วิถีปิดของเวลาในฟิสิกส์ควอนตัมและทฤษฎีสนามควอนตัม

นายเสกสรร สุขะเสนา

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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CLOSED–TIME PATH IN QUANTUM PHYSICS AND QUANTUM FIELD THEORY

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CLOSED–TIME PATH IN QUANTUM PHYSICS AND QUANTUM FIELD THEORY

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หลังจากแสดงที่มาอย่างชัดแจ้งของสูตรสำหรับวิถีปิดของเวลา โดยการใช้หลักพลศาสตร์ควอนตัม โดยการคำนวณหาค่าคาดหมายและค่าความน่าจะเป็นโดยตรงจากหลักดังกล่าว ได้ศึกษากรณีเฉพาะซึ่งเกี่ยวข้องกับการเปลี่ยนแปลงอย่างทันใดสำหรับพลศาสตร์จากด้านบวกไปยังด้านลบของเวลา การวิเคราะห์ทำอย่างเป็นระบบเพื่อคำนวณหาค่าคาดหมายในฟิสิกส์ควอนตัมและในทฤษฎีสนามควอนตัม โดยวิธีปฏิบัติการโดยใช้อนุพันธ์เชิงฟังก์ชันซึ่งบุกเบิกโดยชวิงเกอร์ ในฟิสิกส์ควอนตัมวิธีปฏิบัติการโดยใช้อนุพันธ์เชิงฟังก์ชันได้ถูกพัฒนาผ่านหลักพลศาสตร์ควอนตัมและใช้กับระบบที่มีการเปลี่ยนแปลงคู่ควบกับสิ่งแวดล้อม ในขณะที่ระบบมีการเปลี่ยนแปลงตามสิ่งแวดล้อม การวิเคราะห์ต้องคิดเกี่ยวกับความน่าจะเป็นในการเปลี่ยนสถานะมากกว่าการวิเคราะห์หาค่าแอมพลิจูด นั่นคือการปฏิบัติการโดยอนุพันธ์เชิงฟังก์ชันค่อนข้างมีความเหมาะสมสำหรับการศึกษาในวิทยานิพนธ์นี้ เนื่องจากเกี่ยวข้องกับการคำนวณหาอนุพันธ์เชิงฟังก์ชันเทียบกับแหล่งกำเนิดดั่งเดิมต่างๆ ซึ่งใช้อธิบายฟังก์ชันของระบบเชิงกายภาพที่ได้แยกออกมาจากสิ่งแวดล้อมแล้วในทฤษฎีสนามควอนตัม เมื่อเจาะจงนำมาใช้กับระบบควอนตัมของสนามโน้มถ่วง หรือระบบสนามโน้มถ่วงเชิงควอนตัม ทำให้ได้สูตรใหม่ที่เป็นรูปแบบทั่วไปของตัวแผ่อนุภาคกราวิตอนซึ่งหาได้จากทฤษฎีสนามของลากรางเจียนมีทั้งหมดสามสิบเทอมหาได้โดยการใช้หลักความจริงสำหรับอนุพันธ์เชิงฟังก์ชันของทฤษฎีสนามควอนตัม และการก่อกำเนิดความไม่เป็นเชิงเส้นในอนุภาคกราวิตอนและการทำอันตรกิริยากับมวลสารอื่น แหล่งกำเนิดภายนอกนี้เมื่อมีการคู่ควบกับสนามโน้มถ่วงและกำหนดให้การแปรผันของแหล่งกำเนิดจึงไม่ควรเป็นปริมาณอนุรักษ์ ดังนั้นการแปรผันเทียบกับองค์ประกอบทั้งสิบของสนามจึงสามารถทำได้อย่างอิสระหรือไม่ขึ้นกับองค์ประกอบอื่นของแหล่งกำเนิด ผลลัพธ์ที่ได้คือตัวแผ่ของอนุภาคที่ได้มาจากการคำนวณเชิงฟังก์ชัน ซึ่งผลที่ได้นี้จะไม่สอดคล้องกับการคำนวณโดยใช้วิธีผลลัพธ์ของการคำนวณตามลำดับเวลาของสองสนามซึ่งผลจากอันแรกจะได้เทอมที่มีชื่อว่าชวิงเกอร์เทอมเพิ่มขึ้นมาด้วย การควอนไตซ์ได้ถูกคำนวณออกมาด้วยโดยใช้เกจที่สอดคล้องกันกับสถานะทางกายภาพที่มีอยู่จริง ซึ่งเป็นการมีอยู่สองสถานะโพลาไรเซชัน เพื่อทำให้แน่ใจว่าว่าเมื่อนำไปประยุกต์ใช้จะได้ผลในทางบวก หลังจากการสร้างเงื่อนไขบังคับค่าเชิงบวกและใส่สปิน ใน

ทฤษฎีของอนุภาคกราวิตอนซึ่งทำอันตรกิริยากับแหล่งกำเนิดอื่น โดยแหล่งกำเนิดนี้ไม่จำเป็นต้องมีเทนเซอร์ของพลังงานและโมเมนตัมภายนอกที่คงที่แล้วสมการของค่าคาดหมายของทฤษฎีได้ถูกพัฒนาขึ้นด้วยเงื่อนไขที่อุณหภูมิมีค่าจำกัด ในการคำนวณและโดยใช้อนุพันธ์เชิงฟังก์ชันของทฤษฎีสนามควอนตัม เทคนิกที่จำเป็นในการคำนวณที่จะต้องกำหนดเป็นอันดับแรกคือ เทนเซอร์ของพลังงานและโมเมนตัมภายนอกต้องไม่มีการอนุรักษ์ ค่าโคแวเรียนซ์ของเทนเซอร์ความโค้งรีมันน์ที่ถูกเหนี่ยวนำในสภาพสุญญากาศได้ถูกสร้างขึ้นด้วย แม้ว่าจะใช้สำหรับการควอนไตเซชันในเกจที่กำหนดขึ้นสำหรับกราวิตอนที่มีเพียงสองสถานะกายภาพดังกล่าวข้างต้น สำหรับการประยุกต์ได้เหนี่ยวนำทำให้ได้เมตริกซ์ที่ถูกต้องและนำผลที่ได้ถูกตรวจสอบกับสตริงแบบปิด ซึ่งเกิดขึ้นจากกิริยานัมบุ ซึ่งผลเฉลยของการสั่นของสตริงแบบวงปิดอาจเป็นรูปแบบทั่วไปที่ง่ายที่สุดของวัตถุที่เป็นจุด สุดท้ายได้มีการพิจารณาถึงว่าทำไมความสมมาตรของปริภูมิเวลา อาจขึ้นอยู่กับอุณหภูมิอาจจะเกิดจากการเปลี่ยนแปลงเพื่อความถูกต้องที่เกิดจากการแผ่รังสีและนัยสำคัญทางกายภาพได้ถูกเน้นย้ำในวิทยานิพนธ์นี้ด้วย

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา ____________________

ปีการศึกษา 2550 ลายมือชื่ออาจารย์ที่ปรึกษา ____________________

SECKSON SUKKHASENA : CLOSED–TIME PATH IN QUANTUM PHYSICS AND QUANTUM FIELD THEORY. THESIS ADVISOR : PROF. EDOUARD B. MANOUKIAN, Ph.D. 220 PP.

QUANTUM DYNAMICAL PRINCIPLE/ CLOSED–TIME PATH FORMALISM/ EXPECTATION VALUE FORMALISM/ FUNCTIONAL DIFFERENTIAL TREATMENT OF QUANTUM FIELD THEORY/ COUPLING OF QUANTUM SYSTEMS TO THE ENVIRONMENT/ EXTERNAL SOURCE TECHNIQUES/ QUANTUM GRAVITY/ GRAVITON PROPAGATOR/ NON–CONSERVED EXTERNAL ENERGY–MOMENTUM TENSOR/ EXPECTATION VALUE FORMALISM AT FINITE TEMPERATURE/ SCHWINGER TERMS.

After an explicit derivation is given for the quantum dynamical principle (QDP) for the closed-time path formulation, involved in computing directly expectation values and probabilities, paying special attention to the sudden change of the dynamics from the positive sense to the negative one, a systematic analysis is carried out of the expectation formalism in quantum physics and quantum field theory in the functional *differential* treatment pioneered by Schwinger. In quantum physics, a functional differential treatment is developed, via the QDP, for the *coupling* of quantum systems to the environment. As one is involved in taking the trace over the dynamical variables of environment, the analysis necessarily deals with transition probabilities rather than with amplitudes. It is shown that the functional differential treatment is quite suitable for such a study as it involves in carrying out functional differentiations, with respect to classical sources, on functionals describing decoupled physical systems from the environment. In quantum field theory, with particular emphasis on the quantum aspect of gravitation, that is, on quantum gravity, a general *novel* expression is derived for the graviton propagator from Lagrangian field theory, which includes 30 terms, by taking into account the necessary fact that in the functional differential approach of quantum filed theory, in order to generate non-linearities in gravitation and interactions with matter, the external source $T_{\mu\nu}$, coupled to the gravitational field, should *a priori* not be conserved $\partial^{\mu} T_{\mu\nu} \neq 0$, so variations with respect to its ten components may be varied *independently*. The resulting propagator is the one which arises in the functional differential approach and does *not* coincide with the corresponding time-ordered product of two fields and it includes so-called Schwinger terms. The quantization is carried out in a gauge corresponding to physical states with two polarization states to ensure positivity in quantum applications. After establishing the positivity constraint and spin content of the theory for gravitons interacting with a necessarily, and *a priori*, *non*-conserved external energy-momentum tensor, the expectation value formalism of the theory is developed at *finite* temperature in the functional *differential* treatment of quantum field theory. The necessity of having, *a priori*, a non-conserved external energy-momentum tensor is an obvious technical requirement. The covariance of the *induced* Riemann curvature tensor, in the initial vacuum, is established even for the quantization in a gauge corresponding only to two physical states of the gravitons as established above. As an application, the *induced* correction to the metric and the resulting underlying geometry is investigated due to a closed string arising from the Nambu action as a solution of a circularly oscillating string as, perhaps, the simplest generalization of a limiting point-like object. Finally it is discussed on why the geometry of spacetime may, in general, depend on temperature due to radiative corrections and its physical significance is emphasized.

School of Physics — Student's Signature ______________________

Academic Year 2007 — Advisor's Signature ______________________

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Seckson Sukkhasena

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CHAPTER I

INTRODUCTION

The quantum dynamical principle (QDP), pioneered by Julian Schwinger (Schwinger, 1951a, 1951b, 1953a, 1953b, 1954; see also Lam, 1965; Manoukian, 1985, 2006a; Manoukian et al., 2007), is indisputably recognized as a very powerful indispensable tool for all sorts of investigations in quantum field theory, in describing the underlying dynamics of the fundamental interactions of elementary particles, in carrying out explicit computations as well as testing these interactions in the high-energy regime via the application of renormalization theory (see Manoukian, 1983). The underlying formalism developed via the QDP is properly referred to as the functional *differential* treatment of quantum field theory, as all the propagators, Green functions or correlation functions of the basic quantum fields are obtained simply by functional *differentiations* of the so-called vacuum-to-vacuum transition amplitude as the generating functional of such correlation functions, with the functional differentiations carried out with respect to external sources coupled to the underlying fields in a theory. A rigorous derivation of the QDP is now available (Manoukian et al., 2007). The theory that emerges by unifying quantum physics and relativity is, of course, referred to as quantum field theory and is necessarily a many-particle theory in which particles are created or destroyed in fundamental interactions as purely relativistic phenomenae. It provides the non-phenomenological theory of present elementary particle physics. The QDP has also proved to be equally useful in the formulation of quantum physics for which endless computations and investigations have been carried out (see, Manoukian, 2006a, Ch. 11). The functional *differential formalism, via the application of the QDP*, has been extremely useful, in particular, in the quantization problem of the celebrated non-abelian gauge theories as was shown in : Manoukian (1986a), where the so-called Faddeev - Popov factor (Faddeev and Popov, 1967; Faddeev, 1969) emerges naturally from the QDP, without using symmetry arguments, without making appeal to path integrals, without using commutation relations and without even going to the complicated (Fradkin and Tyutin, 1970) structure of Hamiltonians.

It is worth recalling that the QDP involves functional differentiations with respect to classical functions, while in path integrals one is involved with continual integrals, defined as the infinite product of integrals over spacetime which are often ill defined and quite involved for theories with interactions. It is also far easier to differentiate than to integrate, especially in this context. All of the present theories describing the fundamental interactions in physics are gauge theories, that is theories in which the interactions are mediated by gauge fields. These include: QED (Dirac, 1927; Fermi, 1930; Schwinger, 1984, 1949a, 1949b, 1951c; Feynman, 1949a, 1949b, 1950; Tomonaga, 1948; Dyson, 1949a, 1949b), the Unified Weak-Electromagnetic Theory (Salam, 1948, 1980; Salam and Strathdee, 1972; Weinberg, 1967, 1974, 1980; Glashow, 1959, 1961, 1980); QCD and unified theories involving strong interactions (Bjorken, 1972; Pati and Salam, 1973; George and Glashow, 1974; Gross, 1999; Ioffe, 2001; Gross, Wilczek, Politzer, 2004) and theories attempting to include Einstein's theory of gravitation and modifications thereof (Zumino, 1975; Deser, 1986; 't Hooft, 1986). Unfortunately theories involving gravitation turn out be to non-renormalrizable and no quantum field theory is of practical value if it is not renormalizable (see Manoukian, 1983). This was best demonstrated by the years it took to develop the weak-electromagnetic theory due to its non-renormalizability if the underlying vector bosons mediating the interactions are initially massive. The Quantum Dynamical approach has also the advantages, mentioned above, in comparison to the canonical approach to gauge theories (Utiyama and Sakamoto, 1977; Mohapatra, 1971, 1972). The Quantum Dynamical Principle allows one to obtain directly the vacuum-to-vacuum transition amplitudes in field theory or the transformation functions in quantum physics.

In a classic paper, Schwinger (1961) has generalized the Quantum Dynamical Principle to a new method, referred to as the Closed-Time Path formalism, which bypasses the tedious steps in computing probabilities and expectation values and these physical quantities are directly obtained from this formalism. Schwinger's original formulation (1961) was developed in quantum physics with a very detailed application given to the quantum Brownian motion. It was successfully applied in studying the infrared behavior of QED (Mahanthappa, 1962) and in particle productions by strong external sources (Bakshi and Mahanthappa, 1963a, 1963b; Manoukian, 1988a) and in non-relativistic quantum scattering (Manoukian, 1988b). A derivation of the Closed-Time Formalism in the fully relativistic quantum field theory, including for non-abelian gauge theories, was given in (Manoukian, 1987) and was also extended to finite temperatures (Manoukian, 1991a; Xu, 1995). Multiparticle states and multiparticle collisions were particularly studied in Cooper (1995) and Manoukian (1988a, 1991b). Interesting studies were also carried out in Jordan (1986a, 1986b), Calzetta and Hu (1987), in non-equilibrium phenomena (Keldysh, 1965; Craig, 1968; Korenman, 1969; Hall, 1975; Schmutz, 1978; de Boer and van Weert, 1979) and specific applications were made as well to superconductivity (Volkov and Kogan, 1974), plasma physics (Bezzerides and DuBois, 1972) and to transport properties (Sandström, 1972) emphasizing path integral techniques. The purpose of this thesis is of a systematic analysis of the closed-time path formulation of quantum field theory emphasizing its role in quantum gravity where very little is known due to the complicated structure of the gravitational field as a second rank symmetric tensor field. For reasons that will become apparent the closed-time path formulation is also referred to as the *expectation value formalism*. As in carrying out expectation values one is involved, in the process of the investigation, in taking the product of amplitudes and their *complex conjugates* in a time evolution process. The complex conjugate of an amplitude, while the latter describes time evolution in the forward direction, the former describes time evolution in reversed direction thus ending at the initial time of the process when taking products of the amplitudes and their complex conjugates as described above. The present thesis also applies the formalism to quantum physics as well. In Chapter II, we provide a careful derivation of the QDP for closed-time path as an extension of our previous work (Manoukian et al., 2007) given for open time-path, that is dealing with amplitudes. To this end, we use our earlier approach of having two unitary time-dependent operators which in turn allow an otherwise non-trivial interchange of the orders of parameters variations of transformation functions, or general amplitudes in various descriptions, with specific time-dependent ones . This procedure answers the rather otherwise mysterious question as to why the variation of a transformation function, with respect to given parameters, is given by matrix elements, with respect to the given states defining the transformation function, of the integral of the variation of the Lagrangian with the states in question, which may depend on these parameters, kept non-varied. Particular attention is given to the sudden change of the dynamics from the positive (i.e., forward) to the negative (i.e., reversed) sense encountered in the closed-time path and its relation to a unitarity expansion by extending the paths of the two dynamical processes $t_1 \to t_2$ and $t_2 \to t_1$ to $t_1 \to t_2 + \varepsilon$, $t_2 + \varepsilon \to t_1$ for arbitrarily small ε and imposing *continuity* at time $t = t_2 + \varepsilon$ and then taking the limit $\varepsilon \to 0$. Most importantly, we provide *one single* time-dependent Hamiltonian for the entire closed-time process : $t_1 \to t_2 + \varepsilon$, $t_2 + \varepsilon \to t_1$ with continuity achieved at the return point and having, *a priori*, two different dynamics in the two branches. The key point is to introduce a smooth "step function" generalizing the well known step function. The rules for the application of the QDP for closed-time path are then developed. The closed-time path as a formalism for carrying out expectation values of, say, fields, is then applied to derive the theory for interacting spin 0 particles in the presence of external sources representing emitters and detectors of these particles in Sect. 2.3. The far more complicated problem for gravitons will be investigated in Chapter VI. As the thesis is full of applications of functional differentiation techniques, these methods are first applied to quantum scattering problems in Chapter VII, as a preparation for the more difficult problems encountered in the remaining part of the thesis. It is an adaptation of field theory methods in quantum physics. Chapter V, deals with the important role that the environment surrounding a quantum system has on the latter. Here the closed-time path formalism becomes quite evident as one is dealing with probabilities with time-evolution processes in the forward and reversed senses for amplitudes and their complex conjugates, respectively. Here we use functional calculus methods, via functional differentiations and application of the QDP to investigate the *coupling* of quantum mechanical systems to the environment, understood to be surrounding a physical system, as the former systems, in the real world, are never in isolation from the latter. The incorporation of the environment in quantum mechanical systems has led to much physical insights into such fundamental problems as quantum decoherence, Schrödinger's cat and in measurement theory, in general (see, Manoukian, 2006a, §8.7, §8.9, §12.7; Brune, 1992; Munroe, 1996; Walls, 1985; Zurek, 1991). We will see, that the functional differential approach is quite suitable for studying the coupling of quantum mechanical systems to the environment. It involves in carrying out functional differentiations, with respect to classical sources, of a functional describing "decoupled" systems from the environment. As one is involved in taking the *trace* over the dynamical variables of the environment in studying the response of physical systems to it, the analysis necessarily involves in dealing directly with transition probabilities rather than amplitudes. This is a basic departure from the far simpler case of studying quantum mechanical systems in isolation. In dealing with probabilities and in taking traces, it turns out that two different sets of classical sources, coupled to the dynamical variables of the theory, should, *a priori*, be introduced. The physically relevant probabilities are then recovered in the limit as the two sets of sources coincide and are eventually set equal to zero. The general expression for transition probabilities of quantum mechanical systems, coupled to the environment, is given in Eq. (4.2.8) involving functional differentiations with respect to these two sets of classical sources. Chapter V, deals with the derivation of a *novel* expression of the graviton propagator. In obtaining the expression for the propagator, we were rather surprised that it contains 30 terms and not just 3 terms as it has been used for years. The graviton propagator is the basic ingredient in quantum gravity computations. It provides and mediates the gravitational interaction between all particles and all matter to the leading order in the gravitational coupling constant. In the so-called functional differential treatment (Manoukian, 1986, 1985, 2006; Limboonsong and Manoukian, 2006; Schwinger, 1951) of quantum field theory, referred as the quantum dynamical principle approach, based on functional differentiation techniques with respect to external sources coupled to the underlying fields in a theory, functional derivatives are taken of the so-called vacuum-to-vacuum transition amplitude. The latter generates *n*-point functions by functional differentiations leading finally to transition amplitudes for various physical processes. For higher spin fields such as the electromagnetic vector potential A^{μ}, the gluon field A^{μ}_{a}, and certainly the gravitational field $h^{\mu\nu}$, the respective external sources J_{μ}, J^{a}_{μ}, $T_{\mu\nu}$, coupled to these fields, cannot *a priori* taken to be conserved so that their respective components may be varied *independently*. The consequences of relaxing the conservation of these external sources are highly non-trivial. For one thing the corresponding field propagators become modified. Also they have led to the rediscovery (Manoukian, 1986; Limboonsong and Manoukian, 2006) of Faddeev–Popov (Faddeev, 1967) factors in non-abelian gauge theories and the discovery (Limboonsong and Manoukian, 2006) of even more *generalized* such factors, directly from the functional *differential* treatment, via the application of the quantum dynamical principle, in the presence of external sources, without using commutation rules, and without even going to the well known complicated structures of the underlying Hamiltonians.

We provide, in passing, examples which show a contradictory result is obtained if a conservation law is imposed on $T_{\mu\nu}$. The Lagrangian density is the one of a massless spin-2 particle coupled to, *a priori*, not conserved $T_{\mu\nu}$. Only after functional differentiations of the vacuum-to-vacuum transition amplitude are taken, with respect to $T_{\mu\nu}$, that a conservation law on $T_{\mu\nu}$ may be imposed. Our novel expression for the graviton propagator is given in Eqs. (5.2.149), (5.2.151) and (5.2.152). It is different from the so-called time-ordered product due to the appearance of so-called Schwinger terms. Chapter V and Chapter VI are of central importance in the entire thesis. In Chapter VI, we first establish the positivity constraint of the vacuum-to-vacuum persistence probability as well as the non-trivial spin content of the graviton with only *two* polarization states which is *necessary* for all massless particles (see Manoukian, 2006a) even for *a priori* non-conserved energy-momentum tensor $T_{\mu\nu}$ In order to study gravitational effects such as the induced geometry due to external sources and even due to fluctuating quantum fields, the expectation value formalism turns out to be of practical value. In Sect. 6.4, we develop the expectation value formalism for gravitons interacting with an external energy-momentum tensor $T_{\mu\nu}$ at *finite* temperature with *a priori* not conserved $T_{\mu\nu}$, so that variations with respect to its ten components may be varied independently in order to generate expectation values. *After* all the relevant functional differentiations with respect to $T_{\mu\nu}$ are carried out, the conservation law on $T_{\mu\nu}$ may be then imposed. We establish the covariance of the *induced* Riemann curvature tensor, in the initial vacuum, due to the external source, in spite of the quantization carried out in a gauge which ensures only two polarization states for the graviton. As an application, we investigate the *induced* correction to the metric and the underlying geometry due a closed string arising from the Nambu action (e.g., Kibble, 1989; Sakellariadou, 1990; Goddard, 1995) as a solution of a circularly oscillating string (Manoukian, 1991b, 1992, 1995, 1998) as, perhaps, the simplest generalization of a limiting point-like object. Finally, it is discussed on why the geometry of spacetime may, in general, depend on temperature due to radiative corrections and its physical significance is emphasized. All of our results derived have been published. With the exception of the quantum physics treatments, we use units $\hbar = 1$, $c = 1$ in the quantum field theory analyses as is conventionally done. For the Minkowski metric, we choose $[\eta_{\mu\nu}] = \text{diag}[-1, 1, 1, 1]$. In the concluding chapter (Chapter VII), we summarize our main results and discuss further important points related to our analyses in the thesis.

CHAPTER II

THE CLOSED–TIME PATH

(EXPECTATION VALUE FORMALISM)

AND THE QUANTUM DYNAMICAL PRINCIPLE

2.1 Introduction

The power of the closed-time path formalism, also known as the expectation value formalism, pioneered by Julian Schwinger (1961) with extensions, additional technical details and applications over the years in the functional *differential* formalism (Manoukian, 1987, 1988a, 1988b, 1991, 2008) in abelian as well as non-abelian gauge theories, and in Chapters IV, V and VI of the present thesis, should be noted. In this formalism one is able to obtain *probabilities* and *expectation values* of fields and other objects *without* first deriving expressions for transition *amplitudes*. This is an important and a powerful shortcut to deal more directly with the physics of a quantum system or a quantum field theory involving many relativistic particles in the quantum domain. The closed-time path derivations rests on considering *a priori* two *different* dynamics, i.e., via two different Hamiltonians and external sources, one given from times $t_1 \to t_2$ and another one from $t_2 \to t_1$ for the time *reversed* process *ending up again* at time t_1 - hence the nomenclature a "closed-time path" for a complex process. *Only after all the relevant functional differentiations with respect to external sources are carried*, the corresponding two sets of Hamiltonians with the external sources may be set to be equal.

At time t_2, of the closed-time process a discontinuity arises in time as the Hamiltonians in the forward and reversed time paths are *different* at time t_2. This point is handled rigorously by extending the time paths from $t_1 \to t_2 + \varepsilon$ and $t_2 + \varepsilon \to t_1$, by taking the limit $\varepsilon \to +0$, with *continuity* ensured at the point $t = t_2 + \varepsilon$ for the forward and reversed time path. The present Chapter deals with the derivation of the quantum dynamical principle for Closed-Time Path (CTP).

The very elegant quantum (action) dynamical principle (QDP) (Schwinger, 1951a, 1951b, 1953a, 1953b, 1954, 1972; Lam, 1965; Manoukian, 1985, 1986a, 1987a; Manoukian and Siranan, 2005) is indisputably recognized as a very powerful tool for carrying out explicit computations in quantum field theory. The QDP has been used to quantize gauge theories (Manoukian, 1986a, 1987a) in constructing the vacuum-to-vacuum transition amplitude and the direct generation of the Faddeev-Popov (Faddeev and Popov, 1967) (FP) factor, encountered in non-abelian gauge theories, with no much effort and without making an appeal to path integrals or to commutation rules and without even going into the well known complicated structure of the Hamiltonian (Fradkin and Tyutin, 1970). In particular, it has been shown (Limboonsong and Manoukian, 2006) that the so-called FP factor needs to be modified in more general cases of gauge theories and that a gauge invariant theory does not necessarily imply the familiar FP factor for proper quantization as may be otherwise naïvely expected based on symmetry arguments. On the other hand the QDP corresponding to the so-called closed-time path (CTP) (expectation value formalism) (Schwinger, 1961; Manoukian, 1987a, 1988c, 1991b) has been also very useful as it provides a short cut for computing expectation values of observables directly without first working out specific amplitudes. The CTP had wide applications over the years see, e.g., Kao et al. (2002); Koide (2000); Cooper (1995); Chou et al. (1985); Zhou et al. (1980); deBoer et al. (1979); Garrido et al. (1977); Hall (1974) and Keldysh (1965). We recall that the QDP. provides the variations of amplitudes with respect to external parameters, such as coupling constants and external sources coupled to the quantum fields, which upon integrations of the amplitudes over these parameters yield the expression for the latter (see, e.g., Manoukian (1986a); Limboonsong and Manoukian (2006). The purpose of this work is to provide systematic explicit derivations of the QDP for CTP in quantum field theory. To this end, we introduce, in the process, two unitary time-dependent operators which in turn allow an otherwise non-trivial interchange of the orders of parameters variations of so-called transformation functions with specific time-dependent ones. This procedure answers the rather otherwise mysterious question as to why the variation of a transformation function, with respect to given parameters, is given by matrix elements, with respect to the given states defining the transformation function, of the integral of the variation of the Lagrangian with the states in question, which may depend on these parameters, kept non-varied ! The answer is based, mostly on

$$\langle at| = {}_1\langle at|\, V(t,\lambda), \tag{2.1.1}$$

i.e., the states $\langle at|$ of interest are related to the states ${}_1\langle at|$ which are independent of the parameter λ, and

$$V(t,\lambda) = U_1^\dagger(t) U(t,\lambda), \tag{2.1.2}$$

where $U(t,\lambda)$ is the time evolution operator depends on λ and t, while $U_1^\dagger(t)$ depends only on time t, and a key identity derived given by

$$\begin{aligned} &\mathrm{i}\hbar \frac{d}{d\tau} \left[V(t_2,\lambda) V^\dagger(\tau,\lambda) V(\tau,\lambda') V^\dagger(t_1,\lambda') \right] \\ &\quad = V(t_2,\lambda) \left[U^\dagger(\tau,\lambda)(H(\tau,\lambda') - H(\tau,\lambda)) U(\tau,\lambda') \right] V^\dagger(t_1,\lambda'), \end{aligned} \tag{2.1.3}$$

written in terms of the two unitary time-dependent operators mentioned above. On the other hand, for CTP, the analysis pays special attention to the so-called sudden change of the dynamics from the positive to the negative senses in defining the closed-path and its relation to a unitarity expansion. The derivations are extensions of the corresponding ones in quantum mechanics (Manoukian, 2006a; Manoukian et al., 2007) to the more complicated case of quantum field theory. The QDP for the CTP is derived in Sect. (2.2).

2.2 The Closed–Time Path Formalism

The Quantum Dynamical Principle (QDP) for Closed–Time Path (CTP) is involved in evaluating the expectation value of observables $\mathcal{O}(t)$, say in an initially prepared state $|bt_1\rangle$, with the dynamics evolving from time t_1 to t_2, with $t_1 < t < t_2$, by the QDP variational technique. Unfortunately, unitarity implies that $\langle bt_1 \,|\, bt_1\rangle$ is one and hence its variation is equal to zero and such a procedure seems *a priori* not useful. Schwinger (1961) has solved this problem by assuming a different dynamics for the time evolution from time t_1 to t_2 in the positive sense from the one in the opposite negative sense from t_2 back to t_1, associating different Hamiltonians,

$$H_+(t, \lambda_+),\; H_-(t, \lambda_-),$$

with the two segments $(t_1 \to t_2)$ and $(t_2 \to t_1)$, respectively, such that

$$H_+(t, \lambda_+) = H_-(t, \lambda_-), \tag{2.2.1}$$

for

$$\lambda_+ = \lambda_-, \tag{2.2.2}$$

where $\lambda_\pm$ are coupling constants or external (classical) sources. An immediate obstacle then seems to arise with the application of unitarity, in this case, as the Hamiltonian makes a *sudden* (instantaneous) change at $t = t_2$ upon changing the direction of time evolution.

We remedy the above problem, by introducing a unified Hamiltonian varying continuously during the entire time evolution process from $t_1 \to t_2$ and then from

$t_2 \to t_1$ by a limiting procedure. To this end, we introduce the Hamiltonian:

$$H(t, \lambda_+, \lambda_-) = \frac{1}{2}\Big[H_+(t, \lambda_+) + H_-(t, \lambda_-)\Big] + \frac{1}{2}\Big[H_+(t, \lambda_+) - H_-(t, \lambda_-)\Big]\vartheta_c(t, \varepsilon), \tag{2.2.3}$$

where $\vartheta_c(t, \varepsilon)$ is a continuous function of

$$t : t_1 \to t_2, \quad \text{and} \quad t_2 \to t_1, \tag{2.2.4}$$

extended to $t_2 + \varepsilon$, for arbitrarily small ε, and continuous at $t = t_2 + \varepsilon$. Such a function which *together* with its time derivative, are *continuous*, is rigorously given by

$$\vartheta_c(t, \varepsilon) = \begin{cases} 1 \quad , \quad t : t_1 \to t_2 \\ 1 - \frac{\varepsilon}{(t-t_2)} \mathrm{e} \exp[-\frac{\varepsilon}{(t-t_2)}] \quad , \quad t : t_2 \to t_2 + \varepsilon \\ -1 + \frac{\varepsilon}{(t-t_2)} \mathrm{e} \exp[-\frac{\varepsilon}{(t-t_2)}] \quad , \quad t : t_2 + \varepsilon \to t_2 \\ -1 \quad , \quad t : t_2 \to t_1. \end{cases} \tag{2.2.5}$$

The plot of the smooth "step function", $\vartheta_c(t, \varepsilon)$ for $\varepsilon = 0.1, 0.05, 0.01, 0.001$ as a function of t are shown in the following figures.

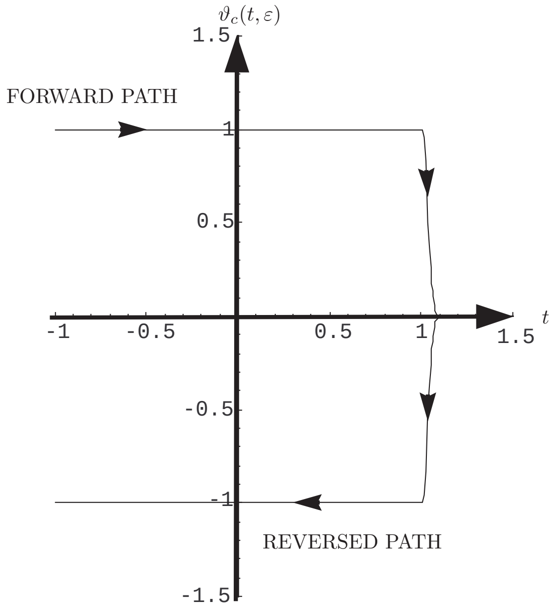

Figure 2.1 Plot of the smooth "step function", $\vartheta_c(t,\varepsilon)$ for $\varepsilon = 0.1$ as a function of t.

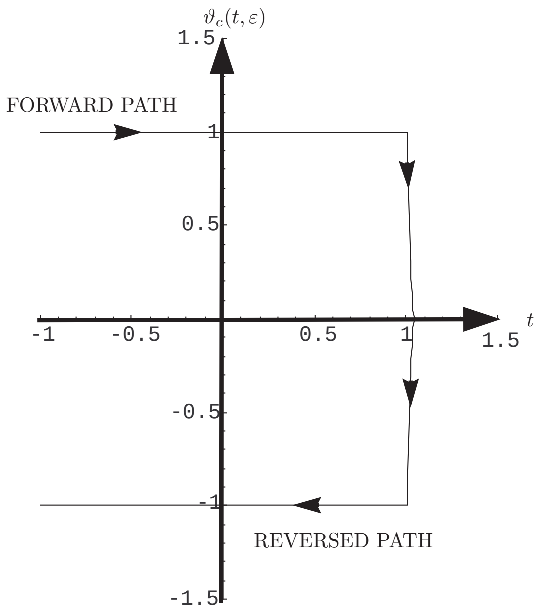

Figure 2.2 Plot of the smooth "step function", $\vartheta_c(t,\varepsilon)$ for $\varepsilon = 0.05$ as a function of t.

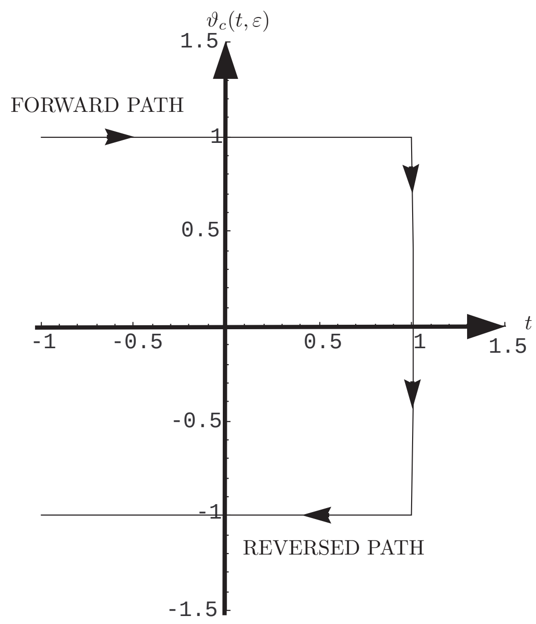

Figure 2.3 Plot of the smooth "step function", $\vartheta_c(t, \varepsilon)$ for $\varepsilon = 0.01$ as a function of t.

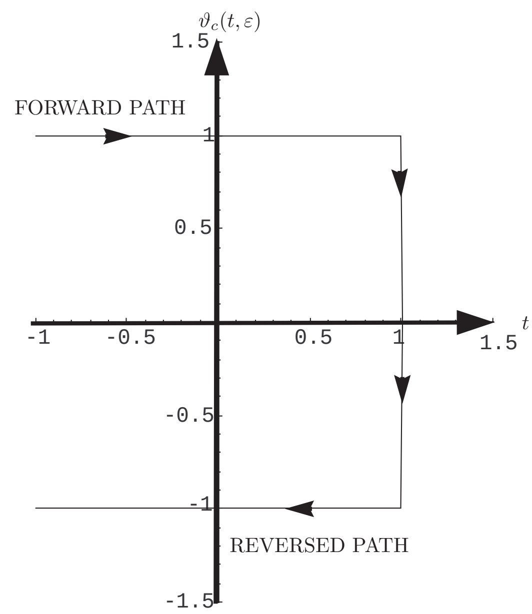

Figure 2.4 Plot of the smooth "step function", $\vartheta_c(t, \varepsilon)$ for $\varepsilon = 0.001$ as a function of t.

