Local bases for generalized cubic splines

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Abstract - Direct and recursive algorithms are proposed for constructing generalized cubic B-splines. Explicit formulae are obtained for generalized B-splines. Properties of series consisting of B-splines are studied. It is shown that generalized B-splines form weak Chebyshev systems. The presented formulae of local approximation are exact for polynomials of the first degree. Examples of generalized B-splines including those with alternating signs are considered.

The tool of generalized cubic splines is widely used in solving problems of isogeometric interpolation. Introducing one or another of parameters into the spline structure, we can preserve such characteristics of the initial data as the convexity, monotonicity, linear and plane pieces, etc. Here the main problem is to develop an algorithm for choosing parameters automatically. The available algorithms are based mainly on the piecewise representation of splines.

The method of local optimization [15], combined with recursive algorithms for calculating polynomial B-splines [1], was found to be efficient in practical applications. Although the theory of generalized B-splines is well developed [21], they are not applied widely for solving problems of isogeometric approximation. This is due to fact that there are no simple and efficient computational algorithms and explicit formulae for generalized B-splines. They are developed only for trigonometric [16], hyperbolic [19], and some special kinds of more general Chebyshev splines [4,14,20], which have a limited field of application.

In this paper we propose two quite universal methods for constructing generalized B-splines. The first method is based on solving directly a system

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of defining relations for B-splines and it allows one to obtain explicit formulae for them. For this case, in [12,13] local bases for rational parabolic and cubic splines, respectively, were constructed and algorithms for isogeometric approximation were developed.

The other method generalizes the results of papers [8–10] dealing with exponential splines and provides recursive algorithms for constructing a fairly wide class of generalized B-splines (rational, exponential, and hyperbolic splines; splines with additional nodes, etc).

The properties of generalized B-splines, in particular, the representation of polynomial, exponents and rational functions with their help are studied. Series of generalized B-splines are considered. It is shown that the series decrease variation and the systems of generalized B-splines are weak Chebyshev systems. One- and three-point formulae are obtained for local approximation by generalized B-splines. The formulae are exact for polynomials of the first degree. The properties of the approximations are considered. Examples of generalized B-splines including those with alternating signs are considered.

1. GENERALIZED CUBIC SPLINES. CONDITIONS OF EXISTENCE AND UNIQUENESS

Assume that the division $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ is given on the segment $[a, b]$. For a fufficiently smooth function $S(x)$ we put $S_i^r = S^{(r)}(x_i)$, $r = 0, 1, 2$, and introduce the notation

$$\Delta_i S = (S_{i+1} - S_i)/h_i, \quad h_i = x_{i+1} - x_i, \quad i = 0, 1, \cdots, N-1,$$

$$\delta_i S = \Delta_i S - \Delta_{i-1} S, \quad i = 1, \cdots, N-1.$$

Definition 1.1. The generalized cubic spline is the function $S(x) \in C^2[a, b]$ which is of the form:

$$\begin{aligned} S(x) = [S_i - \varphi_i(0) h_i^2 S_i''](1-t) + [S_{i+1} - \psi_i(1) h_i^2 S_{i+1}'']t \\ + \varphi_i(t) h_i^2 S_i'' + \psi_i(t) h_i^2 S_{i+1}'' \end{aligned} \tag{1.1}$$

on each subsegment $[x_i, x_{i+1}]$. Here $t = (x - x_i)/h_i$, and the functions $\varphi_i(t)$ and $\psi_i(t)$ are subject to the constraints

$$\varphi_i^{(r)}(1) = \psi_i^{(r)}(0) = 0, \quad r = 0, 1, 2; \quad \varphi_i''(0) = \psi_i''(1) = 1.$$

We denote by S_3^G the set of splines satisfying Definition 1.1. The functions $\varphi_i(t)$ and $\psi_i(t)$ depending on parameters influence essentially the spline properties. Therefore, according to [25], we call them the defining functions. In practice one takes $\varphi_i(t) = \varphi_i(p_i, t)$ and $\psi_i(t) = \varphi_i(q_i, 1-t)$. In particular, at $\varphi_i(t) = (1-t)^3/6$ and $\psi_i(t) = t^3/6$ we have the conventional cubic spline.

According to (1.1), we have

$$\begin{aligned} S'_i &= \Delta_i S + [\varphi_i(0) + \varphi'_i(0)]h_i S''_i - \psi_i(1) h_i S''_{i+1}, \\ S'_{i+1} &= \Delta_i S + \varphi_i(0) S''_i - [\psi_i(1) - \psi'_i(1)] h_i S''_{i+1}. \end{aligned} \tag{1.2}$$

The continuity conditions for $S'(x)$ on Δ and the boundary relations $S'(a) = S'_0$ and $S'(b) = S'_N$ result in the system of linear algebraic equations

$$\left\{ \begin{aligned} &-[\varphi_0(0) + \varphi'_0(0)]h_0 S''_0 + \psi_0(1) h_0 S''_1 = \Delta_0 S - S'_0, \\ &\quad \varphi_{i-1}(0) h_{i-1} S''_{i-1} + [\psi'_{i-1}(1) h_{i-1} - \psi_{i-1}(1) h_{i-1} - \varphi'_i(0) h_i \\ &\qquad - \varphi_i(0) h_i] S''_i + \psi_i(1) h_i S''_{i+1} = \Delta_i S - \Delta_{i-1} S, \\ &\qquad\qquad i = 1, \dots, N-1, \\ &\quad \varphi_{N-1}(0) h_{N-1} S''_{N-1} + [\psi'_{N-1}(1) - \psi_{N-1}(1)] h_{N-1} S''_N \\ &\qquad\qquad = S'_N - \Delta_{N-1} S. \end{aligned} \right. \tag{1.3}$$

Let us find constraints on the defining functions $\varphi_i(t)$ and $\psi_i(t)$ which ensure that the generalized cubic spline $S(x)$ exists and is unique.

Lemma 1.1. If the conditions

$$\begin{aligned} &\varphi_i(0) > 0, \quad \varphi_i(0) + \varphi'_i(0) + \psi_i(1) < 0, \\ &\psi_i(1) > 0, \quad \psi'_i(1) - \psi_i(1) - \varphi_i(0) > 0, \quad i = 0, \dots, N-1 \end{aligned}$$

are satisfied, the generalized cubic spline $S(x)$ exists and is unique.

Proof. By virtue of the lemma conditions, we have

$$\varphi'_i(0) < 0, \quad \psi'_i(1) > 0, \quad i = 0, \dots, N-1$$

and system (1.3) has the diagonal predominance:

$$\begin{aligned} r_0 &= -[\varphi_0(0) + \varphi'_0(0) + \psi_0(1)] h_0 > 0, \\ r_i &= [\psi'_{i-1}(1) - \psi_{i-1}(1) - \varphi_{i-1}(0)] h_{i-1} \\ &\qquad - [\varphi_i(0) + \varphi'_i(0) + \psi_i(1)] h_i > 0, \quad i = 1, \dots, N-1, \\ r_N &= [-\psi_{N-1}(1) + \psi'_{N-1}(1) - \varphi_{N-1}(0)] h_{N-1} > 0. \end{aligned}$$

Thus, according to the Hadamar criterion [26], the matrix of system (1.3) is nonsingular and the generalized cubic spline $S(x)$ exists and is unique. Thus, the lemma is proved.

Further we need relations for $S'(x)$ on the grid Δ. From (1.2) we have

$$\begin{aligned} S_i'' &= \frac{T_i}{h_i}\{\psi_i'(1)\Delta_i S + [\psi_i(1) - \psi_i'(1)]S_i' - \psi_i(1)S_{i+1}'\}, \\ S_{i+1}'' &= \frac{T_i}{h_i}\{\varphi_i'(0)\Delta_i S + \varphi_i(0)S_i' - [\varphi_i(0) + \varphi_i'(0)]S_{i+1}'\} \end{aligned} \tag{1.4}$$

at $T_i^{-1} = [\varphi_i(0) + \varphi_i'(0)][\psi_i(1) - \psi_i'(1)] - \varphi_i(0)\psi_i(1)$.

The continuity conditions for $S''(x)$ on Δ and the boundary relations $S''(a) = S_0''$ and $S''(b) = S_N''$ result in the system of equations

$$\begin{cases} [\psi_0'(1) - \psi_0(1)]S_0' + \psi_0(1)S_1' = \psi_0'(1)\Delta_0 S - h_0 T_0^{-1} S_0'', \\ \varphi_{i-1}(0)\frac{T_{i-1}}{h_{i-1}}S_{i-1}' + \{-[\varphi_{i-1}(0) + \varphi_{i-1}'(0)]\frac{T_{i-1}}{h_{i-1}} \\ \quad +[\psi_i'(1) - \psi_i(1)]\frac{T_i}{h_i}\}S_i' + \psi_i(1)\frac{T_i}{h_i}S_{i+1}' = \\ \quad \psi_i'(1)\frac{T_i}{h_i}\Delta_i S - \varphi_{i-1}'(0)\frac{T_{i-1}}{h_{i-1}}\Delta_{i-1}S, \quad i = 1, \dots, N-1, \\ \varphi_{N-1}(0)S_{N-1}' - [\varphi_{N-1}(0) + \varphi_{N-1}'(0)]S_N' = \frac{h_{N-1}}{T_{N-1}}S_N'' - \varphi_{N-1}'(0)\Delta_{N-1}S. \end{cases} \tag{1.5}$$

Lemma 1.2. If the conditions

$$\begin{aligned} &\varphi_i(0) > 0, \quad 2\varphi_i(0) + \varphi_i'(0) < 0, \\ &\psi_i(1) > 0, \quad \psi_i'(1) - 2\psi_i(1) > 0, \quad i = 0, \dots, N-1 \end{aligned} \tag{1.6}$$

are satisfied, the generalized cubic spline $S(x)$ exists and is unique.

Proof. It follows from the lemma conditions that $\varphi_i'(0) < 0$ and $\psi_i'(1) > 0$, $i = 0, \dots, N-1$. Then

$$\begin{aligned} T_i^{-1} &= \psi_i(1)\varphi_i'(0) - \psi_i'(1)[\varphi_i(0) + \varphi_i'(0)] \\ &= \psi_i(1)\varphi_i'(0) + \psi_i'(1)\varphi_i(0) - \psi_i'(1)[2\varphi_i(0) + \varphi_i'(0)] \\ &> 2\psi_i(1)\varphi_i(0) + \psi_i(1)\varphi_i'(0) - \psi_i'(1)[2\varphi_i(0) + \varphi_i'(0)] \\ &= [2\varphi_i(0) + \varphi_i'(0)][\psi_i(1) - \psi_i'(1)] > 0. \end{aligned}$$

Therefore, by virtue of the lemma conditions, the matrix of system (1.5) has the diagonal predominance:

$$
\begin{aligned}
r_0 &= -2\psi_0(1) + \psi_0'(1) > 0,\\
r_i &= -[2\varphi_{i-1}(0) + \varphi_{i-1}'(0)]\frac{T_{i-1}}{h_{i-1}} + [-2\psi_i(1) + \psi_i'(1)]\frac{T_i}{h_i} > 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, N-1,\\
r_N &= -[2\varphi_{N-1}(0) + \varphi_{N-1}'(0)] > 0.
\end{aligned}
$$

This ensures [26] that the spline $S(x)$ exists and is unique. Thus, the lemma is proved..

The conditions formulated in Lemma 1.2 are satisfied for the majority of splines used in practice. This allows one to construct them, i.e. to solve, in fact, the system (1.5), by the conventional three-point sweeping method [26].

2. CONSTRUCTION OF BASIS SPLINES

Let us construct a basis for the space of generalized cubic splines S_3^G by using functions which have local supports of minimum length. Since

$$dim(S_3^G) = 4N - 3(N-1) = N+3$$

we extend the grid Δ by adding the points x_j, $j = -3, -2, -1, N+1, N+2, N+3$, such that $x_{-3} < x_{-2} < x_{-1} < a$, $b < x_{N+1} < x_{N+2} < x_{N+3}$.

