Discrete GB-Splines and Their Properties

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Abstract. Discrete generalized splines are continuous piecewise defined functions which meet some smoothness conditions for the first and second divided differences at the knots. Direct algorithms and recurrence relations are proposed for constructing discrete generalized B-splines (discrete GB-splines for short). Properties of discrete GB-splines and their series are studied. It is shown that discrete GB-splines form weak Chebyshev systems and that series of discrete GB-splines have a variation diminishing property.

§1. Introduction

The tools of generalized splines and GB-splines are widely used in solving problems of shape preserving approximation (e.g., see [7]). By introducing various parameters into the spline structure, one can preserve characteristics of the initial data such as positivity, monotonicity, convexity, presence of linear sections, etc. Here, the main challenge is to develop algorithms that choose parameters automatically. Recently, in [2] a difference method for constructing shape preserving hyperbolic splines as solutions of multipoint boundary value problems was developed. Such an approach avoids the computation of hyperbolic functions and has substantial other advantages. However, the extension of a mesh solution will be a discrete hyperbolic spline. In this paper we consider more general constructions of discrete generalized splines and discrete GB-splines, and investigate their main properties.

§2. Discrete Generalized Splines

Let a partition $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a,b]$ be given. For fixed $\tau_j^{L_i} > 0$ and $\tau_j^{R_i} > 0$, $j = i, i+1$, and a function S

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which is defined and continuous on the real line $\mathbb{R}$ we introduce the linear difference operators

$$\begin{aligned}D_{i,1}S(x) =&(\lambda_i^{R_i}S[x-\tau_i^{L_i},x]+\lambda_i^{L_i}S[x,x+\tau_i^{R_i}])(1-t)\\&+(\lambda_{i+1}^{R_i}S[x-\tau_{i+1}^{L_i},x]+\lambda_{i+1}^{L_i}S[x,x+\tau_{i+1}^{R_i}])t,\\D_{i,2}S(x) =&2S[x-\tau_i^{L_i},x,x+\tau_i^{R_i}](1-t)+2S[x-\tau_{i+1}^{L_i},x,x+\tau_{i+1}^{R_i}]t,\\&x\in[x_i,x_{i+1}],\quad i=0,\dots,N-1,\end{aligned}$$

where $\lambda_j^{R_i}=1-\lambda_j^{L_i}=\tau_j^{R_i}/(\tau_j^{L_i}+\tau_j^{R_i})$, $j=i,i+1$ and $t=(x-x_i)/h_i$, $h_i=x_{i+1}-x_i$. The square parentheses denote the usual first and second divided differences of the function S.

We associate to Δ a system of functions $\{1,x,\Phi_i,\Psi_i\}$, $i=0,\dots,N-1$, which are defined and continuous on $\mathbb{R}$ and for given i are linearly independent on the interval $[x_i,x_{i+1}]$. The functions Φ_i and Ψ_i are subject to the constraints

$$\begin{aligned}&\Phi_i(x_{i+1}-\tau_{i+1}^{L_i})=\Phi_i(x_{i+1})=\Phi_i(x_{i+1}+\tau_{i+1}^{R_i})=0,\quad D_{i,2}\Phi_i(x_i)=1,\\&\Psi_i(x_i-\tau_i^{L_i})=\Psi_i(x_i)=\Psi_i(x_i+\tau_i^{R_i})=0,\qquad\quad D_{i,2}\Psi_i(x_{i+1})=1.\end{aligned}\tag{1}$$

Any element S_i of the linear space Υ_i spanned by the four functions 1, x, Φ_i, Ψ_i can be uniquely written as

$$\begin{aligned}S_i(x) =&S_i(x_i)(1-t)+S_i(x_{i+1})t+D_{i,2}S_i(x_i)[\Phi_i(x)-\Phi_i(x_i)(1-t)]\\&+D_{i,2}S_i(x_{i+1})[\Psi_i(x)-\Psi_i(x_{i+1})t].\end{aligned}\tag{2}$$

Definition 1. *A function $S:[a,b]\to\mathbb{R}$ is called a* discrete generalized spline *if:*

(i) *for any integer i, $0\le i\le N-1$, there exists a unique function $S_i\in\Upsilon_i$ such that*
$$S(x)\equiv S_i(x),\quad x\in[x_i,x_{i+1}];\tag{3}$$

(ii) *for all integers $i=1,\dots,N-1$, S satisfies the following smoothness conditions*
$$\begin{aligned}S_{i-1}(x_i)&=S_i(x_i),\\D_{i-1,1}S_{i-1}(x_i)&=D_{i,1}S_i(x_i),\\D_{i-1,2}S_{i-1}(x_i)&=D_{i,2}S_i(x_i).\end{aligned}\tag{4}$$

The set of discrete generalized splines satisfying Definition 1 will be denoted by S_4^{DG}. The usual operations of addition of elements from S_4^{DG} and their multiplication by real numbers give again elements in the set S_4^{DG} which hence is a linear space.

