x_1, x_2 \in V,$$

and the map $s \mapsto \langle A(t, x + sz), y \rangle$ is continuous on $[0, 1]$ for all $x, y, z \in V$;

(3) Growth condition: There exists a constant $C_2 > 0$ and a nonnegative function $a_1(\cdot) \in L_q(I)$ such that

$$\|A(t, x)\|_{V^*} \leqslant a_1(t) + C_2 \|x\|_V^{p-1}, \quad \text{for all } x \in V, \text{ a.e. on } I;$$

(4) Coerciveness: There exists a constant $C_3 > 0$ such that

$$\langle A(t, x), x \rangle \geqslant C_3 \|x\|_V^p, \quad \text{for all } x \in V, \text{ a.e. on } I.$$

Without loss of generality, we can assume that $A(t, 0) = 0$ for all $t \in \bar{I}$.

(F1) $f : I \times H \to V^*$ is an operator such that

(1) $t \mapsto f(t, x)$ is measurable;

(2) $x \mapsto f(t, x)$ is continuous;

(3) There exists a nonnegative function $h_1(\cdot) \in L_q(I)$ and a constant $C_4 > 0$ such that

$$\|f(t, x)\|_{V^*} \leqslant h_1(t) + C_4 \|x\|_H^{k-1}, \quad \text{for all } x \in V, \ t \in I,$$

where $1 \leqslant k < p$ is constant;

(4) $f(t,x)$ is Hölder continuous with respect to x with exponent $0 < \alpha \leqslant 1$ in H and uniformly in t. That is, there is a constant L such that

$$\big\| f(t,x_1) - f(t,x_2)\big\|_{V^*} \leqslant L\|x_1 - x_2\|_H^\alpha, \quad \text{for all } x_1, x_2 \in H \text{ and for all } t \in I.$$

It is convenient to rewrite system (2.1) as an operator equation in

$$W_{pq}(T) = \big\{x \in W_{pq}\colon x(0) = x(T)\big\}.$$

For $x \in X$, we set

$$A(x)(t) = A\big(t, x(t)\big), \quad F(x)(t) = f\big(t, x(t)\big), \quad t \in I.$$

It follows from Theorem 30.A of Zeidler [10] that the operator $A : X \to X^*$ is bounded, uniformly monotone, hemicontinuous, and coercive. By using the same technique, one can show that the operator $F : L_p(I,H) \to X^*$ is bounded and satisfies

$$\big\|F(u)\big\|_{X^*} \leqslant M_1 + M_2\|u\|_{L_p(I,H)}^{(k-1)}, \quad \text{for all } u \in L_p(I,H).$$

Lemma 1. *Under assumption* (F1), *the operator* $F : L_p(I,H) \to X^*$ *is Hölder continuous with exponent* α, $0 < \alpha \leqslant 1$, *and* $F(x_n) \to F(x)$ *in* X^* *whenever* $x_n \xrightarrow{W} x$ *in* W_{pq}.

Proof. Let $x_1, x_2 \in L_p(I,H)$. By hypothesis (4) of (F1) and Hölder inequality, we get

$$\begin{aligned}
&\big\|F(x_1) - F(x_2)\big\|_{X^*} \\
&\quad = \left(\int_0^T \big\| f\big(t,x_1(t)\big) - f\big(t,x_2(t)\big)\big\|_{V^*}^q\, dt\right)^{1/q} \\
&\quad \leqslant L\left(\int_0^T \big\|x_1(t) - x_2(t)\big\|_H^{q\alpha}\, dt\right)^{1/q} \\
&\quad \leqslant L\left(\int_0^T \big\|x_1(t) - x_2(t)\big\|_H^{p}\, dt\right)^{\alpha/p} \left(\int_0^T 1^{1-p/q\alpha}\, dt\right)^{\alpha/(q\alpha-1)} \\
&\quad \leqslant L'\|x_1 - x_2\|_{L_p(I,H)}^\alpha
\end{aligned}$$

for some constant L'. This proves that F is Hölder continuous with exponent α in $L_p(I,H)$. Hence F is continuous on $L_p(I,H)$.