The continuity of the smooth "step function" is readily seen in Figs. (2.1) – (2.4) for various small ε. We note that

$$\vartheta_c(t,\varepsilon) \equiv \begin{cases} 1 & : \text{for condition} \quad : t = t_2 \\ 1 & : \text{for condition} \quad : t = t_2 \\ 1 - \mathrm{ee}^{-1} \quad = 0 & : \text{for condition} \quad : t = t_2 + \varepsilon \\ -1 + \mathrm{ee}^{-1} = 0 & : \text{for condition} \quad : t = t_2 + \varepsilon \\ -1 & : \text{for condition} \quad : t = t_2 \\ -1 & : \text{for condition} \quad : t = t_2, \end{cases} \tag{2.2.6}$$

establishing continuities at $t = t_2$, $t = t_2 + \varepsilon$ and back again at $t = t_2$.

Hence

$$H(t, \lambda_+, \lambda_-) = H_+(t, \lambda_+) \quad \text{for} \quad t_1 \leq t \leq t_2, \tag{2.2.7}$$

$$H(t, \lambda_+, \lambda_-) = \frac{1}{2}[H_+(t, \lambda_+) + H_-(t, \lambda_-)] \quad \text{for} \quad t_2 < t \leq t_2 + \varepsilon, \tag{2.2.8}$$

$$H(t, \lambda_+, \lambda_-) = \frac{1}{2}[H_+(t, \lambda_+) + H_-(t, \lambda_-)] \quad \text{for} \quad t : t_2 + \varepsilon \to t_2, \tag{2.2.9}$$

$$H(t, \lambda_+, \lambda_-) = H_-(t, \lambda_-) \quad \text{for} \quad t : t_2 \to t_1, \tag{2.2.10}$$

as a consequence of the continuity of $\vartheta_c(t,\varepsilon)$ for all t of interest, thus establishing the continuity of the Hamiltonian at time $t_2 + \frac{\varepsilon}{2}$, as well.

This allows us to carry out a unitarity expansion as follows:

$$\langle bt_1' \,|\, bt_1' \rangle^{(\varepsilon)} = \sum_{c,\,a} \left\langle bt_1 \middle| c\, t_2 + \frac{\varepsilon}{2} \right\rangle \left\langle c\, t_2 + \frac{\varepsilon}{2} \middle| a\, t_2 - \frac{\varepsilon}{2} \right\rangle \left\langle a\, t_2 - \frac{\varepsilon}{2} \middle| bt_1 \right\rangle, \tag{2.2.11}$$

where

$$\left| c\, t_2 \pm \frac{\varepsilon}{2} \right\rangle \left\langle c\, t_2 \pm \frac{\varepsilon}{2} \right| = U_\pm^\dagger(t_2 \pm \frac{\varepsilon}{2}, \lambda_\pm) \,|c\rangle\, \langle c|\, U_\pm(t_2 \pm \frac{\varepsilon}{2}, \lambda_\pm), \tag{2.2.12}$$

are given by genuine unitary transformations.

For $\varepsilon \rightarrow 0$,

$$\left\langle c\, t_2 + \frac{\varepsilon}{2} \middle| a\, t_2 - \frac{\varepsilon}{2} \right\rangle \rightarrow \delta(c, a),$$

and we obtain the completeness relation

$$\begin{aligned} \langle bt_1 \,|\, bt_1 \rangle &= \sum_a \langle bt_1 \,|\, at_2 \rangle_- \langle at_2 \,|\, bt_1 \rangle_+ \\ &= \sum_a \langle at_2 \,|\, bt_1 \rangle_-^* \langle at_2 \,|\, bt_1 \rangle_+ \,, \end{aligned} \tag{2.2.13}$$

with different dynamics described in the different segments from $t_1 \rightarrow t_2$ and $t_2 \rightarrow t_1$ with corresponding Hamiltonians $H_\pm(t, \lambda_\pm)$, respectively.

The quantum dynamical principle or the Schwinger dynamical (action) principle states (see Manoukian et al. 2007) that

$$\delta \langle at_2 \,|\, bt_1 \rangle = \frac{\mathrm{i}}{\hbar} \left\langle at_2 \middle| \int_{t_1}^{t_2} (\mathrm{d}x)\, \delta\mathcal{L}(x, \lambda) \middle| bt_1 \right\rangle, \tag{2.2.14}$$

written in terms of the Lagrangian, where

$$(\mathrm{d}x) = \mathrm{d}t\, \mathrm{d}^3\mathbf{x},$$

and the variation $\delta\mathcal{L}(x, \lambda)$, with respect to λ, is carried out with the independent fields χ *and* dependent ones η, and their derivatives $\partial_\mu \chi, \ \boldsymbol{\nabla}\eta$, all kept *fixed*.

From Eqs. (2.2.13), (2.2.14), we then obtain ($\varepsilon \rightarrow 0$)

$$\delta \langle bt_1 \,|\, bt_1 \rangle = \frac{\mathrm{i}}{\hbar} \left\langle bt_1 \middle| \int_{t_1}^{t_2} (\mathrm{d}x) [\delta\mathcal{L}_+(x, \lambda_+) - \delta\mathcal{L}_-(x, \lambda_-)] \middle| bt_1 \right\rangle. \tag{2.2.15}$$

This may be also obtained from Eq. (2.2.11) giving

$$\delta \langle bt_1 \,|\, bt_1 \rangle^{(\varepsilon)} = \frac{\mathrm{i}}{\hbar} \left\langle bt_1 \middle| \left[\int_{t_1}^{t_2 - \varepsilon/2} (\mathrm{d}x) \delta\mathcal{L}_+(x, \lambda_+) - \int_{t_1}^{t_2 + \varepsilon/2} (\mathrm{d}x) \delta\mathcal{L}_-(x, \lambda_-) \right] \middle| bt_1 \right\rangle$$

$$- \frac{\mathrm{i}}{\hbar} \left\langle bt_1 \left| \int_{t_2-\varepsilon/2}^{t_2+\varepsilon/2} \mathrm{d}t \, \delta I\!\!H(t, \lambda_+, \lambda_-) \right| bt_1 \right\rangle , \qquad (2.2.16)$$

which reduces to Eq. (2.2.15) for $\varepsilon \to 0$, where $I\!\!H(\tau, \lambda) = U^\dagger\ (\tau, \lambda)\ H(\chi, \pi, \tau, \lambda)$ $U(\tau, \lambda)$, is the Heisenberg representation of $H(\tau, \lambda)$ at time τ.

The equation for the CTP, becomes

$$\begin{aligned} \delta \left\langle bt_1 \right| I\!\!B(\tau, \lambda_+) \left| bt_1 \right\rangle = &- \frac{\mathrm{i}}{\hbar} \left\langle bt_1 \left| \int_{t_1}^{t_2} (\mathrm{d}x) \, \delta \mathcal{L}_-(x, \lambda_-) I\!\!B(\tau, \lambda_+) \right| bt_1 \right\rangle \\ &+ \frac{\mathrm{i}}{\hbar} \left\langle bt_1 \left| \int_{t_1}^{t_2} (\mathrm{d}x) (I\!\!B(\tau, \lambda_+) \, \delta \mathcal{L}_+(x, \lambda_+))_+ \right| bt_1 \right\rangle \\ &+ \left\langle bt_1 \right| \delta I\!\!B(\tau, \lambda_+) \left| bt_1 \right\rangle , \end{aligned} \qquad (2.2.17)$$

where $I\!\!B(\tau, \lambda)$ is an arbitrary function of $\chi(x)$ and $\pi(x)$, and in general, of λ and t, i.e.,

$$B(\chi(x), \pi(x), \lambda, t) \equiv I\!\!B(t, \lambda), \qquad (2.2.18)$$

of the variables indicated, with $\chi(x), \pi(x)$ in the Heisenberg representation defined as

$$\chi(x) = U^\dagger(t, \lambda) \chi(\mathbf{x}) U(t, \lambda), \qquad (2.2.19)$$

$$\pi(x) = U^\dagger(t, \lambda) \pi(\mathbf{x}) U(t, \lambda). \qquad (2.2.20)$$

We may write

$$I\!\!B(t, \lambda) = U^\dagger(t, \lambda) B(\chi(\mathbf{x}), \pi(\mathbf{x}), \lambda, t) U(t, \lambda), \qquad (2.2.21)$$

and

$$\delta \left\langle bt_1 \right| I\!\!B(\tau, \lambda_-) \left| bt_1 \right\rangle = \frac{\mathrm{i}}{\hbar} \left\langle bt_1 \left| \int_{t_1}^{t_2} (\mathrm{d}x) I\!\!B(\tau, \lambda_-) \, \delta \mathcal{L}_+(x, \lambda_+) \right| bt_1 \right\rangle$$

$$- \frac{\mathrm{i}}{\hbar} \left\langle bt_1 \left| \int_{t_1}^{t_2} (\mathrm{d}x)(I\!\!B(\tau, \lambda_-)\, \delta\mathcal{L}_-(x, \lambda_-))_- \right| bt_1 \right\rangle$$

$$+ \langle bt_1 | \delta I\!\!B(\tau, \lambda_-) | bt_1 \rangle \,, \tag{2.2.22}$$

where $(\ldots)_-$, in the second term on the right-hand side of Eq. (2.2.22) stands for the chronological time anti-ordering operation.

2.3 Application to Interacting Spin 0 Particles in the Presence of External Sources

In this section we derive the exact solution for the closed-time path $\langle 0_- \,|\, 0_- \rangle$ of spin–0 particles interacting with an external source K.

We consider the source term as real (K is reals), the $vacuum$ to $vacuum$ ($vac-vac$) transition amplitude in the positive sense of time is

$$\langle 0_+ \,|\, 0_- \rangle^K = \exp \left[\frac{\mathrm{i}}{2} \int (dx)(dx') K(x) \Delta_+(x - x') K(x') \right] , \tag{2.3.1}$$

where

$$\Delta_+(x - x') = \int \frac{(dp)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2 + m^2 - \mathrm{i}\epsilon}, \epsilon \to +0. \tag{2.3.2}$$

Then we can write, from $p^2 = \mathbf{p}^2 - p^{0^2}$,

$$\Delta_+(x - x') = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{x}')} \int \frac{\mathrm{d}p^0}{(2\pi)} \frac{\mathrm{e}^{\mathrm{i}p^0(x^0 - x^{0'})}}{p^2 + m^2 - \mathrm{i}\epsilon} \tag{2.3.3}$$

and from $p^2 = \mathbf{p}^2 - p^{0^2}$, we have

$$p^2 + m^2 - \mathrm{i}\epsilon = \mathbf{p}^2 - p^{0^2} + m^2 - \mathrm{i}\epsilon$$

$$= - \left[p^0 - \left((\mathbf{p}^2 + m^2)^{1/2} - \mathrm{i}\epsilon \right) \right] \left[p^0 + \left((\mathbf{p}^2 + m^2)^{1/2} - \mathrm{i}\epsilon \right) \right] . \tag{2.3.4}$$

Insert the above equation into $\Delta_+(x-x')$, and then by complex integration, and by the application of residue theorem we obtain

$$\Delta_+(x-x') = -\int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}[(\frac{-2\pi i}{2\pi})\frac{\Theta(x^0-x^{0'})e^{-ip^0(x^0-x^{0'})}}{2p^0} + (\frac{2\pi i}{2\pi})\frac{\Theta(x^{0'}-x^0)e^{ip^0(x^0-x^{0'})}}{2p^0}] \tag{2.3.5}$$

Note that $\epsilon\sqrt{\mathbf{p}^2+m^2}$ for $\epsilon \to +0$ is the same as ϵ since $\sqrt{\mathbf{p}^2+m^2} > 0$.

For $x^0 > x^{0'}$, we have

$$\Delta_+(x-x') = i\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2p^0} e^{ip(x-x')}. \tag{2.3.6}$$

Let

$$d\omega_{\mathbf{p}} \equiv \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2p^0}, \tag{2.3.7}$$

$$p^0 = +\sqrt{\mathbf{p}^2+m^2}, \tag{2.3.8}$$

then

$$\Delta_+(x-x') = i\int d\omega_{\mathbf{p}} e^{ip(x-x')}. \tag{2.3.9}$$

The function $\Delta_+(x-x')$ is the propagator of spin-0 that propagate in the positive sense of time.

The Fourier transform of the source, $K(x)$, is defined by

$$K(p) = \int (\mathrm{d}x) \mathrm{e}^{-\mathrm{i}px} K(x). \tag{2.3.10}$$

Let

$$K = K_1 + K_2, \tag{2.3.11}$$

where K_1 is switched on *after* K_2 is switched off. By inserting this expression of source K into Eq. (2.3.1) and use Eq. (2.3.6), we obtain another form of the vacuum-to-vacuum transition amplitude,

$$\langle 0_+ \,|\, 0_- \rangle^K = \langle 0_+ \,|\, 0_- \rangle^{K_1} \exp\left[\int \mathrm{d}\omega_{\mathbf{p}} \mathrm{i} K_1^*(\mathbf{p}) \mathrm{i} K_2(\mathbf{p}) \right] \langle 0_+ \,|\, 0_- \rangle^{K_2}. \tag{2.3.12}$$

By using of the *discretization notation over momenta*, we write for book-keeping purposes (see, Schwinger, 1970)

$$K_p \equiv \sqrt{\mathrm{d}\omega_{\mathbf{p}}} K(p), \tag{2.3.13}$$

to rewrite the integrals in the square bracket in Eq. (2.3.5) as

$$\int \mathrm{d}\omega_{\mathbf{p}} \mathrm{i} K_1^*(p) \mathrm{i} K_2(p) \to \sum_p \mathrm{i} K_{1p}^* \mathrm{i} K_{2p}, \tag{2.3.14}$$

and

$$\begin{aligned} \exp\left[\sum_p \mathrm{i} K_{1p}^* \mathrm{i} K_{2p} \right] &= \prod_p \exp \mathrm{i} K_{1p}^* \mathrm{i} K_{2p} \\ &= \prod_p \sum_{n_p} \frac{(\mathrm{i} K_{1p}^* \mathrm{i} K_{2p})^{n_p}}{n_p!} \end{aligned}$$

$$= \prod_p \sum_{n_p} \frac{(\mathrm{i}K^*_{1p})^{n_p}}{\sqrt{n_p!}} \frac{(\mathrm{i}K_{2p})^{n_p}}{\sqrt{n_p!}}. \tag{2.3.15}$$

Let $n_{\mathbf{q}_1},\ n_{\mathbf{q}_2}, \ldots$ denote number of particles with momenta $\mathbf{q}_1, \mathbf{q}_2, \ldots$ such that

$$n_{\mathbf{q}_1} + n_{\mathbf{q}_2} + \ldots = n, \tag{2.3.16}$$

then

$$\prod_p \sum_{n_p} \frac{(\mathrm{i}K^*_{1\mathbf{p}})^{n_\mathbf{p}}}{\sqrt{n_{\mathbf{p}!}}} \frac{(\mathrm{i}K_{2\mathbf{p}})^{n_\mathbf{p}}}{\sqrt{n_{\mathbf{p}!}}} = \sum_n \sum_{n_{\mathbf{q}1}+n_{\mathbf{q}2}+\ldots=n} \frac{(\mathrm{i}K^*_{1\mathbf{q}1})^{n_{\mathbf{q}1}}}{\sqrt{n_{\mathbf{q}1}!}} \frac{(\mathrm{i}K^*_{1\mathbf{q}2})^{n_{\mathbf{q}2}}}{\sqrt{n_{\mathbf{q}2}!}} \ldots \frac{(\mathrm{i}K_{2\mathbf{q}1})^{n_{\mathbf{q}1}}}{\sqrt{n_{\mathbf{q}1}!}} \frac{(\mathrm{i}K_{2\mathbf{q}2})^{n_{\mathbf{q}2}}}{\sqrt{n_{\mathbf{q}2}!}} \ldots. \tag{2.3.17}$$

Then Eq. (2.3.12) can be written as

$$\langle 0_+ \,|\, 0_- \rangle^K = \sum_n \sum_{n_{\mathbf{q}1}+n_{\mathbf{q}2}+\ldots=n} \left\{ \left[\langle 0_+ \,|\, 0_- \rangle^{K_1} \frac{(\mathrm{i}K^*_{1\mathbf{q}1})^{n_{\mathbf{q}1}}}{\sqrt{n_{\mathbf{q}1}!}} \frac{(\mathrm{i}K^*_{1\mathbf{q}2})^{n_{\mathbf{q}2}}}{\sqrt{n_{\mathbf{q}2}!}} \ldots \right] \left[\frac{(\mathrm{i}K_{2\mathbf{q}1})^{n_{\mathbf{q}1}}}{\sqrt{n_{\mathbf{q}1}!}} \frac{(\mathrm{i}K_{2\mathbf{q}2})^{n_{\mathbf{q}2}}}{\sqrt{n_{\mathbf{q}2}!}} \ldots \langle 0_+ \,|\, 0_- \rangle^{K_2} \right] \right\}. \tag{2.3.18}$$

We compare this with the unitarity expansion, given by

$$\langle 0_+ \,|\, 0_- \rangle^K = \sum_n \sum_{n_{q1}+n_{q2}+\ldots=n} \langle 0_+ \,|\, n; n_{q1}, n_{q2}, \ldots \rangle^{K_1} \langle n; n_{q1}, n_{q2}, \ldots \,|\, 0_- \rangle^{K_2}. \tag{2.3.19}$$

Thus in particular for any sources K, we have

$$\langle n; n_{q1}, n_{q2}, \ldots \,|\, 0_- \rangle^K = \frac{(\mathrm{i}K_{q1})^{n_{q1}}}{\sqrt{n_{q1}!}} \frac{(\mathrm{i}K_{q2})^{n_{q2}}}{\sqrt{n_{q2}!}} \ldots \langle 0_+ \,|\, 0_- \rangle^K. \tag{2.3.20}$$

The next step is to find out for the closed–time path expression for $\langle 0_- \,|\, 0_- \rangle$ in the presence of external sources. As in Eq. (2.3.19), we write $\langle 0_- \,|\, 0_- \rangle$ as a unitarity expansion

$$\langle 0_- \,|\, 0_- \rangle = \sum_n \sum_{n_{q1}+n_{q2}+\ldots=n} \langle 0_- \,|\, n; n_{q1}, n_{q2}, \ldots \rangle^{K_-} \langle n; n_{q1}, n_{q2}, \ldots \,|\, 0_- \rangle^{K_+}. \quad (2.3.21)$$

Using the identity of the complex conjugate, we also have

$$\langle 0_- \,|\, n; n_{q1}, n_{q2}, \ldots \rangle^{K_-} = \left(\langle n; n_{q1}, n_{q2}, \ldots \,|\, 0_- \rangle^{K_-} \right)^*. \quad (2.3.22)$$

Thus $\langle 0_- \,|\, 0_- \rangle$, according to Eqs. (2.3.22) and (2.3.20), is given by

$$\langle 0_- \,|\, 0_- \rangle = \sum_n \sum_{n_{q1}+n_{q2}+\ldots=n} \left(\langle 0_+ \,|\, 0_- \rangle^{K_-} \right)^* \left[\frac{K^*_{-q1} K_{+q1}}{n_{q1}!} \right]^{n_{q1}}$$
$$\left[\frac{K^*_{-q2} K_{+q2}}{n_{q2}!} \right]^{n_{q2}} \ldots \langle 0_+ \,|\, 0_- \rangle^{K_+}. \quad (2.3.23)$$

According to the mutinomial expression

$$\sum_{x_1, x_2, \ldots} \sum_{(x_1+x_2+\ldots=n)} \frac{(a_1)^{x_1}}{x_1!} \cdot \frac{(a_2)^{x_2}}{x_2!} \cdots = \frac{(a_1 + a_2 + a_3 + \cdots)^n}{n!}, \quad (2.3.24)$$

as applied to the present case

$$\sum_{n_{q1}+n_{q2}+\ldots=n} \left(\frac{K^*_{-q1} K_{+q1}}{n_{q1}!} \right)^{n_{q1}} \left(\frac{K^*_{-q2} K_{+q2}}{n_{q2}!} \right)^{n_{q2}} \ldots = \frac{\left[\sum_q \left(K^*_{-q} K_{+q} \right) \right]^n}{n!}, \quad (2.3.25)$$

leads to

$$\langle 0_- \,|\, 0_- \rangle = \left(\langle 0_+ \,|\, 0_- \rangle^K_- \right)^* \exp\left(\sum_q K^*_{-q} K_{+q} \right) \langle 0_+ \,|\, 0_- \rangle^K_+. \quad (2.3.26)$$

Consider the sum in the exponent of the middle term in Eq. (2.3.26). It can be rewritten as an integral giving

$$\sum_q K^*_{-q}K_{+q} \to \int \mathrm{d}\omega_q K^*_-(q)K_+(q)$$

$$= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q^0} K^*_-(q)K_+(q) \quad , q^0 = +\sqrt{\mathbf{q}^2 + m^2}$$

$$= \int (\mathrm{d}x)(\mathrm{d}x')K_-(x)\Delta^{(+)}(x-x')K_+(x'), \tag{2.3.27}$$

where

$$\Delta^{(+)}(x-x') = \int \mathrm{d}\omega_p \mathrm{e}^{\mathrm{i}p(x-x')} \quad , \quad p^0 = +\sqrt{\mathbf{p}^2 + m^2}, \tag{2.3.28}$$

and

$$(\Delta_+(x-x'))^* = \int \frac{(\mathrm{d}p)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2 + m^2 + \mathrm{i}\epsilon} \quad , \quad p^0 = +\sqrt{\mathbf{p}^2 + m^2}$$

$$\equiv \Delta_-(x-x'). \tag{2.3.29}$$

We see that the complex conjugate of $\Delta_+(x-x')$ leads to $\Delta_-(x-x')$ for the time reversed process.

Then the final expression of $\langle 0_- \,|\, 0_- \rangle$ of spin–0 particles interacting with external sources may be written as

$$\langle 0_- \,|\, 0_- \rangle = \exp -\frac{\mathrm{i}}{2} K_-\Delta_-K_- \exp \frac{\mathrm{i}}{2} K_+\Delta_+K_+ \exp K_-\Delta^{(+)}K_+, \tag{2.3.30}$$

where

$$K_+\Delta_+K_+ = \int (\mathrm{d}x)(\mathrm{d}x')K_+(x)\Delta_+(x-x')K_+(x'), \tag{2.3.31}$$

$$\Delta_+(x-x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2 + m^2 - \mathrm{i}\epsilon}, \qquad \epsilon \to +0, \tag{2.3.32}$$

$$\Delta_-(x-x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2+m^2+\mathrm{i}\epsilon}, \qquad \epsilon \to +0, \tag{2.3.33}$$

$$\Delta^{(+)}(x-x') = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2p^0}\mathrm{e}^{\mathrm{i}p(x-x')}, \quad p^0 = +\sqrt{\mathbf{p}^2+m^2}, \tag{2.3.34}$$

which $\Delta_+(x-x')$ is the propagator and $\Delta_-(x-x')$ is its complex conjugate corresponding to the reversed time paths for $\Delta_+(x-x')$.

For example for a theory with interaction Lagrangian density $\mathcal{L}_I = \lambda\phi^4$, the vacuum-to-vacuum amplitude $\langle 0_- \,|\, 0_- \rangle$, for the initial vacuum, for the closed-time path is given by

$$\langle 0_- \,|\, 0_- \rangle = \exp \mathrm{i}\lambda \int (\mathrm{d}x) \bigg[\bigg(-\mathrm{i}\frac{\delta}{\delta K_+(x)} \bigg)^4 - \bigg(\mathrm{i}\frac{\delta}{\delta K_-(x)} \bigg)^4 \bigg] \langle 0_- \,|\, 0_- \rangle_{\lambda=0}\,, \tag{2.3.35}$$

where $\langle 0_- \,|\, 0_- \rangle_{\lambda=0}$ is defined in Eq. (2.3.30) for spin 0 bosons interacting with external sources only, that is, for $\lambda = 0$.

With a system in the initial vacuum state $|0_-\rangle$, the expectation value of the induced field excitation at a given time $x^0 = t$, is then given by

$$\langle 0_- \,|\phi(x)|0_- \rangle \;=\; -\mathrm{i}\frac{\delta}{\delta K_+(x)}\,\langle 0_- \,|\, 0_- \rangle \bigg|_{K_+=K_-\equiv K} \tag{2.3.36}$$

Detail applications of an induced field in the initial vacuum state, as defined in Eq. (2.3.36), will be given to the far more complex situation involving the gravitational field in Chapter VI.

CHAPTER III

APPLICATION OF THE FUNCTIONAL DIFFERENTIAL TREATMENT TO QUANTUM SCATTERING VIA THE DYNAMICAL PRINCIPLE

3.1 Introduction

The underlying technical language involved in the present thesis in dealing with quantum physics and quantum field theory is the so-called functional differentiation treatment. That is, it is involved by differentiations of functions of functions with respect to the latter functions. This chapter deals with a rigorous treatment of the functional *differential* treatment to scattering in quantum physics, and its extension to the expectation value formalism, i.e., to the closed-time one, in quantum physics is carried out in the next chapter. The power and elegance of the functional differential treatment as simply taking derivatives to obtain physical results will be evident.

The purpose of this chapter is to use Schwinger's (Schwinger, 1951, 1953, 1960, 1962; Manoukian, 1985) most elegant quantum dynamical principle to provide a careful functional treatment of quantum scattering. We derive rigorously an expression for the scattering amplitude involving a functional differentiation operation applied to a functional, depending on the potential, written in closed form. The main result of this paper is given in Eq.(3.2.45). In particular, it provides a systematic starting point for studies of deviations from so-called straight-line "trajectories" of particles, with small deviation angles, by mere functional differentiations. An investigation of a time limit of a function related to this expression shows that the latter may be also used to obtain the asymptotic "free" modified Green functions for theories with long range potentials such as for the Coulomb potential with the latter defining the transitional potential between short and long range potentials. Functional methods have been also introduced earlier in the literature (Brenner and Galimzyanov, 1982; Chuluunbaatar et al., 2001; Cambell, 1975; Gelman and Spruch, 1969; Gerry, 1980; Pazma, 1979; Singh, 1975; Zubarev, 1977, 1978; Sukumar, 1984) in quantum scattering dealing with path integrals or variational optimization methods which, however, are not in the spirit of the present paper based on the dynamical principle. The present study is an adaptation of quantum field theory methods (Manoukian, 1988a) to quantum potential scattering.

3.2 Functional Treatment of Scattering

Given a Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}), \tag{3.2.1}$$

for a particle of mass *m* interacting with a potential $V(\mathbf{x})$, we introduce a Hamiltonian $H'(\lambda, \tau)$ involving external sources $\mathbf{F}(\tau)$, $\mathbf{S}(\tau)$ coupled linearly to $\mathbf{x}$ and $\mathbf{p}$ as follows:

$$H'(\lambda, \tau) = \frac{\mathbf{p}^2}{2m} + \lambda V(\mathbf{x}) - \mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau), \tag{3.2.2}$$

where λ is an arbitrary parameter which will be eventually set equal to one. Schwinger's (Schwinger, 1951a, 1953a, 1960a, 1960b, 1962; Manoukian, 1985) **dynamical principle states, that the variation of the transformation function**

$$\langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle \,, \tag{3.2.3}$$

with respect to the parameter λ, for the theory governed by the Hamiltonian $H'(\lambda, \tau)$ **is given by**

$$\delta \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle = \left(-\frac{\mathrm{i}}{\hbar} \right) \int_{t'}^{t} \mathrm{d}\tau \, \delta \left(\lambda V \left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)} \right) \right) \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle \,. \tag{3.2.4}$$

Here

$$V(-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau)), \tag{3.2.5}$$

denotes $V(\mathbf{x})$ with $\mathbf{x}$ in it replaced by

$$-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau). \tag{3.2.6}$$

Eq. (3.2.4) may be readily integrated for

$$\lambda \;=\; 1, \qquad \mathbf{F}(\tau), \mathbf{S}(\tau) \quad \text{set equal to zero,} \tag{3.2.7}$$

that is for the theory governed by the Hamiltonian H in Eq. (3.2.1), to obtain

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\, V\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}\right)\right] \langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle^{(0)}\bigg|_{\mathbf{F}=0,\mathbf{S}=0}. \tag{3.2.8}$$

The transformation function

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle^{(0)}, \tag{3.2.9}$$

corresponds a theory developing in time via the Hamiltonian

$$H'(0,\tau) = \frac{\mathbf{p}^2}{2m} - \mathbf{x}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau), \tag{3.2.10}$$

to which we now pay special attention.

With $\mathbf{p}$ replaced by $\mathrm{i}\hbar\delta/\delta\mathbf{S}(\tau)$, the dynamical principle, exactly as in Eq. (3.2.8), gives

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle^{(0)} = \exp\left[-\frac{\mathrm{i}}{2m\hbar}\int_{t'}^{t} \mathrm{d}\tau\left(\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)^2\right] \langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_0\,, \tag{3.2.11}$$

where the transformation function $\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_0$ is governed by the "Hamiltonian"

$$\hat{H}(\tau) = -\mathbf{x}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau), \tag{3.2.12}$$

involving no kinetic energy term.

The Heisenberg equations corresponding to $\hat{H}(\tau)$ give the equations

$$\mathbf{x}(\tau) = \mathbf{x}(t) - \int_{t'}^{t} \mathrm{d}\tau'\,\Theta(\tau'-\tau)\mathbf{S}(\tau'), \tag{3.2.13}$$

$$\mathbf{p}(\tau) = \mathbf{p}(t') + \int_{t'}^{t} \mathrm{d}\tau'\,\Theta(\tau-\tau')\mathbf{F}(\tau'), \tag{3.2.14}$$

where $\Theta(\tau)$ is the step function

$$\Theta(\tau) = \begin{cases} 1, & \text{for} \quad \tau > 0 \\ 0, & \text{for} \quad \tau < 0 \end{cases}, \tag{3.2.15}$$

Using the relations

$${}_0\langle \mathbf{x}t|\,\mathbf{x}(t) = \mathbf{x}\,\langle \mathbf{x}t|\,, \tag{3.2.16}$$

$$\mathbf{p}(t')\,|\mathbf{p}t'\rangle_0 = |\mathbf{p}t'\rangle\,\mathbf{p}, \tag{3.2.17}$$

and the dynamical principle, we obtain by taking the matrix elements of $\mathbf{x}(\tau)$, $\mathbf{p}(\tau)$ in Eqs. (3.2.13), (3.2.14) between the states

$${}_0\langle \mathbf{x}t|\,, \quad |\mathbf{p}t'\rangle_0\,, \tag{3.2.18}$$

the functional differential equations

$$-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}\,\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_0 = \left[\mathbf{x} - \int_{t'}^{t} \mathrm{d}\tau'\,,\Theta(\tau'-\tau)\mathbf{S}(\tau')\right]\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_0\,, \tag{3.2.19}$$

$$\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\,\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_0 = \left[\mathbf{p} + \int_{t'}^{t} \mathrm{d}\tau'\,\Theta(\tau-\tau')\mathbf{F}(\tau')\right]\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_0\,. \tag{3.2.20}$$

These equations may be integrated to yield

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_0 = \exp\left[\frac{\mathrm{i}}{\hbar}\mathbf{x}\cdot\left(\mathbf{p}+\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{F}(\tau)\right)\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\mathbf{p}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{S}(\tau)\right]$$
$$\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\mathbf{F}(\tau')\Theta(\tau-\tau')\right], \quad (3.2.21)$$

satisfying the familiar boundary condition

$$\exp(\mathrm{i}\mathbf{x}\cdot\mathbf{p}/\hbar) \quad \text{for} \quad \mathbf{F},\,\mathbf{S} \quad \text{set equal to zero.} \quad (3.2.22)$$

Since we are interested in Eq. (3.2.8), in particular, for the case when $\mathbf{S}$ is set equal to zero, the functional differentiation in Eq. (3.2.11) may easily carried out giving

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle^{(0)}\Big|_{\mathbf{S}=\mathbf{0}} = \exp\left[\frac{\mathrm{i}}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}(t-t')\right)\right]$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{F}(\tau)\cdot\left(\mathbf{x}-\frac{\mathbf{p}}{m}(t-\tau)\right)\right]$$
$$\times\exp\left[-\frac{\mathrm{i}}{2m\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{F}(\tau)\cdot\mathbf{F}(\tau')(t-\tau_>)\right], \quad (3.2.23)$$

where $\tau_>$ is the largest of τ and τ' :

$$\tau = \max(\tau,\tau'). \quad (3.2.24)$$

We recall from Eq. (3.2.8) that we eventually set $\mathbf{F}(\tau)$ equal to zero. This allows us to interchange the exponential factor in Eq. (3.2.8) involving the $V(-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau))$ term and the last two exponential factors in Eq. (3.2.23). This gives for $\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle$ in Eq. (3.2.8) the expression

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle = \exp\left[\frac{\mathrm{i}}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}(t-t')\right)\right]$$

$$\times \exp\left[\frac{\mathrm{i}\hbar}{2m}\int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau' \,[t-\tau_>]\,\frac{\delta}{\delta\mathbf{F}(\tau)}\cdot\frac{\delta}{\delta\mathbf{F}(\tau')}\right]$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\, V\left(\mathbf{x}-\frac{\mathbf{p}}{m}(t-\tau)+\mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=\mathbf{0}}. \tag{3.2.25}$$

Since we finally set $\mathbf{F} = \mathbf{0}$ in Eq. (3.2.25), the theory becomes translationally invariant in time and $\langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle$ is a function of

$$t - t' \equiv T. \tag{3.2.26}$$

For $t > t'$, we have the definition of the Green function

$$\langle \mathbf{x}t \,|\, \mathbf{x}'t' \rangle = G_+(\mathbf{x}t, \mathbf{x}'t'), \tag{3.2.27}$$

with

$$G_+(\mathbf{x}t, \mathbf{x}'t') = 0 \quad \text{for} \quad t < t', \tag{3.2.28}$$

and

$$\begin{aligned} \langle \mathbf{x}t \,|\, \mathbf{p}t' \rangle &= G_+(\mathbf{x}t, \mathbf{p}t') \\ &= \int \mathrm{d}^3\mathbf{x}'\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}'/\hbar} G_+(\mathbf{x}t, \mathbf{x}'t'). \end{aligned} \tag{3.2.29}$$

We may now introduce the Fourier transform defined by

$$\begin{aligned} G_+(\mathbf{p}, \mathbf{p}'; p^0) = -\frac{\mathrm{i}}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}(p^0+\mathrm{i}\epsilon)T/\hbar} \\ \times \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\, \langle \mathbf{x}T \,|\, \mathbf{p}'0 \rangle\,, \end{aligned} \tag{3.2.30}$$

for $\epsilon \to +0$, where $\langle \mathbf{x}T \,|\, \mathbf{p}0 \rangle$ is given in Eq. (3.2.25) with $t - t' \equiv T$.

From Eqs. (3.2.30), (3.2.25), we may rewrite $G_+(\mathbf{p},\mathbf{p}';p^0)$ as

$$G_+(\mathbf{p},\mathbf{p}';p^0) = -\frac{\mathrm{i}}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\alpha\,\mathrm{e}^{\mathrm{i}[p^0-E(\mathbf{p}')+\mathrm{i}\epsilon]\alpha/\hbar}$$

$$\times\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha), \qquad (3.2.31)$$

where

$$E(\mathbf{p}) = \mathbf{p}^2/2m, \qquad (3.2.32)$$

$$K(\mathbf{x},\mathbf{p}';\alpha) = \exp\left[\frac{\mathrm{i}\hbar}{2m}\int_{t'}^t \mathrm{d}\tau\int_{t'}^t \mathrm{d}\tau'[t-\tau_>]\frac{\delta}{\delta\mathbf{F}(\tau)}\cdot\frac{\delta}{\delta\mathbf{F}(\tau')}\right]$$

$$\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^t \mathrm{d}\tau\, V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}(t-\tau)+\mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=\mathbf{0}}, \qquad (3.2.33)$$

with

$$t - t' \equiv \alpha, \qquad (3.2.34)$$

playing the role of time $-$ a notation used for it quite often in field theory.

In the α-integrand in the exponential in Eq. (3.2.31), we recognize

$$[p^0 - E(\mathbf{p}) + \mathrm{i}\epsilon], \qquad (3.2.35)$$

as the inverse of the free Green function in the energy-momentum representation.