Definition 1 generalizes the notion of discrete cubic splines in [8]. If $\tau_j^{L_i}\to 0$, $\tau_j^{R_i}\to 0$, $j=i,i+1$ for all i, then as the limiting case we obtain generalized splines in [6]. If $\tau_i^{L_j} = \tau_i^L$ and $\tau_i^{R_j} = \tau_i^R$, $j = i-1, i$, then according to smoothness conditions (4), the values of the functions S_{i-1} and S_i at the three consecutive points $x_i - \tau_i^L$, x_i, $x_i + \tau_i^R$ coincide. Setting $\tau_j^{L_i} = \tau_j^{R_i} = \tau_i$, $j = i, i+1$, we obtain $D_{i,1}S(x) = S[x-\tau_i, x+\tau_i]$ and $D_{i,2}S(x) = 2S[x-\tau_i, x, x+\tau_i]$, which is the case discussed in [2].

According to the conditions (4), the discrete generalized spline S defined by (2) and (3) can be written as

$$\begin{aligned} S(x) =& S(x_i)(1-t) + S(x_{i+1})t + M_i[\Phi_i(x) - \Phi_i(x_i)(1-t)] \\ &+ M_{i+1}[\Psi_i(x) - \Psi_i(x_{i+1})t], \end{aligned} \tag{5}$$

for $x \in [x_i, x_{i+1}]$ and $i = 0, \dots, N-1$, where $M_j = D_{i,2}S_i(x_j)$, $j = i, i+1$.

The functions Φ_i and Ψ_i depend on tension parameters which influence the behaviour of S fundamentally. We call them the defining functions. In practice one takes Φ_i to depend on a parameter p_i, and Ψ_i to depend on a parameter q_i, $0 \le p_i, q_i < \infty$. In the limiting case when $p_i, q_i \to \infty$ we require that $\lim_{p_i \to \infty} \Phi_i(x) = 0$, $x \in (x_i, x_{i+1}]$ and $\lim_{q_i \to \infty} \Psi_i(x) = 0$, $x \in [x_i, x_{i+1})$ so that the function S in formula (5) turns into a linear function. Additionally, we require that if $p_i = q_i = 0$ for all i, then we get a discrete cubic spline.

§3. Construction of Discrete GB-Splines

Let us construct a basis for the space of discrete generalized splines S_4^{DG} by using functions which have local supports of minimum length. Since $\dim(S_4^{DG}) = N+3$ we extend the grid Δ by adding the points x_j, $j = -3, -2, -1, N+1, N+2, N+3$, such that $x_{-3} < x_{-2} < x_{-1} < a$, $b < x_{N+1} < x_{N+2} < x_{N+3}$. As in Section 2, for each interval $[x_i, x_{i+1}]$, $i = -3, -2, -1, N, N+1, N+2$, we introduce the linear space Υ_i. This permits us to define the discrete generalized spline S on the extended interval $[x_{-3}, x_{N+3}]$.

We demand that the discrete GB-splines $\mathrm{B}_{-3}, \dots, \mathrm{B}_{N-1}$ have the properties

$$\begin{aligned} &\mathrm{B}_i(x) > 0, \quad x \in (x_i + \tau_i^{R_i}, x_{i+4} - \tau_{i+4}^{L_{i+3}}), \\ &\mathrm{B}_i(x) \equiv 0, \quad x \notin (x_i, x_{i+4}), \end{aligned} \tag{6}$$

$$\sum_{j=-3}^{N-1} \mathrm{B}_j(x) \equiv 1, \quad x \in [a, b]. \tag{7}$$

According to (5), on the interval $[x_j, x_{j+1}]$, $j = i, \dots, i+3$, for each $i = -3, \dots, N-1$ the discrete GB-spline B_i has the form

$$\mathrm{B}_i(x) \equiv \bar{\mathrm{B}}_{i,j}(x) = P_{i,j}(x) + \Phi_j(x) M_{j,\mathrm{B}_i} + \Psi_j(x) M_{j+1,\mathrm{B}_i}, \tag{8}$$

where $P_{i,j}$ is a linear polynomial and $M_{l,\mathrm{B}_i} = D_{j,2}\bar{\mathrm{B}}_{i,j}(x_l)$, $l = j, j+1$ are constants to be determined.

The smoothness conditions (4) together with the constraints (1) give the relations

$$P_{i,j}(x_j) = P_{i,j-1}(x_j) + z_j M_{j,\mathrm{B}_i},$$
$$D_{j,1}P_{i,j}(x_j) = D_{j-1,1}P_{i,j-1}(x_j) + c_{j-1,2}M_{j,\mathrm{B}_i},$$

where

$$z_j \equiv z_j(x_j) = \Psi_{j-1}(x_j) - \Phi_j(x_j),$$
$$c_{j-1,2} = D_{j-1,1}\Psi_{j-1}(x_j) - D_{j,1}\Phi_j(x_j).$$

Thus

$$P_{i,j}(x) = P_{i,j-1}(x) + [z_j + c_{j-1,2}(x - x_j)]M_{j,\mathrm{B}_i}. \tag{9}$$

According to (4), the condition $\mathrm{B}_i(x) \equiv 0$ for $x \notin (x_i, x_{i+4})$ is satisfied if and only if

$$\bar{\mathrm{B}}_{i,i}(x_i) = D_{i,1}\bar{\mathrm{B}}_{i,i}(x_i) = D_{i,2}\bar{\mathrm{B}}_{i,i}(x_i) = 0,$$
$$\bar{\mathrm{B}}_{i,i+3}(x_{i+4}) = D_{i+3,1}\bar{\mathrm{B}}_{i,i+3}(x_{i+4}) = D_{i+3,2}\bar{\mathrm{B}}_{i,i+3}(x_{i+4}) = 0.$$

