ปัญหาการควบคุมและเงินทุนเริ่มต้นน้อยสุดในประกันวินาศภัย

นายครรชิต เชื้อขำ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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CONTROL AND MINIMUM INITIAL CAPITAL PROBLEMS IN NON-LIFE INSURANCE

Khanchit Chuarkham

A Thesis Submitted in Partial Fulfillment of the Requirements for the

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CONTROL AND MINIMUM INITIAL CAPITAL PROBLEMS IN NON-LIFE INSURANCE

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์ฉบับนี้ได้ศึกษาในกรอบของแบบจำลองความเสี่ยงที่ไม่ต่อเนื่อง (หรือกระบวนการส่วนเกินไม่ต่อเนื่อง) ในประกันวินาศภัย ซึ่งได้แยกออกเป็นสองส่วน ดังนี้

ส่วนที่หนึ่ง ศึกษากระบวนการส่วนเกินไม่ต่อเนื่องในสถานการณ์ที่ว่า กระบวนการส่วนเกินสามารถถูกควบคุมด้วยสองกิจกรรม กิจกรรมแรกคือ การประกันภัยต่อ และกิจกรรมที่สองคือ การเติมเงินของผู้ถือหุ้น เราได้พิสูจน์การมีอยู่ของแผนที่เหมาะสมและได้สร้างสูตรสำหรับฟังก์ชันค่า นอกจากนี้ได้ยกตัวอย่างการหาแผนที่เหมาะสมในกรณีของการประกันภัยต่อแบบสัดส่วน

ส่วนที่สอง ศึกษากระบวนการส่วนเกินไม่ต่อเนื่องภายใต้กฎข้อบังคับที่ว่า บริษัทประกันวินาศภัยต้องสำรองเงินทุนเริ่มต้นให้เพียงพอเพื่อให้ความน่าจะเป็นรูอินไม่เกินปริมาณที่กำหนด ซึ่งกระบวนการส่วนเกินนี้ถูกพิจารณาในสถานการณ์ที่ว่า มีความเป็นไปได้ที่จะเกิดการขาดสภาพคล่อง ณ เวลาที่ต้องจ่ายค่าสินไหมทดแทน เราศึกษาความสัมพันธ์ระหว่างเงินทุนเริ่มต้นและความน่าจะเป็นรูอิน และได้พิสูจน์การมีอยู่ของเงินทุนเริ่มต้นน้อยสุด นอกจากนี้ได้ยกตัวอย่างการประมาณค่าของเงินทุนเริ่มต้นน้อยสุดในกรณีที่ขนาดของค่าสินไหมทดแทนมีการแจกแจงแบบเอกโพเนนเชียล

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา_______________

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KHANCHIT CHUARKHAM : CONTROL AND MINIMUM INITIAL CAPITAL PROBLEMS IN NON-LIFE INSURANCE.
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SURPLUS PROCESS/ VALUE FUNCTION/ RUIN PROBABILITY/ MINIMUM INITIAL CAPITAL/ REINSURANCE/ SHAREHOLDER INPUT.

The study in this thesis was conducted in the framework of the discrete-time risk model (or the discrete-time surplus process) in non-life insurance and was separated into two parts.

In the first part, we studied the discrete-time surplus process in the situation that the surplus process can be controlled by two activities; one is reinsurance and the other is the shareholder input. We had proved the existence of an optimal plan and derived a formula for the value function. Moreover, approximating the optimal plan in the case of proportional reinsurance has been given as an example.

In the second part, we studied the discrete-time surplus process under the regulation that the non-life insurance company has to reserve sufficient initial capital to ensure that ruin probability did not exceed the given quantity. The process is considered in the situation that the possible insolvency can occur only at claim arrival times. We studied the relationships between initial capital and ruin probability, and we had proved the existence of the minimum initial capital. Finally, we give an example in approximating the minimum initial capital in the case of exponential claims.

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CONTENTS

CONTENTS (Continued)

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

PRELIMINARIES ON RISK MODEL

Risk theory has become the natural language for formulating quantitative models of finance markets. In this chapter, we introduce some terminology of insurance, the description of the risk model and research problems. The reader is assumed to have some background knowledge of random variables and probability.

1.1 Mathematical Finance and Risk Theory

The general goal of mathematical finance and risk management is to mathematically quantify the behavior of financial instruments today and under different possible environments in the future. This implies that we have some mathematical or empirical procedure for determining values under various circumstances. While the road is long, and while there has been substantial progress, for many reasons, this goal is only partially achievable in the end and must be tempered with good judgment, especially in the case of problematic and rare extreme events, which are difficult to characterize, in which most of the risk lies.

In fact, financial risk theory can be considered as an indispensable part of Mathematical Finance. The latter was born in 1900 with the contribution of Louis Bachelier (Paris) on speculation in markets, then around the same time, the contribution of Philip Lundberg (Uppsala, Sweden) to the research on actuarial calculations became the cornerstone of the theory of non-life insurance. Since then, a rich theory has been developed for the study in financial risk measurement and management.

1.2 Classical Risk Model

In 1903 the Swedish actuary Fillip Lundberg laid the foundations of modern risk theory. Risk theory is a synonym for non-life insurance mathematics, which deals with the modeling of claims that arrive in a non-life insurance business and which gives advice on how much premium has to be charged in order to avoid insolvency (ruin) of the non-life insurance company.

One of Lundberg's main contributions was the introduction of a simple model which is capable of describing the basic dynamics of a homogeneous insurance portfolio. By this we mean a portfolio of contracts of policies for similar risks such as car insurance for a particular kind of car, insurance against theft in households or insurance against water damage of one-family homes. There are three assumptions in the model:(all processes are defined in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$.)

- Claims happen at the times T_i satisfying $0 \leq T_1 \leq T_2 \leq \cdots$. We call them *claim arrivals* of claim time or, simply *arrivals*.

- The i^{th} claim arriving at time T_i causes the claim size of claim severity Y_i. The sequence of $\{Y_i\}$ constitutes an independent and identically distributed sequence of non-negative random variables.

- The claim size processes $\{Y_i\}$ and the claim arrival processes $\{T_i\}$ are mutually independent.

Now we can define the *claim number process*

$$N(t) = \max\{i \geq 1 : T_i \leq t\},$$

i.e. $N = \{N(t)\}_{t \geq 0}$ is a counting process on $[0, \infty)$: $N(t)$ is the number of claims in the time interval $[0, t]$.

The objective of main interest from the point of view of a non-life insurance company is the *total claim amount process*:

$$S(t) = \sum_{i=1}^{N(t)} Y_i, t \geq 0. \tag{1.1}$$

It is also often called a *compound process*.

Later on in the 1930s, Harald Cramér, the famous Swedish statistician and probabilist, extensively developed collective risk theory by using the total claim amount process $S(t)$ with arrivals T_i which are generated by a Poisson process. The homogeneous Poisson process plays a major role in non-life insurance mathematics. If we specify the claim number process as a homogeneous Poisson process the resulting model which combines claim sizes and claim arrivals is called *Cramér-Lundberg model*:

The Poisson process and the homogeneous Poisson process as mentioned above, are define as follows:

Definition 1.1. *(Poisson process)*

A stochastic process $\{N(t)\}_{t\geq 0}$ is said to be a Poisson process if the following conditions hold:

(i) *The process starts at zero, i.e. $N(0) = 0$ a.s.*

(ii) *The process has independent increments, i.e. $N(t_{i-1}, t_i]$ and $N(t_i, t_{i+1}]$ are indepent where $i = 0, 1, 2, \ldots, n$, $n \geq 1$ and $N(t_{i-1}, t_i] = N(t_i) - N(t_{i-1})$.*

(iii) *The increment $N(s, t]$, $0 < s < t < \infty$, has a Poisson distribution $Poi(\mu(s, t])$ where μ is the mean value function of N, i.e. $N(s, t] \sim Poi(\mu(s, t])$*

(iv) *The sample paths of the process $\{N(t)\}_{t\geq 0}$ are càdlàg, i.e. $N(t)$ is right-continuous for $t \geq 0$ and limits from the left for $t > 0$ exists.*

A *Poisson process* is said to be a *homogeneous Poisson process* if the increment $N(s,t]$, in condition (iii), has a Poisson distribution $Poi(\lambda(s,t])$ where λ is the intensity of N, i.e. $N(s,t] \sim Poi(\lambda(t-s))$.

Next, let $p(t)$ denote the premium income in the time interval $[0,t]$. In the Cramér-Lundberg model, it is assumed that $p(\cdot)$ is a deterministic linear function: that is,

$$p(t) = c_0 t$$

where $c_0 > 0$ is a constant called the *premium rate*. Therefore the quantity

$$X(t) = x + p(t) - S(t) = x + c_0 t - \sum_{i=1}^{N(t)} Y_i \tag{1.2}$$

is the insurer's balance (or surplus) at time $t \geq 0$ with the constant $x \geq 0$ as initial capital. Moreover, the process $\{X(t)\}_{t\geq 0}$ is called the *risk process* (or *surplus process*).

Furthermore, the model (1.2) only at time T_i is the following

$$X(T_i) = x + c_0 T_i - S(T_i). \tag{1.3}$$

From the fact that

$$N(T_n) = \max\{i \geq 1 : T_i \leq T_n\} = n \text{ a.s.},$$

then, for convenience, we set $X(T_n) = X_n$, $S(T_n) = S_n$ and let $Z_n := T_n - T_{n-1}$ which is called $inter - arrival\ time$. Therefore, the model (1.3) can be written in the form:

$$\begin{aligned} X_n &= x + c_0 T_n - S_n \\ &= x + c_0 T_n - c_0 T_{n-1} + c_0 T_{n-1} - S_{n-1} - Y_n \\ &= (x + c_0 T_{n-1} - S_{n-1}) + c_0 (T_n - T_{n-1}) - Y_n \\ &= X_{n-1} + c_0 Z_n - Y_n \\ &= x + c_0 \sum_{k=1}^{n} Z_k - \sum_{k=1}^{n} Y_k \qquad (1.4) \end{aligned}$$

where $X_0 = x$.

The model (1.4) is usually considered under the assumption that $\{Y_k\}_{k\geq 1}$ and $\{Z_k\}_{k\geq 1}$ are independent.

1.3 Ruin Theory

1.3.1 Ruin Probability(Ruin), Net Profit Condition and Expected Value Principle

Definition 1.2. *(Ruin, Ruin probability)*

Ruin is the set of events that X_n falls below zero.That is

$$\begin{aligned} Ruin &= \{\omega \in \Omega \mid X_n(\omega) < 0 \ \ \textit{for some } n \geq 1\} \\ &= \left\{\omega \in \Omega \mid \inf_{n\geq 1} X_n(\omega) < 0\right\} \end{aligned} \tag{1.5}$$

Set

$$T = \inf\{T_n > 0, X_n < 0\}; \tag{1.6}$$

T is called the *ruin time* ; it is the first time the surplus falls below zero. The ruin probability is then given by

$$\Phi(x) = P(T < \infty) = P(X_n < 0 \text{ for some } n \geq 1 | X_0 = x) = P\left(\inf_{n\geq 1} X_n < 0 | X_0 = x\right). \tag{1.7}$$

Note that $\Phi(x)$ depends on the premium rate c_0 as well.

Lemma 1.1. *(Ruin with probability 1)*

If $E[Z_1]$ and $E[Y_1]$ are finite and the condition

$$E[Y_1] - c_0 E[Z_1] \geq 0$$

holds, then ruin occurs with probability 1 for every fixed $x > 0$.

Proof : See Mikosch (2004).

From virtue of Lemma 1.1, any non-life insurance company should choose the premium rate c_0 in such a way that $E[Y_1] - c_0E[Z_1] < 0$. Hence, we make the following definitions:

Definition 1.3. *(Net profit condition)*

The risk process satisfies the net profit condition if

$$E[Y_1] - c_0E[Z_1] < 0.$$

Definition 1.4. *(Expected value principle)*

The risk process satisfies the expected value principle if

$$c_0 = (1+\theta)\frac{E[Y_1]}{E[Z_1]},$$

for some $0 < \theta < 1$ which is called a safety loading.

1.3.2 Bounds for Ruin Probability

In this section we derive an elementary upper bound for the ruin probability $\Phi(x)$.

Definition 1.5. *(Adjustment or Lundberg coefficient)*

Assume that the moment generating function of $Y_1 - c_0Z_1$ exists in some neighborhood $(-d, d),\ d > 0$, of the origin. If a unique positive solution d_0 to the equation

$$m_{Y_1 - c_0Z_1}(\tilde{d}) = E[e^{\tilde{d}(Y_1 - c_0Z_1)}] = 1, \quad \tilde{d} \in (-d, d) \tag{1.8}$$

exists it is called the adjustment or Lundberg coefficient

Theorem 1.2. *(The Lundberg inequality : Mikosch (2004))*

Assume that the risk process satisfies the net profit condition and the adjustment or Lundberg coefficient d_0 exists. Then the following inequality holds for all $x \geq 0$

$$\Phi(x) \leq e^{-d_0x}. \tag{1.9}$$

Proof : Let $W_n = Y_n - c_0 Z_n$ for all $n \geq 1$ and set

$$S_n = \sum_{k=1}^{n} W_k.$$

From equation (1.7), ruin probability can be written in the form:

$$\Phi(x) = P\left(\inf_{n\geq 1}(-S_n) < -x\right) = P\left(\max_{n\geq 1} S_n > x\right). \tag{1.10}$$

Let

$$\Phi_n(x) = P\left(\max_{1\leq k\leq n} S_k > x\right).$$

Thus $\{\Phi_n(x)\}_{n\geq 1}$ is non-decreasing sequence and $\Phi_n(x) \to \Phi(x)$ as $n \to \infty$ for all x. From this, it suffices to prove that

$$\Phi_n(x) \leq e^{-d_0 x} \text{ for all } n \geq 1. \tag{1.11}$$

We prove inequality (1.11) by induction. We start with $n = 1$. By the definition of the adjustment coefficient, we get

$$\begin{aligned}
\Phi_1(x) &= P(W_1 > x) \\
&= P(d_0 W_1 > d_0 x) \\
&= P(e^{d_0 W_1} > e^{d_0 x}) \\
&\leq \frac{E\left[e^{d_0 W_1}\right]}{e^{d_0 x}} \qquad \text{(By Markov's inequality)} \\
&= e^{-d_0 x} m_{W_1}(d_0) \\
&= e^{-d_0 x}.
\end{aligned}$$

This proves for $n = 1$. Now assume that inequality (1.11) holds for $n = k \geq 1$ and let F_{W_1} be the distribution function of W_1. Then

$$\Phi_{k+1}(x) = P\left(\max_{1\leq n\leq k+1} S_n > x\right)$$

$$= P\left(W_1 > x\right) + P\left(\max_{2\leq n\leq k+1} S_n > x, W_1 \leq x\right)$$

$$
\begin{aligned}
&= P(W_1 > x) + P\left(\max_{2\le n\le k+1} (S_n - W_1) + W_1 > x, W_1 \le x\right) \\
&= \int_x^\infty dF_{W_1}(u) + \int_{-\infty}^x P\left(\max_{1\le n\le k} S_n + u > x\right) dF_{W_1}(u) \\
&= \int_x^\infty dF_{W_1}(u) + \int_{-\infty}^x P\left(\max_{1\le n\le k} S_n > x - u\right) dF_{W_1}(u) \\
&= \int_x^\infty dF_{W_1}(u) + \int_{-\infty}^x \Phi_k(x-u) dF_{W_1}(u) \\
&\le \int_x^\infty e^{d_0(u-x)} dF_{W_1}(u) + \int_{-\infty}^x e^{d_0(u-x)} dF_{W_1}(u) \\
&= \int_{-\infty}^\infty e^{d_0(u-x)} dF_{W_1}(u) \\
&= e^{-d_0 x} \int_{-\infty}^\infty e^{d_0 u} dF_{W_1}(u) \\
&= e^{-d_0 x} E\left[e^{d_0 W_1}\right] \\
&= e^{-d_0 x} m_{W_1}(d_0) \\
&= e^{-d_0 x}.
\end{aligned}
$$

This proves inequality (1.11) for $n = k + 1$ and concludes the proof.

1.4 Insurance and Reinsurance

In law and economics, *insurance* is a form of risk management primarily used to hedge against the risk of a contingent, uncertain loss. Insurance is defined as the equitable transfer of the risk of a loss, from one entity to another, in exchange for payment. An *insurer* is a company selling the insurance; an *insured* or *policyholder* is the person or entity buying the insurance policy. The insurance rate is a factor used to determine the amount to be charged for a certain amount of insurance coverage, called the *premium.*

Reinsurance is insurance that is purchased by a non-life insurance company (insurer) from a reinsurer as a means of risk management, to transfer risk from the insurer to the reinsurer.

Reinsurance treaties are mutual agreements between different non-life insurance companies with the aim to reduce the risk in a particular insurance portfolio by sharing the risk of the occurring claims as well as premium in this portfolio. In a sense, reinsurance is insurance for non-life insurance companies. Reinsurance is a necessity for portfolios which are subject to catastrophic risks such as earthquakes, failure of nuclear power stations, major windstorms, industrial fire, flooding, war, riots, etc. There are many types of reinsurance treaties, non-life insurance company handles mostly two types of treaties as follows:

- *Proportional reinsurance.* In a proportional reinsurance treaty each individual claim of size y is divided between insurer and reinsurer to a proportionality factor $b \in [0, 1]$. Hence if we let $h(b, y)$ be a (measurable) function which stands for the part of the claim size y paid by the insurer; then the remaining part $y - h(b, y)$ called *reinsurance recovery* is paid by the reinsurer. In the case of a *proportional reinsurance*, we have:

$$h(b, y) = by.$$

- *Excess-of-loss reinsurance.* In excess of loss (XL) reinsurance, each claim of size y is divided between the insurer and the reinsurer according to priority $b \in [0, \infty]$. Again if we let $h(b, y)$ be a (measurable) function which stands for the part of the claim size y paid by the insurer; then the remaining part $y - h(b, y)$ called *reinsurance recovery* is paid by the reinsurer. In the case of an *excess of loss reinsurance*, we have:

$$h(b, y) = \min\{b, y\}.$$

The above constant b is called the *ceding company's retention level* or *retention level.*

1.5 Shareholder Input

Shareholder input means the amount of money that the shareholders put into the firm.

1.6 An Extension of Risk Model

Usually, the risk process (1.4) was considered under the assumption that $\{Y_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$ are independent. For example, Schal (2004) applied this model with reinsurance and investment as control parameters and proved the existence of an optimal plan for the exponential utility function under the assumption of independence as mentioned above. In 2010, Klongdee, Sattayatham and Sangaroon extended the study of Schal (2004) and proved the existence of the optimal plan for the exponential utility function under the additional assumption that a reinsurer has the opportunity to default.

In Chapter II, we study the model (1.4) together with two controllers, i.e. reinsurance and shareholder input allowing the firm to reach a desired target. Finally, we find an optimal control policy which minimizes a reasonable objective function.

Moreover, the ruin probability for the model (1.4) is interesting as found in Pavlovao and Willmot (2004), Dickson (2005) and Li (2005b). All of these articles study the ruin probability as a function of the initial capital $x \geq 0$. In the opposite direction, Sattayatham, Sangaroon and Klongdee (To be plublished) considered the initial capital for this model via the ruin probability when $Z_n = 1,\ n = 1, 2, 3, \ldots$.

In Chapter III, we extend the study of Sattayatham, Sangaroon and Klongdee (To be plublished) when the claims can be controlled by reinsurance.

We prove the existence of the minimum initial capital and apply the bisection method to approximate the minimum initial capital for exponential claims.