Since the embedding $V \hookrightarrow H$ is compact, the embedding $W_{pq} \hookrightarrow L_p(I,H)$ is compact. That is,

$$x_n \to x \quad \text{in } L_p(I,H) \quad \text{whenever} \quad x_n \overset{W}{\to} x \quad \text{in } W_{pq}.$$

By using the above relation and the continuity of F, we have

$$F(x_n) \to F(x) \quad \text{in } X^* \quad \text{whenever} \quad x_n \overset{W}{\to} x \quad \text{in } W_{pq}. \qquad \square$$

Moreover, we observe that the original problem (2.1) is equivalent to the following operator equation:

$$\begin{cases} \dot{x} + A(x) = F(x), \\ x \in W_{pq}(T). \end{cases} \tag{2.2}$$

Lemma 2. *Assume* (A1) *and* (F1) *are satisfied. Then the set*

$$\mathrm{S1} = \bigl\{x \in W_{pq}(T) \mid \dot{x} + A(x) = \sigma F(x),\ \text{for some } \sigma \in [0,1]\bigr\} \tag{2.3}$$

is bounded in W_{pq}. Moreover, there exists a positive constant M such that

$$\bigl\|A(x)\bigr\|_{X^*} \leqslant M \quad \text{and} \quad \max_{t\in\bar{I}}\bigl\|x(t)\bigr\|_H \leqslant M$$

for all $x \in$ S1.

Proof. Let $x \in$ S1, then we have

$$\langle\langle \dot{x}, x\rangle\rangle + \bigl\langle\bigl\langle A(x), x\bigr\rangle\bigr\rangle = \bigl\langle\bigl\langle \sigma F(x), x\bigr\rangle\bigr\rangle.$$

Since A is coercive (hypothesis (A1)) then

$$C_3\|x\|_X^p \leqslant \bigl\langle\bigl\langle \sigma F(x), x\bigr\rangle\bigr\rangle - \langle\langle \dot{x}, x\rangle\rangle. \tag{2.4}$$

By using integration by part and the relation $x(0) = x(T)$, one can see that the second term on the right-hand side of (2.4) is equal to zero. Hence, inequality (2.4) reduces to

$$\begin{aligned} C_3\|x\|_X^p &\leqslant \sigma\left(\int_0^T \bigl\|f(t,x)\bigr\|_{V^*}^q\,dt\right)^{1/q}\left(\int_0^T \|x\|_V^p\,dt\right)^{1/p} \\ &\leqslant \sigma\left(\int_0^T \bigl(h_1(t) + \|x\|_H^{k-1}\bigr)^q\,dt\right)^{1/q} \|x\|_X \\ &\leqslant \alpha_1\|x\|_X + \alpha_2\|x\|_X^k \end{aligned} \tag{2.5}$$

for some constants $\alpha_1 > 0$ and $\alpha_2 > 0$. Since $1 \leqslant k < p$, thus, by virtue of the inequality (2.5), we can find a constant $M_1 > 0$ such that

$$\|x\|_X \leqslant M_1 \tag{2.6}$$

for all $x \in$ S1.

From (2.6), the boundedness of operators A and F, and the continuous embedding $X \hookrightarrow L_p(I, H)$, we obtain

$$\|A(x)\|_{X^*} \leqslant M_2 \quad \text{and} \quad \|F(x)\|_{X^*} \leqslant M_2, \tag{2.7}$$

for some constant $M_2 > 0$ and all $x \in \mathrm{S1}$. Therefore,

$$\|\dot{x}\|_{X^*} \leqslant \|A(x)\|_{X^*} + \|F(x)\|_{X^*} \leqslant 2M_2, \quad \text{for all } x \in \mathrm{S1}. \tag{2.8}$$

It follows from (2.6) and (2.8) that

$$\|x\|_{W_{pq}} \leqslant M_1 + 2M_2.$$

Hence, S1 is a bounded subset of W_{pq}.

Finally, we note that $W_{pq} \hookrightarrow C(\bar{I}, H)$; then

$$\|x\|_{C(I,H)} \leqslant \alpha \|x\|_{W_{pq}},$$

and hence

$$\max_{t\in I} \|x(t)\|_H \leqslant M_3$$

for some positive constants α, M_3, and for all $x \in \mathrm{S1}$. Choosing $M = \max(M_2, M_3)$, the assertion follows. □

Theorem 3. *Under assumptions* (A1) *and* (F1), Eq. (2.2) *has a solution* $x \in W_{pq}(T)$.