Due to (8) and (1), the latter relations are equivalent to

$$P_{i,i} \equiv 0, \quad M_{i,\mathrm{B}_i} = 0 \quad \text{and} \quad P_{i,i+3} \equiv 0, \quad M_{i+4,\mathrm{B}_i} = 0.$$

Therefore, by repeated use of (9) we obtain

$$P_{i,j}(x) = \sum_{l=i+1}^{j} [z_l + c_{l-1,2}(x - x_l)]M_{l,\mathrm{B}_i} = -\sum_{l=j+1}^{i+3} [z_l + c_{l-1,2}(x - x_l)]M_{l,\mathrm{B}_i}.$$

In particular, the following identity is valid,

$$\sum_{j=i+1}^{i+3} [z_j + c_{j-1,2}(x - x_j)]M_{j,\mathrm{B}_i} \equiv 0,$$

from which one obtains the equalities

$$\sum_{j=i+1}^{i+3} c_{j-1,2}y_j^r M_{j,\mathrm{B}_i} = 0, \quad r = 0, 1, \quad y_j = x_j - \frac{z_j}{c_{j-1,2}}. \tag{10}$$

Thus the formula for the discrete GB-spline B_i takes the form

$$\mathrm{B}_i(x) = \begin{cases} \Psi_i(x)M_{i+1,\mathrm{B}_i}, & x \in [x_i, x_{i+1}), \\ (x - y_{i+1})c_{i,2}M_{i+1,\mathrm{B}_i} + \Phi_{i+1}(x)M_{i+1,\mathrm{B}_i} + \Psi_{i+1}(x)M_{i+2,\mathrm{B}_i}, & \\ & x \in [x_{i+1}, x_{i+2}), \\ (y_{i+3} - x)c_{i+2,2}M_{i+3,\mathrm{B}_i} + \Phi_{i+2}(x)M_{i+2,\mathrm{B}_i} + \Psi_{i+2}(x)M_{i+3,\mathrm{B}_i}, & \\ & x \in [x_{i+2}, x_{i+3}), \\ \Phi_{i+3}(x)M_{i+3,\mathrm{B}_i}, & x \in [x_{i+3}, x_{i+4}), \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

After substituting formula (11) into the normalization condition (7) written for $x \in [x_i, x_{i+1}]$, $i = 0, \dots, N-1$, we obtain

$$\sum_{j=i-3}^{i} \mathrm{B}_j(x) = \Phi_i(x) \sum_{j=i-3}^{i-1} M_{i,\mathrm{B}_j} + \Psi_i(x) \sum_{j=i-2}^{i} M_{i+1,\mathrm{B}_j}$$
$$+ (y_{i+1} - x) c_{i,2} M_{i+1,\mathrm{B}_{i-2}} + (x - y_i) c_{i-1,2} M_{i,\mathrm{B}_{i-1}} \equiv 1.$$

Due to the linear independence of functions 1, x, Φ_i, and Ψ_i on $[x_i, x_{i+1}]$, the latter relation is satisfied if and only if

$$\sum_{j=i-3}^{i-1} M_{i,\mathrm{B}_j} = \sum_{j=i-2}^{i} M_{i+1,\mathrm{B}_j} = 0, \tag{12}$$

$$\begin{aligned} y_{i+1} c_{i,2} M_{i+1,\mathrm{B}_{i-2}} - y_i c_{i-1,2} M_{i,\mathrm{B}_{i-1}} &= 1, \\ c_{i,2} M_{i+1,\mathrm{B}_{i-2}} - c_{i-1,2} M_{i,\mathrm{B}_{i-1}} &= 0. \end{aligned} \tag{13}$$

In particular, from (13) we derive the identity

$$(y_{i+1} - x) c_{i,2} M_{i+1,\mathrm{B}_{i-2}} + (x - y_i) c_{i-1,2} M_{i,\mathrm{B}_{i-1}} \equiv 1.$$

Solving system (13) and using (10) or (12), we obtain

$$M_{j,\mathrm{B}_i} = \frac{y_{i+3} - y_{i+1}}{c_{j-1,2}\, \omega'_{i+1}(y_j)}, \qquad j = i+1, i+2, i+3,$$
$$\omega_{i+1}(x) = (x - y_{i+1})(x - y_{i+2})(x - y_{i+3})$$

or with the notation $c_{j,3} = y_{j+2} - y_{j+1}$, $j = i, i+1$,

$$\begin{aligned} M_{i+1,\mathrm{B}_i} &= \frac{1}{c_{i,2} c_{i,3}}, \\ M_{i+2,\mathrm{B}_i} &= -\frac{1}{c_{i+1,2}} \Big(\frac{1}{c_{i,3}} + \frac{1}{c_{i+1,3}} \Big), \\ M_{i+3,\mathrm{B}_i} &= \frac{1}{c_{i+2,2} c_{i+1,3}}. \end{aligned} \tag{14}$$

§4. Properties of Discrete GB-Splines

The functions $\mathrm{B}_{-3}, \dots, \mathrm{B}_{N-1}$ possess many of the properties inherent in the usual discrete polynomial B-splines. To provide inequality (6), in what follows we need to impose additional conditions on the functions Φ_j and Ψ_j which, as the reader may readily check, are satisfied by all the defining functions given in Section 8. The proofs of the following four assertions repeat those given in [5].

