การทำสมการเชิงอนุพันธ์สามัญอันดับสองและอันดับสาม ให้เป็นเชิงเส้นโดยการแปลงแบบทั่วไปของซันด์แมน

นางสาววริษา นาคพิมพ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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LINEARIZATION OF SECOND-ORDER AND THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY GENERALIZED SUNDMAN TRANSFORMATIONS

Warisa Nakpim

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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LINEARIZATION OF SECOND-ORDER AND THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY GENERALIZED SUNDMAN TRANSFORMATIONS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์ฉบับนี้ ศึกษาปัญหาการทำให้เป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับสองและอันดับสามโดยการแปลงแบบทั่วไปของซันด์แมน ในส่วนแรกมีการประยุกต์ใช้การแปลงแบบทั่วไปของซันด์แมนสำหรับสมการเชิงอนุพันธ์สามัญอันดับสอง ผลลัพธ์ของส่วนนี้ยืนยันว่าผลการศึกษาปัญหาของ Duarte Moreira and Santos ที่ใช้สมการเชิงเส้นในรูปแบบของลาแกร์ (Laguerre form) ยังไม่สมบูรณ์และมีตัวอย่างที่แสดงว่า รูปแบบของลาแกร์ (Laguerre form) ไม่เพียงพอสำหรับปัญหาการทำให้เป็นเชิงเส้นโดยการแปลงแบบทั่วไปของซันด์แมน นั่นคือ สำหรับปัญหาการทำให้เป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับสอง ต้องศึกษารูปแบบทั่วไปของสมการเชิงเส้นในรูป $u''+\beta u'+\alpha u=\gamma$ แทนสมการเชิงเส้นในรูปแบบของลาแกร์ (Laguerre form) ในส่วนที่สองได้นำเกณฑ์ของการทำให้เป็นเชิงเส้นโดยการแปลงแบบทั่วไปของซันด์แมนมาประยุกต์ใช้กับสมการเชิงอนุพันธ์สามัญอันดับสาม ทั้งนี้ได้ทำการศึกษาและนำเสนอเงื่อนไขที่จำเป็นและเพียงพอสำหรับการทำให้เป็นเชิงเส้นในรูป $u'''+\alpha u=0$

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WARISA NAKPIM : LINEARIZATION OF SECOND-ORDER AND THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY GENERALIZED SUNDMAN TRANSFORMATIONS. THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D. 72 PP.

LINEARIZATION PROBLEM / GENERALIZED SUNDMAN TRANSFORMATIONS / NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

This thesis is devoted to the study of the linearization problem of second-order and third-order ordinary differential equations via generalized Sundman transformations. The first problem considered in the thesis is related with the application of generalized Sundman transformations to second-order ordinary differential equations. The results obtained demonstrate that the solution given by Duarte, Moreira and Santos using the Laguerre form is not complete. We also give examples which show that the Laguerre form is not sufficient for the linearization problem via generalized Sundman transformations. The equation $u''+\beta u'+\alpha u = \gamma$ should be used as the canonical linear equation for the linearization problem instead of the Laguerre form. The second part of the thesis applies generalized Sundman transformations to third-order ordinary differential equations. Necessary and sufficient conditions for a third-order ordinary differential equation to be linearizable into a linear equation $u''' + \alpha u = 0$ are obtained.

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Warisa Nakpim

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CHAPTER I

INTRODUCTION

The basic problem in the modeling of physical phenomena is to find solutions of differential equations. In general, these equations are very difficult to solve explicitly. Many solution methods make use of a change of variables that transforms a given differential equation into another equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there arises the question whether a given differential equation can be transformed into a linear equation. This problem is called the linearization problem.

Transformations used for solving the linearization problem considered in the literature employ point transformations, contact transformations, reduction of order, differentiation, differential substitutions and generalized Sundman transformations.

1.1 Historical review

The problem of linearizing a second-order ordinary differential equation via point transformations was solved by Lie (1883). He showed that any linearizable second-order equation can be at most cubic in the first-order derivative, and provided a linearization test in terms of its coefficients. Lie's approach has also been applied to third-order and fourth-order ordinary differential equations: In 1997, Grebot studied the linearization of third-order ordinary differential equations by means of a restricted class of point transformations, namely $t = \varphi(x), u = \psi(x, y)$.

Complete criteria for linearization by means of point transformations were obtained in (Ibragimov and Meleshko, 2005). The linearization of fourth-order ordinary differential equations via point transformations was discussed by Ibragimov, Meleshko and Suksern (2008). Two distinctly different classes for linearization are provided.

Another approach was developed by Cartan (1924). He used differential geometry for solving the linearization problem. In 1940, Chern obtained conditions for a third-order ordinary differential equation to be equivalent to the equations $u''' = 0$ and $u''' + u = 0$ by using Cartan's approach.

Lie (1883) also noted that all second-order ordinary differential equations can be mapped into each other by means of contact transformations. Hence, the solution of the linearization problem via contact transformations is trivial. Linearization of third-order ordinary differential equations with respect to contact transformations was studied by Neut and Petitot (2002). Ibragimov and Meleshko (2005) presented the explicit form of the linearization criteria. In 2005, Dridi and Neut solved a particular linearization problem for a fourth-order ordinary differential equation. They found conditions for a fourth-order ordinary differential equation to be equivalent to $u^{(4)} = 0$ under contact transformations. Complete criteria for fourth-order ordinary differential equations to be linearizable via contact transformations were given by Suksern, Meleshko and Ibragimov (2009).

The generalized Sundman transformation takes an intermediate place between point and contact transformations. Since it is weaker than contact transformations it can be applied to the linearization problem of second-order ordinary differential equations. The generalized Sundman transformation was earlier considered for second-order ordinary differential equations by Duarte, Moreira and Santos (1994) using the Laguerre form. The generalized Sundman transformation was also applied in Euler, Wolf, Leach and Euler (2003) for obtaining necessary and sufficient conditions for a third-order ordinary differential equation to be equivalent to the equation $u''' = 0$. Some applications of generalized Sundman transformations to ordinary differential equations were considered in Berkovich (2001) and earlier papers, which are summarized in Berkovich (2002).

Sundman symmetries were first introduced in Euler, Wolf, Leach and Euler (2003). They discovered that all third-order ordinary differential equations that can be linearized to the equation

$$u''' = 0$$

by the generalized Sundman transformation

$$u(t) = F(x, y), \quad dt = G(x, y)dx, \quad (F_y G \neq 0)$$

admit the symmetry

$$F(\tilde{x}, \tilde{y}) = F^{-1}(x, y), \quad G(\tilde{x}, \tilde{y})d\tilde{x} = F^{-3/2}(x, y)G(x, y)dx$$

called a Sundman symmetry transformation. In 2004, Euler and Euler investigated the Sundman symmetries of second-order autonomous equations

$$u'' + a_2(u)(u')^2 + a_1(u)u' + a_0(u) = 0$$

where a_0, a_1 and a_2 are differentiable functions. Moreover, they found the Sundman symmetries of third-order autonomous equations

$$u''' + a_5(u)(u'')^2 + a_4(u)u'u'' + a_3(u)(u')^3 + a_2(u)(u')^2 + a_1(u)u' + a_0(u) = 0$$

where a_j (j=0,1,...,5) are differentiable functions.

There are other approaches for solving the linearization problem of ordinary differential equations. Ibragimov and Meleshko (2007) gave criteria for a second-order ordinary differential equation to be linearizable by increasing the order of the equation using either differentiation of the equation or the Ricatti substitution. A new algorithm for linearization of a third-order ordinary differential equation was presented by Meleshko (2006). The algorithm consists of the composition of two operations: first reducing the order of the equation, and then applying the Lie linearization test to the obtained second-order ordinary differential equation. There are several papers dealing with the increase of the order of an ordinary differential equation (Ferapontov and Svirshchevskii, 2007; Andriopoulos and Leach, 2007) or using a combination of reduction and increase of the order (Abraham-Shrauner, 1993).

Many applications of group analysis employ the use of Lie point symmetries. For ordinary differential equations which are invariant under Lie point symmetries, the order of the ordinary differential equation can be reduced by using the known order-reduction processes. In 2001, Muriel and Romero introduced a new class of symmetries and gave a reduction process for ordinary differential equations, using the invariance of the equations under these symmetries.

1.2 Results obtained in the thesis

The studies considered in the thesis are related with the application of the generalized Sundman transformation to the linearization problem of second-order and third-order ordinary differential equations.

The first study presented in the thesis demonstrates that the equation $u'' = 0$ does not define the class of all equations which are linearizable by the generalized Sundman transformation. Thus, the linearization problem considered by Duarte, Moreira and Santos (1994) via the generalized Sundman transformation is not completely studied. The examples in this thesis show that in contrast to point transformations, for the linearization problem via generalized Sundman transformations one needs to use the general form of a linear second-order ordinary differential equation instead of the Laguerre form.

The second part of the thesis is devoted to applying generalized Sundman transformations to third-order ordinary differential equations. We have obtained necessary and sufficient conditions which allow the most general third-order ordinary differential equation to be mapped into the form,

$$u''' + \alpha u = 0, \tag{1.1}$$

where $\alpha \neq 0$ is constant. Note that according to the Laguerre theorem, one of the canonical forms of a linear third-order ordinary differential equation is (1.1). If $\alpha'(t) \neq 0$, then equation (1.1) is mapped by the generalized Sundman transformation into a functional equation, which is not a differential equation.

The thesis is organized as follows. In chapter II, we introduce the background knowledge of point transformations, contact transformations, generalized Sundman transformations and the main tools for the solving linearization problem, which are necessary for our study. In chapter III, we demonstrate that the solution of the linearization problem via generalized Sundman transformations of second-order ordinary differential equations given by Duarte, Moreira and Santos (1994) only gives particular criteria for linearizable equations. Complete analysis of compatibility of the arising equations is given for the case $F_x = 0$. We also give examples which show that the Laguerre form is not sufficient for the linearization problem via the generalized Sundman transformation. In chapter IV, necessary and sufficient conditions which allow the most general third-order ordinary differential equation $y''' = f(x, y, y', y'')$ to be transformed to $u''' + \alpha u = 0$ under a generalized Sundman transformation

$$u = F(x, y), \quad dt = G(x, y)dx, \quad (F_y G \neq 0),$$

are obtained. Here $\alpha \neq 0$ is constant. The conclusion of the thesis is presented in the last chapter.

For solving the problem in the thesis, we have to solve the compatibility problem, considering an overdetermined system of partial differential equations. Compatibility analysis requires cumbersome symbolic calculations: prolongations of a system, substitution of complicated expressions, and matrix calculations. These operations consist of a large amount of analytical calculations. For this purpose it is necessary to use a computer system for symbolic calculations. Hence, the REDUCE system (Hearn, 1987) was used. REDUCE is a system for carrying out algebraic operations accurately, no matter how complicated the expressions become.

CHAPTER II

PRELIMINARY BACKGROUND

The generalized Sundman transformation can be considered as one of the methods for solving ordinary differential equations. In this thesis, we apply the generalized Sundman transformation to second-order and third-order ordinary differential equations. Let us consider the main tools used in the thesis for solving the linearization problem.

2.1 Point transformations

Definition 2.1. *A transformation*

$$t = \varphi(x, y),$$
$$u = \psi(x, y) \tag{2.1}$$

is called a point transformation. Here it is assumed that $\varphi_x\psi_y - \varphi_y\psi_x \neq 0$.

2.1.1 The mapping of a function by a point transformation

Assume that $y_0(x)$ is a given function. To obtain the transformed function $u_0(t)$, start with the equation

$$t = \varphi(x, y_0(x)).$$

Using Inverse Function Theorem, we can express x as $x = \alpha(t)$. Substituting x into the function $\psi(x, y_0(x))$, we get the transformed function

$$u_0(t) = \psi(\alpha(t), y_0(\alpha(t))).$$

Conversely, we have to change $u_0(t)$ to $y_0(x)$. Applying the Inverse Function Theorem to point transformations (2.1), we obtain

$$x = \tilde{\varphi}(t, u),$$
$$y = \tilde{\psi}(t, u). \tag{2.2}$$

Let $u_0(t)$ be a given function of t. The first equation of (2.2) becomes

$$x = \tilde{\varphi}(t, u_0(t)).$$

Using the Inverse Function Theorem, we find $t = H(x)$. Substituting t into the function $\tilde{\psi}(t, u_0(t))$, the transformed function $y_0(x) = \tilde{\psi}(H(x), u_0(H(x)))$ is obtained.

2.2 Tangent transformations

Let us consider the transformations of the independent, dependent variables and their derivatives

$$\bar{x} = f(x, u, p), \quad \bar{u} = \phi(x, u, p), \quad \bar{p} = \psi(x, u, p). \tag{2.3}$$

Here p is the vector of derivatives of the function u with respect to x: $p_k = u^{(k)}, (k = 1, 2, \ldots, s)$.

Definition 2.2. *A transformation (2.3) is called a tangent transformation if it preserves the tangent conditions*

$$d\bar{u} - \bar{p}_1 d\bar{x} = 0, \quad d\bar{p}_k - \bar{p}_{k+1} d\bar{x} = 0.$$

Contact transformations are a special case of tangent transformations, for which the transformation of the independent, dependent variables and the first order partial derivatives are defined through the independent, dependent variables and the first order partial derivatives:

2.2.1 Contact transformations

Definition 2.3. *A transformation*

$$
\begin{aligned}
t &= \varphi(x, y, y'),\\
u &= \psi(x, y, y'),\\
s &= g(x, y, y')
\end{aligned}
\tag{2.4}
$$

is called a contact transformation if it obeys the contact condition

$$s = u' = \frac{du}{dt}.$$

Let us explain how contact transformations map one function into another.