Proof. We denote $S = L_P(I, H)$. Define $G : S \times [0, 1] \to S$ by $G(u, \sigma) = v$, where v is the solution of the following problem:

$$\begin{cases} \dot{v} + A(v) = \sigma F(u), \\ v(0) = v(T). \end{cases} \tag{2.9}$$

Since A is uniformly monotone, then A is strictly monotone. By Theorem 32.D of [10], for any $u \in S$, problem (2.9) has a unique solution $v \in W_{pq} \subset S$. So G is well defined.

(1) We now assert that $G : S \times [0, 1] \to S$ is continuous and compact.

In fact, for any sequence $(u_n, \sigma_n) \subset S \times [0, 1]$ such that

$$(u_n, \sigma_n) \to (u, \sigma) \quad \text{in } S \times [0, 1],$$

we denote v_n to be the solution of the problem

$$\begin{cases} \dot{v}_n + A(v_n) = \sigma_n F(u_n), \\ v_n(0) = v_n(T), \end{cases}$$

and v to be the solution of the problem

$$\begin{cases} \dot{v} + A(v) = \sigma F(u), \\ v(0) = v(T). \end{cases}$$

Therefore,

$$\langle\langle \dot{v}_n - \dot{v}, v_n - v\rangle\rangle + \big\langle\big\langle A(v_n) - A(v), v_n - v\big\rangle\big\rangle = \big\langle\big\langle \sigma_n F(u_n) - \sigma F(u), v_n - v\big\rangle\big\rangle. \tag{2.10}$$

Using integration by parts and the monotonicity of the operator A, we obtain from (2.10) that

$$\begin{aligned} &\frac{1}{2}\big(\big\|v_n(T) - v(T)\big\|_H^2 - \big\|v_n(0) - v(0)\big\|_H^2\big) + C_1\|v_n - v\|_X^p \\ &\quad\leqslant \big\langle\big\langle \sigma_n F(u_n) - \sigma F(u), v_n - v\big\rangle\big\rangle \\ &\quad\leqslant \int_0^T \big\|\sigma_n f(t, u_n) - \sigma f(t, u)\big\|_{V^*}\big\|v_n(t) - v(t)\big\|_V\,dt \\ &\quad\leqslant \frac{\varepsilon}{p}\|v_n - v\|_X^p + \frac{\varepsilon^{-q/p}}{q}\big\|\sigma_n F(u_n) - \sigma F(u)\big\|_{X^*}^q, \\ &\quad\text{for all } \varepsilon > 0. \end{aligned} \tag{2.11}$$

Since $F(u)$ is Hölder continuous with exponent α, $F : L_p(I, H) \to X^*$ is bounded, by choosing ε in (2.11) small enough, then

$$\begin{aligned} M_1\|v_n - v\|_X^p &\leqslant M_2\big\|\sigma_n F(u_n) - \sigma F(u)\big\|_{X^*}^q \\ &= M_2\big\|\sigma_n F(u_n) - \sigma_n F(u) + \sigma_n F(u) - \sigma F(u)\big\|_{X^*}^q \\ &\leqslant M_3\big(\big\|F(u_n) - F(u)\big\|_{X^*}^q + |\sigma_n - \sigma|^q\big\|F(u)\big\|_{X^*}^q\big) \\ &\leqslant M_4\|u_n - u\|_S^{q\alpha} + M_5|\sigma_n - \sigma|^q\big(1 + \|u\|_S^{(k-1)}\big)^q, \end{aligned}$$

for some positive constants M_1, M_2, M_3, M_4, and M_5. Noting that the embedding $L_p(I, V) \hookrightarrow L_p(I, H)$ is continuous, we have

$$\|v_n - v\|_S \leqslant M'\big(\|u_n - u\|_S^{q\alpha/p} + |\sigma_n - \sigma|^{q/p}\big(1 + \|u\|_S^{(k-1)}\big)^{q/p}\big),$$

for some constant $M' > 0$. Hence, $G : S \times [0, 1] \to S$ is continuous.