The scattering amplitude $f(\mathbf{p},\mathbf{p}')$ for scattering of the particle with initial and final momenta $\mathbf{p}'$, $\mathbf{p}$, respectively, is defined by

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{p}''\,V(\mathbf{p}-\mathbf{p}'')G_+(\mathbf{p}'',\mathbf{p}';p^0)[p^0-E(\mathbf{p}')]\Bigg|_{p^0=E(\mathbf{p}')}, \qquad (3.2.36)$$

where

$$V(\mathbf{p}) = \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\mathbf{p}/\hbar} V(\mathbf{x}). \tag{3.2.37}$$

This suggests to multiply Eq. (3.2.31) by $[p^0 - E(\mathbf{p}')]$ giving

$$G_+(\mathbf{p}, \mathbf{p}'; p^0)[p^0 - E(\mathbf{p}')] = -\frac{1}{(2\pi\hbar)^3} \int_0^\infty \mathrm{d}\alpha \left(\frac{\partial}{\partial\alpha} \mathrm{e}^{\mathrm{i}\alpha[p^0 - E(\mathbf{p}') + \mathrm{i}\epsilon]/\hbar} \right)$$
$$\times \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x}, \mathbf{p}'; \alpha). \tag{3.2.38}$$

From the fact that

$$\langle \mathbf{x} \,|\, \mathbf{p} \rangle = \exp(\mathrm{i}\mathbf{x}\cdot\mathbf{p}/\hbar), \tag{3.2.39}$$

and the definition of $K(\mathbf{x}, \mathbf{p}'; \alpha)$ in Eq. (3.2.33), we have

$$K(\mathbf{x}, \mathbf{p}'; 0) = 1. \tag{3.2.40}$$

We now consider the cases for which

$$\lim_{\alpha\to\infty} \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x}, \mathbf{p}'; \alpha), \tag{3.2.41}$$

exists. This, in particular, implies that $(\epsilon > 0)$

$$\lim_{\alpha\to\infty} \mathrm{e}^{-\epsilon\alpha} \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x}, \mathbf{p}'; \alpha) = 0. \tag{3.2.42}$$

We may then integrate over α in Eq. (3.2.38) to obtain simply

$$G_+(\mathbf{p}, \mathbf{p}'; p^0)[p^0 - E(\mathbf{p}')]\Big|_{p^0 = E(\mathbf{p}')}$$

$$= \lim_{\alpha\to\infty} \frac{1}{(2\pi\hbar)^3} \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x},\mathbf{p}';\alpha), \tag{3.2.43}$$

on the energy shell

$$p^0 = E(\mathbf{p}), \tag{3.2.44}$$

and for the scattering amplitude, in Eq. (3.2.36), *after* integrating over $\mathbf{p}''$, the expression

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2} \lim_{\alpha\to\infty} \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} V(\mathbf{x}) K(\mathbf{x},\mathbf{p}';\alpha), \tag{3.2.45}$$

with $K(\mathbf{x},\mathbf{p}';\alpha)$ defined in Eq. (3.2.33). Here we recognize that the formal replacement of $K(\mathbf{x},\mathbf{p}';\alpha)$ by one gives the celebrated Born approximation. On the other hand, part of the argument

$$[\mathbf{x} - \mathbf{p}'(t-\tau)/m], \tag{3.2.46}$$

of

$$V\Big(\mathbf{x} - \mathbf{p}'(t-\tau)/m + \mathbf{F}(\tau)\Big), \tag{3.2.47}$$

in Eq. (3.2.33), represents a "straight line trajectory" of a particle, with the functional differentiations with respect to $\mathbf{F}(\tau)$, as defined in Eq. (3.2.33), leading to deviations of the dynamics from such a straight line trajectory. With a straight line approximation, ignoring all of the functional differentiations, with respect to $\mathbf{F}(\tau)$ and setting the latter equal to zero, gives the following explicit expression for the scattering amplitude $f(\mathbf{p},\mathbf{p}')$ in Eq. (3.2.45):

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} V(\mathbf{x})$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^{\infty} \mathrm{d}\alpha\, V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right]. \tag{3.2.48}$$

This modifies the Born approximation by the presence of an additional phase factor in the integrand in Eq. (3.2.48), depending on the potential, accumulated during the scattering process. Here one recognizes the expression which leads to scattering with small deflections at high energies (the so-called eikonal approximation) obtained from the straight line trajectory approximation discussed above. Deviations from this approximation may be then systematically obtained by carrying out a functional power series expansion of

$$V\left(\mathbf{x} - \mathbf{p}'(t-\tau)/m + \mathbf{F}(\tau)\right), \tag{3.2.49}$$

in $\mathbf{F}(\tau)$ and performing the functional differential operation as dictated by the first exponential in Eq. (3.2.33) and finally setting $\mathbf{F}(\tau)$ equal to zero.

We note that formally that the τ-integral, involving the potential V, in Eq. (3.2.33) increases with no bound for $\alpha \to \infty$ for the Coulomb potential and for potentials of longer range with the former potential defining the transitional potential between long and short range potentials. And in case that the limit in Eq. (3.2.41) does not exist, as encountered for the Coulomb potential, Eq. (3.2.38) cannot be integrated by parts. This is discussed in the next section.

3.3 Asymptotic “free” Green Function

In case the $\alpha \to \infty$ limit in Eq. (3.2.41) does not exist, one may study the behaviour of $G_+(\mathbf{p}, \mathbf{p}'; p^0)$ near the energy shell

$$p^0 \simeq \mathbf{p}'^2/2m, \tag{3.3.1}$$

directly from Eq. (3.2.31). To this end, we introduce the integration variable

$$z = \frac{\alpha}{\hbar}\left[p^0 - E(\mathbf{p}')\right], \tag{3.3.2}$$

in Eq. (3.2.31), to obtain

$$G_+(\mathbf{p},\mathbf{p}';p^0)[p^0 - E(\mathbf{p}')] = -\frac{\mathrm{i}}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}z\,\mathrm{e}^{\mathrm{i}z(1+\mathrm{i}\epsilon)}$$
$$\times \int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}\,K\left(\mathbf{x},\mathbf{p}';\frac{z\hbar}{p^0 - E(\mathbf{p}')}\right), \tag{3.3.3}$$

for $\epsilon \to +0$.

For

$$p^0 - E(\mathbf{p}') \gtrsim 0, \tag{3.3.4}$$

i.e., near the energy shell, we may substitute

$$K\left(\mathbf{x},\mathbf{p}';z\hbar/(p^0 - E(\mathbf{p}'))\right) \simeq \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^{z\hbar/(p^0-E(\mathbf{p}'))}\mathrm{d}\alpha\,V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right], \tag{3.3.5}$$

in Eq. (3.3.3) to obtain for the following integral

$$\int \mathrm{d}^3\mathbf{p}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}/\hbar}G_+(\mathbf{p},\mathbf{p}';p^0) \simeq$$
$$\frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{p}'/\hbar}}{[p^0 - E(\mathbf{p}') + \mathrm{i}\epsilon]}\int_0^\infty \mathrm{d}z\,\mathrm{e}^{\mathrm{i}z(1+\mathrm{i}\epsilon)}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^{z\hbar/(p^0-E(\mathbf{p}'))}\mathrm{d}\alpha\,V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right], \tag{3.3.6}$$

For the Coulomb potential

$$V(\mathbf{x}) = \lambda/|\mathbf{x}|, \tag{3.3.7}$$

$$\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha\, V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}\alpha\right)$$

$$\simeq \frac{\lambda m}{|\mathbf{p}'|}\ln\left(\frac{2|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))|\mathbf{x}|(1-\cos\theta)}\right), \tag{3.3.8}$$

where

$$\cos\theta = \mathbf{p}'\cdot\mathbf{x}/|\mathbf{p}'||\mathbf{x}|. \tag{3.3.9}$$

Hence

$$\exp -\frac{\mathrm{i}}{\hbar}\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha\, V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}\alpha\right) \simeq$$

$$\frac{1}{[p^0-E(\mathbf{p}')+\mathrm{i}\epsilon]^{-\mathrm{i}\gamma}} \times \exp -\mathrm{i}\gamma\ln\left(\frac{2p'^2 z\hbar}{m(p'x-\mathbf{p}'\cdot\mathbf{x})}\right), \tag{3.3.10}$$

where

$$\gamma = \lambda m/\hbar p'. \tag{3.3.11}$$

Finally using the integral

$$\int_0^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}z(1+\mathrm{i}\epsilon)}(z)^{-\mathrm{i}\gamma} = \mathrm{i}\mathrm{e}^{\pi\gamma/2}\Gamma(1-\mathrm{i}\gamma), \tag{3.3.12}$$

for $\epsilon \to +0$, where Γ is the gamma function, we obtain from Eq. (3.3.6)

$$\int \mathrm{d}^3\mathbf{p}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}/\hbar} G_+(\mathbf{p},\mathbf{p}';p^0) \;\simeq$$

$$\mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{p}'/\hbar}\frac{\mathrm{e}^{-\mathrm{i}\gamma\ln(2p'^2/m)}}{[p^0-E(\mathbf{p}')+\mathrm{i}\epsilon]^{1-\mathrm{i}\gamma}}\exp \mathrm{i}\gamma\ln\left(\frac{p'x-\mathbf{p}'\cdot\mathbf{x}}{\hbar}\right)\mathrm{e}^{\pi\gamma/2}\Gamma(1-\mathrm{i}\gamma), \tag{3.3.13}$$

to be compared with earlier results (e.g., Papanicolaou, 1976), and for the asymptotic "free" Green function, in the energy-momentum representation, the expression.

$$G_{+}^{0}(\mathbf{p})=\frac{\mathrm{e}^{-\mathrm{i}\gamma\ln(2p^{2}/m)}}{[p^{0}-E(\mathbf{p})+\mathrm{i}\epsilon]^{1-\mathrm{i}\gamma}}\mathrm{e}^{\pi\gamma/2}\Gamma(1-\mathrm{i}\gamma), \tag{3.3.14}$$

showing on obvious modification from the Fourier transform of the free Green function $[p^{0}-E(\mathbf{p})+\mathrm{i}\epsilon]^{-1}$.

CHAPTER IV
COUPLING OF QUANTUM SYSTEMS TO THE ENVIRONMENT: APPLICATION OF THE EXPECTATION VALUE FORMALISM

4.1 Introduction

This chapter involves with a rigorous treatment of the expectation value formalism as is applied to quantum mechanics and the role of the environment on a quantum mechanical system and of quantum decoherence. As one is involved here directly with *propabilities*, such as persistence of a system to remain in the same state after a time evolution process, *instead of* dealing first with *amplitudes*, the closed-time path formalism appears quite naturally. This chapter deals with the interesting story of the environment, surrounding a quantum mechanical system, and the resulting *response* of the latter system to the environment.

The functional differential treatment (Limboonsong and Manoukian, 2006; Manoukian, 1985, 1986a, 1987a, 1987c, 2006a, 2006b; Manoukian and Siranan, 2005; Schwinger, 1951b, 1953a, 1972), via the quantum dynamical principle, has been a very powerful tool for investigating properties of quantum systems and for carrying out explicit computations. In this regard, it has been quite successful in gauge theories and of the generation of essential modifications (Limboonsong and Manoukian, 2006; Manoukian, 1985, 1986a, 1987a, 1987c) needed for their proper quantization with no much effort. For a pedagogical treatment of the theory and for several applications of the functional differential method, via the quantum dynamical principle, in quantum mechanics, the reader may wish to refer to Chapter. 11 in "Quantum theory: a wide spectrum" in Manoukian (2006a). The purpose of this work is to carry out an analysis, using the functional differential approach, of the *coupling* of quantum mechanical systems to the environment, understood to be surrounding a physical system, as the former systems, in the real world, are never in isolation from the latter. The incorporation of the environment in quantum mechanical systems has led to much physical insights into such fundamental problems as quantum decoherence, Schrödinger's cat and in measurement theory, in general (see e.g., §8.7, §8.9, §12.7 in Manoukian, 2006a; Brune, Haroche and Raimond, 1992; Munroe, Meekhof et al., 1996; Walls and Milburn, 1985; Zurek, 1991). We will see, that the functional differential approach is quite suitable for studying the coupling of quantum mechanical systems to the environment. It involves in carrying out functional differentiations, with respect to classical sources, of a functional describing "decoupled" systems from the environment. As one is involved in taking the *trace* over the dynamical variables of the environment in studying the response of physical systems to it, the analysis necessarily involves in dealing directly with transition probabilities rather than amplitudes. This is a basic departure from the far simpler case of studying quantum mechanical systems in isolation. In dealing with probabilities and in taking traces, it turns out that two different sets of classical sources, coupled to the dynamical variables of the theory, should, *a priori*, be introduced. The physically relevant probabilities are then recovered in the limit as the two sets of sources coincide and are eventually set equal to zero. The general expression for transition probabilities of quantum mechanical systems, coupled to the environment, is given in Eq. (4.2.11) involving functional differentiations with respect to these two sets of classical sources. The method used in this chapter generalizes to quantum field theory and will be studied in a forthcoming chapter.

4.2 Transition Probabilities and the Role of the Environment

Typically a quantum mechanical system may be described by a Hamiltonian

$$H_1(t) = H(q, p, t) - qF(t) + pS(t), \tag{4.2.1}$$

written in terms of dynamical variables in the (q, p) language, where $F(t)$ and $S(t)$ are classical source functions introduced to generate functions of q and p, respectively. For simplicity of the notation, we have in Eq. (4.2.1), suppressed indices in q and p reflecting the dimensionality of space and of the number of particles involved in the theory. In most applications, the classical sources $F(t)$, $S(t)$ are set equal to zero *after* all relevant functional differentiations in the theory, with respect to them, are carried out, with $H(q, p; t)$ finally emerging as the Hamiltonian describing the actual physical system into consideration.

The quantum dynamical principle states, see Manoukian (2006a), Ch. 11, that a transition amplitude $\langle at_2 \,|\, bt_1 \rangle$ for the system governed by Eq. (4.2.1), from time t_1 to time t_2, is given by

$$\langle at_2 \,|\, bt_1 \rangle = \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t_1}^{t_2} \mathrm{d}\tau H \left[-\mathrm{i}\hbar \frac{\delta}{\delta F(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta S(\tau)}, \tau \right] \right) \langle at_2 \,|\, bt_1 \rangle_0 \,, \qquad (4.2.2)$$

where H in Eq. (4.2.2) is obtained from $H(q, p, \tau)$ by simply replacing q and p in the latter by the operators of functional differentiations $-\mathrm{i}\hbar\delta/\delta F(\tau)$, $\mathrm{i}\hbar\delta/\delta S(\tau)$, respectively, and $\langle at_2 \,|\, bt_1 \rangle_0$ denotes the transition amplitude governed by the simple "Hamiltonian"

$$[-qF(t) + pS(t)], \qquad (4.2.3)$$

only.

To investigate the role of the environment on the quantum mechanical system, governed initially by the Hamiltonian $H(t)$ in Eq. (4.2.1), one modifies the latter Hamiltonian by including, in the Hamiltonian, the contribution of the environment and of its interaction with the physical system at hand. Of particular interest is in the response of the physical system to the environment. Accordingly, one takes a trace over the dynamical variables of the environment in the manner to be spelled out below. The Hamiltonian of the combined system is taken to be of the form

$$H(t) = H_1(q_1, p_1, t) - q_1 F_1(t) + p_1 S_1(t) + H_2(q_2, p_2, t)$$

$$- q_2 F_2(t) + p_2 S_2(t) + H_I(q_1, p_1, q_2, p_2, t), \tag{4.2.4}$$

where the indices 1, 2 correspond, respectively, to the physical system and the environment, and H_I specifies the interaction term between them.

The transition amplitude for the combined system to evolve from a state, say,

$$|a, A; 0\rangle\,, \quad \text{initially at time} \quad t\,=\,0, \tag{4.2.5}$$

to a state, say,

$$|b, B; t\rangle\,, \text{at time} \quad t\,>\,0, \tag{4.2.6}$$

is then given by

$$\langle b, B; t\,|\,a, A; 0\rangle = \exp\Big(- \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau H_I\big(- \mathrm{i}\hbar\frac{\delta}{\delta F_1(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta S_1(\tau)},$$

$$- \mathrm{i}\hbar\frac{\delta}{\delta F_2(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta S_2(\tau)}, t\big)\Big) \langle b; t\,|\,a; 0\rangle^{F_1, S_1} \langle B; t\,|\,A; 0\rangle^{F_2, S_2}, \tag{4.2.7}$$

as in Eq. (4.2.2), where

$$\langle b; t\,|\,a; 0\rangle^{F_1, S_1}, \quad \langle B; t\,|\,A; 0\rangle^{F_2, S_2}, \tag{4.2.8}$$

are the transition amplitudes of the *decoupled* subsystems in the presence of their respective classical sources.

To find the response of the physical system, described by the Hamiltonian $H_1(q_1, p_1, t)$ in Eq. (4.2.4), to the environment, it is necessary to work with transition *probabilities*, corresponding to the process associated with the expression in Eq. (4.2.7), rather than with amplitudes as done in the latter equation, and “trace out" over the environment. To this end, let

$$\{|B_n;t\rangle\}, \tag{4.2.9}$$

denote a complete set of states pertaining to the environment, then the probability for the physical system to make a transition from an initial state

$$|a;0\rangle \ \text{ to a state } \ |b;t\rangle \text{ in time } t, \tag{4.2.10}$$

responding in the process to the environment, emerges as

$$\begin{aligned}\text{Prob}[(a;0)\rightarrow(b;t)]_E = \mathcal{O}(\mathcal{O}')^*\,\langle b;t\,|\,a;0\rangle^{F_1,S_1}\,(\langle b;t\,|\,a;0\rangle^{F_1',S_1'})^*\\ \mathscr{F}[F_2,S_2;F_2',S_2']\Big|,\end{aligned} \tag{4.2.11}$$

where

$$\mathcal{O} = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau H_I\big(-\mathrm{i}\hbar\frac{\delta}{\delta F_1(\tau)},\mathrm{i}\hbar\frac{\delta}{\delta S_1(\tau)},-\mathrm{i}\hbar\frac{\delta}{\delta F_2(\tau)},\mathrm{i}\hbar\frac{\delta}{\delta S_2(\tau)},\tau\big)\right), \tag{4.2.12}$$

with $\mathcal{O}'$ defined similarly with

$$F_1,S_1,F_2,S_2, \tag{4.2.13}$$

replaced by

$$F_1',S_1',F_2',S_2', \tag{4.2.14}$$

respectively, and the presence of the letter E attached to the probability on the left-hand side of Eq. (4.2.11) is to emphasize the coupling of the environment to the physical system as the latter evolves in time. The functional $\mathscr{F}$ is given by

$$\mathscr{F}[F_2, S_2; F_2', S_2'] = \sum_n \langle B_n; t\,|\, A; 0\rangle^{F_2,S_2} \left(\langle B_n; t\,|\, A; 0\rangle^{F_2',S_2'}\right)^*, \qquad (4.2.15)$$

where the closed-time path concept is emphasized in the above expression, *where* for time 0 to t we have sources F_2, S_2, which in the reversed path form t to 0, we have *a priori* different set of sources F_2' and S_2'. At the end of all manipulations F_2' will be set equal to F_2 and S_2' will be set equal to S_2 which will all taken to be zero.

We note that Eq. (4.2.15) reduces to the trace over the environment in the special case for which

$$F_2' \quad \text{is set equal to} \quad F_2, \quad \text{and} \quad S_2' \quad \text{to} \quad S_2. \qquad (4.2.16)$$

One cannot, *a priori*, set such equalities until the functional differentiations, with respect to these sources, as accomplished by the operators $\mathcal{O}, (\mathcal{O}')^*$, are independently carried out.

The vertical bar sign on the right-hand side of Eq. (4.2.11) refers to the fact that finally one is to set

$$F \;=\; F' \;=\; 0, \quad S \;=\; S' \;=\; 0, \qquad (4.2.17)$$

after all the operations of functional differentiations have been done.

Eq. (4.2.11) gives the general expression for the transition probability of a physical system, as it evolves in time, in response to the environment.

4.3 Exponential Decay versus Vacuum Persistence Probabilities and the Role of the Environment

Of significance importance is for systems written in terms of creation and annihilation operators, which most conveniently describe processes of transitions between their allowed states. Such a typical example is given by the Hamiltonian

$$H(t) = H_1(t) + H_2(t) + H_{12}(t), \tag{4.3.1}$$

with

$$H_1(t) = \hbar\omega a^\dagger a - a^\dagger F(t) - F^*(t)a, \tag{4.3.2}$$

$$H_2(t) = \sum_k \hbar\omega_k b_k^\dagger b_k - \sum_k \left(K_k(t) b_k^\dagger + b_k K_k^*(t) \right), \tag{4.3.3}$$

$$H_{12}(t) = a^\dagger \sum_k \lambda_k b_k + a \sum_k \lambda_k^* b_k^\dagger, \tag{4.3.4}$$

and $(a, a^\dagger)$, $(b_k, b_k^\dagger)$, pertaining to the physical system in consideration and the environment, respectively,

$$[a, a^\dagger] = 1, \quad [b_k, b_{k'}^\dagger] = \delta_{kk'}, \tag{4.3.5}$$

for the corresponding commutators.

Suppose that the environment is initially in the ground-state $|0;0\rangle_2$. Let $U_2(t)$ denote the time evolution unitary operator describing the time evolution of the environment in the absence of the physical system. The so-called Heisenberg operator $b_k(t)$

associated with b_k is given by

$$b_k(t) = U_2^\dagger(t) b_k U_2(t), \tag{4.3.6}$$

which works out to be

$$b_k(t) = b_k \mathrm{e}^{-\mathrm{i}\omega_k t} + \frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}\tau \, K_k(\tau) \mathrm{e}^{-\mathrm{i}\omega_k (t-\tau)}. \tag{4.3.7}$$

The quantum dynamical principle (Limboonsong and Manoukian, 2006; Manoukian, 1985, 1986a, 1987a, 1987c, 2006a; Manoukian and Siranan, 2005; Schwinger; 1951c, 1953a, 1972) for the vacuum-to-vacuum transition amplitude $\langle 0; t \,|\, 0; 0 \rangle_2^K$ gives

$$-\mathrm{i}\hbar \frac{\delta}{\delta K_k^*(t')} \langle 0; t \,|\, 0; 0 \rangle_2^K \quad = \quad \langle 0; t \,| b_k(t') | 0; 0 \rangle_2^K \,, \tag{4.3.8}$$

for $0 < t' < t$.

From the expression in Eqs. (4.3.7), (4.3.8) simplifies to

$$-\mathrm{i}\hbar \frac{\delta}{\delta K_k^*(t')} \langle 0; t \,|\, 0; 0 \rangle_2^K \quad = \quad \frac{\mathrm{i}}{\hbar} \langle 0; t \,|\, 0; 0 \rangle_2^K \int_0^{t'} \mathrm{d}\tau \, K_k(\tau) \mathrm{e}^{-\mathrm{i}\omega_k (t'-\tau)}, \tag{4.3.9}$$

which integrates out to

$$\langle 0; t \,|\, 0; 0 \rangle_2^K = \exp\left(-\frac{1}{\hbar^2} \sum_k \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \mathrm{e}^{-\mathrm{i}\omega_k (\tau-\tau')} K_k^*(\tau) \Theta(\tau - \tau') K_k(\tau') \right), \tag{4.3.10}$$

where $\Theta(\tau - \tau')$ is the step function.

The functional $\mathscr{F}[K, K']$, corresponding to the one in Eq. (4.2.15), may be worked out in closed form. To this end, set

$$K(t) = K_1(t) + K_2(t), \tag{4.3.11}$$

with $K_1(t), K_2(t)$ localized in time between $(0, t)$, such that $K_2(t)$ is "switched on" after the source $K_1(t)$ is "switched off". That is, in particular, $K_2(t)$ and $K_1(t)$ do not overlap in time.

From Eq. (4.3.10) we may then write

$$\langle 0; t \,|\, 0; 0\rangle_2^{K_1+K_2} = \langle 0; t \,|\, 0; 0\rangle_2^{K_2} \, \exp\left[\sum_k \left(\int_{-\infty}^{\infty} \mathrm{d}\tau \, \mathrm{e}^{-\mathrm{i}\omega_k \tau} \frac{\mathrm{i}}{\hbar} K_2^*(\tau)\right) \times \left(\int_{-\infty}^{\infty} \mathrm{d}\tau' \, \mathrm{e}^{\mathrm{i}\omega_k \tau} \frac{\mathrm{i}}{\hbar} K_1(\tau)\right)\right] \langle 0; t \,|\, 0; 0\rangle_2^{K_1} \,, \tag{4.3.12}$$

where due to the fact that $K_1(\tau), K_2(\tau')$ are localized in time, we have extended the time integrations in the middle exponential from $-\infty$ to ∞.

Let

$$|n; n_{k_1}, n_{k_2}, \ldots\rangle_2 \,, \tag{4.3.13}$$

denote a state of n excitations, n_{k_1} of which in the state k_1, n_{k_2} of which in state k_2,

and so on, i.e., such that

$$n = n_{k_1} + n_{k_2} + \ldots . \tag{4.3.14}$$

Then upon introducing the unitarity completeness property

$$\langle 0; t \,|\, 0; 0\rangle_2^{K_1+K_2} = \sum_{n=0}^{\infty} \sum_{(n_{k_1}+n_{k_2}+\ldots=n)} \langle 0; t \,|\, n; n_{k_1}, n_{k_2}, \ldots\rangle_2^{K_2} \, \langle n; n_{k_1}, n_{k_2}, \ldots \,|\, 0\rangle_2^{K_1} \,, \tag{4.3.15}$$

where the intermediate states are evaluated at any time after the switching off of source

K_1 and before the switching on of source K_2, and the Fourier transform

$$K_i(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, K_i(\omega)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{4.3.16}$$

we obtain by expanding the middle exponential in powers of the source functions

$$K_{1k}(\omega_k), \quad K_{2k}(\omega_k), \tag{4.3.17}$$

the expression

$$\langle n; n_{k_1}, n_{k_2}, \ldots ; t \,|\, 0; 0\rangle_2^K = \langle 0; t \,|\, 0; 0\rangle_2^K \, \frac{\left(\frac{\mathrm{i}}{\hbar} K_{k_1}(\omega_{k_1})\right)^{n_{k1}}}{\sqrt{n_{k_1}!}} \frac{\left(\frac{\mathrm{i}}{\hbar} K_{k_2}(\omega_{k_2})\right)^{n_{k2}}}{\sqrt{n_{k_2}!}} \ldots, \tag{4.3.18}$$

for a given source $K(t)$.

The functional $\mathscr{F}[K, K']$, corresponding to the one in Eq. (4.2.15), is then given by

$$\mathscr{F}[K, K'] = \sum_{n=0}^{\infty} \sum_{(n_{k_1}+n_{k_2}+\ldots=n)} \langle n; n_{k_1}, n_{k_2}, \ldots ; t \,|\, 0; 0\rangle_2^K \times \left(\langle n; n_{k_1}, n_{k_2}, \ldots ; t \,|\, 0; 0\rangle_2^{K'}\right)^*. \tag{4.3.19}$$

Here again we have a priori different sources K_2 and K_2' for the initial and reversed time paths, and may be summed exactly over n giving

$$\mathscr{F}[K, K'] = \langle 0; t \,|\, 0; 0\rangle_2^K \exp\left[\frac{1}{\hbar^2} \sum_k \left(\int_0^t \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega_k \tau} K_k(\tau)\right) \times \left(\int_0^t \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}\omega_k \tau'} K_k'^*(\tau')\right)\right] \left(\langle 0; t \,|\, 0; 0\rangle_2^{K'}\right)^*, \tag{4.3.20}$$

which cannot be expressed as the product of two functionals one depending on K and the other on K', as expected.

Formally one checks the *unitarity condition* :

$$\mathscr{F}[K,K] \quad = \quad 1, \tag{4.3.21}$$

directly from Eq. (4.3.20).

Suppose that the physical system is initially in the ground-state, i.e., the vacuum-state $|0;0\rangle$. The vacuum persistence amplitude of the physical system, in isolation from the environment, but in the presence of the external sources $F(t), F^*(t)$ in Eq. (4.3.2), may be then inferred from Eq. (4.3.10) to be

$$\langle 0;t\,|\,0;0\rangle_1^F = \exp\left(-\frac{1}{\hbar^2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \mathrm{e}^{-\mathrm{i}\omega(\tau-\tau')} F^*(\tau)\Theta(\tau-\tau')F(\tau')\right). \tag{4.3.22}$$

From our general expression in Eq. (4.2.11), we then obtain for the vacuum persistence probability of the physical system, in response to the environment,

$$\text{Prob}[(0;0)\to(0;t)]_E = \mathcal{O}\,(\mathcal{O}')^*\,\langle 0;t\,|\,0;0\rangle_1^F\left(\langle 0;t\,|\,0;0\rangle_1^{F'}\right)^*\mathscr{F}[K,K']\Big|, \tag{4.3.23}$$

where

$$\mathcal{O} = \exp -\frac{\mathrm{i}}{\hbar}\sum_k \int_0^t \mathrm{d}\tau\,\left[\lambda_k\,\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta F(\tau)}\,\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta K_k^*(\tau)} + \lambda_k^*\,\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta F^*(\tau)}\,\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta K_k(\tau)}\right], \tag{4.3.24}$$

and $\mathcal{O}'$ similarly defined with

$$F, F^*, K_k, K_k^*,$$

replaced, respectively, by

$$F',\ F'^*, \quad K'_k,\ K'^*_k,$$

and $\mathscr{F}[K, K']$ given by the explicit expression in Eq. (4.3.20).

To evaluate the expression on the right-hand side of (4.3.23), we use, in the process, the identity

$$\mathrm{e}^{A}\mathrm{e}^{B} = \exp\left(\mathrm{e}^{A}B\mathrm{e}^{-A}\right)\mathrm{e}^{A}, \tag{4.3.25}$$

for two operators A, B. We note that

$$\delta/\delta F(\tau), \quad \delta/\delta F^{*}(\tau),$$

in Eq. (4.3.24), give rise to translation operators, via $\mathcal{O}$, to functionals of F and F^{*} as given, for example, in Eq. (4.3.22), and similarly for

$$\delta/\delta K(\tau), \quad \delta/\delta K^{*}(\tau).$$

The functional differentiations operations in Eq. (4.3.23) are then readily carried for a physical system weakly coupled to the environment, and after setting the classical sources equal to zero, we obtain for the survival probability the expression

$$\text{Prob}[(0;0) \to (0;t)]_{E} = \exp\left(-2\sum_{k}\frac{|\lambda_{k}|^{2}}{\hbar^{2}}\int_{0}^{t}\mathrm{d}\tau\int_{0}^{\tau}\mathrm{d}\tau'\cos[(\omega-\omega_{k})(\tau'-\tau)]\right). \tag{4.3.26}$$

For the environment described by an infinite set of degrees of freedom, we replace the sum over k by an integral over the frequency $\omega_{k} \to \omega$, and in turn introduce a frequency density $n(\omega')$ to rewrite Eq. (4.3.26) as

$$\text{Prob}[(0;0) \to (0;t)]_{E} = \exp\left(-\frac{1}{\hbar^{2}}\int_{0}^{\infty}\mathrm{d}\omega'|\lambda(\omega')|^{2}\, n(\omega')\frac{\sin^{2}(\omega'-\omega)\dfrac{t}{2}}{(\omega'-\omega)^{2}/4}\right). \tag{4.3.27}$$

Upon introducing the integration variable

$$x = (\omega' - \omega)t/2, \tag{4.3.28}$$

one may rewrite the integral in Eq. (4.3.27) as

$$2t \int_{-\omega t/2}^{\infty} \mathrm{d}x \left| \lambda(\omega(1 + \frac{x}{\omega t})) \right|^2 n(\omega(1 + \frac{x}{\omega t})) \frac{\sin^2 x}{x^2}. \tag{4.3.29}$$

If one makes the Markov approximation by assuming that

$$|\lambda(\omega')|^2 n(\omega'), \tag{4.3.30}$$

is slowly varying around the point

$$\omega' = \omega, \tag{4.3.31}$$

and hence for $\omega t \gg \pi$, it may be taken outside the integral evaluated at ω, one gets for the integral in Eq. (4.3.29)

$$2t|\lambda(\omega)|^2 n(\omega) \int_{-\omega t/2}^{\infty} \mathrm{d}x \, \frac{\sin^2 x}{x^2}, \tag{4.3.32}$$

with increasing accuracy for $\omega t \gg \pi$. And for $\omega t \gg \pi$, we obtain from Eqs. (4.3.28), (4.3.32) the familiar exponential law

$$\text{Prob}[(0;0) \rightarrow (0;t)]_E \quad = \quad \mathrm{e}^{-\gamma t}, \tag{4.3.33}$$

where γ is the decay constant,

$$\gamma = 2\pi|\lambda(\omega)|^2 n(\omega)/\hbar^2. \tag{4.3.34}$$

This expression is strictly valid for

$$\pi/\omega \ll t \ll 1/\gamma, \tag{4.3.35}$$

consistent with the property of the decay of quantum systems and the Paley-Wiener Theorem (cf. Manoukian, 2006a in "Quantum Theory: A Wide Spectrum", §3.5), that the exponential law may be valid for intermediate values of t and not in the truly asymptotic limit $t \to \infty$.

CHAPTER V

THE GRAVITON PROPAGATOR

WITH A PRIORI NON-CONSERVED

EXTERNAL GENERATING SOURCE

5.1 Introduction

In the functional *differential* treatment of quantum field theory, it has been emphasized for years (Manoukian, 1986a, 1986b, 1987a, 1987b, 1987c, 1988a, 1988b, 1991b, 1991c, 1998) that, *a priori*, *no* conservation law may be imposed on the external sources coupled to higher spin fields *so that* one may vary the components of these sources independently. Only after the relevant functional differentiations with respect to these sources are carried out to generate Green functions, expectation values and so on such conservation law may be imposed. The consequences of this will be discussed below. For the time being note that a conservation law leads to a restriction between the components of a source. That is, a variation of one component of one or more components of the source leads automatically to the variations of other components, and these various components may not be varied independently.

Two examples may be readily given for the contradictions that may arise otherwise – one from elementary calculus and another one which is in the heart of the matter for this thesis. These are

(1).

$$\frac{\partial}{\partial x} f(x,y)\bigg|_{x=y} \neq \frac{\partial}{\partial x} f(x,x). \tag{5.1.1}$$

(2). The functional derivative of an expression like

$$[a_{\mu\nu}(x) + b(x)\partial_\mu\partial_\nu]T^{\mu\nu}(x), \tag{5.1.2}$$

with respect to $T^{\sigma\lambda}(x')$ is

$$\frac{1}{2}\left[a_{\mu\nu}(x) + b(x)\partial_\mu\partial_\nu\right]\left(\delta_\sigma{}^\mu\delta_\lambda{}^\nu + \delta_\lambda{}^\mu\delta_\sigma{}^\nu\right)\delta^4(x, x'), \tag{5.1.3}$$

where $a_{\mu\nu}(x)$, $b(x)$, may depend on x, and *not*

$$\frac{1}{2}a_{\mu\nu}(x)\left[\delta_\sigma{}^\mu\delta_\lambda{}^\nu + \delta_\lambda{}^\mu\delta_\sigma{}^\nu\right]\delta^4(x, x'), \tag{5.1.4}$$

if we first impose a conservation law $\partial_\mu T^{\mu\nu} = 0$, before functional differentiation. *Afterwards*, if we impose the conservation law $\partial_\mu T^{\mu\nu}(x) = 0$, then we get Eq. (5.1.3) and not Eq. (5.1.4).

By *a priori* relaxing the conservation law of the energy-momentum tensor $T^{\mu\nu}$ *coupled* to the gravitational field, we have succeeded in deriving *a novel* expression for the graviton propagator. It involves 30 terms as opposed to the one used for years involving only 3 terms. This is no surprise as with careful mathematical we cannot go wrong. The present chapter deals with the fascinating story of this development and our contribution the graviton propagator as mediating the gravitational interaction between all particles and everything else in our universe. It remains an intriguing problem as to what consequence of this explicit structure of the graviton propagator has on the problem of renormalizability of quantum gravity. Other consequences which follow by *a priori* relaxing the conservation of external sources coupled to higher spin fields will be discussed below.