CHAPTER II

CONTROL PROBLEM

In this chapter, we study the control problem of a discrete-time surplus process under two controllers, i.e., reinsurance and shareholder input which allow a firm to reach a desired target. Moreover, we have proved the existence of an optimal plan and we obtain a formula for the value function which gives an optimal control policy. An example shows some numerical calculations for getting an optimal plan.

Furthermore, we assume that all the processes are defined in a probability space $(\Omega, \mathcal{F}, P)$.

2.1 Model Description

Firstly, we recall the discrete-time surplus process without control which consists of the claim size process $\{Y_n\}_{n\geq 1}$ and the claim arrival process $\{T_n\}_{n\geq 0}$. The inter-arrival process $\{Z_n\}_{n\geq 1}$ is defined by

$$Z_n := T_n - T_{n-1},$$

is the length of time between the $(n-1)^{th}$ claim and the n^{th} claim. By *period* n, we shall mean the random interval $[T_{n-1}, T_n), n \geq 1$.

Now let the constant c_0 represent the premium rate for one unit time. The random variable $c_0 \sum_{i=1}^{n+1} Z_i = c_0 T_{n+1}$ describes the inflow of capital into the business in $[0, T_{n+1}]$, and $\sum_{i=1}^{n+1} Y_i$ describes the outflow of capital due to payments for claims occurring in $[0, T_{n+1}]$. Therefore, the quantity

$$X_{n+1} = x + c_0 \sum_{i=1}^{n+1} Z_i - \sum_{i=1}^{n+1} Y_i, \ n = 0, 1, 2, \ldots \tag{2.1}$$

is the insurer's balance (or surplus) at time T_{n+1} with the constant $x \geq 0$ as initial capital.

In summary, the discrete-time surplus process (2.1) can be written in the form:

$$X_0 = x, \ X_{n+1} = X_n + c_0 Z_{n+1} - Y_{n+1}, \ n = 0, 1, 2, \ldots . \tag{2.2}$$

Usually, this model was considered under the assumptions that $\{Y_n\}_{n \geq 1}$ and $\{Z_n\}_{n \geq 1}$ are independent. In 2004, Schal applied this model with reinsurance and investment as control parameters and proved the existence of an optimal plan for the exponential utility function under the assumption of independence as mentioned above. Recently, Klongdee, Sattayatham and Sangaroon (2010) applied this model with reinsurance and investment as control parameters and proved the existence of an optimal plan for the exponential utility function under the additional assumption that a reinsurer has opportunity to default.

In this chapter, we studied this model together with two controllers, i.e., reinsurance and shareholder input allowing the firm to reach a desired target. Moreover, we find an optimal control policy which minimizes a reasonable objective function.

Now let $\{X_n\}_{n \geq 0}$ be the surplus process which can be controlled by choosing a retention level $b \in [\underline{b}, \overline{b}], 0 \leq \underline{b} \leq b \leq \overline{b} \leq \infty$, of reinsurance for one period. Next, for each retention level b, an insurer pays a premium rate to a reinsurer which is deducted from c_0. As a result, the insurer's income rate will be represented by the function $c(b)$. The level $\overline{b}$ stands for the control action without reinsurance, so that $c_0 = c(\overline{b})$ and the level $\underline{b}$ is the smallest retention level which can be chosen.

As a consequence, we obtain the *net income rate* $c(b)$ where $0 \leq c(b) \leq c_0$ for all $b \in [\underline{b}, \bar{b}]$ and $c(b)$ is non-decreasing. By the *expected value principle*, c_0 and $c(b)$ can be calculated as follows :

$$c_0 = (1+\theta_0)\frac{E[Y]}{E[Z]} \quad \text{and} \quad c(b) = c_0 - (1+\theta_1)\frac{E[Y - h(b,Y)]}{E[Z]} \tag{2.3}$$

where Y is a claim size, Z is an inter-arrival time, and $0 < \theta_0 < 1$, $0 < \theta_1 < 1$ are the *safety loading* of the insurer and reinsurer respectively. The measurable function $h(b, y)$ is the part of the claim size y paid by the insurer, and the remaining part $y - h(b, y)$ which is called *reinsurance recovery* is paid by the reinsurer. In the case of an *excess of loss reinsurance*, we have:

$$h(b, y) = \min\{b, y\} \ \text{ with retention level } \ 0 \leq \underline{b} \leq b \leq \bar{b} = \infty.$$

In the case of a *proportional reinsurance*, we have:

$$h(b, y) = by \ \text{ with retention level } \ 0 \leq \underline{b} \leq b \leq \bar{b} = 1.$$

Furthermore, the surplus process can also be controlled by shareholder input, i.e., the insurance company can ask its shareholders for input their money $\delta \in [0, \infty)$, so that the firm can reach a desired target A. Hence the two controllers for the surplus process, b and δ will stand for reinsurance and shareholder input respectively. Note that, we can interpret the target A as an initial capital for supporting the growth of various policies in the future.

Let b_n and δ_n be the two control actions at the time T_n. Therefore, the surplus process (2.2) can be modified to be the following:

$$X_{n+1} = X_n + \delta_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1}), \ n = 0, 1, 2, \ldots \tag{2.4}$$

where $X_0 = x$. It is convenient to rewrite (2.4) into an equivalent form

$$X_{n+1} = X_n + L(b_n, \delta_n, Y_{n+1}, Z_{n+1}), \ n = 0, 1, 2, \ldots \tag{2.5}$$

where $L(b, \delta, y, z) = \delta + c(b)z - h(b, y)$. We see that the process $\{X_n\}_{n\geq 0}$ is driven by the sequence of the control actions $\{(b_n, \delta_n)\}_{n\geq 0}$, the sequence of inter-arrival times $\{Z_n\}_{n\geq 1}$ and the sequence of claims $\{Y_n\}_{n\geq 1}$. Let us assume that $\{Z_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ are independent and identically distributed (iid) sequences of random variables with finite variance, i.e., we make the following assumption:

Assumption 2.1. Independence Assumption (IA)

The sequence of inter-arrival times $\{Z_n\}_{n\geq 1}$ and the sequence of claims $\{Y_n\}_{n\geq 1}$ are iid sequences with finite variances. Moreover, for each $n \in \{1, 2, 3, \ldots\}$, Z_n and Y_n are independent.

We immediately get from Assumption 2.1 that $\{h(b_n, Y_{n+1})\}_{n\geq 0}$ is an independent sequence.

Remark 2.1. *Let $n \in \{0, 1, 2, \ldots\}$ and f be the density of Y_{n+1}, then the variance of $h(b_n, Y_{n+1})$ is finite.*

Proof: We prove by cases :

Case 1. $h(b_n, Y_{n+1}) = b_n Y_{n+1}$. We get

$$\begin{aligned} Var[h(b_n, Y_{n+1})] &= Var[b_n Y_{n+1}] \\ &= b_n^2 Var[Y_{n+1}]. \end{aligned}$$

Since $b_n < \infty$ and Y_{n+1} has finite variance, then $Var[h(b_n, Y_{n+1})] < \infty$.

Case 2. $h(b_n, Y_{n+1}) = \min\{b_n, Y_{n+1}\}$. We get

$$\begin{aligned} &Var[h(b_n, Y_{n+1})] \\ &= E\left[h^2(b_n, Y_{n+1})\right] - \left(E[h(b_n, Y_{n+1})]\right)^2 \\ &= E\left[b_n^2 1_{Y_{n+1}>b_n} + Y_{n+1}^2 1_{Y_{n+1}\leq b_n}\right] - \left(E[b_n 1_{Y_{n+1}>b_n} + Y_{n+1} 1_{Y_{n+1}\leq b_n}]\right)^2 \\ &= b_n^2 P[Y_{n+1} > b_n] + \int\limits_{y\leq b_n} y^2 f(y)dy - \left(b_n P[Y_{n+1} > b_n] + \int\limits_{y\leq b_n} y f(y)dy\right)^2. \end{aligned}$$

Since Y_{n+1} has finite variance, then $\int\limits_{y \le b_n} y^2 f(y)dy < \infty$ and $0 \le \int\limits_{y \le b_n} yf(y)dy < \infty$. Thus $Var[h(b_n, Y_{n+1})] < \infty$ and this proves case 2.

From case 1 and 2, Remark 2.1 holds.

2.2 A Value Function with Finite Horizon

Let $\{X_n\}_{n \ge 0}$ be a surplus process with value in a state space (S, Ξ) which is a measurable space. The surplus process can be controlled at the beginning of every period $[T_n, T_{n+1}),\ n = 0, 1, 2, \ldots$ on a measurable space $(U, \mho)$ which is called a *control space*. In addition, the model is further specified by the following quantities:

- $N \in \{2, 3, 4, \ldots\}$ is a *time horizon* (number of periods);
- T_N is a *time at the time horizon* N;
- $\alpha_N \in (0, 1]$ is a *positive real constant*;
- $g : S \times U \to (-\infty, \infty]$ is a $one - period\ cost\ function$, which is measurable and bounded from below;
- $\widehat{V} : S \to (-\infty, \infty]$ is a $cost\ function$ for time horizon N, which is measurable and bounded from below.

Definition 2.1. *A plan for the time horizon N over a control space U is a (finite) sequence $\pi = \{u_n\}_{n=0}^{N-1}$ of $u_0 = (b_0, \delta_0) = (b_0, 0)$ and $u_n = (b_n, \delta_n) \in U$ for $n = 1, 2, 3, \ldots, N-1$. The set of all plans for the time horizon N over the space U is denoted by $\mathcal{P}(N, U)$. A plan $\pi \in \mathcal{P}(N, U)$ is said to be stationary, if $b_0 = b_1$ and $(b_n, \delta_n) = (b_1, \delta_1)$ for $n = 1, 2, 3, \ldots, N-1$.*

For each initial state $x \in S$ and plan $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$, the surplus process (2.5) can be written in the form

$$X_{n+1} = x + \sum_{k=0}^{n} L(b_k, \delta_k, Y_{k+1}, Z_{k+1}), \; n = 0, 1, 2, \ldots, N-1 \tag{2.6}$$

with $X_0 = x$.

Definition 2.2. *Let $x \in S$ be an initial state and $\pi = \{u_n\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ where N is the time horizon. The total cost function $\Phi_N(x, \pi)$ and the value function $V_N(x)$ for the time horizon N are defined by*

$$\begin{aligned} \Phi_N(x, \pi) &= E\left[\sum_{n=0}^{N-1} g(X_n, u_n) + \alpha_N \widehat{V}(X_N)|_{X_0=x}\right] \qquad \textit{and} \\ V_N(x) &= \inf_{\pi \in \mathcal{P}(N,U)} \Phi_N(x, \pi) \qquad \textit{respectively} \end{aligned} \tag{2.7}$$

when the X_n are random variables which satisfy equation (2.6). A plan $\tilde{\pi} \in \mathcal{P}(N, U)$ is said to be optimal, if $\inf_{\pi \in \mathcal{P}(N,U)} \Phi_N(x, \pi) = \Phi_N(x, \tilde{\pi})$.

2.3 Main Results

Firstly, we note that it is natural to assume that the target A should satisfy the condition

$$A \geq E\left[X_N|_{X_0=x}\right], \tag{2.8}$$

where X_N is the random variable which satisfies equation (2.2). The above expectation can be calculated as follows:

$$E\left[X_N|_{X_0=x}\right] = E\left[X_{N-1} + c_0 Z_N - Y_N|_{X_0=x}\right] = x + c_0 \sum_{n=1}^{N} E[Z_n] - \sum_{n=1}^{N} E[Y_n].$$

Since the goal of this chapter is to find retention level and shareholder input that can make the firm reach a desired target A, the case of shareholder input greater than A is uninteresting. So, we assume that

$$S = \mathcal{R} \;\text{ and }\; U = [\underline{b}, \overline{b}] \times [0, A] \tag{2.9}$$

are the state space and the control space respectively.

In this section, we studied the surplus model (2.6) when the insurance company is controlled by choosing retention level b_n and shareholder input δ_n at the beginning of the period $[T_n, T_{n+1})$ in order to reach a desired target A at the time horizon N.

We studied the cost function under the assumption that the insurance company is solvent (not ruined) and we look for a control policy that ensures the minimization of the distance from the surplus at the time horizon N to the target A. Therefore, we define the $one-period\ cost\ function$ and the $cost\ function$ at the time horizon N respectively, as follows:

$$g(x,u) = g(x,(b,\delta)) = \delta^2 \quad \text{and} \quad \widehat{V}(x) = (x-A)^2$$

where $u = (b,\delta) \in U$ and $x \in S$. Thus, we obtain the $total\ cost\ function$ of model (2.7) as

$$\Phi_N(x,\pi) = \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N E\left[(X_N - A)^2|_{X_0=x}\right] \tag{2.10}$$

where $\pi = \{(b_n,\delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$.

Remark 2.2. *By substituting* $\pi = \{(b_n,\delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ *into equation (2.10), one gets*

$$\Phi_N(x,\pi) = \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N \left\{ \sum_{n=0}^{N-1} \left\{ c^2(b_n) Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})] \right\} + G_N^2(x,\pi) \right\} \tag{2.11}$$

where $G_N(x,\pi) = x - A + \sum_{n=0}^{N-1} E[L(b_n,\delta_n,Y_{n+1},Z_{n+1})]$. *Moreover* $0 \le \Phi_N(x,\pi) < \infty$.

Proof: Let $\pi = \{(b_n,\delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$. Then

$$\Phi_N(x,\pi)$$
$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N E\left[(X_N - A)^2|_{X_0=x}\right]$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N E \left(x + \sum_{n=0}^{N-1} L(b_n, \delta_n, Y_{n+1}, Z_{n+1}) - A \right)^2$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N Var \left(x + \sum_{n=0}^{N-1} L(b_n, \delta_n, Y_{n+1}, Z_{n+1}) - A \right)$$

$$+\alpha_N \left(x + \sum_{n=0}^{N-1} E[L(b_n, \delta_n, Y_{n+1}, Z_{n+1})] - A \right)^2$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N Var \left(x - A + \sum_{n=0}^{N-1} \{\delta_n + c(b_n) Z_{n+1} - h(b_n, Y_{n+1})\} \right)$$

$$+\alpha_N \left(x - A + \sum_{n=0}^{N-1} E[L(b_n, \delta_n, Y_{n+1}, Z_{n+1})] \right)^2$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N \left\{ \sum_{n=0}^{N-1} \left\{ c^2(b_n) Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})] \right\} + G_N^2(x, \pi) \right\}$$

where $G_N(x, \pi) = x - A + \sum_{n=0}^{N-1} E[L(b_n, \delta_n, Y_{n+1}, Z_{n+1})]$.

Finally, we shall prove that $0 \leq \Phi_N(x, \pi) < \infty$. By definition 2.2, $\Phi_N(x, \pi) \geq 0$. Next, we will show that $\Phi_N(x, \pi)$ is finite. By Assumption 2.1 and Remark 2.1, Z_{n+1} and $h(b_n, Y_{n+1})$ have a finite variance $(n = 0, 1, 2, \ldots, N-1)$. Then we have

$$\sum_{n=0}^{N-1} \left\{ c^2(b_n) Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})] \right\} < \infty, \ \sum_{n=1}^{N-1} \delta_n^2 < \infty \text{ and}$$

$$\begin{aligned} G_N(x, \pi) &= x - A + \sum_{n=0}^{N-1} E[L(b_n, \delta_n, Y_{n+1}, Z_{n+1})] \\ &= x - A + \sum_{n=0}^{N-1} \{\delta_n + c(b_n) E[Z_{n+1}] - E[h(b_n, Y_{n+1})]\} \\ &< \infty. \end{aligned}$$

This proves Remark 2.2.

Remark 2.3. *Define a subset* $\mathcal{P}^*(N, U)$ *of* $\mathcal{P}(N, U)$ *by*

$$\mathcal{P}^*(N, U) = \{\pi \in \mathcal{P}(N, U) | G_N(x, \pi) = 0\}.$$

We have

(i) $\mathcal{P}^*(N,U)$ *is not empty.*

(ii) $\mathcal{P}^*(N,U)$ *contains an element of the form* $\pi := \{(b_n,\delta_n)\}_{n=0}^{N-1}$ *where* $\delta_1 = \delta_2 = \cdots = \delta_{N-1}$.

Proof of (i): We choose an arbitrary finite sequence $b_n \in [\underline{b}, \bar{b}]$, $n = 0,1,2,\ldots,N-1$ and $\delta_0 = 0$, $\delta_n = \dfrac{A - x - c(b_0)E[Z_1] + E[h(b_0,Y_1)]}{N-1} - c(b_n)E[Z_{n+1}] + E[h(b_n,Y_{n+1})]$, $n = 1,2,3,\ldots,N-1$. From inequality (2.8), we have

$$A \geq x + \sum_{n=1}^{N} c_0 E[Z_n] - \sum_{n=1}^{N} E[Y_n] = x + \sum_{n=0}^{N}(c_0E[Z_n] - E[Y_n]). \tag{2.12}$$

Since $E[h(b_n,Y_{n+1})] \leq E[Y_{n+1}]$ for each $b_n, n = 0,1,2,\ldots,$ and $\theta_1 > 0$, then it follows from the expected value principle and equation (2.3) that

$$\begin{aligned}
0 &< c(b_n)E[Z_{n+1}] - E[h(b_n,Y_{n+1})] \\
&= E[Z_{n+1}]\left(c_0 - (1+\theta_1)\frac{E[Y_{n+1} - h(b_n,Y_{n+1})]}{E[Z_{n+1}]}\right) - E[h(b_n,Y_{n+1})] \\
&= c_0E[Z_{n+1}] - (1+\theta_1)E[Y_{n+1} - h(b_n,Y_{n+1})] - E[h(b_n,Y_{n+1})] \\
&= c_0E[Z_{n+1}] - (1+\theta_1)E[Y_{n+1}] + \theta_1E[h(b_n,Y_{n+1})] \\
&= c_0E[Z_{n+1}] - E[Y_{n+1}] + \theta_1(E[h(b_n,Y_{n+1})] - E[Y_{n+1}]) \\
&\leq c_0E[Z_{n+1}] - E[Y_{n+1}].
\end{aligned} \tag{2.13}$$

From inequality (2.12) and summing both sides of inequality (2.13), one gets

$$0 < x + \sum_{n=0}^{N-1}(c(b_n)E[Z_{n+1}] - E[h(b_n,Y_{n+1})]) \leq x + \sum_{n=0}^{N-1}(c_0E[Z_{n+1}] - E[Y_{n+1}]) \leq A. \tag{2.14}$$

Claim that $0 \leq \delta_n \leq A$, $n = 1,2,\ldots,N-1$.

Firstly, assume that there exists $\delta_m < 0$ for some $m \in \{1,2,\ldots,N-1\}$. It follows from the definition of δ_m that

$$\frac{A - x - c(b_0)E[Z_1] + E[h(b_0,Y_1)]}{N-1} < c(b_m)E[Z_{m+1}] - E[h(b_m,Y_{m+1})]. \tag{2.15}$$

Putting $n = 0$ in inequality (2.13), we get

$$-c_0E[Z_1] + E[Y_1] \le -c(b_0)E[Z_1] + E[h(b_0, Y_1)].$$

Hence

$$A - x - c_0E[Z_1] + E[Y_1] \le A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)] \tag{2.16}$$

Since the sequences $\{Y_n\}_{n\ge1}$ and $\{Z_n\}_{n\ge1}$ satisfy iid property, then $E[Z_{m+1}] - E[Y_{m+1}] = E[Z_1] - E[Y_1]$. Hence

$$\begin{aligned}
\frac{A - x - c_0E[Z_1] + E[Y_1]}{N-1} &\le \frac{A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]}{N-1} && \text{(By (2.16))}\\
&< c(b_m)E[Z_{m+1}] - E[h(b_m, Y_{m+1})] && \text{(By (2.15))}\\
&\le c_0E[Z_{m+1}] - E[Y_{m+1}] && \text{(By (2.13))}\\
&= c_0E[Z_1] - E[Y_1].
\end{aligned}$$

Thus $A < x + N(c_0E[Z_1] - E[Y_1])$. It follows from inequality (2.14) and iid property of the sequences $\{Y_n\}_{n\ge1}$ and $\{Z_n\}_{n\ge1}$, that

$$x + N(c_0E[Z_1] - E[Y_1]) = x + \sum_{n=0}^{N-1}(c_0E[Z_{n+1}] - E[Y_{n+1}]) \le A.$$

This is a contradiction and then $\delta_n \ge 0,\ n = 1, 2, \ldots, N-1$.