We demand that the basis splines $B_i(x)$, $i = -1, \dots, N+1$ have the properties

$$
\begin{aligned}
&B_i(x) > 0, \quad x \in (x_{i-2}, x_{i+2}),\\
&B_i(x) \equiv 0, \quad x \notin (x_{i-2}, x_{i+2}),\\
&\sum_{j=-1}^{N+1} B_j(x) \equiv 1, \quad x \in [a,b].
\end{aligned}
\tag{2.1}
$$

For a spline $B_i(x)$ different from zero only on the interval (x_{i-2}, x_{i+2}) the system (1.3) is transformed to

$$
\begin{aligned}
B_i(x_{i-1}) &= \psi_{i-2}(1)h_{i-2}^2 B_i''(x_{i-1})\\
B_i(x_{i+1}) &= \varphi_{i+1}(0)h_{i+1}^2 B_i''(x_{i+1})
\end{aligned}
\tag{2.2}
$$

$$
\left\{
\begin{aligned}
(h_{i-1}u_{i-1} + v_{i-1})B_i''(x_{i-1}) + \psi_{i-1}(1)h_{i-1}^2 B_i''(x_i) &= B_i(x_i),\\
u_{i-1}B_i''(x_{i-1}) + u_i B_i''(x_i) + u_{i+1}B_i''(x_{i+1}) &= 0,\\
\varphi_i(0)h_i^2 B_i''(x_i) + (h_i u_{i+1} - v_{i+1})B_i''(x_{i+1}) &= B_i(x_i),
\end{aligned}
\right.
\tag{2.3}
$$

where

$$u_i = \psi'_{i-1}(1)h_{i-1} - \varphi'_i(0)h_i, \quad v_i = \psi_{i-1}(1)h_{i-1}^2 - \varphi_i(0)h_i^2. \tag{2.4}$$

According to (1.2) and (1.5), we also have

$$\begin{aligned} B'_i(x_{i-1}) &= \psi'_{i-2}(1)h_{i-2}B''_i(x_{i-1}), \\ B'_i(x_{i+1}) &= \varphi'_{i+1}(0)h_{i+1}B''_i(x_{i+1}), \\ B'_i(x_i) &= -\frac{1}{h_i}B_i(x_i) + [\varphi_i(0) + \varphi'_i(0)]h_iB''_i(x_i) - \frac{v_{i+1}}{h_i}B''_i(x_{i+1}). \end{aligned} \tag{2.5}$$

Let us write the norming conditions (2.1) and the corollaries of them as

$$\sum_{j=i-1}^{i+1} B_j^{(r)}(x_i) = \delta_{0r}, \quad r = 0, 1, 2, \tag{2.6}$$

where δ_{0r} is the Kronecker's symbol.

Taking into account (2.2) and (2.5), we can rewrite (2.6) as

$$\begin{cases} \varphi_i(0)h_i^2B''_{i-1}(x_i) + B_i(x_i) + \psi_{i-1}(1)h_{i-1}^2B''_{i+1}(x_i) = 1, \\ \varphi'_i(0)h_iB''_{i-1}(x_i) - \frac{1}{h_i}B_i(x_i) + [\varphi_i(0) + \varphi'_i(0)]h_iB''_i(x_i) \\ \qquad - \frac{v_{i+1}}{h_i}B''_i(x_{i+1}) + \psi'_{i-1}(1)h_{i-1}B''_{i+1}(x_i) = 0, \\ B''_{i-1}(x_i) + B''_i(x_i) + B''_{i+1}(x_i) = 0. \end{cases}$$

Eliminating $B''_{i-1}(x_i)$ and $B''_{i+1}(x_i)$, we have

$$(h_iu_i + v_i)[B_i(x_i) - \varphi_i(0)h_i^2B''_i(x_i)] + v_iv_{i+1}B''_i(x_{i+1}) = h_iu_i.$$

The equation obtained, together with the last equation in the system (2.3), yields

$$B''_i(x_{i+1}) = \frac{1}{u_{i+1}(y_{i+1} - y_i)}, \quad y_j = x_j - \frac{v_j}{u_j}. \tag{2.7}$$

Subtracting the third equation of the system (2.3) from the first in the same system, we find

$$\begin{cases} u_{i-1}(x_i - y_{i-1})B''_i(x_{i-1}) + u_i(x_i - y_i)B''_i(x_i) = -u_{i+1}(x_i - y_{i+1})B''_i(x_{i+1}), \\ u_{i-1}B''_i(x_{i-1}) + u_iB''_i(x_i) = -u_{i+1}B''_i(x_{i+1}), \end{cases}$$

and hence taking into account (2.7), we have

$$B_i''(x_{i-1}) = \frac{1}{u_{i-1}(y_i - y_{i-1})}, \; B_i''(x_i) = -\frac{1}{u_i}\left(\frac{1}{y_i - y_{i-1}} + \frac{1}{y_{i+1} - y_i}\right). \quad (2.8)$$

Thus,

$$B_i''(x_j) = \frac{y_{i+1} - y_{i-1}}{u_j \omega_{i-1}'(y_j)}, \quad j = i-1, i, i+1, \quad (2.9)$$

where $\omega_{i-1}(x) = (x - y_{i-1})(x - y_i)(x - y_{i+1})$.

If we also denote

$$W_j^k = u_{j+k} u_j^{-1} u_{j+1}^{-1} (y_{j+1} - y_j)^{-1}, \quad j = i-1, i, \quad k = 0, 1$$

then

$$B_i''(x_{i-1}) = W_{i-1}^1, \quad B_i''(x_i) = -(W_{i-1}^0 + W_i^1), \quad B_i''(x_{i+1}) = W_i^0. \quad (2.10)$$

The obtained formulae allow us to write explicitly the spline $B_i(x)$ and its derivatives at the nodes of the grid Δ.

Table 1.

x	$B_i(x)$	$B_i'(x)$	$B_i''(x)$
x_{i-1}	$\psi_{i-2}(1)h_{i-2}^2 W_{i-1}^1$	$\psi_{i-2}'(1)h_{i-2}W_{i-1}^1$	W_{i-1}^1
x_i	$1 - \psi_{i-1}(1)h_{i-1}^2 W_i^1$ $-\varphi_i(0)h_i^2 W_{i-1}^0$	$-\psi_{i-1}'(1)h_{i-1}W_i^1$ $-\varphi_i'(0)h_i W_{i-1}^0$	$-(W_{i-1}^0 + W_i^1)$
x_{i+1}	$\varphi_{i+1}(0)h_{i+1}^2 W_i^0$	$\varphi_{i+1}'(0)h_{i+1}W_i^0$	W_i^0

The following statement can be checked directly.

Lemma 2.1. The relations

$$u_i B_{i+1}''(x_i) = u_{i+1} B_i''(x_{i+1}), \quad i = -1, \ldots, N$$

hold for generalized cubic B-splines.

Now let us find an explicit formula for the spline $B_i(x)$. If $x \in [x_i, x_{i+1}]$ then, according to (2.2), (2.3), and (2.7), in formula (1.1) we have

$$l_i(x) = (1-t)[B_i(x_i) - \varphi_i(0)h_i^2 B_i''(x_i)] + t[B_i(x_{i+1}) - \psi_i(1)h_i^2 B_i''(x_{i+1})]$$

$$= (1-t)(h_i u_{i+1} - v_{i+1})B_i''(x_{i+1}) + t[\varphi_{i+1}(0)h_{i+1}^2 - \psi_i(1)h_i^2]B_i''(x_{i+1}) =$$

$$[(1-t)(h_i u_{i+1} - v_{i+1}) - t v_{i+1}]B_i''(x_{i+1}) = u_{i+1}(y_{i+1} - x)B_i''(x_{i+1}) = \frac{y_{i+1} - x}{y_{i+1} - y_i}.$$

Similarly, if $x \in [x_{i-1}, x_i]$, then, according to (1.1), (2.2), (2.3), and (2.8), we have

$$l_{i-1}(x) = \frac{x_i - x}{h_{i-1}}\Big[B_i(x_{i-1}) - \varphi_{i-1}(0)h_{i-1}^2 B_i''(x_{i-1})\Big] + \frac{x - x_{i-1}}{h_{i-1}}\Big[B_i(x_i) - \psi_{i-1}(1)h_{i-1}^2 B_i''(x_i)\Big]$$

$$= \left[\frac{x_i - x}{h_{i-1}} v_{i-1} + \frac{x - x_{i-1}}{h_{i-1}}(h_{i-1}u_{i-1} + v_{i-1})\right]B_i''(x_{i-1})$$

$$= u_{i-1}(x - y_{i-1})B_i''(x_{i-1}) = \frac{x - y_{i-1}}{y_i - y_{i-1}}.$$

Using the notation (2.10), from (1.1) we finally obtain

$$B_i(x) = \begin{cases} \Psi_{i-2}(x)W_{i-1}^1, \quad x \in [x_{i-2}, x_{i-1}], \\ \dfrac{x - y_{i-1}}{y_i - y_{i-1}} + \Phi_{i-1}(x)W_{i-1}^1 - \Psi_{i-1}(x)(W_{i-1}^0 + W_i^1), \\ \qquad\qquad x \in [x_{i-1}, x_i], \\ \dfrac{y_{i+1} - x}{y_{i+1} - y_i} - \Phi_i(x)(W_{i-1}^0 + W_i^1) + \Psi_i(x)W_i^0, \\ \qquad\qquad x \in [x_i, x_{i+1}], \\ \Phi_{i+1}(x)W_i^0, \quad x \in [x_{i+1}, x_{i+2}], \\ 0, \quad x \notin [x_{i-2}, x_{i+2}], \end{cases} \tag{2.11}$$

where

$$\Phi_j(x) = \varphi_j\left(\frac{x - x_j}{h_j}\right)h_j^2, \quad \Psi_j(x) = \psi_j\left(\frac{x - x_j}{h_j}\right)h_j^2.$$

The formula is very convenient for practical purposes. Its characteristic feature is that the linear part in it is explicitly separated, which is due to the representation (1.1).

Figures 1a and 1b present the graphs of rational B-splines $B_i(x)$ on a uniform grid with step $h = 1$ for

$$\varphi_j(t) = (1-t)^3/[1+p_j t(1-t)]P_j, \ P_j^{-1} = 2(1+p_j)(3+p_j), \ \psi_j(t) = \varphi_j(q_j, 1-t)$$

(a) (b)

Figure 1. The basis splines on a uniform grid with step $h = 1$, (a) $p = 0$, $p = 1$, and $p = 5$; (b) $p = 10$, $p = 20$, and $p = 50$.

and $p_j = q_j = p$, $j = i-2, \dots, i+1$. The solid, dotted, and dashed lines represent the graphs of B-splines for $p = 0$, $p = 1$, and $p = 5$, respectively (Fig. 1a), and for $p = 10$, $p = 20$, and $p = 50$, respectively (Fig. 1b). If $p = 0$ we have the conventional cubic B-spline. Passing to the limit as $p \to \infty$, we obtain the piecewise linear function of the Schauder basis having the support $[x_{i-1}, x_{i+1}]$.

3. ANOTHER WAY OF DETERMINING BASIS SPLINES

In (2.11) the expressions for $B_i(x)$ differ in the subsegments $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$, $j = i-2, \dots, i+2$ by

$$-\Phi_{j-1}(x)B_i''(x_{j-1}) + [\Phi_j(x) - \Psi_{j-1}(x) + u_j(x-y_j)]B_i''(x_j) + \Psi_j(x)B_i''(x_{j+1}).$$

Summing up these 'jumps', we arrive at the representation

$$B_i(x) = \sum_{j=i-1}^{i+1} B_i''(x_j)\Omega_j(x), \tag{3.1}$$

where

$$\Omega_j(x) = \Psi_{j-1}(x)\theta(x-x_{j-1}) + [\Phi_j(x) - \Psi_{j-1}(x) + u_j(x-y_j)]\theta(x-x_j)$$

$$-\Phi_j(x)\theta(x - x_{j+1}),$$

$$\theta(x - x_j) = \begin{cases} 1, & x \geq x_j, \\ 0, & x < x_j. \end{cases}$$

Since $\theta(x - x_j) = 1 - \theta(x_j - x)$, we can rewrite (3.1) as

$$B_i(x) = -\sum_{j=i-1}^{i+1} B_i''(x_j)\hat{\Omega}_j(x) + R_i(x). \tag{3.2}$$

Here $\hat{\Omega}_j(x)$ can be obtained from $\Omega_j(x)$ by replacing $\theta(x - x_j)$ with $\theta(x_j - x)$. For $R_i(x)$, taking into account (2.9), we have

$$R_i(x) = \sum_{j=i-1}^{i+1} B_i''(x_j)u_j(x - y_j) = (y_{i+1} - y_{i-1}) \sum_{j=i-1}^{i+1} \frac{x - y_j}{\omega_{i-1}'(y_j)} =$$

$$= (y_{i+1} - y_{i-1})g[x; y_{i-1}, y_i, y_{i+1}] \equiv 0, \quad g(x, y) = x - y.$$

The square brackets denote the second divided difference of the function $g(x, y)$ with respect to its argument $y = y_j$, $j = i-1, i, i+1$.