A basic ingredient in quantum gravity computations is the graviton propagator (Schwinger, 1976; Manoukian, 1990, 2005; Sivaram, 1999; Weinberg, 1965). The latter mediates the gravitational interaction between all particles to the leading order in the gravitational coupling constant. In the so-called functional differential treatment (Manoukian, 1985, 1986; Limboonsong and Manoukian, 2006; Schwinger, 1951 and Manoukian, 2006a) of quantum field theory, referred as the quantum dynamical principle approach, based on functional derivative techniques with respect to external sources coupled to the underlying fields in a theory, functional derivatives are taken of the so-called vacuum-to-vacuum transition amplitude. The latter generates *n*-point functions by functional differentiations leading finally to transition amplitudes for various physical processes. For higher spin fields such as the electromagnetic vector potential A^μ, the gluon field A_a^μ, and certainly the gravitational field $h^{\mu\nu}$, the respective external sources J_μ, J_μ^a, $T_{\mu\nu}$, coupled to these fields, cannot *a priori* taken to be conserved so that their respective components may be varied *independently*. The consequences of relaxing the conservation of these external sources are highly non-trivial. For one thing the corresponding field propagators become modified. Also they have led to the rediscovery (Manoukian, 1986; Limboonsong and Manoukian, 2006) of Faddeev–Popov (Faddeev and Popov, 1967) factors in non-abelian gauge theories and the discovery (Limboonsong and Manoukian, 2006) of even more generalized such factors, directly from the functional *differential* treatment, via the application of the quantum dynamical principle, in the presence of external sources, without using commutation rules, and without even going to the well known complicated structures of the underlying Hamiltonians. A brief account of this is given in the concluding section for the convenience of the reader.

For higher spin fields, the propagator and the time-ordered product of two fields do *not* coincide as the former includes so-called Schwinger terms which, in general, lead to a simplification of the expression for the propagator over the time-ordered one. This is well known for spin 1 , and, as shown below, is also true for the graviton propagator. Let $h^{\mu\nu}$ denote the gravitational field (see Sect. 5.2).

We work in a gauge

$$\partial_i h^{i\nu} = 0, \tag{5.1.5}$$

where $i = 1, 2, 3;\ \nu = 0, 1, 2, 3$, which guarantees that only two states of polarization occur with the massless particle and ensures positivity in quantum applications avoiding non-physical states. Let $T_{\mu\nu}$ denote an external source coupled to the gravitational field $h^{\mu\nu}$ (see Sect. 5.2), and let $\langle 0_+ \,|\, 0_- \rangle^T$ denote the vacuum-to-vacuum transition amplitude in the presence of the external source. The propagator of the gravitational field is then defined by

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \mathrm{i}\left((-\mathrm{i})\frac{\delta}{\delta T_{\mu\nu}(x)}(-\mathrm{i})\frac{\delta}{\delta T_{\sigma\lambda}(x')}\, \langle 0_+ \,|\, 0_- \rangle^T \right) \Big/ \langle 0_+ \,|\, 0_- \rangle^T \,, \tag{5.1.6}$$

in the limit of the vanishing of the external source $T_{\mu\nu}$. In more detail we may rewrite Eq. (5.1.6) as

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \mathrm{i}\frac{\left\langle 0_+ \middle| \left(h^{\mu\nu}(x) h^{\sigma\lambda}(x') \right)_+ \middle| 0_- \right\rangle^T}{\langle 0_+ \,|\, 0_- \rangle^T} + \frac{\left\langle 0_+ \middle| \dfrac{\delta}{\delta T_{\mu\nu}(x)} h^{\sigma\lambda}(x') \middle| 0_- \right\rangle^T}{\langle 0_+ \,|\, 0_- \rangle^T}, \tag{5.1.7}$$

in the limit of vanishing $T_{\mu\nu}$, where the first term on the right-hand side, up to the i factor, denotes the time-ordered product. In the second term, the functional derivative with respect to the external source $T_{\mu\nu}(x)$ is taken by keeping the independent field components of $h^{\sigma\lambda}(x')$ fixed. The dependent field components depend on the external source and lead to extra terms on the right-hand side of Eq. (5.1.7) in addition to the time-ordered product and may be referred to as Schwinger terms. For a detailed derivation of the general identity in Eq. (5.1.7) see Ref. Manoukian et al. (2007) (see also Manoukian, 2006a). These additional terms lead to a simplification of the expression for the propagator over the time-ordered product. Accordingly, the propagator and the time-ordered product do *not* coincide and it is the propagator $\Delta_+^{\mu\nu;\sigma\lambda}$ that appears in the functional approach and not the time-ordered product. The derivation of the explicit expression for $\Delta_+^{\mu\nu;\sigma\lambda}(x,x')$ follows by relaxing the conservation of $T_{\mu\nu}$ and it includes 30 terms in contrast to the well known case involving only 3 terms when a conservation law of $T_{\mu\nu}$ is imposed. It is important to emphasize that our interest here is in the propagator, the basic component which appears in the theory, and not the time-ordered product. At the end of Section 5.2, some additional pertinent comments are made regarding our expression for the propagator. We also include a section on attempts to detect gravitational waves. Our notation for the Minkowski meter is as always $[\eta^{\mu\nu}] = \text{diag}[-1, 1, 1, 1]$, also quite generally we set $i, j, k, l = 1, 2, 3$, $a, b = 1, 2$, while $\mu, \nu, \sigma, \lambda = 0, 1, 2, 3$.

5.2 The Graviton Propagator

The Lagrangian density of the gravitational field $h^{\mu\nu}$ coupled to an external source $T_{\mu\nu}$, is

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h^\sigma{}_\sigma \partial_\alpha h^\beta{}_\beta - \partial^\alpha h_{\alpha\mu}\partial^\mu h^\sigma{}_\sigma \\ &+\frac{1}{2}\partial_\alpha h^{\alpha\nu}\partial^\beta h_{\beta\nu} + \frac{1}{2}\partial_\alpha h^\mu{}_\nu \partial_\mu h^{\alpha\nu} + h^{\mu\nu}T_{\mu\nu},\end{aligned} \tag{5.2.1}$$

where $h^{\mu\nu} = h^{\nu\mu}$, and as a result $T_{\mu\nu}$ is chosen to be symmetric. We consider the ten components of $T_{\mu\nu}$ to be independent by, *a priori*, not imposing a conservation law for $T_{\mu\nu}$. The action corresponding to the Lagrangian density in Eq. (5.2.1), in the absence of the external source $T_{\mu\nu}$, is invariant under the gauge transformation (see Chapter VI for a demonstration of this).

$$h^{\mu\nu} \rightarrow h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu + \partial^\mu\partial^\nu \xi, \tag{5.2.2}$$

with a gauge constraint is

$$\partial_i h^{i\nu} = 0. \tag{5.2.3}$$

Then the effective Lagrangian can be written as

$$\mathscr{L}_{eff} = -\,h_{0i}\boldsymbol{\partial}^2 h_{0i} + h_{00}\boldsymbol{\partial}^2 h_{ii} + \frac{1}{2}h^{ij}\Box h_{ij} - \frac{1}{2}h_{ii}h_{jj}$$

$$+\,h_{00}T^{00} + 2h_{0i}T^{0i} + h_{ij}T^{ij}. \tag{5.2.4}$$

From Eq. (5.2.3) the gauge constraint allows us to solve, $h_{3\nu}$, in terms of other components:

$$\partial_1 h^{10} + \partial_2 h^{20} + \partial_3 h^{30} = 0$$

$$\partial_3 h^{30} = -\partial_a h^{a0}$$

$$h^{30} = -(\partial_3^{-1})\partial_a h^{a0}, \tag{5.2.5}$$

or

$$h_{30} = -(\partial_3^{-1})\partial_a h_{a0}, \tag{5.2.6}$$

where $a = 1, 2$.

From

$$\partial_i h^{ij} = 0, \tag{5.2.7}$$

then

$$\partial_1 h^{1j} + \partial_2 h^{2j} + \partial_3 h^{3j} = 0$$

$$\partial_3 h^{3j} = -\partial_b h^{bj}, \tag{5.2.8}$$

where $b = 1, 2$, and

$$\partial_3 h^{3a} + \partial_3 h^{33} = -\partial_b h^{ba} - \partial_b h^{b3}. \tag{5.2.9}$$

Then we have

$$\partial_3 h_{3a} = -\partial_b h_{ba} \tag{5.2.10}$$

$$h_{3a} = -(\partial_3)^{-1} \partial_b h_{ba}, \tag{5.2.11}$$

and

$$\partial_3 h_{33} = -\partial_a h_{a3}, \tag{5.2.12}$$

$$h_{33} = -(\partial_3)^{-1} \partial_a h_{a3}. \tag{5.2.13}$$

Replacing h^{3a} from Eq. (5.2.11) into Eq. (5.2.13), we get

$$h_{33} = -(\partial_3)^{-2} \partial_a \partial_b h_{ab}, \tag{5.2.14}$$

where $a,\, b = 1,\, 2$. Substituting the expressions for $h_{3\nu}$ in Eq. (5.2.4) and varying the Lagrangian with respect to h_{ab}, we obtain

$$\begin{aligned} \frac{\partial \mathscr{L}_{eff}}{\partial h_{ab}} &= \boldsymbol{\partial}^2 \left(\delta_{ab} + \frac{\partial_a \partial_b}{\partial_3^2} \right) h_{00} + \Box h_{ab} + 2\partial_3^{-2} \partial_d \Box \partial_f h_{ed} \\ &+ \partial_3^{-2} \partial_a \partial_b \Box \partial_3^{-2} \partial_c \partial_d h_{cd} - \left(\delta_{ab} - \frac{\partial_a \partial_b}{\partial_3^2} \right) \Box h_{ii} \\ &+ T^{ab} + \partial_3^{-2} \partial_a \partial_b T^{33} - 2\frac{\partial_b}{\partial_3} T^{a3}, \end{aligned} \tag{5.2.15}$$

we have

$$\partial_3^{-2}\partial_c\partial_d h_{cd} = h_{33}, \tag{5.2.16}$$

and then the fourth term in Eq. (5.2.15) becomes

$$\partial_3^{-2}\partial_a\partial_b\Box\partial_3^{-2}\partial_c\partial_d h_{cd} = \frac{\partial_a\partial_b}{\partial_3^2}\Box h_{33}. \tag{5.2.17}$$

Consider the third term in Eq. (5.2.15),

$$\begin{aligned}
2\partial_3^{-2}\partial_d\Box\partial_f h_{ed} &= 2\frac{\partial_d}{\partial_3}\Box\frac{\partial_f}{\partial_3}h_{ed} \\
&= 2\frac{\partial_d}{\partial_3}\Box\frac{1}{2\partial_3}(\partial_f h_{ed} + \partial_e h_{fd}) \\
&= \frac{\partial_d}{\partial_3}\Box\frac{\partial_f}{\partial_3}h_{ed} + \frac{\partial_d}{\partial_3}\Box\frac{\partial_e}{\partial_3}h_{fd} \\
&= \frac{\partial_f}{\partial_3}\Box\frac{\partial_d}{\partial_3}h_{ed} + \frac{\partial_e}{\partial_3}\Box\frac{\partial_d}{\partial_3}h_{fd} \\
&= \frac{-\partial_f}{\partial_3}\Box h_{e3} - \frac{\partial_e}{\partial_3}\Box h_{f3} \\
&= \frac{-\partial_b}{\partial_3}\Box h_{a3} - \frac{\partial_a}{\partial_3}\Box h_{b3},
\end{aligned} \tag{5.2.18}$$

and

$$2\frac{\partial_b}{\partial_3}T_{a3} = \frac{\partial_b}{\partial_3}T_{a3} + \frac{\partial_a}{\partial_3}T_{b3}. \tag{5.2.19}$$

Then insert Eqs. (5.2.17) - (5.2.19) into Eq. (5.2.10), and rearrange all terms, to obtain

$$(\Box h_{ab} + T_{ab}) - \frac{\partial_b}{\partial_3}(\Box h_{a3} + T_{a3}) - \frac{\partial_a}{\partial_3}(\Box h_{b3} + T_{b3})$$

$$+\frac{\partial_a\partial_b}{\partial_3^2}(\Box h_{33}+T_{33})+\left[\delta_{ab}+\frac{\partial_a\partial_b}{\partial_3^2}\right](\boldsymbol{\partial}^2 h_{00}-\Box h_{ii})=0, \tag{5.2.20}$$

where $a,\, b = 1,\, 2$. Upon multiplying Eq. (5.2.20) by $(\delta_{ab} - \partial_a\partial_b/\boldsymbol{\partial}^2)$, where $\boldsymbol{\partial}^2 = \partial^i\partial_i, \quad i = 1,\, 2,\, 3$, by considering of multiplication term by term, first by multiplying the h-terms, then we may write

$$(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2})\Bigg(\Box h_{ab}-\frac{\partial_b}{\partial_3}\Box h_{a3}-\frac{\partial_a}{\partial_3}\Box h_{b3}+\frac{\partial_a\partial_b}{\partial_3^2}\Box h_{33}$$
$$+[\delta_{ab}+\frac{\partial_a\partial_b}{\partial_3^2}](\boldsymbol{\partial}^2 h_{00}-\Box h_{ii})\Bigg). \tag{5.2.21}$$

By expanding of Eq. (5.2.21) we get

$$\Box h_{aa}-\frac{\partial_a}{\partial_3}\Box h_{a3}-\frac{\partial_b}{\partial_3}\Box h_{b3}+\frac{\partial_a\partial_a}{\partial_3^2}\Box h_{33}$$
$$-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\Box h_{ab}+\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_b}{\partial_3}\Box h_{a3}+\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_a}{\partial_3}\Box h_{b3}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_a\partial_b}{\partial_3^2}\Box h_{33}$$
$$+(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2})[\delta_{ab}+\frac{\partial_a\partial_b}{\partial_3^2}](\boldsymbol{\partial}^2 h_{00}-\Box h_{ii}). \tag{5.2.22}$$

Consider $[C]$, the third line of Eq. (5.2.22), where

$$[C]=(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2})[\delta_{ab}+\frac{\partial_a\partial_b}{\partial_3^2}](\boldsymbol{\partial}^2 h_{00}-\Box h_{ii})$$
$$=(\delta_a{}^a+\frac{\partial_a\partial^a}{\partial_3^2}-\frac{\partial_a\partial^a}{\boldsymbol{\partial}^2}-\frac{\partial_a\partial_a\partial_b\partial_b}{\boldsymbol{\partial}^2\partial_3^2})(\boldsymbol{\partial}^2 h_{00}-\Box h_{ii})$$
$$=(\delta_a{}^a+\partial_a\partial^a(\frac{1}{\partial_3^2}-\frac{1}{\boldsymbol{\partial}^2})-\frac{\partial_a\partial_a\partial_b\partial_b}{\boldsymbol{\partial}^2\partial_3^2})(\boldsymbol{\partial}^2 h_{00}-\Box h_{ii})$$
$$=(\delta_a{}^a+\frac{\partial_a\partial^a(\boldsymbol{\partial}^2-\partial_3^2)}{\boldsymbol{\partial}^2\partial_3^2})-\frac{\partial_a\partial_a\partial_b\partial_b}{\boldsymbol{\partial}^2\partial_3^2}(\boldsymbol{\partial}^2 h_{00}-\Box h_{ii})$$
$$=(\delta_a{}^a+\frac{\partial_a\partial_a\partial_b\partial_b}{\boldsymbol{\partial}^2\partial_3^2}-\frac{\partial_a\partial_a\partial_b\partial_b}{\boldsymbol{\partial}^2\partial_3^2})(\boldsymbol{\partial}^2 h_{00}-\Box h_{ii}). \tag{5.2.23}$$

The second and third term are canceled, where $\delta_a{}^a = 2$, then we get $[C]$ as

$$[C] = 2(\boldsymbol{\partial}^2 h_{00} - \Box h_{ii}). \tag{5.2.24}$$

From Eq. (5.2.11), we have $h_{b3} = -(\partial_3)^{-1}\partial_a h_{ab}$, replacing h_{a3} and h_{b3} into Eq. (5.2.22) and consider

$$\begin{aligned}[A] = \Box h_{aa} - \frac{\partial_a}{\partial_3}\Box(-\partial_3^{-1})\partial_b h_{ba} - \frac{\partial_b}{\partial_3}\Box(-\partial_3^{-1}\partial_a h_{ab}) + \frac{\partial_a\partial_a}{\partial_3^2}\Box h_{33} \\ - \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\Box h_{ab} + \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_b}{\partial_3}\Box(-\partial_3^{-1}\partial_b h_{ba}) \\ + \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_a}{\partial_3}\Box(-\partial_3^{-1}\partial_a h_{ab}) - \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_a\partial_b}{\partial_3^2}\Box h_{33}.\end{aligned} \tag{5.2.25}$$

We consider term by term in Eq. (5.2.25) defined as follows:

$$[1] = \Box h_{aa}, \tag{5.2.26}$$

$$[2] = \frac{-\partial_a}{\partial_3}\Box(-\partial_3^{-1}\partial_b h_{ba}) = \frac{\partial_a}{\partial_3}\Box\frac{\partial_b}{\partial_3}h_{ba} = \Box\frac{\partial_a\partial_b}{\partial_3^2}h_{ab} = \Box h_{33}, \tag{5.2.27}$$

$$[3] = \frac{-\partial_b}{\partial_3}\Box(-\partial_3^{-1}\partial_a h_{ab}) = \frac{\partial_b}{\partial_3}\Box\frac{\partial_a}{\partial_3}h_{ab} = \Box\frac{\partial_a\partial_b}{\partial_3^2}h_{ab} = \Box h_{33}, \tag{5.2.28}$$

$$[4] = \frac{\partial_a\partial_a}{\partial_3^2}\Box h_{33}, \tag{5.2.29}$$

$$[5] = \frac{-\partial_a\partial_b}{\boldsymbol{\partial}^2}\Box h_{ab} = \frac{-\partial_3^2}{\partial_3^2}\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\Box h_{ab} = \frac{-\partial_3^2}{\boldsymbol{\partial}^2}\Box h_{33}, \tag{5.2.30}$$

$$[6] = \frac{\partial_a \partial_b}{\boldsymbol{\partial}^2} \frac{\partial_b}{\partial_3} \Box(-\partial_3^{-1} \partial_b h_{ab}) = \frac{-\partial_b \partial_b}{\boldsymbol{\partial}^2} \Box \frac{\partial_a \partial_b}{\partial_3^2} h_{ab} = \frac{-\partial_b \partial_b}{\boldsymbol{\partial}^2} \Box h_{33}, \tag{5.2.31}$$

$$[7] = \frac{\partial_a \partial_b}{\boldsymbol{\partial}^2} \frac{\partial_a}{\partial_3} \Box(-\partial_3^{-1} \partial_a h_{ab}) = \frac{-\partial_a \partial_a}{\boldsymbol{\partial}^2} \Box \frac{\partial_a \partial_b}{\partial_3^2} h_{ab} = \frac{-\partial_a \partial_a}{\boldsymbol{\partial}^2} \Box h_{33}, \tag{5.2.32}$$

$$[8] = \frac{-\partial_a \partial_b}{\boldsymbol{\partial}^2} \frac{\partial_a \partial_b}{\partial_3^2} \Box h_{33} = \frac{-\partial_a \partial_a \partial_b \partial_b}{\boldsymbol{\partial}^2 \partial_3^2} \Box h_{33}. \tag{5.2.33}$$

Consider $[4] + [6]$

$$\begin{aligned} [4] + [6] &= \frac{\partial_a \partial_a}{\partial_3^2} \Box h_{33} - \frac{\partial_b \partial_b}{\boldsymbol{\partial}^2} \Box h_{33} \\ &= \partial_a \partial_a \left(\frac{1}{\partial_3^2} - \frac{1}{\boldsymbol{\partial}^2} \right) \Box h_{33} \\ &= \partial_a \partial_a \left(\frac{\boldsymbol{\partial}^2 - \partial_3^2}{\partial_3^2 \boldsymbol{\partial}^2} \right) \Box h_{33} \\ &= \frac{\partial_a \partial_a \partial_b \partial_b}{\partial_3^2 \boldsymbol{\partial}^2} \Box h_{33}. \end{aligned} \tag{5.2.34}$$

Then

$$[4] + [6] + [8] = 0, \tag{5.2.35}$$

and

$$[2] + [3] = 2 \Box h_{33}. \tag{5.2.36}$$

Consider

$$[5] + [7] = \frac{-\partial_3^2}{\boldsymbol{\partial}^2} \Box h_{33} - \frac{\partial_a \partial_a}{\boldsymbol{\partial}^2} \Box h_{33}$$

$$= \frac{-\Box}{\boldsymbol{\partial}^2}(\partial_3^2 + \partial_a\partial_a)h_{33}$$

$$= \frac{-\Box}{\boldsymbol{\partial}^2}\boldsymbol{\partial}^2 h_{33}$$

$$= -\Box h_{33}. \tag{5.2.37}$$

So

$$[2] + [3] + [5] + [7] = \Box h_{33}. \tag{5.2.38}$$

Since

$$[1] + [2] + [3] + [5] + [7] = \Box h_{aa} + \Box h_{33}$$

$$= \Box h_{ii}, \tag{5.2.39}$$

and

$$(5.2.39) + (5.2.24) = \Box h_{ii} + 2\boldsymbol{\partial}^2 h_{00} - 2\Box h_{ii}$$

$$= 2\boldsymbol{\partial}^2 h_{00} - \Box h_{ii}, \tag{5.2.40}$$

or

$$(\delta_{ab} - \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2})\left(\Box h_{ab} - \frac{\partial_b}{\partial_3}\Box h_{a3} - \frac{\partial_a}{\partial_3}\Box h_{b3} + \frac{\partial_a\partial_b}{\partial_3^2}\Box h_{33}\right.$$

$$\left. + [\delta_{ab} + \frac{\partial_a\partial_b}{\partial_3^2}](\boldsymbol{\partial}^2 h_{00} - \Box h_{ii})\right)$$

$$= 2\boldsymbol{\partial}^2 h_{00} - \Box h_{ii}. \tag{5.2.41}$$

Next we will consider the multiplication of the T-terms

$$(\delta_{ab} - \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2})(T_{ab} - \frac{\partial_b}{\partial_3}T_{a3} - \frac{\partial_a}{\partial_3}T_{b3} + \frac{\partial_a\partial_b}{\partial_3^2}T_{33})$$

$$= T_{aa} - \frac{\partial_a}{\partial_3}T_{a3} - \frac{\partial_b}{\partial_3}T_{b3} + \frac{\partial_a\partial_a}{\partial_3^2}T_{33}$$

$$- \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}T_{ab} + \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_b}{\partial_3}T_{a3} + \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_a}{\partial_3}T_{b3} - \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_a\partial_b}{\partial_3^2}T_{33}. \tag{5.2.42}$$

Let

$$[D_1] = T_{aa}, \tag{5.2.43}$$

$$[D_2] + [D_3] = \frac{-\partial_a}{\partial_3}T_{a3} - \frac{\partial_b}{\partial_3}T_{b3}, \tag{5.2.44}$$

$$[D_8] = \frac{-\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_a\partial_b}{\partial_3^2}T_{33} = \frac{-\partial_a\partial_a\partial_b\partial_b}{\boldsymbol{\partial}^2\partial_3^2}T_{33} = \frac{-\partial_a\partial_a(\boldsymbol{\partial}^2 - \partial_3^2)}{\boldsymbol{\partial}^2\partial_3^2}T_{33},$$

$$= \frac{-\partial_a\partial_a\boldsymbol{\partial}^2}{\boldsymbol{\partial}^2\partial_3^2}T_{33} + \frac{\partial_a\partial_a\partial_3^2}{\boldsymbol{\partial}^2\partial_3^2}T_{33},$$

$$= \frac{-\partial_a\partial_a}{\partial_3^2}T_{33} + \frac{\partial_a\partial_a}{\boldsymbol{\partial}^2}T_{33}, \tag{5.2.45}$$

then

$$[D_4] + [D_8] = \frac{\partial_a\partial_a}{\partial_3^2}T_{33} - \frac{\partial_a\partial_a}{\partial_3^2}T_{33} - \frac{\partial_a\partial_a}{\boldsymbol{\partial}^2}T_{33},$$

$$= \frac{\partial_a\partial_a}{\boldsymbol{\partial}^2}T_{33} = \frac{\boldsymbol{\partial}^2 - \partial_3^2}{\boldsymbol{\partial}^2}T_{33} = T_{33} - \frac{\partial_3^2}{\boldsymbol{\partial}^2}T_{33}, \tag{5.2.46}$$

and

$$[D_6] + [D_7] = \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\frac{\partial_b}{\partial_3}T_{a3} + \frac{\partial_a\partial_b\partial_a}{\boldsymbol{\partial}^2\partial_3}T_{b3},$$

$$= \frac{\partial_a}{\boldsymbol{\partial}^2}\frac{\nabla_b^2}{\partial_3}T_{a3} + \frac{\partial_b \nabla_a^2}{\boldsymbol{\partial}^2 \partial_3}T_{b3}. \tag{5.2.47}$$

Then Eq. (5.2.42) can be written as

$$[B] = T_{aa} - \frac{\partial_a}{\partial_3}T_{a3} - \frac{\partial_b}{\partial_3}T_{b3} + T_{33} - \frac{\partial_3^2}{\boldsymbol{\partial}^2}T_{33} - \frac{\partial_a \partial_b}{\boldsymbol{\partial}^2}T_{ab}$$

$$+ \frac{\partial_a \partial_b \partial_b}{\boldsymbol{\partial}^2 \partial_3}T_{a3} + \frac{\partial_b \partial_a \partial_a}{\boldsymbol{\partial}^2 \partial_3}T_{b3}. \tag{5.2.48}$$

Consider

$$\begin{aligned}
\frac{\partial_a \partial_b \partial_b}{\boldsymbol{\partial}^2 \partial_3}T_{a3} &= \frac{\partial_a}{\boldsymbol{\partial}^2}\frac{(\boldsymbol{\partial}^2 - \partial_3^2)}{\partial_3}T_{a3} \\
&= \frac{\partial_a \boldsymbol{\partial}^2}{\boldsymbol{\partial}^2 \partial_3}T_{a3} - \frac{\partial_a \partial_3^2}{\boldsymbol{\partial}^2 \partial_3}T_{a3} \\
&= \frac{\partial_a}{\partial_3}T_{a3} - \frac{\partial_a \partial_3}{\boldsymbol{\partial}^2}T_{a3},
\end{aligned} \tag{5.2.49}$$

and is also true for

$$\begin{aligned}
\frac{\partial_b \partial_a \partial_a}{\boldsymbol{\partial}^2 \partial_3}T_{b3} &= \frac{\partial_b}{\boldsymbol{\partial}^2}\frac{(\boldsymbol{\partial}^2 - \partial_3^2)}{\partial_3}T_{3b} \\
&= \frac{\partial_b \boldsymbol{\partial}^2}{\boldsymbol{\partial}^2 \partial_3}T_{3b} - \frac{\partial_b \partial_3^2}{\boldsymbol{\partial}^2 \partial_3}T_{3b} \\
&= \frac{\partial_b}{\partial_3}T_{b3} - \frac{\partial_b \partial_3}{\boldsymbol{\partial}^2}T_{3b}.
\end{aligned} \tag{5.2.50}$$

Then, replace Eqs. (5.2.49) and (5.2.50) into $[B]$ to get

$$[B] = T_{aa} - \frac{\partial_a}{\partial_3}T_{a3} - \frac{\partial_b}{\partial_3}T_{b3} + T_{33} - \frac{\partial_3^2}{\boldsymbol{\partial}^2}T_{33} - \frac{\partial_a \partial_b}{\boldsymbol{\partial}^2}T_{ab}$$

$$+ \frac{\partial_a}{\partial_3}T_{a3} - \frac{\partial_a \partial_3}{\boldsymbol{\partial}^2}T_{a3} + \frac{\partial_b}{\partial_3}T_{3b} - \frac{\partial_b \partial_3}{\boldsymbol{\partial}^2}T_{3b}$$

$$= T_{aa} + T_{33} - \left(\frac{\partial_a \partial_b}{\boldsymbol{\partial}^2} T_{ab} + \frac{\partial_a \partial_3}{\boldsymbol{\partial}^2} T_{a3} + \frac{\partial_3 \partial_b}{\boldsymbol{\partial}^2} T_{3b} + \frac{\partial_3^2}{\boldsymbol{\partial}^2} T_{33} \right). \qquad (5.2.51)$$

We see that the first two terms combine to T_{ii}, and, the terms in the round brackets is $\frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} T_{ij}$. Then Eq. (5.2.51) can be rewritten as shown below

$$[B] = T_{ii} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} T_{ij}$$

$$= \left(\delta_{ij} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} T_{ij} \right). \qquad (5.2.52)$$

Combine Eqs. (5.2.40) and (5.2.52) to obtain

$$2\boldsymbol{\partial}^2 h_{00} - \Box h_{ii} + \left(\delta_{ij} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} T_{ij} \right) = 0, \qquad (5.2.53)$$

and

$$-\boldsymbol{\partial}^2 h_{00} = -\frac{1}{2} \Box h_{ii} + \frac{1}{2} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} T_{ij} \right). \qquad (5.2.54)$$

Replacing h_{30}, h_{3a} and h_{33} into Eq. (5.2.4) and variations with respect to h_{00} we will get

$$\boldsymbol{\partial}^2 h_{ii} + T_{00} = 0, \qquad (5.2.55)$$

or

$$-\boldsymbol{\partial}^2 h_{ii} = T_{00}. \qquad (5.2.56)$$

The variation Eq. (5.2.4) with respect to h_{0a}, considering is obtained by the terms in the effective Lagrangian containing h_{0a}:

$$-h_{0i}\boldsymbol{\partial}^2 h_{0i} = -h_{0a}\boldsymbol{\partial}^2 h_{0a} - h_{03}\boldsymbol{\partial}^2 h_{03}$$

$$= -h_{0a}\boldsymbol{\partial}^2 h_{0a} - (-\partial_3^{-1})\partial_a h_{0a}\boldsymbol{\partial}^2(-\partial_3^{-1})\partial_a h_{0a}. \tag{5.2.57}$$

Then varying Eq. (5.2.57) with respect to h_{0a} we get

$$\frac{\delta}{\delta h_{0a}}(-h_{0i}\boldsymbol{\partial}^2 h_{0i}) = -2\boldsymbol{\partial}^2 h_{0a} + 2\frac{\partial_a}{\partial_3}\boldsymbol{\partial}^2 h_{03}, \tag{5.2.58}$$

and consider

$$2h_{0i}T^{0i} = 2h_{0a}T^{0a} + 2h_{03}T^{03}$$

$$= 2h_{0a}T^{0a} + 2(-\partial_3^{-1})\partial_a h_{0a}T^{03}. \tag{5.2.59}$$

Then a variation with respect to h_{0a} gives

$$\frac{\delta}{\delta h_{0a}}(2h_{0i}T^{0i}) = 2T^{0a} + 2(-\partial_3^{-1})\partial_a T^{03}$$

$$= 2T^{0a} - \frac{2\partial_a}{\partial_3}T^{03}. \tag{5.2.60}$$

Combine Eqs. (5.2.58) and (5.2.60) to get

$$-\boldsymbol{\partial}^2 h_{0a} + \frac{\partial_a}{\partial_3}\boldsymbol{\partial}^2 h_{03} = -T^{0a} + \frac{\partial_a}{\partial_3}T^{03}, \tag{5.2.61}$$

or the variation with respect to h_{0a} gives finally

$$-\boldsymbol{\partial}^2 h_{0a} + \frac{\partial_a}{\partial_3}\boldsymbol{\partial}^2 h_{03} = T_{0a} - \frac{\partial_a}{\partial_3}T_{03}. \tag{5.2.62}$$

We substitute the expressions for $h_{3\nu}$ in Eq. (5.2.4), and vary h_{ab}. First consider, the expression

$$h_{ii} = h_{aa} + h_{33}, \tag{5.2.63}$$

where $a = 1, 2$. Replace $h_{33} = (\partial_3)^{-2}\partial_a\partial_b h_{ab}$ into Eq. (5.2.21), to get

$$\begin{aligned} h_{ii} &= h_{aa} + (\partial_3)^{-2}\partial_a\partial_b h_{ab} \\ &= \left(\delta_{ab} - \frac{\partial_a\partial_b}{\partial_3^2}\right) h_{ab}. \end{aligned} \tag{5.2.64}$$

Consider the term

$$h_{00}\boldsymbol{\partial^2} h_{ii} = h_{00}\boldsymbol{\partial^2}\left(\delta_{ab} + (\partial_3)^{-2}\partial_a\partial_b\right) h_{ab}. \tag{5.2.65}$$

Then

$$\frac{\partial}{\partial h_{ab}} h_{00}\boldsymbol{\partial^2} h_{ii} = \boldsymbol{\partial^2}\left(\delta_{ab} + \frac{\partial_a\partial_b}{\partial_3^2}\right) h_{00}. \tag{5.2.66}$$

From

$$\begin{aligned} \frac{1}{2}h^{ij}\Box h_{ij} &= \frac{1}{2}h^{aj}\Box h_{aj} + \frac{1}{2}h^{3j}\Box h_{3j} \\ &= \frac{1}{2}h^{ab}\Box h_{ab} + \frac{1}{2}h^{a3}\Box h_{a3} + \frac{1}{2}h^{3a}\Box h_{3a} + \frac{1}{2}h^{33}\Box h_{33}, \end{aligned} \tag{5.2.67}$$

thus

$$\frac{1}{2}h^{ij}\Box h_{ij} = \frac{1}{2}h^{ab}\Box h_{ab} + h^{a3}\Box h_{a3} + \frac{1}{2}h^{33}\Box h_{33}. \tag{5.2.68}$$

Replace h_{3a}, h_{33} into Eq. (5.2.68), to obtain

$$\frac{1}{2}h^{ij}\Box h_{ij} = \frac{1}{2}h^{ab}\Box h_{ab} + \partial_3^{-1}\partial_b h_{ab}\Box\partial_3^{-1}\partial_b h_{ad}$$

$$+\frac{1}{2}\partial_3^{-2}\partial_a\partial_b h_{ab}\Box\partial_3^{-2}\partial_a\partial_b h_{ab}. \tag{5.2.69}$$

Differentiate Eq. (5.2.69) with respect to h_{ab}, to derive

$$\frac{1}{2}\frac{\partial}{\partial h_{ab}}h^{ij}\Box h_{ij} = \Box h_{ab} + 2\partial_3^{-2}\partial_d\Box\partial_f h_{ed} + \partial_3^{-2}\partial_a\partial_b\Box\partial_3^{-2}\partial_c\partial_d h_{cd}. \tag{5.2.70}$$

Consider the term $h_{ij}T^{ij}$ which works out to be

$$\begin{aligned} h_{ij}T^{ij} &= h_{aj}T^{aj} + h_{3j}T^{3j} \\ &= h_{ab}T^{ab} + h_{a3}T^{a3} + h_{3a}T^{3a} + h_{33}T^{33} \\ &= h_{ab}T^{ab} + 2h_{a3}T^{a3} + h_{33}T^{33}. \end{aligned} \tag{5.2.71}$$

By differentiation of Eq. (5.2.71) with respect to h_{cd}, we get

$$\begin{aligned} &\frac{\partial}{\partial h_{cd}}\left(h_{ab}T^{ab} + 2h_{a3}T^{a3} + h_{33}T^{33}\right) \\ &= T^{cd} + \left(\frac{\partial}{\partial h_{cd}}h_{33}\right)T^{33} + 2\left(\frac{\partial}{\partial h_{cd}}h_{a3}\right)T^{a3} \\ &= T^{cd} + \frac{\partial}{\partial h_{cd}}(\partial_3^{-2}\partial_a\partial_b)T^{33} + 2\frac{\partial}{\partial h_{cd}}(-\partial_3^{-1}\partial_b h_{ab})T^{a3} \\ &= T^{cd} + \partial_3^{-2}\partial_c\partial_d T^{33} - 2\partial_3^{-1}\partial_d T^{a3}. \end{aligned} \tag{5.2.72}$$

Upon changing $c \to a$ and $d \to b$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial h_{cd}}\left(h_{ab}T^{ab} + 2h_{a3}T^{a3} + h_{33}T^{33}\right) \\ &\qquad = T^{ab} + \partial_3^{-2}\partial_a\partial_b T^{33} - 2\partial_3^{-1}\partial_b T^{a3}. \end{aligned} \tag{5.2.73}$$

Consider the term $\frac{1}{2}h_{ii}\Box h_{jj}$. From Eq. (5.2.64) we have

$$\frac{\partial}{\partial h_{cd}}\frac{1}{2}h_{ii}\Box h_{jj}$$

$$= \frac{\partial}{\partial h_{cd}}\frac{1}{2}\left(\delta_{ab} - \frac{\partial_a\partial_b}{\partial_3^2}\right)h_{ab}\Box\left(\delta_{a'b'} - \frac{\partial_{a'}\partial_{b'}}{\partial_3^2}\right)h_{a'b'}$$

$$= 2\cdot\frac{1}{2}\left(\delta_{ab} - \frac{\partial_a\partial_b}{\partial_3^2}\right)\Box h_{ii}. \tag{5.2.74}$$