Next assume that there exists $\delta_m > A$ for some $m \in \{1, 2, \ldots, N-1\}$. Again by definition of δ_m we have

$$\frac{A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]}{N-1} > A + c(b_m)E[Z_{m+1}] - E[h(b_m, Y_{m+1})]. \tag{2.17}$$

Since $c(b_m)E[Z_{m+1}] - E[h(b_m, Y_{m+1})] > 0$ for all m, thus inequality (2.17) satisfy

$$\frac{A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]}{N-1} > A$$

Thus

$$A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)] > (N-1)A.$$

Hence $-(N-2)A > x + c(b_0)E[Z_1] - E[h(b_0, Y_1)] \geq x \geq 0$. This is a contradiction since $-(N-2)A$ is negative and cannot be greater than zero. Therefore we have the claim and then the plan $\pi := \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$. Moreover, one can see that

$$
\begin{aligned}
&G_N(x, \pi) \\
&= x - A + \sum_{n=1}^{N-1} \delta_n + \sum_{n=0}^{N-1} (c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) \\
&= x - A + \sum_{n=1}^{N-1} \left(\frac{A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]}{N-1} \right) \\
&- \sum_{n=1}^{N-1} (c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) + \sum_{n=0}^{N-1} (c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) \\
&= x - A + (A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]) \\
&- \sum_{n=1}^{N-1} (c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) + \sum_{n=0}^{N-1} (c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) \\
&= x - A + (A - x) \\
&- \sum_{n=0}^{N-1} (c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) + \sum_{n=0}^{N-1} (c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) = 0.
\end{aligned}
$$

Then $\mathcal{P}^*(N, U)$ is not empty. This proves (i).

Proof of (ii): By choosing

$$\delta_0 := 0, \ \delta_n := \frac{A - x - \sum_{k=0}^{N-1} (c(b_k)E[Z_{k+1}] - E[h(b_k, Y_{k+1})])}{N-1}, \ n = 1, 2, \ldots, N-1.$$

Hence $\delta_1 = \delta_2 = \cdots = \delta_{N-1}$. By the same proof as in case (i), we have $0 \leq \delta_n \leq A, \ n = 0, 1, 2, ..., N-1$. Thus the plan $\pi := \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$. Again we have $G_N(x, \pi) = 0$. Hence the plan $\pi := \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N, U)$ which this proves (ii).

Lemma 2.4. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$ and let $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N, U)$ be such that $\delta_1 = \delta_2 = \cdots = \delta_{N-1} > 0$. Then*

$$\Phi_N(x, \pi) < \Phi_N(x, ((b_0, 0), (b_1, 0), ..., (b_{N-1}, 0))).$$

Proof: Let $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N, U)$ be such that $\delta_1 = \delta_2 = \cdots = \delta_{N-1} > 0$. Hence $G_N(x, \pi) = 0$. It follows from equation (2.11) and the iid property of $Z_1, Z_2, \ldots, Z_N$ (Assumption 2.1) that

$$\begin{aligned}\Phi_N(x,\pi) &= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N \sum_{n=0}^{N-1} \left\{c^2(b_n) Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})]\right\} \\ &= (N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\}.\end{aligned}$$

Next, we consider

$$\begin{aligned}
&\Phi_N(x, ((b_0,0),(b_1,0),...,(b_{N-1},0))) \\
&= \alpha_N E\left(x - A + \sum_{n=0}^{N-1} c(b_n) Z_{n+1} - \sum_{n=0}^{N-1} h(b_n, Y_{n+1})\right)^2 \\
&= \alpha_N Var\left(x - A + \sum_{n=0}^{N-1} \{c(b_n) Z_{n+1} - h(b_n, Y_{n+1})\}\right) \\
&+ \alpha_N \left(x - A + \sum_{n=0}^{N-1} \{c(b_n) E[Z_{n+1}] - E[h(b_n, Y_{n+1})]\}\right)^2 \\
&= \alpha_N \sum_{n=0}^{N-1} \left\{c^2(b_n) Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})]\right\} \\
&+ \alpha_N \left(x - A + (N-1)\delta_1 - (N-1)\delta_1 + \sum_{n=0}^{N-1} \{c(b_n) E[Z_{n+1}] - E[h(b_n, Y_{n+1})]\}\right)^2 \\
&= \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\} + \alpha_N \left(G_N(x,\pi) - (N-1)\delta_1\right)^2 \\
&= \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\} + \alpha_N \left((N-1)\delta_1\right)^2 \\
&= \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\} + \alpha_N (N-1)(N-1)\delta_1^2.
\end{aligned}$$

Since $(N-1)\alpha_N > 1$, we obtain

$$\begin{aligned}
&\Phi_N(x, ((b_0,0),(b_1,0),...,(b_{N-1},0))) \\
&> \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\} + (N-1)\delta_1^2 = \Phi_N(x,\pi).
\end{aligned}$$

The proof is complete.

Theorem 2.5. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$. Then there exists $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in$*

$\mathcal{P}(N,U) - \mathcal{P}^*(N,U)$ *such that* $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$ *and*

$$\Phi_N(x,\tilde{\pi}) < \Phi_N(x,((\tilde{b}_0,0),(\tilde{b}_1,0),...,(\tilde{b}_{N-1},0))).$$

Proof: Let $\pi = \{(b_n,\delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N,U)$ be such that $\delta_1 = \delta_2 = \cdots = \delta_{N-1} > 0$. From equation (2.11) and the iid property of $Z_1, Z_2, ..., Z_N$ (Assumption 2.1), we get

$$\Phi_N(x,\pi) = (N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1]c^2(b_n) + Var[h(b_n,Y_{n+1})]\right\} \text{ and } G_N(x,\pi) = 0.$$

Choose a plan $\tilde{\pi} = \{(\tilde{b}_n,\tilde{\delta}_n)\}_{n=0}^{N-1}$ defined by

$$\tilde{\delta}_n = \frac{N-1}{N}\delta_n \ \text{ and } \tilde{b}_n = b_n,\ n = 0,1,2...,N-1.$$

Obviously, $\tilde{\delta}_0 = 0, \tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$ and

$$G_N(x,\tilde{\pi}) = x - A + \sum_{n=1}^{N-1}\tilde{\delta}_n + \sum_{n=0}^{N-1}(c(\tilde{b}_n)E[Z_{n+1}] - E[h(\tilde{b}_n,Y_{n+1})]) \neq 0.$$

Hence $\tilde{\pi} \in \mathcal{P}(N,U) - \mathcal{P}^*(N,U)$. Next, we shall show that

$$\Phi_N(x,\tilde{\pi}) < \Phi_N(x,((\tilde{b}_0,0),(\tilde{b}_1,0),...,(\tilde{b}_{N-1},0))).$$

From equation (2.11) and the iid property of $Z_1, Z_2, \ldots, Z_N$, we get

$$\begin{aligned}
&\Phi_N(x,\tilde{\pi})\\
&= \sum_{n=1}^{N-1}\tilde{\delta}_n^2 + \alpha_N \sum_{n=0}^{N-1}\left\{Var[Z_1]c^2(\tilde{b}_n) + Var[h(\tilde{b}_n,Y_{n+1})]\right\} + \alpha_N G_N^2(x,\tilde{\pi})\\
&= (N-1)\tilde{\delta}_1^2 + \alpha_N \sum_{n=0}^{N-1}\left\{Var[Z_1]c^2(\tilde{b}_n) + Var[h(\tilde{b}_n,Y_{n+1})]\right\}\\
&+ \alpha_N \left(x - A + (N-1)\tilde{\delta}_1 + \sum_{n=0}^{N-1}\left\{E[Z_1]c(\tilde{b}_n) - E[h(\tilde{b}_n,Y_{n+1})]\right\}\right)^2\\
&= (N-1)\frac{(N-1)^2}{N^2}\delta_1^2 + \alpha_N \sum_{n=0}^{N-1}\left\{Var[Z_1]c^2(b_n) + Var[h(b_n,Y_{n+1})]\right\}\\
&+ \alpha_N \left(x - A + (N-1)\frac{(N-1)}{N}\delta_1 + \frac{(N-1)}{N}\delta_1 - \frac{(N-1)}{N}\delta_1\right.\\
&\left. + \sum_{n=0}^{N-1}\left\{E[Z_1]c(b_n) - E[h(b_n,Y_{n+1})]\right\}\right)^2
\end{aligned}$$

$$= (N-1)\frac{(N-1)^2}{N^2}\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\}$$
$$+ \alpha_N \left(G_N(x,\pi) - \frac{(N-1)}{N}\delta_1\right)^2$$
$$= (N-1)\frac{(N-1)^2}{N^2}\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\}$$
$$+ \alpha_N \left(\frac{(N-1)}{N}\delta_1\right)^2$$
$$\leq (N-1)\frac{(N-1)^2}{N^2}\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\}$$
$$+ \left(\frac{(N-1)}{N}\delta_1\right)^2$$
$$= \left\{\frac{(N-1)^2}{N^2} + \frac{N-1}{N^2}\right\}(N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\}$$
$$= \frac{N-1}{N}(N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\}$$
$$< (N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \left\{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\right\} = \Phi_N(x,\pi).$$

By virtue of Lemma 2.4, we have

$$\Phi(x,\pi) < \Phi_N(x, ((b_0,0),(b_1,0),...,(b_{N-1},0))) = \Phi_N(x, ((\tilde{b}_0,0),(\tilde{b}_1,0),...,(\tilde{b}_{N-1},0))).$$

Thus

$$\Phi_N(x,\tilde{\pi}) < \Phi_N(x, ((\tilde{b}_0,0),(\tilde{b}_1,0),...,(\tilde{b}_{N-1},0))).$$

The proof is now complete.

Lemma 2.6. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$. If $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ is an optimal plan, then $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$.*

Proof: Let $\tilde{\pi} = \{\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ be an optimal plan. From equation (2.11), we have

$$\Phi_N(x,\tilde{\pi}) = \sum_{n=1}^{N-1} \tilde{\delta}_n^2 + \alpha_N \left\{\sum_{n=0}^{N-1}\{c^2(\tilde{b}_n)Var[Z_{n+1}] + Var[h(\tilde{b}_n, Y_{n+1})]\} + G_N^2(x,\tilde{\pi})\right\} \tag{2.18}$$

where $G_N(x,\tilde{\pi}) = x - A + \sum_{n=1}^{N-1}\tilde{\delta}_n + \sum_{n=0}^{N-1}(c(\tilde{b}_n)E[Z_{n+1}] - E[h(\tilde{b}_n, Y_{n+1})])$.

First, we show that $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1}$. We work by a contradiction. Assume that $\tilde{\delta}_i \neq \tilde{\delta}_{i+1}$ for some $i \in \{1,2,...,N-2\}$. Let a plan $\pi_0 = \{(b_n,\delta_n)\}_{n=0}^{N-1}$ be defined by

$$\delta_n = \begin{cases} \dfrac{\tilde{\delta}_i + \tilde{\delta}_{i+1}}{2} & , \ n = i, i+1 \\ \tilde{\delta}_n & , \ n \neq i, i+1 \end{cases} \quad \text{and} \quad b_n = \tilde{b}_n, \ n = 0,1,2,\ldots,N-1.$$

Obviously, $\pi_0 \in \mathcal{P}(N,U)$ and

$$\begin{aligned}
&G_N(x,\pi_0) \\
&= x - A + \sum_{n=1}^{N-1}\delta_n + \sum_{n=0}^{N-1}c(b_n)E[Z_{n+1}] - \sum_{n=0}^{N-1}E[h(b_n,Y_{n+1})] \\
&= x - A + \sum_{n=1,n\neq i,i+1}^{N-1}\delta_n + \delta_i + \delta_{i+1} + \sum_{n=0}^{N-1}c(b_n)E[Z_{n+1}] - \sum_{n=0}^{N-1}E[h(b_n,Y_{n+1})] \\
&= x - A + \sum_{n=1,n\neq i,i+1}^{N-1}\tilde{\delta}_n + 2(\frac{\tilde{\delta}_i + \tilde{\delta}_{i+1}}{2}) + \sum_{n=0}^{N-1}c(\tilde{b}_n)E[Z_{n+1}] - \sum_{n=0}^{N-1}E[h(\tilde{b}_n,Y_{n+1})] \\
&= x - A + \sum_{n=1}^{N-1}\tilde{\delta}_n + \sum_{n=0}^{N-1}c(\tilde{b}_n)E[Z_{n+1}] - \sum_{n=0}^{N-1}E[h(\tilde{b}_n,Y_{n+1})] \\
&= G_N(x,\tilde{\pi}). \qquad (2.19)
\end{aligned}$$

Moreover, $\sum_{n=1}^{N-1}\delta_n^2 < \sum_{n=1}^{N-1}\tilde{\delta}_n^2$. To see this, we note that since $\tilde{\delta}_i \neq \tilde{\delta}_{i+1}$, then $\tilde{\delta}_i$ and $\tilde{\delta}_{i+1}$can not be equal to zero at the same time. Hence $2\tilde{\delta}_i\tilde{\delta}_{i+1} < \tilde{\delta}_i^2 + \tilde{\delta}_{i+1}^2$. Thus

$$\begin{aligned}
&\sum_{n=1}^{N-1}\delta_n^2 \\
&= \sum_{n=1,n\neq i,i+1}^{N-1}\delta_n^2 + \delta_i^2 + \delta_{i+1}^2 \\
&= \sum_{n=1,n\neq i,i+1}^{N-1}\tilde{\delta}_n^2 + 2\left(\frac{\tilde{\delta}_i + \tilde{\delta}_{i+1}}{2}\right)^2 \\
&= \sum_{n=1,n\neq i,i+1}^{N-1}\tilde{\delta}_n^2 + \frac{1}{2}(\tilde{\delta}_i^2 + 2\tilde{\delta}_i\tilde{\delta}_{i+1} + \tilde{\delta}_{i+1}^2) \\
&< \sum_{n=1,n\neq i,i+1}^{N-1}\tilde{\delta}_n^2 + \frac{1}{2}(\tilde{\delta}_i^2 + \tilde{\delta}_i^2 + \tilde{\delta}_{i+1}^2 + \tilde{\delta}_{i+1}^2)
\end{aligned}$$

$$= \sum_{n=1, n\neq i, i+1}^{N-1} \tilde{\delta}_n^2 + \tilde{\delta}_i^2 + \tilde{\delta}_{i+1}^2$$

$$= \sum_{n=1}^{N-1} \tilde{\delta}_n^2. \tag{2.20}$$

It follows from inequality (2.18), (2.19) and (2.20) that $\Phi_N(x, \pi_0) < \Phi_N(x, \tilde{\pi})$ which is a contradiction. Hence $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1}$. Next, we try show that $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$. Assume that there exists $\tilde{\delta}_m = 0$ for some $m \in \{1, 2, \ldots, N-1\}$. Then $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} = 0$. By Theorem 2.5, there exists $\pi' = \{(b'_n, \delta'_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ such that $\delta'_1 = \delta'_2 = \cdots = \delta'_{N-1} > 0$ and $\Phi_N(x, \pi') < \Phi_N(x, \tilde{\pi})$. This contradicts the optimal plan of $\tilde{\pi}$ and then the proof is complete.

Next, we prove the existence of $\min_{\pi \in \mathcal{P}(N,U)} \Phi_N(x, \pi)$. We note that $\mathcal{P}(N, U)$ is a compact subset of the Euclidean space $(R^2)^N$. We can easily see this by utilizing the compactness $U = [\underline{b}, \overline{b}] \times [0, A]$ in R^2. What remaining is to prove the continuity of $\Phi_N(x, \pi)$ on $\mathcal{P}(N, U)$.

Lemma 2.7. *The function $c(b)$ and $h(b, y)$ are continuous on $[\underline{b}, \overline{b}]$ for each y.*

Proof: First, we show that $h(b, y)$ is continuous on $[\underline{b}, \overline{b}]$ for each y.

Let $y \geq 0$ be fixed and $b_0 \in [\underline{b}, \overline{b}]$ be arbitrary. We proof by cases:

Case 1. $h(b, y) = by$. We get $\lim_{b \to b_0} h(b, y) = \lim_{b \to b_0} by = b_0 y = h(b_0, y)$.

Case 2. $h(b, y) = \min\{b, y\}$. We get

$$\lim_{b \to b_0} h(b, y) = \lim_{b \to b_0} \min\{b, y\} = \lim_{b \to b_0} \frac{1}{2}(b + y - |b - y|) = \frac{1}{2}(b_0 + y - |b_0 - y|) = h(b_0, y).$$

From cases 1 and 2, we have $h(b, y)$ is continuous on $[\underline{b}, \overline{b}]$.

Next, we show that $c(b)$ is continuous on $[\underline{b}, \overline{b}]$. Note that

$$\begin{aligned} c(b) &= c_0 - (1+\theta_1)\frac{E[Y - h(b,Y)]}{E[Z]} \\ &= c_0 - (1+\theta_1)\frac{E[Y]}{E[Z]} + (1+\theta_1)\frac{E[h(b,Y)]}{E[Z]} \\ &= c_0 - (1+\theta_1)\frac{E[Y]}{E[Z]} + \frac{1+\theta_1}{E[Z]}\int_\Omega h(b,Y(\omega))dP(\omega) \\ &= c_0 - (1+\theta_1)\frac{E[Y]}{E[Z]} + \frac{1+\theta_1}{E[Z]}\int_{-\infty}^{\infty} h(b,y)f_Y(y)dy \end{aligned} \tag{2.21}$$

where f_Y is the density function of Y.

From equation (2.21) it suffices to show that $\int_{-\infty}^{\infty} h(b,y)f_Y(y)dy$ is continuous on $[\underline{b},\overline{b}]$. Let $b_0 \in [\underline{b},\overline{b}]$ be arbitrary and $\tilde{g}$ be a function on R defined by

$$\tilde{g}(y) = yf_Y(y).$$

By Assumption 2.1, we have

$$\int_{-\infty}^{\infty} \tilde{g}(y)dy = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_\Omega Y(\omega))dP(\omega) = E[Y] < \infty.$$

Since $h(b,y) \leq y$, then $h(b,y)f_Y(y) \leq yf_Y(y) = \tilde{g}(y)$. By Lebesgue Dominated Convergence Theorem (F. Jones, 1993, page 153) and the continuity of $h(b,y)$ on $[\underline{b},\overline{b}]$, we get

$$\lim_{b\to b_0}\int_{-\infty}^{\infty} h(b,y)f_Y(y)dy = \int_{-\infty}^{\infty} \lim_{b\to b_0} h(b,y)f_Y(y)dy = \int_{-\infty}^{\infty} h(b_0,y)f_Y(y)dy.$$

Hence $\int_{-\infty}^{\infty} h(b,y)f_Y(y)dy$ is continuous on $[\underline{b},\overline{b}]$ and so $c(b)$ is continuous on $[\underline{b},\overline{b}]$.

Lemma 2.8. *The mapping $F : U \mapsto R$ define by*

$$F(b,\delta) = \delta^2 + \alpha_N E\left[\frac{x-A}{N} + \delta + c(b)Z_1 - h(b,Y_1)\right]^2$$

is continuous on U.