It follows from formulae (3.1) and (3.2) that $B_i(x) \equiv 0$ for $x \notin (x_{i-2}, x_{i+2})$. Any of the formulae can be taken as the definition of the generalized cubic spline.

For the ordinary cubic spline [at $\varphi_i(t) = (1-t)^3/6$ and $\psi_i(t) = t^3/6$], in (3.1) we have

$$\Omega_j(x) = \frac{1}{6}(x_{j+1} - x_{j-1})g[x; x_{j-1}, x_j, x_{j+1}], \quad g(x, y) = (x - y)_+^3,$$

where $z_+ = \max(0, z)$ and (3.1) takes the form:

$$B_i(x) = (x_{i+2} - x_{i-2})g[x; x_{i-2}, \ldots, x_{i+2}].$$

4. RECURRENCE FORMULAE FOR GENERALIZED B-SPLINES

Let us consider the splines

$$B_{j,1}(x) = \begin{cases} \Psi_j''(x), & x_j \leq x < x_{j+1}, \\ \Phi_{j+1}''(x), & x_{j+1} \leq x < x_{j+2}, \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

where $j = i-2, i-1, i$. We assume that the functions $\Psi_j''(x)$ and $\Phi_{j+1}''(x)$ are monotonous. The splines $B_{j,1}(x)$ are generalizations of 'functions-roof' for the polynomial B-splines. They are nonnegative and, besides, $B_{j,1}(x_{j+1}) = 1$ and $B_{j,1}(x_l) = 0$ for $l \neq j+1$.

Using $B_{j,1}(x)$, we define recurrently the splines

$$B_{j,k}(x) = \int_{x_j}^{x} \frac{B_{j,k-1}(\tau)}{c_{j,k-1}} d\tau - \int_{x_{j+1}}^{x} \frac{B_{j+1,k-1}(\tau)}{c_{j+1,k-1}} d\tau \qquad (4.2)$$

and

$$c_{j,k-1} = \int_{x_j}^{x_{j+k}} B_{j,k-1}(\tau) d\tau, \quad j = i-2, i-k+1, \quad k = 2, 3. \qquad (4.3)$$

Formula (4.2) can also be represented as

$$B_{j,k}(x) = -\int_{x}^{x_{j+k-1}} \frac{B_{j,k-1}(\tau)}{c_{j,k-1}} d\tau + \int_{x}^{x_{j+k}} \frac{B_{j+1,k-1}(\tau)}{c_{j+1,k-1}} d\tau.$$

Simple calculations result in

$$c_{j,1} = u_{j+1}, \quad j = i-2, i-1, i; \quad c_{j,2} = y_{j+2} - y_{j+1}, \quad j = i-2, i-1, \qquad (4.4)$$

which makes clear the geometric sense of these quantities. Differentiating (4.2), we also have

$$B'_{j,k}(x) = \frac{B_{j,k-1}(x)}{c_{j,k-1}} - \frac{B_{j+1,k-1}(x)}{c_{j+1,k-1}}, \quad k = 2, 3. \qquad (4.5)$$

The splines $B_{j,k}(x)$, $k = 2, 3$, can be written explicitly. Due to (4.1) and (4.2), for $j = i-2, i-1$ we have

$$B_{j,2}(x) = \begin{cases} \Psi_j'(x)/u_{j+1}, \quad x_j \le x < x_{j+1}, \\ 1 + \Phi_{j+1}'(x)/u_{j+1} - \Psi_{j+1}'(x)/u_{j+2}, \quad x_{j+1} \le x < x_{j+2}, \\ -\Phi_{j+2}'(x)/u_{j+2}, \quad x_{j+2} \le x < x_{j+3}, \\ 0, \quad \text{otherwise}. \end{cases} \qquad (4.6)$$

Note that $B'_{j,2}(x_{j+l+1}) = (-1)^l u_{j+l+1}^{-1}$, $l = 0, 1$. The expression for $B_{j,3}(x)$ coincides with that for $B_j(x)$ in (2.11) if we renumber the splines with respect to the support centre.

(a) (b)

(c)

Figure 2. The rational B-splines of order $k = 1, 2, 3$ (from left to right) with various vectors of parameters, (a) $p_j = 0$ for all j. (b) $p0 = (1, 5)$, $p = (10, 100, 0)$, and $p = (1, 5, 10, 10)$; (c) $p_j = 50$ for all j.

Figures 2a, 2b, and 2c present the graphs of rational B-splines $B_{j,k}(x)$, $k = 1, 2, 3$ (from left to right, respectively) on a uniform grid with step $h = 1$ for

$$\varphi_i(t) = (1-t)^3/[1+p_i t(1-t)]P_i, \quad P_i^{-1} = 2(1+p_i)(3+p_i), \quad \psi_i(t) = \varphi_i(q_i, 1-t)$$

and $p_i = q_i$, $i = j, \ldots, j+3$. For all i the case $p_i = 0$ (Fig. 2a) corresponds to the conventional polynomial B-splines. In Fig. 2b the vectors of parameters are $(1,5)$, $(10,100,0)$, and $(1,5,10,10)$ for $B_{j,k}(x)$, $k=1,2,3$, respectively. Finally, in Fig. 2c the parameters are $p_i = 50$ for all i. Passing to the limit as $p_i \to \infty$, we obtain an impulse function for $B_{j,1}(x)$, a 'function-step' for $B_{j,2}(x)$, and a 'function-roof' for $B_{j,3}(x)$ (all the functions are of unit height).

5. PROPERTIES OF GENERALIZED B-SPLINES

The functions $B_{j,k}(x)$, $k=2,3$, have many properties inherent in conventional polynomial B-splines.

Theorem 5.1. The functions $B_{j,k}(x)$, $k=2,3$, have the following properties.

(1) $B_{j,k}(x) > 0$ for $x \in (x_j, x_{j+k+1})$ and $B_{j,k}(x) \equiv 0$ for $x \notin (x_j, x_{j+k+1})$.

(2) The spline $B_{j,k}(x)$ is $(k-1)$-times continuously differentiable.

(3) $\sum_{j=-2}^{N-1} B_{j,2}(x) \equiv 1$ for $x \in [a,b]$;
$\Phi_j'(x) = u_j B_{j-2,2}(x)$ and $\Psi_j'(x) = u_{j+1} B_{j,2}(x)$
for $x \in [x_j, x_{j+1}]$, $j = 0, \ldots, N-1$.

(4) $\sum_{j=-3}^{N-1} y_{j+2}^r B_{j,3}(x) \equiv x^r$, $r = 0,1$, for $x \in [a,b]$;
$\Phi_j(x) = u_j(y_j - y_{j-1}) B_{j-3,3}(x)$ and $\Psi_j(x) = u_{j+1}(y_{j+2} - y_{j+1}) B_{j,3}(x)$
for $x \in [x_j, x_{j+1}]$, $j = 0, \ldots, N-1$.

Proof. First let us consider the splines $B_{j,2}(x)$. The values u_l, $l = j+1, j+2$, used in (4.6) are positive, since they are integrals of nonnegative B-splines $B_{l,1}(x)$. The functions $\Psi_l'(x)$ and $-\Phi_l'(x)$ are nonnegative and monotonous. Therefore, according to (4.6), the spline $B_{j,2}(x)$ is positive, increases monotonously on the interval (x_j, x_{j+1}), and decreases monotonously on the interval (x_{j+2}, x_{j+3}). Since

$$B_{j,2}''(x) = \frac{\Phi_{j+1}'''(x)}{u_{j+1}} - \frac{\Psi_{j+1}'''(x)}{u_{j+2}} < 0, \quad x \in [x_{j+1}, x_{j+2}]$$

we see that the function $B_{j,2}(x)$ is convex upwards on this interval and thus it is also positive.

To prove the other properties of the splines $B_{j,2}(x)$, which are formulated in Theorem 5.1, it is sufficient to use formula (4.6).

Since the splines $B_{j,2}(x)$ are nonnegative, the integral of them over the supports (x_j, x_{j+3}), i.e. the constants $c_{j,2} = y_{j+2} - y_{j+1}$ in (4.4), are positive. Therefore, according to (2.9),

$$B_{j,3}''(x_{j+1}) > 0, \quad B_{j,3}''(x_{j+2}) < 0, \quad B_{j,3}''(x_{j+3}) > 0 \tag{5.1}$$

and by virtue of (2.2) and (2.5),

$$B_{j,3}^{(r)}(x_{j+1}) > 0, \quad (-1)^r B_{j,3}^{(r)}(x_{j+3}) > 0, \quad r = 0, 1. \tag{5.2}$$

Taking into account that the functions $\Phi_l(x)$ and $\Psi_l(x)$ are nonnegative and monotonous on $[x_l, x_{l+1}]$ and using formula (2.11), we see that the spline $B_{j,3}(x)$ is positive and increases monotonously on the interval (x_j, x_{j+1}). Similarly, $B_{j,3}(x) > 0$ decreases monotonously on the interval (x_{j+3}, x_{j+4}).

For $x \in [x_l, x_{l+1}]$, $l = j+1, j+2$, we have

$$B_{j,3}'''(x) = \Phi_l'''(x) B_{j,3}''(x_l) + \Psi_l'''(x) B_{j,3}''(x_{l+1}).$$

Here $\Phi_l'''(x) < 0$ and $\Psi_l'''(x) > 0$, since they are derivatives of monotonous functions. Hence, by virtue of (5.1), the function $B_{j,3}''(x)$ decreases monotonously on $[x_{j+1}, x_{j+2}]$ and increases monotonously on $[x_{j+2}, x_{j+3}]$. Then there exists two points $x^* \in [x_{j+1}, x_{j+2}]$ and $x^{**} \in [x_{j+2}, x_{j+3}]$ such that $B_{j,3}''(x^*) = B_{j,3}''(x^{**}) = 0$. Taking into account (5.2), we find that the spline $B_{j,3}(x)$ is positive, increases monotonously on $[x_{j+1}, x^*]$, and decreases monotonously on $[x^{**}, x_{j+3}]$. The function $B_{j,3}(x)$ is convex upwards and also positive on the segment $[x^*, x^{**}]$.

All the other properties of the splines $B_{j,3}(x)$, which are formulated in (2) and (4) of Theorem 5.1, follow directly from formula (2.11). Thus, the theorem is proved.

The quantities y_j, $j = i-1, i, i+1$, are of considerable importance in constructing the splines $B_i(x) = B_{i-2,3}(x)$. The quantities are also important, because they enter into relations (4) of Theorem 5.1. The following statement allows us to estimate their values.

Lemma 5.1. If the following relations hold:

$$\begin{aligned} \psi_{j-1}(1) &> 0, \quad \psi_{j-1}'(1) - 2\psi_{j-1}(1) > 0, \\ \varphi_j(0) &> 0, \quad 2\varphi_j(0) + \varphi_j'(0) < 0 \end{aligned}$$

then the inequalities hold:

$$x_j - h_{j-1}/2 < y_j < x_j + h_j/2.$$

Proof. Since $y_j = x_j - v_j/u_j$, by using formulae (2.4) for u_j and v_j we can rewrite the above inequalities as

$$\begin{aligned} 0 &< h_j[-\varphi_j'(0)h_{j-1} + 2\varphi_j(0)h_j] + h_{j-1}^2[-2\psi_{j-1}(1) + \psi_{j-1}'(1)], \\ 0 &< -h_j^2[2\varphi_j(0) + \varphi_j'(0)] + h_{j-1}[2\psi_{j-1}(1)h_{j-1} + \psi_{j-1}'(1)h_j]. \end{aligned}$$

Evidently the last ones hold by virtue of the lemma conditions.