Thus

$$-\frac{1}{2}\frac{\partial}{\partial h_{cd}}\frac{1}{2}h_{ii}\Box h_{jj} = -\left(\delta_{ab} - \frac{\partial_a\partial_b}{\partial_3^2}\right)\Box h_{ii}. \tag{5.2.75}$$

Consider Eq. (5.2.62). If we replace a by 3 where $a = 1, 2$, we will get

$$-\boldsymbol{\partial}^2 h_{03} + \frac{\partial_3}{\partial_3}\boldsymbol{\partial}^2 h_{03} = T_{03} - \frac{\partial_3}{\partial_3}T_{03}$$

$$-\boldsymbol{\partial}^2 h_{03} + \boldsymbol{\partial}^2 h_{03} = T_{03} - T_{03}$$

$$0 = 0. \tag{5.2.76}$$

Thus we can rewrite Eq. (5.2.62) as

$$-\boldsymbol{\partial}^2 h_{0i} + \frac{\partial_i}{\partial_3}\boldsymbol{\partial}^2 h_{03} = T_{0i} - \frac{\partial_i}{\partial_3}T_{03}, \tag{5.2.77}$$

where $i = 1, 2, 3$. Upon taking the divergence ∂^i of Eq. (5.2.77), we get

$$-\partial^i\boldsymbol{\partial}^2 h_{0i} + \frac{\partial^i\partial_i}{\partial_3}\boldsymbol{\partial}^2 h_{03} = \partial^i T_{0i} - \partial^i\frac{\partial_i}{\partial_3}T_{03}. \tag{5.2.78}$$

From the constraint in Eq. (5.2.3) we have

$$-\boldsymbol{\partial}^2 \partial^i h_{i0} + \frac{\partial^i \partial_i}{\partial_3} \boldsymbol{\partial}^2 h_{03} = \partial^i T_{0i} - \frac{\partial^i \partial_i}{\partial_3} T_{03}. \tag{5.2.79}$$

Divide Eq. (5.2.79) by $\boldsymbol{\partial}^2$ to obtain

$$\frac{\partial_i}{\partial_3} \boldsymbol{\partial}^2 h_{03} = \frac{\partial_j}{\partial_i} - \frac{\partial_i}{\partial_3} T_{03}, \tag{5.2.80}$$

or

$$\frac{\partial_i}{\partial_3} \boldsymbol{\partial}^2 h_{03} = \frac{\partial_i}{\boldsymbol{\partial}^2} \left(\partial_j T_{0j} - \frac{\boldsymbol{\partial}^2}{\partial_3} T_{03} \right). \tag{5.2.81}$$

By inserting Eq. (5.2.81) into Eq. (5.2.77), we get

$$-\boldsymbol{\partial}^2 h_{0i} + \frac{\partial_i}{\boldsymbol{\partial}^2} \left(\partial_j T_{0j} - \frac{\boldsymbol{\partial}^2}{\partial_{03}} T_{03} \right) = T_{0i} - \frac{\partial_i}{\partial_3} T_{03}, \tag{5.2.82}$$

and

$$-\boldsymbol{\partial}^2 h_{0i} + \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} T_{0j} - \frac{\partial_i}{\partial_3} T_{03} = T_{0i} - \frac{\partial_i}{\partial_3} T_{03}. \tag{5.2.83}$$

Then

$$-\boldsymbol{\partial}^2 h_{0i} = T_{0i} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} T_{0j}, \tag{5.2.84}$$

or

$$-\boldsymbol{\partial}^2 h_{0i} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} \right) T_{0j}. \tag{5.2.85}$$

Consider

$$-\boldsymbol{\partial}^2 h_{00} = -\frac{1}{2} \Box h_{ii} + \frac{1}{2} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2} \right) T_{ij}, \tag{5.2.86}$$

and

$$-\boldsymbol{\partial}^2 h_{ii} = T_{00}. \tag{5.2.87}$$

Also upon substituting Eq. (5.2.87) into (5.2.86) and using the fact that $\Box = \boldsymbol{\partial}^2 - {\partial_0}^2$, we obtain for Eq. (5.2.86)

$$\begin{aligned}
-\boldsymbol{\partial}^2 h_{00} &= -\frac{1}{2}(\boldsymbol{\partial}^2 - \partial_0^2)h_{ii} + \frac{1}{2}(\delta_{ij} - \frac{\partial_i\partial_j}{\boldsymbol{\partial}^2})T_{ij} \\
&= -\frac{1}{2}\boldsymbol{\partial}^2 h_{ii} + \frac{1}{2}\partial_0^2 h_{ii} + \frac{1}{2}(\delta_{ij} - \frac{\partial_i\partial_j}{\boldsymbol{\partial}^2})T_{ij} \\
&= \frac{1}{2}T_{00} + \frac{1}{2}\partial_0^2 h_{ii} + \frac{1}{2}T_{ii} - \frac{1}{2}\frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}T_{ij}.
\end{aligned} \tag{5.2.88}$$

Consider

$$\frac{1}{2}\partial_0^2 \frac{\boldsymbol{\partial}^2}{\boldsymbol{\partial}^2} h_{ii} = -\frac{1}{2}\frac{\partial_0^2}{\boldsymbol{\partial}^2}T_{00}, \tag{5.2.89}$$

then Eq. (5.2.88) can be written as

$$\begin{aligned}
-\boldsymbol{\partial}^2 h_{00} &= \frac{1}{2}T_{00} - \frac{1}{2}\frac{\partial_0^2}{\boldsymbol{\partial}^2}T_{00} + \frac{1}{2}T_{ii} - \frac{1}{2}\frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}T_{ij} \\
&= \frac{1}{2}T_{00} + \frac{1}{2}T_{ii} - \frac{1}{2\boldsymbol{\partial}^2}\left(\partial^0\partial^0 T_{00} + \partial_i\partial_j T_{ij}\right).
\end{aligned} \tag{5.2.90}$$

Consider

$$T = \eta^{\mu\nu}T_{\mu\nu} = T^{\nu}{}_{\nu}, \tag{5.2.91}$$

and

$$T^{\nu}{}_{\nu} = T^0{}_0 + T^i{}_i = T^0{}_0 + T_{ii} = -T_{00} + T_{ii}, \tag{5.2.92}$$

which leads to the useful relation

$$\frac{T}{2} = -\frac{T_{00}}{2} + \frac{T_{ii}}{2}, \tag{5.2.93}$$

and

$$\frac{1}{2}T_{00} + \frac{1}{2}T_{ii} - \frac{1}{2}T_{00} + \frac{1}{2}T_{00} = T_{00} + \frac{T}{2}. \tag{5.2.94}$$

Thus, for the expression of $-\boldsymbol{\partial}^2 h_{00}$, we have

$$-\boldsymbol{\partial}^2 h_{00} = T_{00} + \frac{T}{2} - \frac{1}{2\boldsymbol{\partial}^2}\left(\partial^0\partial^0 T_{00} + \partial_i\partial_j T_{ij}\right), \tag{5.2.95}$$

where $T = \eta^{\mu\nu}T_{\mu\nu} = T^{\nu}{}_{\nu}$.

Equations (5.2.87), (5.2.85), (5.2.95) are not equations of motion as they involve no time derivatives of the corresponding fields and they yield to constraints which together with the gauge condition in Eq. (5.2.3) give rise to two degrees of freedom corresponding to two polarization states for the graviton as it should be. This is shown in Sect. 6.4.

We now substitute the expression for $-\boldsymbol{\partial}^2 h_{00}$, as given in Eq. (5.2.95), in Eq. (5.2.20) and use Eq. (5.2.87) to obtain an equation involving h_{ij}, $i, j = 1, 2, 3$. Upon multiplying the resulting equation from Eq. (5.2.20) by $\partial_a\partial_b$ and using the expressions for h_{33} in Eq. (5.2.13) we derive Eq. (5.2.108) after some very tedious algebra which follows.

$$(\Box h_{ab} + T_{ab}) - \frac{\partial_b}{\partial_3}(\Box h_{a3} + T_{a3}) - \frac{\partial_a}{\partial_3}(\Box h_{b3} + T_{b3})$$

$$+ \frac{\partial_a \partial_b}{\partial_3^2}(\Box h_{33} + T_{33}) + \left[\delta_{ab} + \frac{\partial_a \partial_b}{\partial_3^2}\right] \times$$

$$\times \left(- T_{00} - \frac{T}{2} + \frac{1}{2\boldsymbol{\partial}^2}\left(\partial^0\partial^0 T_{00} + \partial_i\partial_j T_{ij}\right) - \Box h_{ii} \right) = 0. \tag{5.2.96}$$

Consider the expression in last round brackets. Use the fact that $-\boldsymbol{\partial}^2 h_{ii} = T_{00}$, and insert this into the equation, to obtain

$$\left(- T_{00} - \frac{T}{2} + \frac{1}{2\boldsymbol{\partial}^2}\left(\partial^0\partial^0 T_{00} + \partial_i\partial_j T_{ij}\right) - \Box h_{ii} \right)$$

$$= \left(- T_{00} - \frac{T}{2} + \frac{1}{2\boldsymbol{\partial}^2}\left(\partial^0\partial^0(-\boldsymbol{\partial}^2 h_{ii}) + \partial_i\partial_j T_{ij}\right) - \Box h_{ii} \right) \tag{5.2.97}$$

$$= \left(- T_{00} - \frac{T}{2} - \frac{1}{2}\left(\partial^0\partial^0(h_{ii})\right) + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i\partial_j T_{ij}\right) - \Box h_{ii} \right). \tag{5.2.98}$$

According to the definition $\Box = \boldsymbol{\partial}^2 - \partial^{0^2}$, Eq. (5.2.98) may be rewritten as

$$\left(- T_{00} - \frac{T}{2} - \frac{1}{2}\left(\partial^0\partial^0(h_{ii})\right) + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i\partial_j T_{ij}\right) - (\boldsymbol{\partial}^2 - \partial^{0^2}) h_{ii} \right)$$

$$= \left(- T_{00} - \frac{T}{2} - \frac{1}{2}\left(\partial^0\partial^0(h_{ii})\right) + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i\partial_j T_{ij}\right) - \boldsymbol{\partial}^2 h_{ii} + \partial^{0^2} h_{ii} \right)$$

$$= \left(- T_{00} - \frac{T}{2} + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i\partial_j T_{ij}\right) - \boldsymbol{\partial}^2 h_{ii} + \frac{1}{2}\partial^{0^2} h_{ii} \right)$$

$$= \left(- T_{00} - \frac{T}{2} + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i\partial_j T_{ij}\right) + T_{00} + \frac{1}{2}\partial^{0^2} h_{ii} \right)$$

$$= \left(- \frac{T}{2} + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i\partial_j T_{ij}\right) + \frac{1}{2}\partial^{0^2} h_{ii} \right). \tag{5.2.99}$$

Then we obtain the important field equation

$$(\Box h_{ab} + T_{ab}) - \frac{\partial_b}{\partial_3}(\Box h_{a3} + T_{a3}) - \frac{\partial_a}{\partial_3}(\Box h_{b3} + T_{b3})$$

$$+ \frac{\partial_a \partial_b}{\partial_3^2}(\Box h_{33} + T_{33}) + \left[\delta_{ab} + \frac{\partial_a \partial_b}{\partial_3^2}\right]\left(-\frac{T}{2} + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i \partial_j T_{ij}\right) + \frac{1}{2}\partial^{0^2} h_{ii}\right)$$

$$= 0. \tag{5.2.100}$$

Multiplying Eq. (5.2.100) by $\partial_a \partial_b$ we get

$$\partial_a \partial_b \Box h_{ab} + \partial_a \partial_b T_{ab} - \frac{\partial_b \partial_a \partial_b}{\partial_3}\Box h_{a3} - \frac{\partial_b \partial_a \partial_b}{\partial_3}\Box T_{a3}$$

$$- \frac{\partial_a \partial_a \partial_b}{\partial_3}\Box h_{b3} - \frac{\partial_a \partial_a \partial_b}{\partial_3}\Box T_{b3} + \frac{(\partial_a \partial_b)^2}{(\partial_3)^2}(\Box h_{33} + T_{33})$$

$$\left[\delta_{ab} + \frac{\partial_a \partial_b}{(\partial_3)^2}\right](\partial_a \partial_b)\left(-\frac{T}{2} + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i \partial_j T_{ij}\right) + \frac{1}{2}\partial^{0^2} h_{ii}\right) = 0. \tag{5.2.101}$$

Consider the third and forth terms in Eq. (5.2.101), to get

$$-\frac{\partial_b \partial_a \partial_b}{\partial_3}\Box h_{a3} - \frac{\partial_b \partial_a \partial_b}{\partial_3}\Box T_{a3} = -\partial_b \partial_b \Box \frac{\partial_a}{\partial_3} h_{a3} - \frac{\partial_a \partial_b \partial_b}{\partial_3} T_{a3}$$

$$= \partial_b \partial_b \Box h_{33} - \frac{\partial_a \partial_b \partial_b}{\partial_3} T_{a3}, \tag{5.2.102}$$

where we have used the identity that $h_{33} = -\partial_a (\partial_3)^{-1} h_{a3}$.

Then the fourth and sixth terms in Eq. (5.2.101), lead to

$$-\partial_a \partial_a \Box \frac{\partial_b}{\partial_3} h_{b3} - \frac{\partial_a \partial_a \partial_b}{\partial_3} T_{b3} = \partial_a \partial_a \Box h_{33} - \frac{\partial_a \partial_a \partial_b}{\partial_3} T_{b3}, \tag{5.2.103}$$

where we have also used $h_{33} = -\partial_b (\partial_3)^{-1} h_{b3}$.

Then adding the expression in Eqs. (5.2.102) and (5.2.103) gives

$$2\partial\partial_a\partial_a\Box h_{33} - \frac{\partial_a\partial_b\partial_b}{\partial_3}T_{a3} - \frac{\partial_b\partial_a\partial_a}{\partial_3}T_{b3}. \tag{5.2.104}$$

Combining all the above equations together, gives

$$\partial_a\partial_b\Box h_{ab} + \partial_a\partial_b T_{ab} + 2\partial_a\partial_a\Box h_{33} - \frac{\partial_a\partial_b\partial_b}{\partial_3}T_{a3} - \frac{\partial_b\partial_a\partial_a}{\partial_3}T_{b3} - \frac{\partial_b\partial_a\partial_b}{\partial_3}\Box T_{a3}$$
$$- \frac{\partial_a\partial_a\partial_b}{\partial_3}\Box h_{b3} - \frac{\partial_a\partial_a\partial_b}{\partial_3}\Box T_{b3} + \frac{(\partial_a\partial_b)^2}{(\partial_3)^2}(\Box h_{33} + T_{33})$$
$$+ \left[\delta_{ab} + \frac{\partial_a\partial_b}{(\partial_3)^2}\right](\partial_a\partial_b)\Bigg(- \frac{T}{2} + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i\partial_j T_{ij}\right) + \frac{1}{2}\partial^{0^2}h_{ii}\Bigg). \tag{5.2.105}$$

Consider the terms

$$\begin{aligned}
2\partial_a\partial_a\Box h_{33} + \frac{(\partial_a\partial_b)^2}{(\partial_3)^2}\Box h_{33} &= (2\partial_a\partial_a + \frac{(\partial_a\partial_b)^2}{(\partial_3)^2})\Box h_{33} \\
&= \partial_a\partial_a(2 + \frac{\partial_b\partial_b}{(\partial_3)^2})\Box h_{33} \\
&= \frac{\partial_a\partial_a}{(\partial_3)^2}(2(\partial_3)^2 + \partial_b\partial_b)\Box h_{33} \\
&= \frac{\partial_a\partial_a}{(\partial_3)^2}((\partial_3)^2 + (\partial_3)^2 + \partial_b\partial_b)\Box h_{33} \\
&= \frac{\partial_a\partial_a}{(\partial_3)^2}((\partial_3)^2 + \boldsymbol{\partial}^2)\Box h_{33} \\
&= \partial_a\partial_a(1 + \frac{\boldsymbol{\partial}^2}{(\partial_3)^2})\Box h_{33}.
\end{aligned} \tag{5.2.106}$$

Then Eq. (5.2.105) becomes

$$\partial_a\partial_b\Box h_{ab} + \partial_a\partial_b T_{ab} + \partial_a\partial_a(1 + \frac{\boldsymbol{\partial}^2}{(\partial_3)^2})\Box h_{33} - \frac{\partial_a\partial_b\partial_b}{\partial_3}T_{a3} - \frac{\partial_b\partial_a\partial_a}{\partial_3}T_{b3} - \frac{\partial_b\partial_a\partial_b}{\partial_3}\Box T_{a3}$$

$$- \frac{\partial_a\partial_a\partial_b}{\partial_3}\Box h_{b3} - \frac{\partial_a\partial_a\partial_b}{\partial_3}\Box T_{b3} + \frac{(\partial_a\partial_b)^2}{(\partial_3)^2}(T_{33})$$

$$+ \left[\delta_{ab} + \frac{\partial_a\partial_b}{(\partial_3)^2}\right](\partial_a\partial_b)\Bigg(- \frac{T}{2} + \left(\frac{1}{2\boldsymbol{\partial}^2}\partial_i\partial_j T_{ij}\right) + \frac{1}{2}\partial^{0^2} h_{ii}\Bigg). \tag{5.2.107}$$

This finally gives the field equation for h_{33}

$$(\Box h_{33} + T_{33}) - \frac{1}{2}\left(1 - \frac{(\partial_3)^2}{\boldsymbol{\partial}^2}\right)T + \frac{1}{2\boldsymbol{\partial}^2}\left(-\partial^0\partial^0 T_{00} + \partial^i\partial^j T_{ij}\right)$$

$$-\frac{2}{\boldsymbol{\partial}^2}\partial^i\partial_3 T_{i3} + \frac{(\partial_3)^2}{2\boldsymbol{\partial}^2}\left(\frac{\partial^i\partial^j}{\boldsymbol{\partial}^2}T_{ij} + \frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}\right) = 0. \tag{5.2.108}$$

Similarly, upon multiplying Eq. (5.2.20) by ∂_a and using the expression for h_{b3} in Eq. (5.2.11), we obtain

$$(\Box h_{b3} + T_{b3}) -$$

$$\frac{1}{\boldsymbol{\partial}^2}\left[\partial_3\partial^i T_{ib} + \partial_b\partial^i T_{i3} - \frac{\partial_b\partial_3}{2}\left(\frac{\partial^i\partial^j}{\boldsymbol{\partial}^2}T_{ij} + \frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00} + T\right)\right] = 0. \tag{5.2.109}$$

To obtain the equation for h_{ab}, we substitute Eqs. (5.2.108), (5.2.109) in Eq. (5.2.20), to obtain after some lengthy algebra

$$(\Box h_{ab} + T_{ab}) - \frac{1}{2}\left(\delta_{ab} - \frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\right)T - \frac{1}{\boldsymbol{\partial}^2}(\partial_a\partial^i T_{ib} + \partial_b\partial^i T_{ia})$$

$$+ \frac{\delta_{ab}}{2\boldsymbol{\partial}^2}\left(-\partial^0\partial^0 T_{00} + \partial^i\partial^j T_{ij}\right) + \frac{\partial_a\partial_b}{2(\boldsymbol{\partial}^2)^2}\left(\partial^i\partial^j T_{ij} + \partial^0\partial^0 T_{00}\right) = 0. \tag{5.2.110}$$

Equations (5.2.108) - (5.2.110) may be now combined in the form

$$-\Box h_{ij} = T_{ij} - \frac{1}{2}\left(\delta_{ij} - \frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}\right)\left(T + \frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}\right)$$

$$- \frac{1}{\boldsymbol{\partial}^2}\left[\partial_i\partial^k T_{kj} + \partial_j\partial^k T_{ki} - \frac{1}{2}\left(\delta_{ij} + \frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}\right)\partial^k\partial^l T_{kl}\right], \tag{5.2.111}$$

where $i, j, k, l = 1, 2, 3$. For example, we check that Eq. (5.2.111) leads to Eq. (5.2.108) by setting $i = j = 3$. The analysis follows.

$$-\Box h_{33} = T_{33} - \frac{1}{2}(\delta_{33} - \frac{\partial_3\partial_3}{\boldsymbol{\partial}^2})(T + \frac{\partial_0\partial_0}{\boldsymbol{\partial}^2}T_{00})$$
$$- \frac{1}{\boldsymbol{\partial}}\left[\partial_3\partial^k T_{k3} + \partial_3\partial^k T_{k3} - \frac{1}{2}(\delta_{33} + \frac{\partial_3\partial_3}{\boldsymbol{\partial}^2})\partial^k\partial^l T_{kl}\right], \quad (5.2.112)$$

, then we get

$$\Box h_{33} + T_{33} - \frac{1}{2}\left(1 - \frac{(\partial_3)^2}{\boldsymbol{\partial}^2}\right)T + \frac{1}{2\boldsymbol{\partial}^2}(-(\partial^0)^2 T_{00} + \partial^k\partial^l T_{kl})$$
$$- \frac{2}{\boldsymbol{\partial}^2}\partial_3\partial^k T_{k3} + \frac{(\partial_3)^2}{2\boldsymbol{\partial}^2}\left(\frac{\partial^k\partial^l}{\boldsymbol{\partial}^2}T_{kl} + \frac{(\partial^0)^2}{\boldsymbol{\partial}^2}\right) = 0. \quad (5.2.113)$$

By setting $i = b$, $j = 3$, where $b = 1, 2$, into Eq. (5.2.111) we obtain

$$-\Box h_{b3} = T_{b3} - \frac{1}{2}(\delta_{b3} - \frac{\partial_b\partial_3}{\boldsymbol{\partial}^2})(T + \frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00})$$
$$- \frac{1}{\boldsymbol{\partial}^2}[\partial_b\partial_k T_{k3} + \partial_3\partial^k T_{kb} - \frac{1}{2}(\delta_{b3} + \frac{\partial_b\partial_3}{\boldsymbol{\partial}^3})\partial^k\partial^l T_{kl}] = 0. \quad (5.2.114)$$

Thus

$$\Box h_{b3} + T_{b3} - \frac{1}{2}(\delta_{b3} - \frac{\partial_b\partial_3}{\boldsymbol{\partial}^2})(T + \frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00})$$
$$- \frac{1}{\boldsymbol{\partial}^2}[\partial_b\partial^k T_{k3} + \partial_3\partial^k T_{kb} - \frac{1}{2}(\delta_{b3} + \frac{\partial_b\partial_3}{\boldsymbol{\partial}^3})\partial^k\partial^l T_{kl}] = 0. \quad (5.2.115)$$

Expanding the third term which is the product of 2 rounded brackets, Eq. (5.2.115) leads to

$$\Box h_{b3} + T_{b3} - \frac{1}{2}(\delta_{b3} - \frac{\partial_b\partial_3}{\boldsymbol{\partial}^2})T - \frac{1}{2}(\delta_{b3} - \frac{\partial_b\partial_3}{\boldsymbol{\partial}^2})(\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00})$$

$$-\frac{1}{\boldsymbol{\partial}^2}\left[\partial_b\partial^k T_{k3}+\partial_3\partial^k T_{kb}-\frac{\partial_b\partial_3}{2}\partial^k\partial^l T_{kl}\right]+\frac{1}{2\boldsymbol{\partial}^2}\delta_{b3}\partial^k\partial^l$$

$$=0, \tag{5.2.116}$$

or

$$\Box h_{b3}+T_{b3}-\frac{1}{\boldsymbol{\partial}^2}\left[\partial_3\partial^k T_{kb}+\partial_b\partial^k T_{k3}-\frac{\partial_b\partial_3}{2}\left(\frac{\partial^k\partial^l}{\boldsymbol{\partial}^2}T_{kl}+\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}+T\right)\right]$$

$$-\frac{1}{2}\delta_{b3}T-\frac{1}{2}\delta_{b3}\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}+\frac{1}{2\boldsymbol{\partial}^2}\delta_{b3}\partial^k\partial^l T_{kl}$$

$$=0. \tag{5.2.117}$$

Let's consider the second line in Eq. (5.2.117) to get

$$-\frac{1}{2}\delta_{b3}T-\frac{1}{2}\delta_{b3}\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}+\frac{1}{2\boldsymbol{\partial}^2}\delta_{b3}\partial^k\partial^l T_{kl}$$

$$=-\frac{1}{2}\delta_{b3}T+\frac{1}{2\boldsymbol{\partial}^2}\delta_{b3}(\partial^k\partial^l T_{kl}-\partial^0\partial^0 T_{00})$$

$$=-\frac{1}{2}\delta_{b3}T+\frac{1}{2\boldsymbol{\partial}^2}\delta_{b3}[\boldsymbol{\partial}^2 T]$$

$$=-\frac{1}{2}\delta_{b3}T+\frac{1}{2}\delta_{b3}T$$

$$=0. \tag{5.2.118}$$

We see from Eq. (5.2.118) and Eq. (5.2.117) lead to obtain the equation of Eq. (5.2.109)

By setting $i=a$, $j=b$, $a,b=1,2$, in Eq. (5.2.111), we obtain

$$-\Box h_{ab}=T_{ab}-\frac{1}{2}(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2})(T+\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00})$$

$$-\frac{1}{\boldsymbol{\partial}^2}\left[\partial_a\partial^k T_{kb}+\partial_b\partial^k T_{ka}-\frac{1}{2}(\delta_{ab}+\frac{\partial_a\partial^b}{\boldsymbol{\partial}^2})\partial^k a\partial^l T_{kl}\right], \qquad (5.2.119)$$

$$\Box h_{ab}+T_{ab}-\frac{1}{2}(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2})(T+\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00})$$
$$-\frac{1}{\boldsymbol{\partial}^2}\left[\partial_a\partial^k T_{kb}+\partial_b\partial^k T_{ka}-\frac{1}{2}(\delta_{ab}+\frac{\partial_a\partial^b}{\boldsymbol{\partial}^2})\partial^k a\partial^l T_{kl}\right]$$
$$=0. \qquad (5.2.120)$$

By expansion of all terms in Eq. (5.2.120) we obtain

$$(\Box h_{ab}+T_{ab})-\frac{1}{2}\left(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\right)T-\frac{1}{2}(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2})\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}$$
$$+\frac{\delta_{ab}}{2\boldsymbol{\partial}^2}(\partial^k\partial^l T_{kl})+\frac{1}{2\boldsymbol{\partial}^2}\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\partial^k\partial^l T_{kl}-\frac{1}{\boldsymbol{\partial}^2}(\partial_a\partial^k T_{ib}T_{kb}+\partial_b\partial^k T_{ka})$$
$$=0, \qquad (5.2.121)$$

then it leads to the equation in Eq. (5.2.110)

$$(\Box h_{ab}+T_{ab})-\frac{1}{2}\left(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\right)T-\frac{1}{\boldsymbol{\partial}^2}(\partial_a\partial^i T_{ib}+\partial_b\partial^i T_{ia})$$
$$+\frac{\delta_{ab}}{2\boldsymbol{\partial}^2}\left(-\partial^0\partial^0 T_{00}+\partial^i\partial^j T_{ij}\right)+\frac{\partial_a\partial_b}{2(\boldsymbol{\partial}^2)^2}\left(\partial^i\partial^j T_{ij}+\partial^0\partial^0 T_{00}\right)$$
$$=0. \qquad (5.2.122)$$

Equations (5.2.111), (5.2.95), (5.2.85) give the equations for the various components of $h_{\mu\nu}$. To obtain the unifying equation for $h_{\mu\nu}$, we note that we may write

$$h^{\mu\nu}=\eta^{\mu i}h_{ij}\eta^{j\nu}+\eta^{\mu i}h_{i0}\eta^{0\nu}+\eta^{\mu 0}h_{0j}\eta^{j\nu}+\eta^{\mu 0}h_{00}\eta^{0\nu}, \qquad (5.2.123)$$

with $i, j = 1, 2, 3; \mu,\ \nu = 0, 1, 2, 3$, and use, in the process, the identity

$$\eta^{\mu i}\partial_i = (\partial^\mu + N^\mu \partial_0), \tag{5.2.124}$$

where N^μ is the unit time-like vector ($N^\mu N_\mu = -1$)

$$(N^\mu) = (\eta^\mu{}_0) = (1, 0, 0, 0). \tag{5.2.125}$$

This proof of Eqs. (5.2.123) and (5.2.124) follows, we first use the identity

$$\begin{aligned} h^{\mu\nu} &= \eta^{\mu\sigma} h_{\sigma\lambda} \eta^{\lambda\nu} \\ &= \eta^{\mu 0} h_{0\lambda} \eta^{\lambda\nu} + \eta^{\mu i} h_{i\lambda} \eta^{\lambda\nu} \\ &= \eta^{\mu 0} h_{00} \eta^{0\nu} + \eta^{\mu 0} h_{0i} \eta^{i\nu} + \eta^{\mu i} h_{i0} \eta^{0\nu} + \eta^{\mu i} h_{ij} \eta^{j\nu}, \end{aligned} \tag{5.2.126}$$

which verifies Eq. (5.2.123). On the other hand

$$\begin{aligned} \eta^{\mu i}\partial_i &= \eta^{\mu 0}\partial_0 + \eta^{\mu i}\partial_i - \eta^{\mu 0}\partial_0 \\ &= \eta^{\mu\nu}\partial_\nu + N^\mu \partial_0 \\ &= \partial_\mu + N^\mu \partial_0, \end{aligned} \tag{5.2.127}$$

which also verifies Eq. (5.2.124).

Finally, we use the identity relating a tensor $A_{\lambda\sigma}$, e.g., to the components A_{ij} as follows:

$$\eta^{\mu i} A_{ij} \eta^{j\nu} = \left[\eta^{\mu\lambda}\eta^{\nu\sigma} + N^\mu N^\lambda \eta^{\sigma\nu} + N^\nu N^\sigma \eta^{\lambda\mu} + N^\mu N^\nu N^\lambda N^\sigma\right] A_{\lambda\sigma}, \tag{5.2.128}$$

and the fact that $\Box = \boldsymbol{\partial}^2 - \partial^{0^2}$. To prove Eq. (5.2.128), we note that

$$\begin{aligned}
\eta^{\mu i} A_{ij} \eta^{j\nu} &= (\eta^{\mu i} A_{ij} + \eta^{\mu 0} A_{0j}) \eta^{j\nu} - \eta^{\mu 0} A_{0j} \eta^{j\nu} \\
&= \eta^{\mu\sigma} A_{\sigma j} \eta^{j\nu} + N^{\mu} A_{0j} \eta^{j\nu} \\
&= \eta^{\mu\sigma} (A_{\sigma j} \eta^{j\nu} + A_{\sigma 0} \eta^{0\nu}) - \eta^{\mu\sigma} A_{\sigma 0} \eta^{0\nu} + N^{\mu} (A_{0j} \eta^{j\nu} + A_{00} \eta^{\lambda 0}) - N^{\mu} A_{00} \eta^{0\nu} \\
&= \eta^{\mu\sigma} A_{\sigma\lambda} \eta^{\lambda\nu} + \eta^{\mu\sigma} A_{\sigma 0} N^{\nu} + N^{\mu} A_{0\lambda} \eta^{\lambda\nu} + N^{\mu} A_{00} N^{\nu} \\
&= \left[\eta^{\mu\sigma} \eta^{\nu\lambda} - \eta^{\mu\sigma} \eta^{0\lambda} N^{\nu} - N^{\mu} \eta^{0\sigma} \eta^{\lambda\nu} + N^{\mu} \eta^{0\lambda} \eta^{0\sigma} N^{\nu} \right] A_{\sigma\lambda} \\
&= \left[\eta^{\mu\sigma} \eta^{\nu\lambda} + N^{\lambda} N^{\nu} \eta^{\mu\sigma} + N^{\mu} N^{\sigma} \eta^{\lambda\nu} + N^{\mu} N^{\lambda} N^{\sigma} N^{\nu} \right] A_{\sigma\lambda}. \qquad (5.2.129)
\end{aligned}$$

A lengthy analysis from Eqs. (5.2.85), (5.2.95), (5.2.111) follows which will allows us to combine all the components $h_{\mu\nu}$ and derive the resulting field equation.