Proof: By Assumption 2.1, the random variables Y_1 and Z_1 are independent, then we have

$$\begin{aligned}F(b,\delta) &= \delta^2 + \alpha_N E\left[\frac{x-A}{N} + \delta + c(b)Z_1 - h(b,Y_1)\right]^2 \\ &= \int_{\Omega}\left(\delta^2 + \alpha_N\left[\frac{x-A}{N} + \delta + c(b)Z_1(\omega) - h(b,Y_1(\omega))\right]^2\right)dP(\omega) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\delta^2 + \alpha_N\left[\frac{x-A}{N} + \delta + c(b)z - h(b,y)\right]^2\right) f_{Y_1}(y)f_{Z_1}(z)dydz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f((y,z),(b,\delta))dydz \qquad (2.22)\end{aligned}$$

where $f((y,z),(b,\delta)) = \left(\delta^2 + \alpha_N\left[\frac{x-A}{N} + \delta + c(b)z - h(b,y)\right]^2\right) f_{Y_1}(y)f_{Z_1}(z)$ and, f_{Y_1} and f_{Z_1} are the density function of Y_1 and Z_1 respectively.

Let $\hat{g}$ be a function on R^2 defined by

$$\hat{g}(y,z) = (5A^2 + 2A(c_0 z + y) + c_0^2 z^2 + y^2)f_{Y_1}(y)f_{Z_1}(z).$$

Now, we consider

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{g}(y,z))dydz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(5A^2 + 2A(c_0 z + y) + c_0^2 z^2 + y^2)f_{Y_1}(y)f_{Z_1}(z)dydz \\ &= 5A^2 + 2A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(c_0 z + y)f_{Y_1}(y)f_{Z_1}(z)dydz + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(c_0^2 z^2 + y^2)f_{Y_1}(y)f_{Z_1}(z)dydz \\ &= 5A^2 + 2A\int_{\Omega}(c_0 Z_1(\omega) + Y_1(\omega))dP(\omega) + \int_{\Omega}(c_0^2 Z_1^2(\omega) + Y_1^2(\omega))d(\omega) \\ &= 5A^2 + 2A(c_0 E[Z_1] + E[Y_1]) + c_0^2 E[Z_1^2] + E[Y_1^2].\end{aligned}$$

Since Y_1 and Z_1 have finite variances (by Assumption 2.1), we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{g}(y,z)dydz < \infty. \qquad (2.23)$$

From equation (2.9) and inequality (2.22), we have

$$
\begin{aligned}
&f((y,z),(b,\delta))\\
&= \left(\delta^2 + \alpha_N \left[\frac{x-A}{N} + \delta + c(b)z - h(b,y)\right]^2\right) f_{Y_1}(y) f_{Z_1}(z)\\
&\leq \left(\delta^2 + \left[\frac{x-A}{N} + \delta + c(b)z - h(b,y)\right]^2\right) f_{Y_1}(y) f_{Z_1}(z)\\
&= \left(\delta^2 + \left[\frac{x-A}{N} + \delta\right]^2\right.\\
&\left. +2\left[\frac{x-A}{N} + \delta\right][c(b)z - h(b,y)] + [c(b)z - h(b,y)]^2\right) f_{Y_1}(y) f_{Z_1}(z)\\
&\leq \left(\delta^2 + \left[\frac{x-A}{N} + \delta\right]^2\right.\\
&\left. +2\left[\frac{x-A}{N} + \delta\right][c(b)z + h(b,y)] + [c(b)z]^2 + [h(b,y)]^2\right) f_{Y_1}(y) f_{Z_1}(z)\\
&= (A^2 + 4A^2 + 2A[c_0 z + y] + c_0^2 z^2 + y^2) f_{Y_1}(y) f_{Z_1}(z)\\
&= \left(5A^2 + 2A[c_0 z + y] + c_0^2 z^2 + y^2\right) f_{Y_1}(y) f_{Z_1}(z) = \hat{g}(y,z). \qquad (2.24)
\end{aligned}
$$

By Lemma 2.7, for each fixed y and z, the function $f((y,z),(b,\delta))$ is continuous in the variable (b,δ) on U. So, by Lebesgue Dominated Convergence Theorem (F. Jones, 1993, page 153), we obtain $F(b,\delta)$ is continuous on U.

Theorem 2.9. *Let $x \in S$ be fixed and A be the target at the time horizon N, then $\Phi_N(x,\pi)$ is continuous on $\mathcal{P}(N,U)$.*

Proof : From equation (2.11) and iid property of Y_n and Z_n, we have

$$
\begin{aligned}
&\Phi_N(x,\pi)\\
&= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N E\left[x - A + \sum_{n=1}^{N-1}\delta_n + \sum_{n=0}^{N-1} c(b_n) Z_{n+1} - \sum_{n=0}^{N-1} h(b_n, Y_{n+1})\right]^2\\
&= \sum_{n=0}^{N-1} \delta_n^2 + \alpha_N E\left[x - A + \sum_{n=0}^{N-1}\delta_n + \sum_{n=0}^{N-1} c(b_n) Z_{n+1} - \sum_{n=0}^{N-1} h(b_n, Y_{n+1})\right]^2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\sum_{n=0}^{N-1}\delta_n^2 + \alpha_N\left[x - A + \sum_{n=0}^{N-1}\delta_n + \sum_{n=0}^{N-1} c(b_n) z_{n+1}\right.\right.\\
&\left.\left. - \sum_{n=0}^{N-1} h(b_n, y_{n+1})\right]^2\right) f_{Y_1}(y_1) f_{Z_1}(z_1) \ldots f_{Y_1}(y_N) f_{Z_1}(z_N) dy_1 dz_1 \ldots dy_N dz_N
\end{aligned}
$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f((y_1,\ldots,y_N,z_1,\ldots,z_N,\pi)dy_1dz_1\ldots dy_Ndz_N$$

where $f((y_1,\ldots,y_N,z_1,\ldots,z_N)),\pi)$

$$= \left(\sum_{n=0}^{N-1}\delta_n^2+\alpha_N\left[x-A+\sum_{n=0}^{N-1}\delta_n+\sum_{n=0}^{N-1}c(b_n)z_{n+1}\right.\right.$$
$$\left.\left.-\sum_{n=0}^{N-1}h(b_n,y_{n+1})\right]^2\right) f_{Y_1}(y_1)f_{Z_1}(z_1)\ldots f_{Y_1}(y_N)f_{Z_1}(z_N)$$

and, f_{Y_1} and f_{Z_1} are the density function of Y_1 and Z_1 respectively.

Let g^* be a function on R^{2N} defined by

$$g^*(y_1,\ldots,y_N,z_1,\ldots,z_N)$$
$$= \left((N+N^2)A^2+2NA\sum_{n=1}^{N}(c_0z_n+y_n)\right.$$
$$\left.+\sum_{n=1}^{N}(c_0^2z_n^2+y_n^2)+\sum_{m,n=1:n\neq m}^{N}(c_0^2z_mz_n+y_my_n)\right) f_{Y_1}(y_1)f_{Z_1}(z_1)\ldots f_{Y_1}(y_N)f_{Z_1}(z_N).$$

Since the sequences $\{Y_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$ are iid and have finite variances, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(y_1,\ldots,y_N,z_1,\ldots,z_N)dy_1dz_1\ldots dy_Ndz_N<\infty.$$

By the same proof as in inequality (2.24), one gets

$$f((y_1,\ldots,y_N,z_1,\ldots,z_N),\pi)\leq g^*(y_1,\ldots,y_N,z_1,\ldots,z_N.)$$

Thus, by Lebesgue Dominated Convergence Theorem, we obtain Theorem 2.9.

Theorem 2.10. *Let $x\in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N>1$. Then there exists an optimal plan $\tilde{\pi}=\{(\tilde{b}_n,\tilde{\delta}_n)\}_{n=0}^{N-1}\in\mathcal{P}(N,U)$ such that $\tilde{\delta}_1=\tilde{\delta}_2=\cdots=\tilde{\delta}_{N-1}>0$ and*

$$V_N(x)=(N-1)\tilde{\delta}_1^2+\alpha_N\left\{\sum_{n=0}^{N-1}Var[Z_1]c^2(\tilde{b}_n)+\sum_{n=0}^{N-1}Var[h(\tilde{b}_n,Y_{n+1})]+G_N^2(x,\tilde{\pi})\right\}$$

where $G_N(x,\tilde{\pi})=x-A+(N-1)\tilde{\delta}_1+\sum_{n=0}^{N-1}E[Z_1]c(\tilde{b}_n)-\sum_{n=0}^{N-1}E[h(\tilde{b}_n,Y_{n+1})].$

Proof: From Theorem 2.9, we have $\Phi_N(x,\pi)$ is continuous on $\mathcal{P}(N,U)$. Since $\mathcal{P}(N,U)$ is a compact subset of $(R^2)^N$, then there exists a plan $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ such that $\inf_{\pi\in\mathcal{P}(N,U)} \Phi_N(x,\pi) = \Phi_N(x,\tilde{\pi})$. Hence $\tilde{\pi}$ is an optimal plan. By Lemma 2.6, we have $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$. From equation (2.7) and (2.11), we have

$$V_N(x) = (N-1)\tilde{\delta}_1^2 + \alpha_N \left\{ \sum_{n=0}^{N-1} Var[Z_1]c^2(\tilde{b}_n) + \sum_{n=0}^{N-1} Var[h(\tilde{b}_n, Y_{n+1})] + G_N^2(x,\tilde{\pi}) \right\}$$

where $G_N(x,\tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + \sum_{n=0}^{N-1} E[Z_1]c(\tilde{b}_n) - \sum_{n=0}^{N-1} E[h(\tilde{b}_n, Y_{n+1})]$. This proves Theorem 2.10.

Corollary 2.11. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$ and $h(b,y)$ is the proportional reinsurance. Then there exists an optimal plan $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ such that $\tilde{\pi}$ is stationary and*

$$V_N(x) = (N-1)\tilde{\delta}_1^2 + \alpha_N \left\{ Nc^2(\tilde{b}_0)Var[Z_1] + N\tilde{b}_0^2 Var[Y_1] + G_N^2(x,\tilde{\pi}) \right\} \quad (2.25)$$

where $G_N(x,\tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + Nc(\tilde{b}_0)E[Z_1] - N\tilde{b}_0 E[Y_1]$.

Proof: From Theorem 2.10, there exists an optimal plan $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ such that $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$ and

$$\Phi_N(x,\tilde{\pi}) = (N-1)\tilde{\delta}_1^2 + \alpha_N \left\{ Var[Z_1] \sum_{n=0}^{N-1} c^2(\tilde{b}_n) + Var[Y_1] \sum_{n=0}^{N-1} \tilde{b}_n^2 + G_N^2(x,\tilde{\pi}) \right\} \quad (2.26)$$

where $G_N(x,\tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + E[Z_1] \sum_{n=0}^{N-1} c(\tilde{b}_n) - E[Y_1] \sum_{n=0}^{N-1} \tilde{b}_n$.

Next, we shall show that $\tilde{\pi}$ is a stationary plan. Since $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1}$, we are left to show that $\tilde{b}_0 = \tilde{b}_1 = \cdots = \tilde{b}_{N-1}$. We work by a contradiction. Assume that $\tilde{b}_i \neq \tilde{b}_{i+1}$ for some $i \in \{0,1,2,...,N-2\}$. Let a plan $\pi_0 = \{(b_n, \delta_n)\}_{n=0}^{N-1}$ be defined by

$$b_n = \begin{cases} \dfrac{\tilde{b}_i + \tilde{b}_{i+1}}{2} & , n = i, i+1 \\ \\ \tilde{b}_n & , n \neq i, i+1 \end{cases} \quad \text{and } \delta_n = \tilde{\delta}_n \text{ for } n = 0, 1, 2, ..., N-1.$$

Obviously, $\pi_0 \in \mathcal{P}(N, U)$ and

$$\begin{aligned}
& G_N(x, \pi_0) \\
&= x - A + (N-1)\delta_1 + E[Z_1] \sum_{n=0}^{N-1} c(b_n) - E[Y_1] \sum_{n=0}^{N-1} b_n \\
&= x - A + (N-1)\delta_1 + E[Z_1] \left\{ \sum_{n=0, n\neq i, i+1}^{N-1} c(b_n) + c(b_i) + c(b_{i+1}) \right\} \\
&- E[Y_1] \left\{ \sum_{n=0, n\neq i, i+1}^{N-1} b_n + b_i + b_{i+1} \right\} \\
&= x - A + (N-1)\tilde{\delta}_1 + E[Z_1] \left\{ \sum_{n=0, n\neq i, i+1}^{N-1} c(\tilde{b}_n) + 2c\left(\frac{\tilde{b}_i + \tilde{b}_{i+1}}{2} \right) \right\} \\
&- E[Y_1] \left\{ \sum_{n=0, n\neq i, i+1}^{N-1} \tilde{b}_n + 2\left(\frac{\tilde{b}_i + \tilde{b}_{i+1}}{2} \right) \right\} \\
&= x - A + (N-1)\tilde{\delta}_1 \\
&+ E[Z_1] \left\{ \sum_{n=0, n\neq i, i+1}^{N-1} c(\tilde{b}_n) + 2\left\{ c_0 - (1+\theta_1)\left(1 - \frac{\tilde{b}_i + \tilde{b}_{i+1}}{2} \right) \frac{E[Y_1]}{E[Z_1]} \right\} \right\} \\
&- E[Y_1] \sum_{n=0}^{N-1} \tilde{b}_n \\
&= x - A + (N-1)\tilde{\delta}_1 \\
&+ E[Z_1] \left\{ \sum_{n=0, n\neq i, i+1}^{N-1} c(\tilde{b}_n) + \left\{ 2c_0 - (1+\theta_1)(2 - \tilde{b}_i - \tilde{b}_{i+1}) \frac{E[Y_1]}{E[Z_1]} \right\} \right\} \\
&- E[Y_1] \sum_{n=0}^{N-1} \tilde{b}_n \\
&= x - A + (N-1)\tilde{\delta}_1 \\
&+ E[Z_1] \left\{ \sum_{n=0, n\neq i, i+1}^{N-1} c(\tilde{b}_n) + \left\{ 2c_0 - (1+\theta_1)(1 - \tilde{b}_i + 1 - \tilde{b}_{i+1}) \frac{E[Y_1]}{E[Z_1]} \right\} \right\} \\
&- E[Y_1] \sum_{n=0}^{N-1} \tilde{b}_n \\
&= x - A + (N-1)\tilde{\delta}_1 \\
&+ E[Z_1] \left\{ \sum_{n=0, n\neq i, i+1}^{N-1} c(\tilde{b}_n) + \left\{ 2c_0 - (1+\theta_1)(1 - \tilde{b}_i) \frac{E[Y_1]}{E[Z_1]} \right. \right.
\end{aligned}$$

$$-(1+\theta_1)(1-\tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\Bigg\}\Bigg\} - E[Y_1]\sum_{n=0}^{N-1}\tilde{b}_n$$

$$= x - A + (N-1)\tilde{\delta}_1 + E[Z_1]\left\{\sum_{n=0,n\neq i,i+1}^{N-1} c(\tilde{b}_n) + c(\tilde{b}_i) + c(\tilde{b}_{i+1})\right\} - E[Y_1]\sum_{n=0}^{N-1}\tilde{b}_n$$

$$= x - A + (N-1)\tilde{\delta}_1 + E[Z_1]\sum_{n=0}^{N-1} c(\tilde{b}_n) - E[Y_1]\sum_{n=0}^{N-1}\tilde{b}_n$$

$$= G_N(x,\tilde{\pi}). \tag{2.27}$$

Moreover, we have $\sum_{n=0}^{N-1} b_n^2 < \sum_{n=0}^{N-1}\tilde{b}_n^2$ and $\sum_{n=0}^{N-1} c^2(b_n) < \sum_{n=0}^{N-1} c^2(\tilde{b}_n)$. To see this, we note that since $\tilde{b}_i \neq \tilde{b}_{i+1}$ (i.e., $\tilde{b}_i$ and $\tilde{b}_{i+1}$ can not be equal to zero at the same time) and $c(\tilde{b}_i), c(\tilde{b}_{i+1}) > 0$, then $2\tilde{b}_i\tilde{b}_{i+1} < \tilde{b}_i^2 + \tilde{b}_{i+1}^2$ and $2c(\tilde{b}_i)c(\tilde{b}_{i+1}) < c^2(\tilde{b}_i) + c^2(\tilde{b}_{i+1})$ respectively. This implies that

$$\sum_{n=0}^{N-1} b_n^2$$

$$= \sum_{n=0,n\neq i,i+1}^{N-1} b_n^2 + b_i^2 + b_{i+1}^2$$

$$= \sum_{n=0,n\neq i,i+1}^{N-1} \tilde{b}_n^2 + 2\left(\frac{\tilde{b}_i + \tilde{b}_{i+1}}{2}\right)^2$$

$$= \sum_{n=0,n\neq i,i+1}^{N-1} \tilde{b}_n^2 + \frac{1}{2}(\tilde{b}_i^2 + 2\tilde{b}_i\tilde{b}_{i+1} + \tilde{b}_{i+1}^2)$$

$$< \sum_{n=0,n\neq i,i+1}^{N-1} \tilde{b}_n^2 + \frac{1}{2}(\tilde{b}_i^2 + \tilde{b}_i^2 + \tilde{b}_{i+1}^2 + \tilde{b}_{i+1}^2)$$

$$= \sum_{n=0,n\neq i,i+1}^{N-1} \tilde{b}_n^2 + \tilde{b}_i^2 + \tilde{b}_{i+1}^2$$

$$= \sum_{n=0}^{N-1} \tilde{b}_n^2 \tag{2.28}$$

and

$$\sum_{n=0}^{N-1} c^2(b_n)$$

$$= \sum_{n=0,n\neq i,i+1}^{N-1} c^2(b_n) + c^2(b_i) + c^2(b_{i+1})$$

$$= \sum_{n=0,n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) + 2c^2\left(\frac{\tilde{b}_i + \tilde{b}_{i+1}}{2}\right)$$

$$
\begin{aligned}
&= \sum_{n=0, n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) + 2\left(c_0 - (1+\theta_1)(1 - \frac{\tilde{b}_i + \tilde{b}_{i+1}}{2})\frac{E[Y_1]}{E[Z_1]}\right)^2 \\
&= \sum_{n=0, n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(2c_0 - (1+\theta_1)(2 - \tilde{b}_i - \tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right)^2 \\
&= \sum_{n=0, n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(2c_0 - (1+\theta_1)(1 - \tilde{b}_i + 1 - \tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right)^2 \\
&= \sum_{n=0, n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) \\
&+ \frac{1}{2}\left(2c_0 - (1+\theta_1)(1 - \tilde{b}_i)\frac{E[Y_1]}{E[Z_1]} - (1+\theta_1)(1 - \tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right)^2 \\
&= \sum_{n=0, n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(c(\tilde{b}_i) + c(\tilde{b}_{i+1})\right)^2 \\
&= \sum_{n=0, n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(c^2(\tilde{b}_i) + 2c(\tilde{b}_i)c(\tilde{b}_{i+1}) + c^2(\tilde{b}_{i+1})\right) \\
&< \sum_{n=0, n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(c^2(\tilde{b}_i) + c^2(\tilde{b}_i) + c^2(\tilde{b}_{i+1}) + c^2(\tilde{b}_{i+1})\right) \\
&= \sum_{n=0, n\neq i,i+1}^{N-1} c^2(\tilde{b}_n) + c^2(\tilde{b}_i) + c^2(\tilde{b}_{i+1}) \\
&= \sum_{n=0}^{N-1} c^2(\tilde{b}_n). \qquad (2.29)
\end{aligned}
$$

It follows from inequality (2.26), (2.27), (2.28) and (2.29) that $\Phi_N(x, \pi_0) < \Phi_N(x, \tilde{\pi})$ which contradicts the optimal plan of $\tilde{\pi}$ and then the proof is complete.