Note that the inequalities in Lemma 5.1 are also used in Lemma 1.2 to prove the existence and uniqueness of the generalized cubic spline. Therefore, if the conditions of Lemma 1.2 are satisfied, then $u_j > 0$ for $j = i-1, i, i+1$, and $y_{j+1} - y_j > 0$ for $j = i-1, i$. According to formulae (4.3) and (4.4), the integrals of B-splines $B_{j,k}(x)$, $k = 1, 2$, are positive even if the functions $\Phi_j''(x)$ and $\Psi_j''(x)$ are not monotonous. According to (2.7) and (2.8), the inequalities (5.1) hold.

The known properties of polynomial B-splines [26] can be generalized in the following way.

Lemma 5.2. The splines $B_{i,k}(x)$, $k = 1, 2, 3$, have supports of minimum length.

Proof. It is clear that the spline $B_{i,3}(x)$ cannot be different from zero only on a part of the subsegment $[x_j, x_{j+1}]$, $j = i, i+3$. If we suppose that $B_{i,3}(x)$ is not zero only on the segment $[x_{i+1}, x_{i+4}]$, then due to the continuity of $B''_{i,3}(x)$, we have $B''_{i,3}(x_{i+1}) = 0$. But then it follows from the system (2.3) that $B''_{i,3}(x_{i+2}) = B''_{i,3}(x_{i+3}) = 0$, and according to (2.11) we obtain $B_{i,3}(x) \equiv 0$. If we suppose that $B_{i,3}(x) \not\equiv 0$ only on the segment $[x_i, x_{i+3}]$, we arrive at the same result.

It follows from explicit formulae (4.1) and (4.6) that the supports of the splines $B_{j,k}(x)$, $k = 1, 2$, cannot be diminished. Thus, the lemma is proved.

Denote by S_k^G the set of splines $S(x) \in C^{k-1}[a, b]$ such that they are formed by linear combinations of the functions $\{1, \ldots, x^{k-2}, \Phi_i^{(3-k)}(x), \Psi_i^{(3-k)}(x)\}$, $k = 1, 2, 3$, on each subsegment $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$.

Theorem 5.2. The splines $B_{i,k}(x)$, $i = -k, \ldots, N-1$; $k = 1, 2, 3$, are linearly independent and form a basis in the space S_k^G of generalized splines.

Proof. Let us assume the opposite. Assume there exist constant $c_{i,k}$, $i = -k, \ldots, N-1$; $k = 1, 2, 3$, which are not all equal to zero and such that

$$c_{-k,k} B_{-k,k}(x) + \cdots + c_{N-1,k} B_{N-1,k}(x) = 0, \quad x \in [a, b]. \qquad (5.3)$$

According to formula (1.1), a spline $S(x)$ is formed by $(k+1)$ linearly independant functions on each subsegment $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$. Only (k+1) terms, which have the subscripts $i-k, \ldots, i$,remain in the sum (5.3) for $x \in [x_i, x_{i+1}]$ and hence taking into account formulae (4.1), (4.6), and (2.11), we have $c_{i-k,k} = \cdots = c_{i,k} = 0$. Continuing this process, we find that $c_{i,k} = 0$ for all i.

Since $dim(S_k^G) = N + k$, we see that the splines $B_{i,k}(x) \in S_k^G$, $i = -k, \ldots, N-1$, form a basis in this space. Thus, the theorem is proved.

6. SERIES OF GENERALIZED B-SPLINES

The statements of this section are formulated mainly for series of generalized cubic B-splines $B_{j,3}(x)$, since they are of the most interest to us. Many statements, however, can easily be reformulated for the splines $B_{j,k}(x)$, $k = 1, 2$. For simplicity sake, we do not do this, because such splines are auxiliary in this paper.

According to Theorem 5.2, any generalized cubic spline $S(x) \in S_3^G$ can be represented uniquely as

$$S(x) = \sum_{j=-1}^{N+1} b_j B_j(x), \quad x \in [a, b] \tag{6.1}$$

where b_j are constant coefficients.

Let us study how a spline $S(x)$ behaves depending on the coefficients b_j. Since B-splines are local, from (6.1) we obtain the inequalities

$$\min_{i-1 \le j \le i+2} b_j \le S(x) = \sum_{j=i-1}^{i+2} b_j B_j(x) \le \max_{i-1 \le j \le i+2} b_j, \quad x_i \le x \le x_{i+1}. \tag{6.2}$$

Hence it follows that the way the spline $S(x)$ behaves on the segment $[x_i, x_{i+1}]$ is determined by the coefficients $b_{i-1}, \cdots, b_{i+2}$. In particular, in order for a spline $S(x)$ to be zero at a point of the segment $[x_i, x_{i+1}]$, it is necessary that $b_j b_{j+1} \le 0$ for some $i - 1 \le j \le i + 1$.

Estimate (6.2) can be essentially improved. Applying the differentiation formula (4.5), we obtain

$$S^{(k)}(x) = \sum_{j=-3+k}^{N-1} b_j^{(k)} B_{j,3-k}(x), \tag{6.3}$$

where

$$b_j^{(k)} = \begin{cases} b_{j+2}, & k = 0, \\ \dfrac{b_j^{(k-1)} - b_{j-1}^{(k-1)}}{c_{j,3-k}}, & k = 1, 2. \end{cases}$$

Lemma 6.1. If $b_j \ge 0$, $j = -1, \ldots, N + 1$, then $S(x) \ge 0$ for all x.

The lemma statement is obvious, because the B-splines $B_j(x)$ are non-negative.

Lemma 6.2. If $b_{j+1} > b_j$ $(b_{j+1} < b_j)$, $j = -1, \dots, N$, then the function $S(x)$ increases (decreases) monotonously.

Proof. According to formulae (4.4) and (6.3), if we denote

$$\tilde{\Delta}_j b = \frac{b_{j+1} - b_j}{y_{j+1} - y_j}$$

we have

$$S'(x) = \sum_{j=-2}^{N-1} \tilde{\Delta}_{j+1} b B_{j,2}(x).$$

Because the splines $B_{j,2}(x)$ are nonnegative, from the above formula and Lemma 6.1 follows the lemma statement. Thus, the lemma is proved.

Lemma 6.3. If $\tilde{\Delta}_{j+1} b - \tilde{\Delta}_j b > 0$ (< 0), $j = -1, \dots, N-1$, then the function $S(x)$ is convex downwards (upwards).

Proof. By virtue of (4.4) and (6.3), we have

$$S''(x) = \sum_{j=-1}^{N-1} (\tilde{\Delta}_{j+1} b - \tilde{\Delta}_j b) u_{j+1}^{-1} B_{j,1}(x). \tag{6.4}$$

Because the splines $B_{j,1}(x)$ are nonnegative, taking into account Lemma 6.1 we obtain the lemma statement. Thus, the lemma is proved.

Let $Z_{[a,b]}(f(x))$ be the number of isolated zeros of a function $f(x)$ on the segment $[a, b]$.

Lemma 6.4. If the spline

$$S(x) = \sum_{j=-1}^{N+1} b_j B_j(x)$$

does not become zero at all points of any subsegment of $[a, b]$, then

$$Z_{[a,b]}(S(x)) \le N + 2.$$

Proof. According to (6.4) and (4.1), the function $S''(x)$ has no more than one zero on $[x_i, x_{i+1}]$, because the function $\Phi_i''(x)$ and $\Psi_i''(x)$ are monotonous and nonnegative on this subsegment. Hence $Z_{[a,b]}(S''(x)) \le N$. Then, according to the Roll theorem [21], we find $Z_{[a,b]}(S(x)) \le N + 2$. Thus, the lemma is proved.

Denote by $supp\, B_i(x) = \{x \mid B_i(x) \ne 0\}$ the support of the spline $B_i(x)$, i.e. the interval (x_{i-2}, x_{i+2}).

Theorem 6.1. Assume that $\tau_{-1} < \tau_0 < \cdots < \tau_{N+1}$. Then

$$D = det(B_i(\tau_j)) \neq 0, \quad i,j = -1, \dots, N+1$$

if and only if

$$\tau_i \in supp\, B_i(x), \quad i = -1, \dots, N+1. \tag{6.5}$$

If condition (6.5) is satisfied, then $D > 0$.

Proof. Let us prove the theorem by induction. It is clear that the theorem holds for one basic function. Assume that it also holds for (l-1) basic functions. Let us show that if (6.5) is satisfied, then $D \neq 0$ for l basis functions.

Let $\tau_l \notin supp\, B_l(x)$. If τ_l lies to the left (right) with respect to the support of $B_l(x)$ then the last column (line) of the determinant D consists of zeros, i.e. $D = 0$. If $\tau_l \in supp\, B_l(x)$ and $D = 0$, then there exists a nonzero vector $\mathbf{c} = (c_{-1}, \cdots, c_{l-2})$ such that

$$S(\tau_k) = \sum_{j=-1}^{l-2} c_j B_j(\tau_k) = 0, \quad k = -1, \dots, l-2,$$

i.e. the spline $S(x)$ has l isolated zeros. But this contradicts Lemma 6.4, which states that $S(x)$ can have no more than $(l-1)$ isolated zeros. Hence $\mathbf{c} = 0$ and $D \neq 0$.

Now it only remains to prove that $D > 0$ if (6.5) is satisfied. Let us choose $x_{k-2} < \tau_k < x_{k-1}$ for all k. Then the diagonal elements of D are positive and all the elements above the main diagonal are zero, i.e. $D > 0$. It is clear that D depends continuously on τ_k, $k = -1, \dots, l-2$, and $D \neq 0$ for $\tau_k \in supp\, B_k(x)$. Hence the determinant D is positive, if condition (6.5) is satisfied. Thus, the problem is proved.

The following three statements follows immediately from the theorem.

Corollary 6.1. The system of generalized cubic B-splines $\{B_j(x)\}$, $j = -1, \dots, N+1$, is a weak Chebyshev system according to the definition given in [7], i.e. for any $\tau_{-1} < \tau_0 < \cdots < \tau_{N+1}$ we have $D \geq 0$ and $D > 0$ if and only if condition (6.5) is satisfied. If the latter is satisfied, the generalized spline $S(x) = \sum_{j=-1}^{N+1} b_j B_j(x)$ has no more than $N+2$ isolated zeros.

Corollary 6.2. If the condition of Theorem 6.1 are satisfied, the solution of the interpolation problem

$$S(\tau_i) = f_i, \quad i = -1, \dots, N+1, \quad f_i \in \mathbb{R} \tag{6.6}$$

exists and is unique.

Let $A = \{a_{ij}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ be a rectangular $m \times n$ matrix with $m \le n$. The matrix A is said to be totally nonnegative (totally positive) [5] if all the minors of any order of the matrix are nonnegative (positive), i.e. for all $1 \le p \le m$ we have

$$det(a_{i_k j_l}) \ge 0 \ (> 0) \quad \text{for all} \quad \begin{array}{l} 1 \le i_1 < \cdots < i_p \le m, \\ 1 \le j_1 < \cdots < j_p \le n. \end{array}$$

Corollary 6.3. For arbitrary integers $-1 \le \nu_{-1} < \cdots < \nu_{p-2} \le N+1$ and $\tau_{-1} < \tau_0 < \cdots < \tau_{p-2}$ we have

$$D_p = det\{B_{\nu_i}(\tau_j)\} \ge 0, \quad i,j = -1, \ldots, p-2$$

and $D_p > 0$ if and only if

$$\tau_i \in supp\, B_{\nu_i}(x), \quad i = -1, \ldots, p-2$$

i.e. the matrix $\{B_j(\tau_i)\}$, $i, j = -1, \ldots, N+1$, is totally nonnegative.

The last statement is proved by induction on the basis of Theorem 6.1 and the recurrence relations for the minors of the matrix $\{B_j(\tau_i)\}$. The proof does not differ from that cited by Schumaker [21].

Since the suport of B-splines are finite, the matrix of the system (6.6) is bandwise and has seven nonzero diagonal in the general case. The matrix is tridiagonal if $\tau_i = x_i$, $i = -1, \ldots, N+1$.