2.2.2 The mapping of a function by a contact transformation

Let $y_0(x)$ be a given function. The transformed function $u_0(t)$ is found from the equations

$$t = \varphi\left(x, y_0(x), y_0'(x)\right),$$

$$u = \psi\left(x, y_0(x), y_0'(x)\right).$$

Using the Inverse Function Theorem, the first equation gives $x = \tau(t)$. Substituting x into the second equation, we obtain the transformed function

$$u_0(t) = \psi\left(\tau(t), y_0(\tau(t)), y_0'(\tau(t))\right).$$

It is assumed that $D_x\varphi \neq 0$. The derivative is

$$u_0'(t) = \frac{D_x\psi}{D_x\varphi}\left(\tau(t), y_0(\tau(t)), y_0'(\tau(t)), y_0''(\tau(t))\right),$$

where

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + \cdots$$

is the total derivative with respect to x.

The contact conditions require that

$$g(x,y,y') = \frac{D_x\psi}{D_x\varphi}(x,y,y',y'') = \frac{\psi_x + y'\psi_y + y''\psi_{y'}}{\varphi_x + y'\varphi_y + y''\varphi_{y'}}. \tag{2.5}$$

Thus equation (2.5) can be represented in the form

$$(g(\varphi_x + y'\varphi_y) - (\psi_x + y'\psi_y)) + y''(g\varphi_{y'} - \psi_{y'}) = 0.$$

Since the contact condition is satisfied for any y'', we obtain

$$\begin{aligned} g(\varphi_x + y'\varphi_y) &= \psi_x + y'\psi_y, \\ g\varphi_{y'} &= \psi_{y'}. \end{aligned} \tag{2.6}$$

2.3 Generalized Sundman transformations

Definition 2.4. *A non-point transformation*

$$\begin{aligned} u &= F(x,y), \\ dt &= G(x,y)dx \end{aligned} \tag{2.7}$$

where $F_yG \neq 0$ is called a generalized Sundman transformation.

2.3.1 The mapping of a function by a generalized Sundman transformation

Let us explain how a generalized Sundman transformation maps one function into another.

Assume that $y_0(x)$ is a given function. Integrating the second equation of (2.7), we obtain $t = Q(x)$, where

$$Q(x) = t_0 + \int_{x_0}^{x} G(s, y_0(s))\, ds$$

with some initial conditions t_0 and x_0. Using the inverse function theorem, we find $x = Q^{-1}(t)$. Substituting x into the function $F(x, y_0(x))$, we get the transformed function

$$u_0(t) = F\left(Q^{-1}(t), y_0(Q^{-1}(t))\right).$$

Conversely, let $u_0(t)$ be a given function of t. Using the inverse function theorem we solve the equation

$$u_0(t) = F(x, y)$$

with respect to y: $y = \phi(x, t)$. Solving the ordinary differential equation

$$\frac{dt}{dx} = G(x, \phi(x, t)),$$

we find $t = H(x)$. The function $H(x)$ can be written as an action of a functional $H = \mathcal{L}(u_0)$. Substituting $t = H(x)$ into the function $\phi(x, t)$, the transformed function $y_0(x) = \phi(x, H(x))$ is obtained.

Notice that for the case $G_y = 0$ the action of the functional $\mathcal{L}$ does not depend on the function $u_0(t)$. In this case the generalized Sundman transformation becomes a point transformation. Conversely, since for a point transformation the value dt in the generalized Sundman transformation is the total differential of t, then the compatibility condition for dt to be a total differential leads to the equation $G_y = 0$. Hence, the generalized Sundman transformation is a point transformation if and only if $G_y = 0$.

Formulae (2.7) also allows us to obtain the derivatives of $u_0(t)$ through the derivatives of the function $y_0(x)$, and vice versa.

Hence, using transformation (2.7), we can relate the solutions of two differential equations $Q(x, y, y', \ldots, y^{(n)}) = 0$ and $P(t, u, u', \ldots, u^{(n)}) = 0$. Therefore the knowledge of the general solution of one of them gives the general solution of the other equation, up to solving one ordinary differential equation of first-order and finding two inverse functions.

2.4 The equivalence problem

Definition 2.5. *Two equations are called equivalent if there exists an invertible transformation such that one of the equations is transformed into the other.*

Definition 2.6. *The problem of finding all equations which are equivalent to a given equation is called the equivalence problem. If the given equation is a linear equation, then the equivalence problem is called the linearization problem.*

2.5 The Inverse function theorem

Theorem 2.7. *(Inverse Function Theorem). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on some open set containing a, and suppose $\det Jf(a) \neq 0$, where J is the Jacobian matrix. Then there is some open set V containing a and an open set W containing $f(a)$ such that $f : V \longrightarrow W$ has a continuous inverse $f^{-1} : W \longrightarrow V$ which is differentiable for all $y \in W$.*

2.6 Compatibility theory

There are two approaches for studying compatibility. These approaches are related to the works of E. Cartan and C. H. Riquier. In this thesis the Riquier approach is used.

2.6.1 The Cartan approach

The Cartan approach is based on the calculus of exterior differential forms. The problem of the compatibility of a system of partial differential equations is then reduced to the problem of the compatibility of a system of exterior differential forms. Cartan studied the formal algebraic properties of systems of exterior forms. For their description he introduced special integer numbers, called characters. With the help of the characters he formulated a criterion for a given system of partial differential equations to be involutive.

2.6.2 The Riquier approach

The Riquier approach has a different theory of establishing the involution. This method can be found in (Kuranashi, 1967) and (Pommaret, 1978). The main advantage is that there is no necessity to reduce the system of partial differential equations being studied to exterior differential forms. The calculations in the Riquier approach are shorter than in the Cartan approach. The main operations of the study of compatibility in the Riquier approach are prolongations of a systems of a partial differential equations and the study of the ranks of some matrices.

Remark 2.1. In the thesis, the problem of obtaining sufficient condition of linearization is to analyse compatibility of the overdetermined system. Analysis of the compatibility of this system consists of comparing mixed derivatives.

Remark 2.2. Roughly writing, a system is involutive if it does not produce new equations using differentiation and their linear combinations.

2.6.3 Completely integrable systems

One class of overdetermined systems, for which the problem of compatibility is solved, is the class of completely integrable systems.

Definition 2.8. *A system*

$$\frac{\partial z^i}{\partial a^j} = f^i_j(a, z), \; (i = 1, 2, ..., N; \; j = 1, 2, ..., r) \tag{2.8}$$

is called completely integrable if it has a solution for any initial values a_0, z_0 *in some open domain* D.

Theorem 2.9. *A system of the type (2.8) is completely integrable if and only if all of the mixed derivatives equalities*

$$\frac{\partial f^i_j}{\partial a^\beta} + \sum_{\gamma=1}^{N} f^\gamma_\beta \frac{\partial f^i_j}{\partial z^\gamma} = \frac{\partial f^i_\beta}{\partial a^j} + \sum_{\gamma=1}^{N} f^\gamma_j \frac{\partial f^i_\beta}{\partial z^\gamma}, \; (i = 1, 2, ..., N; \; \beta, j = 1, 2, ..., r) \tag{2.9}$$

are identically satisfied with respect to the variables $(a, z) \in D$.

Corollary 2.10. *If in an overdetermined system of partial differential equations all derivatives of order* n *are defined and comparison of all mixed derivatives of order* $n+1$ *does not produce new equations of order less or equal to* n*, then this system is compatible.*

2.7 Laguerre canonical form

According to the Laguerre theorem, in any linear ordinary differential equation the two terms of orders next below the highest can be simultaneously removed by a point transformation.

Theorem 2.11. *(Laguerre). Any linear kth-order ordinary differential equation*

$$y^{(k)} + \sum_{i=0}^{k-1} a_i(x) y^{(i)} = 0, \quad k \geq 3$$

can be transformed by a point transformation to an equation of the form

$$y^{(k)} + \sum_{i=0}^{k-3} a_i(x) y^{(i)} = 0.$$

Notice that the Laguerre forms of second-order and third-order ordinary differential equations are the linear equations $y'' = 0$ and $y''' + a_0 y = 0$, respectively.

2.8 The method of solving the linearization problem

One of the classical methods for solving ordinary differential equations is the Lie classical method. The first linearization problem for ordinary differential equations was solved by Lie (1883). He showed that any second-order ordinary differential equation $y'' = F(x, y, y')$ obtained from a linear equation $u'' = 0$ by a change of the independent and dependent variables,

$$t = \varphi(x, y), \quad u = \psi(x, y), \tag{2.10}$$

is cubic in the first-order derivative:

$$y'' + a(x, y) y'^3 + b(x, y) y'^2 + c(x, y) y' + d(x, y) = 0, \tag{2.11}$$

where

$$\begin{aligned}
a &= \Delta^{-1} \left(\varphi_y \psi_{yy} - \varphi_{yy} \psi_y \right), \\
b &= \Delta^{-1} \left(\varphi_x \psi_{yy} - \varphi_{yy} \psi_x + 2 \left(\varphi_y \psi_{xy} - \varphi_{xy} \psi_y \right) \right), \\
c &= \Delta^{-1} \left(\varphi_y \psi_{xx} - \varphi_{xx} \psi_y + 2 \left(\varphi_x \psi_{xy} - \varphi_{xy} \psi_x \right) \right), \\
d &= \Delta^{-1} \left(\varphi_x \psi_{xx} - \varphi_{xx} \psi_x \right).
\end{aligned}$$

Here the Jacobian of the change of variables is

$$\Delta = \varphi_x \psi_y - \varphi_y \psi_x \neq 0.$$

Moreover, a second-order ordinary differential equation is linearizable if and only if it has the form (2.11) with the coefficients satisfying the conditions

$$H = 3a_{xx} - 2b_{xy} + c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a = 0,$$
$$K = b_{xx} - 2c_{xy} + 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y a - 3d_x a + 3d_y b = 0.$$

For ordinary differential equations of higher order, the necessary form of an equation to be linearizable by point transformations is *

$$y^{(i)} + y^{(i-1)}[A_1 y' + A_0] + ... = 0,$$

or

$$y^{(i)} + y^{(i-1)} \frac{1}{y'+r}[-y'' \frac{i(i+1)}{2} + F_2 y'^{\,2} + F_1 y' + F_0] + ... = 0,$$

where $i \geq 3$ is the order of the equation, $F_j = F_j(x,y)$, $A_j = A_j(x,y)$, and ... denotes terms involving derivatives of order less than $i-1$.

The linearization problem for second-order ordinary differential equations via generalized Sundman transformations was investigated in Duarte, Moreira and Santos (1994). They obtained that any second-order linearizable ordinary differential equation which can be mapped into the equation $u'' = 0$ via a generalized Sundman transformation has to be of the form

$$y'' + \lambda_2(x,y)y'^2 + \lambda_1(x,y)y' + \lambda_0(x,y) = 0. \tag{2.12}$$

Using the functions

$$\lambda_3 = \lambda_{1y} - 2\lambda_{2x}, \quad \lambda_4 = 2\lambda_{0yy} - 2\lambda_{1xy} + 2\lambda_0\lambda_{2y} - \lambda_{1y}\lambda_1 + 2\lambda_{0y}\lambda_2 + 2\lambda_{2xx},$$

they showed that equation (2.12) can be mapped into the equation $u'' = 0$ via a generalized Sundman transformation provided that the coefficients $\lambda_i(x,y), (i = 0, 1, 2)$ satisfy the conditions:

*A proof for third-order ODEs can be found in (Meleshko, 2005), a proof for the general case is given in (Ibragimov, Meleshko and Suksern, 2008).

(a) if $\lambda_3 = 0$, then $\lambda_4 = 0$;

(b) if $\lambda_3 \neq 0$, then $\lambda_4 \neq 0$ and the following equations have to be satisfied

$$\lambda_4^2 + 2\lambda_{3x}\lambda_4 - 2\lambda_3^2\lambda_{1x} + 4\lambda_3^2\lambda_{0y} + 4\lambda_3^2\lambda_0\lambda_2 - 2\lambda_3\lambda_{4x} - \lambda_3^2\lambda_1^2 = 0,$$

$$\lambda_{3y}\lambda_4 + \lambda_3^2\lambda_{1y} - 2\lambda_3^2\lambda_{2x} - \lambda_3\lambda_{4y} = 0.$$

Generalized Sundman transformations were also applied in Euler, Wolf, Leach and Euler (2003) by using the well-known method for obtaining necessary and sufficient conditions for a third-order ordinary differential equation to be equivalent to the equation $u''' = 0$.

CHAPTER III

LINEARIZATION OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY GENERALIZED SUNDMAN TRANSFORMATIONS

In this chapter, generalized Sundman transformations are applied to a second-order ordinary differential equation. Because of the nature of generalized Sundman transformations, the composition of a point transformation with a generalized Sundman transformation is not necessarily a generalized Sundman transformation. This means that for the linearization problem via generalized Sundman transformations, it is not sufficient to use the Laguerre form. The calculations in this chapter demonstrate that the solution given by Duarte, Moreira and Santos (1994) is only a particular linearizability criterion.

3.1 Necessary conditions for linearization

We start with obtaining necessary conditions for the linearization problem.

First, we find the general form of a second-order ordinary differential equation

$$y'' = H(x, y, y'),$$

which can be mapped via a generalized Sundman transformation

$$u = F(x,y),\qquad dt = G(x,y)dx \tag{3.1}$$

into the linear equation

$$u'' + \beta u' + \alpha u = \gamma, \tag{3.2}$$

where $\alpha(t), \beta(t)$ and $\gamma(t)$ are some functions and $F_yG \neq 0$. Notice that the Laguerre form of a linear second-order ordinary differential equation corresponds to $\alpha = 0, \beta = 0$ and $\gamma = 0$.

The function u and its derivatives u' and u'' are defined by the first formula (3.1) and its derivatives with respect to x:

$$u'G = F_x + F_yy',\qquad u''G^2 + u'(G_x + G_yy') = F_yy'' + 2F_{xy}y' + F_{yy}y'^2 + F_{xx}. \tag{3.3}$$

The independent variable t is defined by the functional $\mathcal{L}(u)$. As noted above, if $G_y \neq 0$, then the action of the functional $\mathcal{L}$ depends on the function u. Hence, if one of the coefficients in (3.2) is not constant and $G_y \neq 0$, then the substitution of t into equation (3.2) gives a functional equation. Since the case $G_y = 0$ reduces the generalized Sundman transformation to a point transformation, the generalized Sundman transformation maps equation (3.2) into a differential equation only for constant coefficients α, β and γ.