2.4 Example

In this section, we give an example of an optimal plan which can make a surplus approaching to the target A at a time horizon N. We begin by assuming that $h(b_0, y)$ is the proportional reinsurance with retention level b_0, an initial capital $x = 10$, a time horizon $N = 100$, the target $A = 60$ and $\alpha_N = 0.05,\ 0.10,\ 0.20$. We consider the safety loading of the insurer and reinsurer in three cases as follows: (a). $\theta_0 = 0.20, \theta_1 = 0.25$ (b). $\theta_0 = 0.25, \theta_1 = 0.25$ and (c). $\theta_0 = 0.30, \theta_1 = 0.25$.

Suppose that the error of this estimate is $e = 0.1$. By Corollary 2.11, we know that the optimal plan π is stationary, thus it suffices to find b_0 and δ_1. We assume that $\{Y_n\}_{n=1}^{100}$ is a sequence of claims with iid exponential $Exp(1.2)$ and $\{Z_n\}_{n=1}^{100}$ is a sequence of inter-arrival times with iid exponential $Exp(1)$. We solved for b_0, δ_1 under the conditions that $|E[X_N] - A| \leq 0.1$ (or equivalently, $|G_N(x,\pi)| \leq 0.1$) and $\Phi_N(x,\pi)$ is minimum. We get several optimal plans which satisfy the given parameters and the error $e = 0.1$.

Case (a): If $\theta_0 = 0.20$ and $\theta_1 = 0.25$, the optimal plan is as follows:

$$\pi = \{(b_0, \delta_0) = (0.88, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.88, 0.361)\} \text{ for } \alpha_N = 0.05.$$

$$\pi = \{(b_0, \delta_0) = (0.51, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.51, 0.439)\} \text{ for } \alpha_N = 0.10.$$

$$\pi = \{(b_0, \delta_0) = (0.31, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.31, 0.481)\} \text{ for } \alpha_N = 0.20.$$

Case (b): If $\theta_0 = 0.25$ and $\theta_1 = 0.25$, the optimal plan is as follows:

$$\pi = \{(b_0, \delta_0) = (0.78, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.78, 0.340)\} \text{ for } \alpha_N = 0.05.$$

$$\pi = \{(b_0, \delta_0) = (0.49, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.49, 0.401)\} \text{ for } \alpha_N = 0.10.$$

$$\pi = \{(b_0, \delta_0) = (0.25, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.25, 0.452)\} \text{ for } \alpha_N = 0.20.$$

Case (c): If $\theta_0 = 0.30$ and $\theta_1 = 0.25$, the optimal plan is as follows:

$$\pi = \{(b_0, \delta_0) = (0.68, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.68, 0.319)\} \text{ for } \alpha_N = 0.05.$$

$$\pi = \{(b_0, \delta_0) = (0.40, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.40, 0.378)\} \text{ for } \alpha_N = 0.10.$$

$$\pi = \{(b_0, \delta_0) = (0.21, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.21, 0.418)\} \text{ for } \alpha_N = 0.20.$$

Note that, $V_N(x)$ can be calculated by putting b_0, δ_1 and $c(b_0)$, ($c(b_0) := \theta_0 - \theta_1 + (1+\theta_1)b_0$) into equation (2.25). The value of these parameters for each case was shown in Table 2.1:

Table 2.1 The Value of b_0, δ_1 and V_N

	Case (a)	Case (b)	Case (c)
	$\theta_0 = 0.20, \theta_1 = 0.25$	$\theta_0 = 0.25, \theta_1 = 0.25$	$\theta_0 = 0.30, \theta_1 = 0.25$
	$b_0 : \delta_1 : V_N$	$b_0 : \delta_1 : V_N$	$b_0 : \delta_1 : V_N$
$\alpha_N = 0.05$	0.88:0.361:19.4192	0.78:0.340:16.8581	0.68:0.319:14.4928
$\alpha_N = 0.1$	0.51:0.439:23.2832	0.49:0.401:20.1928	0.40:0.378:17.3579
$\alpha_N = 0.2$	0.31:0.481:25.8231	0.25:0.452:22.4509	0.21:0.418:19.2677

The numerical results in Table 2.1 show a minimum value $V_N(10) = 19.4192$ satisfy an optimal plan $\pi = \{(b_0, \delta_0) = (0.88, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.88, 0.361)\}$ for $\alpha = 0.05$, $\theta_0 = 0.20$ and $\theta_1 = 0.25$ etc.

Finally, suppose that the time horizon N and the safety loading of the reinsurer θ_1 are fixed. By virtue of Remark 2.3, for each fixed retention level, we can find a shareholder input which satisfies the condition that $|E[X_N] - A| \leq e$ for a given error e. Hence, if the error of estimate is decreased we can still find an optimal plan according to Corollary 2.11.

CHAPTER III

MINIMUM INITIAL CAPITAL PROBLEM

In this chapter, we studied a minimum initial capital problem of the discrete-time surplus process (2.1) when the inter-arrival times $Z_n = 1, n \in \{1, 2, 3, \ldots\}$ and the claim can be controled by reinsurance. Moreover, we consider the relationship between ruin probability and initial capital.

We assume that all processes are defined in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

3.1 Model Description

Now, we recall the discrete-time surplus process (2.1) in the situation that the possible insolvency can occur only at claim arrival times $T_n = n, n \in \{1, 2, 3, \ldots\}$. Hence

$$Z_n = 1, n \in \{1, 2, 3, \ldots\}.$$

Therefore, the surplus process (2.1) can modify as follows:

$$\begin{aligned} X_n &= x + c_0 \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n} Y_i \\ &= x + nc_0 - \sum_{i=1}^{n} Y_i \end{aligned} \tag{3.1}$$

where $X_0 = x \geq 0$ and $c_0 > 0$ are the initial capital and the premium rate for one unit time respectively as mentioned in the previous chapter.

The general approach for studying the ruin probability in the discrete-time surplus process (3.1) is through the so-called $Gerber - Shiu$ *discounted penalty* $function$; as found in, Pavlovao and Willmot (2004), Dickson (2005) and Li

(2005a-b). These articles study the ruin probability as a function of the initial capital x.

In the opposite direction, Sattayatham, Sangaroon and Klongdee (To be published) considered the initial capital for the discrete-time surplus process (3.1) via a function of the ruin probability.

In this chapter, we extend the model (3.1) when the claims can be controlled by reinsurance.

Let $\{X_n\}_{n\geq 0}$ be the surplus process as mentioned in equation (3.1) which can be controlled by choosing a retention level $b \in [\underline{b}, \overline{b}]$ of reinsurance for one period as mentioned in the previous chapter. Moreover, by the *net profit condition*, the premium rate for one unit time c_0 and the net income rate $c(b)$ satisfy the following:

$$c_0 > \frac{E[Y]}{E[Z]} \quad \text{and} \quad c(b) > \frac{E[h(b,Y)]}{E[Z]}, \tag{3.2}$$

where Y is claim size and Z is inter-arrival time.

For each $n \in \{1, 2, 3, \ldots\}$, let b_{n-1} be a retention level (control action) at the time T_{n-1}. Therefore, we can modify the surplus process (3.1) to be the following:

$$X_n = x + \sum_{i=1}^{n} c(b_{i-1}) - \sum_{i=1}^{n} h(b_{i-1}, Y_i) \tag{3.3}$$

where $X_0 = x$.

We see that the process $\{X_n\}_{n\geq 0}$ is driven by the sequence of retention levels (control actions) $\{b_{n-1}\}_{n\geq 1}$ and the sequence of claims $\{Y_n\}_{n\geq 1}$. So, we make the following assumption:

Assumption 3.1. Independence Assumption (IA)

The sequence of claims $\{Y_n\}_{n\geq 1}$ is a sequence of independent and identically distributed (iid) random variables.

From Assumption IA, it follows that $\{h(b_{n-1}, Y_n)\}_{n\geq 1}$ is an independent sequence.

Definition 3.1. *Let* $N \in \{1, 2, 3, \ldots\}$ *be a time horizon (number of periods). A plan for the time* N *is a (finite) sequence* $\pi = \{b_{n-1}\}_{n=1}^{N}$ *of* $b_{n-1} \in [\underline{b}, \overline{b}]$ *for* $n = 1, 2, 3, \ldots, N$. *A set of all plans for the time horizon* N *over control space* $[\underline{b}, \overline{b}]$ *is denoted by* $\mathcal{P}(N, [\underline{b}, \overline{b}])$. *A plan* $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ *is said to be stationary, if* $b_0 = b_1 = \cdots = b_{N-1}$.

3.2 Main Results

In this section, we consider the finite-time ruin probabilities of the discrete-time surplus process as in equation (3.3) where the sequence of claims $\{Y_N\}_{n\geq 1}$ satisfies Assumption IA. Let F_{Y_1} be the distribution function of Y_1, i.e.,

$$F_{Y_1}(y) = P(Y_1 \leq y).$$

Let $N \in \{1, 2, 3, \ldots\}$ be a time horizon and $x \geq 0$ be an initial capital. The *survival probability* at a time $n \in \{1, 2, 3, \ldots, N\}$ is defined by

$$\varphi_n(x, \pi) := P(X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, \ldots, X_n \geq 0 | X_0 = x) \tag{3.4}$$

where $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$. Moreover, the *ruin probability* at a time $n \in \{1, 2, 3, \ldots, N\}$ is defined by

$$\Phi_n(x, \pi) = 1 - \varphi_n(x, \pi). \tag{3.5}$$

Definition 3.2. *Let* $\{X_n\}_{n\geq 0}$ *be the surplus process as in equation (3.3), driven by the sequence of control actions* $\{b_{n-1}\}_{n\geq 1}$ *and the sequence of claims* $\{Y_n\}_{n\geq 1}$. *Let* $\{c(b_{n-1})\}_{n\geq 1}$ *be a sequence of net income rates and* $x \geq 0$ *be an initial capital. For each the time horizon* $N \in \{1, 2, 3, \ldots\}$, *let* $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ *and* $\alpha \in (0, 1)$. *If* $\Phi_N(x, \pi) \leq \alpha$, *then* x *is called an* **acceptable initial capital** *corresponding to* $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$. *Particularly, if*

$$x^* = \min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}$$

exists, x^* is called the **minimum initial capital** *corresponding to* $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$ *and is written as*

$$x^* := \mathbf{MIC}(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1}).$$

3.2.1 Ruin and Survival Probability

We defined the *total claim process* by

$$S_n := h(b_0, Y_1) + h(b_1, Y_2) + \cdots + h(b_{n-1}, Y_n)$$

for all $n \in \{1, 2, 3, \ldots\}$. The survival probability at the time horizon N as mentioned in equation (3.4) can be expressed as follows:

$$\begin{aligned} \varphi_N(x, \pi) &= P\left(S_1 \leq x + c(b_0), S_2 \leq x + \sum_{n=1}^{2} c(b_{n-1}), \ldots, S_N \leq x + \sum_{n=1}^{N} c(b_{n-1})\right) \\ &= P\left(\bigcap_{n=1}^{N} \left\{S_n \leq x + \sum_{k=1}^{n} c(b_{k-1})\right\}\right) \end{aligned} \tag{3.6}$$

From equation (3.6), we have

$$\varphi_N(x, \pi) = E\left[\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - x\right)\right], \tag{3.7}$$

where

$$1_A(x) = \begin{cases} 1 & , \ x \in A \\ 0 & , \ \text{else} \ , \end{cases}$$

for all $A \subseteq R$. For each $a \in R$ and $x \geq 0$, we obtain

$$1_{(-\infty,0]}(a - x) = \begin{cases} 1 & , \ x \geq a, \\ 0 & , \ x < a. \end{cases}$$

Then, $1_{(-\infty,0]}(a - x)$ is non-decreasing in x and right continuous on $(0, \infty]$. This implies that $\prod_{n=1}^{N} 1_{(-\infty,0]}(a_n - x)$ is also non-decreasing in x and right continuous on

$(0, \infty]$ where $a_n \in R, n = 1, 2, 3, \ldots, N$. For each plan $\pi = \{b_0, b_1, b_2, \ldots, b_{N-1}\}$, by the Dominated Convergence Theorem, we get

$$\begin{aligned}
\lim_{u \to x^+} \varphi_N(u, \pi) &= \lim_{u \to x^+} E\left[\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - u\right)\right] \\
&= E\left[\lim_{u \to x^+} \prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - u\right)\right] \\
&= E\left[\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - x\right)\right] \\
&= \varphi_N(x, \pi).
\end{aligned}$$

Therefore, $\varphi_N(x, \pi)$ is non-decreasing in x and right continuous on $(-\infty, \infty)$. This implies that $\Phi_N(x, \pi) = 1 - \varphi_N(x, \pi)$ is non-increasing in x and also right continuous on $(-\infty, \infty)$.

Theorem 3.1. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$, and let $x \geq 0$ be given. Then*

$$\lim_{x \to \infty} \varphi_N(x, \pi) = 1 \text{ and } \lim_{x \to \infty} \Phi_N(x, \pi) = 0.$$

Proof: Let $\omega \in \Omega$ be fixed and $f(x, \omega) = \prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n(\omega) - \sum_{k=1}^{n} c(b_{k-1}) - x\right)$. From equation (3.7), we have $\varphi_N(x, \pi) = E[f(x, \omega)]$. From the definition of $\Phi_N(x, \pi)$, its suffice to show that $\lim_{x \to \infty} E[f(x, \omega)] = 1$.

For each $n \in \{1, 2, 3, \cdots, N\}$, there exists $X_n(\omega)$ such that $S_n(\omega) - \sum_{k=1}^{n} c(b_{k-1}) - x < 0$ for all $x \geq X_n(\omega)$, i.e.,

$$\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n(\omega) - \sum_{k=1}^{n} c(b_{k-1}) - x\right) = 1 \tag{3.8}$$

for all $x \geq X_n(\omega)$.

Let $X_0^*(\omega) = \max\{X_1(\omega), X_2(\omega), \ldots, X_n(\omega)\}$. Then $f(x, \omega) = 1$ for all $x \geq X_0^*(\omega)$. We have

$$\lim_{x \to \infty} f(x, \omega) = 1 \tag{3.9}$$

for all $x \geq X_0^*(\omega)$. By Monotone Convergence Theorem, we have

$$\lim_{x\to\infty} E[f(x,\omega)] = E\left[\lim_{x\to\infty} f(x,\omega)\right] = 1. \tag{3.10}$$

The proof is now complete.

Corollary 3.2. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$, $\alpha \in (0,1)$. Then there exists smallest $\tilde{x} \geq 0$ such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.*

Proof: Let

$$\tilde{x} = sup\{x \geq 0 \mid \Phi_N(x,\pi) > \alpha\}. \tag{3.11}$$

Case 1. $\Phi_N(\tilde{x},\pi) > \alpha$. Since $\Phi_N(x,\pi)$ is non-increasing in x, by (3.11) we have $\Phi_N(x,\pi) \leq \alpha$ for all $x > \tilde{x}$, i.e., $\Phi_N(x,\pi) \leq \alpha$ on $(\tilde{x},\infty)$. Thus

$$\lim_{x\to\tilde{x}^+} \Phi_N(x,\pi) \leq \alpha.$$

Since $\Phi_N(x,\pi)$ right continuous on $(\tilde{x},\infty)$ and non-increasing in x, then

$$\alpha < \Phi_N(\tilde{x},\pi) = \lim_{x\to\tilde{x}^+} \Phi_N(x,\pi) \leq \alpha.$$

Hence, $\Phi_N(\tilde{x},\pi) = \alpha$. As a result $\tilde{x}$ is a smallest real constant such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to
$(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.

Case 2. $\Phi_N(\tilde{x},\pi) \leq \alpha$. Since $\Phi_N(x,\pi)$ is non-increasing in x, by (3.11) we have $\Phi_N(x,\pi) > \alpha$ for all $x < \tilde{x}$ and $\Phi_N(x,\pi) \leq \alpha$ for all $x \geq \tilde{x}$, i.e., $\tilde{x}$ is a smallest real constant such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.

3.2.2 Bounds of the Ruin Probability

In this part, we describe the upper bound of the ruin probability with negative exponential. In order to prove the following lemma, we shall use an equivalent definition of the ruin probability which will be given as follows:

$$\Phi_n(x,\pi) = P\left(\max_{1\le k\le n}\left(\sum_{i=1}^{k}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x\right),\ n=1,2,3,\ldots,N.$$

Lemma 3.3. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$ be stationary, $\alpha \in (0,1)$, and let $x \ge 0$ be given. Then the ruin probability at the time N satisfies the following equation*

$$\Phi_N(x,\pi) = \Phi_1(x,\pi) + \int\limits_{\{y\in R:0\le h(b_0,y)\le x+c(b_0)\}} \Phi_{N-1}(x+c(b_0)-h(b_0,y),\pi)dF_{Y_1}(y) \tag{3.12}$$

where $\Phi_0(x,\pi)=0$.