An important specific case of the problem, in which $S'(x_i) = f'_i$, $i = 0, N$, can be obtained by passing to the limit as $\tau_{-1} \to \tau_0$, $\tau_{N+1} \to \tau_N$.

De Boor and Pinkus [2] proved that linear systems with totally nonnegative matrices can be solved by the Gauss method without choosing a leading element. Thus, the system (6.6) can be solved effectively by the conventional sweeping method.

Denote by $S^-(\mathbf{v})$ the number of sign changes (variations) in the sequence of components of the vector $\mathbf{v} = (v_1, \cdots, v_n)$, with zeros being neglected. Karlin [7] showed that if a matrix A is totally nonnegative then it decreases the variation, i.e.

$$S^-(A\mathbf{v}) \le S^-(\mathbf{v}).$$

By virtue of Corollary 6.3, the totally nonnegative matrix $\{B_j(\tau_i)\}$, $i, j = -1, \ldots, N+1$, formed by generalized cubic B-splines decreases the variation.

For a bounded real function $f(x)$ we denote by $S^-(f) \equiv S^-(f(x))$ the number of sign changes of the function on the real axis $\mathbb{R}$, with zeros being neglected. Thus, we have

$$S^-(f(x)) = \sup_n S^-[f(\tau_1), \dots, f(\tau_n)], \quad \tau_1 < \tau_2 < \dots < \tau_n.$$

Theorem 6.2. The spline

$$S(x) = \sum_{j=-1}^{N+1} b_j B_j(x)$$

is a function decreasing the variation, i.e. the number of sign changes of the spline $S(x)$ is no more than the number of sign changes in the sequence of the spline coefficients:

$$S^-_{\mathbb{R}}\Big(\sum_{j=-1}^{N+1} b_j B_j(x)\Big) \le S^-(\overline{\mathbf{b}}), \quad \overline{\mathbf{b}} = (b_{-1}, \dots, b_{N+1}).$$

Proof. We use the approach proposed by Schumaker [21]. Let $S^-(\overline{b}) = d - 1$. Let us divide the coefficients b_j into d groups:

$$b_{-1}, \dots, b_{k_2}, b_{k_2+1}, \dots, b_{k_3}, \dots, b_{k_d+1}, \dots, b_{N+1}.$$

In this case in each group at least one coefficient is not zero and all the nonzero coefficients have the same sign.

Putting $k_1 = -2$ and $k_{d+1} = N + 1$, we define the function

$$\tilde{B}_j(x) = \sum_{i=k_j+1}^{k_{j+1}} |b_i| B_i(x), \quad j = 1, \dots, d.$$

Then for arbitrary $\tau_1 < \tau_2 < \dots < \tau_d$ we have

$$\det(\tilde{B}_j(\tau_i))_{i,j=1}^d = \sum_{\nu_1=k_1+1}^{k_2} \dots \sum_{\nu_d=k_d+1}^{k_{d+1}} |b_{\nu_1}| \dots |b_{\nu_d}| \det(B_j(\tau_i)) \ge 0,$$

$$i = 1, \dots, d; \quad j = \nu_1, \dots, \nu_d.$$

by virtue of Corollary 6.3 and because at least one coefficient b_i is not zero in each group. It is clear that we can choose $\tau_1 < \tau_2 < \ldots < \tau_d$ such that $det(\tilde{B}_j(\tau_i)) > 0$. Hence the function $\tilde{B}_j(x)$ are linearly independent.

Assume that $\delta = \pm 1$ is the sign of the first group of the coefficients b_i. Let us take $\tilde{b}_i = (-1)^{i-1}\delta$, $i = 1, 2, \ldots, d$. Then

$$\tilde{S}(x) = \sum_{i=1}^{d} \tilde{b}_i \tilde{B}_i(x) \equiv S(x) = \sum_{j=-1}^{N+1} b_j B_j(x).$$

Applying Lemma 6.4, we obtain

$$Z\left(\sum_{j=-1}^{N+1} b_j B_j(x)\right) = Z\left(\sum_{i=1}^{d} \tilde{b}_i B_i(x)\right) \le d - 1 = S^-(b_{-1}, \ldots, b_{N+1}).$$

Thus, the theorem is proved.

The statement of Theorem 6.2 can be refined, namely we can point out a relation between the point at which the spline changes its sign and the corresponding spline coefficient. The coefficient corresponds to the B-spline whose support includes the point of the sign change [see (6.2)].

Theorem 6.3. Assume that the inequalities $(-1)^j S(\tau_j) > 0$, $j = 1, 2 \ldots, d$, are valid for the spline

$$S(x) = \sum_{j=-1}^{N+1} b_j B_j(x)$$

at some $\tau_1 < \tau_2 < \cdots < \tau_d$. Then there exist $-1 \le i_1 < i_2 < \cdots < i_d \le N + 1$ such that

$$(-1)^j b_{i_j} B_{i_j}(\tau_j) > 0, \quad j = 1, 2, \ldots, d.$$

The proof of this statement does not differ from the proof of the corresponding theorem for polynomial B-splines [21].

7. TRANSFORMATION OF A SPLINE REPRESENTATION INTO ANOTHER ONE

If b_j are known in (6.1), then by virtue of (2.11), we can write a expression in an easy-to-use form for the generalized cubic spline $S(x)$ on the subsegment $[x_i, x_{i+1}]$:

$$S(x) = b_i + \tilde{\Delta}_i b(x - y_i) + c_i h_i^2 \varphi_i(t) + c_{i+1} h_i^2 \psi_i(t), \tag{7.1}$$

where

$$c_j = (\tilde{\Delta}_j b - \tilde{\Delta}_{j-1} b) u_j^{-1}, \quad j = i, i+1, \qquad \tilde{\Delta}_j b = \frac{b_{j+1} - b_j}{y_{j+1} - y_j}.$$

From this, in particular, we obtain the formulae

$$\begin{aligned} S(x_i) &= b_i + \frac{1}{u_i}[\psi_{i-1}(1) h_{i-1}^2 \tilde{\Delta}_i b - \varphi_i(0) h_i^2 \tilde{\Delta}_{i-1} b], \\ S'(x_i) &= \frac{1}{u_i}[\psi'_{i-1}(1) h_{i-1} \tilde{\Delta}_i b - \varphi'_i(0) h_i \tilde{\Delta}_{i-1} b], \\ S''(x_i) &= \frac{1}{u_i}(\tilde{\Delta}_i b - \tilde{\Delta}_{i-1} b). \end{aligned} \tag{7.2}$$

Let us also write the inverse transformation

$$\begin{pmatrix} b_{i-1} \\ b_i \\ b_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & y_{i-1} - x_i & \vartheta_i(y_{i-1}) \\ 1 & y_i - x_i & \kappa_i(y_i) \\ 1 & y_{i+1} - x_i & \kappa_i(y_{i+1}) \end{pmatrix} \begin{pmatrix} S(x_i) \\ S'(x_i) \\ S''(x_i) \end{pmatrix}, \tag{7.3}$$

where

$$\begin{aligned} \vartheta_i(y_j) &= -h_{i-1}[\psi_{i-1}(1) h_{i-1} + (y_j - x_i)\psi'_{i-1}(1)], \\ \kappa_i(y_j) &= -h_i[\varphi_i(0) h_i + (y_j - x_i)\varphi'_i(0)], \\ \kappa_i(y_j) &= \vartheta_i(y_j) + u_i(y_j - y_i), \quad j = i-1, i, i+1. \end{aligned}$$

Therefore in (7.1) we have

$$b_j = \begin{cases} S(x_i) + (y_j - x_i) S'(x_i) + \vartheta_i(y_j) S''(x_i), & j = i-1, i, \\ S(x_{i+1}) + (y_j - x_{i+1}) S'(x_{i+1}) + \kappa_{i+1}(y_j) S''(x_{i+1}), & j = i+1, i+2. \end{cases}$$

Formula (7.1) allows one to express the coefficients of the spline $S(x)$ in the representation (6.1) as

$$b_j = \begin{cases} S(y_j) - S''(x_{j-1}) \Phi_{j-1}(y_j) - S''(x_j) \Psi_{j-1}(y_j), & y_j < x_j, \\ S(y_j) - S''(x_j) \Phi_j(y_j) - S''(x_{j+1}) \Psi_j(y_j), & y_j \geq x_j. \end{cases} \tag{7.4}$$

Multiplying (7.3) from the left by the matrix $B = \{B_{rj} | B_{rj} = B_j^{(r)}(x_i)\}$, $r = 0, 1, 2$, $j = i-1, i, i+1$, we arrive at the identities

$$\sum_{j=i-1}^{i+1} B_j^{(r)}(x_i) = \delta_{r0}, \qquad \sum_{j=i-1}^{i+1} (y_j - x_i) B_j^{(r)}(x_i) = \delta_{r1},$$

$$\sum_{j=i-1}^{i+1} \kappa_i(y_j)B_j^{(r)}(x_i) + u_i(y_i - y_{i-1})B_{i-1}^{(r)}(x_i) \tag{7.5}$$

$$= \sum_{j=i-1}^{i+1} \vartheta_i(y_j)B_j^{(r)}(x_i) + u_i(y_{i+1} - y_i)B_{i+1}^{(r)}(x_i) = \delta_{r2},$$

where δ_{rk} is the Kronecker's symbol.

By using Table 1, we can rewrite the last relations in (7.5) as

$$\sum_{j=i-1}^{i+1} \kappa_i(y_j)B_j^{(r)}(x_i) = -\varphi_i^{(r)}(0)h_i^{2-r}, \quad r = 0, 1,$$

$$\sum_{j=i-1}^{i+1} \vartheta_i(y_j)B_j^{(r)}(x_i) = -\psi_{i-1}^{(r)}(1)h_{i-1}^{2-r}, \quad r = 0, 1,$$

$$\sum_{j=i-1}^{i+1} \kappa_i(y_j)B_j''(x_i) = \sum_{j=i-1}^{i+1} \vartheta_i(y_j)B_j''(x_i) = 0.$$

Now if we demand that $y_i = x_i$, $i = 0, \ldots, N$, then we should put $v_i = \psi_{i-1}(1)h_{i-1}^2 - \varphi_i(0)h_i^2 = 0$. In this case formulae (7.3) for the coefficients b_i, $i = -1, \ldots, N+1$, are essentially simplified and are of the form:

$$\begin{aligned} b_{i-1} &= S(x_i) - h_{i-1}S'(x_i) + h_{i-1}^2[-\psi_{i-1}(1) + \psi_{i-1}'(1)]S''(x_i), \\ b_i &= S(x_i) - h_i^2\varphi_i(0)S''(x_i), \\ b_{i+1} &= S(x_i) + h_iS'(x_i) - h_i^2[\varphi_i(0) + \varphi_i'(0)]S''(x_i), \\ & \qquad i = 0, \ldots, N. \end{aligned} \tag{7.6}$$

8. FORMULAE FOR LOCAL APPROXIMATION BY GENERALIZED CUBIC SPLINES

Representations (6.1) and (7.1) allows us to find a simple and efficient way to approximate a pointwise given function $f(x)$.

Lemma 8.1. For $b_j = f(y_j)$, $j = -1, \ldots, N+1$, formula (6.1) is exact for polynomials of the first degree and provides the local smoothing.

Proof. We should prove, in fact, that the identities

$$\sum_{j=-1}^{N+1} y_j^r B_j(x) \equiv x^r, \quad r = 0, 1 \tag{8.1}$$

hold for $x \in [a,b]$. Using formula (7.1) with the coefficients $b_j = 1$, y_j, $j = i-1, i, i+1, i+2$, for an arbitrary subsegment $[x_i, x_{i+1}]$, we find that identities (8.1) hold.

For $b_j = f(y_j)$ formula (7.1) can be rewritten as

$$\begin{aligned} S(x) =& f(y_i) + f[y_i, y_{i+1}](x - y_i) + (y_{i+1} - y_{i-1}) f[y_{i-1}, y_i, y_{i+1}] u_i^{-1} h_i^2 \varphi_i(t) \\ &+ (y_{i+2} - y_i) f[y_i, y_{i+1}, y_{i+2}] u_{i+1}^{-1} h_i^2 \psi_i(t), \quad x \in [x_i, x_{i+1}], \end{aligned} \tag{8.2}$$

where the square brackets denote the divided differences of the function $f(x)$ with respect to its arguments y_j, $j = i-1, i, i+1, i+2$. This formula is the formula of local smoothing.