Let v be the solution of problem (2.9) with $\|u\|_S < b_1$ for some $\sigma \in [0, 1]$, where $b_1 > 0$ is a constant. Similar to the proof of Lemma 2, one can show that there exists constant $b_2 > 0$ such that

$$\|v\|_{W_{pq}} \leqslant b_2.$$

Hence, G maps bounded sets in $S \times [0, 1]$ into bounded sets in W_{pq}. Since the embedding $W_{pq} \hookrightarrow S$ is compact, $G : S \times [0, 1] \to S$ is also compact.

(2) Next, we must show that the set

$$\big\{u \in S \mid u = G(u, \sigma) \text{ for some } 0 \leqslant \sigma \leqslant 1\big\}$$

is bounded in S.

Assume that $u \in S$ and $u = G(u, \sigma)$. Then $u \in W_{pq}$ and satisfies the problem

$$\begin{cases} \dot{u} + A(u) = \sigma F(u), \\ u(0) = u(T). \end{cases}$$

By Lemma 2, we get $\|u\|_{W_{pq}} \leqslant M$. Again, since the embedding $W_{pq} \hookrightarrow S$ is compact, we get

$$\|u\|_S \leqslant B$$

for some constant $B > 0$.

(3) $G(u, 0) = 0$, for any $u \in S$,

For any $u \in S$, set $G(u, 0) = v_0$, where v_0 satisfies

$$\begin{cases} \dot{v}_0 + A(v_0) = 0, \\ v_0(0) = v_0(T). \end{cases} \tag{2.12}$$

By uniqueness of the solution of Eq. (2.12), we get from $A(0) = 0$ that

$$v_0 = 0, \quad \text{in } W_{pq}.$$

Since the imbedding $W_{pq} \hookrightarrow S$ is continuous, we get

$$v_0 = 0, \quad \text{in } S,$$

that is,

$$G(u, 0) = 0, \quad \text{for any } u \in S.$$

(4) Applying Leray–Schauder fixed point theorem (see [3, p. 231]) in the space S, there is one fixed point $x \in S$ such that

$$x = G(x, 1)$$

and $x \in S \cap W_{pq}$. That is, x is a solution of problem (2.2). Since the problem (2.1) is equivalent to the problem (2.2), there exists a periodic solution for nonlinear evolution equation (2.1). □

3. Examples

In this section, to illustrate the applicability of our work, we prove the existence of a periodic solution for a quasi-linear parabolic partial differential equations of order $2m$.

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, $Q_T = (0, T) \times \Omega$, $0 < T < \infty$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index with $\{\alpha_i\}$ nonnegative integers and $|\alpha| = \sum_i^n \alpha_i$. Suppose $p \geqslant 2$ and $q = p/(p-1)$, $W^{m,p}(\Omega)$ denotes the standard Sobolev space with the usual norm:

$$\|\varphi\|_{W^{m,p}} = \left(\sum_{|\alpha| \leqslant m} \|D^\alpha \varphi\|^p_{L^p(\Omega)} \right)^{1/p}, \quad m = 0, 1, 2, \ldots.$$

Let $W_0^{m,p}(\Omega) = \{\varphi \in W^{m,p} \mid D^\beta \varphi|_{\partial\Omega} = 0,\ |\beta| \leqslant m-1\}$. It is well known that $C_0^\infty(\Omega) \hookrightarrow W_0^{m,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-m,p}(\Omega)$ and the embedding $W_0^{m,p}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Denote $V \equiv W_0^{m,p}(\Omega)$ and $H \equiv L_2(\Omega)$; then $V^* \equiv W^{-m,q}(\Omega)$.

Example 1. We consider the time-periodic solutions for $2m$-order quasi-linear parabolic equations

$$\begin{cases} \frac{\partial}{\partial t} x(t,z) + \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,z,\eta(x)(t,z)) \\ \quad = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha(t,z,x(t,z)) \quad \text{on } Q_T, \\ D^\beta x(t,z) = 0 \quad \text{on } [0,T]\times\partial\Omega \text{ for all } \beta\colon |\beta| \leqslant m-1, \\ x(0,z) = x(T,z) \quad \text{on } \Omega, \end{cases} \tag{3.1}$$

where $\eta(x) \equiv \{(D^\gamma x),\ |\gamma| \leqslant m\}$ and $M = (n+m)!/(n!m!)$.