Proof: We prove equation (3.12) by induction. We start with $N=1$. Since $\Phi_0(x,\pi)=0$ for all $x\ge 0$ then

$$\int\limits_{\{y\in R:0\le h(b_0,y)\le x+c(b_0)\}} \Phi_0(x+c(b_0)-h(b_0,y),\pi)dF_{Y_1}(y) = 0.$$

This proves equation (3.12) for $N=1$. Now assume that equation (3.12) holds for $1 < n \le N-1$. Then

$$\Phi_N(x,\pi) = P\left(\max_{1\le n\le N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x\right)$$

$$= P\left(\left\{\max_{1\le n\le N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x\right\}\bigcap\Omega\right)$$

$$= P\left(\left\{\max_{1\le n\le N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x\right\}\right.$$
$$\left.\bigcap\{\{h(b_0,Y_1)-c(b_0)>x\}\bigcup\{h(b_0,Y_1)-c(b_0)\le x\}\}\right)$$

$$= P\left(\max_{1\le n\le N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x, h(b_0,Y_1)-c(b_0)>x\right)$$
$$+ P\left(\max_{1\le n\le N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x, h(b_0,Y_1)-c(b_0)\le x\right).$$

Since π is stationary and $\{Y_n\}_{n\geq 1}$ is an iid sequence, then

$$\left\{\omega \in \Omega : \max_{1\leq n\leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)(\omega)-c(b_{i-1}))\right) > x, h(b_0,Y_1)(\omega)-c(b_0) > x\right\}$$
$$= \{\omega \in \Omega : h(b_0,Y_1)(\omega)-c(b_0) > x\}.$$

This result implies

$$\Phi_N(x,\pi) = P(h(b_0,Y_1)-c(b_0) > x) +$$
$$P\left(\max_{2\leq n\leq N}\left(h(b_0,Y_1)-c(b_0)+\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x, h(b_0,Y_1)-c(b_0)\leq x\right)$$
$$= \Phi_1(x,\pi) +$$
$$P\left(h(b_0,Y_1)-c(b_0)+\max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x, h(b_0,Y_1)-c(b_0)\leq x\right)$$
$$= \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)-c(b_0)\leq x,\ h(b_0,Y_1)-c(b_0)+\max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)>x}\right]$$
$$= \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)-c(b_0)\leq x}\cdot 1_{h(b_0,Y_1)-c(b_0)+\max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)>x}\right]$$
$$= \Phi_1(x,\pi)$$
$$+ E\left[E\left[1_{h(b_0,Y_1)-c(b_0)\leq x}\cdot 1_{h(b_0,Y_1)-c(b_0)+\max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)>x}|\sigma(Y_1)\right]\right]$$
$$= \Phi_1(x,\pi)$$
$$+ E\left[1_{h(b_0,Y_1)\leq x+c(b_0)}\cdot E\left[1_{\max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)+(h(b_0,Y_1)-x-c(b_0))>0}|\sigma(Y_1)\right]\right]$$

$$= \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\leq x+c(b_0)}\cdot E\left[1_{(0,\infty)}(Z+W|\sigma(Y_1))\right]\right] \tag{3.13}$$

where $Z = \max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)$ and $W = h(b_0,Y_1)-x-c(b_0)$. Since $\{h(b_{n-1},Y_n)\}_{n\geq 1}$ is an independent sequence, then Z and W are independent. It follows from Dudley, R. M. (2002, exercise 9, page 341) that

$$E\left[1_{(0,\infty)}(Z+W)|\sigma(Y_1)\right] = \int\limits_{\omega\in\ \Omega} 1_{(0,\infty)}(Z(\omega)+W|\sigma(Y_1))dP_Z(\omega)$$
$$= \int\limits_{\omega\in\ \Omega} 1_{(0,\infty)}(Z(\omega)+W)dP_Z(\omega)$$
$$= \int\limits_{R} 1_{(0,\infty)}(z+W)dF_Z(z).$$

This implies that

$$\Phi_N(x,\pi) = \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\leq x+c(b_0)} \cdot \left(\int\limits_R 1_{(0,\infty)}(z+W)dF_Z(z)\right)\right]$$
$$= \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\leq x+c(b_0)} \cdot \left(\int\limits_R 1_{(0,\infty)}(z+h(b_0,Y_1)-x-c(b_0))dF_Z(z)\right)\right]$$
$$= \Phi_1(x,\pi)$$
$$+ \int\limits_{\{\omega\in\Omega:h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}} \left(\int\limits_R 1_{(0,\infty)}(z+h(b_0,Y_1)(\omega)-x-c(b_0))dF_Z(z)\right) dP(\omega)$$
$$= \Phi_1(x,\pi) + \int\limits_{\{\omega\in\Omega:h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}} E\left[1_{Z>x+c(b_0)-h(b_0,Y_1)(\omega)}\right] dP(\omega)$$
$$= \Phi_1(x,\pi) + \int\limits_{\{\omega\in\Omega:h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}} P\left(Z > x+c(b_0)-h(b_0,Y_1)(\omega)\right) dP(\omega)$$
$$= \Phi_1(x,\pi) + \int\limits_{\{y\in R:0\leq h(b_0,y)\leq x+c(b_0)\}} P\left(Z > x+c(b_0)-h(b_0,y)\right) dF_{Y_1}(y). \qquad (3.14)$$

Since π is stationary and $\{Y_n\}_{n\geq 1}$ is an iid sequence, then

$$P\left(Z > x+c(b_0)-h(b_0,y)\right)$$
$$= P\left(\max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x+c(b_0)-h(b_0,y)\right)$$
$$= P\left(\max_{1\leq n\leq N-1}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right) > x+c(b_0)-h(b_0,y)\right)$$
$$= \Phi_{N-1}(x+c(b_0)-h(b_0,y),\pi). \qquad (3.15)$$

From equation (3.14) and (3.15), we get

$$\Phi_N(x,\pi) = \Phi_1(x,\pi) + \int\limits_{\{y\in R:0\le h(b_0,y)\le x+c(b_0)\}} \Phi_{N-1}(x+c(b_0)-h(b_0,y),\pi)dF_{Y_1}(y).$$

This proves equation (3.12).

Remark 3.4. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$ be stationary, $\alpha \in (0,1)$. Assume that $\{Y_n, n \ge 1\}$ is an iid sequence of exponential distribution with intensity $\lambda > 0$, i.e., Y_1 has the probability density function*

$$f(y) = \lambda e^{-\lambda y}.$$

By Lemma 3.3, the ruin probability can be written in a recursive form as follows:

Case 1: For $h(b_0,y) = \min\{b_0,y\}$. Assume $b_0 \ge x + c(b_0)$. We get

$$\begin{aligned} \Phi_0(x,\pi) &= 0 \text{ and} \\ \Phi_n(x,\pi) &= \Phi_{n-1}(x,\pi) + \frac{[\lambda(x+nc(b_0))]^{n-1}}{(n-1)!} e^{-\lambda[x+nc(b_0)]} \frac{x+c(b_0)}{x+nc(b_0)} \end{aligned} \quad (3.16)$$

for $n = 1,2,3,\ldots,N$. We will use mathematic induction and Lemma 3.3 to show the recursive form (3.16) holds. Now as $b_0 \ge x + c(b_0)$, then

$$\begin{aligned} \Phi_1(x,\pi) &= P(h(b_0,Y_1) > x + c(b_0)) \\ &= P(Y_1 > x + c(b_0)) \\ &= \int\limits_{x+c(b_0)}^{\infty} \lambda e^{-\lambda y} dy \\ &= e^{-\lambda(x+c(b_0))}. \end{aligned}$$

This proves equation (3.16) for $n = 1$. Let $1 < k \le N-1$. Assume that equation (3.16) holds for $1 < n \le k$. By Lemma 3.3 and the inductive assumption, we get $\Phi_{n+1}(x,\pi)$

$$\begin{aligned} &= \Phi_1(x,\pi) + \int\limits_{\{y\in R:0\le h(b_0,y)\le x+c(b_0)\}} \Phi_n(x+c(b_0)-h(b_0,y),\pi)dF_{Y_1}(y) \\ &= \Phi_1(x,\pi) + \int\limits_{\{y\in R:0\le h(b_0,y)\le x+c(b_0)\}} \Phi_{n-1}(x+c(b_0)-h(b_0,y),\pi)dF_{Y_1}(y) \end{aligned}$$

$$
\begin{aligned}
&+ \int\limits_{\{y\in R: 0\le h(b_0,y)\le x+c(b_0)\}} \frac{[\lambda(x+(n+1)c(b_0)-h(b_0,y))]^{n-1}}{(n-1)!} \\
&\cdot \quad e^{-\lambda[x+(n+1)c(b_0)-h(b_0,y)]}\frac{x+2c(b_0)-h(b_0,y)}{x+(n+1)c(b_0)-h(b_0,y)}dF_{Y_1}(y) \\
&= \Phi_n(x,\pi) + \int\limits_{\{y\in R: 0\le y\le x+c(b_0)\}} \frac{[\lambda(x+(n+1)c(b_0)-y)]^{n-1}}{(n-1)!} \\
&\cdot \quad e^{-\lambda[x+(n+1)c(b_0)-y]}\frac{x+2c(b_0)-y}{x+(n+1)c(b_0)-y}\lambda e^{-\lambda y}dy \\
&= \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!}\cdot \int\limits_{\{y\in R: 0\le y\le x+c(b_0)\}} [x+(n+1)c(b_0)-y]^{n-1} \\
&\cdot \quad \frac{x+2c(b_0)-y}{x+(n+1)c(b_0)-y}dy \\
&= \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!}\cdot \int\limits_{\{y\in R: 0\le y\le x+c(b_0)\}} [x+(n+1)c(b_0)-y]^{n-1} \\
&\cdot \quad \frac{x+(n+1)c(b_0)-y-(n+1)c(b_0)+2c(b_0)}{x+(n+1)c(b_0)-y}dy \\
&= \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!}\cdot \int\limits_{\{y\in R: 0\le y\le x+c(b_0)\}} [x+(n+1)c(b_0)-y]^{n-1} \\
&\cdot \quad \left[1-\frac{(n-1)c(b_0)}{x+(n+1)c(b_0)-y}\right]dy \\
&= \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!}\cdot \left\{ \int_0^{x+c(b_0)} [x+(n+1)c(b_0)-y]^{n-1}dy \right. \\
&- \quad \left. (n-1)c(b_0)\int_0^{x+c(b_0)} [x+(n+1)c(b_0)-y]^{n-2}dy \right\}.
\end{aligned}
$$

Let $w = x+(n+1)c(b_0)-y$. We get

$\Phi_{n+1}(x,\pi)$

$$
= \quad \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!}
$$

$$
\begin{aligned}
& \cdot \left\{ \int\limits_{nc(b_0)}^{x+(n+1)c(b_0)} w^{n-1}dw - (n-1)c(b_0) \int\limits_{nc(b_0)}^{x+(n+1)c(b_0)} w^{n-2}dw \right\} \\
= \; & \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!} \\
& \cdot \left\{ \left[\frac{[x+(n+1)c(b_0)]^n}{n} - \frac{[nc(b_0)]^n}{n} \right] \right. \\
- \; & \left. (n-1)c(b_0) \left[\frac{[x+(n+1)c(b_0)]^{n-1}}{n-1} - \frac{[nc(b_0)]^{n-1}}{n-1} \right] \right\} \\
= \; & \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!} \\
& \cdot \left\{ \frac{[x+(n+1)c(b_0)]^n}{n} - n^{n-1}c(b_0)^n - c(b_0)[x+(n+1)c(b_0)]^{n-1} + n^{n-1}c(b_0)^n \right\} \\
= \; & \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!} \\
& \cdot \left\{ \frac{[x+(n+1)c(b_0)]^n}{n} - c(b_0)[x+(n+1)c(b_0)]^{n-1} \right\} \\
= \; & \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!} \\
& \cdot \left\{ \frac{[x+(n+1)c(b_0)]^n}{n} - \frac{nc(b_0)[x+(n+1)c(b_0)]^n}{n[x+(n+1)c(b_0)]} \right\} \\
= \; & \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{(n-1)!} \frac{[x+(n+1)c(b_0)]^n}{n} \left\{ 1 - \frac{nc(b_0)}{x+(n+1)c(b_0)} \right\} \\
= \; & \Phi_n(x,\pi) + \frac{\lambda^n e^{-\lambda[x+(n+1)c(b_0)]}}{n!} [x+(n+1)c(b_0)]^n \frac{x+c(b_0)}{x+(n+1)c(b_0)}.
\end{aligned}
$$

This proves case 1.

Case 2: For $h(b_0, y) = b_0 y$. By the same proof as in case 1, we get

$$\Phi_0(x,\pi) = 0 \text{ and}$$

$$\Phi_n(x,\pi) = \Phi_{n-1}(x,\pi) + \frac{1}{(n-1)!} \left[\frac{\lambda}{b_0}(x+nc(b_0)) \right]^{n-1} e^{-\frac{\lambda}{b_0}(x+nc(b_0))} \frac{x+c(b_0)}{x+nc(b_0)} \quad (3.17)$$

for all $n = 1, 2, 3, \ldots, N$. Further, for $b_0 = \bar{b}_0 = 1$, we also obtain the recursive form as

$$\Phi_0(x,\pi) = 0 \text{ and } \Phi_n(x,\pi) = \Phi_{n-1}(x,\pi) + \frac{1}{(n-1)!} \left[\lambda(x+nc_0) \right]^{n-1} e^{-\lambda(x+nc_0)} \frac{x+c_0}{x+nc_0}$$

for all $n = 1, 2, 3, \ldots, N$.

Definition 3.3. *(Sub-adjustment coefficient). Let $s > 0$ and Y be a non-negative random variable. If there exists $d_0 > 0$ such that*

$$E\left[e^{d_0 Y}\right] \leq e^{d_0 s}, \tag{3.18}$$

then d_0 is called a sub-adjustment coefficient of (s, Y). Specifically, if (3.18) is an equality then d_0 is called an adjustment coefficient of (s, Y).

Theorem 3.5. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ be stationary, and let $c(b_0) > 0$ be a net income rate. If $d_0 > 0$ is a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$, then*

$$\Phi_n(x, \pi) \leq e^{-d_0 x}, \tag{3.19}$$

for all $x \geq 0$ and all $n = 1, 2, 3, \ldots, N$.

Proof: Let $h(b_0, y) = \min\{b_0, y\}$, $x \geq 0$ and $d_0 > 0$ be a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$, i.e.,

$$E\left[e^{d_0 h(b_0, Y_1)}\right] \leq e^{d_0 c(b_0)}.$$

We prove this theorem by induction. We start with $n = 1$,

$$\begin{aligned}
\Phi_1(x, \pi) &= P(h(b_0, Y_1) > x + c(b_0)) \\
&= P(d_0 h(b_0, Y_1) > d_0(x + c(b_0))) \\
&= P(e^{d_0 h(b_0, Y_1)} > e^{d_0(x + c(b_0))}) \\
&\leq \frac{E\left[e^{d_0 h(b_0, Y_1)}\right]}{e^{d_0(x + c(b_0))}} \qquad \text{(By Markov's inequality)} \\
&\leq \frac{e^{d_0 c(b_0)}}{e^{d_0(x + c(b_0))}} \\
&= e^{-d_0 x}.
\end{aligned}$$

Let $k \leq N - 1$. Assume that inequality (3.19) holds for $1 < n \leq k$. Next, we show that inequality (3.19) holds for $n = k + 1$. By Lemma 3.3 we get

$$\Phi_{k+1}(x,\pi) \quad = \quad \Phi_1(x,\pi) + \int\limits_{\{y\in R:0\le h(b_0,y)\le x+c(b_0)\}} \Phi_k(x+c(b_0)-h(b_0,y),\pi)dF_{Y_1}(y). \tag{3.20}$$

Firstly, we consider the second term of the right-hand side of equation (3.20). By using the inductive assumption, we have

$$\int\limits_{\{y\in R:0\le h(b_0,y)\le x+c(b_0)\}} \Phi_k(x+c(b_0)-h(b_0,y),\pi)dF_{Y_1}(y)$$

$$\le \int\limits_{\{y\in R:0\le h(b_0,y)\le x+c(b_0)\}} e^{-d_0(x+c(b_0)-h(b_0,y))}dF_{Y_1}(y). \tag{3.21}$$

Next, we calculate the first term of right-hand side of equation (3.20).

$$\begin{aligned}
&\Phi_1(x,\pi)\\
&= P(h(b_0,Y_1) > x+c(b_0))\\
&= P\left(e^{d_0 h(b_0,Y_1)} 1_{(x+c(b_0),\infty)}(h(b_0,Y_1)) > e^{d_0(x+c(b_0))}\right)\\
&\le \frac{E\left[e^{d_0 h(b_0,Y_1)} 1_{(x+c(b_0),\infty)}(h(b_0,Y_1))\right]}{e^{d_0(x+c(b_0))}} \qquad \text{(By Markov's inequality)}\\
&= \frac{\int\limits_R e^{d_0 h(b_0,y)} 1_{(x+c(b_0),\infty)}(h(b_0,y))dF_{Y_1}(y)}{e^{d_0(x+c(b_0))}}\\
&= \frac{\int\limits_{\{y\in R:x+c(b_0)<h(b_0,y)<\infty\}} e^{d_0 h(b_0,y)}dF_{Y_1}(y)}{e^{d_0(x+c(b_0))}}\\
&= \int\limits_{\{y\in R:x+c(b_0)<h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))}dF_{Y_1}(y).
\end{aligned} \tag{3.22}$$

From inequality (3.21) and (3.22), the equation (3.20) can be modified to be the following

$$\Phi_{k+1}(x,\pi)$$

$$\leq \int\limits_{\{y\in R: x+c(b_0)<h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$+ \int\limits_{\{y\in R: 0\leq h(b_0,y)\leq x+c(b_0)\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$= \int\limits_{\{y\in R: 0\leq h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$= \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} \int\limits_{R} e^{d_0 h(b_0,y)} dF_{Y_1}(y)$$

$$= \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} E\left[e^{d_0 h(b_0,Y_1)}\right]$$

$$\leq \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} e^{d_0 c(b_0)}$$

$$= e^{-d_0 x}.$$

This proves equation (3.19) for $n = k+1$ in the case $h(b_0, y) = \min\{b_0, y\}$. By the same proof of this case, we also get equation (3.19) holds for $h(b_0, y) = b_0 y$.

Corollary 3.6. *Let* $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$ *be stationary,* $\alpha \in (0, 1)$, *and let* $c(b_0) > 0$ *be a net income rate. Assume that* $d_0 > 0$ *is a sub-adjustment coefficient of* $(c(b_0), h(b_0, Y_1))$, *then there exists an acceptable initial capital* $x (x \geq 0)$ *corresponding to* $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{h(b_0, Y_n)\}_{n\geq 1})$ *such that*

$$0 \leq x \leq -\frac{ln\alpha}{d_0} \quad or \quad \alpha \leq e^{-d_0 x}.$$

Proof : Let $d_0 > 0$ be a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$. By Theorem 3.5, we have

$$\Phi_N(u, \pi) \leq e^{-d_0 u},$$

for all $u \geq 0$. Let $\alpha \in (0, 1)$. By Corollary 3.2, there exists $v \geq 0$ which is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{h(b_0, Y_n)\}_{n\geq 1})$. By Definition 3.2, we have

$$\Phi_N(v, \pi) \leq \alpha.$$

Since $\Phi_N(v,\pi)$ is non-increasing in v for each π, then there exists $0 \le x \le v$ such that $\alpha = \Phi_N(x,\pi) \le e^{-d_0 x}$. Hence x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\ge1}, \{h(b_0, Y_n)\}_{n\ge1})$. The proof is now complete.

Note: We know that a smaller ruin probability can be controlled by a larger initial capital. However, an insurance company usually does not possess unlimited initial capital, but only a small initial capital that must be sufficient for a predetermined solvency (not ruin) condition for the firm is preferable. If an acceptable ruin probability is fixed, the firm can find an interval of acceptable initial capital by virtue of Corollary 3.6.

Example 3.1. *(Exponential claims under the proportional reinsurance). We assume that $\{Y_n\}_{n\ge1}$ is a sequence of claims with iid exponential $Exp(1)$, and $\{X_n\}_{n\ge0}$ is a sequence of surplus which satisfies the model (3.3). Let $N \in \{1,2,3,\ldots\}$, and $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$ be stationary. Suppose that $h(b_0, y)$ is the proportional reinsurance with retention level b_0, and $c(b_0) > 0$ is a net income rate which is calculated by the expected value principle, i.e.,*

$$c(b_0) = c_0 - (1+\theta_1)E[Y_1 - h(b_0, Y_1)] = \theta_0 - \theta_1 + b_0(1+\theta_1). \tag{3.23}$$

Assume that $\alpha = 0.05$, $\theta_0 = \theta_1 = 0.1$, and $b_0 = 0.6$. Then there exists an adjustment coefficient $d_0 = 0.2935569060$ of $(c(b_0), b_0 Y_1)$ such that

$$0 \le x \le \frac{-ln0.05}{0.2935569060} = 10.20494566$$

which is an interval of acceptable initial capital with corresponding to $(1, N, \{c(b_{n-1}) = c(b_0)\}_{n\ge1}, \{b_0 Y_n\}_{n\ge1})$

Let

$$f(d) := E\left[e^{d b_0 Y_1}\right] - e^{d c(b_0)}.$$

Note that

$$E\left[e^{db_0Y_1}\right] = \int_0^\infty e^{db_0y} f_{Y_1}(y)dy = \int_0^\infty e^{db_0y} e^{-y} dy = \frac{1}{1-db_0} \quad \text{and}$$

$$e^{dc(b_0)} = e^{db_0(1+\theta_1)}. \tag{3.24}$$

By Definition 3.3, d_0 is an adjustment coefficient of $(c(b_0), b_0Y_1)$ if $f(d_0) = 0$. Hence $E\left[e^{d_0b_0Y_1}\right] = e^{d_0c(b_0)}$. By substituting b_0 and θ_1 into equation (3.24), we get

$$\frac{1}{1-0.6d_0} = e^{0.66d_0}.$$

Solving for d_0, we get $d_0 = 0.2935569060$. By Corollary 3.6, we get

$$0 \leq x \leq \frac{-ln0.05}{0.2935569060} = 10.20494566$$

which is an interval of acceptable initial capital corresponding to $(0.05, N, \{c(b_{n-1}) = 0.66\}_{n\geq1}, \{0.6Y_n\}_{n\geq1})$. This means that $\Phi_N(x,\pi) \leq 0.05$ for all $0 \leq x \leq 10.20494566$.