In particular, we have

$$-\boldsymbol{\partial}^2 h_{00} = T_{00} + \frac{T}{2} - \frac{1}{2\boldsymbol{\partial}^2} (\partial^0 \partial^0 T_{00} + \partial_i \partial_j T_{ij}), \qquad (5.2.130)$$

thus

$$\begin{aligned}
h_{00} &= -\frac{1}{\boldsymbol{\partial}^2} \left[T_{00} + \frac{T}{2} - \frac{1}{2\boldsymbol{\partial}^2} (\partial^0 \partial^0 T_{00} + \partial_i \partial_j T_{ij}) \right] \\
&= -\frac{1}{\boldsymbol{\partial}^2} \left[\eta_{0\sigma} \eta_{0\lambda} T^{\sigma\lambda} + \frac{1}{2} \eta_{\sigma\lambda} T^{\sigma\lambda} - \frac{1}{2\boldsymbol{\partial}^2} (\partial^0 \partial^0 \eta_{0\sigma\eta_{0\lambda}} + \partial_i \partial_j \eta_{i\sigma} \eta_{j\lambda}) T^{\sigma\lambda} \right] \\
&= -\frac{1}{\boldsymbol{\partial}^2} \left[N_{\sigma} N_{\lambda} + \frac{1}{2} \eta_{\sigma\lambda} - \frac{1}{2\boldsymbol{\partial}^2} ((\partial^0)^2 N_{\sigma} N_{\lambda} + \partial_i \partial_j \eta_{i\sigma} \eta_{j\lambda}) \right] T^{\sigma\lambda} \\
&= -\frac{1}{\boldsymbol{\partial}^2} \left[N_{\sigma} N_{\lambda} + \frac{1}{2} \eta_{\sigma\lambda} - \frac{1}{2\boldsymbol{\partial}^2} ((\boldsymbol{\partial}^2 - \Box) N_{\sigma} N_{\lambda} \right.
\end{aligned}$$

$$+ (\partial^\sigma + N^\sigma \partial_0)(\partial^\lambda + N^\lambda \partial_0))\Big] T^{\sigma\lambda}$$

$$= -\frac{1}{\boldsymbol{\partial}^2}\Big[N_\sigma N_\lambda + \frac{1}{2}\eta_{\sigma\lambda} - \frac{1}{2}N_\sigma N_\lambda + \frac{\Box}{2\boldsymbol{\partial}^2}N_\sigma N_\lambda$$

$$-\frac{1}{2\boldsymbol{\partial}^2}(\partial^\sigma\partial^\lambda + N^\lambda\partial^\sigma\partial_0 + N^\sigma\partial^\lambda\partial_0 + N^\sigma N^\lambda(\partial^0)^2)\Big] T^{\sigma\lambda}$$

$$= -\frac{1}{\boldsymbol{\partial}^2}\Big[\frac{1}{2}N_\sigma N_\lambda + \frac{1}{2}\eta_{\sigma\lambda} + \frac{\Box}{2\boldsymbol{\partial}^2}N_\sigma N_\lambda$$

$$-\frac{1}{2\boldsymbol{\partial}^2}(\partial^\sigma N^\lambda + \partial^\lambda N_\sigma)\partial_0 - \frac{1}{2\boldsymbol{\partial}^2}\partial^\sigma\partial^\lambda - \frac{1}{2\boldsymbol{\partial}^2}N^\sigma N^\lambda(\partial^0)^2\Big] T^{\sigma\lambda}$$

$$= -\frac{1}{\boldsymbol{\partial}^2}\Big[\frac{N_\sigma N_\lambda \Box}{\boldsymbol{\partial}^2} + \frac{1}{2}(\eta_{\sigma\lambda} - \frac{\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2})$$

$$- \frac{1}{2\boldsymbol{\partial}^2}(\partial^\sigma N^\lambda + \partial^\lambda N_\sigma)\partial_0\Big] T^{\sigma\lambda}, \tag{5.2.131}$$

and finally we obtain

$$\eta^{\mu 0} h_{00} \eta^{0\nu} = \frac{N^\mu N^\nu}{-\boldsymbol{\partial}^2}\Big[\frac{N_\sigma N_\lambda \Box}{\boldsymbol{\partial}^2} + \frac{1}{2}(\eta_{\sigma\lambda} - \frac{\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2})$$

$$- \frac{1}{2\boldsymbol{\partial}^2}(\partial^\sigma N^\lambda + \partial^\lambda N_\sigma)\partial_0\Big] T^{\sigma\lambda}. \tag{5.2.132}$$

Next consider the term h_{0i}

$$h_{0i} = -\frac{1}{\boldsymbol{\partial}^2}(T_{0j} - \frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}T_{0i}), \tag{5.2.133}$$

then

$$\eta^{\mu 0} h_{0j} \eta^{j\nu}$$

$$= (\frac{N^\mu}{\boldsymbol{\partial}^2})\left(\eta^{j\nu}T_{j0} - \frac{(\eta^{j\nu}\partial_j)}{\boldsymbol{\partial}^2}\partial_i T_{0i}\right)$$

$$= \frac{N^\mu}{\boldsymbol{\partial}^2}\Bigg[\eta^{\nu\sigma}T_{\sigma 0} - \eta^{\nu 0}T_0 0$$

$$- \frac{(\partial^\nu + N^\nu\partial_0)}{\boldsymbol{\partial}^2}(\partial_i T_{0i} - \partial_0 T_{00} + \partial_0 T_{00})\Bigg]$$

$$= \frac{N^\mu}{\boldsymbol{\partial}^2}\Bigg[-\eta^{\nu\sigma}\eta^{\lambda 0}T_{\sigma\lambda} - \eta^{\nu 0}\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda} - \frac{(\partial^\nu + N^\nu\partial_0)}{\boldsymbol{\partial}^2}(\partial^\sigma T_{0\sigma} + \partial_0 T_{00})\Bigg]$$

$$= \frac{N^\mu}{\boldsymbol{\partial}^2}\Bigg[-\eta^{\nu\sigma}\eta^{\lambda 0}T_{\sigma\lambda} - \eta^{\nu 0}\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda}$$

$$- \frac{(\partial^\nu + N^\nu\partial_0)}{\boldsymbol{\partial}^2}(-\partial^\sigma\eta^{\lambda 0}T_{\sigma\lambda} + \partial_0\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda})\Bigg]$$

$$= \frac{N^\mu}{\boldsymbol{\partial}^2}\Bigg[-\eta^{\nu\sigma}\eta^{\lambda 0} - \eta^{\nu 0}\eta^{\sigma 0}\eta^{\lambda 0}$$

$$- \frac{(\partial^\nu + N^\nu\partial_0)}{\boldsymbol{\partial}^2}(-\partial^\sigma\eta^{\lambda 0} + \partial_0\eta^{\sigma 0}\eta^{\lambda 0})\Bigg]T_{\sigma\lambda}$$

$$= \frac{N^\mu}{\boldsymbol{\partial}^2}\Bigg[\eta^{\nu\sigma}N^\lambda + N^\nu N^\sigma N^\lambda - \frac{(\partial^\nu + N^\nu\partial_0)}{\boldsymbol{\partial}^2}(\partial^\sigma N^\lambda + \partial_0 N^\sigma N^\lambda)\Bigg]T_{\sigma\lambda}$$

$$= \frac{N^\mu}{\boldsymbol{\partial}^2}\Bigg[\eta^{\nu\sigma}N^\lambda + N^\nu N^\sigma N^\lambda - \frac{\partial^\nu}{\boldsymbol{\partial}^2}(\partial^\sigma N^\lambda + \partial_0 N^\sigma N^\lambda)$$

$$+ \frac{N^\nu N^\lambda\partial^\sigma\partial^0}{\boldsymbol{\partial}^2} - \frac{N^\nu N^\sigma N^\lambda \partial^{0^2}}{\boldsymbol{\partial}^2}\Bigg]T_{\sigma\lambda}$$

$$= \frac{N^\mu}{\boldsymbol{\partial}^2}\Bigg[\eta^{\nu\sigma}N^\lambda + N^\nu N^\sigma N^\lambda - \frac{\partial^\nu}{\boldsymbol{\partial}^2}(\partial^\sigma N^\lambda + \partial_0 N^\sigma N^\lambda)$$

$$+ N^\nu N^\sigma N^\lambda\frac{\Box}{\boldsymbol{\partial}^2} - N^\nu N^\sigma N^\lambda - \frac{N^\nu N^\lambda\partial^\sigma\partial^0}{\boldsymbol{\partial}^2}\Bigg]. \qquad (5.2.134)$$

Thus we finally obtain the expression for $\eta^{\mu 0}h_{0j}\eta^{j\nu}$

$$\eta^{\mu 0}h_{0j}\eta^{j\nu} = \frac{N^\mu}{\boldsymbol{\partial}^2}\Bigg[\eta^{\nu\sigma}N^\lambda + (\partial^\nu N^\sigma - N^\nu\partial^\sigma)\frac{\partial^0 N^\lambda}{\boldsymbol{\partial}^2}$$

$$+ \frac{\partial^\nu}{\boldsymbol{\partial}^2}N^\lambda N^\sigma + N^\nu N^\sigma N^\lambda\frac{\Box}{\boldsymbol{\partial}^2}\Bigg]T_{\sigma\lambda}. \qquad (5.2.135)$$

Consider the h_{ij} term

$$-\Box h_{ij} = T_{ij} - \frac{1}{2}(\delta_{ij} - \frac{\partial_i\partial_j}{\boldsymbol{\partial}^2})(T + \frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00})$$

$$- \frac{1}{\boldsymbol{\partial}^2}\Big[\partial_i\partial^k T_{kj} + \partial_j\partial^k T_{ki} - \frac{1}{2}(\delta_{ij} + \frac{\partial_i\partial_j}{\boldsymbol{\partial}^2})\partial^k\partial^l T_{kl}\Big]. \qquad (5.2.136)$$

This allows us to find $\eta^{\mu i}h_{ij}\eta^{j\nu}$ as follows. Let's consider term by term,

$$[E_1] = -\frac{1}{\Box}\eta^{\mu i}\eta^{j\nu}T_{ij}$$

$$= -\frac{1}{\Box}\eta^{\mu i}\Big[\eta^{\lambda}\nu T_{i\lambda} - \eta^{0\nu}T_{i0}\Big]$$

$$= -\frac{1}{\Box}\Big[\eta^{\lambda\nu}\eta^{\mu i}T_{i\lambda} - \eta^{0\nu}\eta^{\mu i}T_{i0}\Big]$$

$$= -\frac{1}{\Box}\Big[\eta^{\lambda\nu}(\eta^{\mu\sigma}T_{\sigma\lambda} - \eta^{\mu 0}T_{0\lambda}) - \eta^{0\nu}(\eta^{\mu\sigma}T_{\sigma 0} - \eta^{\mu 0}T_{00})\Big]$$

$$= -\frac{1}{\Box}\Big[\eta^{\lambda\nu}\eta^{\mu\sigma}T_{\sigma\lambda} - \eta^{\lambda\nu}\eta^{\mu 0}T_{0\lambda} - \eta^{0\nu}\eta^{\mu\sigma}\eta^{\lambda 0}T_{\sigma\lambda} + \eta^{0\nu}\eta^{\mu 0}\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda}\Big]$$

$$= -\frac{1}{\Box}\Big[\eta^{\lambda\nu}\eta^{\mu\sigma} - \eta^{\lambda\nu}\eta^{\mu 0} - \eta^{0\nu}\eta^{\mu\sigma}\eta^{\lambda 0} + \eta^{0\nu}\eta^{\mu 0}\eta^{\sigma 0}\eta^{\lambda 0}\Big]T_{\sigma\lambda}, \qquad (5.2.137)$$

$$[E_2] = -\frac{1}{\Box}\eta^{\mu i}\eta^{j\nu}\Big[-\frac{1}{2}\delta_{ij}T\Big]$$

$$= \frac{1}{2\Box}\eta^{\mu i}\eta^{j\nu}\delta_{ij}\eta^{\sigma\lambda}T_{\sigma\lambda}$$

$$= \frac{1}{2\Box}\eta^{\mu j}\eta^{j\nu}\eta^{\sigma\lambda}T_{\sigma\lambda}$$

$$= \frac{1}{2\Box}\Big[\eta^{\mu\alpha}\eta^{\alpha\nu} - \eta^{\mu 0}\eta^{0\nu}\Big]\eta^{\sigma\lambda}T_{\sigma\lambda}$$

$$= \frac{1}{2\Box}\Big[\eta^{\mu\alpha}\eta^{\alpha\nu}\eta^{\sigma\lambda} - \eta^{\mu 0}\eta^{0\nu}\eta^{\sigma\lambda}\Big]T_{\sigma\lambda}$$

$$= \frac{1}{2\Box}\Big[\eta^{\mu\alpha}\eta^{\alpha\nu}\eta^{\sigma\lambda} - \eta^{\mu 0}\eta^{\nu 0}\eta^{\sigma\lambda}\Big]T_{\sigma\lambda}$$

$$= \frac{1}{2\Box}\Big[\eta^{\mu\alpha}\eta^{\alpha\nu}\eta^{\sigma\lambda} - N^{\mu}N^{\nu}\eta^{\sigma\lambda}\Big]T_{\sigma\lambda}, \tag{5.2.138}$$

$$[E_3] = -\frac{1}{\Box}\eta^{\mu i}\eta^{j\nu}\Big[-\frac{1}{2}\delta_{ij}\frac{(\partial^0)^2}{\boldsymbol{\partial}^2}T_{00}\Big]$$

$$= \frac{1}{2\Box}\eta^{\mu i}\eta^{j\nu}\delta_{ij}\frac{(\partial^0)^2}{\boldsymbol{\partial}^2}T_{00}$$

$$= \frac{1}{2\Box}\eta^{\mu j}\eta^{\nu j}\frac{(\partial^0)^2}{\boldsymbol{\partial}^2}T_{00}$$

$$= \frac{1}{2\Box}\Big[\eta^{\mu\alpha}\eta^{\alpha\nu} - \eta^{\mu 0}\eta^{\nu 0}\Big]\frac{(\partial^0)^2}{\boldsymbol{\partial}^2}\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda}$$

$$= \frac{1}{2\Box}\frac{(\partial^0)^2}{\boldsymbol{\partial}^2}\Big[\eta^{\mu\alpha}\eta^{\alpha\nu}\eta^{\sigma 0}\eta^{\lambda 0} - \eta^{\mu 0}\eta^{\nu 0}\eta^{\sigma 0}\eta^{\lambda 0}\Big]T_{\sigma\lambda}$$

$$= \frac{1}{2\Box}\frac{(\partial^0)^2}{\boldsymbol{\partial}^2}\Big[\eta^{\mu\alpha}\eta^{\alpha\nu}N^{\sigma}N^{\lambda} - N^{\mu}N^{\nu}N^{\sigma}N^{\lambda}\Big]T_{\sigma\lambda}, \tag{5.2.139}$$

$$[E_4] = -\frac{1}{\Box}\eta^{\mu i}\eta^{j\nu}\Big[-\frac{1}{2}\Big(\frac{-\partial_i\partial_j}{\boldsymbol{\partial}^2}\Big)T\Big]$$

$$= -\frac{1}{2\Box\boldsymbol{\partial}^2}\eta^{\mu i}\eta^{j\nu}\partial_i\partial_j\eta^{\sigma\lambda}T_{\sigma\lambda}$$

$$= -\frac{1}{2\Box\boldsymbol{\partial}^2}\eta^{\mu i}\partial_i\eta^{j\nu}\partial_j\eta^{\sigma\lambda}T_{\sigma\lambda}$$

$$= -\frac{1}{2\Box\boldsymbol{\partial}^2}(\partial^{\mu} + N^{\mu}\partial_0)(\partial^{\nu} + N^{\nu}\partial_0)\eta^{\sigma\lambda}T_{\sigma\lambda}$$

$$= -\frac{1}{2\Box\boldsymbol{\partial}^2}\Big[\partial^{\mu}\partial^{\nu} + \partial^{\mu}N^{\nu}\partial_0 + N^{\mu}\partial_0\partial^{\nu} + N^{\mu}N^{\nu}(\partial_0)^2\Big]\eta^{\sigma\lambda}T_{\sigma\lambda}$$

$$= -\frac{1}{2\Box\boldsymbol{\partial}^2}\Big[\partial^{\mu}\partial^{\nu}\eta^{\sigma\lambda} + \partial^{\mu}\partial_0\eta^{\sigma\lambda}N^{\nu}$$

$$+ \partial_0\partial^{\nu}\eta^{\sigma\lambda}N^{\mu} + (\partial_0)^2\eta^{\sigma\lambda}N^{\mu}N^{\nu}\Big]T_{\sigma\lambda}, \tag{5.2.140}$$

$$
\begin{aligned}
[E_5] &= -\frac{1}{\Box}\eta^{\mu i}\eta^{j\nu}\Big[-\frac{1}{2}\Big(-\frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}\Big)\frac{(\partial^0)^2}{\boldsymbol{\partial}^2}T_{00}\Big] \\
&= -\frac{1}{2\Box(\boldsymbol{\partial}^2)^2}\eta^{\mu i}\eta^{j\nu}\partial_i\partial_j(\partial^0)^2T_{00} \\
&= -\frac{1}{2\Box(\boldsymbol{\partial}^2)^2}\eta^{\mu i}\partial_i\eta^{j\nu}\partial_j(\partial^0)^2T_{00} \\
&= -\frac{1}{2\Box(\boldsymbol{\partial}^2)^2}(\partial^\mu + N^\mu\partial_0)(\partial^\nu + N^\nu\partial_0)(\partial^0)^2\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda} \\
&= -\frac{1}{2\Box(\boldsymbol{\partial}^2)^2}\Big[\partial^\mu\partial^\nu + \partial^\mu N^\nu\partial_0 + N^\mu\partial_0\partial^\nu \\
&\qquad + N^\mu\partial_0 N^\nu\partial_0\Big](\partial^0)^2\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda} \\
&= -\frac{T_{\sigma\lambda}(\partial^0)^2}{2\Box(\boldsymbol{\partial}^2)^2}\Big[\partial^\mu\partial^\nu\eta^{\sigma 0}\eta^{\lambda 0} + \partial^\mu\partial_0\eta^{\sigma 0}\eta^{\lambda 0}N^\nu \\
&\qquad + \partial_0\partial^\nu\eta^{\sigma 0}\eta^{\lambda 0}N^\mu + (\partial_0)^2\eta^{\sigma 0}\eta^{\lambda 0}N^\mu N^\nu\Big] \\
&= -\frac{T_{\sigma\lambda}}{2\Box}\Big(\frac{\partial^0}{\boldsymbol{\partial}^2}\Big)^2\Big[\partial^\mu\partial^\nu N^\sigma N^\lambda + \partial^\mu\partial_0 N^\sigma N^\lambda N^\nu \\
&\qquad + \partial_0\partial^\nu N^\sigma N^\lambda N^\mu + (\partial_0)^2 N^\sigma N^\lambda N^\mu N^\nu\Big],
\end{aligned} \tag{5.2.141}
$$

and from Eq. (5.2.136), we let

$$
[E_6] = -\frac{\eta^{\mu i}\eta^{j\nu}}{\boldsymbol{\partial}^2}\Big[\partial_i\partial^k T_{kj} + \partial_j\partial^k T_{ki} - \frac{1}{2}(\delta_{ij} + \frac{\partial_i\partial_j}{\boldsymbol{\partial}^2})\partial^k\partial^l T_{kl}\Big], \tag{5.2.142}
$$

and

$$
\begin{aligned}
[E_{6.1}] &= -\frac{\eta^{\mu i}\eta^{j\nu}}{\boldsymbol{\partial}^2}\partial_i\partial^k T_{kj} \\
&= -\frac{\partial^k}{\boldsymbol{\partial}^2}\eta^{\mu i}\partial_i\eta^{j\nu}T_{kl}
\end{aligned}
$$

$$= -\frac{\partial^k}{\boldsymbol{\partial}^2}(\partial^\mu + N^\mu\partial_0)\Big[\eta^{\nu\lambda}T_{k\lambda} - \eta^{\nu 0}T_{k0}\Big]$$

$$= -\frac{(\partial^\mu + N^\mu\partial_0)}{\boldsymbol{\partial}^2}\Big[\partial^k\eta^{\nu\lambda}T_{k\lambda} - \partial^k\eta^{\nu 0}T_{k0}\Big]$$

$$= -\frac{(\partial^\mu + N^\mu\partial_0)}{\boldsymbol{\partial}^2}\Big[\eta^{\nu\lambda}(\partial^\sigma T_{\sigma\lambda} - \partial^0 T_{0\lambda}) - \eta^{\nu 0}(\partial^\sigma T_{\sigma 0} - \partial^0 T_{00})\Big]$$

$$= -\frac{(\partial^\mu + N^\mu\partial_0)}{\boldsymbol{\partial}^2}\Big[\eta^{\nu\lambda}\partial^\sigma T_{\sigma\lambda} - \eta^{\nu\lambda}\partial^0\eta^{\sigma 0}T_{\sigma\lambda} - \eta^{\nu 0}\partial^\sigma\eta^{\lambda 0}T_{\sigma\lambda}$$

$$+ \eta^{\nu 0}\partial^0\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda}\Big]$$

$$= -\frac{(\partial^\mu + N^\mu\partial_0)}{\boldsymbol{\partial}^2}\Big[\eta^{\nu\lambda}\partial^\sigma + \eta^{\nu\lambda}\partial^0 N^\sigma - \partial^\sigma N^\nu N^\lambda - \partial^0 N^\nu N^\sigma N^\lambda\Big]T_{\sigma\lambda}$$

$$= -\frac{1}{\boldsymbol{\partial}^2}\Big[\eta^{\nu\lambda}\partial^\mu\partial^\sigma + \eta^{\nu\lambda}\partial^0\partial^\mu N^\sigma - \partial^\mu\partial^\sigma N^\nu N^\lambda$$

$$- \partial^0\partial^\mu N^\nu N^\sigma N^\lambda + \eta^{\nu\lambda}\partial_0\partial^\sigma N^\mu - \eta^{\nu\lambda}(\partial^0)^2 N^\mu N^\sigma$$

$$- \partial_0\partial^\sigma N^\mu N^\nu N^\lambda + (\partial^0)^2 N^\mu N^\nu N^\sigma N^\lambda\Big]T_{\sigma\lambda}, \qquad (5.2.143)$$

$$[E_{6.2}] = \frac{\eta^{\mu i}\eta^{j\nu}}{-\boldsymbol{\partial}^2}(\partial_j\partial^k T_{ki})$$

$$= \frac{\eta^{\nu j}\partial_j}{-\boldsymbol{\partial}^2}\partial^k\eta^{\mu i}T_{ki}$$

$$= \frac{(\partial^\nu + N^\nu\partial_0)}{-\boldsymbol{\partial}^2}\partial^k(\eta^{\mu\lambda}T_{k\lambda} - \eta^{\mu 0}T_{k0})$$

$$= \frac{(\partial^\nu + N^\nu\partial_0)}{-\boldsymbol{\partial}^2}(\eta^{\mu\lambda}\partial^k T_{k\lambda} - \eta^{\mu 0}\partial^k T_{k0})$$

$$= \frac{(\partial^\nu + N^\nu\partial_0)}{-\boldsymbol{\partial}^2}\Big[\eta^{\mu\lambda}(\partial^\sigma T_{\sigma\lambda} - \partial^0 T_{0\lambda}) - \eta^{\mu 0}(\partial^\sigma T_{\sigma 0} - \partial^0 T_{00})\Big]$$

$$= \frac{(\partial^\nu + N^\nu \partial_0)}{-\boldsymbol{\partial}^2}\Big[\eta^{\mu\lambda}\partial^\sigma T_{\sigma\lambda} - \eta^{\mu\lambda}\partial^0\eta^{\sigma 0}T_{\sigma\lambda} - \eta^{\mu 0}\partial^\sigma\eta^{\lambda 0}T_{\sigma\lambda} + \eta^{\mu 0}\partial^0\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda}\Big]$$

$$= \frac{(\partial^\nu + N^\nu \partial_0)}{-\boldsymbol{\partial}^2}\Big[\eta^{\mu\lambda}\partial^\sigma + \eta^{\mu\lambda}\partial^0 N^\sigma - \partial^\sigma N^\mu N^\lambda - \partial_0 N^\mu N^\sigma N^\lambda\Big]T_{\sigma\lambda}$$

$$= -\frac{1}{\boldsymbol{\partial}^2}\Big[\partial^\nu\partial^\sigma\eta^{\mu\lambda} + \partial^0\partial^\nu\eta^{\mu\lambda}N^\sigma - \partial^\nu\partial^\sigma N^\mu N^\lambda - \partial^0\partial^\nu N^\mu N^\sigma N^\lambda$$

$$+ \partial_0\partial^\sigma\eta^{\mu\lambda}N^\nu - (\partial^0)^2\eta^{\mu\lambda}N^\nu N^\sigma - \partial_0\partial^\sigma N^\mu N^\nu N^\lambda + (\partial^0)^2 N^\mu N^\nu N^\sigma N^\lambda\Big], \tag{5.2.144}$$

$$[E_{6.3}] = \frac{\eta^{\mu i}\eta^{j\nu}}{-\boldsymbol{\partial}^2}(-\frac{1}{2}\delta_{ij}\partial^k\partial^l T_{kl})$$

$$= \frac{\eta^{\mu i}\eta^{j\nu}}{2\boldsymbol{\partial}^2}\partial^k\partial^l T_{kl}$$

$$= \frac{(\eta^{\mu\lambda}\eta^{\lambda\nu} - \eta^{\mu 0}\eta^{0\nu})}{2\boldsymbol{\partial}^2}\partial^k(\partial^\lambda T_{k\lambda} - \partial^0 T_{k0})$$

$$= \frac{(\eta^{\mu\lambda}\eta^{\lambda\nu} - \eta^{\mu 0}\eta^{0\nu})}{2\boldsymbol{\partial}^2}(\partial^\lambda\partial^k T_{k\lambda} - \partial^0\partial^k T_{k0})$$

$$= \frac{(\eta^{\mu\lambda}\eta^{\lambda\nu} - \eta^{\mu 0}\eta^{0\nu})}{2\boldsymbol{\partial}^2}\Big[\partial^\lambda(\partial^\sigma T_{\sigma\lambda} - \partial^0 T_{0\lambda}) - \partial^0(\partial^\sigma T_{\sigma 0} - \partial^0 T_{00})\Big]$$

$$= \frac{(\eta^{\mu\lambda}\eta^{\lambda\nu} - \eta^{\mu 0}\eta^{0\nu})}{2\boldsymbol{\partial}^2}\Big[\partial^\lambda T_{\sigma\lambda} - \partial^\lambda\partial^0\eta^{\sigma 0}T_{\sigma\lambda} - \partial^0\partial^\sigma\eta^{\lambda 0}T_{\sigma\lambda} + (\partial^0)^2\eta^{\sigma 0}\eta^{\lambda 0}T_{\sigma\lambda}\Big]$$

$$= \frac{(\eta^{\mu\lambda}\eta^{\lambda\nu} - \eta^{\mu 0}\eta^{0\nu})}{2\boldsymbol{\partial}^2}\Big[\partial^\lambda\partial^\sigma + \partial^\lambda\partial^0 N^\sigma + \partial^0\partial^\sigma N^\lambda + (\partial^0)^2 N^\sigma N^\lambda\Big]T_{\sigma\lambda}$$

$$= \frac{1}{2\boldsymbol{\partial}^2}\Big[\eta^{\mu\lambda}\eta^{\lambda\nu}\partial^\lambda\partial^\sigma + \eta^{\mu\lambda}\eta^{\lambda\nu}\partial^\lambda\partial^0 N^\sigma + \eta^{\mu\lambda}\eta^{\lambda\nu}\partial^0\partial^\sigma N^\lambda + \eta^{\mu\lambda}\eta^{\lambda\nu}(\partial^0)^2 N^\sigma N^\lambda$$

$$- \partial^\lambda\partial^\sigma N^\mu N^\nu - \partial^\lambda\partial^0 N^\mu N^\nu N^\sigma - \partial^0\partial^\sigma N^\mu N^\nu N^\lambda - (\partial^0)^2 N^\mu N^\nu N^\sigma N^\lambda\Big]T_{\sigma\lambda}, \tag{5.2.145}$$

$$[E_{6.4}] = \frac{\eta^{\mu i}\eta^{j\nu}}{-\boldsymbol{\partial}^2}(-\frac{1}{2}\frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}\partial^k\partial^l T_{kl})$$

$$= \frac{\eta^{\mu i}\partial_i\eta^{\nu j}\partial_j}{2(\boldsymbol{\partial}^2)^2}\partial^k\partial^l T_{kl}$$

$$= \frac{(\partial^\mu + N^\mu\partial_0)(\partial^\nu + N^\nu\partial_0)}{2(\boldsymbol{\partial}^2)^2}\partial^k(\partial^\lambda T_{k\lambda} - \partial^0 T_{k0})$$

$$= \frac{(\partial^\mu\partial^\nu + \partial^\mu\partial^0 N^\nu + \partial_0\partial^\nu N^\mu + \partial_0^2 N^\mu N^\nu)}{2(\boldsymbol{\partial}^2)^2}(\partial^\lambda\partial^k T_{k\lambda} - \partial^0\partial^k T_{k0})$$

$$= \frac{(\partial^\mu\partial^\nu + \partial^\mu\partial^0 N^\nu + \partial_0\partial^\nu N^\mu + \partial_0^2 N^\mu N^\nu)}{2(\boldsymbol{\partial}^2)^2}\Big[\partial^\lambda(\partial^\sigma T_{\sigma\lambda} - \partial^0 T_{0\lambda})$$

$$- \partial^0(\partial^\sigma T_{\sigma 0} - \partial^0 T_{00})\Big]$$

$$= \frac{(\partial^\mu\partial^\nu + \partial^\mu\partial^0 N^\nu + \partial_0\partial^\nu N^\mu + \partial_0^2 N^\mu N^\nu)}{2(\boldsymbol{\partial}^2)^2}\Big[\partial^\lambda\partial^\sigma - \partial^\lambda\partial^0\eta^{\sigma 0}$$

$$- \partial^0\partial^\sigma\eta^{\lambda 0} + (\partial^0)^2\eta^{\sigma 0}\eta^{\lambda 0}\Big]T_{\sigma\lambda}$$

$$= \frac{1}{2(\boldsymbol{\partial}^2)^2}\Big[\partial^\mu\partial^\nu\partial^\sigma\partial^\lambda + \partial^\mu\partial^\nu\partial^0\partial^\lambda N^\sigma + \partial^\mu\partial^\nu\partial^0\partial^\sigma N^\lambda + (\partial^0)^2\partial^\mu\partial^\nu N^\sigma N^\lambda$$

$$+ \partial^\mu\partial_0\partial^\sigma N^\lambda N^\nu + (\partial^0)^2\partial^\mu\partial^\lambda N^\nu N^\sigma - (\partial^0)^2\partial^\mu\partial^\sigma N^\nu N^\lambda + (\partial^0)^2\partial^\mu\partial_0 N^\nu N^\sigma N^\lambda$$

$$+ \partial_0\partial^\nu\partial^\sigma\partial^\lambda N^\mu - (\partial^0)^2\partial^\nu\partial^\lambda N^\mu N^\sigma - (\partial^0)^2\partial^\nu\partial^\sigma N^\mu N^\lambda + (\partial^0)^2\partial_0\partial^\nu N^\mu N^\sigma N^\lambda$$

$$+ (\partial^0)^2\partial^\sigma\partial^\lambda N^\mu N^\nu - (\partial^0)^2\partial^0\partial^\lambda N^\mu N^\nu N^\sigma$$

$$+ (\partial^0)^2\partial^0\partial^\sigma N^\mu N^\nu N^\lambda + (\partial^0)^4 N^\mu N^\nu N^\sigma N^\lambda\Big], \tag{5.2.146}$$

The previous $\eta^{(..)}h_{(..)}\eta^{(..)}$ could be written as following from Eq. (5.2.123)

$$h_{\mu\nu} = [E_1] + [E_2] + [E_3] + [E_4] + [E_5] - \frac{1}{\Box}\{[E_{6.1}] + [E_{6.2}] + [E_{6.3}] + [E_{6.4}]\}. \tag{5.2.147}$$

The following explicit expression for $h_{\mu\nu}$ is obtained, by simplifying the previous equation:

$$\begin{aligned}
&\langle 0_+ | h^{\mu\nu} | 0_- \rangle \\
&= \frac{1}{(-\Box - \mathrm{i}\epsilon)} \Bigg\{ \frac{\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\sigma\lambda}}{2} \\
&+ \frac{1}{2\boldsymbol{\partial}^2}\Big[\eta^{\mu\nu}\partial^\sigma\partial^\lambda + \eta^{\sigma\lambda}\partial^\mu\partial^\nu - \eta^{\nu\sigma}\partial^\mu\partial^\lambda - \eta^{\nu\lambda}\partial^\mu\partial^\sigma \\
&- \eta^{\mu\sigma}\partial^\nu\partial^\lambda - \eta^{\mu\lambda}\partial^\nu\partial^\sigma + \frac{\partial^\mu\partial^\nu\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2}\Big] \\
&+ \frac{1}{2}\left(\eta^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{\boldsymbol{\partial}^2}\right)\left(\frac{N^\sigma\partial^\lambda + N^\lambda\partial^\sigma}{\boldsymbol{\partial}^2}\right)\partial_0 + \frac{1}{2}\left(\eta^{\sigma\lambda} + \frac{\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2}\right)\left(\frac{N^\nu\partial^\mu + N^\mu\partial^\nu}{\boldsymbol{\partial}^2}\right)\partial_0 \\
&\qquad - \frac{1}{2}\Big[\eta^{\nu\sigma}(N^\mu\partial^\lambda + N^\lambda\partial^\mu) + \eta^{\nu\lambda}(N^\mu\partial^\sigma + N^\sigma\partial^\mu) \\
&\qquad + \eta^{\mu\sigma}(N^\nu\partial^\lambda + N^\lambda\partial^\nu) + \eta^{\mu\lambda}(N^\nu\partial^\sigma + N^\sigma\partial^\nu)\Big]\frac{\partial_0}{\boldsymbol{\partial}^2} \\
&\qquad\quad + \frac{\partial^\mu\partial^\nu}{\boldsymbol{\partial}^2}N^\sigma N^\lambda + \frac{\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2}N^\mu N^\nu\Bigg\} T_{\sigma\lambda}\langle 0_+ | 0_- \rangle \\
&\qquad + \frac{1}{\boldsymbol{\partial}^2}\left\{\frac{\partial^\mu\partial^\nu}{\boldsymbol{\partial}^2}N^\sigma N^\lambda + \frac{\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2}N^\mu N^\nu\right\} T_{\sigma\lambda}\langle 0_+ | 0_- \rangle,
\end{aligned} \tag{5.2.148}$$

$\epsilon \to +0$, where we have taken the matrix element of the field $h_{\mu\nu}$ between the vacuum states $|0_+\rangle$, $|0_-\rangle$.

From Eq. (5.2.148) the explicit expression for the graviton propagator

$\Delta_+^{\mu\nu;\sigma\lambda}(x,x')$ emerges as,

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \int \frac{(\mathrm{d}k)}{(2\pi)^4} \mathrm{e}^{\mathrm{i}k(x-x')} \left[\frac{\Delta_1^{\mu\nu;\sigma\lambda}(k)}{k^2 - \mathrm{i}\epsilon} + \frac{\Delta_2^{\mu\nu;\sigma\lambda}(k)}{\mathbf{k}^2} \right], \tag{5.2.149}$$

$\epsilon \to +0$, where

$$(\mathrm{d}k) = \mathrm{d}k^0 \mathrm{d}k^1 \mathrm{d}k^2 \mathrm{d}k^3, \qquad \text{and} \quad k^2 = \mathbf{k}^2 - k^{0^2}, \tag{5.2.150}$$

and

$$\begin{aligned}
\Delta_1^{\mu\nu;\lambda\sigma}(k) = {} & \frac{(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\sigma\lambda})}{2} \\
& + \frac{1}{2\mathbf{k}^2}\Big[\eta^{\mu\nu}k^\sigma k^\lambda + \eta^{\sigma\lambda}k^\mu k^\nu - \eta^{\nu\sigma}k^\mu k^\lambda - \eta^{\nu\lambda}k^\mu k^\sigma \\
& - \eta^{\mu\sigma}k^\nu k^\lambda - \eta^{\mu\lambda}k^\nu k^\sigma + \frac{k^\mu k^\nu k^\sigma k^\lambda}{\mathbf{k}^2}\Big] \\
& - \frac{1}{2}\left(\eta^{\mu\nu} + \frac{k^\mu k^\nu}{\mathbf{k}^2}\right)\left(\frac{N^\sigma k^\lambda + N^\lambda k^\sigma}{\mathbf{k}^2}\right) k^0 \\
& - \frac{1}{2}\left(\eta^{\sigma\lambda} + \frac{k^\sigma k^\lambda}{\mathbf{k}^2}\right)\left(\frac{N^\nu k^\mu + N^\mu k^\nu}{\mathbf{k}^2}\right) k^0 \\
& + \frac{1}{2}\Big[\eta^{\nu\sigma}(N^\mu k^\lambda + N^\lambda k^\mu) + \eta^{\nu\lambda}(N^\mu k^\sigma + N^\sigma k^\mu) \\
& + \eta^{\mu\sigma}(N^\nu k^\lambda + N^\lambda k^\nu) + \eta^{\mu\lambda}(N^\nu k^\sigma + N^\sigma k^\nu)\Big] \frac{k^0}{\mathbf{k}^2} \\
& + \frac{k^\mu k^\nu}{\mathbf{k}^2} N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{\mathbf{k}^2} N^\mu N^\nu,
\end{aligned} \tag{5.2.151}$$

$$\Delta_2^{\mu\nu;\lambda\sigma}(k) = \frac{k^\mu k^\nu}{\mathbf{k}^2} N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{\mathbf{k}^2} N^\mu N^\nu. \tag{5.2.152}$$

The vacuum-to-vacuum transition amplitude for the gravitational field coupled to an external source is then given by

$$\langle 0_+ | 0_- \rangle^T = \exp\left[\frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}x')T_{\mu\nu}(x)\Delta_+^{\mu\nu;\sigma\lambda}(x,x')T_{\sigma\lambda}(x')\right], \tag{5.2.153}$$

which follows upon using the action principle which states that

$$-\mathrm{i}\frac{\delta}{\delta T_{\mu\nu}(x)}\langle 0_+ | 0_- \rangle^T = \langle 0_+ | h^{\mu\nu}(x) | 0_- \rangle . \tag{5.2.154}$$

The graviton propagator given by the explicit expression in Eqs. (5.2.149) - (5.2.152). Applications of the above results are given in the next chapter. We have derived a novel expression for the graviton propagator, from Lagrangian field theory, valid for the case when the external source $T_{\mu\nu}$ coupled to the gravitational field is not necessarily conserved, by working in a gauge where only two polarization physical states of the graviton arise to ensure positivity in the quantum treatment thus avoiding non-physical states. That such a conservation should *a priori* not to be imposed is a necessary mathematical requirement so that all the ten components of the external source $T_{\mu\nu}$ may be varied independently in order to generate interactions of the gravitational field with matter and produce non-linearity of the gravitational field itself in the functional procedure. The latter requirement arises by noting that such interactions are generated by the application (Manoukian, 1986a; Limboonsong and Manoukian, 2006) of some functional $F[-\mathrm{i}\delta/\delta T_{\mu\nu}]$ to $\langle 0_+ | 0_- \rangle^T$, where $\langle 0_+ | 0_- \rangle$ corresponding to other particles, as well as functional derivatives of their corresponding sources in F, have been suppressed to simplify the notation. Accordingly, to vary the ten components of $T_{\mu\nu}$ independently, no conservation may *a priori* be imposed. The $1/\mathbf{k^2}$ terms in Eqs. (5.2.149) - (5.2.152) are apparent singularities due to the sufficient powers in k in the corresponding denominators and the three-dimensional character of space, in the same way that this happens for the photon propagator in the Coulomb gauge in quantum electrodynamics, and give rise to static $1/r$ type interactions complicated by the tensorial character of a spin two object. It is important to note that for a conserved $T_{\mu\nu}$, i.e., for

$\partial^\mu T_{\mu\nu} = 0$, all the terms in the propagators in Eq. (5.2.149), with the exception of the terms $(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\sigma\lambda})/2$, do not contribute in Eq. (5.2.153) since *all* the other terms in Eqs. (5.2.151), (5.2.152) involve derivatives of $T_{\mu\nu}$ and the graviton propagator $\Delta_+^{\mu\nu;\sigma\lambda}(x, x')$ effectively *goes over* to the well documented expression

$$\frac{1}{(-\Box - \mathrm{i}\epsilon)} \frac{(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\sigma\lambda})}{2}, \tag{5.2.155}$$

which has been known for years (Schwinger, 1970, 1976; Manokian, 1990, 1997). This is unlike the corresponding time-ordered product which does not go over to the result in Eq. (5.2.155) for $\partial^\mu T_{\mu\nu} = 0$. This may be shown by solving for the time-ordered product in Eq. (5.1.7) in terms of the propagator and carrying out explicitly, say, the functional derivatives $\delta h^{0i}/\delta T_{\mu\nu}$, $\delta h^{00}/\delta T_{\mu\nu}$, as arising on the right-hand side of Eq. (5.1.7), by using, in the process, Eqs. (5.2.61), (5.2.62). In any case, it is the propagator $\Delta_+^{\mu\nu;\sigma\lambda}$, as given in Eq. (5.2.149), is the one that appears in the theory and not the time-ordered product as is often naïvely assumed. After all the functional derivatives with respect to $T_{\mu\nu}$ are carried out in the theory, one may impose a conservation law on $T_{\mu\nu}$ or even set $T_{\mu\nu}$ equal to zero if required on physical grounds. Such methods have led to the discovery (Manoukian, 1986a; Limboonsong and Manoukian, 2006), in the functional quantum dynamical principle differential approach, of Faddeev–Popov (FP) factors, and of their generalizations, in non-abelian gauge theories such as in QCD and in other theories.