For the cubic spline at $q_{j-1} = p_j = 0$, $j = i-1, \ldots, i+2$, we have

$$y_j = (x_{j-1} + x_j + x_{j+1})/3, \quad u_j = (x_{j+1} - x_{j-1})/2.$$

If, in addition, we assume that the grid Δ is uniform, i.e. $h_j = h$ for all j, then formula (8.2) takes the form:

$$S(x) = f_i(1-t) + f_{i+1}t + \frac{h}{6}[(1-t)^3 \delta_i f + t^3 \delta_{i+1} f],$$

where $\delta_j f = \Delta_j f - \Delta_{j-1} f$, $j = i, i+1$. In particular, $S(x_i) = (f_{i-1} + 4f_i + f_{i+1})/6$. Thus, the lemma is proved.

Corollary 8.1. Putting

$$b_j = f_j - \frac{1}{u_j}[\psi_{j-1}(1) h_{j-1}^2 \Delta_j f - \varphi_j(0) h_j^2 \Delta_{j-1} f], \tag{8.3}$$

in (6.1), we obtain the formula of three-point local approximation, which is exact for polynomials of the first degree.

To prove the corollary, it is sufficient to take the monomials $1, x$ as $f(x)$. Then according to (8.3), we obtain $b_j = 1$ and $b_j = y_j$ and it only remains to use identities (8.1).

By virtue of Theorem 6.2, the spline

$$S_f(x) = \sum_{j=-1}^{N+1} f(y_j) B_j(x)$$

decreases the variation. This allows us to write the inequalities

$$S^-(S_f(x)) \le S^-(\overline{\mathbf{f}}) \le S^-(f(x)),$$

where $\overline{\mathbf{f}} = (f(y_{-1}), \ldots, f(y_{N+1}))$. Since the locally approximating spline $S_f(x)$ is also exact for polynomials $l(x)$ of the first degree, we arrive at the inequality

$$S^-(S_f(x) - l(x)) = S^-(S_{f-l}(x)) \le S^-(f(x) - l(x)).$$

Thus, the following statement is true.

Theorem 8.1. If $b_j = f(y_j)$, $j = -1, \dots, N+1$, then the locally approximating spline $S_f(x)$ intersects an arbitrary straight line no more times than the function $f(x)$ does.

Note that according to (7.4),

$$b_j = S(y_j) + O(\overline{h}_j^2), \quad \overline{h}_j = \max(h_{j-1}, h_j).$$

Hence it follows that the test polygon converges quadratically to the fubction $f(x)$ both at $b_j = f(y_j)$ and if formula (8.3) is used.

9. EXAMPLES OF GENERALIZED B-SPLINES

Let us consider the defining functions $\varphi_i(t)$ and $\psi_i(t)$ in (1.1), which are in most common use. In the examples given below they depend on the parsameters:

$$\varphi_i(t) = \varphi_i(p_i, t), \quad \psi_i(t) = \varphi_i(q_i, 1-t), \quad 0 \le p_i, q_i < \infty.$$

(1) Rational splines with a linear denominator [23]:

$$\varphi_i(t) = (1-t)^3/(1+p_i t)P_i, \quad P_i^{-1} = 2(3+3p_i+p_i^2),$$
$$\psi_i(t) = t^3/[1+q_i(1-t)]Q_i, \quad Q_i^{-1} = 2(3+3q_i+q_i^2).$$

The conditions of Lemma 1.2 are satisfied at $-1 < p_i, q_i < \infty$, $i = 0, \dots, N-1$, and thus the interpolation spline exists and is unique. Lemma 1.1 holds, if, for example, we demand additionally that $p_i = q_i$, $i = 0, \dots, N-1$.

Formula (3.1) allows us to write a compact representation of the rational spline [13]:

$$B_i(x) = (y_{i+1} - y_{i-1}) \sum_{j=i-1}^{i+1} \frac{1}{u_j \omega'_{i-1}(y_j)} \Big\{ P_{j-1}\varphi[x - x_{j-1}, (x-x_j)(1+p_j)] - Q_j \varphi[(x-x_j)(1+q_j), x - x_{j+1}] \Big\}, \tag{9.1}$$

where $\varphi[z_1, z_2]$ denotes the first divided difference of the truncated power function $\varphi(z) = z_+^3 = [\max(0,z)]^3$ with respect to its argument $z = z_1, z_2$.

It is easy to check that the 'averaged' nodes y_j of the B-spline $B_i(x)$ satisfy the constraints

$$x_j - h_{j-1}/3 < y_j < x_j + h_j/3, \quad j = i-1, i, i+1. \tag{9.2}$$

If $p_j = q_j = 0$, $j = i-2, \dots, i+1$, then formula (9.1) is transformed to $B_i(x) = (x_{i+2} - x_{i-2})\varphi[x; x_{i-2}, \dots, x_{i+2}]$ for $\varphi(x,y) = (x-y)_+^3$, which corresponds to the normalized cubic B-spline [26]. Passing to the limit as $p_j, q_j \to \infty$, $j = i-2, \dots, i+1$, we obtain $B_i(x) = (x_{i+1} - x_{i-1})\varphi[x; x_{i-1}, x_i, x_{i+1}]$ for $\varphi(x,y) = (x-y)_+$, i.e. a piecewise linear function of the Schauder basis.

(2) Rational splines with a quadratic denominator [23]:

$$\varphi_i(t) = (1-t)^3/[1 + p_i t(1-t)]P_i, \quad P_i^{-1} = 2(1+p_i)(3+p_i).$$

Here the conditions for Lemmas 1.1 and 1.2 to hold are the same as in (1). Formula (3.1) yields

$$B_i(x) = (y_{i+1} - y_{i-1}) \sum_{j=i-1}^{i+1} \frac{\Omega_j(x)}{u_j \omega_{i-1}'(y_j)},$$

where

$$\begin{aligned}\Omega_j(x) =& Q_{j-1} h_{j-1} \frac{(z_{j-1})_+^3 - (1 + 3q_{j-1} + q_{j-1}^2)(z_j)_+^3 - h_{j-1} Q_{j-1}^{-1} (z_j)_+^2}{h_{j-1}^2 - q_{j-1} z_{j-1} z_j} \\ &+ P_j h_j \frac{P_j^{-1} h_j (z_j)_+^2 - (1 + 3p_j + p_j^2)(z_j)_+^3 + (z_{j+1})_+^3}{h_j^2 - p_j z_j z_{j+1}},\end{aligned}$$

$$z_k = x - x_k, \quad k = j-1, j, j+1.$$

(3) Exponential splines [22,23]:

$$\varphi_i(t) = (1-t)^3 e^{-p_i t}/(6 + 6p_i + p_i^2).$$

(4) Hyperbolic splines (see [10] and numerous references there):

$$\varphi_i(t) = \frac{\sinh p_i(1-t) - p_i(1-t)}{p_i^2 \sinh(p_i)}.$$

(5) Splines with additional nodes [18]:

$$\varphi_i(t) = \frac{1}{6(1+p_i)^2}[1 - (1+p_i)t]_+^3.$$

If we take $\alpha_i = (1+p_i)^{-1}$ and $\beta_i = 1 - (1+q_i)^{-1}$, then the points $x_{i1} = x_i + \alpha_i h_i$ and $x_{i2} = x_i + \beta_i h_i$ fix the positions of two additional nodes

(a) (b)

Figure 3. Rational basis splines with multiple nodes. (a) x_j = (0,1,2,4,4), x_j = (1,2,5,5,5), and x_j = (2,6,6,6,6); (b) x_j = (0,0,2,3,4), x_j = (1,1,1,4,5), and x_j = (2,2,2,2,6).

of the spline on the segment $[x_i, x_{i+1}]$. Moving them, we can go from a cubic spline to a piecewise linear interpolation [18].

Note that in the cases (2)–(5) the 'averaged' nodes y_j, $j = i-1, i, i+1$, of the splines $B_i(x)$ also satisfy the inequalities (9.2).

Figures 3a, 3b, and 4a present the graphs of rational B-splines with a quadratic denominator at $p_j = q_j = p$, $j = i-2, \ldots, i+1$, and with multiple nodes. The solid, dotted, and dashed lines represent the graphs of B-splines for $p = 0$, $p = 5$, and $p = 15$, respectively. The sequences of the nodes are (0,1,2,4,4), (1,2,5,5,5), and (2,6,6,6,6) in Fig. 3a; (0,0,2,3,4), (1,1,1,4,5,), and (2,2,2,2,6) in Fig. 3b; and (0,0,2,4,4), (1,1,1,5,5), and (2,4,4,4,6) in Fig. 4a. Figure 4b duplicates Fig. 4a for splines with additional nodes (5).

Various generalizations of parabolic splines [24], which can easily be included in our scheme, prove useful in practical calculations. In order to determine them, we introduce an additional grid $\overline{\Delta} = \{\overline{x}_i \mid i = -2, \ldots, N+3\}$, where $x_{i-1} < \overline{x}_i < x_i$ and $\overline{x}_{i+1} = x_i + \alpha_i h_i = x_{i+1} - \beta_i h_i$. Using the representation (1.1), we can propose the following variants of defining functions $\varphi_i(t) = \varphi_i(\alpha_i, p_i, t)$ and $\psi_i(t) = \varphi_i(\beta_i, q_i, 1-t)$:

(a) $\varphi_i(t) = P_i(\alpha_i - t)_+^2/(1 + p_i t)$, $P_i^{-1} = 2(1 + \alpha_i p_i)^2$;

(b) $\varphi_i(t) = P_i(\alpha_i - t)_+^2/[1 + p_i t(1-t)]$, $P_i^{-1} = 2[(1 + \alpha_i p_i)^2 + \alpha_i^2 p_i]$;

(c) $\varphi_i(t) = P_i e^{-p_i t}(\alpha_i - t)_+^2$, $P_i^{-1} = (2 + \alpha_i p_i)^2 - 2$;

(a) (b)

Figure 4. Rational B-splines (a) and B-splines with additional nodes (b). Multiple nodes are x_j = (0,0,2,4,4), x_j = (1,1,1,5,5), and x_j = (2,4,4,4,6).

(d) $\varphi_i(t) = [\alpha_i/(1+p_i) - t]_+^2/2$.

Here the conditions of Lemma 1.2 are satisfied at $p_i, q_i \geq 0$, $i = 0,\dots,N-1$, and thus the corresponding interpolating splines exist and are unique. Lemma 1.1 holds if, for example, we demand additionally that $p_i = q_i$ and $\alpha_i = \beta_i$, $i = 0,\dots,N-1$.

Formulae for generalized parabolic B-splines $B_{j,k}(x)$, $k = 1,2,3$, are no different formally from the corresponding formulae for generalized cubic splines (4.1), (4.6), and (2.11). It should be noted, however, that the parabolic splines belong to the smoothness class $C^{k-2}[a,b]$ (they have a discontinuity at $k = 1$) and their supports are narrower. A generalized parabolic spline $B_{j,k}(x)$, $k = 1,2,3$, differs from zero only on the interval $(\overline{x}_{j+1}, \overline{x}_{j+k+1})$. The 'averaged' nodes y_j of such splines are subject to the constraints $\overline{x}_j < y_j < \overline{x}_{j+1}$, $j = i-1, i, i+1$.

The rational parabolic spline (a) was proposed in [11] for solving the geometric interpolation problem. We give its representation similar to (9.1) in terms of truncated rational functions:

$$B_i(x) = (y_{i+1} - y_{i-1}) \sum_{j=i-1}^{i+1} \frac{1}{u_j \omega'_{i-1}(y_j)} \Big\{ Q_{j-1}\beta_{j-1}^{-1} h_{j-1} \varphi\big[x - \overline{x}_j, (x - x_j) \\ \times (1 + \beta_{j-1} q_{j-1})\big] + P_j \alpha_j^{-1} h_j \varphi\big[(x - x_j)(1 + \alpha_j p_j), x - \overline{x}_{j+1}\big] \Big\}. \qquad (9.3)$$

Here $\varphi(z) = z_+^2$. For a conventional parabolic spline, where $p_j = q_j = 0$ for all j, we have $y_j = (\overline{x}_j + \overline{x}_{j+1})/2$ and formula (9.3) is of the form:

$$B_i(x) = (\overline{x}_{i+2} - \overline{x}_{i-1})\varphi[x; \overline{x}_{i-1}, \ldots, \overline{x}_{i+2}]$$

for $\varphi(x, y) = (x - y)_+^2$.