Example 3.2. *(Exponential claims under the excess of loss reinsurance). We assume that $\{Y_n\}_{n\geq1}$ and $\{X_n\}_{n\geq0}$ are the sequences given in example 3.1. Let $N \in \{1, 2, 3, \ldots\}$, and $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ be stationary. Suppose that $h(b_0, y)$ is the excess of loss reinsurance with retention level b_0. By the expected value principle, the net income rate $c(b_0)$ satisfies the following equation*

$$c(b_0) = c_0 - (1+\theta_1)E[Y_1 - h(b_0, Y_1)] = \theta_0 - \theta_1 + (1+\theta_1)[1-e^{-b_0}]. \tag{3.25}$$

Assume that $\alpha = 0.05$, $\theta_0 = \theta_1 = 0.1$ and $b_0 = 100$. Then there exists a sub-adjustment coefficient $d_0 = 0.17$ of $(c(b_0), h(b_0, Y_1))$ such that

$$0 \leq x \leq -\frac{ln0.05}{0.17} = 17.6220$$

which is an interval of acceptable initial capital with corresponding to $(0.05, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq1}, \{h(b_0, Y_n)\}_{n\geq1})$

Let

$$f(d) := E\left[e^{dh(b_0,Y_1)}\right] - e^{dc(b_0)}.$$

Note that

$$E\left[e^{dh(b_0,Y_1)}\right] = \int_0^\infty e^{dh(b_0,y)}e^{-y}dy = \int_0^{b_0} e^{dy}e^{-y}dy + \int_{b_0}^\infty e^{b_0 d}e^{-y}dy = \frac{de^{b_0(d-1)}-1}{d-1},$$

$$\text{and} \quad e^{dc(b_0)} = e^{d(1+\theta_1)\left[1-e^{-b_0}\right]}. \tag{3.26}$$

By Definition 3.3, d_0 is a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$ if $f(d_0) \leq 0$. Hence $E\left[e^{d_0 h(b_0,Y_1)}\right] \leq e^{d_0 c(b_0)}$. By substituting b_0, θ_0 and θ_1 into equation (3.26), we get

$$\frac{d_0 e^{100(d_0-1)}-1}{d_0-1} \leq e^{1.1d_0\left[1-e^{-100}\right]}.$$

Solving for d_0, we get $d_0 = 0.17$. By Corollary 3.6, we get

$$0 \leq x \leq -\frac{ln 0.05}{0.17} = 17.6220$$

which is an interval of acceptable initial capital with corresponding to $(0.05, N, \{c(b_{n-1}) = 1.1\}_{n\geq 1}, \{h(100, Y_n)\}_{n\geq 1})$. This means that $\Phi_N(x,\pi) \leq 0.05$ for all $0 \leq x \leq 17.6220$.

3.2.3 Existence of Minimal Capital

Let $\alpha \in (0,1)$. As a result of Corollary 3.2 that $\{x \geq 0 : \Phi_N(x,\pi) \leq \alpha\}$ is a non-empty set. Since the set $\{x \geq 0 : \Phi_N(x,\pi) \leq \alpha\}$ is an infinite set, then there are many acceptable initial capitals corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$. In this section, we will prove the existence of a minimum initial capital that correspond to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$. , i.e.,

$$\min_{x\geq 0}\{x : \Phi_N(x,\pi) \leq \alpha\} = x^*.$$

The following lemma and theorems are proved in Klongdee' Ph.D. dissertation (2010).

Lemma 3.7. *Let a, b and α be real numbers such that $a \leq b$. If f is non-increasing and right continuous on $[a, b]$ and $\alpha \in [f(b), f(a)]$, then there exists $d \in [a, b]$ such that*

$$d = \min\{x \in [a, b] : f(x) \leq \alpha\}.$$

Proof: Let

$$S := \{x \in [a, b] : f(x) \leq \alpha\}. \tag{3.27}$$

Since $\alpha \in [f(b), f(a)]$, i.e., $f(b) \leq \alpha \leq f(a)$, then we have $b \in S$. Hence S is a non empty set. Since S is a subset of the closed and bounded interval $[a, b]$, then there exists $d \in [a, b]$ such that $d = \inf S$. Next, we consider the following cases:

Case 1. $d = b$. We know that $b \in S$, thus $b = \min S$.

Case 2. $a \leq d < b$. Since $d = \inf S$, then there exists $d_n \in S$ such that

$$d \leq d_n < d + 1/n$$

for all $n \in \{1, 2, 3, \ldots\}$. Since f is non-increasing and $d_n \in S$, then

$$f(d_n) \leq \alpha.$$

Since f is right continuous at d, we have

$$f(d) = \lim_{n \to \infty} f(d_n) \leq \alpha.$$

Therefore, $d \in S$, i.e., $d = \min S$. This completes the proof.

Theorem 3.8. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and let $\alpha \in (0, 1)$. Then there exists $x^* \geq 0$ such that*

$$x^* = \min_{x \geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}.$$

Proof: Let $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$. We consider by case.

Case 1: For $\Phi_N(0,\pi) \leq \alpha$. We get $\min_{x\geq 0}\{x : \Phi_N(x,\pi) \leq \alpha\} = 0$.

Case 2: For $\Phi_N(0,\pi) > \alpha$. By Corollary 3.2, there exists $\tilde{x} > 0$ such that $\Phi_N(\tilde{x},\pi) \leq \alpha$. Hence $\alpha \in [\Phi_N(\tilde{x},\pi), \Phi_N(0,\pi)]$. Since $\Phi_N(x,\pi)$ is non-increasing in x and right continuous on $[0,\infty)$. Then ,by Lemma 3.7, there exists $x^* \in [0,\tilde{x}]$ such that

$$x^* = \min_{x\in[0,\tilde{x}]}\{x : \Phi_N(x,\pi) \leq \alpha\} = \min_{x\in[0,\infty)}\{x : \Phi_N(x,\pi) \leq \alpha\}.$$

From case 1 and 2, we have $x^* = \min_{x\geq 0}\{x : \Phi_N(x,\pi) \leq \alpha\}$.

Next, we approximate the minimal initial capital x^* by the bisection method.

Theorem 3.9. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$ and let $\alpha \in (0,1)$. Assume that $v_0, x_0 \geq 0$ such that $v_0 < x_0$. Let $\{v_m\}_{m\geq 1}$ and $\{x_m\}_{m\geq 1}$ be two real sequences defined by*

$$\begin{cases} v_m = v_{m-1} \quad \text{and} \quad x_m = \dfrac{x_{m-1}+v_{m-1}}{2}, & \text{if } \Phi_N\left(\dfrac{x_{m-1}+v_{m-1}}{2},\pi\right) \leq \alpha \\ v_m = \dfrac{v_{m-1}+x_{m-1}}{2} \quad \text{and} \quad x_m = x_{m-1}, & \text{if } \Phi_N\left(\dfrac{x_{m-1}+v_{m-1}}{2},\pi\right) > \alpha \end{cases}$$

for all $m = 1,2,3,\ldots$. If $\Phi_N(x_0,\pi) \leq \alpha < \Phi_N(v_0,\pi)$, then

$$\lim_{m\to\infty} x_m = \min_{x\geq 0}\{x : \Phi_N(x,\pi) \leq \alpha\} = x^*.$$

Proof: Obviously, $\{x_m\}_{m\geq 1}$ is non-increasing and $\{v_m\}_{m\geq 1}$ is non-decreasing. Moreover, $v_m \leq x_m$ for all $m \in \{1,2,3,\ldots\}$. Thus, $\{x_m\}_{m\geq 1}$ and $\{v_m\}_{m\geq 1}$ are convergent. Since

$$0 \leq x_m - v_m = \frac{x_0 - v_0}{2^m} \to 0 \text{ as } m \to \infty,$$

then there exists $x^* \in [v_0, x_0]$ such that

$$\lim_{m\to\infty} x_m = \lim_{m\to\infty} v_m := x^*.$$

Since $\Phi_N(x, \pi)$ is right continuous in x for each π and $\Phi_N(x_m, \pi) \leq \alpha$ for all m, then

$$\Phi_N(x^*, \pi) = \lim_{m \to \infty} \Phi_N(x_m, \pi) \leq \alpha.$$

Since $\Phi_N(x, \pi)$ is non-increasing in x for each π and $\Phi_N(v_m, \pi) > \alpha$ for all m, then $\Phi_N(x, \pi) > \alpha$ for $x < x^*$. Therefore

$$x^* = \min_{x \geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}. \tag{3.28}$$

This completes the proof.

3.3 Numerical Results

In this section, we provide numerical illustration of main results. We approximate the minimal initial capital of the discrete-time surplus process (3.3) by using Theorem 3.9 according to the following cases:

3.3.1 Proportional Reinsurance Case

We assume that $\{Y_n\}_{n\geq 1}$ is a sequence of claims with iid exponential $Exp(1)$ and $h(b_0, y)$ is the proportional reinsurance with retention level b_0. Let $N \in \{1, 2, 3, \ldots\}$ be the time horizon and $\pi = \{b_{n-1} = 0.6\}_{n=1}^N$ be stationary. We choose model parameters as follows: $\theta_0 = \theta_1 = 0.10$ which gives $c(b_0) = 0.66$ and $\theta_0 = \theta_1 = 0.25$ which gives $c(b_0) = 0.75$. Moreover, we choose $\alpha = 0.05, \alpha = 0.1$ and $\alpha = 0.2$. As a result, we get the table of the minimum initial capital below:

Table 3.1 Minimum Initial Capital in the Proportional Reinsurance Case

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.2$
N	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$
10	3.3909 : 2.7854	2.5919 : 2.0384	1.7358 : 1.2562
20	4.4983 : 3.3728	3.4846 : 2.4796	2.3918 : 1.5524
30	5.2438 : 3.6605	4.0747 : 2.6854	2.8148 : 1.6829
40	5.8067 : 3.8215	4.5137 : 2.7963	3.1233 : 1.7504
50	6.2558 : 3.9175	4.8593 : 2.8605	3.3619 : 1.7884
100	7.6364 : 4.0664	5.8902 : 2.9559	4.0471 : 1.8426
200	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
300	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
400	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
500	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
$1,000$	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
$5,000$	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
$10,000$	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497

Table 3.1 shows an approximation of $\min_{x \geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}$ with $m = 25$, $v_0 = 0, x_0 = 20$ as mentioned in Theorem 3.9 and $\Phi_N(x, \pi)$ is computed by using the recursive form as mentioned in equation (3.17). The numerical results in Table 3.1 show a minimum initial capital $x = 3.3909$ for $\alpha = 0.05$, $N = 10$ and $\theta_0 = \theta_1 = 0.1$ etc.

3.3.2 Excess of Loss Reinsurance Case

Again we assume that $\{Y_n\}_{n\geq 1}$ is a sequence of claims with iid exponential $Exp(1)$ and $h(b_0, y)$ is the excess of loss reinsurance with retention level $b_0 = 100$. Let $N \in \{1, 2, 3, \ldots\}$ be the time horizon and $\pi = \{b_{n-1} = 100\}_{n=1}^{N}$ be stationary. We choose model parameters as follows: $\theta_0 = \theta_1 = 0.10$ which give $c(b_0) = 1.1$ and $\theta_0 = \theta_1 = 0.25$ which give $c(b_0) = 1.25$. Moreover, we choose $\alpha = 0.05, \alpha = 0.1$ and $\alpha = 0.2$. As a result, we get the table of the minimum initial capital as below:

Table 3.2 Minimum Initial Capital in the Excess of Loss Reinsurance Case

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.2$
N	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$
10	5.6515 : 4.6424	4.3198 : 3.3973	2.8930 : 2.0936
20	7.4972 : 5.6213	5.8076 : 4.1327	3.9863 : 2.5874
30	8.7396 : 6.1009	6.7911 : 4.4756	4.6913 : 2.8048
40	9.6779 : 6.3692	7.5229 : 4.6605	5.2054 : 2.9174
50	10.4264 : 6.5291	8.0989 : 4.7675	5.6031 : 2.9806
100	12.7273 : 6.7773	9.8169 : 4.9265	6.7452 : 3.0709
200	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
300	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
400	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
500	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
1,000	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
5,000	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
10,000	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828

Table 3.2 shows an approximation of $\min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}$ with $m = 25$,

$v_0 = 0, x_0 = 20$ as mentioned in Theorem 3.9 and $\Phi_N(x, \pi)$ is computed by using the recursive form as mentioned in equation (3.16). The numerical results in Table 3.2 show a minimum initial capital $x = 5.6516$ for $\alpha = 0.05$, $N = 10$ and $\theta_0 = 0.1$, etc.

CHAPTER IV

CONCLUSIONS

This thesis is devoted to the study of the two different discrete-time surplus processes: the classical surplus process is considered under the conditions of reinsurance and shareholder input, and the classical surplus process is considered under the condition of reinsurance and the inter-arrival times $Z_n = 1, n \in \{1, 2, 3, \ldots\}$. Therefore, the results obtained are separated into two parts.

In the first part, we find formula for the control problem of the discrete-time surplus process

$$X_{n+1} = X_n + \delta_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1}),\ n = 0, 1, 2, \ldots \tag{4.1}$$

where $X_0 = x$ is an initial capital, $\{(b_n, \delta_n)\}_{n\geq 0}$ is the sequence of control actions, $\{Z_n\}_{n\geq 1}$ is the sequence of inter-arrival times and $\{Y_n\}_{n\geq 1}$ is the sequence of claims.

We start by imposing assumptions, and we define a plan, the total cost function and the value function for the time horizon N as follows:

Assumption 4.1. Independence Assumption (IA)

The sequence of inter-arrival times $\{Z_n\}_{n\geq 1}$ and the sequence of claims $\{Y_n\}_{n\geq 1}$ are iid sequences with finite variances. Moreover, for each $n \in \{1, 2, 3, \ldots\}$, Z_n and Y_n are independent.

Definition 4.1. *A plan for the time horizon N over a control space U is a (finite) sequence $\pi = \{u_n\}_{n=0}^{N-1}$ of $u_0 = (b_0, \delta_0) = (b_0, 0)$ and $u_n = (b_n, \delta_n) \in U$ for $n = 1, 2, 3, \ldots, N-1$. The set of all plans for the time horizon N over the space U is denoted by $\mathcal{P}(N, U)$. A plan $\pi \in \mathcal{P}(N, U)$ is said to be stationary, if $b_0 = b_1$ and $(b_n, \delta_n) = (b_1, \delta_1)$ for $n = 1, 2, 3, \ldots, N-1$.*

Definition 4.2. *Let $x \in S$ be an initial state and $\pi = \{u_n\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ where N is the time horizon. The total cost function $\Phi_N(x,\pi)$ and the value function $V_N(x)$ for the time horizon N are defined by*

$$\begin{aligned} \Phi_N(x,\pi) &= E\left[\sum_{n=0}^{N-1} g(X_n,u_n) + \alpha_N \widehat{V}(X_N)|_{X_0=x}\right] \qquad \textit{and} \\ V_N(x) &= \inf_{\pi \in \mathcal{P}(N,U)} \Phi_N(x,\pi) \qquad \textit{respectively} \end{aligned} \tag{4.2}$$

when $\alpha_N \in (0,1]$, X_n's are random variables which satisfy equation (4.1), $g(\cdot,\cdot)$ is a one − period cost function and $\widehat{V}(\cdot)$ is a cost function for time horizon N. A plan $\tilde{\pi} \in \mathcal{P}(N,U)$ is said to be optimal, if $\inf_{\pi \in \mathcal{P}(N,U)} \Phi_N(x,\pi) = \Phi_N(x,\tilde{\pi})$.

The main results of this part are summarized as follows:

Remark 4.1. *Let $\pi = \{(b_n,\delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$, one gets*

$$\Phi_N(x,\pi) = \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N \left\{ \sum_{n=0}^{N-1} \left\{ c^2(b_n) Var[Z_{n+1}] + Var[h(b_n,Y_{n+1})] \right\} + G_N^2(x,\pi) \right\} \tag{4.3}$$

where $G_N(x,\pi) = x - A + \sum_{n=0}^{N-1} E[L(b_n,\delta_n,Y_{n+1},Z_{n+1})]$.

Lemma 4.2. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$ and let $\pi = \{(b_n,\delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N,U)$ be such that $\delta_1 = \delta_2 = \cdots = \delta_{N-1} > 0$. Then*

$$\Phi_N(x,\pi) < \Phi_N(x,((b_0,0),(b_1,0),...,(b_{N-1},0)))$$

where $\mathcal{P}^*(N,U) = \{\pi \in \mathcal{P}(N,U) | G_N(x,\pi) = 0\}$.

Theorem 4.3. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$. Then there exists $\tilde{\pi} = \{(\tilde{b}_n,\tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U) - \mathcal{P}^*(N,U)$ such that $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$ and*

$$\Phi_N(x,\tilde{\pi}) < \Phi_N(x,((\tilde{b}_0,0),(\tilde{b}_1,0),...,(\tilde{b}_{N-1},0))).$$

Lemma 4.4. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$. If $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ is an optimal plan, then $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$.*

Theorem 4.5. *Let $x \in S$ be fixed and A be the target at the time horizon N, then $\Phi_N(x,\pi)$ is continuous on $\mathcal{P}(N,U)$.*

Theorem 4.6. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$. Then there exists an optimal plan $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ such that $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$ and*

$$V_N(x) = (N-1)\tilde{\delta}_1^2 + \alpha_N \left\{ \sum_{n=0}^{N-1} Var[Z_1]c^2(\tilde{b}_n) + \sum_{n=0}^{N-1} Var[h(\tilde{b}_n, Y_{n+1})] + G_N^2(x,\tilde{\pi}) \right\}$$

where $G_N(x,\tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + \sum_{n=0}^{N-1} E[Z_1]c(\tilde{b}_n) - \sum_{n=0}^{N-1} E[h(\tilde{b}_n, Y_{n+1})]$.

Corollary 4.7. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$ and $h(b,y)$ is the proportional reinsurance. Then there exists an optimal plan $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N,U)$ such that $\tilde{\pi}$ is stationary and*

$$V_N(x) = (N-1)\tilde{\delta}_1^2 + \alpha_N \left\{ Nc^2(\tilde{b}_0)Var[Z_1] + N\tilde{b}_0^2 Var[Y_1] + G_N^2(x,\tilde{\pi}) \right\}$$

where $G_N(x,\tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + Nc(\tilde{b}_0)E[Z_1] - N\tilde{b}_0 E[Y_1]$.

In the second part, we find the relationship between the initial capital and ruin probability of the discrete-time surplus process

$$X_n = x + \sum_{i=1}^{n} c(b_{i-1}) - \sum_{i=1}^{n} h(b_{i-1}, Y_i) \tag{4.4}$$

where $X_0 = x$ is an initial capital.

Again we start by imposing assumption, define a plan for the time horizon N and define the minimum initial capital as follows:

Assumption 4.2. Independence Assumption (IA)

The claims $\{Y_n\}_{n\geq 1}$ form an independent and identically distributed (iid) sequence of random variables.

Definition 4.3. *Let $N \in \{1, 2, 3, \ldots\}$ be a time horizon (number of periods). A plan for the time N is a (finite) sequence $\pi = \{b_{n-1}\}_{n=1}^{N}$ of $b_{n-1} \in [\underline{b}, \overline{b}]$ for $n = 1, 2, 3, \ldots, N$. A set of all plans for the time horizon N over control space $[\underline{b}, \overline{b}]$ is denoted by $\mathcal{P}(N, [\underline{b}, \overline{b}])$. A plan $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ is said to be stationary, if $b_0 = b_1 = \cdots = b_{N-1}$.*

Define the *survival probability* and the *ruin probability* at a time $n \in \{1, 2, 3, \ldots, N\}$ as follows :

$$\varphi_n(x, \pi) := P(X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, \ldots, X_n \geq 0 | X_0 = x) \tag{4.5}$$

and

$$\Phi_n(x, \pi) = 1 - \varphi_n(x, \pi). \tag{4.6}$$

where $N \in \{1, 2, 3, \ldots\}$ is a time horizon, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and $x \geq 0$ is an initial capital.