Re-iterating the discussion above, the relevance of the analysis and the explicit expression derived for the graviton propagator for, *a priori*, not conserved external source $T_{\mu\nu} : \partial^\mu T_{\mu\nu} \neq 0$ is immediate. If, in contrast, a conservation law is *a priori*, imposed then variations with respect to one of the components of $T_{\mu\nu}$ would automatically imply, via such a conservation law, variations with respect some of its *other* components as well. A problem that may arise otherwise, may be readily seen from a simple example. The functional derivative of an expression like $[a_{\mu\nu}(x) + b(x)\partial_\mu\partial_\nu]T^{\mu\nu}(x)$, with respect to a component $T^{\sigma\lambda}(x')$ is $(1/2)[a_{\mu\nu}(x) +$

$b(x)\partial_\mu\partial_\nu](\delta_\sigma{}^\mu\delta_\lambda{}^\nu + \delta_\lambda{}^\mu\delta_\sigma{}^\nu)\delta^4(x,x')$, where $a_{\mu\nu}(x)$, $b(x)$, for example, depend on x, and not $(1/2)a_{\mu\nu}(x)(\delta_\sigma{}^\mu\delta_\lambda{}^\nu + \delta_\lambda{}^\mu\delta_\sigma{}^\nu)\delta^4(x,x')$ as one may naïvely assume by, *a priori* imposing a conservation law. Also, as mentioned above, the present method, based on the functional differential treatment, as applied to non-abelian gauge theories such as QCD (Manoukian, 1986a; Limboonsong and Manoukian, 2006) leads automatically to the presence of the FP determinant modifying naïve Feynman rules. The *physical* relevance of such a factor is important as its omission would lead to a violation of unitarity. For the convenience of the reader we briefly review, before closing the concluding section, on how the FP determinant arises in the functional differential treatment (Manoukian, 1986a; Limboonsong and Manoukian, 2006).

Consider, for simplicity of the demonstration, the non-abelian gauge theory with Lagrangian density

$$\mathscr{L} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + J^\mu_a A^a_\mu \tag{5.2.156}$$

where J^μ_a is an external source taken, *a priori*, not to be conserved. Here

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_o f^{abc} A^b_\mu A^c_\nu \tag{5.2.157}$$

We work in the Coulomb gauge. The gauge field propagator, in analogy to the graviton one in Eqs. (5.2.150), (5.2.151), is given by

$$D^{\mu\nu}_{ab} = \delta_{ab}[g^{\mu\nu} - \frac{(\partial^\mu\partial^\nu + N^\mu\partial^\nu\partial_0 + N^\nu\partial^\mu\partial_0)}{\boldsymbol{\partial}^2}]\frac{1}{-\Box - \mathrm{i}\varepsilon} \tag{5.2.158}$$

with $k = 1, 2, 3$.

The quantum dynamical principle states that

$$\frac{\partial}{\partial g_o}\langle 0_+ \,|\, 0_- \rangle = \mathrm{i}\left\langle 0_+ \left| \int (\mathrm{d}x) \frac{\partial}{\partial g_o}\mathscr{L}(x) \right| 0_- \right\rangle \tag{5.2.159}$$

where, with $k = 1, 2, 3$,

$$\frac{\partial}{\partial g_o}\mathscr{L}(x) = -f^{abc}A_k^b(A_0^c G_a^{k0} + \frac{1}{2}A_l^c G_a^{kl}) \tag{5.2.160}$$

and G_a^{kl} may be expressed in terms of independent fields, that is, for which the canonical conjugate momenta do not vanish. On the other hand, G_a^{k0} depends on the dependent field A_a^0. By using the identity

$$(-\mathrm{i})\frac{\delta}{\delta J_a^\mu(x')}\langle 0_+|\mathscr{O}|0_-\rangle = \left\langle 0_+\left|(A_\mu^a(x')\mathscr{O}(x))_+\right|0_-\right\rangle - \mathrm{i}\left\langle 0_+\left|\frac{\delta}{\delta J_a^\mu(x')}\mathscr{O}(x)\right|0_-\right\rangle \tag{5.2.161}$$

for an operator $\mathscr{O}(x)$, where $(...)_+$ denotes the time-ordered product, and the functional derivative $\delta\mathscr{O}(x)/\delta J_a^\mu(x')$ in the second term on the right-hand side of Eq. (5.2.161) is taken by keeping the independent fields and their canonical conjugate kept fixed in $\mathscr{O}(x)$, after the latter is expressed in terms of these fields, together, possibly, in terms of the dependent fields and the external current (Limboonsong and Manoukian, 2006; Manoukian et al., 2007).

From the Lagrangian density in Eq. (5.2.156), the following relation follows

$$G_a^{k0} = \pi_a^k - \partial^k D_{ab} J_b^0 \tag{5.2.162}$$

as a matrix equation, where π_a^k denotes the canonical conjugate momentum of A_a^k, and D_{ab} is the Green operator satisfying

$$[\delta^{ac}\boldsymbol{\partial}^2 + g_o f^{abc}A_k^b\partial^k]D^{cd}(x, x'; g_o) = \delta^4(x, x')\delta^{ad} \tag{5.2.163}$$

Accordingly, with, *a priori*, non-conserved $J_a^\mu(x')$, we may vary each of its components independently to obtain from Eq. (5.2.162)

$$\frac{\delta}{\delta J_a^\mu(x')}G_a^{k0}(x) = -\delta_\mu{}^0\partial^k D_{ac}(x, x'; g_o) \tag{5.2.164}$$

Hence from Eqs. (5.2.160), (5.2.161), and (5.2.164), we may write

$$\langle 0_+|\frac{\partial}{\partial g_o}\mathscr{L}(x)|0_-\rangle = [(\frac{\partial}{\partial g_o}\mathscr{L})' + \mathrm{i}f^{bca}A'^{b}_{k}\partial^k D'^{ac}(x,x;g_o)]\,\langle 0_+\,|\,0_-\rangle \tag{5.2.165}$$

where the primes mean to replace $A^c_\mu(x)$ in the corresponding expressions by the functional differential operator $(-\mathrm{i})\delta/\delta J^\mu_c(x)$.

Clearly, upon an elementary integration over g_o in Eq. (5.2.159) by using, in the process, Eq. (5.2.165) and the equation for D^{ac} in (5.2.163), we *obtain* the FP determinant

$$\exp \mathrm{Tr}\ln[1 - \mathrm{i}g_o\frac{1}{\boldsymbol{\partial}^2}A'_k\partial^k] \tag{5.2.166}$$

as a multiplicative modifying differential operating factor in $\langle 0_+\,|\,0_-\rangle$. For additional related details see Manoukian (1986a); Limboonsong and Manoukian (2006) and also for further generalizations of the occurrence of such factors in field theory.

It is interesting to extend such analyses (Manoukian, 1986a; Limboonsong and Manoukian, 2006), as well as of gauge transformations (Manoukian, 1986a), and covariance (Manoukian, 1987b), to theories involving gravity. This would be exponentially much harder to do and will be attempted in further investigations. In this regard, our ultimate interest is in aspects of renormalizability (Manoukian, 1983) and rules for physical applications that would follow from our, *a priori*, systematic analysis carried out at the outset, in a quantum setting with the newly modified propagator, by a functional *differential* treatment, in the presence of external sources, to generate non-linearities in gravitation and interactions with matter.

5.3 On the Detection of Gravitational Waves

Gravitations are the particles associated with the gravitational field in the same way as photons are associated with the electromagnetic field. One of the most fascinating predictions of relativity theory is the emission of radiation of massive objects in accelerating motion, known as gravitational radiation, describing a wave motion in the curvature of space-time known as "gravitational waves".

The detection of gravitational waves in a ground-base has not yet been succeeded, but Hulse and Taylor (Weisberge, Taylor and Fowler, 1981; Jeffries et al., 1987; Hewish, 1968) have convinced us that this type of radiation actually exists. This is because the orbiting period of the pulsar (PSR1913 + 16) around its companion (binary system) gradually diminished with time. For their work, Hulse and Taylor were awarded the Nobel Prize in 1993. The general theory of relativity has predicted as a result of the emission of gravitational waves. When a heavy star has used up all its nuclear fuel, it is destroyed in the emergence of a supernova explosion. A small star may be left and its gravity is so strong that causes electrons fall down into protons in the atomic nuclei thus forming neutrons. A neutron star has a very high density even if it is only about 20 km in diameter, and it weighs at least as much as the Sun in our solar system. The pulsar is a rapidly rotating neutron star with a strong magnetic field and the radio waves are emitted at their magnetic poles. Gravitational radiation is the last fundamental

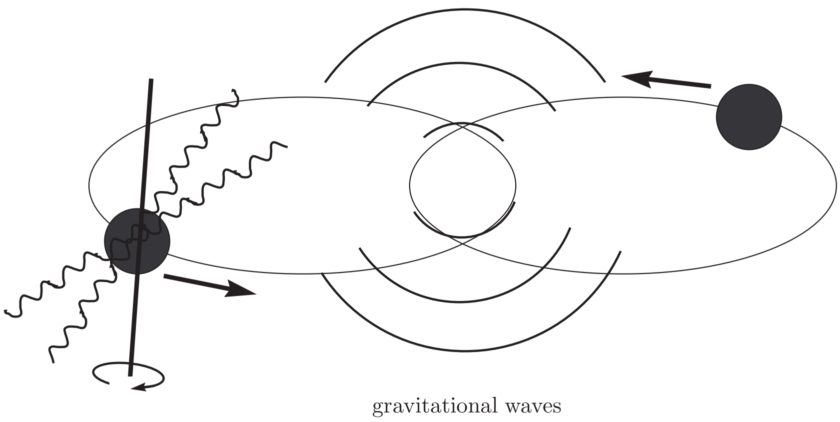

Figure 5.1 Hulse-Taylor binary pulsar (PSR1913+16).

prediction of Einstein's general relativity that has not yet been directly verified.

The problem for the experimental physicist is that the predicted magnitudes of the strains in space caused by gravitational waves are of the order of 10^{-21} or lower. Indeed, present theoretical models suggest that in order to detect a few events per year from coalescing neutron star binary systems for example a sensitivity close to 10^{-22} is required.

The small signal levels mean that limiting noise sources resulting from the thermal motion of molecules in the detector (thermal noise), from seismic or other mechanical disturbances, and from noise associated with the detector readout, whether electronic or optical, must be reduced to a very low level.

For signals above $\sim 10Hz$ ground-based experiments are possible, but for lower frequencies where local fluctuating gravitational gradients and seismic noise on Earth become a problem, it is best to consider developing detectors to be used in space.

Initial detectors and their development in the earliest experiments in the field were ground based and were carried out by Joseph Weber of the University of Maryland about 30 years ago. Having looked for evidence of excitation of the normal modes of the Earth by very low-frequency gravitational waves, Weber then moved on to look for tidal strains in aluminum bars which were at room temperature and were well isolated from ground vibrations and acoustic noise in the laboratory. The bars were resonant at 1600 Hz, a frequency where the energy spectrum of the signals from collapsing stars was predicted to peak. Despite the fact that Weber observed coincident excitations of his detectors placed up to 1000 km apart, at a rate of approximately one event per day, his results were not substantiated by similar experiments carried out in several other laboratories in the USA, Germany, Britain and Russia. It seems unlikely that Weber was observing gravitational wave signals because, although his detectors were very sensitive, being able to detect strains of around 10^{-15} over millisecond timescales, their sensitivity was far away from what was predicted to be required theoretically.

Development of Weber bar type detectors has continued with the emphasis being on cooling to reduce the noise levels, and currently systems at the Universities of Rome, Padua, Louisiana and Perth (Western Australia) are achieving sensitivity levels better than 10^{-18} for millisecond pulses. Bar detectors have a disadvantage, however, of being sensitive only to signals that have significant spectral energy in a narrow band around their resonant frequency.

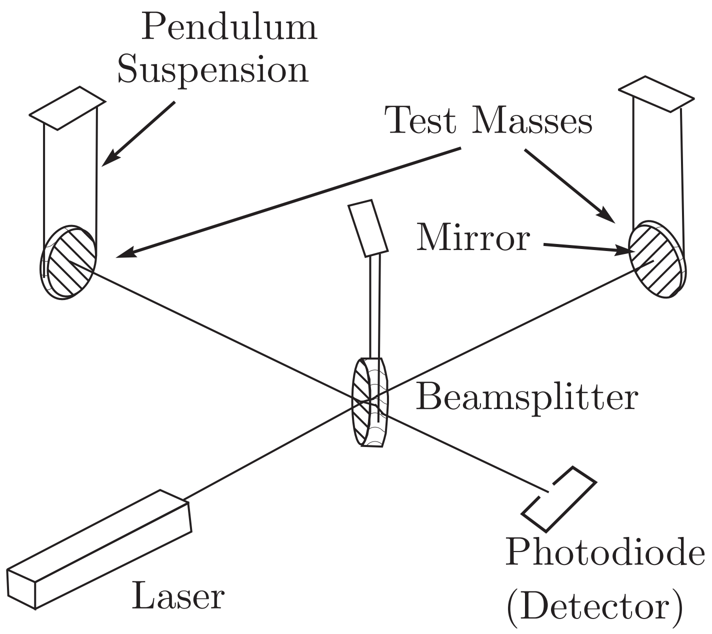

Figure 5.2 Schematic of gravitational wave detector using laser interferometry.

An alternative design of gravitational wave detector offers the possibility of very high sensitivities over a wide range of frequency. This uses test masses placed a long distance apart and freely suspended as pendulums to isolate against seismic noise and reduce the effects of thermal noise; laser interferometry provides a means of sensing the motion of the masses produced as they interact with a gravitational wave. This technique is based on the Michelson interferometer and is particularly suited to the detection of gravitational waves as they have a quadrupole nature. Waves propagating perpendicular to the plane of the interferometer will result in one arm of the interferometer being increased in length while the other arm is decreased and vice versa. Gravitational wave strengths are characterized by the gravitational wave amplitude h, the measure of the strain in space induced by a gravitational wave.

The induced change in the length of the interferometer arms results in a small change in the interference pattern of the light observed at the interferometer output. A typical design specification to allow a reasonable probability for detecting sources requires a noise floor in strain smaller than $2\times10^{-23}Hz^{-1/2}$ to be achieved. The distance between test masses possible on Earth is limited to a few km by geographical and cost factors. If we assume an arm length of 3 km the above specification sets the requirement that the residual motion of each test mass is smaller than $3 \times 10^{-20} mHz^{-1/2}$ over the operating range of the detector, which may be from $3 \sim 10$ Hz to a few kHz. It requires that the optical detection system at the output of the interferometer must be good enough to detect such small changing of motions.

CHAPTER VI

GRAVITONS, INDUCED GEOMETRY AND EXPECTATION VALUE FORMALISM AT FINITE TEMPERATURE

6.1 Introduction

The present chapter is of central importance in the entire thesis. It deals with the novel expression for the graviton propagator derived at length in the previous chapter and its role in the expectation value formalism which is the subject matter of investigation of the present thesis. This analysis will set up a careful formalism to confront the challenges of quantum gravity which has certainly been problematic over the years since the successful treatments of the other interactions in physics involved with the electromagnetic (QED) interaction, the electro-weak (Salam-Weinberg) interaction, the strong (QCD) and more general grand unified field theories embracing these interactions. The most important points to note in this chapter are the following :

(i). As always, the external energy-momentum tensor $T_{\mu\nu}$ coupled to the gravitational field is *a priori* taken to be not conserved so that variations with respect to its ten components may be carried out independently. Only after all the functional differentiations with respect to $T_{\mu\nu}$ are carried out a conservation law for $T_{\mu\nu}$ may be imposed.

(ii). A careful treatment of the gauge problem is considered which ensures that the Riemann curverture tensor of the underlying theory is gauge invariant and covariant in spite of the fact that we work in a Coulomb-like gauge for the gravitational field which in turn ensures only two degrees of polarization for the graviton.

(iii). With the quantization of the gravitational field carried out in the Coulomb-like gauge and *a priori not* conserved energy-momentum tensor, *positivity* of the underlying theory via the vacuum persistence probability $|\langle 0_+ \,|\, 0_- \rangle|^2 \leq 1$ is proved.

(iv). The relationship between the nomenclatures "closed-time-path" and "expectation value formalism" is emphasized as a process which begins with the vacuum state $|0_-\rangle$ and ends up in the *initial* vacuum state $|0_-\rangle$ with *a priori* different sets of external energy-momentum tensors $T^1_{\mu\nu}$, $T^2_{\mu\nu}$. Only after all the relevant functional differentiations with respect to the external source, say, $T^1_{\mu\nu}$ are carried out to generate expectation value, we set them to be equal.

(v). A detailed analysis is carried out to develop the formalism at non-zero temperatures via the well known Boltzmann factor in thermodynamics.

(vi). Demonstration as to show that the Minkowski metric $\eta_{\mu\nu}$ is modified to a general metric in a Riemannian geometry with a new metric $g_{\mu\nu} \neq \eta_{\mu\nu}$ in the presence of a source $T_{\mu\nu}$ of the gravitational field.

(vii). Explicit evaluation of the energy-momentum tensor for a given closed string and investigation of the modification of the underlying geometry from that of a flat Minkowski space.

(viii). Investigations of the structure of the metric at observation points away from the string and show, in particular, how time slows down due to the presence of the source of gravitation as a deformation not only of configuation space but of time itself with an explicit evaluation of such a time slowing *factor*.

(ix). The emphasis of the dependence of geometry on temperature due to radiative corrections and the initial presence of a background of gravitons as extra sources of gravitation due to the non-linearities of the structure of gravitation – as the gravitons themselves carry energies and hence are sources of gravitational field.

The graviton propagator (Manoukian, 1990, 1997, 2005, 2007; Schwinger, 1976 and Sivaram, 1999) plays a central role in the quantum field theory treatment of gravitation. It mediates the gravitational interaction between all particles to the leading order in the gravitational coupling constant. It is well known that in the functional *differential* formalism of quantum field theory, pioneered by Schwinger (1951), functional derivatives (e.g., Schwinger, 1951; Manoukian, 1986b; Limboonsong and Manoukian, 2006; Manoukian, Sukkhasena and Siranan, 2007) are taken of the so-called vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle$ with respect to external sources, via the application, in the process, of the quantum dynamical (action) principle (e.g., Manoukian, 1986b; Manoukian, 2006; Manoukian, Sukkhasena and Siranan, 2007) to generate non-linearities (interactions) in the theory and *n*-point functions leading finally to transition amplitudes for various physical processes. [For a recent modern and a detailed derivation of the quantum dynamical principle see Manoukian, Sukkhasena and Siranan (2007)]. For higher spin fields such as the electromagnetic vector potential A^μ, the gluon field A^μ_a, and, of course, the gravitational field $h^{\mu\nu}$, the respective external sources J_μ, J^a_μ, $T_{\mu\nu}$, coupled to these fields, cannot *a priori* taken to be conserved so that their respective components may be varied *independently* in the functional differentiations process. A problem that may arise otherwise, may be readily seen from a simple example given in Manoukian (2007): The functional derivative of an expression like

$$\Big[a_{\mu\nu}(x) + b(x)\partial_\mu \partial_\nu\Big] T^{\mu\nu}(x), \tag{6.1.1}$$

with respect to $T^{\sigma\lambda}(x')$ is

$$(1/2)\Big[a_{\mu\nu}(x) + b(x)\partial_\mu \partial_\nu\Big]\Big(\delta_\sigma{}^\mu \delta_\lambda{}^\nu + \delta_\lambda{}^\mu \delta_\sigma{}^\nu\Big)\delta^4(x, x'), \tag{6.1.2}$$

where $a_{\mu\nu}(x), b(x)$, for example, depend on *x*, and *not*

$$(1/2)a_{\mu\nu}(x)\Big(\delta_\sigma{}^\mu \delta_\lambda{}^\nu + \delta_\lambda{}^\mu \delta_\sigma{}^\nu\Big)\delta^4(x, x'), \tag{6.1.3}$$

as one may naïvely assume by, *a priori*, imposing a conservation law on $T^{\mu\nu}(x)$ *prior* to functional differentiation. The consequences of relaxing the conservation of the such external sources are highly non-trivial. For one thing the corresponding field propagators become modified. Also they have led to the rediscovery (Manoukian, 1986b; Limboonsong and Manoukian, 2006) of Faddeev-Popov (FP), (Faddeev and Popov, 1967)-like factors in non-abelian gauge theories (Manoukian , 1986b; Limboonsong and Manoukian, 2006) and the discovery of even further *generalizations* (Limboonsong and Manoukian, 2006) of such factors, directly from the functional *differential* treatment, via the application of the quantum dynamical principle (Manoukian, Sukkhasena and Siranan 2007), in the presence of external sources, without making an appeal to path integrals, without using symmetry arguments which may be broken, and without even going into the well known complicated structures of the underlying Hamiltonians. An account of this procedure, which is also pedagogical, was given in the concluding section of Manoukian and Sukkhasena (2007b) for the convenience of the reader and needs not to be repeated.

For higher spin fields, the propagator and time-ordered product of two fields do not, in general, coincide as the former includes so-called Schwinger terms which, in general, lead to a simplification for the propagator over the time-ordered one. This is well known for spin 1 and is also true for the graviton propagator (Manoukian and Sukkhasena, 2007b). Let $h^{\mu\nu}$ denote the gravitational field. We work in a gauge

$$\partial_i h^{i\nu} = 0, \tag{6.1.4}$$

where $i = 1, 2, 3;\ \nu = 0, 1, 2, 3$, which, as established in Sect. 6.3, guarantees that only two states of polarization occur for the graviton even with a non-conserved external source $T_{\mu\nu}$ in the theory.

If we denote the vacuum-to-vacuum transition amplitude for the interaction of gravitons with the external source $T_{\mu\nu}$ by $\langle 0_+ \,|\, 0_- \rangle^T$, then the propagator of the gravitational field is defined by

$$\Delta_{+}^{\mu\nu;\sigma\lambda}(x,x') = \mathrm{i}\left((-\mathrm{i})\frac{\delta}{\delta T_{\mu\nu}(x)}(-\mathrm{i})\frac{\delta}{\delta T_{\sigma\lambda}(x')}\,\langle 0_+\,|\,0_-\rangle^T\right)\Big/\,\langle 0_+\,|\,0_-\rangle^T\,, \tag{6.1.5}$$

in the limit of the vanishing of the external source $T_{\mu\nu}$. In more detail we may rewrite Eq. (6.1.5) as

$$\Delta_{+}^{\mu\nu;\sigma\lambda}(x,x') = \mathrm{i}\frac{\left\langle 0_+\left|\left(h^{\mu\nu}(x)h^{\sigma\lambda}(x')\right)_+\right|0_-\right\rangle^T}{\langle 0_+\,|\,0_-\rangle^T} + \frac{\left\langle 0_+\left|\dfrac{\delta}{\delta T_{\mu\nu}(x)}h^{\sigma\lambda}(x')\right|0_-\right\rangle^T}{\langle 0_+\,|\,0_-\rangle^T}, \tag{6.1.6}$$

in the limit of vanishing $T_{\mu\nu}$, where the first term on the right-hand side, up to the i factor, denotes the time-ordered product. In the second term, the functional derivative with respect to $T_{\mu\nu}(x)$ is taken by keeping the independent field components of $h^{\sigma\lambda}(x')$ fixed. The dependent field components depend on the external source and lead to extra terms on the right-hand side of Eq. (6.1.6) in addition to the time-ordered product and may be referred to as Schwinger terms. A detailed derivation of the general identity in Eq. (6.1.6) is given in Manoukian, Sukkhasena and Siranan (2007) (see also Manoukian, 2006a)). It is the propagator $\Delta_{+}^{\mu\nu;\sigma\lambda}$ that appears in this formalism and not the time-ordered product. The propagator $\Delta_{+}^{\mu\nu;\sigma\lambda}(x,x')$ has been derived in Manoukian and Sukkhasena (2007b) and will be elaborated upon in Sect.6.2. It includes 30 terms in contrast to the well known one involving only 3 terms when a conservation law of $T_{\mu\nu}$ is imposed. The positivity constraint of the vacuum persistence probability

$$|\,\langle 0_+\,|\,0_-\rangle\,|^2 \leq 1, \tag{6.1.7}$$

as well as the correct spin content of the theory is established in Sect.6.3 for, *a priori*, *non*-conserved external energy-momentum tensor.

The expectation value formalism, pioneered by Schwinger (1961), also known as the closed-time path formalism, in quantum field theory has been a useful tool in performing expectation values without first evaluating transition amplitudes. For a partial list of studies of the expectation value formalism, the reader may refer to Manoukian (1987c, 1988b, 1988d, 1991b) in the functional differential formalism. See also related work in Keldysh (1965); Craig (1968), Hall (1975); Kao, Nayak and Greiner (2002), emphasizing on non-equilibrium phenomenae and Jordan (1986); Calzetta and Hu (1988); Cooper (1998), emphasizing Feynman path integrals.

In order to study gravitational effects such as the induced geometry due to external sources and even due to fluctuating quantum fields, the expectation value formalism turns out to be of practical value. In Sect. 6.5, we develop the expectation value formalism for gravitons interacting with an external energy-momentum tensor $T_{\mu\nu}$ at *finite* temperature with *a priori* not conserved $T_{\mu\nu}$, so that variations with respect to its ten components may be varied independently in order to generate expectation values. *After* all the relevant functional differentiations with respect to $T_{\mu\nu}$ are carried out, the conservation law on $T_{\mu\nu}$ may be then imposed. We establish the covariance of the *induced* Riemann curvature tensor, in the initial vacuum, due to the external source, in spite of the quantization carried out in a gauge which ensures only two polarization states for the graviton. As an application, we investigate the *induced* correction to the metric and the underlying geometry due a closed string arising from the Nambu action (e.g., Kibble and Turok, 1982; Albrecht and Turok, 1989; Sakellariadou, 1990; Goddard et al.,1995), as a solution of a circularly oscillating string (Manoukian, 1991, 1992; Manoukian, Ungkitchanukit and Eab, 1995; Manoukian and Sattayatham, 1998), as, perhaps, the simplest generalization of a limiting point-like object. Finally, it is discussed on why the geometry of spacetime may, in general, depend on temperature due to radiative corrections and its physical significance is emphasized.

The Minkowski metric is denoted by

$$[\eta_{\mu\nu}] = \text{diag}[-1, 1, 1, 1], \tag{6.1.8}$$

and we use units such that $\hbar = 1, c = 1$.

6.2 Graviton Propagator and Vacuum-to-Vacuum Transition Amplitude

The action for the gravitational field $h^{\mu\nu}$ coupled to an external energy-momentum tensor source $T_{\mu\nu}$ is taken to be

$$A = \frac{1}{8\pi G}\int(\mathrm{d}x)\mathscr{L}(x) + \int(\mathrm{d}x)h^{\mu\nu}(x)T_{\mu\nu}(x), \tag{6.2.1}$$

with

$$\mathscr{L} = -\frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h^\sigma{}_\sigma\partial_\alpha h^\beta{}_\beta - \partial^\alpha h_{\alpha\mu}\partial^\mu h^\sigma{}_\sigma$$
$$+\frac{1}{2}\partial_\alpha h^{\alpha\nu}\partial^\beta h_{\beta\nu} + \frac{1}{2}\partial_\alpha h^\mu{}_\nu\partial_\mu h^{\alpha\nu}, \tag{6.2.2}$$

and G is Newton's gravitational constant. The action part $\int(\mathrm{d}x)\mathscr{L}(x)$ is invariant under gauge transformations

$$h^{\mu\nu}(x) \to h^{\mu\nu}(x) + \partial^\mu\xi^\nu(x) + \partial^\nu\xi^\mu(x) + \partial^\mu\partial^\nu\xi(x). \tag{6.2.3}$$

Consider first the $\partial^\mu\partial^\nu\xi$ term. To the above end

$$[1] = -\frac{1}{2}\int(\mathrm{d}x)(\partial^\alpha\partial^\mu\partial^\nu\xi)\partial_\alpha h_{\mu\nu} - \frac{1}{2}\int(\mathrm{d}x)\partial^\alpha h^{\mu\nu}\partial_\alpha\partial_\mu\partial_\nu\xi$$
$$= \frac{1}{2}\int(\mathrm{d}x)\xi\partial^\alpha\partial^\mu\partial^\nu\partial_\alpha h_{\mu\nu} + \frac{1}{2}\int(\mathrm{d}x)[\partial_\alpha\partial_\mu\partial_\nu\partial^\alpha h^{\mu\nu}]\xi$$
$$= \frac{1}{2}\int(\mathrm{d}x)\xi\partial^\mu\partial^\nu\Box h_{\mu\nu} + \frac{1}{2}\int(\mathrm{d}x)\partial_\mu\partial_\nu\Box h^{\mu\nu}\xi$$
$$= \int(\mathrm{d}x)(\partial_\mu\partial_\nu\Box h^{\mu\nu})\xi, \tag{6.2.4}$$

$$[2] = \frac{1}{2}\int(\mathrm{d}x)\partial^\alpha\Box\xi\partial_\alpha h^\beta{}_\beta + \frac{1}{2}\int(\mathrm{d}x)\partial^\alpha h^\sigma{}_\sigma\partial_\alpha\Box\xi$$

$$= -\frac{1}{2}\int(\mathrm{d}x)\Box\xi\Box h^\beta{}_\beta - \frac{1}{2}\int(\mathrm{d}x)\Box h^\sigma{}_\sigma\Box\xi$$

$$= -\frac{1}{2}\int(\mathrm{d}x)\xi\Box\Box h^\beta{}_\beta - \frac{1}{2}\int(\mathrm{d}x)\Box\Box h^\sigma{}_\sigma\xi$$

$$= -\int(\mathrm{d}x)\xi(\Box\Box h^\beta{}_\beta), \tag{6.2.5}$$

$$[3] = -\int(\mathrm{d}x)\Box\partial_\mu\xi\partial^\mu h^\sigma{}_\sigma - \int(\mathrm{d}x)\partial^\alpha h_{\alpha\mu}\partial^\mu\Box\xi$$

$$= \int(\mathrm{d}x)\xi(\Box\Box h^\sigma{}_\sigma) + \int(\mathrm{d}x)(\Box\partial^\alpha\partial^\mu h_{\alpha\mu})\xi, \tag{6.2.6}$$

$$[4] = \frac{1}{2}\int(\mathrm{d}x)\Box\partial^\nu\xi\partial^\beta h_{\beta\nu} + \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha h^{\alpha\nu}\Box\partial_\nu\xi$$

$$= -\frac{1}{2}\int(\mathrm{d}x)\xi(\Box\partial^\nu\partial^\beta h_{\beta\nu}) + \frac{1}{2}\int(\mathrm{d}x)(\Box\partial_\nu\partial_\alpha h^{\alpha\nu})\xi, \tag{6.2.7}$$

$$[5] = \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha\partial^\mu\partial_\nu\xi\partial_\mu h^{\alpha\nu} + \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha h^\mu{}_\nu\partial_\mu\partial^\alpha\partial^\nu\xi$$

$$= \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha\Box\partial_\nu h^{\alpha\nu}\xi + \frac{1}{2}\int(\mathrm{d}x)\Box h^\mu{}_\nu\partial_\mu\partial^\nu\xi$$

$$= -\frac{1}{2}\int(\mathrm{d}x)(\Box\partial_\nu\partial_\alpha h^{\alpha\nu})\xi + \frac{1}{2}\int(\mathrm{d}x)(\Box\partial_\mu\partial^\nu h^\mu{}_\nu)\xi, \tag{6.2.8}$$

thus adding all terms together to obtain

$$[1] + [2] + [3] + [4] + [5] = \int(\mathrm{d}x)(\partial_\mu\partial_\nu\Box h^{\mu\nu})\xi - \int(\mathrm{d}x)\xi(\Box\Box h^\beta{}_\beta)$$

$$+\int(\mathrm{d}x)\xi(\Box\Box h^{\sigma}{}_{\sigma})+\int(\mathrm{d}x)(\Box\partial^{\alpha}\partial^{\mu}h_{\alpha\mu})\xi$$

$$-\frac{1}{2}\int(\mathrm{d}x)\xi(\Box\partial^{\nu}\partial^{\beta}h_{\beta\nu})+\frac{1}{2}\int(\mathrm{d}x)(\Box\partial_{\nu}\partial_{\alpha}h^{\alpha\nu})\xi$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\Box\partial_{\nu}\partial_{\alpha}h^{\alpha\nu})\xi+\frac{1}{2}\int(\mathrm{d}x)(\Box\partial_{\mu}\partial^{\nu}h^{\mu}{}_{\nu})\xi. \quad (6.2.9)$$