Lemma 1. *Let the conditions*

$$0 < 2\Phi_j(x_j) < -h_i D_{j,1}\Phi_j(x_j), \quad 0 < 2\Psi_j(x_{j+1}) < h_j D_{j,1}\Psi_j(x_{j+1}),$$

$j = i+1, i+2, i+3$, *be satisfied. Then in (14)*

$$c_{j,k} > 0, \quad j = i,\dots,i+4-k; \; k = 2,3,$$

and, therefore,

$$(-1)^{j-i-1} M_{j,\mathrm{B}_i} > 0, \quad j = i+1, i+2, i+3.$$

Theorem 1. *Let the conditions of Lemma 1 be satisfied, the functions* Φ_j *and* Ψ_j *be convex and* $D_{j,2}\Phi_j$ *and* $D_{j,2}\Psi_j$ *be strictly monotone on the interval* $[x_j, x_{j+1}]$ *for all* j. *Then the functions* $\mathrm{B}_{-3},\dots,\mathrm{B}_{N-1}$ *have the following properties:*

(a) $\mathrm{B}_j(x) > 0$ *for* $x \in (x_j + \tau_j^{R_j}, x_{j+4} - \tau_{j+4}^{L_{j+3}})$, *and* $\mathrm{B}_j(x) \equiv 0$ *if* $x \notin (x_j, x_{j+4})$;

(b) B_j *satisfies the smoothness conditions (4);*

(c) $\Phi_j(x) = c_{j-1,2}c_{j-2,3}\mathrm{B}_{j-3}(x)$, $\Psi_j(x) = c_{j,2}c_{j,3}\mathrm{B}_j(x)$ *for* $x \in [x_j, x_{j+1}]$, $j = 0,\dots,N-1$, *and*

$$\sum_{j=-3}^{N-1} y_{j+2}^r \mathrm{B}_j(x) \equiv x^r, \quad r = 0,1 \quad \text{for} \quad x \in [a,b]. \tag{15}$$

Lemma 2. *The functions* $\mathrm{B}_{-3},\dots,\mathrm{B}_{N-1}$ *are splines from* S_4^{DG} *with finite supports of minimal length.*

Theorem 2. *The functions* $\mathrm{B}_{-3},\dots,\mathrm{B}_{N-1}$ *are linearly independent on* $[a,b]$ *and form a basis of the space of discrete generalized splines* S_4^{DG}.

§5. Local Approximation by Discrete GB-Splines

According to Theorem 2, any discrete generalized spline $S \in S_4^{DG}$ can be uniquely written in the form

$$S(x) = \sum_{j=-3}^{N-1} b_j \mathrm{B}_j(x), \quad x \in [a,b] \tag{16}$$

for some coefficients b_j.

If the coefficients b_j in (16) are known, then by virtue of formula (11) we can write out an expression for the discrete generalized spline S on the interval $[x_i, x_{i+1}]$, which is convenient for calculations,

$$S(x) = b_{i-2} + b_{i-1}^{(1)}(x - y_i) + b_{i-1}^{(2)}\Phi_i(x) + b_i^{(2)}\Psi_i(x), \tag{17}$$

where

$$b_j^{(k)} = \frac{b_j^{(k-1)} - b_{j-1}^{(k-1)}}{c_{j,4-k}}, \quad k = 1,2; \quad b_j^{(0)} = b_j. \tag{18}$$

The representations (16) and (17) allow us to find a simple and effective way to approximate a given continuous function f from its samples.

Theorem 3. *Let a continuous function f be given by its samples $f(y_j)$, $j=-1,\dots,N+1$, where y_j is defined in (10). Then for $b_j=f(y_{j+2})$, $j=-3,\dots,N-1$, formula (16) is exact for linear polynomials and provides a formula for local approximation.*

Proof: It suffices to employ the identities (15). Inserting the coefficients $b_{j-2}=1$ and $b_{j-2}=y_j$ in formula (16), and using the identities (15), we prove the first assertion of the theorem.

For $b_{j-2}=f(y_j)$, formula (17) can be rewritten as

$$\begin{aligned} S(x) =& f(y_i)+f[y_i,y_{i+1}](x-y_i)+(y_{i+1}-y_{i-1})f[y_{i-1},y_i,y_{i+1}]c_{i-1,2}^{-1}\Phi_i(x) \\ &+(y_{i+2}-y_i)f[y_i,y_{i+1},y_{i+2}]c_{i,2}^{-1}\Psi_i(x), \qquad x\in[x_i,x_{i+1}]. \end{aligned}$$

This is the formula of local approximation. The theorem is thus proved. □

Corollary 1. *Let a continuous function f be given by its samples $f_j=f(x_j)$, $j=-2,\dots,N+2$. Then by setting*

$$b_{j-2}=f_j-\big(\Psi_{j-1}(x_j)f[x_j,x_{j+1}]-\Phi_j(x_j)f[x_{j-1},x_j]\big)c_{j-1,2}^{-1} \tag{19}$$

in (16), we obtain a formula of three-point local approximation, which is exact for linear polynomials.