Definition 4.4. *Let $\{X_n, n \geq 0\}$ be the surplus process as in equation (5.4) which is driven by the sequence of control actions $\{b_{n-1}, n \geq 1\}$ and the sequence of claims $\{Y_n, n \geq 1\}$. Let $\{c(b_{n-1})\}_{n\geq 1}$ be the sequence of the net income rates and $x \geq 0$ be an initial capital. For each the time horizon $N \in \{1, 2, 3, \ldots\}$, let $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and $\alpha \in (0, 1)$. If $\Phi_N(x, \pi) \leq \alpha$, then x is called an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$. Particularly, if*

$$x^* = \min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}$$

exists, x^* is called the minimum initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$ and is written as

$$x^* := \mathbf{MIC}(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1}).$$

Definition 4.5. *(Sub-adjustment coefficient). Let $s > 0$ and Y be a non-negative random variable. If there exists $d_0 > 0$ such that*

$$E\left[e^{d_0 Y}\right] \le e^{d_0 s}, \tag{4.7}$$

then d_0 is called a sub-adjustment coefficient of (s, Y). Specifically, if (5.5) is an equality then d_0 is called an adjustment coefficient of (s, Y).

The main results of this part are summarized as follows:

Theorem 4.8. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$, and let $x \ge 0$ be given. Then*

$$\lim_{x\to\infty} \varphi_N(x, \pi) = 1 \text{ and } \lim_{x\to\infty} \Phi_N(x, \pi) = 0.$$

Corollary 4.9. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$, $\alpha \in (0, 1)$. Then there exists the smallest $\tilde{x} \ge 0$ such that, for all $x \ge \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\ge1}, \{h(b_{n-1}, Y_n)\}_{n\ge1})$.*

Lemma 4.10. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ be stationary, $\alpha \in (0, 1)$, and let $x \ge 0$ be given. Then the ruin probability at the time N satisfies the following equation*

$$\Phi_N(x, \pi) = \Phi_1(x, \pi) + \int\limits_{\{y: 0 \le h(b_0, y) \le x + c(b_0)\}} \Phi_{N-1}(x + c(b_0) - h(b_0, y), \pi) dF_{Y_1}(y) \tag{4.8}$$

where $\Phi_0(x, \pi) = 0$.

Theorem 4.11. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ be stationary, and let $c(b_0) > 0$ be a net income rate. If $d_0 > 0$ is a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$, then*

$$\Phi_n(x, \pi) \le e^{-d_0 x}, \tag{4.9}$$

for all $x \ge 0$ and all $n = 1, 2, 3, \ldots, N$.

Corollary 4.12. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ be stationary, $\alpha \in (0, 1)$, and let $c(b_0) > 0$ be a net income rate. Assume that $d_0 > 0$ is a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$, then there exists an acceptable initial capital $x(x \geq 0)$ corresponding to $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{h(b_0, Y_n)\}_{n\geq 1})$ such that*

$$0 \leq x \leq -\frac{ln\alpha}{d_0} \text{ or } \alpha \leq e^{-d_0 x}.$$

Theorem 4.13. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and let $\alpha \in (0, 1)$. Then there exists $x^* \geq 0$ such that*

$$x^* = \min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}.$$

Theorem 4.14. *Let $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and let $\alpha \in (0, 1)$. Assume that $v_0, x_0 \geq 0$ such that $v_0 < x_0$. Let $\{v_m\}_{m\geq 1}$ and $\{x_m\}_{m\geq 1}$ be two real sequences defined by*

$$\begin{cases} v_m = v_{m-1} \quad \text{and} \quad x_m = \dfrac{x_{m-1} + v_{m-1}}{2}, & \text{if } \Phi_N\left(\dfrac{x_{m-1} + v_{m-1}}{2}, \pi\right) \leq \alpha \\ v_m = \dfrac{v_{m-1} + x_{m-1}}{2} \quad \text{and} \quad x_m = x_{m-1}, & \text{if } \Phi_N\left(\dfrac{x_{m-1} + v_{m-1}}{2}, \pi\right) > \alpha \end{cases}$$

for all $m = 1, 2, 3, \ldots$. If $\Phi_N(x_0, \pi) \leq \alpha < \Phi_N(v_0, \pi)$, then

$$\lim_{m\to\infty} x_m = \min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\} = x^*.$$

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APPENDIX

APPENDIX A

PRELIMINARY ON FINANCIAL RISK MANAGEMENT

A.1 Financial Risk

Financial risk has traditionally been separated into market risk, credit risk, operational risk and liquidity risk. Moreover, at present, as the insurance industry (non-life and life insurance) has grown at a faster pace in many countries over the world, for insurance business we also need to be concerned about insurance risk.

- *Market risk* is the risk due to the fluctuations of market variables. It is the best known type of risk in the banking and securities industry. It is the risk of a change in value of a financial position due to changes in the value of the underlying assets on which that position depends, such as cash products (equities, bond), derivatives (plain vanilla, exotics), interest rates, foreign exchange (FX) rates, commodities, etc. Each area will have its own specific risk management requirement.

- *Credit risk*, or default risk, is the possibility that a counterparty will be unable to satisfy the contracts. In the U.S., many default risks are monitored by credit rating firms such as Standard & Poor's, Moody's, or some other rating agencies. Investors control default risk by monitoring the ratings of the bonds they hold or consider for a purchase.

- *Operational risk*: A better definition is provided by the Basel Committee of

Banking Supervision. *The risk of loss resulting from inadequate or failed internal processes, people and systems or from external events.* This definition includes legal risk, but excludes strategic and reputational risk.

- *Liquidity risk* means risk resulting from a financial institution's failure to pay its debts and obligations because of its inability to convert assets into cash, or its failure to procure enough funds, or if it can, that the funds come with an exceptionally high cost that may effect the institution's income and capital fund now or in the future.

- *Insurance risk* is concerned with the possibility that an insurance company does not have enough funds to pay compensations to its costomers, and cause for insolvency occurs when its surplus becomes negative.

Risk models have attracted much attention in the insurance business, in connection with the possible insolvency and the capital reserve of a insurance company.

A.2 Regulatory Framework on Finance

*In the past one would rely to a large extent on self-regulating or local regulations, since there were rules. However, 20^{th} century has seen key developments leading to the present regulatory risk management framework.

Much of the regulatory drive originated from the Basel Committee of Banking Supervision. In 1974, this committee was established by the Central-Bank Governors of the Group of Ten (G-10) which consists of Belgium, Canada, France, Italy, Germany, Japan, The Netherlands, Sweden, The United Kingdom and The United States. The Group of Ten is made up (oddly) of eleven industrial countries

*This exposition follows McNeil et al.(2005)

which consult and cooperate on economic, monetary and financial matters. The Basel Committee does not possess any formal supranational supervising authority, and hence its conclusions do not have legal force. Rather, it formulates broad supervisory standards and guidelines and recommends statements of best practice in the expectation that an individual authorities will take steps to implement them through detailed arrangements–statutory or otherwise–which are best suited to their own national systems.

A.2.1 The First Basel Accord (Basel I)

In 1988, the first Basel Accord (Basel I) was born by the Basel Committee of Banking Supervision. Basel I took an important step towards an international minimum capital standard. Its main emphasis was on credit risk, by then clearly the most important source of risk in the banking industry. In hindsight, however, Basel I took an approach which was fairly coarse and measured risk in an insufficiently differentiated way. Also its treatment of derivatives was considered unsatisfactory.

In 1993, the G-30 (an influential international body consisting of senior representatives of the private and public sectors and academia) published a seminal report addressing for the first time so called *off-balance-sheet* products in a systematic way, as presented below.

Balance Sheet	
Asset	Liability or Reserve
	Capital or Surplus

Figure A.1 Balance Sheet

Around the same time, the banking industry clearly saw the need for a proper risk management of these new products. At J.P. Morgan, for instance, the famous Weatherstone 4.15 report asked for a one-day, one-page summary of the bank's market risk to be delivered to the chief executive officer (CEO). *Value-at-Risk* (VaR) as a market risk measure was born.

In a highly dynamic world with round-the-clock market activities, the need for instant market valuation of trading positions (known as *marking-to-market*) became a necessity. Moreover, in markets where so many positions (both long and short) were written on the same underlying, managing risks based on simple aggregation of nominal positions became unsatisfactory. Banks are pushed to consider *netting* effects, i.e. the composition of long versus short positions on the same underlying.

In 1996, an important Amendment to Basel I prescribed a so called *standardized* model for market risk, but at the time allowed bigger (more sophisticated) banks to opt for an *internal*, VaR based model. Legal implementation was to be achieved by the year 2000. The coarseness problem for the credit risk remained unresolved and banks continued to claim that they were not given enough incentives to diversify credit portfolios and that the regulatory capital rules currently in place were far too risk insensitive. Because of overcharging on the regulatory capital side of certain credit positions, banks started shifting business away from certain market segments that they perceived offering a less attractive risk-return profile.

A.2.2 The Second Basel Accord (Basel II)

In 2001, the second Basel Accord (Basel II) was initiated. The main theme of this accord consisted of three pillars which cover market risk, credit risk, operational risk, liquidity risk and insurance risk as follows :

- *Pillar I.* Banks are required to calculate a *minimum capital charge*, referred to as regulatory capital, with the aim of bringing the quantification of this minimal capital more in line with the bank's economic loss potential. Under the Basel II framework, there will be a capital charge for credit risk, market risk, insurance risk, liquidity risk, and operational risk. Whereas the treatment of market risk is unchanged relative to the 1996 Amendment of the Basel I capital accord, the capital charge for credit risk was revised substantially. In computing the capital charge for credit risk and operational risk, banks may choose between three approaches of increasing risk sensitivity and complexity.

- *Pillar II.* A quantitative approach to risk management should be embedded in a well-functioning corporate governance structure. The best practice risk management imposes clear constraints on the organization of the institution, i.e. the board of directors, management, employees, and internal and external audit processes. In particular, the board of directors assumes the ultimate responsibility for oversight of the risk landscape and the formulation of the company's risk appetite.

  It should be note that this pillar is related to the *supervisory review process,* local regulators review the various checks and balances put into place. This pillar recognizes the necessity of an effective overview of the bank's internal assessments of their overall risk, and ensures that management requites effective and had set aside adequate capital for the various risks.

- *Pillar III.* This pillar seeks to establish *market discipline* through a better public disclosure of risk measures and other information relevant to risk management. In particular, banks have to offer greater insight into the adequacy of their capitalization.

The three-pillar concept is a key conceptual change within the Basel II framework. In spite of this concept, the Basel Committee aims to achieve a more holistic approach to risk management that focuses on the interaction between the different risk categories. At the same time the three-pillar concept clearly signals the existing difference between quantifiable and non-quantifiable risks.

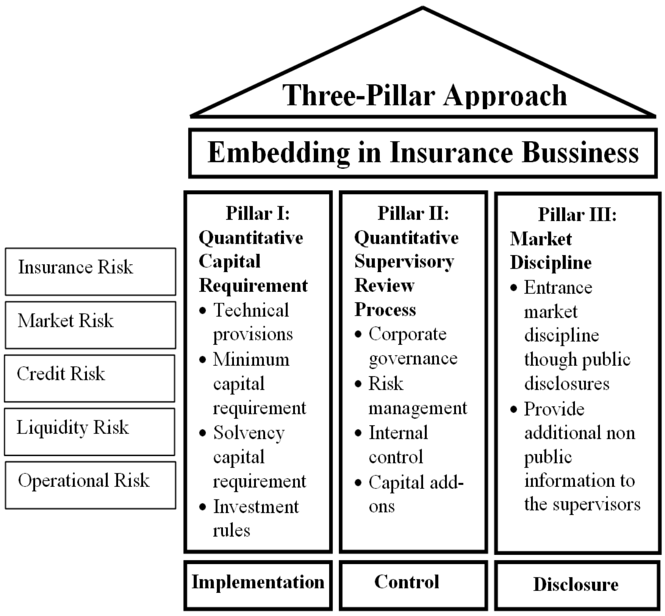

Figure A.2 The Three-Pillars in the Basel II Framework

A.2.3 Solvency I and II

[†]The first EU non-life and life directives on solvency margins appeared around 1970 and in Basel I was defined as an extra capital buffer against unforeseen events such as higher than expected claims levels or unfavourable investment results. In 1997, the Muller report appeared under the heading *"Solvency of insurance undertaking"* leading to a review of the solvency rules and initiated the Solvency I project, which was completed in 2002 and came into force in 2004. Meanwhile, Solvency II was initiated in 2001 with the publication of the influential Sharma report; the detailed technical rules of Solvency II are currently being worked out.

At the heart of Solvency II lies a risk-oriented assessment of overall solvency, honoring the three-pillar concept from Basel II. Insurers are encouraged to measure and manage their risks based on internal models. Consistency between Solvency II (Insurance) and Basel II (Banking) is adhered to as much as possible. The new framework should allow an efficient supervision of insurance groups (holding) and financial conglomerates (bank-assurance).

The EU Insurance Solvency Sub-Committee (2001) focuses on the differences between the Basel II and Solvency II framework as illustrated in the following statement:

The difference between the two prudential regimes goes further in that their actual objectives differ. The prudential objective of the Basel Accord is to reinforce the soundness and stability of the international banking system. To that end, the initial Basel Accord and the draft New Accord are directed primarily at banks that are internationally active. The draft New Accord attaches particular importance to the self-regulating mechanisms of a market where practitioners are dependent on one

[†]This exposition follows McNeil et al.(2005)

another. In the insurance sector, the purpose of prudential supervision is to protect policyholders against the risk of (isolated) bankruptcy facing every insurance company. The systematic risk, assuming that it exists in the insurance sector, has not been deemed to be of sufficient concern to warrant minimum harmonisation of prudential supervisory regimes at international level; nor has it been the driving force behind European harmonisation in this field.

More so than in the case of banking regulation, the regulatory framework for insurance companies has a strong local flavour since many local statutory rules prevail. The various solvency committees in EU member countries and beyond are trying to come up with some global principles which would be binding on a larger geographical scale. Furthermore, the difference between the Basel II and Solvency II framework results in an increased harmonization of supervisory methodology between the different legislative entities, based on a wide international cooperation with actuarial, financial, and accounting bodies.

In principle on solvency II, all risks under Basel II are to be analysed including underwriting; credit risk, market risk, insurance risk, operational risk and liquidity risk. Strong emphasis is put on the modelling of interdependencies and a detailed analysis of street tests. The system should be as much as possible *principle based* rather than *rule based* and there should be *prudent regulation* which focus on the total balance sheet, handing assets and liabilities in a single common framework.

The final decision on Solvency I and II is based on a two-tier procedure. This involves setting a first safety barrier at the level of the so-called *target capital* based on risk sensitive, market-consistent valuation. Breaches of this early-warning level would trigger regulatory intervention. The second and final tier is the *minimum capital level* calculated with solvency rules satisfy the following :

- *Tier I* is a capital which meets the following criteria in full:

  (i). Subordination to policyholder liabilities.

  (ii). Fully available to absorb losses in the event of a winding up.

  (iii). Fully available to absorb losses in a going concern situation.

  (iv). Of substantially sufficient duration given the nature of the liabilities.

  (v). Free of mandatory servicing costs.

- *Tier II* is a capital which meet other points except the once listed above. The amount of Tier II capital is not to exceed Tier I capital.

A.3 The Shareholder's View

[‡]It is widely believed that proper financial risk management can increase the value of a corporation and hence shareholder value. In fact, this is the main reason why corporations which are not subject to regulation by financial supervisory authorities engage in risk management activities. Understanding the relationship between shareholder value and financial risk management also has important implications for the design of risk-management (RM) systems. Questions to be answered include the followings.

- When does RM increase the value of a firm, and which risks should be managed?

- How should RM consider factors concerning investment policy and capital budgeting?

[‡]This exposition follows McNeil et al.(2005)

There is a rather extensive corporate finance literature on the issue of *corporate risk management and shareholder value.* We briefly discuss some of the main arguments. In this way we hope to alert the reader to the fact that there is more to RM than the mainly technical questions related to the implementation of RM strategies dealt.

The first thing to note is that from a corporate-finance perspective, it is by no means obvious that in a world with perfect capital markets RM enhances shareholder value. While individual investors are typically risk averse and should therefore manage the risk in their portfolios, it is not clear that RM or risk reduction at the *corporate level,* such as hedging a foreign-currency exposure or holding a certain amount of risk capital, increase the value of a corporation. The rationale for this-at first surprising-observation is simple. If investors have access to perfect capital markets, they can do the RM transactions they deem necessary via their own trading and diversification. The following statement from the chief investment officer of an insurance company exemplifies this line of reasoning:

If our shareholders believe that our investment portfolio is too risky, they should short futures on major stock market indices.

The potential irrelevance of corporate RM for the value of a corporation is an immediate consequence of the famous *Modigliani-Miller Theorem* (Modigliani and Miller, 1958). This result, marking the beginning of modern corporate finance theory, that, in an ideal world without taxes, bankruptcy cost and informational asymmetries, with frictionless and arbitrage-free capital markets, the financial structure of a firm and hence also its RM decisions-are irrelevant for the firm's value. Hence, in order to find reasons for corporate RM, one has to "turn the Modigliani-Miller Theorem upside down" and identify situations where RM enhances the value of a firm by deviating from the unrealistically strong assumptions of the theorem. This leads to the following rationale for RM.

- RM can reduce *tax costs.* Under a typical tax regime the amount of tax to be paid by a corporation is a *convex* function of its profits. By reducing the variability in a firm's cash flow, RM can therefore lead to a higher expected after-tax profit.
- RM can be beneficial, since a company may (and usually will) have better access to capital markets than individual investors.
- RM can increase a firm's value in the presence of *bankruptcy costs,* as it makes bankruptcy less likely.
- RM can reduce the impact of *costly external financing* on the firm's value, as it facilitates the achievement of optimal investment.

The last two points merit a more detailed discussion. Bankruptcy costs consist of direct bankruptcy costs, such as the cost of lawsuits, and the more important indirect bankruptcy costs. The latter may include liquidation costs, which can be substantial in the case of intangibles like research and development (*R&D*) and know-how. This is why high *R&D* spending appears to be positively related to the use of RM techniques. Moreover, increased likelihood of bankruptcy often has a negative effect on key employees, management, and customer relations, in particular in areas where a client wants a long-term business relationship. For instance, few customers would want to enter into a life insurance contract with an insurance company which is known to be close to bankruptcy. On a related note, banks which are close to bankruptcy might be faced with the unpalatable prospect of a bank run, where depositors try to withdraw their money simultaneously.

A.4 Economic Capital

[§]Economic capital is the capital that shareholders should invest in the company in order to limit the probability of default to a given confidence level over a given time horizon. More broadly, economic capital offers a firm-wide language for discussing and pricing risk related directly to the principle concerns of management and other key stakeholders; namely, institutional solvency and profitability. In this broader sense, economic capital represents the emerging best practice for measuring and reporting all kinds of risk across a financial organization.

Economic capital is so called because it measures risk in term of *economic* realities rather than potentially misleading regulatory or accounting rules. Moreover, part of the measurement process involves converting a risk distribution into the amount of *capital* that is required to support the risk, in line with the institution's target financial strength (e.g. credit rating). Hence, the calculation of economic capital is a process that begins with the quantification of the risks that any given company faces over a given time period. These risks include those that are well-defined by a regulatory point of view, such as credit risk, market risk, insurance risk and operational risk, and also include other categories like liquidity risk, reputational risk and strategic or business risk.

[§]This exposition follows McNeil et al.(2005)

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