Finding the derivatives u', u'' from (3.3), and substituting them into (3.2) with constant coefficients, we have the following equation

$$y'' + \lambda_2(x,y)y'^2 + \lambda_1(x,y)y' + \lambda_0(x,y) = 0, \tag{3.4}$$

where the coefficients $\lambda_i(x,y)$ $(i=0,1,2)$ are related to the functions F and G:

$$\lambda_2 = (F_{yy}G - F_yG_y)/K, \tag{3.5}$$

$$\lambda_1 = (2F_{xy}G - F_xG_y - F_yG_x + F_y\beta G^2)/K, \tag{3.6}$$

$$\lambda_0 = (F_{xx}G - F_xG_x + F_x\beta G^2 + \alpha FG^3 - G^3\gamma)/K, \tag{3.7}$$

and $K = GF_y \neq 0$.

Equation (3.4) presents the necessary form of a second-order ordinary differential equation which can be mapped into a linear equation (3.2) via a generalized Sundman transformation.

3.2 Sufficient conditions for linearization

For obtaining sufficient conditions, we have to solve the compatibility problem, considering (3.5)-(3.7) as an overdetermined system of partial differential equations for the functions F and G with the given coefficients $\lambda_i(x,y), (i=0,1,2)$. Notice that the compatibility conditions (3.5)-(3.7) for the particular case $\alpha = 0, \beta = 0$ and $\gamma = 0$ were obtained in Duarte, Moreira and Santos (1994). This case corresponds to the Laguerre form of a linear second-order ordinary differential equation. It is shown here that for the linearization problem via generalized Sundman transformations it is not sufficient to use the Laguerre form.

The compatibility analysis depends on the value of F_x. A complete study of all cases is cumbersome. Here a complete solution is given for the case where $F_x = 0$.

Remark 3.1. The motivation of this chapter is to show that for generalized Sundman transformations, in contrast to point or contact transformations, one has to use equation (3.2) as the goal for linearization. The complete study of compatibility of equations (3.5)-(3.7) was not the objective of this research.

Solving equations (3.5)-(3.7) with respect to F_{yy}, β and γ, we find

$$F_{yy} = (G_y F_y + F_y G\lambda_2)/G, \tag{3.8}$$

$$\beta = (G_x + G\lambda_1)/G^2, \tag{3.9}$$

$$\gamma = (-F_y\lambda_0 + \alpha F G^2)/G^2. \tag{3.10}$$

Since $F_x = 0$, then differentiating F_{yy} with respect to x, we obtain

$$GG_{xy} - G_x G_y + \lambda_{2x} G^2 = 0. \tag{3.11}$$

Differentiating (3.9) and (3.10) with respect to x and y, we get the following equations

$$G_{xx} = (2G_x^2 + G_x G\lambda_1 - \lambda_{1x} G^2)/G, \tag{3.12}$$

$$G_{xy} = G\lambda_3 - G_y\lambda_1, \tag{3.13}$$

$$2G_x\lambda_0 - \lambda_{0x} G = 0, \tag{3.14}$$

$$\alpha = (-G_y\lambda_0 + G(\lambda_{0y} + \lambda_0\lambda_2))/G^3, \tag{3.15}$$

where

$$\lambda_3 = \lambda_{1y} - 2\lambda_{2x}.$$

Substituting (3.13) into (3.11), it becomes

$$G_x G_y + G_y G\lambda_1 - G^2(\lambda_{2x} + \lambda_3) = 0. \tag{3.16}$$

Comparing the mixed derivatives $(G_{xy})_x = (G_{xx})_y$, we find the equation

$$G_x\lambda_3 - G(\lambda_{2xx} + \lambda_{2x}\lambda_1 + \lambda_{3x}) = 0. \tag{3.17}$$

Differentiating α with respect to x and y, we have

$$2G_x(\lambda_{0y} + \lambda_0\lambda_2) + G_y(\lambda_{0x} + 2\lambda_0\lambda_1) - G(\lambda_{0xy} + \lambda_{0x}\lambda_2 + 4\lambda_{2x}\lambda_0 + 2\lambda_0\lambda_3) = 0, \tag{3.18}$$

$$2GG_{yy}\lambda_0 - 6G_y^2\lambda_0 + 2G_y G(3\lambda_{0y} + 2\lambda_0\lambda_2) - G^2(\lambda_4 + 2\lambda_5 - \lambda_1\lambda_3) = 0, \tag{3.19}$$

where

$$\lambda_4 = 2\lambda_{0yy} - 2\lambda_{1xy} + 2\lambda_0\lambda_{2y} - \lambda_{1y}\lambda_1 + 2\lambda_{0y}\lambda_2 + 2\lambda_{2xx},$$

$$\lambda_5 = \lambda_{2xx} + \lambda_{2x}\lambda_1 + \lambda_{3x} + \lambda_1\lambda_3.$$

Further analysis of the compatibility depends on λ_3.

3.2.1 Case $\lambda_3 \neq 0$

From equation (3.17), we find

$$G_x = G(\lambda_{2xx} + \lambda_{2x}\lambda_1 + \lambda_{3x})/\lambda_3. \tag{3.20}$$

Substituting G_x into equations (3.14), (3.16), (3.12) and (3.13), we obtain the equations

$$\lambda_{0x} = 2\lambda_0(-\lambda_1\lambda_3 + \lambda_5)/\lambda_3, \tag{3.21}$$

$$\lambda_{2xxy} = -\lambda_{2xy}\lambda_1 - \lambda_{3xy} - 2\lambda_{2x}^2 - 2\lambda_{2x}\lambda_3 - \lambda_{3y}\lambda_1 + (\lambda_{3y}\lambda_5)\lambda_3^{-1}, \tag{3.22}$$

$$\lambda_{2xxx} = -\lambda_{3xx} - \lambda_{1x}\lambda_{2x} - \lambda_{1x}\lambda_3 + \lambda_{2x}\lambda_1^2 + \lambda_1^2\lambda_3 - 2\lambda_1\lambda_5 + \lambda_3^{-1}\lambda_5(\lambda_{3x} + \lambda_5), \tag{3.23}$$

$$G_y\lambda_5 - G\lambda_3(\lambda_{2x} + \lambda_3) = 0. \tag{3.24}$$

Case $\lambda_5 \neq 0$

Equation (3.24) gives

$$G_y = G\lambda_3(\lambda_{2x} + \lambda_3)/\lambda_5. \tag{3.25}$$

Substituting G_y into equations (3.13), (3.18) and (3.19) and comparing the mixed derivatives $(G_x)_y = (G_y)_x$, we get

$$\begin{aligned}&\lambda_3\lambda_5(6\lambda_{0y}\lambda_{2x} + 2\lambda_{2xy}\lambda_0 + 4\lambda_{2x}\lambda_0\lambda_2 + 2\lambda_{3y}\lambda_0 + 4\lambda_0\lambda_2\lambda_3 + \lambda_1\lambda_5)\\ &-\lambda_3^2(6\lambda_{2x}^2\lambda_0 + 12\lambda_{2x}\lambda_0\lambda_3 - 6\lambda_{0y}\lambda_5 + 6\lambda_0\lambda_3^2) - \lambda_4\lambda_5^2 - 2\lambda_5^3 = 0.\end{aligned} \tag{3.26}$$

Case $\lambda_5 = 0$

Equations (3.21), (3.24), (3.22), (3.23), (3.18) and (3.19) become

$$\lambda_{0x} = -2\lambda_0\lambda_1, \tag{3.27}$$

$$\lambda_{2x} = -\lambda_3, \tag{3.28}$$

$$2GG_{yy}\lambda_0 - 6G_y^2\lambda_0 + 2G_yG(3\lambda_{0y} + 2\lambda_0\lambda_2) - G^2(\lambda_4 - \lambda_1\lambda_3) = 0. \tag{3.29}$$

If $\lambda_0 \neq 0$, then equation (3.29) defines

$$G_{yy} = (6G_y^2\lambda_0 - 2G_yG(3\lambda_{0y} + 2\lambda_0\lambda_2) + G^2(\lambda_4 - \lambda_1\lambda_3))/(2G\lambda_0). \tag{3.30}$$

In this case, $(G_{yy})_x = (G_{xy})_y$ and $(G_x)_{yy} = (G_{yy})_x$ are satisfied. Hence, there are no other compatibility conditions. Thus, if $\lambda_3 \neq 0$, $\lambda_5 = 0$ and $\lambda_0 \neq 0$, then conditions (3.27) and (3.28) are sufficient for equation (3.4) to be linearizable by a generalized Sundman transformation.

If $\lambda_0 = 0$, there are no other conditions.

Remark 3.2. If $\lambda_5 = 0$, equations (3.21), (3.22), (3.23), (3.24) and (3.26) become conditions (3.27) and (3.28) respectively.

Thus, sufficient conditions for equation (3.4) in the case $\lambda_3 \neq 0$ to be linearizable by generalized Sundman transformation are (3.21), (3.22), (3.23) and (3.26).

3.2.2 Case $\lambda_3 = 0$

Notice that the particular case $\lambda_3 = 0$ and $\lambda_4 = 0$ was studied in Duarte, Moreira and Santos (1994). Here the case $\lambda_3 = 0$ and $\lambda_4 \neq 0$ is considered.

Equation (3.19) for $\lambda_3 = 0$ becomes

$$2GG_{yy}\lambda_0 - 6G_y^2\lambda_0 + 2G_yG(3\lambda_{0y} + 2\lambda_0\lambda_2) - G^2\lambda_4 = 0. \tag{3.31}$$

The assumption $\lambda_0 = 0$ leads to the contradiction that $\lambda_4 = 0$. Hence, we have to assume that $\lambda_0 \neq 0$.

Equations (3.17), (3.14) and (3.18) become

$$\lambda_{2xx} = -\lambda_{2x}\lambda_1, \tag{3.32}$$

$$G_x = (G\lambda_{0x})/(2\lambda_0), \tag{3.33}$$

$$G_y\lambda_0\lambda_6 - G(\lambda_{6y}\lambda_0 - \lambda_{0y}\lambda_6) = 0, \tag{3.34}$$

where

$$\lambda_6 = \lambda_{0x} + 2\lambda_0\lambda_1.$$

Substituting G_x into equations (3.13) and (3.12), we get

$$\lambda_{6y} = (\lambda_{0y}\lambda_6 + 2\lambda_{2x}\lambda_0^2)/\lambda_0, \tag{3.35}$$

$$\lambda_{6x} = (3\lambda_6(\lambda_6 - 2\lambda_0\lambda_1))/(2\lambda_0). \tag{3.36}$$

Case $\lambda_6 \neq 0$.

From equations (3.34), we find

$$G_y = G(-\lambda_{0y}\lambda_6 + \lambda_{6y}\lambda_0)/(\lambda_0\lambda_6).$$

Substituting G_y into equations (3.13) and (3.31), and comparing the mixed derivatives $(G_x)_y = (G_y)_x$, we obtain

$$\lambda_{4x} = (-24\lambda_{2x}^2\lambda_0^3 - 4\lambda_0\lambda_1\lambda_4\lambda_6 + \lambda_4\lambda_6^2)/(2\lambda_0\lambda_6). \tag{3.37}$$

Case $\lambda_6 = 0$.

In this case equation (3.34) is satisfied. We need to check the only condition $(G_{yy})_x = (G_x)_{yy}$, which is

$$\lambda_{4x} = -2\lambda_1\lambda_4, \tag{3.38}$$

Equation (3.35) becomes

$$\lambda_{2x} = 0. \tag{3.39}$$

Remark 3.3. If $\lambda_6 = 0$, equations (3.35) becomes condition (3.39).

All obtained results can be summarized to a theorem.

Theorem 3.1. *Sufficient conditions for equation (3.4) to be linearizable via a generalized Sundman transformation with $F_x = 0$ are as follows.*

(a) If $\lambda_3 \neq 0$, then the conditions are (3.21), (3.22), (3.23) and (3.26).

(b) If $\lambda_3 = 0$, $\lambda_6 \neq 0$, then the conditions are (3.32), (3.35), (3.36) and (3.37).

(c) If $\lambda_3 = 0$, $\lambda_6 = 0$, then the conditions are (3.32), (3.35), (3.36) and (3.38).

Remark 3.4. These conditions extend the criteria obtained in Duarte, Moreira and Santos (1994).

3.3 Examples

Here, we will give examples demonstrating the obtained results. The equations in the examples are not linearizable by point transformations and also do not satisfy the conditions of Duarte, Moreira and Santos (1994).

Example 3.1. Consider the nonlinear ordinary differential equation

$$y'' + (1/y)y'^2 + yy' + 1/2 = 0. \tag{3.40}$$

Since this equation does not satisfy the Lie criteria (Lie, 1883) for linearization it is not linearizable by point transformations. Equation (3.40) is of the form (3.4) with coefficients

$$\lambda_2 = 1/y\,,\ \lambda_1 = y\,,\ \lambda_0 = 1/2. \tag{3.41}$$

It is straightforward to check that the coefficients (3.41) obey the conditions (3.21), (3.22), (3.23) and (3.26). Thus, equation (3.40) is linearizable via generalized Sundman transformation.