Proof: It suffices to check the result for the monomials 1 and x. Then according to (19), we obtain $b_{j-2}=1$ and $b_{j-2}=y_j$, and it only remains to make use of the identities (15). This proves the corollary. □

Equation (17) permits us to write the coefficients of the spline S in its representation (16) in the form

$$b_{j-2}=\begin{cases} S(y_j)-D_{j-1,2}S(x_{j-1})\Phi_{j-1}(y_j)-D_{j,2}S(x_j)\Psi_{j-1}(y_j), & y_j<x_j, \\ S(y_j)-D_{j,2}S(x_j)\Phi_j(y_j)-D_{j+1,2}S(x_{j+1})\Psi_j(y_j), & y_j\ge x_j. \end{cases}$$

According to this formula we have $b_{j-2}=S(y_j)+O(\overline{h}_j^2)$, $\overline{h}_j=\max(h_{j-1},h_j)$. Hence it follows that the control polygon (e.g., see [3]) converges quadratically to the function f when $b_{j-2}=f(y_j)$, or if the formula (19) is used.

§6. Recurrence Formulae for Discrete GB-Splines

Let us define functions

$$\mathrm{B}_{j,2}(x)=\begin{cases} D_{j,2}\Psi_j(x), & x\in[x_j,x_{j+1}), \\ D_{j+1,2}\Phi_{j+1}(x), & x\in[x_{j+1},x_{j+2}], \\ 0, & \text{otherwise}, \end{cases} \qquad j=i,i+1,i+2. \tag{20}$$

We assume that the functions $D_{j,2}\Psi_j$ and $D_{j+1,2}\Phi_{j+1}$ are strictly monotone on $[x_j, x_{j+1})$ and $[x_{j+1}, x_{j+2}]$, respectively. The splines $\mathrm{B}_{j,2}$ are a generalization of the "hat-functions" for polynomial B-splines. They are nonnegative, and furthermore, $\mathrm{B}_{j,2}(x_j) = \mathrm{B}_{j,2}(x_{j+2}) = 0$, $\mathrm{B}_{j,2}(x_{j+1}) = 1$. Let us denote

$$\begin{matrix} D_1S(x) \equiv D_{i,1}S_i(x), \\ D_2S(x) \equiv D_{i,2}S_i(x), \end{matrix} \quad x \in [x_i, x_{i+1}], \quad i = 0, \dots, N-1;$$

then from (4) D_1S and D_2S are well defined if $S \in S_4^{DG}$. With the previous notation, according to (11), (14) and (20) we obtain

$$\begin{aligned} D_2\mathrm{B}_i(x) &= \sum_{j=i+1}^{i+3} M_{j,\mathrm{B}_i}\mathrm{B}_{j-1,2}(x) \\ &= \frac{1}{c_{i,3}}\Big(\frac{\mathrm{B}_{i,2}(x)}{c_{i,2}} - \frac{\mathrm{B}_{i+1,2}(x)}{c_{i+1,2}}\Big) - \frac{1}{c_{i+1,3}}\Big(\frac{\mathrm{B}_{i+1,2}(x)}{c_{i+1,2}} - \frac{\mathrm{B}_{i+2,2}(x)}{c_{i+2,2}}\Big). \end{aligned} \tag{21}$$

In addition, the function $D_1\mathrm{B}_i$ satisfies the relation

$$D_1\mathrm{B}_i(x) = \frac{\mathrm{B}_{i,3}(x)}{c_{i,3}} - \frac{\mathrm{B}_{i+1,3}(x)}{c_{i+1,3}}, \tag{22}$$

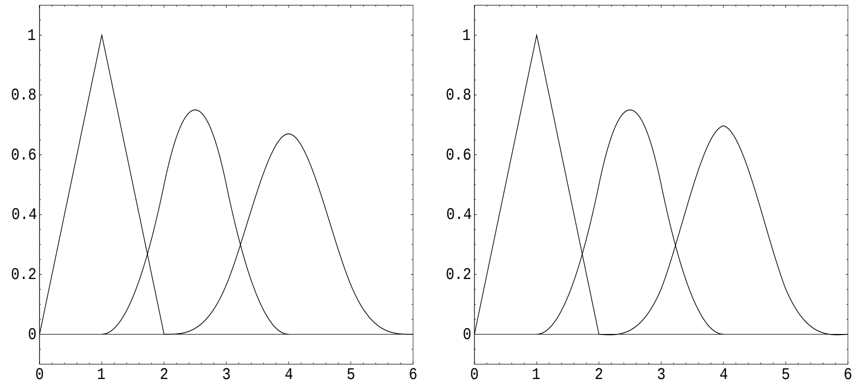
where

$$\mathrm{B}_{j,3}(x) = \begin{cases} \dfrac{D_{j,1}\Psi_j(x)}{c_{j,2}}, & x \in [x_j, x_{j+1}), \\ 1 + \dfrac{D_{j+1,1}\Phi_{j+1}(x)}{c_{j,2}} - \dfrac{D_{j+1,1}\Psi_{j+1}(x)}{c_{j+1,2}}, & x \in [x_{j+1}, x_{j+2}), \\ -\dfrac{D_{j+2,1}\Phi_{j+2}(x)}{c_{j+1,2}}, & x \in [x_{j+2}, x_{j+3}), \\ 0, & \text{otherwise}. \end{cases} \tag{23}$$

Using formula (23), it is easy to show that the functions $\mathrm{B}_{-2}, \dots, \mathrm{B}_{N-1}$ satisfy the first and second smoothness conditions in (4), have supports of minimum length, are linearly independent and form a partition of unity:

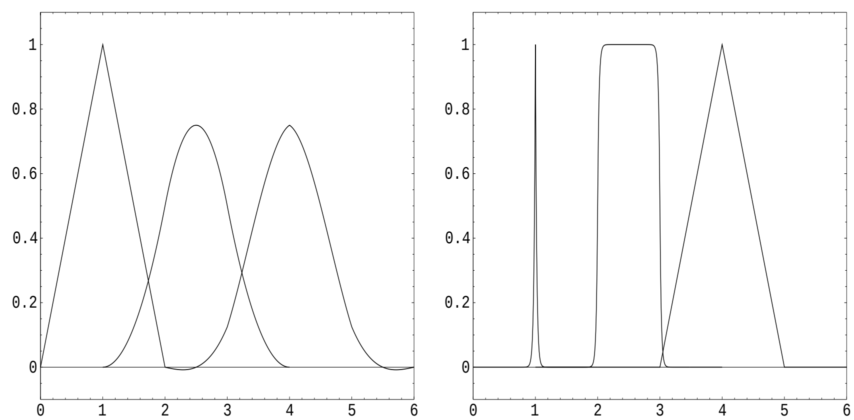
$$\sum_{j=1}^{N-1} \mathrm{B}_{j,3}(x) \equiv 1, \quad x \in [a, b].$$

Figures 1 and 2 show the graphs of the discrete GB-splines $\mathrm{B}_{j,2}$, $\mathrm{B}_{j,3}$, and B_j (from left to right) on a uniform mesh with step size $h_i = 1$ and with $\tau_j^{L_i} = \tau_j^{R_i} = \tau$, $j = i, i+1$, for all i. We have chosen discretization



(a) (b)

Fig. 1. The discrete GB-splines $\mathrm{B}_{j,k}$, $k = 2, 3, 4$ (from left to right) on a uniform mesh with step size $h_i = 1$, no tension and discretization parameter $\tau = 0.1$ (a) and $\tau = 0.33$ (b).



(a) (b)

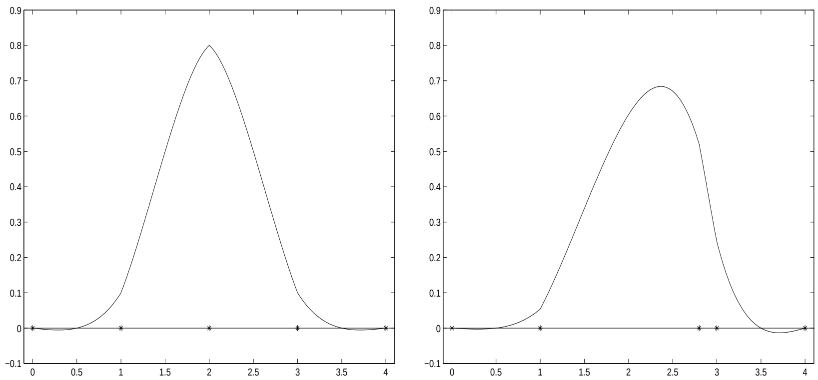
Fig. 2. Same as Figure 1, but with discretization parameter $\tau = 0.5$ (a) and with tension parameters $q_i = 50$, for all i (b).

parameter $\tau = 0.1$ (Fig. 1(a) and Fig. 2(b)), $\tau = 0.33$ (Fig. 1(b)) and $\tau = 0.5$ (Fig. 2(a)) for

$$\Psi_i(x) = \psi_i(q_i, t)h_i^2 = \frac{\hat{\tau}_i \sinh q_i t - t \sinh(q_i \hat{\tau}_i)}{\frac{4}{\hat{\tau}_i} \sinh^2 \frac{q_i \hat{\tau}_i}{2} \sinh q_i} h_i^2, \quad \hat{\tau}_i = \frac{\tau}{h_i},$$

$$\Phi_i(x) = \psi_i(q_i, 1-t)h_i^2.$$

This is a special case of Example 4 in Section 8. In Figures 1 and 2(a) we



(a) (b)

Fig. 3. The discrete GB-splines $\mathrm{B}_{j,4}$ on a uniform mesh (a) and on a nonuniform mesh (b). The asterisk $*$ denotes the x_i. For both plots $q_i = 2$ and $\tau = 0.5$.

have parameters $q_i = 0$, i.e. we have conventional discrete cubic B-splines (e.g., see [8]). Visually, the presence of intervals where the B-spline B_j is negative is more visible with growing discretization parameter τ. In Figure 2(b) the tension parameters are $q_i = 50$ for all i, whence the shape of the graphs is practically unchanged when τ increases from 0.1 to 0.5. As the limit for $q_i \to \infty$ we obtain the pulse function for $\mathrm{B}_{j,2}$, the "step-function" for $\mathrm{B}_{j,3}$ and the "hat-function" for B_j (all of height 1).

Figure 3 shows the graphs of discrete GB-splines $\mathrm{B}_{j,4}$ on a uniform mesh (left) and on a nonuniform mesh (right), where the asterisk $*$ denotes the x_i. For both plots $q_i = 2$ and $\tau = 0.5$.

Applying formulae (21) and (22) to the representation (16), we also obtain

$$D_1 S(x) = \sum_{j=-2}^{N-1} b_j^{(1)} \mathrm{B}_{j,3}(x), \quad D_2 S(x) = \sum_{j=-1}^{N-1} b_j^{(2)} \mathrm{B}_{j,2}(x), \tag{24}$$

where $b_j^{(k)}$, $k = 1, 2$ are defined in (18).