We need the following hypotheses on the data of (3.1):

(A′) The functions $A_\alpha(|\alpha| \leqslant m) : Q_T \times R^M \to R$ are functions such that
- (1) $(t,z) \to A_\alpha(t,z,\eta)$ is measurable on Q_T for $\eta \in R^M$, $\eta \to A_\alpha(t,z,\eta)$ is continuous on R^M for almost all $(t,z) \in Q_T$;
- (2) $|A_\alpha(t,z,\eta)| \leqslant a_1(t,z) + c_1(z) \sum_{|\gamma|\leqslant m} |\eta_\gamma|^{p-1}$ with $a_1(\cdot,\cdot) \in L_q(Q_T)$ and $c_1(\cdot) \in L^\infty(\Omega)$ for almost all $t \in (0,T)$;
- (3) $\sum_{|\alpha|\leqslant m} (A_\alpha(t,z,\eta) - A_\alpha(t,z,\tilde\eta))(\eta_\alpha - \widetilde{\eta_\alpha}) \geqslant c_2(z) \sum_{|\gamma|\leqslant m} |\eta_\gamma - \widetilde{\eta_\gamma}|^p$ with $c_2(\cdot) \in L^\infty(\Omega)$ for almost all $t \in (0,T)$;
- (4) $A_\alpha(t,z,0) = 0$ for all $(t,z) \in Q_T$.

(F′) $f_\alpha : Q_T \times R \to R$ are functions such that
- (1) $(t,z) \to f_\alpha(t,z,x)$ is measurable on Q_T for $x \in R$, $x \to f_\alpha(t,z,x)$ is continuous on R for almost all $(t,z) \in Q_T$;
- (2) $|f_\alpha(t,z,x)| \leqslant a_2(t,z) + c_4(z)|x|^{k-1}$ with $1 \leqslant k < p$, $a_2(\cdot,\cdot) \in L_q(Q_T)$, and $c_4(\cdot) \in L^\infty(\Omega)$ for almost all $t \in (0,T)$;
- (3) $f_\alpha(t,z,x)$ is Hölder continuous with respect x and exponent $0 < \alpha \leqslant 1$; that is, there is a constant L

$$\left| f_\alpha(t,z,x_1) - f_\alpha(t,z,x_2) \right| \leqslant L|x_1 - x_2|^\alpha$$

for any $x_1, x_2 \in R$, $(t,z) \in Q_T$.

For $x, y \in W_0^{m,p}$, $t \in I$, we define

$$a(t,x,y) = \int_\Omega \sum_{|\alpha|\leqslant m} A_\alpha\big(t,z,\eta(x)(t,z)\big) D^\alpha y\, dz.$$

It is not difficult to verify that under the above assumption (A′), for each $x \in V$ and $t \in I$, $y \to a(t,x,y)$ is a continuous linear form on V. Hence there exists an operator $A : I \times V \to V^*$ such that

$$\langle A(t,x), y\rangle = a(t,x,y).$$

Under the given assumption (A′), it is easy to verify that A satisfies our hypotheses (A1) of Section 2.

Next, by using the time-varying Dirichlet form $f : I \times H \times V \to R$ defined by

$$f(t,x,y) = \int_\Omega \sum_{|\beta|\leqslant m} f_\beta\big(t,z,x(t,z)\big) D^\beta y(z)\,dz.$$

Then $y \to f(t,x,y)$ is a continuous linear form on V. Hence there exists an operator $F : I \times H \to V^*$ such that

$$f(t,x,y) = \langle F(t,x), y\rangle.$$

Under the given hypotheses (F′), we obtain that F satisfies our hypotheses (F1) of Section 2.

Using the operators A and F as defined above, Eq. (3.1) can be written in an abstract form:

$$\begin{cases} \dot{x} + A(t,x) = F(t,x), \\ x(0) = x(T). \end{cases} \tag{3.2}$$

So applying Theorem 3, we get the following theorem.

Theorem 4. *If hypotheses* (A′) *and* (F′) *hold, then there exists a periodic solution* $x \in L_p(I, W_0^{m,p}(\Omega))$, $\partial x/\partial t \in L_q(I, W^{-m,q}(\Omega))$ *of* Eq. (3.1).

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