For finding the functions F and G we have to solve equations (3.8), (3.20) and (3.25), which become

$$F_x = 0\,,\ F_{yy} = (2F_y)/y\,,\ G_x = 0\,,\ G_y = G/y.$$

We take the simplest solution, $F = y^3$ and $G = y$, which satisfies (3.8), (3.20) and (3.25). We obtain the transformation

$$u = y^3, \quad dt = ydx. \tag{3.42}$$

Equations (3.9), (3.10) and (3.15) give

$$\beta = 1\,,\ \gamma = -3/2\,,\ \alpha = 0.$$

Hence equation (3.40) is mapped by the transformation (3.42) into the linear equation

$$u'' + u' + 3/2 = 0. \tag{3.43}$$

The general solution of equation (3.43) is

$$u = c_1 + c_2 e^{-t} - 3t/2,$$

where c_1, c_2 are arbitrary constants. Applying the generalized Sundman transformation (3.42) to equation (3.40) we obtain that the general solution of equation (3.40) is

$$y(x) = (c_1 + c_2 e^{-\phi(x)} - 3\phi(x)/2)^{1/3},$$

where the function $t = \phi(x)$ is a solution of the equation

$$\frac{dt}{dx} = (c_1 + c_2 e^{-t} - 3t/2)^{1/3}.$$

For example, if $c_1 = c_2 = 0$, then we obtain the particular solution of equation (3.40)

$$y = (-x)^{1/2}.$$

Example 3.2. Consider the nonlinear ordinary differential equation

$$y'' + xy'^2 + yy' + 1/e^{2xy} = 0. \quad (3.44)$$

Equation (3.44) is of the form (3.4) with coefficients

$$\lambda_2 = x\,,\ \lambda_1 = y\,,\ \lambda_0 = 1/e^{2xy}. \quad (3.45)$$

One easily checks that the coefficients (3.45) do not satisfy the conditions of linearizability by point transformations, but they obey the conditions (3.27) and (3.28). Thus, equation (3.44) is linearizable via a generalized Sundman transformation.

For finding the functions F and G we have to solve equations (3.8), (3.20) and (3.30), which become

$$F_x = 0\,,\ F_{yy} = (G_yF_y + F_yGx)/G\,,$$

$$G_x = -yG\,,\ G_{yy} = (3G_y^2 + 4G_yGx + 2G^2x^2)/G.$$

We take the simplest solution, $F = y$ and $G = e^{-xy}$, which satisfies (3.8), (3.20) and (3.30). The linearizing generalized Sundman transformation is

$$u = y, \quad dt = e^{-xy}dx. \quad (3.46)$$

Equations (3.9), (3.10) and (3.15) give

$$\beta = 0\,,\ \gamma = -1\,,\ \alpha = 0.$$

Hence equation (3.44) is mapped by the transformation (3.46) into the linear equation

$$u'' + 1 = 0. \quad (3.47)$$

The general solution of equation (3.47) is

$$u = -t^2/2 + c_1 t + c_2,$$

where c_1, c_2 are arbitrary constants. Applying the generalized Sundman transformation (3.46) to equation (3.44) we obtain that the general solution of equation (3.44) is

$$y(x) = -\phi(x)^2/2 + c_1\phi(x) + c_2,$$

where the function $t = \phi(x)$ is a solution of the equation

$$\frac{dt}{dx} = e^{-x(-t^2/2 + c_1 t + c_2)}.$$

Example 3.3. Consider the nonlinear second-order ordinary differential equation

$$y'' + \mu_3 y^{k_3} y'^2 + \mu_2 y^{k_2} y' + \mu_1 y^{k_1} = 0, \tag{3.48}$$

where k_1, k_2, k_3, μ_1, μ_2 and $\mu_3 \neq 0$ are arbitrary constants. The Lie criteria (Lie, 1883) show that the nonlinear equation (3.48) is linearizable by a point transformation if and only if $\mu_1 = 0$ and $\mu_2 = 0$.

From equation (3.48), the coefficients are

$$\begin{gathered} \lambda_0 = \mu_1 y^{k_1},\ \lambda_1 = \mu_2 y^{k_2},\ \lambda_2 = \mu_3 y^{k_3},\ \lambda_3 = \mu_2 k_2 y^{k_2}/y, \\ \lambda_4 = 2\mu_1 y^{(k_1+k_3)+1}(k_1\mu_3 + k_3\mu_1) + 2\mu_1 y^{k_1}(k_1^2 - k_1) - k_2\mu_2^2 y^{2k_2+1}/y^2, \\ \lambda_5 = k_2\mu_2^2 y^{2k_2}/y. \end{gathered} \tag{3.49}$$

If $\mu_2 \neq 0$ and $\mu_1 = 0$, then $\lambda_3 \neq 0$ and $\lambda_5 \neq 0$. We can check that the coefficients obey the conditions (3.21), (3.22), (3.23) and (3.26). Thus, equation

$$y'' + \mu_3 y^{k_3} y'^2 + \mu_2 y^{k_2} y' = 0 \tag{3.50}$$

is linearizable by a generalized Sundman transformation.

For finding the functions F and G we have to solve equations (3.8), (3.20) and (3.25), which become

$$F_x = 0\,,\ F_{yy} = F_y(\mu_3 y^{k_3+1} + k_2)/y\,,\ G_x = 0\,,\ G_y = Gk_2/y.$$

For example, if $k_2 = k_3$, we take the simplest solution, $F = \frac{1}{\mu_3} e^{\frac{\mu_3 y^{k_2+1}}{k_2+1}}$ and $G = y^{k_2}$, and the generalized Sundman transformation becomes

$$u = \frac{1}{\mu_3} e^{\frac{\mu_3 y^{k_2+1}}{k_2+1}}, \quad dt = y^{k_2} dx. \tag{3.51}$$

Equations (3.9), (3.10) and (3.15) give

$$\beta = \mu_2, \quad \gamma = 0, \quad \alpha = 0.$$

Hence equation (3.50) is mapped by the transformation (3.51) into the linear equation

$$u'' + \mu_2 u' = 0. \tag{3.52}$$

If $\mu_3 = 0$, then equation (3.48) is

$$y'' + \mu_2 y^{k_2} y' + \mu_1 y^{k_1} = 0, \tag{3.53}$$

where $\mu_2 \neq 0$. The Lie criteria (Lie, 1883) show that the nonlinear equation (3.53) is linearizable by a point transformation if and only if $k_1 = 3$, $k_2 = 1$ and $\mu_1 = (\mu_2/3)^2$. In the particular case, $k_1 = 3$, $k_2 = 1$, $\mu_1 = 1$ and $\mu_2 = 3$, we have the equation

$$y'' + 3yy' + y^3 = 0. \tag{3.54}$$

Equation (3.54) arises in many areas. Some of these are the analysis of the fusion of pellets, the theory of univalent functions, the stability of gaseous spheres, operator Yang-Baxter equations, motion of a free particle in a space of constant curvature, the stationary reduction of the second member of the Burgers hierarchy (Karasu and Leach, 2009).

Remark 3.5. Equation (3.54) is linearizable by a point transformation and by a generalized Sundman transformation into the equations $u'' = 0$ and $u'' + 3u' + 2u = 0$, respectively.

Without loss of the generality*, we can assume that $\mu_2 = 1$. Hence, equation (3.53) becomes

$$y'' + y^{k_2}y' + \mu_1 y^{k_1} = 0. \tag{3.55}$$

For this equation the coefficients are

$$\lambda_0 = \mu_1 y^{k_1},\ \lambda_1 = y^{k_2},\ \lambda_2 = 0,\ \lambda_3 = k_2 y^{k_2-1},$$
$$\lambda_4 = \mu_1 k_1(k_1-1)y^{k_1-2} - k_2 y^{2k_2-1},\ \lambda_5 = k_2 y^{2k_2+1}.$$

If $k_2 = 0$, then $\lambda_5 = 0$ and equation (3.55) is linearizable by a generalized Sundman transformation.

If $k_2 \neq 0$, then $\lambda_5 \neq 0$ and conditions (3.21), (3.22), (3.23), (3.26) are reduced to

$$\mu_1(2k_2 + 1 - k_1)(k_2 - k_1) = 0. \tag{3.56}$$

If conditions (3.56) are satisfied, then equation (3.55) is linearizable by a generalized Sundman transformation. Notice that in the case $\mu_1(k_2 - k_1) = 0$, equation (3.55) is trivially integrated by using the substitution $y' = H(y)$. A nontrivial case is $k_1 = 2k_2 + 1$. In this case the functions F and G are solutions of the compatible overdetermined system of equations

$$F_x = 0,\ F_{yy} = k_2 F_y/y,\ G_x = 0,\ G_y = k_2 G/y. \tag{3.57}$$

The general solution of equations (3.57) depends on the value of the constant k_2. For example, if $k_2 \neq -1$, then a particular solution of system (3.57) is

$$F = y^{k_2+1}, \quad G = y^{k_2}.$$

*For example, scaling of the independent variable: $\overline{x} = \mu_2 x$.

Thus, the generalized Sundman transform reduces equation (3.55) into the linear equation

$$u'' + u' + (\mu_1(k_2 + 1))u = 0. \tag{3.58}$$

Remark 3.6. Since equations (3.40), (3.48) and (3.53) are autonomous, their order can be reduced by the substitution $y' = f(y)$. It is worth to note that for equations (3.48) and (3.53) the difficulties in using the generalized Sundman transformation are similar to solving the original equation by this reduction.

CHAPTER IV

LINEARIZATION OF THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY GENERALIZED SUNDMAN TRANSFORMATIONS

In this chapter we focus on the necessary and sufficient conditions which allow the most general third-order ordinary differential equation

$$y''' = f(x, y, y', y'') \tag{4.1}$$

to be mapped into the equation

$$u''' + \alpha u = 0, \tag{4.2}$$

where $\alpha \neq 0$ is constant. The linerization is considered with respect to the generalized Sundman transformation

$$\begin{aligned} u &= F(x, y), \\ dt &= G(x, y)dx \end{aligned} \tag{4.3}$$

where $F_y G \neq 0$.

Recall that the linearization problem via generalized Sundman transformations for third-order ordinary differential equations has been investigated by Euler, Wolf, Leach and Euler (2003). They found conditions for all equations (4.1) which are equivalent to a linear equation $u''' = 0$. This means that they only gave a particular criterion for applying the generalized Sundman transformation for the linearization problem. The main motivation of the study in this chapter is to extend the equivalence classes of linearizable third-order ordinary differential equations.

4.1 Necessary conditions for linearization

This section is devoted to finding a representation of a third-order ordinary differential equation (4.1) which can be obtained from a linear equation (4.2) by applying a generalized Sundman transformation.

The function u and its derivatives u' and u'' are defined by the first formula (4.3) and its derivatives with respect to x:

$$\begin{gathered} u'G = F_x + F_y y', \\ u''G^2 + u'(G_x + G_y y') = F_y y'' + 2F_{xy}y' + F_{yy}y'^2 + F_{xx}, \\ u'''G^3 + 3Gu''(G_x + G_y y') + u'(G_y y'' + 2G_{xy}y' + G_{yy}y'^2 + G_{xx}) \\ = F_y y''' + 3F_{yy}y'y'' + 3F_{xy}y'' + 3F_{xyy}y'^2 + 3F_{xxy}y' + F_{yyy}F_{xxx}y'^3. \end{gathered} \tag{4.4}$$

Finding the derivatives u', u'', u''' from (4.4) and substituting them into (4.2), we obtain the following equation

$$\begin{gathered} y''' + \lambda_5(x,y)y'' + \lambda_4(x,y)y'y'' + \lambda_3(x,y)y'^3 + \lambda_2(x,y)y'^2 \\ +\lambda_1(x,y)y' + \lambda_0(x,y) = 0, \end{gathered} \tag{4.5}$$

where the coefficients $\lambda_i(x,y),(i=0,1,...,5)$ are related to the functions F and G:

$$\lambda_5 = (3F_{xy}G - F_xG_y - 3F_yG_x)/(F_yG), \quad (4.6)$$

$$\lambda_4 = (3F_{yy}G - 4F_yG_y)/(F_yG), \quad (4.7)$$

$$\lambda_3 = (F_{yyy}G^2 - 3F_{yy}G_yG - F_yG_{yy}G + 3F_yG_y^2)/(F_yG^2), \quad (4.8)$$

$$\lambda_2 = (3F_{xyy}G^2 - 6F_{xy}G_yG - F_xG_{yy}G + 3F_xG_y^2 - 3F_{yy}G_xG - 2F_yG_{xy}G + 6F_yG_xG_y)/(F_yG^2), \quad (4.9)$$

$$\lambda_1 = (-6F_{xy}G_xG + 3F_{xxy}G^2 - 3F_{xx}G_yG - 2F_xG_{xy}G + 6F_xG_xG_y - F_yG_{xx}G + 3F_yG_x^2)/(F_yG^2), \quad (4.10)$$

$$\lambda_0 = (F_{xxx}G^2 - 3F_{xx}G_xG - F_xG_{xx}G + 3F_xG_x^2 + FG^5\alpha)/(F_yG^2). \quad (4.11)$$

Equation (4.5) presents the necessary form of a third-order ordinary differential equation which can be mapped to a linear equation (4.2) via a generalized Sundman transformation.

Notice that if $\alpha = 0$, then equations (4.6)-(4.11) coincide with the corresponding equations of Euler, Wolf, Leach and Euler (2003).

4.2 Sufficient conditions for linearization

For obtaining sufficient conditions we have to solve the compatibility problem by considering equations (4.6)-(4.11) as an overdetermined system of partial differential equations for the functions F and G with the given coefficients $\lambda_i(x,y),(i=0,1,...,5)$. Complete criteria for third-order ordinary differential equations to be linearizable to equation (4.2) via the generalized Sundman transformation are obtained here.