§7. Series of Discrete GB-Splines (Uniform Case)

Let us suppose that each step size $h_i = x_{i+1} - x_i$ of the mesh $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ is an integer multiple of the same tabulation step, τ, of some uniform mesh refinement on $[a, b]$. For $\theta \in \mathbb{R}$, $\tau > 0$ define $\mathbb{R}_{\theta\tau} = \{\theta + i\tau : \ i \text{ is an integer}\}$ and let $\mathbb{R}_{\theta 0} = \mathbb{R}$. For any $a, b \in \mathbb{R}$ and $\tau > 0$ let $[a, b]_\tau = [a, b] \cap \mathbb{R}_{a\tau}$.

The functions $\mathrm{B}_{j,2}$, $\mathrm{B}_{j,3}$, and B_j with $\tau_j^{L_i} = \tau_j^{R_i} = \tau$, $j = i, i+1$ for all i are nonnegative on the discrete interval $[a,b]_\tau$. This permits us to reprove the main results for discrete polynomial splines in [9] for series of discrete generalized splines. In particular, if in (16) and (24) we have coefficients $b_j^{(k)} > 0$, $k = 0,1,2$, $j = -3+k,\dots,N-1$, then the spline S will be a positive, monotonically increasing and convex function on $[a,b]_\tau$.

Denote by $\operatorname{supp}_\tau \mathrm{B}_i = \{x \in \mathbb{R}_{a,\tau} | \mathrm{B}_i(x) > 0\}$ the discrete support of the spline B_i, i.e. the discrete set $(x_i + \tau, x_{i+4} - \tau)_\tau$.

Theorem 4. *Assume that $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ are prescribed points on the discrete line $\mathbb{R}_{a,\tau}$. Then*

$$D = \det\left(\mathrm{B}_i(\zeta_j)\right) \geq 0, \quad i,j = -3,\dots,N-1$$

and strict positivity holds if and only if

$$\zeta_i \in \operatorname{supp}_\tau \mathrm{B}_i, \quad i = -3,\dots,N-1. \tag{25}$$

The proof of this theorem repeats that of Theorem 8.66 in [9, p. 355]. The following three statements follow immediately from Theorem 4.

Corollary 2. *The system of discrete GB-splines $\mathrm{B}_{-3},\dots,\mathrm{B}_{N-1}$ associated with knots on $\mathbb{R}_{a,\tau}$ is a weak Chebyshev system according to the definition given in* [9, p. 36], *i.e. for any $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ in $\mathbb{R}_{a,\tau}$ we have $D \geq 0$ and $D > 0$ if and only if condition (25) is satisfied. In the latter case the discrete generalized spline $S(x) = \sum_{j=-3}^{N-1} b_j \mathrm{B}_j(x)$ has no more than $N+2$ zeros.*

Corollary 3. *If the conditions of Theorem 4 are satisfied, then the solution of the interpolation problem*

$$S(\zeta_i) = f_i, \quad i = -3,\dots,N-1, \quad f_i \in \mathbb{R} \tag{26}$$

exists and is unique.

Let $A = \{a_{ij}\}$, $i = 1,\dots,m$, $j = 1,\dots,n$, be a rectangular $m \times n$ matrix with $m \leq n$. The matrix A is said to be totally nonnegative (totally positive) (e.g., see [4]) if the minors of all order of the matrix are nonnegative (positive), i.e. for all $1 \leq p \leq m$, we have

$$\det\left(a_{i_k j_l}\right) \geq 0 \ (>0) \quad \text{for all} \quad \begin{array}{l} 1 \leq i_1 < \cdots < i_p \leq m, \\ 1 \leq j_1 < \cdots < j_p \leq n. \end{array}$$

Corollary 4. *For arbitrary integers $-3 \leq \nu_{-3} < \cdots < \nu_{p-4} \leq N-1$ and $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{p-4}$ in $\mathbb{R}_{a,\tau}$ we have*

$$\bar{D}_p = \det\left\{\mathrm{B}_{\nu_i}(\zeta_j)\right\} \geq 0, \quad i,j = -3,\dots,p-4$$

and strict positivity holds if and only if

$$\zeta_i \in \operatorname{supp}_{\tau} \mathrm{B}_{\nu_i}, \quad i = -3, \ldots, p-4$$

i.e. the matrix $\{\mathrm{B}_j(\zeta_i)\}_{i,j=-3,\ldots,N-1}$ *is totally nonnegative.*

The last statement is proved by induction based on Theorem 4 and the recurrence relations for the minors of the matrix $\{\mathrm{B}_j(\zeta_i)\}$. The proof does not differ from that of Theorem 8.67 described in [9, p. 356].

Since the supports of discrete GB-splines are finite, the matrix of system (26) is banded and has seven nonzero diagonals in general. The matrix is tridiagonal if $\zeta_i = x_{i+2}$, $i = -3, \ldots, N-1$.

An important particular case of the problem in which $S'(x_i) = f_i'$, $i = 0, N$, can be obtained by passing to the limit as $\zeta_{-3} \to \zeta_{-2}$, $\zeta_{N-1} \to \zeta_{N-2}$.

de Boor and Pinkus [1] proved that linear systems with totally nonnegative matrices can be solved by Gaussian elimination without pivoting. Thus, the system (26) can be solved effectively by the conventional Gauss method.