(a) (b)

Figure 5. Rational parabolic B-splines of orders $k = 1, 2, 3$ (from left to right) with various vectors of parameters. (a) $\mathbf{p} = 0$ for all j; (b) $\mathbf{p} =$ (1,5), $\mathbf{p} =$ (10,100,0), and $\mathbf{p}$ = (1,5,10,10).

Figures 5a and 5b present the graphs of rational parabolic splines with a quadratic denominator (b) on a uniform grid with step $h = 1$ at $\overline{x}_j = (x_j + x_{j+1})/2$. The vectors of the parameters p_j and q_j are the same as in Figs. 2a and 2b. Here the supports of the splines $B_{j,k}(x)$, $k = 1, 2, 3$, are the intervals (1/2,3/2), (3/2,7/2), and (5/2,11/2). Passing to the limit as $p_j, q_j \to \infty$, we again obtain the impulse function, the 'function-step' and the 'function-roof', each having the unit height (Fig. 2c).

10. HERMITIAN REPRESENTATION OF GENERALISED CUBIC SPLINES

The following representation is often used in constructing generalized cubic splines. For a subsegment $[x_i, x_{i+1}]$ we have

$$S(x) = (1 - t)S_i + tS_{i+1} + \varphi_{1i}(t)h_i(S_i' - \Delta_i S) + \psi_{1i}(t)h_i(S_{i+1}' - \Delta_i S), \tag{10.1}$$

where $\varphi_{1i}(0) = \varphi_{1i}^{(k)}(1) = \psi_{1i}(1) = \psi_{1i}^{(k)}(0) = 0$, $k = 0, 1$, and $\varphi'_{1i}(0) = \psi'_{1i}(1) = 1$. For $\varphi_{1i}(t) = t(1-t)^2$ and $\psi_{1i}(t) = -t^2(1-t)$, in particular, we have the conventional cubic spline [26].

Since, according to (10.1), we have

$$\begin{aligned} S''_i &= \varphi''_{1i}(0)h_i^{-1}(S'_i - \Delta_i S) + \psi''_{1i}(0)h_i^{-1}(S'_{i+1} - \Delta_i S), \\ S''_{i+1} &= \varphi''_{1i}(1)h_i^{-1}(S'_i - \Delta_i S) + \psi''_{1i}(1)h_i^{-1}(S'_{i+1} - \Delta_i S), \end{aligned} \tag{10.2}$$

then the continuity conditions $S''(x_i - 0) = S''(x_i + 0)$, $i = 1, \ldots, N-1$, and the boundary conditions $S''(a) = S''_0$ and $S''(b) = S''_N$ result in the system of equations

$$\begin{cases} \varphi''_{10}(0)S'_0 + \psi''_{10}(0)S'_1 = h_0 S''_0 + [\varphi''_{10}(0) + \psi''_{10}(0)]\Delta_0 S, \\ \varphi''_{i-1}(1)h_{i-1}^{-1}S'_{i-1} + [\psi''_{1i-1}(1)h_{i-1}^{-1} - \varphi''_{1i}(0)h_i^{-1}]S'_i - \psi''_{1i}(0)h_i S''_{i+1} \\ \quad = [\varphi''_{1i-1}(1) + \psi''_{1i-1}(1)]h_{i-1}^{-1}\Delta_{i-1}S - [\varphi''_{1i}(0) + \psi''_{1i}(0)]h_i^{-1}\Delta_i S, \\ \qquad\qquad i = 1, \ldots, N-1, \\ \varphi''_{1N-1}(1)S'_{N-1} - \psi''_{1N+1}(1)S'_N = h_{N-1}S''_N \\ \qquad\qquad + [\varphi''_{1N-1}(1) + \psi''_{1N-1}(1)]\Delta_{N-1}S. \end{cases} \tag{10.3}$$

Lemma 10.1. If the conditions

$$\varphi''_{1i}(0) < \psi''_{1i}(0) < 0, \quad \psi''_{1i}(1) > \varphi''_{1i}(1) > 0, \quad i = 0, \ldots, N-1$$

are satisfied, the generalized cubic spline $S(x)$ exists and is unique.

Proof. According to the lemma conditions, the matrix of the system (10.3) has the diagonal predominance:

$$\begin{aligned} r_0 &= -\varphi''_{10}(0) + \psi''_{10}(0) > 0, \\ r_i &= [\psi''_{1i-1}(1) - \varphi''_{i-1}(1)]h_{i-1}^{-1} + [\psi''_{1i}(0) - \varphi''_{1i}(0)]h_i^{-1} > 0, \\ &\qquad i = 1, \ldots, N-1, \\ r_N &= \psi''_{1N-1}(1) - \varphi''_{1N-1}(1) > 0. \end{aligned}$$

Hence, according to the Hadamard criterion [26], the determinant of the system (10.3) is not zero and therefore the generalized cubic spline exists and is unique. Thus, the lemma is proved.

It follows from relations (10.2) that

$$\begin{aligned} S'_i &= \Delta_i S - T_{1i}h_i[\psi''_{1i}(1)S''_i - \psi''_{1i}(0)S''_{i+1}], \\ S'_{i+1} &= \Delta_i S + T_{1i}h_i[\varphi''_{1i}(1)S''_i - \varphi''_{1i}(0)S''_{i+1}], \\ T_{1i}^{-1} &= \varphi''_{1i}(1)\psi''_{1i}(0) - \varphi''_{1i}(0)\psi''_{1i}(1). \end{aligned} \tag{10.4}$$

The continuity conditions $S'(x_i - 0) = S'(x_i + 0)$, $i = 0, \dots, N-1$, and the boundary conditions $S'(a) = S'_0$ and $S'(b) = S'_N$ result in the system of equations

$$\begin{cases} \psi''_{10}(1)S''_0 - \psi''_{10}(0)S''_1 = T_{10}^{-1}h_0^{-1}(\Delta_0 S - S'_0), \\ T_{1i-1}\varphi''_{1i-1}(1)h_{i-1}S''_{i-1} + [-T_{1i-1}\varphi''_{1i-1}(0)h_{i-1} + T_{1i}\psi''_{1i}(1)h_i]S''_i \\ \qquad - T_{1i}\psi''_{1i}(0)h_i S''_{i+1} = \Delta_i S - \Delta_{i-1}S, \quad i = 1, \dots, N-1, \\ \varphi''_{1N-1}(1)S''_{N-1} - \varphi''_{1N-1}(0)S''_N = T_{1N-1}^{-1}h_{N-1}^{-1}(S'_N - \Delta_{N-1}S). \end{cases} \tag{10.5}$$

Lemma 10.2. If the conditions

$$\varphi''_{1i}(0) < -\varphi''_{1i}(1) < 0, \quad \psi''_{1i}(1) > -\psi''_{1i}(0) > 0, \quad i = 0, \dots, N-1$$

are satisfied, the generalized cubic spline $S(x)$ exists and is unique.

Proof. Since under the lemma conditions we have $\varphi''_{1i}(0) < 0$ and $\psi''_{1i}(1) > 0$, then according to (10.4), we get

$$\begin{aligned} T_{1i}^{-1} &= \varphi''_{1i}(1)[\psi''_{1i}(0) + \psi''_{1i}(1)] - \psi''_{1i}(1)[\varphi''_{1i}(0) + \varphi''_{1i}(1)] \\ &= -\varphi''_{1i}(0)[\psi''_{1i}(0) + \psi''_{1i}(1)] + \psi''_{1i}(0)[\varphi''_{1i}(0) + \varphi''_{1i}(1)] > 0. \end{aligned}$$

Then the system (10.5) has the diagonal predominance:

$$\begin{aligned} r_0 &= \psi_{10}(1) + \psi_{10}(0) > 0, \\ r_i &= -T_{i-1}[\varphi''_{1i-1}(0) + \varphi''_{1i-1}(1)]h_{i-1} + T_i[\psi''_{1i}(0) + \psi''_{1i}(1)]h_i > 0, \\ & \qquad i = 1, \dots, N-1, \\ r_N &= -\varphi''_{1N-1}(0) - \varphi''_{1N-1}(1) > 0, \end{aligned}$$

which ensures that the spline $S(x)$ exists and is unique.

The conditions of Lemmas 10.1 and 10.2 are satisfied for the defining functions $\varphi_{1i}(t)$ and $\psi_{1i}(t)$ used in practice. The systems (10.3) and (10.5) have diagonal predominance and can be solved by the Gauss method without rotations.

Let us write formulae for transforming the spline representation (10.1) into (1.1):

$$\begin{aligned} \varphi_i(t) &= T_{1i}\{-\psi''_{1i}(1)\varphi_{1i}(t) + \varphi''_{1i}(1)[\psi_{1i}(t) + 1 - t]\}, \\ \psi_i(t) &= T_{1i}\{\psi''_{1i}(0)[\varphi_{1i}(t) - t] - \varphi''_{1i}(0)\psi_{1i}(t)\}, \end{aligned} \tag{10.6}$$

[here T_{1i} is taken from (10.4)] and vice versa

$$\begin{aligned} \varphi_{1i}(t) &= T_i\{[\psi_i(1) - \psi'_i(1)][\varphi_i(t) - \varphi_i(0)(1-t)] + \varphi_i(0)[\psi_i(t) - \psi_i(1)t]\}, \\ \psi_{1i}(t) &= T_i\{\psi_i(1)[\varphi_i(0)(1-t) - \varphi_i(t)] + [\varphi_i(0) + \varphi'_i(0)][\psi_i(1)t - \psi_i(t)]\}, \\ T_i^{-1} &= [\varphi_i(0) + \varphi'_i(0)][\psi_i(1) - \psi'_i(1)] - \varphi_i(0)\psi_i(1). \end{aligned}$$

Using the formulae, it is easy to check directly that Lemmas 1.1 and 1.2 are equivalent to Lemmas 10.2 and 10.1, respectively.

11. B-SPLINES WITH ALTERNATING SIGNS

Let us show that the representation (10.1) results in B-splines which can change sign on the support. To use the formulae for the B-sp[lines $B_{j,k}(x)$, $k = 1, 2, 3$, it is sufficient to express the functions $\varphi_i(t)$ and $\psi_i(t)$ in (1.1) in terms of $\varphi_{1i}(t)$ and $\psi_{1i}(t)$ in (10.1) by using the transformation (10.6).

The following defining functions are widely used in the representation (10.1) for $\varphi_{1i}(t) = \varphi_{1i}(p_i, t)$ and $\psi_{1i}(t) = -\varphi_{1i}(q_i, 1-t)$, $0 \le p_i, q_i < \infty$.

(1) Rational cubic splines with a linear denominator [23]:

$$\varphi_{1i}(t) = t(1-t)^2/(1+p_i t).$$

In this case the conditions of Lemmas 10.1 and 10.2 are satisfied for any $p_i, q_i \ge -1/2$ and the interpolating spline exists and is unique.

Applying the transformation (10.6), we obtain

$$\varphi_i(t) = \frac{T_i(1-t)^3}{2(1+p_i)(1+p_i t)} \left[1 - \frac{(1+q_i)(p_i + p_i q_i + q_i)t}{1+q_i(1-t)}\right],$$
$$T_i^{-1} = (2+p_i)(2+q_i) - (1+p_i)^{-1}(1+q_i)^{-1}.$$

This function, generally speaking, is not monotonous on the segment $[0,1]$ and can have here a unique root, whose location is governed by the parameters p_i and q_i. Using formulae (4.1), (4.6), and (2.11), we obtain B-splines with alternating signs. Figures 6a, 6b, and 6c present (from left to right) the graphs of such B-splines $B_{j,k}(x)$, $k = 1, 2, 3$, on a uniform grid with step $h = 1$ for various vectors of the parameters. The parameters in Fig. 6a are the same as those in Fig. 2b. For Figs. 6b and 6c the parameter vectors were taken as ((1,5), (1,5)), ((5,5),(5,5),(5,5)), ((15,0),(15,0),(0,15),(0,15)), and ((1,1),(1,1)), ((5,0), (0,0),(0,5)), ((50,0),(0,0),(0,0),(0,50)), respectively.