From equations (4.6)-(4.11), we can find the derivatives of F and G:

$$
\begin{aligned}
F_{xy} &= (F_xG_y + 3F_yG_x + F_yG\lambda_5)/(3G), && (4.12)\\
F_{yy} &= (F_y(4G_y + 4\lambda_4G))/(3G), && (4.13)\\
F_{xxx} &= (3F_{xx}G_xG + F_xG_{xx}G - 3F_xG_x^2 - FG^5\alpha + \lambda_0F_yG^2)/(G^2), && (4.14)\\
G_{yy} &= (F_{yyy}G^2 - \lambda_3F_yG^2 - 3F_{yy}G_yG + 3F_yG_y^2)/(F_yG), && (4.15)\\
G_{xy} &= (3F_{xyy}G^2 - \lambda_2F_yG^2 - 6F_{xy}G_yG - F_xG_{yy}G + 3F_xG_y^2 && (4.16)\\
&\quad -3F_{yy}G_xG + 6F_yG_xG_y)/(2F_yG), &&\\
G_{xx} &= (3F_{xxy}G^2 - \lambda_1F_yG^2 - 6F_xG_xG - 3F_{xx}G_yG && (4.17)\\
&\quad -2F_xG_{xy}G + 6F_xG_xG_y + 3F_yG_x^2)/(F_yG). &&
\end{aligned}
$$

The right hand sides of equations (4.12)-(4.17) can be written through the first-order derivatives of the functions F, G and the derivative F_{xx}. For example, after substituting F_{yyy}, found by differentiating (4.13) with respect to y, into (4.15) we find the expression of the derivative G_{yy} through first-order derivatives of the functions F and G. Later we refer to equations (4.12)-(4.17) as expressions of the derivatives presented in the left hand sides through the first-order derivatives of the functions F, G and the derivative F_{xx}.

Comparing the mixed derivatives

$$
\begin{aligned}
(F_{xy})_y = (F_{yy})_x, (F_{xxx})_y = (F_{xy})_{xx},\\
(G_{xy})_x = (G_{xx})_y, (G_{xy})_y = (G_{yy})_x,
\end{aligned}
\qquad (4.18)
$$

new equations for the functions, F and G, are obtained. One of these equations is

$$F_x\lambda_6 + F_y\lambda_7 = 0, \qquad (4.19)$$

where

$$\lambda_6 = -3\lambda_{4y} + 9\lambda_3 - \lambda_4^2,$$

$$\lambda_7 = -3\lambda_{4x} + 6\lambda_{5y} - 3\lambda_2 + \lambda_4\lambda_5.$$

Further analysis of the compatibility depends on the value of λ_6.

4.2.1 Case $\lambda_6 \neq 0$

Assuming that $\lambda_6 \neq 0$, equation (4.19) gives

$$F_x = -F_y\lambda_7/\lambda_6. \tag{4.20}$$

Substituting F_x into (4.12) and (4.14), we obtain the following equations

$$G_x = (-3G_y\lambda_6\lambda_7 + G(3\lambda_{6y}\lambda_7 - 3\lambda_{7y}\lambda_6 - \lambda_4\lambda_6\lambda_7 - \lambda_5\lambda_6^2))/(3\lambda_6^2), \tag{4.21}$$

$$\alpha = (F_y\lambda_8)/(9FG^3\lambda_6^4), \tag{4.22}$$

where

$$\begin{aligned}\lambda_8 = {} & 9\lambda_6^2(\lambda_{7xx}\lambda_6 - \lambda_{6xx}\lambda_7) + (3\lambda_{5x} + 2\lambda_5^2)\lambda_6^3\lambda_7 + 3\lambda_{5y}\lambda_6^2\lambda_7^2 \\ & +3\lambda_{6x}(6\lambda_{6x}\lambda_6\lambda_7 + 3\lambda_{6y}\lambda_7^2 - 6\lambda_{7x}\lambda_6^2 - 3\lambda_{7y}\lambda_6\lambda_7 - \lambda_4\lambda_6\lambda_7^2 - 3\lambda_5\lambda_6^2\lambda_7) \\ & -3\lambda_{6y}(3\lambda_{7x}\lambda_6\lambda_7 + \lambda_5\lambda_6\lambda_7^2) + 3\lambda_{7x}(\lambda_4\lambda_6^2\lambda_7 + 3\lambda_5\lambda_6^3) + \lambda_6^2(3\lambda_{7y}\lambda_5\lambda_7 \\ & +9\lambda_0\lambda_6^2 - 3\lambda_2\lambda_7^2 + 2\lambda_4\lambda_5\lambda_7^2).\end{aligned}$$

Since the case $\alpha = 0$ was studied in Euler, Wolf, Leach and Euler (2003), further study is considered for $\alpha \neq 0$. From equation (4.22) we get that $\lambda_8 \neq 0$.

Substituting G_x into (4.16) and (4.17), we obtain the conditions

$$\begin{gathered}9\lambda_{2y}\lambda_6 - 27\lambda_{3y}\lambda_6 + 15\lambda_{5y}\lambda_4\lambda_6 + 3\lambda_{6y}\lambda_6 - 6\lambda_{6y}\lambda_7 + 9\lambda_{7y}\lambda_6 - 12\lambda_2\lambda_4\lambda_6 \\ -9\lambda_3\lambda_5\lambda_6 + 5\lambda_4^2\lambda_5\lambda_6 + 4\lambda_4\lambda_6\lambda_7 - \lambda_5\lambda_6^2 = 0,\end{gathered} \tag{4.23}$$

$$\begin{gathered}-18\lambda_{5xy}\lambda_6^2 - \lambda_6^2(18\lambda_{1y} - 36\lambda_{2y} + 12\lambda_{5x}\lambda_4 + 24\lambda_{5y}\lambda_5 + 12\lambda_{7x} - 24\lambda_2\lambda_5 \\ +8\lambda_4\lambda_5^2 + 9\lambda_5\lambda_7) + 9\lambda_{5y}(2\lambda_{6y}\lambda_7 - 2\lambda_{7y}\lambda_6 + \lambda_4\lambda_6\lambda_7) - 6\lambda_{6y}(3\lambda_2\lambda_7 \\ -\lambda_4\lambda_5\lambda_7) + 3\lambda_6\lambda_{7y}(6\lambda_2 - 2\lambda_4\lambda_5\lambda_7) + \lambda_6\lambda_7(-6\lambda_2\lambda_4\lambda_6\lambda_7 \\ -9\lambda_3\lambda_5 + 3\lambda_4^2\lambda_5 + 2\lambda_4 + 9\lambda_{2y} - 27\lambda_{3x} + 9\lambda_{6x}) = 0.\end{gathered} \tag{4.24}$$

Differentiating α with respect to x and y, we obtain

$$\begin{gathered}-4\lambda_{6x}\lambda_6\lambda_8 - 6\lambda_{6y}\lambda_7\lambda_8 + 2\lambda_{7y}\lambda_6\lambda_8 + \lambda_{8x}\lambda_6^2 + \lambda_{8y}\lambda_6\lambda_7 \\ +\lambda_8\lambda_6(\lambda_4\lambda_7 + \lambda_5\lambda_6) = 0,\end{gathered} \tag{4.25}$$

$$F_y = -5G_yF\lambda_6\lambda_8 + FG(-12\lambda_{6y}\lambda_8 + 3\lambda_{8y}\lambda_6 + \lambda_4\lambda_6\lambda_8)/(3G\lambda_6\lambda_8). \quad (4.26)$$

Substituting F_y into (4.20), we get

$$F_x = 5G_yF\lambda_6\lambda_7\lambda_8 + FG\lambda_7(12\lambda_{6y}\lambda_8 - 3\lambda_{8y}\lambda_6 - \lambda_4\lambda_6\lambda_8)/(3G\lambda_6^2\lambda_8). \quad (4.27)$$

Because the derivatives, F_{xy} and F_{yy}, have been found from equations (4.12) and (4.13), we need to consider the equations

$$(F_x)_y - F_{xy} = 0,$$
$$(F_y)_y - F_{yy} = 0.$$

These equations become

$$35\lambda_6^2\lambda_8(\frac{G_y}{G})^2 - 14\lambda_{10}\lambda_6(\frac{G_y}{G}) - \lambda_9 = 0, \quad (4.28)$$

$$\begin{aligned} &9\lambda_{7yy}\lambda_6^2 - 9\lambda_{6yy}\lambda_6\lambda_7 + 18\lambda_{6y}^2\lambda_7 + \lambda_6^3(-6\lambda_{5y} + 9\lambda_2 - 3\lambda_4\lambda_5\lambda_6^3 - 2\lambda_7) \\ &+\lambda_6^2(3\lambda_{7y}\lambda_4 + 9\lambda_3\lambda_7 - \lambda_4^2\lambda_7) - 3\lambda_{6y}\lambda_6(6\lambda_{7y} + \lambda_4\lambda_7) = 0, \end{aligned} \quad (4.29)$$

where

$$\lambda_9 = 36\lambda_{6yy}\lambda_6\lambda_8 - 9\lambda_{8yy}\lambda_6^2 - 180\lambda_{6y}^2\lambda_8$$
$$+12\lambda_{6y}(6\lambda_{8y}\lambda_6 + \lambda_4\lambda_6\lambda_8) + \lambda_6^2(-3\lambda_{8y}\lambda_4 - 9\lambda_3\lambda_8 + \lambda_4^2\lambda_8) - 6\lambda_6^3\lambda_8,$$
$$\lambda_{10} = (3\lambda_{8y} + \lambda_4\lambda_8)\lambda_6 - 12\lambda_{6y}\lambda_8.$$

Equation (4.28) leads to the condition that $7\lambda_{10}^2 + 5\lambda_8\lambda_9 \geq 0$.

Differentiating (4.28) with respect to x and y, we obtain

$$\begin{aligned} &70\lambda_6^3\lambda_8(3\lambda_{10}\lambda_2 - \lambda_{10}\lambda_4\lambda_5 - 3\lambda_{5y}\lambda_{10}) + 15\lambda_6^2\lambda_8(\lambda_{9x} + \lambda_5\lambda_9) \\ &-10\lambda_8\lambda_9(9\lambda_{6x}\lambda_6 + 6\lambda_{6y}\lambda_7 - 6\lambda_{7y}\lambda_6 - 2\lambda_4\lambda_6\lambda_7) + (9\lambda_{10}\lambda_9 + \lambda_{11})\lambda_7 = 0, \end{aligned} \quad (4.30)$$

$$18G_y\lambda_6\lambda_{12} - G\lambda_{11} = 0, \quad (4.31)$$

where

$$\lambda_{11} = -90\lambda_{6y}\lambda_8\lambda_9 + 15\lambda_{9y}\lambda_6\lambda_8 - 70\lambda_{10}\lambda_6^3\lambda_8 - 9\lambda_{10}\lambda_9 - 5\lambda_4\lambda_6\lambda_8\lambda_9,$$

$$\lambda_{12} = 7\lambda_{10}^2 + 5\lambda_8\lambda_9.$$

Further study depends on λ_{12}.

Case $\lambda_{12} = 0$

From equation (4.31) we have the condition

$$\lambda_{11} = 0. \tag{4.32}$$

Thus we have shown that, if $\lambda_6 \neq 0$ and $\lambda_{12} = 0$, then conditions (4.23), (4.24), (4.25) and (4.32) are sufficient for equation (4.5) to be linearizable by a generalized Sundman transformation. These conditions guarantee that the overdetermined system of equations (4.21), (4.26), (4.27) and (4.28) for the functions F and G is compatible.

Case $\lambda_{12} \neq 0$

From equation (4.31) we find

$$G_y = G\lambda_{11}/(18\lambda_6\lambda_{12}). \tag{4.33}$$

Substituting G_y into (4.15), (4.16), (4.28), (4.21), (4.26) and (4.27), we get

$$\begin{gathered} 27\lambda_{11y}\lambda_{12}\lambda_6 - 27\lambda_{12y}\lambda_{11}\lambda_6 - 27\lambda_{6y}\lambda_{11}\lambda_{12} - \lambda_{11}^2 \\ -9\lambda_{11}\lambda_{12}\lambda_4\lambda_6 + 162\lambda_{12}^2\lambda_6^3 = 0, \end{gathered} \tag{4.34}$$

$$\begin{gathered} \lambda_{11x}\lambda_{12}\lambda_6^2 + \lambda_{11y}\lambda_{12}\lambda_6\lambda_7 - \lambda_{12x}\lambda_{11}\lambda_6^2 - \lambda_{12y}\lambda_{11}\lambda_6\lambda_7 \\ +18\lambda_{5y}\lambda_{12}^2\lambda_6^3 - \lambda_{6x}\lambda_{11}\lambda_{12}\lambda_6 - 2\lambda_{6y}\lambda_{11}\lambda_{12}\lambda_7 + \lambda_{7y}\lambda_{11}\lambda_{12}\lambda_6 \\ -18\lambda_{12}^2\lambda_2\lambda_6^3 + 6\lambda_{12}^2\lambda_4\lambda_5\lambda_6^3 - 6\lambda_{12}^2\lambda_6^3\lambda_7 = 0, \end{gathered} \tag{4.35}$$

$$\begin{gathered} 3528\lambda_{11}\lambda_{12}\lambda_6^3(\lambda_{10}\lambda_{12} - 7\lambda_{10}^3) + 756\lambda_{11}\lambda_{12}\lambda_6(14\lambda_{10y}\lambda_{10}\lambda_9 - \lambda_{12y}\lambda_9) \\ +7\lambda_{11}\lambda_{12}\lambda_9(1080\lambda_{6y}\lambda_{12} - 7560\lambda_{6y}\lambda_{10}^2 + 324\lambda_{10}\lambda_9 + 41\lambda_{11}) \\ -245\lambda_{10}^2\lambda_{11}^2\lambda_9 - 1620\lambda_{12}^2\lambda_9^3 = 0, \end{gathered} \tag{4.36}$$

$$\begin{gathered} G_x = G(18\lambda_{6y}\lambda_{12}\lambda_7 - 18\lambda_{7y}\lambda_{12}\lambda_6 - \lambda_{11}\lambda_7 - 6\lambda_{12}\lambda_4\lambda_6\lambda_7 \\ -6\lambda_{12}\lambda_5\lambda_6^2)/(18\lambda_{12}\lambda_6^2), \end{gathered} \tag{4.37}$$

$$F_x = F\lambda_7(-126\lambda_{10}^3 - 90\lambda_{10}\lambda_8\lambda_9 + 5\lambda_{11}\lambda_8)/(54\lambda_6^2\lambda_8\lambda_{12}), \tag{4.38}$$

$$F_y = F(126\lambda_{10}^3 + 90\lambda_{10}\lambda_8\lambda_9 - 5\lambda_{11}\lambda_8)/(54\lambda_6\lambda_8\lambda_{12}). \tag{4.39}$$

Thus we have shown that, if $\lambda_6 \neq 0$ and $\lambda_{12} \neq 0$, then conditions (4.23), (4.24), (4.25), (4.34), (4.35) and (4.36) are sufficient for equation (4.5) to be linearizable by a generalized Sundman transformation. These conditions guarantee that the overdetermined system of equations (4.33), (4.37), (4.38) and (4.39) for the functions F and G is compatible.