Denote by $S^-(\mathbf{v})$ the number of sign changes (variations) in the sequence of components of the vector $\mathbf{v} = (v_1, \cdots, v_n)$, with zeros being neglected. Karlin [4] showed that if a matrix A is totally nonnegative, then it decreases the variation, i.e.

$$S^-(A\mathbf{v}) \le S^-(\mathbf{v}).$$

By Corollary 4, the totally nonnegative matrix $\{\mathrm{B}_j(\zeta_i)\}_{i,j=-3,\ldots,N-1}$ formed by discrete GB-splines decreases the variation.

For a bounded real function f, let $S^-(f)$ be the number of sign changes of the function f on the real axis $\mathbb{R}$, without taking into account the zeros

$$S^-(f) = \sup_n S^-[f(\zeta_1), \ldots, f(\zeta_n)], \quad \zeta_1 < \zeta_2 < \cdots < \zeta_n.$$

Theorem 5. *The discrete generalized spline* $S(x) = \sum_{j=-3}^{N-1} b_j \mathrm{B}_j(x)$ *is a variation diminishing function, i.e. the number of sign changes of* S *does not exceed that in the sequence of its coefficients:*

$$S^-\left(\sum_{j=-3}^{N-1} b_j \mathrm{B}_j\right) \le S^-(\mathbf{b}), \quad \mathbf{b} = (b_{-3}, \ldots, b_{N-1}).$$

The proof of this statement is the same as that of Theorem 8.68 for discrete polynomial B-splines in [9, p. 356].

§8. Examples of Defining Functions

Let us give some choices of the defining functions Φ_i and Ψ_i for discrete generalized splines that conform to the sufficiency conditions derived earlier in the paper. Putting

$$\Psi_i(x) = \psi_i(t)h_i^2 = \psi(q_i, \hat\tau_i^{L_i}, \hat\tau_i^{R_i}, t)h_i^2, \quad \Phi_i(x) = \psi(p_i, \hat\tau_{i+1}^{R_i}, \hat\tau_{i+1}^{L_i}, 1-t)h_i^2,$$
$$\hat\tau_j^{L_i} = \tau_j^{L_i}/h_i, \quad \hat\tau_j^{R_i} = \tau_j^{R_i}/h_i; \quad j = i, i+1; \quad 0 \le p_i, q_i < \infty,$$

we consider some possibilities for choosing the functions ψ_i which, due to the constraints (1), satisfy the conditions

$$\psi_i(-\hat\tau_i^{L_i}) = \psi_i(0) = \psi_i(\hat\tau_i^{R_i}) = 0, \; D_{i+1,2}\psi_i(1) = h_i^{-2}. \tag{27}$$

1) Discrete rational spline with linear denominator:

$$\psi_i(t) = C_i \frac{(t+\hat\tau_i^{L_i})t(t-\hat\tau_i^{R_i})}{1+q_i(1-t)}.$$

2) Discrete rational spline with quadratic denominator:

$$\psi_i(t) = C_i \frac{(t+\hat\tau_i^{L_i})t(t-\hat\tau_i^{R_i})}{1+q_i t(1-t)}.$$

3) Discrete exponential spline:

$$\psi_i(t) = C_i(t+\hat\tau_i^{L_i})t(t-\hat\tau_i^{R_i})exp\big(-q_i(1-t)\big).$$

4) Discrete hyperbolic spline:

$$\psi_i(t) = C_{i,1}\Big[\sinh q_i t - t\frac{\sinh q_i\hat\tau_i^{R_i}}{\hat\tau_i^{R_i}}\Big] + C_{i,2}\Big[\cosh q_i t - 1 - t\frac{\cosh q_i\hat\tau_i^{R_i} - 1}{\hat\tau_i^{R_i}}\Big].$$

5) Discrete cubic spline with additional knots:

$$\psi_i(t) = \frac{1}{2}\frac{(t-\beta_i+\hat\tau_i^{L_i})(t-\beta_i)_+(t-\beta_i-\hat\tau_i^{R_i})}{3(1-\beta_i)+\hat\varepsilon_{i+1}-\hat\varepsilon_i},$$
$$\hat\varepsilon_j = \hat\tau_j^{R_i} - \hat\tau_j^{L_i}, \quad j = i, i+1; \quad \beta_i = 1-(1+q_i)^{-1}, \quad E_+ = \max(0, E).$$

The points $x_i + \alpha_i h_i$ ($\alpha_i = (1+p_i)^{-1}$) and $x_i + \beta_i h_i$ fix the position of two additional knots of the spline on the interval $[x_i, x_{i+1}]$. By moving these knots one can perform a transfer from a discrete cubic spline to piecewise linear interpolation.

6) Discrete spline of variable order:

$$\psi_i(t) = C_i(t+\hat\tau_i^{L_i})t^{k_i}(t-\hat\tau^{R_i}), \quad k_i = 1+q_i.$$

The constants C_i in the expressions for the function ψ_i above are calculated from the condition (27) for the second divided difference of ψ_i. To find $C_{i,k}$, $k = 1, 2$, one needs additionally to use the condition $\psi_i(-\hat\tau_i^{L_i}) = 0$. It is easy to check that in all cases we get the corresponding defining functions in [5] by setting $\hat\tau_j^{L_i} = \hat\tau_j^{R_i} = 0$, $j = i, i+1$.

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