(2) Rational cubic splines with a quadratic denominator [6]:

$$\varphi_{1i}(t) = t(1-t)^2/[1+p_i t(1-t)].$$

In this case the systems (10.3) and (10.5) have the diagonal predominance at $p_i, q_i > -1$, $i = 0, \dots, N-1$. The functions

$$\varphi_i(t) = \frac{T_i(1-t)^3}{2[1+q_i t(1-t)]} \left[1 + \frac{(p_i - q_i)(2+q_i)t^2}{1+p_i t(1-t)}\right],$$
$$T_i = [(2+p_i)(2+q_i) - 1]^{-1}$$

(a) (b)

(c)

Figure 6. Alternating rational B-splines of orders k=1,2,3 (from left to right) with various vectors of parameters. (a) $\mathbf{p} = \mathbf{q} =$ (1,5), $\mathbf{p} = \mathbf{q} =$ (10,100,0), and $\mathbf{p} = \mathbf{q} =$ (1,5,10,10); (b) $(\mathbf{p}, \mathbf{q}) =$ ((1,5),(1,5)), $(\mathbf{p}, \mathbf{q}) =$ ((5,5),(5,5),(5,5)), and $(\mathbf{p}, \mathbf{q}) =$ ((15,0), (15,0),(0,15),(0,15)). (c) $(\mathbf{p}, \mathbf{q}) =$ ((1,1),(1,1)), $(\mathbf{p}, \mathbf{q}) =$ ((5,0),(0,0),(0,5)), and $(\mathbf{p}, \mathbf{q}) =$ ((50,0),(0,0), (0,0),(0,50)).

can have a unique root on the segment $[0, 1]$. If $p_i = q_i$, $i = j, \ldots, j + 3$, the basis splines $B_{j,k}(x)$, $k = 1, 2, 3$, are nonnegative and coincide with those obtained on the basis of the representation (1.1).

If formula (10.1) is used, all the other generalized cubic and parabolic B-splines (including exponential splines and those with additional nodes) also have alternating signs. Let us cite the corresponding defining functions.

(3) Exponential cubic splines [23]:

$$\varphi_{1i}(t) = t(1-t)^2 e^{-p_i t}.$$

(4) Cubic splines with additional nodes [17]:

$$\varphi_{1i}(t) = \frac{1}{1+p_i}\{(1-t)^2 - \frac{2}{3}(1-t)^3 - \frac{1}{3}[1-(1+p_i)t]_+^3\}.$$

For generalized parabolic B-splines we have considered the following defining functions $\varphi_{1i}(t) = \varphi_{1i}(\alpha_i, p_i, t)$ and $\psi_{1i}(t) = -\varphi_{1i}(\beta_i, q_i, 1-t)$, $0 < \alpha_i = 1 - \beta_i < 1$:

(a) $\varphi_{1i}(t) = g_i(t)(1+p_i t)^{-1}$;

(b) $\varphi_{1i}(t) = g_i(t)[1+p_i t(1-t)]^{-1}$;

(c) $\varphi_{1i}(t) = g_i(t)e^{-p_i t}$;

(d) $\varphi_{1i}(t) = [2\gamma_i(1-\gamma_i)]^{-1}[\gamma_i^2(1-t)^2 - (\gamma_i - t)_+^2], \gamma_i = \alpha_i(1+p_i)^{-1}$,

where $g_i(t) = (2\alpha_i\beta_i)^{-1}[\alpha_i^2(1-t)^2 - (\alpha_i - t)_+^2]$.

Let us study some characteristic properties of B-splines with alternating signs. The following statement is an analogue to Lemma 5.1 for the representation (10.1).

Lemma 11.1. If the conditions

$$\varphi''_{1j}(0) < \psi''_{1j}(0) < 0, \quad \psi''_{1j}(1) > \varphi''_{1j}(1) > 0, \quad j = i-1, i$$

are satisfied, the inequalities

$$x_i - h_{i-1}/2 < y_i < x_i + h_i/2 \tag{11.1}$$

hold.

Proof. According to (10.6), we have

$$\begin{aligned} \psi_{i-1}(1) &= -T_{1i-1}\psi''_{1i-1}(0), \quad & \psi'_{i-1}(1) &= -T_{1i-1}[\varphi''_{1i-1}(0) + \psi''_{1i-1}(0)], \\ \varphi_i(0) &= T_{1i}\varphi''_{1i}(1), & \varphi'_i(0) &= -T_{1i}[\varphi''_{1i}(1) + \psi''_{1i}(1)], \end{aligned}$$

where T_{1j}, $j = i-1, i$, is taken from (10.4).

Under the lemma conditions, we get

$$T_{1j}^{-1} = \varphi''_{1j}(1)[\psi''_{1j}(0) - \varphi''_{1j}(0)] - \varphi''_{1j}(0)[\psi''_{1j}(1) - \varphi''_{1j}(1)] > 0, \quad j = i-1, i$$

and hence

$$\psi_{i-1}(1) > 0, \quad \psi'_{i-1}(1) > 0, \quad \varphi_i(0) > 0, \quad \varphi'_i(0) < 0.$$

Therefore

$$u_i = \psi'_{i-1}(1)h_{i-1} - \varphi'_i(0)h_i > 0$$

and according to (2.4) and (2.7), we can rewrite inequalities (11.1) as

$$\begin{aligned} &0 < h_{i-1}^2[\psi'_{i-1}(1) - 2\psi_{i-1}(1)] + h_i[-h_{i-1}\varphi'_i(0) + 2h_i\varphi_i(0)], \\ &0 < h_{i-1}[h_i\psi'_{i-1}(1) + 2h_{i-1}\psi_{i-1}(1)] - h_i^2[\varphi'_i(0) + 2\varphi_i(0)]. \end{aligned} \tag{11.2}$$

Since

$$\begin{aligned} \psi'_{i-1}(1) - 2\psi_{i-1}(1) &= T_{1i-1}[\psi''_{1i}(0) - \varphi''_{1i}(0)] > 0, \\ -\varphi'_i(0) - 2\varphi_i(0) &= T_{1i}[\psi''_{1i}(1) - \varphi''_{1i}(1)] > 0, \end{aligned}$$

clearly inequalities (11.2) hold. Thus, the lemma is proved.

Based on Lemma 11.1 and formulae (2.9), (4.3), and (4.4), we can formulate the following statement.

Corollary 11.1. If the conditions of Lemma 10.1 are satisfied for $i = j, \ldots, j+3$, then the inequalities

$$\int_{x_i}^{x_{i+k+1}} B_{i,k}(\tau)\, d\tau > 0, \quad i = j, \ldots, j+3-k; \quad k = 1, 2,$$

$$(-1)^l B''_{j,3}(x_{j+l+1}) > 0, \quad l = 0, 1, 2$$

hold.

The next statement allows us to specify the form of B-splines with alternating signs.

Theorem 11.1. Assume that the defining functions $\varphi_i(t)$ and $\psi_i(t)$, $i = j, \ldots, j+3$, satisfy the inequalities (1.6), and the derivatives $\varphi''_i(t)$ and $\psi''_i(t)$ are convex downwards for $t \in [0,1]$. Then the splines are either monotonous on the segments $[x_j, x_{j+1}]$ and $[x_{j+k}, x_{j+k+1}]$, or have a unique point of minimum on these segments and, besides, $B_{j,k}(x) > 0$ for $x \in [x_{j+1}, x_{j+k}]$, $k = 2, 3$.

Proof. First let us consider the splines $B_{j,2}(x)$. Under the condition of the theorem, the values u_l, $l = j+1, j+2$, in formula (4.6) are positive.

Therefore according to (4.6), the spline $B_{j,2}(x)$ behaves on the subsegments $[x_j, x_{j+1}]$ and $[x_{j+2}, x_{j+3}]$ just like the functions $\Psi'_j(x)$ and $-\Phi'_{j+2}(x)$ do. They are either monotonous or have a unique point of minimum.

If $\beta''_i \in [0,1]$ is a root of the function $\varphi''_i(t)$, then the function $\varphi_i(t)$ is convex downwards on the segment $[0, \beta''_i]$ and

$$\frac{\varphi_i(t) - \varphi_i(0)}{t} \geq \varphi'_i(0), \quad 0 \leq t \leq \beta''_i.$$

Hence it follows that the root β_i of the function $\varphi_i(t)$ should satisfy the inequality $-\varphi_i(0) \geq \beta_i \varphi'_i(0)$, i.e. according to (1.6), $1/2 \leq \beta_i \leq 1$. Similarly, by virtue of (1.6), if $\psi_i(\alpha_i) = 0$, then $\alpha_i \in [0, 1/2]$. Under the conditions of the theorem, for the roots of the functions $\psi''_i(t)$ and $\varphi''_i(t)$ we have $\alpha''_i \in [0, \alpha_i]$ and $\beta''_i \in [\beta_i, 1]$. Therefore the function

$$B'_{j,2}(x) = \Phi''_{j+1}(x) u^{-1}_{j+1} - \Psi''_{j+1}(x) u^{-1}_{j+2}, \quad x \in [x_{j+1}, x_{j+2}]$$

is positive on the segment $[x_{j+1}, x^*]$ and is negative on $[x^{**}, x_{j+2}]$, where $x^* = x_{j+1} + \alpha''_{j+1} h_{j+1}$ and $x^{**} = x_{j+1} + \beta''_{j+1} h_{j+1}$. Since according to (1.6), $B_{j,2}(x_{j+l}) > 0$, $l = 1, 2$, we see that the spline $B_{j,2}(x)$ is positive, increases monotonously on $[x_{j+1}, x^*]$ and decreases monotonously on $[x^{**}, x_{j+2}]$. Since $B''_{j,2}(x) < 0$ for $x \in [x^*, x^{**}]$, we see that here the function $B_{j,2}(x)$ is convex upwards and is also positive.

Lemmas 1.2 and 10.1 are equivalent. Therefore if the inequalities (1.6) hold, then Corollary 11.1 holds and $(-1)^l B''_{j,3}(x_{j+1+l}) > 0$, $l = 0, 1, 2$. Taking into account formula (2.11) and properties of the functions $\Phi_j(x)$ and $\Psi_j(x)$, we find that the spline $B_{j,4}(x)$ is either monotonous or has a unique point of minimum on the subsegments $[x_j, x_{j+1}]$ and $[x_{j+3}, x_{j+4}]$.

Since

$$B''_{j,3}(x) = \Phi''_l(x) B''_{j,3}(x_l) + \Psi''_l(x) B''_{j,3}(x_{l+1}),$$

for $x \in [x_l, x_{l+1}]$, $l = j+1, j+2$, then using the above line of reasoning we find that there exist two points $x' \in [x_{j+1}, x_{j+2}]$ and $x'' \in [x_{j+2}, x_{j+3}]$ such that $B''_{j,3}(x') = B''_{j,3}(x'') = 0$ and $B''_{j,3}(x) > 0$ for $x \in [x_{j+1}, x') \cup (x'', x_{j+3}]$. According to (1.6), (2.2), and (2.5), we also have $B^{(r)}_{j,3}(x_{j+1}) > 0$ and $(-1)^r B^{(r)}_{j,3}(x_{j+3}) > 0$, $r = 0, 1$. Therefore the spline $B_{j,3}(x)$, is positive, increases monotonously on $[x_{j+1}, x']$ and decreases monotonously on $[x'', x_{j+3}]$. Besides, we have $B''_{j,3}(x) < 0$ on the subsegment $[x', x'']$ and thus here the spline $B_{j,3}(x)$ is positive, as it is a convex function upwaeds. Thus, the theorem is proved.

The B-splines with alternating signs possess many properties inherent in conventional B-splines. In particular, statements (2)–(4) of Theorem 5.1 hold for them. They have supports of minimum length and form a basis in the space of generalized splines S_k^G. The proofs of these aand other properties are similar to those cited for positive B-splines.

B-splines with alternating signs can be used in constructing orthogonal bases of B-splines with supports of minimum lengths like wavelets [3]. This allows one to develop methods of spline approximation like the method of Fourier series.

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