4.2.2 Case $\lambda_6 = 0$

Since $F_y \neq 0$, from (4.19), we get

$$\lambda_7 = 0. \tag{4.40}$$

From the last two equations of (4.18), we obtain

$$\begin{gathered} -3\lambda_{13}(F_xG_y - F_yG_x) + F_yG(-9\lambda_{1y} - 3\lambda_{13x} + 9\lambda_{2x} - 3\lambda_{5x}\lambda_4 - \lambda_{13}\lambda_5 \\ +3\lambda_2\lambda_5 - \lambda_4\lambda_5^2) = 0, \end{gathered} \tag{4.41}$$

$$9\lambda_{2y} - 27\lambda_{3x} + 15\lambda_{5y}\lambda_4 - 12\lambda_2\lambda_4 - 9\lambda_3\lambda_5 + 5\lambda_4^2\lambda_5 = 0, \tag{4.42}$$

where

$$\lambda_{13} = 3\lambda_{5y} - 3\lambda_2 + \lambda_4\lambda_5.$$

Case $\lambda_{13} = 0$

Equation (4.41) gives the condition

$$-9\lambda_{1y} + 9\lambda_{2x} - 3\lambda_{5x}\lambda_4 + 3\lambda_2\lambda_5 - \lambda_4\lambda_5^2 = 0. \tag{4.43}$$

Thus we have shown that, if $\lambda_6 = 0$ and $\lambda_{13} = 0$, then conditions (4.40), (4.42) and (4.43) are sufficient for equation (4.5) to be linearizable by a generalized Sundman transformation. These conditions guarantee that the overdetermined system of equations (4.12)-(4.17) for the functions F and G is compatible.

Case $\lambda_{13} \neq 0$

From equation (4.41) we find

$$G_x = (3F_xG_y\lambda_{13} + F_yG(9\lambda_{1y} + 3\lambda_{13x} - 9\lambda_{2x} + 3\lambda_{5x}\lambda_4 + \lambda_{13}\lambda_5 - 3\lambda_2\lambda_5 + \lambda_4\lambda_5^2))/(3F_y\lambda_{13}). \tag{4.44}$$

Substituting G_x into (4.16) and (4.17), we obtain the following equations

$$-27\lambda_{1yy} + 9\lambda_{1y}\lambda_4 - 18\lambda_{13x}\lambda_4 - 27\lambda_{2x}\lambda_4 + 81\lambda_{3xx} + 54\lambda_{3x}\lambda_5 + 9\lambda_{5x}\lambda_4^2 - 16\lambda_{13}^2 - 24\lambda_{13}\lambda_2 + 2\lambda_{13}\lambda_4\lambda_5 - 9\lambda_2\lambda_4\lambda_5 + 3\lambda_4^2\lambda_5^2 = 0, \tag{4.45}$$

$$F_x = F_y\lambda_{14}/3\lambda_{13}^3, \tag{4.46}$$

where

$$\begin{aligned}\lambda_{14} = {} & 18\lambda_{13}(3\lambda_{1xy} + \lambda_{13xx} - 3\lambda_{2xx} + \lambda_{5xx}\lambda_4) \\ & +18\lambda_{1y}(9\lambda_{2x} - 6\lambda_{13x} - 3\lambda_{5y}\lambda_4 - \lambda_{13}\lambda_5 + 3\lambda_2\lambda_5 - \lambda_4\lambda_5^2) \\ & +12\lambda_{13x}(9\lambda_{2x} - 3\lambda_{5y}\lambda_4 + 3\lambda_2\lambda_5 - \lambda_4\lambda_5^2) + 18\lambda_{2x}(3\lambda_{5x}\lambda_4 - 3\lambda_2\lambda_5 + \lambda_4\lambda_5^2) \\ & -3\lambda_{5x}(3\lambda_{5x}\lambda_4^2 - 9\lambda_{13}^2 - 6\lambda_2\lambda_4\lambda_5 + 2\lambda_4^2\lambda_5^2) - 9\lambda_1\lambda_{13}^2 + 6\lambda_{13}^2\lambda_5^2 + 12\lambda_{13}\lambda_2\lambda_5^2 \\ & -4\lambda_{13}\lambda_4\lambda_5^3 - 9\lambda_2^2\lambda_5^2 + 6\lambda_2\lambda_4\lambda_5^3 - 6\lambda_{1x}\lambda_{13}\lambda_5 - 81\lambda_{1y}^2 - 27\lambda_{13x}^2 - \lambda_4^2\lambda_5^4.\end{aligned}$$

Substituting F_x into (4.12) and (4.14) we find that

$$\alpha = (F_y\lambda_{15})/(81FG^3\lambda_{13}^5),$$

where

$$\begin{aligned}\lambda_{15} = {} & 9\lambda_{13}(9\lambda_{13xx}\lambda_{14} - 3\lambda_{14xx}\lambda_{13}) - 27\lambda_{1y}(9\lambda_{13x}\lambda_{14} - 3\lambda_{14x}\lambda_{13} - \lambda_{13}\lambda_{14}\lambda_5) \\ & -9\lambda_{13x}(45\lambda_{14} - 21\lambda_{14x}\lambda_{13} - 27\lambda_{2x}\lambda_{14} + 9\lambda_{5x}\lambda_{14}\lambda_4 - 4\lambda_{13}\lambda_{14}\lambda_5 - 9\lambda_{14}\lambda_2\lambda_5 \\ & +3\lambda_{14}\lambda_4\lambda_5^2) - 9\lambda_{13}\lambda_{14x}(9\lambda_{2x} - 3\lambda_{5x}\lambda_4 + \lambda_{13}\lambda_5 + 3\lambda_2\lambda_5 - \lambda_4\lambda_5^2) \\ & -3\lambda_{13}\lambda_{14}(9\lambda_{2x}\lambda_5 + 3\lambda_{5x}\lambda_{13} - 3\lambda_{5x}\lambda_4\lambda_5 + 3\lambda_2\lambda_5^2 - \lambda_4\lambda_5^3) + 81\lambda_0\lambda_{13}^5 + \lambda_{14}^2.\end{aligned}$$

By virtue of the condition $\alpha \neq 0$, we have that $\lambda_{15} \neq 0$.

Differentiating α with respect to x and y, we obtain the equations

$$\begin{gathered}-3\lambda_{13}^2\lambda_{15}(18\lambda_{1y} + 21\lambda_{13x} - 18\lambda_{2x}\lambda_{15} + 6\lambda_{5x}\lambda_4 - 6\lambda_2\lambda_5 + 2\lambda_4\lambda_5^2) \\ +3\lambda_{13}^3(3\lambda_{15x} - \lambda_{15}\lambda_5) - \lambda_{14}(3\lambda_{15y} - 4\lambda_{15}\lambda_4) = 0,\end{gathered} \tag{4.47}$$

$$G_y = -3F_yG\lambda_{15} + FG(3\lambda_{15y} - 4\lambda_{15}\lambda_4)/5F\lambda_{15}. \tag{4.48}$$

Substituting G_y into (4.15) and (4.16), we get

$$F_y = \beta F\lambda_{16}/(3\lambda_{15}), \tag{4.49}$$

where $\beta^2 = 1$ and

$$\lambda_{16}^2 = (-45\lambda_{15yy}\lambda_{15} + 63\lambda_{15y}^2 - 33\lambda_{15y}\lambda_{15}\lambda_4 + 180\lambda_{15}^2\lambda_3 - 8\lambda_{15}^2\lambda_4^2)/7.$$

Substituting F_y into (4.12) and (4.13), we find the conditions

$$\begin{gathered}-5\lambda_{13}^2\lambda_{15}\lambda_{16}(9\lambda_{1y} + 8\lambda_{13y} - 9\lambda_{2x} + 3\lambda_{5x}\lambda_4 + \lambda_{13}\lambda_5 - 3\lambda_2\lambda_5 + \lambda_4\lambda_5^2) \\ -3\lambda_{15y}\lambda_{14}\lambda_{16} + 5\lambda_{16x}\lambda_{13}^3\lambda_{15} + \lambda_{14}\lambda_{16}(4\lambda_{15}\lambda_4 + \lambda_{16}) = 0,\end{gathered} \tag{4.50}$$

$$-27\lambda_{15y}\lambda_{16} + 15\lambda_{16y}\lambda_{15} + 11\lambda_{15}\lambda_{16}\lambda_4 + 9\lambda_{16}^2 = 0. \tag{4.51}$$

Thus we have shown that, if $\lambda_6 = 0$ and $\lambda_{13} \neq 0$, conditions (4.40), (4.42), (4.45), (4.47), (4.50) and (4.51) are sufficient for equation (4.5) to be linearizable by a generalized Sundman transformation. These conditions guarantee that the overdetermined system of equations (4.44), (4.46), (4.48) and (4.49) for the functions F and G is compatible.

Combining all results obtained, the following theorem is proven.

Theorem 4.1. *Sufficient conditions for equation (4.5) to be linearizable to equation (4.2) by a generalized Sundman transformation are following.*

(a) If $\lambda_6 \neq 0$ *and* $\lambda_{12} = 0$*, then these conditions are (4.23), (4.24), (4.25) and (4.32).*

(b) If $\lambda_6 \neq 0$ *and* $\lambda_{12} \neq 0$*, then these conditions are (4.23), (4.24), (4.25), (4.34), (4.35) and (4.36).*

(c) If $\lambda_6 = 0$*,* $\lambda_{13} = 0$*, then these conditions are (4.40), (4.42) and (4.43).*

(d) If $\lambda_6 = 0$*,* $\lambda_{13} \neq 0$*, then these conditions are (4.40), (4.42), (4.45), (4.47), (4.50) and (4.51).*

4.3 Example

Here, we give examples demonstrating the obtained results.

Example 4.1.

Consider the nonlinear third-order ordinary differential equation

$$y''' - \frac{4}{y}y'y'' + \frac{3}{y^2}y'^3 + y^4 = 0. \tag{4.52}$$

It is an equation of the form (4.5) with the coefficients

$$\lambda_0 = y^4\,,\ \lambda_1 = 0\,,\ \lambda_2 = 0\,,\ \lambda_3 = \tfrac{3}{y^2}\,,\ \lambda_4 = \tfrac{-4}{y}\,,\ \lambda_5 = 0.$$

Notice that

$$\lambda_6 = -y^{-2} \neq 0,$$

$$\lambda_{12} = (81(-1440y^{10} + 827y^8 - 48384y^4 + 145152))/y^{22} \neq 0.$$

One easily checks that these coefficients obey conditions (b) of the theorem. Thus equation (4.52) is linearizable via a generalized Sundman transformation. For the functions F and G we have to solve equations (4.33), (4.37), (4.38) and (4.39), which become

$$G_y = G/y, \quad G_x = 0,$$

$$F_x = 0, \quad F_y = F/y.$$

We take its simplest solution, $F = y$ and $G = y$, which satisfies (4.33), (4.37), (4.38) and (4.39). We obtain the transformation

$$u = y, \quad dt = ydx. \tag{4.53}$$

Since $\lambda_8 = 9/y^4$, equation (4.22) gives

$$\alpha = 1.$$

Hence equation (4.52) is mapped by the transformation (4.53) into the linear equation

$$u''' + u = 0. \tag{4.54}$$

The general solution of equation (4.54) has the form

$$u = c_1 e^{-t} + e^{\frac{t}{2}}(c_2 \cos(\frac{\sqrt{3}}{2}t) + c_3 \sin(\frac{\sqrt{3}}{2}t)),$$

where c_1, c_2 and c_3 are arbitrary constants. Applying the generalized Sundman transformation (4.53) to equation (4.52), we obtain that the general solution of equation (4.52) is

$$y(x) = c_1 e^{-\phi(x)} + e^{\frac{\phi(x)}{2}}(c_2 \cos(\frac{\sqrt{3}}{2}\phi(x)) + c_3 \sin(\frac{\sqrt{3}}{2}\phi(x))),$$

where the function $t = \phi(x)$ is a solution of the equation

$$\frac{dt}{dx} = c_1 e^{-t} + e^{\frac{t}{2}}(c_2 \cos(\frac{\sqrt{3}}{2}t) + c_3 \sin(\frac{\sqrt{3}}{2}t)).$$

For example, if $c_2 = c_3 = 0$ and $c_1 \neq 0$, then we obtain the particular solutions of equation (4.52):

$$y = \frac{1}{x + c_0},$$

where c_0 is constant.

CHAPTER V

CONCLUSIONS

This thesis is devoted to the study of the linearization problem of second-order and third-order ordinary differential equations via the generalized Sundman transformation.

5.1 Problems

It is known that all second-order ordinary differential equations can be mapped to another by means of contact transformations. Comparing with the set of contact transformations, the set of generalized Sundman transformation is weaker: not every second-order ordinary differential equation can be transformed to the linear equation. Hence, it is interesting to study how generalized Sundman transformations can be applied to the linearization problem of second-order as well as higher-order ordinary differential equations.

Since the composition of a point transformation and a generalized Sundman transformation is not necessarily a generalized Sundman transformation, then the Laguerre form does not define the class of all linearizable equations by the generalized Sundman transformation. The first problem studied in the thesis was to demonstrate that the equation

$$u'' + \beta u' + \alpha u = \gamma, \tag{5.1}$$

should be used as the canonical linear equation for the linearization problem via generalized Sundman transformations instead of the Laguerre form usually used.

Here α, β and γ are constants.

The second problem in the thesis deals with the application of the generalized Sundman transformations to third-order ordinary differential equations. We investigated the necessary and sufficient conditions for a third-order ordinary differential equation to be linearizable by a generalized Sundman transformation into the more general linear equation

$$u''' + \alpha u = 0, \tag{5.2}$$

where $\alpha \neq 0$ is constant.

5.2 Results

The results obtained in the thesis are separated into two parts.

In the first part, the application of the generalized Sundman transformation for the linearization problem of second-order ordinary differential equations was analyzed. The general form of second-order ordinary differential equations that are linearizable via generalized Sundman transformations to linear equations is (3.4). Theorem 3.1 provides sufficient conditions for linearization. In particular, our examples, which consist of equations are not linearizable by point transformations, show that for a linearization problem via the generalized Sundman transformation one needs to use the general form of a linear second-order ordinary differential equation instead of the Laguerre form. The results obtained in this part warn that a researcher has to be careful when using the well-known method for the linearization problem.

The second part deals with the linearization of third-order ordinary differential equations by the generalized Sundman transformation. The general form of third-order ordinary differential equations that are linearizable to a linear equation via generalized Sundman transformations is (4.5). Conditions which guarantee that equations (4.5) can be linearized by a generalized Sundman transformation are provided by Theorem 4.1.

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APPENDICES

APPENDIX A

REMARK TO POINT AND CONTACT TRANSFORMATIONS

Let us explain how to transform the derivatives in point and contact transformations.

Point transformations

Let $y(x)$ be a given function. First of all, we have to change $y(x)$ to $u(t)$ by using point transformations

$$t = \varphi(x, y), \quad u = \psi(x, y). \tag{A.1}$$

The transformed function $u(t)$ is found from equation

$$t = \varphi\left(x, y\left(x\right)\right).$$

Using Inverse Function Theorem, we find $x = \alpha\left(t\right)$. Thus, we obtain

$$u\left(t\right) = \psi\left(\alpha\left(t\right), y\left(\alpha\left(t\right)\right)\right).$$

The first-order derivative is transformed by the formula

$$u'\left(t\right) = \frac{du}{dt} = \frac{\partial\psi}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} = \left(\psi_x + y'\psi_y\right)\frac{d\alpha}{dt}. \tag{A.2}$$

Since $t = \varphi\left(\alpha\left(t\right), y\left(\alpha\left(t\right)\right)\right)$ then

$$\begin{aligned} \frac{dt}{dt} &= \frac{\partial\varphi}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} \\ 1 &= \left(\varphi_x + y'\varphi_y\right)\frac{d\alpha}{dt} \\ \frac{d\alpha}{dt} &= \frac{1}{\left(\varphi_x + y'\varphi_y\right)}. \end{aligned} \tag{A.3}$$

Substituting equation (A.3) into equation (A.2), we get

$$u'(t) = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} = \frac{D_x\psi}{D_x\varphi} = \psi_1\left(x, y(x), y'(x)\right).$$

Notice that $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots$ is the total derivative with respect to x.

Next, we find the transformation of the second-order derivative. Consider

$$\begin{aligned} u''(t) &= \frac{d^2u}{dt^2} \\ &= \frac{\partial\psi_1}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\psi_1}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} + \frac{\partial\psi_1}{\partial y'}\frac{dy'}{dx}\frac{d\alpha}{dt} \\ &= \left(\psi_{1x} + y'\psi_{1y} + y''\psi_{1y'}\right)\frac{d\alpha}{dt} \\ &= \frac{\psi_{1x} + y'\psi_{1y} + y''\psi_{1y'}}{\varphi_x + y'\varphi_y} \\ &= \frac{D_x\psi_1}{D_x\varphi} \\ &= \psi_2\left(x, y(x), y'(x), y''(x)\right). \end{aligned}$$

In general, we can write

$$u^{(k+1)}(t) = \frac{d^{k+1}u}{dt^{k+1}} = \frac{D_x\psi_k}{D_x\varphi} = \psi_{k+1}(x, y, y', y'', y''', ..., y^{(k+1)}), \quad (k = 0, 1, 2, ...).$$

Notice that $\psi_0 = \psi$.

Contact transformations

Recall that the contact transformations

$$t = \varphi(x, y, y'), \quad u = \psi(x, y, y'), \quad s = g(x, y, y')$$

satisfy the condition $s = u' = \frac{du}{dt}$.

Let $y(x)$ be a given function. The transformed function $u(t)$ is found from the equations

$$t = \varphi\left(x, y(x), y'(x)\right),$$

$$u = \psi\left(x, y(x), y'(x)\right).$$

By virtue of the Inverse Function Theorem, the first equation gives $x = \tau(t)$, and then

$$u(t) = \psi(\tau(t), y(\tau(t)), y'(\tau(t))).$$

The first-order derivative is transformed by the formula

$$\begin{aligned} u'(t) &= \frac{du}{dt} \\ &= \frac{\partial\psi}{\partial x}\frac{d\tau}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}\frac{d\tau}{dt} + \frac{\partial\psi}{\partial y'}\frac{dy'}{dx}\frac{d\tau}{dt} \\ &= (\psi_x + y'\psi_y + y''\psi_{y'})\frac{d\tau}{dt}. \end{aligned} \tag{A.4}$$

Since $t = \varphi(\tau(t), y(\tau(t)), y'(\tau(t)))$ then

$$\begin{aligned} \frac{dt}{dt} &= \frac{\partial\varphi}{\partial x}\frac{d\tau}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx}\frac{d\tau}{dt} + \frac{\partial\varphi}{\partial y'}\frac{dy'}{dx}\frac{d\tau}{dt} \\ 1 &= (\varphi_x + y'\varphi_y + y''\varphi_{y'})\frac{d\tau}{dt} \\ \frac{d\tau}{dt} &= \frac{1}{(\varphi_x + y'\varphi_y + y''\varphi_{y'})}. \end{aligned} \tag{A.5}$$

Substituting equation (A.5) into equation (A.4), we obtain

$$u'(t) = \frac{\psi_x + y'\psi_y + y''\psi_{y'}}{\varphi_x + y'\varphi_y + y''\varphi_{y'}} = \frac{D_x\psi}{D_x\varphi}(\tau(t), y(\tau(t)), y'(\tau(t)), y''(\tau(t))).$$

The contact condition requires

$$g(x, y, y') = \frac{D_x\psi}{D_x\varphi}(x, y, y', y''). \tag{A.6}$$

Thus, the second-order derivative is transformed by the formula

$$\begin{aligned} u''(t) &= \frac{d^2u}{dt^2} \\ &= \frac{\partial g}{\partial x}\frac{d\tau}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dx}\frac{d\tau}{dt} + \frac{\partial g}{\partial y'}\frac{dy'}{dx}\frac{d\tau}{dt} \\ &= (g_x + y'g_y + y''g_{y'})\frac{d\tau}{dt} \\ &= \frac{g_x + y'g_y + y''g_{y'}}{\varphi_x + y'\varphi_y + y''\varphi_{y'}} \\ &= \frac{D_xg}{D_x\varphi} \\ &= g_1(x, y, y', y''). \end{aligned}$$

In general, we can write

$$u^{(k+1)}(t) = \frac{d^{(k+1)}u}{dt^{(k+1)}} = \frac{D_x g_{(k-1)}}{D_x \varphi} = g_k(x, y, y', y'', y''', ..., y^{(k+1)}), \quad (k = 1, 2, ...).$$

Notice that $g_0 = g$.

APPENDIX B

THE LIE LINEARIZATION TEST

Since the method used in the thesis is similar to the Lie method, let us describe Lie's method in detail.

Lie found necessary and sufficient conditions for a second-order ordinary differential equation

$$y'' = f(x, y, y')$$

to be linearizable by a change of the independent and dependent variables

$$t = \varphi(x, y), \qquad u = \psi(x, y)$$

into the simplest linear form of a second-order ordinary differential equation

$$u'' = 0.$$

One starts with obtaining necessary conditions for the linearization problem. One begins with the general form of a second-order ordinary differential equation

$$y'' = F\left(x, y, y'\right).$$

The derivatives are changed as follows

$$\begin{aligned} \tfrac{du}{dt} &= \psi_1 = \tfrac{D_x\psi}{D_x\varphi} = \tfrac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}, \\ \tfrac{d^2u}{dt^2} &= \psi_2 = \tfrac{D_x\psi_1}{D_x\varphi} = \tfrac{\psi_{1x} + y'\psi_{1y} + y''\psi_{1y'}}{\varphi_x + y'\varphi_y} \\ &= \tfrac{1}{(\varphi_x + y'\varphi_y)^3}[y''(\varphi_x\psi_y - \varphi_y\psi_x) + y'^3(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y) \\ &\quad + y'^2(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)) \\ &\quad + y'(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x)) + \varphi_x\psi_{xx} - \varphi_{xx}\psi_x], \end{aligned} \tag{B.1}$$

where

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + \cdots$$

is the total derivative with respect to x. Finding the derivatives u' and u'' from (B.1), and substituting them into $u'' = 0$, one obtains the following equation

$$y'' + a\,(x,y)\,y'^3 + b\,(x,y)\,y'^2 + c\,(x,y)\,y' + d\,(x,y) = 0, \tag{B.2}$$

where

$$a = \Delta^{-1}\left(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y\right), \tag{B.3}$$

$$b = \Delta^{-1}\left(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\left(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y\right)\right), \tag{B.4}$$

$$c = \Delta^{-1}\left(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2\left(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x\right)\right), \tag{B.5}$$

$$d = \Delta^{-1}\left(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x\right), \tag{B.6}$$

and $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ is the Jacobian of the change of variables.

Equation (B.2) presents the necessary form of a second-order ordinary differential equation which can be mapped to a linear equation $u'' = 0$ via a point transformation.

For obtaining sufficient conditions, one has to solve the compatibility problem, considering equations (B.3)-(B.6) as an overdetermined system of partial differential equations for the functions φ and ψ with given coefficients $a(x,y)$, $b(x,y)$, $c(x,y)$ and $d(x,y)$. Let us analyze the compatibility of this system.

Case $\varphi_y \neq 0$

From equations (B.3)-(B.6), one obtains the derivatives

$$\psi_{yy} = (\varphi_{yy}\psi_y - a\Delta)/\varphi_y,$$

$$\psi_{xy} = (2\varphi_{xy}\varphi_y\psi_y - a\varphi_x\Delta - \varphi_{yy}\Delta + b\varphi_y\Delta)/(2\varphi_y^2),$$

$$\psi_{xx} = (2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2(\psi_y d - \psi_x c))/\varphi_y^2,$$

$$\varphi_{xx} = (2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^3 a - \varphi_x^2\varphi_{yy} + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d)/\varphi_y^2.$$

The first three equations define all second-order derivatives of the function ψ. Comparing the mixed derivatives

$$(\psi_{xy})_y = (\psi_{yy})_x, \quad (\psi_{xy})_x = (\psi_{xx})_y,$$

one finds the following equations

$$\varphi_{yyy} = (3\varphi_x^2 a^2 - 2\varphi_x\varphi_y(a_y + ba) + 6\varphi_x\varphi_{yy}a + \varphi_y^2(2b_y - 4a_x + 4ca - b^2)$$
$$-6\varphi_y\varphi_{xy}a + 3\varphi_{yy}^2)/(2\varphi_y),$$

$$\varphi_{xyy} = (3\varphi_x^3 a^2 + 3\varphi_x\varphi_y^2(-2a_x + 2ac - b^2) + 6\varphi_x\varphi_y\varphi_{yy}b - 3\varphi_x\varphi_{yy}^2$$
$$+2\varphi_y^3(2c_y - b_x + 3ad) - 6\varphi_y^2\varphi_{xy}b + 12\varphi_y\varphi_{xy}\varphi_{yy})/(6\varphi_y^2).$$

Forming the mixed derivatives

$$(\varphi_{xyy})_y = (\varphi_{yyy})_x, \quad (\varphi_{xx})_{yy} = (\varphi_{xyy})_x,$$

one gets the conditions

$$\begin{aligned} 6d_y a - 3c_x a + c_{yy} - c_y b - 2b_{xy} + 2b_x b + 3a_{xx} - 3a_x c + 3a_y d = 0, \\ -3d_x a + 3d_{yy} + 3d_y b - 2c_{xy} - 2c_y c + b_{xx} + b_x c + 3b_y d - 6a_x d = 0. \end{aligned} \quad \text{(B.7)}$$

Thus, it follows that if $\varphi_y \neq 0$, then conditions (B.7) guarantee that the equation (B.2) can be linearized by a point transformation.

Case $\varphi_y = 0$

From equations (B.3)-(B.6), one obtains

$$a = 0, \tag{B.8}$$

$$\psi_{yy} = \psi_y b, \quad \psi_{xy} = (\psi_y \varphi_{xx} + \psi_y \varphi_x c))/(2\varphi_x),$$

$$\psi_{xx} = (\varphi_{xx}\psi_x + \varphi_x \psi_y d)/\varphi_x.$$

Comparing the mixed derivatives

$$(\psi_{xy})_y = (\psi_{yy})_x, \quad (\psi_{xy})_x = (\psi_{xx})_y,$$

one gets the following conditions

$$\begin{gathered} c_y = 2b_x, \\ d_{yy} - b_{xx} - b_x c + d_y b + b_y d = 0. \end{gathered} \tag{B.9}$$

Thus, it follows that if $\varphi_y = 0$, then the conditions (B.8) and (B.9) guarantee that the equation (B.2) can be linearized by a point transformation.

Notice that (B.9) is a particular case of (B.7) assuming that $a = 0$.

APPENDIX C

A LINEARIZATION PROBLEM OF SECOND-ORDER ODEs UNDER CONTACT TRANSFORMATIONS

Recall that the contact transformations

$$t = \varphi(x, y, y'), \quad u = \psi(x, y, y'), \quad s = g(x, y, y')$$

satisfy the conditions

$$\begin{aligned} g\,(\varphi_x + y'\varphi_y) &= \psi_x + y'\psi_y, \\ g\varphi_{y'} &= \psi_{y'}. \end{aligned} \tag{C.1}$$

Lie showed that all second-order ordinary differential equations

$$y'' = f(x, y, y')$$

can be mapped into

$$u'' = 0$$

with respect to contact transformations. Let us consider it in detail.

Since $u'' = \frac{Dg}{D\varphi}$, one needs to find functions $\varphi(x, y, y')$, $\psi(x, y, y')$ and $g(x, y, y')$ which satisfy (C.1) and the equation $Dg = 0$, which is

$$g_x + y'g_y + fg_{y'} = 0. \tag{C.2}$$

Notice that the Jacobian of the transformation is

$$\triangle = (\psi_y - g\varphi_y)\, g_{y'}\, (\varphi_x + y'\varphi_y + f\varphi_{y'}) \neq 0.$$

Without loss of generality it is assumed that $f \neq 0$.

Assume that $g(x, y, y')$ is some solution of (C.2) such that $g_{y'} \neq 0$. Since $f \neq 0$, then

$$g_x + y' g_y \neq 0.$$

Let us denote

$$\alpha = \psi - \varphi g.$$

The conditions (C.1) become

$$(\alpha_x + \varphi_x g - \varphi g_x) + y'(\alpha_y + \varphi_y g + \varphi g_y) - g\varphi_x - g\varphi_y y' = 0,$$
$$g\varphi_{y'} - (\alpha_{y'} + \varphi g_{y'} + g\varphi_{y'}) = 0.$$

One obtains that

$$\alpha_x + y'\alpha_y - \varphi f g_{y'} = 0, \qquad \alpha_{y'} + \varphi g_{y'} = 0. \tag{C.3}$$

The second equation of (C.3) becomes

$$\alpha_{y'} = -\varphi g_{y'}.$$

Substituting $\alpha_{y'}$ into the first equation of (C.3), the function $\alpha(x, y, y')$ has to satisfy the equation

$$\alpha_x + y'\alpha_y + f\alpha_{y'} = 0. \tag{C.4}$$

Notice that the requirement $\Delta \neq 0$ leads to

$$\alpha_y g_{y'} - \alpha_{y'} g_y \neq 0. \tag{C.5}$$

Since $g_{y'} \neq 0$, then for solving equation (C.4) one can change the independent variables (x, y, y') into (x, y, g).

Let $y' = h(x, y, g), \alpha = H(x, y, g)$. Thus, one gets

$$\alpha_x = H_x + H_g g_x, \qquad \alpha_y = H_y + H_g g_y, \qquad \alpha_{y'} = H_g g_{y'}.$$

Substituting α_x, α_y and $\alpha_{y'}$ into equation (C.4), one obtains

$$H_x + hH_y = 0. \tag{C.6}$$

From the condition (C.5) and $g_{y'} \neq 0$, one gets

$$H_y \neq 0.$$

Finding any solution $H(x, y, g)$ of equation (C.6) satisfying this condition one finds a contact transformation mapping the equation $y'' = f(x, y, y')$ into the equation $u'' = 0$.

APPENDIX D

A PARTICULAR LINEARIZATION PROBLEM OF SECOND-ORDER ODEs UNDER GENERALIZED SUNDMAN TRANSFORMATIONS

In 1994, Duarte, Moreira and Santos solved a particular linearization problem of second-order ordinary differential equation Sundman transformation. Let us analyze the solution of their problem.

One starts with obtaining necessary conditions for a second-order ordinary differential equation

$$y'' = F\left(x, y, y'\right), \tag{D.1}$$

which can be mapped via the generalized Sundman transformation

$$\begin{aligned} u &= F(x, y), \\ dt &= G(x, y)dx \end{aligned} \tag{D.2}$$

to the equation

$$u'' = 0.$$

The function u and its derivatives u' and u'' are defined by the first formula (D.2) and its derivatives with respect to x:

$$\begin{aligned} u'G &= F_x + F_y y', \\ u''G^2 + u'(G_x + G_y y') &= F_y y'' + 2F_{xy}y' + F_{yy}y'^2 + F_{xx}. \end{aligned} \tag{D.3}$$

Finding the derivatives u', u'' from (D.3), and substituting them into $u'' = 0$, one has the following equation

$$y'' + \lambda_2(x,y)y'^2 + \lambda_1(x,y)y' + \lambda_0(x,y) = 0, \tag{D.4}$$

where the functions λ_i are related to the functions F and G:

$$\lambda_2 = (F_{yy}G - F_yG_y)/K, \tag{D.5}$$

$$\lambda_1 = (2F_{xy}G - F_xG_y - F_yG_x)/K, \tag{D.6}$$

$$\lambda_0 = (F_{xx}G - F_xG_x)/K, \tag{D.7}$$

with $K = GF_y \neq 0$.

Equation (D.4) presents the necessary form of a second-order ordinary differential equation which can be mapped to a linear equation $u'' = 0$ via the generalized Sundman transformation.

For obtaining sufficient conditions, one has to solve the compatibility problem, considering equations (D.5)-(D.7) as an overdetermined system of partial differential equations for the functions F and G with given coefficients $\lambda_i(x,y), (i = 0,1,2)$. These conditions are obtained as follows.

From equations (D.5)-(D.7), one can find the derivatives of F:

$$F_{yy} = (F_y(G_y + G\lambda_2))/G, \tag{D.8}$$

$$F_{xy} = (F_xG_y + F_yG_x + F_yG\lambda_1)/(2G), \tag{D.9}$$

$$F_{xx} = (F_xG_x + F_yG\lambda_0)/G. \tag{D.10}$$

Comparing the mixed derivatives

$$(F_{xy})_y = (F_{yy})_x, \quad (F_{xx})_y = (F_{xy})_x, \tag{D.11}$$

one obtains the derivatives of G

$$\begin{aligned} G_{xy} = (&-3F_xG_y^2 - 2F_xG_yG\lambda_2 + 2F_xGG_{yy} + 3F_yG_xG_y + F_yG_yG\lambda_1 \\ &+2F_yG2(\lambda_{1y} - 2\lambda_{2x}))/(2F_yG), \end{aligned} \tag{D.12}$$

$$G_{xx} = (-3F_x^2G_y^2 - 2F_x^2G_yG\lambda_2 + 2F_x^2GG_{yy} + 2F_xF_yG^2(\lambda_{1y} - 2\lambda_{2x}) + 3F_y^2G_x^2 + 2F_y^2G_yG\lambda_0 + F_y^2G^2(4\lambda_{0y} - 2\lambda_{1x} + 4\lambda_0\lambda_2 - \lambda_1^2))/(2F_y^2G). \quad \text{(D.13)}$$

Comparing the mixed derivatives $(G_{xy})_x = (G_{xx})_y$, one gets the equation

$$F_xG_yS_1 - F_yG_xS_1 - F_yGS_2 = 0, \quad \text{(D.14)}$$

where

$$S_1 = \lambda_{1y} - 2\lambda_{2x},$$
$$S_2 = 2\lambda_{0yy} - 2\lambda_{1xy} + 2\lambda_0\lambda_{2y} - \lambda_{1y}\lambda_1 + 2\lambda_{0y}\lambda_2 + 2\lambda_{2xx}.$$

Case $S_1 = 0$

Equation (D.14) becomes

$$S_2 = 0. \quad \text{(D.15)}$$

Thus, we have shown that if $S_1 = 0$, then the condition (D.15) is sufficient for equation (D.4) to be linearizable by a generalized Sundman transformation.

Case $S_1 \neq 0$

From equation (D.14), one can find

$$G_x = (F_xG_yS_1 - F_yGS_2)/(F_yS_1).$$

Substituting G_x into (D.12) and (D.13), one obtains conditions

$$-S_{1y}S_2 + S_{2y}S_1 + S_1^3 = 0,$$
$$4\lambda_{0y}S_1^2 - 2\lambda_{1x}S_1^2 - 2S_{1x}S_2 + 2S_{2x}S_1 + 4\lambda_0\lambda_2S_1^2 - \lambda_1^2S_1^2 + S_2^2 = 0. \quad \text{(D.16)}$$

Thus, we have shown that if $S_1 \neq 0$, then the condition (D.16) are sufficient for equation (D.4) to be linearizable by a generalized Sundman transformation.

Let us observe that the second-order ordinary differential equations

$$y'' + (1/y)y'^2 + yy' + 1/2 = 0,$$

$$y'' + xy'^2 + yy' + 1/e^{2xy} = 0.$$

can be linearized via generalized Sundman transformations as shown in chapter III. One can check that the coefficients of these equations do not obey the conditions of Duarte, Moreira and Santos (1994). Thus, the result obtained in the thesis extend the linearization conditions obtained in Duarte, Moreira and Santos (1994) to the general form of a linear second-order ordinary differential equation.

APPENDIX E

THE APPLICATION OF GENERALIZED SUNDMAN TRANSFORMATIONS

In 2010, Muriel and Romero showed that the class of nonlinear second-order equations

$$y'' = M(x, y, y') \tag{E.1}$$

that are linearizable by means of generalized Sundman transformations into the Laguerre form of a linear second-order ordinary differential equation

$$u'' = 0$$

is identified as the class of equations admitting first integrals that are polynomials of first degree in the first-order derivative.

Recall that Duarte, Moreira and Santos (1994) studied the linearization problem of second-order ordinary differential equations via a generalized Sundman transformation

$$\begin{aligned} u &= F(x, y), \\ dt &= G(x, y)dx \end{aligned} \tag{E.2}$$

where $F_y G \neq 0$. They obtained that any second-order ordinary differential equation which can be mapped into the equation $u'' = 0$ via such a transformation has to be of the form

$$y'' + \lambda_2(x, y)y'^2 + \lambda_1(x, y)y' + \lambda_0(x, y) = 0, \tag{E.3}$$

where the functions λ_i are related to the functions F and G:

$$\lambda_2 = (F_{yy}G - F_yG_y)/(F_yG) = (\frac{F_y}{G})_y(\frac{F_y}{G})^{-1}, \quad (E.4)$$
$$\lambda_1 = (2F_{xy}G - F_xG_y - F_yG_x)/(F_yG) = [(\frac{F_x}{G})_y + (\frac{F_y}{G})_x](\frac{F_y}{G})^{-1}, \quad (E.5)$$
$$\lambda_0 = (F_{xx}G - F_xG_x)/(F_yG) = (\frac{F_x}{G})_x(\frac{F_y}{G})^{-1}. \quad (E.6)$$

Using the functions

$$S_1 = \lambda_{1y} - 2\lambda_{2x}, \quad S_2 = 2\lambda_{0yy} - 2\lambda_{1xy} + 2\lambda_0\lambda_{2y} - \lambda_{1y}\lambda_1 + 2\lambda_{0y}\lambda_2 + 2\lambda_{2xx},$$

they showed that equation (E.3) can be mapped into the equation $u'' = 0$ via a generalized Sundman transformation if the coefficients $\lambda_i(x, y), (i = 0, 1, 2)$ satisfy the conditions:

(a) if $S_1 = 0$, then $S_2 = 0$;

(b) if $S_1 \neq 0$, then $S_2 \neq 0$ and the following equations have to be satisfied

$$-S_{1y}S_2 + S_{2y}S_1 + S_1^3 = 0,$$

$$4\lambda_{0y}S_1^2 - 2\lambda_{1x}S_1^2 - 2S_{1x}S_2 + 2S_{2x}S_1 + 4\lambda_0\lambda_2S_1^2 - \lambda_1^2S_1^2 + S_2^2 = 0.$$

In their paper, the authors showed that if a second-order ordinary differential equation which has the form (E.3) is linearizable via a generalized Sundman transformation into the equation $u'' = 0$, then equation (E.3) admits a first integral of the form

$$\omega(x, y, y') = A(x, y)y' + B(x, y). \quad (E.7)$$

If a linearizing generalized Sundman transformation (E.2) is known, then a first integral (E.7) is defined by

$$A(x, y) = \frac{F_y}{G}, \quad B(x, y) = \frac{F_x}{G}. \quad (E.8)$$

It is clear that

$$A(y'' + \lambda_2(x, y)y'^2 + \lambda_1(x, y)y' + \lambda_0(x, y)) = D_x(\omega(x, y, y')),$$

where

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + \cdots$$

is the total derivative operator. Therefore ω, defined by (E.7), is a first integral of (E.3) and $A = \omega_{y'} = \frac{F_y}{G}$ is an integrating factor of equation (E.3).

Conversely, let us suppose that (E.1) has a first integral of the form

$$\omega(x, y, y') = A(x, y)y' + B(x, y).$$

Then

$$M(x, y, y') = -\frac{A_y}{A}y'^2 - \frac{B_y + A_x}{A}y' - \frac{B_x}{A} \tag{E.9}$$

and hence (E.1) must be of the form (E.3). To prove that this case (E.3) is linearizable by the generalized Sundman transformation (E.2), one first tries to find a function F such that

$$BF_y - AF_x = 0.$$

This is a first-order linear partial differential equation whose characteristic equation is

$$y' = -\frac{B}{A}. \tag{E.10}$$

If $I(x, y) = K, \quad K \in R$, is any solution of (E.10) then one chooses F to be any non-constant function of the form

$$F(x, y) = \varphi(I(x, y)). \tag{E.11}$$

Then G is uniquely determined by

$$G(x, y) = \frac{F_y}{A}. \tag{E.12}$$

or

$$G(x, y) = \frac{F_x}{B}, \tag{E.13}$$

if $B \neq 0$.

The results can be summarized as a theorem:

Theorem

The ordinary differential equation (E.1) is linearizable by a generalized Sundman transformation to Laguerre form if and only if (E.1) admits a first integral of the form $\omega(x, y, y') = A(x, y)y' + B(x, y)$. In this case, (E.1) has the form (E.3). If a linearizing generalized Sundman transformation (E.2) is known then a first integral $\omega = Ay' + B$ of (E.3) is defined by (E.8). Conversely, if a first integral $\omega = Ay' + B$ of (E.3) is known then a linearizing generalized Sundman transformation can be determined by (E.11)-(E.13), where $I(x, y)$ is a first integral of (E.10).

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