All the terms depending on ξ are cancel out. Next we consider another terms for, i.e., $\partial^{\mu}\xi^{\nu}+\partial^{\nu}\xi^{\mu}$, thus from Eqs. (6.2.2) and (6.2.3) we have

$$[B_1]=-\frac{1}{2}\int(\mathrm{d}x)\partial^{\alpha}(\partial^{\mu}\xi^{\nu}+\partial^{\nu}\xi^{\mu})\partial_{\alpha}h_{\mu\nu}-\frac{1}{2}\int(\mathrm{d}x)\partial^{\alpha}h^{\mu\nu}\partial_{\alpha}(\partial_{\mu}\xi_{\nu}+\partial_{\nu}\xi_{\mu})$$

$$=-\frac{1}{2}\int(\mathrm{d}x)\partial^{\alpha}\partial^{\mu}\xi^{\nu}\partial_{\alpha}h_{\mu\nu}-\frac{1}{2}\int(\mathrm{d}x)\partial^{\alpha}\partial^{\nu}\xi^{\mu}\partial_{\alpha}h_{\mu\nu}$$

$$-\frac{1}{2}\int(\mathrm{d}x)\partial^{\alpha}h^{\mu\nu}\partial_{\alpha}\partial_{\mu}\xi_{\nu}-\frac{1}{2}\int(\mathrm{d}x)\partial^{\alpha}h^{\mu\nu}\partial_{\alpha}\partial_{\nu}\xi_{\mu}$$

$$=-\frac{1}{2}\int(\mathrm{d}x)\xi^{\nu}\partial^{\alpha}\partial^{\mu}\partial_{\alpha}h_{\mu\nu}-\frac{1}{2}\int(\mathrm{d}x)\xi^{\mu}\partial^{alpha}\partial^{\nu}\partial_{\alpha}h_{\mu\nu}$$

$$+\frac{1}{2}\int(\mathrm{d}x)h^{\mu\nu}\partial^{\alpha}\partial_{\alpha}\partial_{\mu}\xi_{\nu}+\frac{1}{2}\int(\mathrm{d}x)h^{\mu\nu}\partial^{\alpha}\partial_{\alpha}\partial_{\nu}\xi_{\mu}$$

$$=-\frac{1}{2}\int(\mathrm{d}x)\xi^{\nu}\partial^{\mu}\partial^{\alpha}\partial_{\alpha}h_{\mu\nu}-\frac{1}{2}\int(\mathrm{d}x)\xi^{\mu}\partial^{\nu}\partial^{\alpha}\partial_{\alpha}h_{\mu\nu}$$

$$+\frac{1}{2}\int(\mathrm{d}x)h^{\mu\nu}\Box\partial_{\mu}\xi_{\nu}+\frac{1}{2}\int(\mathrm{d}x)h^{\mu\nu}\Box\partial_{\nu}\xi_{\mu}$$

$$=-\frac{1}{2}\int(\mathrm{d}x)\xi^{\nu}\partial^{\mu}\Box h_{\mu\nu}-\frac{1}{2}\int(\mathrm{d}x)\xi^{\mu}\partial^{\nu}\Box h_{\mu\nu}$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\Box\partial_{\mu}h^{\mu\nu})\xi_{\nu}-\frac{1}{2}\int(\mathrm{d}x)(\Box\partial_{\nu}h^{\mu\nu})\xi_{\mu}$$

$$=-\frac{1}{2}\int(\mathrm{d}x)\xi^{\nu}\partial^{\mu}\Box h_{\mu\nu}-\frac{1}{2}\int(\mathrm{d}x)\xi^{\nu}\partial^{\mu}\Box h_{\mu\nu}$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\Box\partial_{\nu}h^{\mu\nu})\xi_{\mu}-\frac{1}{2}\int(\mathrm{d}x)(\Box\partial_{\nu}h^{\mu\nu})\xi_{\mu}$$

$$= -\int (\mathrm{d}x)\xi^\mu(\partial^\nu \Box h_{\mu\nu}) - \int (\mathrm{d}x)\Box\partial_\nu h^{\mu\nu}\xi_\mu$$

$$= -\int (\mathrm{d}x)\xi^\mu(\partial^\nu \Box h_{\mu\nu}) - \int (\mathrm{d}x)\xi^\mu(\Box\partial^\nu h_{\mu\nu})$$

$$= -2\int (\mathrm{d}x)\xi_\mu(\Box\partial_\nu h^{\mu\nu}), \tag{6.2.10}$$

$$[B_2] = +\frac{1}{2}\int (\mathrm{d}x)\partial^\alpha(2\partial^\sigma\xi_\sigma)\partial_\alpha h^\beta{}_\beta + \frac{1}{2}\int (\mathrm{d}x)\partial^\alpha h^\sigma{}_\sigma\partial_\alpha(2\partial^\beta\xi_\beta)$$

$$= +\int (\mathrm{d}x)\partial^\alpha(\partial^\sigma\xi_\sigma)\partial_\alpha h^\beta{}_\beta + \int (\mathrm{d}x)\partial^\alpha h^\sigma{}_\sigma\partial_\alpha\partial^\beta\xi_\beta$$

$$= +\int (\mathrm{d}x)\xi_\sigma\partial^\alpha\partial^\sigma\partial_\alpha h^\beta{}_\beta - \int (\mathrm{d}x)h^\sigma{}_\sigma\partial^\alpha\partial_\alpha\partial^\beta\xi_\beta$$

$$= +\int (\mathrm{d}x)\xi_\sigma\partial^\sigma\Box h^\beta{}_\beta - \int (\mathrm{d}x)h^\sigma{}_\sigma\partial^\beta\Box\xi_\beta$$

$$= +\int (\mathrm{d}x)\xi_\sigma\partial^\sigma\Box h^\beta{}_\beta + \int (\mathrm{d}x)\xi_\beta\partial^\beta\Box h^\sigma{}_\sigma$$

$$= 2\int (\mathrm{d}x)\xi_\beta(\partial^\beta\Box h^\sigma{}_\sigma), \tag{6.2.11}$$

$$[B_3] = -\int (\mathrm{d}x)\partial^\alpha(\partial_\alpha\xi_\mu + \partial_\mu\xi_\alpha)\partial^\mu h^\sigma{}_\sigma - \int (\mathrm{d}x)\partial^\alpha h_{\alpha\mu}\partial^\mu(2\partial^\sigma\xi_\sigma)$$

$$= -\int (\mathrm{d}x)\partial^\alpha\partial_\alpha\xi_\mu\partial^\mu h^\sigma{}_\sigma + \int (\mathrm{d}x)\partial_\mu\xi_\alpha\partial^\mu h^\sigma{}_\sigma - \int (\mathrm{d}x)\partial^\alpha h_{\alpha\mu}\partial^\mu(2\partial^\sigma\xi_\sigma)$$

$$= -\int (\mathrm{d}x)\xi_\mu\Box\partial^\mu h^\sigma{}_\sigma - \int (\mathrm{d}x)\xi_\alpha\partial^\alpha\Box h^\sigma{}_\sigma - 2\int (\mathrm{d}x)\partial^\alpha\partial^\mu\partial^\sigma h_{\alpha\mu}\xi_\sigma$$

$$= -\int (\mathrm{d}x)\xi_\mu\partial^\mu\Box h^\sigma{}_\sigma - \int (\mathrm{d}x)\xi_\alpha\partial^\alpha\Box h^\sigma{}_\sigma - \int (\mathrm{d}x)2\xi_\sigma(\partial^\alpha\partial^\mu\partial^\sigma h_{\alpha\mu})$$

$$= -\int (\mathrm{d}x)2\xi_\alpha(\partial^\alpha\Box h^\sigma{}_\sigma) - 2\int (\mathrm{d}x)\xi_\sigma(\partial^\alpha\partial^\mu\partial^\sigma h_{\alpha\mu}), \tag{6.2.12}$$

$$[B_4] = +\frac{1}{2}\int(\mathrm{d}x)\partial_\alpha(\partial^\alpha\xi^\nu + \partial^\nu\xi^\alpha)\partial^\beta h_{\beta\nu} + \frac{1}{2}\int(\mathrm{d}x)\partial^\alpha h^{\alpha\nu}\partial^\beta(\partial_\beta\xi_\nu + \partial_\nu\xi_\beta)$$

$$= +\frac{1}{2}\int(\mathrm{d}x)\partial_\alpha\partial^\alpha\xi^\nu\partial^\beta h_{\beta\nu} + \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha\partial^\nu\xi^\alpha\partial^\beta h_{\beta\nu}$$

$$+ \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha h^{\alpha\nu}\partial^\beta\partial_\beta\xi_\nu + \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha h^{\alpha\nu}\partial^\beta\partial_\nu\xi_\beta$$

$$= +\frac{1}{2}\int(\mathrm{d}x)\xi^\nu\Box\partial^\beta h_{\beta\nu} + \frac{1}{2}\int(\mathrm{d}x)\xi^\alpha\partial_\alpha\partial^\nu\partial^\beta h_{\beta\nu}$$

$$- \frac{1}{2}\int(\mathrm{d}x)h^{\alpha\nu}\partial_\alpha\Box\xi_\nu - \frac{1}{2}\int(\mathrm{d}x)h^{\alpha\nu}\partial_\alpha\partial^\beta\partial_\nu\xi_\beta$$

$$= +\frac{1}{2}\int(\mathrm{d}x)\xi^\nu(\Box\partial^\beta h_{\beta\nu}) + \frac{1}{2}\int(\mathrm{d}x)\xi^\alpha(\partial_\alpha\partial^\nu\partial^\beta h_{\beta\nu})$$

$$+ \frac{1}{2}\int(\mathrm{d}x)\xi_\nu(\partial_\alpha\Box h^{\alpha\nu}) + \frac{1}{2}\int(\mathrm{d}x)\xi_\beta(\partial_\alpha\partial^\beta\partial_\nu h^{\alpha\nu})$$

$$= \int(\mathrm{d}x)\xi_\nu(\Box\partial_\beta h^{\beta\nu}) + \int(\mathrm{d}x)\xi_\alpha(\partial^\alpha\partial_\nu\partial_\beta h^{\beta\nu}), \tag{6.2.13}$$

$$[B_5] = +\frac{1}{2}\int(\mathrm{d}x)\partial_\alpha(\partial^\mu\xi_\nu + \partial_\nu\xi^\mu)\partial_\mu h^{\alpha\nu} + \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha h^\mu{}_\nu\partial_\mu(\partial^\alpha\xi^\nu + \partial^\nu\xi^\alpha)$$

$$= +\frac{1}{2}\int(\mathrm{d}x)\partial_\alpha\partial^\mu\xi_\nu\partial_\mu h^{\alpha\nu} + \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha\partial_\nu\xi^\mu\partial_\mu h^{\alpha\nu}$$

$$+ \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha h^\mu{}_\nu\partial_\mu(\partial^\alpha\xi^\nu) + \frac{1}{2}\int(\mathrm{d}x)\partial_\alpha h^\mu{}_\nu\partial_\mu\partial^\nu\xi^\alpha$$

$$= +\frac{1}{2}\int(\mathrm{d}x)\xi_\nu\partial_\alpha\Box h^{\alpha\nu} + \frac{1}{2}\int(\mathrm{d}x)\xi^\mu\partial_\alpha\partial_\nu\partial_\mu h^{\mu\nu}$$

$$+ \frac{1}{2}\int(\mathrm{d}x)h^\mu{}_\nu\partial_\alpha\partial_\mu\partial^\alpha\xi^\nu + \frac{1}{2}\int(\mathrm{d}x)h^\mu{}_\nu\partial_\alpha\partial_\mu\partial^\nu\xi^\alpha$$

$$= +\frac{1}{2}\int(\mathrm{d}x)\xi_\nu\partial_\alpha\Box h^{\alpha\nu} + \frac{1}{2}\int(\mathrm{d}x)\xi^\mu\partial_\alpha\partial_\nu\partial_\mu h^{\mu\nu}$$

$$+ \frac{1}{2}\int(\mathrm{d}x)h^\mu{}_\nu\partial_\mu\Box\xi^\nu + \frac{1}{2}\int(\mathrm{d}x)\xi^\alpha\partial_\alpha\partial_\mu\partial^\nu h^\mu{}_\nu$$

$$= +\frac{1}{2}\int(\mathrm{d}x)\xi_\nu\partial_\alpha\Box h^{\alpha\nu} + \frac{1}{2}\int(\mathrm{d}x)\xi^\mu\partial_\alpha\partial_\nu\partial_\mu h^{\mu\nu}$$

$$+ \frac{1}{2}\int(\mathrm{d}x)\xi^\nu\partial_\mu\Box h^\mu{}_\nu + \frac{1}{2}\int(\mathrm{d}x)\xi^\alpha\partial_\alpha\partial_\mu\partial^\nu h^\mu{}_\nu$$

$$= \int(\mathrm{d}x)\xi_\nu(\Box\partial_\mu h^{\mu\nu}) + \int(\mathrm{d}x)\xi^\alpha(\partial_\alpha\partial_\mu\partial_\nu h^{\mu\nu}), \tag{6.2.14}$$

Then adding $[B_1]$ to $[B_5]$ together, gives

$$[B_1] + [B_2] + [B_3] + [B_4] + [B_5] \equiv [C_1]$$

$$= -2\int(\mathrm{d}x)\xi_\mu(\Box\partial_\nu h^{\mu\nu}) + 2\int(\mathrm{d}x)\xi_\beta(\partial^\beta\Box h^\sigma{}_\sigma)$$

$$- 2\int(\mathrm{d}x)\xi_\alpha(\partial^\alpha\Box h^\sigma{}_\sigma) - 2\int(\mathrm{d}x)\xi_\sigma(\partial^\alpha\partial^\mu\partial^\sigma h_{\alpha\mu})$$

$$+ \int(\mathrm{d}x)\xi_\nu(\Box\partial_\beta h^{\beta\nu}) + \int(\mathrm{d}x)\xi_\alpha(\partial^\alpha\partial_\nu\partial_\beta h^{\beta\nu})$$

$$+ \int(\mathrm{d}x)\xi_\nu(\Box\partial_\mu h^{\mu\nu}) + \int(\mathrm{d}x)\xi^\alpha(\partial_\alpha\partial_\mu\partial_\nu h^{\mu\nu})$$

$$= -2\int(\mathrm{d}x)\xi_\mu(\Box\partial_\nu h^{\mu\nu}) + 2\int(\mathrm{d}x)\xi_\beta(\partial^\beta\Box h^\sigma{}_\sigma) - 2\int(\mathrm{d}x)\xi_\alpha(\partial^\alpha\Box h^\sigma{}_\sigma)$$

$$+ 2\int(\mathrm{d}x)\xi_\nu(\Box\partial_\mu h^{\mu\nu}) - 2\int(\mathrm{d}x)\xi_\sigma(\partial^\sigma\partial_\alpha\partial_\mu h^{\alpha\mu})$$

$$+ \int(\mathrm{d}x)\xi_\alpha(\partial^\alpha\partial_\nu\partial_\beta h^{\beta\nu}) + \int(\mathrm{d}x)\xi_\alpha(\partial^\alpha\partial_\mu\partial_\nu h^{\mu\nu})$$

$$= -\int(\mathrm{d}x)2\xi_\sigma(\partial^\sigma\partial_\alpha\partial_\mu h^{\alpha\mu}) + \int(\mathrm{d}x)\xi_\alpha(\partial^\alpha\partial_\mu\partial_\nu h^{\mu\nu})$$

$$= 0, \tag{6.2.15}$$

which together with Eq. (6.2.9) and establish the gauge invariance of $\int(\mathrm{d}x)\mathscr{L}(x)$, i.e.,

$$\delta\int(\mathrm{d}x)\mathscr{L}(x) = 0, \tag{6.2.16}$$

corresponding to the gauge transformations in Eq. (6.2.16) for arbitrary infinitesimal change with the factors $\xi^{\mu}(x)$ and $\xi(x)$.

As mentioned above the external energy-momentum tensor $T_{\mu\nu}$ is, *a priori*, taken to be *not* conserved so that variations of its respective ten components may be varied independently - a necessary *technical* requirement. Details on dependent fields due to the gauge constraints are spelled out in Manoukian, Sukkhasena and Siranan (2007) as well as in Manoukian and Sukkhasena (2007b). The vacuum-to-vacuum transition amplitude is then given by Manoukian and Sukkhasena (2007b),

$$\langle 0_+\,|\,0_-\rangle^T = \exp\left[4\pi G\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')T_{\mu\nu}(x)\Delta_+^{\mu\nu;\sigma\lambda}(x,x')T_{\sigma\lambda}(x')\right], \tag{6.2.17}$$

$$(\mathrm{d}x) = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3. \tag{6.2.18}$$

Here we note that the exponent is scaled by the factor $8\pi G$ to satisfy the boundary condition that the gravitational attraction of two widely separated static sources is given by Newton's law (Schwinger, 1976). The graviton propagator $\Delta_+^{\mu\nu;\sigma\lambda}(x,x')$ contains 30 terms and *not* only just the first 3 terms as may be naïvely expected, and is given by

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \int\frac{(\mathrm{d}k)}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-x')}\left[\frac{\Delta_1^{\mu\nu;\sigma\lambda}(k)}{k^2-\mathrm{i}\epsilon}+\frac{\Delta_2^{\mu\nu;\sigma\lambda}(k)}{\mathbf{k}^2}\right], \tag{6.2.19}$$

$\epsilon\to+0,\quad$ where

$$(\mathrm{d}k) = \mathrm{d}k^0\mathrm{d}k^1\mathrm{d}k^2\mathrm{d}k^3, \tag{6.2.20}$$

$$k^2 = \mathbf{k}^2 - k^{0^2}, \tag{6.2.21}$$

and

$$\Delta_1^{\mu\nu;\lambda\sigma}(k) = \frac{(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\sigma\lambda})}{2}$$
$$+\frac{1}{2\mathbf{k}^2}\Big[\eta^{\mu\nu}k^\sigma k^\lambda + \eta^{\sigma\lambda}k^\mu k^\nu - \eta^{\nu\sigma}k^\mu k^\lambda - \eta^{\nu\lambda}k^\mu k^\sigma$$
$$-\eta^{\mu\sigma}k^\nu k^\lambda - \eta^{\mu\lambda}k^\nu k^\sigma + \frac{k^\mu k^\nu k^\sigma k^\lambda}{\mathbf{k}^2}\Big]$$
$$-\frac{1}{2}\left(\eta^{\mu\nu} + \frac{k^\mu k^\nu}{\mathbf{k}^2}\right)\left(\frac{N^\sigma k^\lambda + N^\lambda k^\sigma}{\mathbf{k}^2}\right)k^0$$
$$-\frac{1}{2}\left(\eta^{\sigma\lambda} + \frac{k^\sigma k^\lambda}{\mathbf{k}^2}\right)\left(\frac{N^\nu k^\mu + N^\mu k^\nu}{\mathbf{k}^2}\right)k^0$$
$$+\frac{1}{2}\Big[\eta^{\nu\sigma}(N^\mu k^\lambda + N^\lambda k^\mu) + \eta^{\nu\lambda}(N^\mu k^\sigma + N^\sigma k^\mu)$$
$$+\eta^{\mu\sigma}(N^\nu k^\lambda + N^\lambda k^\nu) + \eta^{\mu\lambda}(N^\nu k^\sigma + N^\sigma k^\nu)\Big]\frac{k^0}{\mathbf{k}^2}$$
$$+\frac{k^\mu k^\nu}{\mathbf{k}^2}N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{\mathbf{k}^2}N^\mu N^\nu, \tag{6.2.22}$$

$$\Delta_2^{\mu\nu;\lambda\sigma}(k) = \frac{k^\mu k^\nu}{\mathbf{k}^2}N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{\mathbf{k}^2}N^\mu N^\nu. \tag{6.2.23}$$

Here

$$(N^\mu) = (\eta^\mu{}_0) = (1, 0, 0, 0), \tag{6.2.24}$$

is a time-like vector. The $\mathrm{i}\epsilon$ factor in Eq. (6.2.19) corresponds to the Schwinger-Feynman boundary condition.

6.3 Positivity Constraint

It is far from obvious that with a *non-conserved* energy-momentum tensor, the vacuum-to-vacuum amplitude $\langle 0_+ \,|\, 0_- \rangle$ in Eq. (6.2.17) satisfies the positivity constraint

$$| \, \langle 0_+ \,|\, 0_- \rangle \, |^2 \leq 1. \tag{6.3.1}$$

The proof of this follows. We rewrite the vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle$ in Eq. (6.2.17) as

$$\langle 0_+ \,|\, 0_- \rangle^T = \exp\left[4\pi G\mathrm{i} \int (\mathrm{d}x) T_{\mu\nu}(x) H^{\mu\nu}(x)\right], \tag{6.3.2}$$

with

$$T_{\mu\nu} H^{\mu\nu} = T_{00} H^{00} + 2T_{0i} H^{0i} + T_{ij} H^{ij}, \tag{6.3.3}$$

$i, j = 1, 2, 3$, and we may infer from Eq. (5.2.95) that

$$H^{00} = -\frac{1}{\boldsymbol{\partial}^2}\left[T^{00} + \frac{T}{2} - \frac{1}{2\boldsymbol{\partial}^2}\left(\partial^0\partial^0 T_{00} + \partial^i\partial^j T_{ij}\right)\right], \tag{6.3.4}$$

$$T = T_{ii} - T_{00}, \tag{6.3.5}$$

and H^{00} is *real*. Also from Eq. (5.2.85), we may infer that

$$H^{0i} = -\frac{1}{\boldsymbol{\partial}^2}\left[\delta^{ij} - \frac{\partial^i\partial^j}{\boldsymbol{\partial}^2}\right] T_{0j}, \tag{6.3.6}$$

which is again *real*. That is,

$$\exp\left[4\pi G\mathrm{i} \int (\mathrm{d}x)\left(T_{00}(x) H^{00}(x) + 2T_{0i}(x) H^{0i}(x)\right)\right], \tag{6.3.7}$$

is a phase factor.

On the other hand, we may infer from Eq. (5.2.111) that

$$H^{ij} = \frac{1}{(-\Box - \mathrm{i}\epsilon)} A^{ij,lm} T_{lm} - \frac{1}{2}\frac{1}{\boldsymbol{\partial}^2}\left(\delta^{ij} - \frac{\partial^i\partial^j}{\boldsymbol{\partial}^2}\right) T_{00}, \tag{6.3.8}$$

and the second term above involving T_{00} is *real*, while $A^{ij,lm}$ is given by

$$\begin{aligned} A^{ij,lm} = & \frac{(\delta^{il}\delta^{jm} + \delta^{im}\delta^{jl} - \delta^{ij}\delta^{lm})}{2} \\ & - \frac{1}{2\boldsymbol{\partial}^2}\Bigg[\partial^i\partial^l\delta^{jm} + \partial^i\partial^m\delta^{jl} + \partial^j\partial^l\delta^{im} + \partial^j\partial^m\delta^{il} \\ & + \partial^i\partial^j\delta^{lm} - \delta^{ij}\partial^l\partial^m - \frac{\partial^i\partial^j\partial^l\partial^m}{\boldsymbol{\partial}^2}\Bigg], \end{aligned} \tag{6.3.9}$$

where $i,\, j,\, l,\, m \,=\, 1,\, 2,\, 3$.

Accordingly, from Eqs. (6.3.2), (6.3.6) - (6.3.7), we may rewrite

$$\langle 0_+ \,|\, 0_- \rangle^T = \mathrm{e}^{\mathrm{i}G[T]} \exp\left[4\pi G \mathrm{i} \int (\mathrm{d}x) T_{ij}(x) \frac{1}{(-\Box - \mathrm{i}\epsilon)} A^{ij,lm} T_{lm}(x)\right], \tag{6.3.10}$$

where $\exp \mathrm{i}G[T]$ is a phase factor, i.e., $\Big|\exp \mathrm{i}G[T]\Big| = 1$.

By using the facts that the reality of $T_{ij}(x)$ implies that

$$T_{ij}(k)^* = T_{ij}(-k), \tag{6.3.11}$$

where

$$(k^\mu) = (k^0, \mathbf{k}), \tag{6.3.12}$$

and the identity

$$\frac{\mathrm{i}}{2}\left(\frac{1}{k^2 - \mathrm{i}\epsilon} - \frac{1}{k^2 + \mathrm{i}\epsilon}\right) = -\pi\delta(k^2)$$

$$= -\frac{\pi}{|\mathbf{k}|}\Big[\delta(k^0 - |\mathbf{k}|) + \delta(k^0 + |\mathbf{k}|)\Big], \tag{6.3.13}$$

for $\epsilon \to +0$, in the sense of distributions, which follows from the property :

$$\frac{1}{k^2 \pm \mathrm{i}\varepsilon} = \mathrm{P}\frac{1}{k^2} \mp \mathrm{i}\pi\delta(k^2), \tag{6.3.14}$$

where as an integral over k^0 for a given function $f(k)$ of k, the principal part P of an integral is defined by:

$$\begin{aligned}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}k^0}{k^2}f(k) = &-\int_{-\infty}^{-|\mathbf{k}|-\varepsilon}\frac{\mathrm{d}k^0}{(k^0-|\mathbf{k}|)(k^0+|\mathbf{k}|)}f(k)\\ &-\int_{-|\mathbf{k}|+\varepsilon}^{|\mathbf{k}|-\varepsilon}\frac{\mathrm{d}k^0}{(k^0-|\mathbf{k}|)(k^0+|\mathbf{k}|)}f(k)\\ &-\int_{|\mathbf{k}|+\varepsilon}^{\infty}\frac{\mathrm{d}k^0}{(k^0-|\mathbf{k}|)(k^0+|\mathbf{k}|)}f(k),\end{aligned} \tag{6.3.15}$$

in the limit $\varepsilon \to +0$. Thus we obtain

$$\begin{aligned}\left(\frac{1}{k^2+\mathrm{i}\varepsilon} - \frac{1}{k^2+\mathrm{i}\varepsilon}\right) &= \mathrm{P}\frac{1}{k^2} - \mathrm{P}\frac{1}{k^2} + \mathrm{i}\pi\delta(k^2) - (-\mathrm{i}\pi\delta(k^2))\\ &= \mathrm{i}\pi\delta(k^2) + \mathrm{i}\pi\delta(k^2)\\ &= 2\mathrm{i}\pi\delta(k^2),\end{aligned} \tag{6.3.16}$$

and

$$\begin{aligned}\frac{\mathrm{i}}{2}\left(\frac{1}{k^2+\mathrm{i}\varepsilon} - \frac{1}{k^2+\mathrm{i}\varepsilon}\right) &= \frac{\mathrm{i}}{2}2\mathrm{i}\pi\delta(k^2)\\ &= -\pi\delta(k^2),\end{aligned} \tag{6.3.17}$$

since $k^2 = \mathbf{k}^2 - k^{0^2}$, this gives

$$\delta(k^2) = \delta(\mathbf{k}^2 - k^{0^2})$$

$$= \delta[(\mathbf{k} - k^0)(\mathbf{k} + k^0)]$$

$$= \frac{\delta(k^0 - |\mathbf{k}|) + \delta(k^0 + |\mathbf{k}|)}{|\mathbf{k}|}. \tag{6.3.18}$$

Finally, this leads to

$$-\pi\delta(k^2) = -\frac{\pi}{|\mathbf{k}|}[\delta(k^0 - |\mathbf{k}|) + \delta(k^0 + |\mathbf{k}|)]. \tag{6.3.19}$$

Accordingly we obtain the equality

$$\left|\langle 0_+ \,|\, 0_- \rangle^T\right|^2 = \exp\left[-8\pi G \int \mathrm{d}\omega_{\mathbf{k}} T^*_{ij}(k) B^{ij,lm}(k) T_{lm}(k)\right], \tag{6.3.20}$$

where now

$$k^0 = +|\mathbf{k}|, \qquad \mathrm{d}\omega_{\mathbf{k}} = \mathrm{d}^3\mathbf{k}/(2\pi)^3 2|\mathbf{k}|, \tag{6.3.21}$$

and

$$B^{ij,lm}(k) = \frac{1}{2}\Big[(\delta^{il} - \frac{k^i k^l}{\mathbf{k}^2})(\delta^{jm} - \frac{k^j k^m}{\mathbf{k}^2}) + (\delta^{im} - \frac{k^i k^m}{\mathbf{k}^2})(\delta^{jl} - \frac{k^j k^l}{\mathbf{k}^2})$$
$$- (\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2})(\delta^{lm} - \frac{k^l k^m}{\mathbf{k}^2})\Big], \tag{6.3.22}$$

with $i,\, j,\, l,\, m \,=\, 1,\, 2,\, 3$ as before.

The verification of the equivalence of the expressions in Eqs. (6.3.22) and (6.3.9) follows explicitly from

$$B^{ij,lm}(k) = \frac{1}{2}\Big[(\delta^{il} - \frac{k^i k^l}{\mathbf{k}^2})(\delta^{jm} - \frac{k^j k^m}{\mathbf{k}^2})$$

$$+ (\delta^{im} - \frac{k^i k^m}{\mathbf{k}^2})(\delta^{jl} - \frac{k^j k^l}{\mathbf{k}^2})$$

$$- (\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2})(\delta^{lm} - \frac{k^l k^m}{\mathbf{k}^2})\Bigg]$$

$$= \frac{1}{2}\Bigg[\delta^{il}\delta^{jm} - \delta^{il}\frac{k^j k^m}{\mathbf{2}^2} - \delta^{jm}\frac{k^i k^l}{\mathbf{k}^2} + \frac{k^i k^l}{\mathbf{k}^2}\frac{k^j k^m}{\mathbf{k}^2}$$

$$+ \delta^{im}\delta^{jl} - \delta^{im}\frac{k^j k^l}{\mathbf{k}^2} - \delta^{jl}\frac{k^i k^m}{\mathbf{k}^2} + \frac{k^i k^m}{\mathbf{k}^2}\frac{k^j k^l}{\mathbf{k}^2} - \delta^{ij}\delta^{lm}$$

$$+ \delta^{ij}\frac{k^l k^m}{\mathbf{k}^2} + \delta^{lm}\frac{k^i k^j}{\mathbf{k}^2} - \frac{k^i k^j}{\mathbf{k}^2}\frac{k^l k^m}{\mathbf{k}^2}\Bigg]. \tag{6.3.23}$$

The fourth and the last term of Eq. (6.3.23) cancel each other, and we have

$$B^{ij,lm}(k) = \frac{\delta^{il}\delta^{jm} + \delta^{im}\delta^{jl} - \delta^{ij}\delta^{lm}}{2}$$

$$- \frac{1}{2\mathbf{k}^2}\Bigg[k^i k^l + k^i k^m \delta^{jl} + k^j k^l \delta^{im} + k^j k^m \delta^{il}$$

$$- k^i k^j \delta^{lm} - k^l k^m \delta^{ij} - \frac{k^i k^j k^l k^m}{\mathbf{k}^2}\Bigg], \tag{6.3.24}$$

which corresponds to Eq. (6.3.9).

6.4 Spin Content

For a given 3-vector $\mathbf{k}$, we introduce two orthonormal complex 3-vectors $\mathbf{e}_+$, $\mathbf{e}_-$,

$$\begin{aligned} &\mathbf{e}_+ \cdot \mathbf{e}_+^* = 1 = \mathbf{e}_- \cdot \mathbf{e}_-^*, \\ &\mathbf{e}_+ \cdot \mathbf{e}_-^* = 0, \end{aligned} \tag{6.4.1}$$

such that $\mathbf{k}/|\mathbf{k}|$, $\mathbf{e}_+$, $\mathbf{e}_-$ constitute three mutually orthonormal vectors. That is, in addition to the conditions in Eq. (6.4.1),

$$\mathbf{k}\cdot\mathbf{e}_+ = 0, \qquad \mathbf{k}\cdot\mathbf{e}_- = 0. \tag{6.4.2}$$

Upon writing

$$\mathbf{k} = |\mathbf{k}|\Big(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta\Big), \tag{6.4.3}$$

we may set

$$\mathbf{e}_+ = \frac{1}{\sqrt{2}}\Big(\cos\phi\cos\theta - \mathrm{i}\sin\phi, \sin\phi\cos\theta + \mathrm{i}\cos\phi, -\sin\theta\Big), \tag{6.4.4}$$

$$\mathbf{e}_- = \frac{1}{\sqrt{2}}\Big(\cos\phi\cos\theta + \mathrm{i}\sin\phi, \sin\phi\cos\theta - \mathrm{i}\cos\phi, -\sin\theta\Big), \tag{6.4.5}$$

and note that

$$\mathbf{e}_- = \mathbf{e}_+^* \quad . \tag{6.4.6}$$

The above orthogonality relations is established below:

$$\mathbf{e}_+\cdot\mathbf{e}_+^* = \frac{1}{2}\Bigg[\Bigg(\cos\phi\cos\theta - \mathrm{i}\sin\phi, \sin\phi\cos\theta + \mathrm{i}\cos\phi, \sin\theta\Bigg)\times$$

$$\Bigg(\cos\phi\cos\theta + \mathrm{i}\sin\phi, \sin\phi\cos\theta - \mathrm{i}\cos\phi, -\sin\theta\Bigg)\Bigg]$$

$$= \frac{1}{2}\Bigg[\cos^2\phi\cos^2\theta + \cos\phi\cos\theta\mathrm{i}\sin\phi - \cos\phi\cos\theta\mathrm{i}\sin\phi$$

$$+\sin^2\phi + \sin^2\phi\cos^2\theta - \sin\phi\cos\theta\mathrm{i}\cos\phi$$

$$+\sin\phi\cos\theta\mathrm{i}\cos\phi+\cos^2\phi+\sin^2\theta\Big]$$

$$=\frac{1}{2}\Big[\cos^2\phi\cos^2\theta+\sin^2\phi\cos^2\theta+\sin^2\phi+\cos^2\phi+\sin^2\theta\Big]$$

$$=\frac{1}{2}\Big[\cos^2\theta(\cos^2\phi+\sin^2\phi)+1+\sin^2\theta\Big]$$

$$=\frac{1}{2}\Big[\cos^2\theta+\sin^2\theta+1\Big]$$

$$=1, \tag{6.4.7}$$

$$\mathbf{e}_+\cdot\mathbf{e}_-^*=\frac{1}{2}\Big[\Big(\cos\phi\cos\theta-\mathrm{i}\sin\phi,\sin\phi\cos\theta+\mathrm{i}\cos\phi,\sin\theta\Big)\times$$

$$\Big(\cos\phi\cos\theta-\mathrm{i}\sin\phi,\sin\phi\cos\theta+\mathrm{i}\cos\phi,\sin\theta\Big)\Big]$$

$$=\frac{1}{2}\Big[\cos^2\phi\cos^2\theta-\cos\phi\cos\theta\mathrm{i}\sin\phi-\cos\phi\cos\theta\mathrm{i}\sin\phi$$

$$-\sin^2\phi+\sin^2\phi\cos^2\theta+\sin\phi\cos\theta\mathrm{i}\cos\phi$$

$$+\sin\phi\cos\theta\mathrm{i}\cos\phi-\cos^2\phi+\sin^2\theta\Big]$$

$$=\frac{1}{2}[\cos^2\theta(\cos^2\phi+\sin^2\phi)+\sin^2\theta-(\cos^2\phi+\sin^2\phi)]$$

$$=\frac{1}{2}[(\cos^2\theta+\sin^2\theta)-(\cos^2\phi+\sin^2\phi)]$$

$$=0, \tag{6.4.8}$$

$$\mathbf{e}_-\cdot\mathbf{e}_-^*=\frac{1}{2}\Big[\Big(\cos\phi\cos\theta+\mathrm{i}\sin\phi,\sin\phi\cos\theta-\mathrm{i}\cos\phi,-\sin\theta\Big)\times$$

$$\left(\cos\phi\cos\theta+\mathrm{i}\sin\phi,\sin\phi\cos\theta-\mathrm{i}\cos\phi,-\sin\theta\right)\Big]$$

$$=\frac{1}{2}\Big[\cos^2\phi\cos^2\theta-\cos\phi\cos\theta\mathrm{i}\sin\phi+\cos\phi\cos\theta\mathrm{i}\sin\phi$$

$$+\sin^2\phi+\cos^2\phi+\sin^2\phi\cos^2\theta+\sin\phi\cos\theta\mathrm{i}\cos\phi$$

$$-\sin\phi\cos\theta\mathrm{i}\cos\phi+\sin^2\theta\Big]$$

$$=\frac{1}{2}\Big[\cos^2\phi\cos^2\theta+\sin^2\phi+\sin^2\phi\cos^2\theta+\cos^2\phi+\sin^2\theta\Big]$$

$$=\frac{1}{2}\Big[\cos^2\theta(\cos^2\phi+\sin^2\phi)+\sin^2\phi+\cos^2\phi+\sin^2\theta\Big]$$

$$=\frac{1}{2}\Big[\cos^2\theta+\sin^2\theta+1\Big]$$

$$=1, \tag{6.4.9}$$

$$\mathbf{k}\cdot\mathbf{e}_+=\frac{|\mathbf{k}|}{\sqrt{2}}\Big[\left(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta\right)\times$$

$$\left(\cos\phi\cos\theta-\mathrm{i}\sin\phi,\sin\phi\cos\theta+\mathrm{i}\cos\phi,\sin\theta\right)\Big]$$

$$=\frac{|\mathbf{k}|}{\sqrt{2}}\Big[\cos^2\phi\sin\theta\cos\theta-\cos\phi\sin\theta\mathrm{i}\sin\phi$$

$$+\sin^2\phi\sin\theta\cos\theta+\sin\phi\sin\theta\mathrm{i}\cos\phi-\cos\theta\sin\theta\Big]$$

$$=\frac{|\mathbf{k}|}{\sqrt{2}}[\sin\theta\cos\theta(\cos^2\phi+\sin^2\phi)-\cos\theta\sin\theta]$$

$$=\frac{|\mathbf{k}|}{\sqrt{2}}(0)$$

$$= 0, \tag{6.4.10}$$

$$\begin{aligned}
\mathbf{k} \cdot \mathbf{e}_- &= \frac{|\mathbf{k}|}{\sqrt{2}} \Bigg[\Bigg(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \Bigg) \times \\
&\qquad \Bigg(\cos\phi\cos\theta + \mathrm{i}\sin\phi, \sin\phi\cos\theta - \mathrm{i}\cos\phi, -\sin\theta \Bigg) \Bigg] \\
&= \frac{|\mathbf{k}|}{\sqrt{2}} \Bigg[\cos^2\phi \sin\theta\cos\theta + \cos\phi\sin\theta \mathrm{i}\sin\phi + \sin^2\phi\sin\theta\cos\theta \\
&\qquad - \sin\phi\sin\theta\mathrm{i}\cos\phi - \cos\theta\sin\theta \Bigg] \\
&= \frac{|\mathbf{k}|}{\sqrt{2}} \Bigg[\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \cos\theta\sin\theta \Bigg] \\
&= \frac{|\mathbf{k}|}{\sqrt{2}}(0) \\
&= 0.
\end{aligned} \tag{6.4.11}$$

The above allows us to introduce the completeness relation

$$\begin{aligned}
\delta^{ij} &= \sum_{\lambda=\pm} e^i_\lambda e^{*j}_\lambda + \frac{k^i k^j}{|\mathbf{k}|^2} \\
&= \sum_{\lambda=\pm} e^{i*}_\lambda e^j_\lambda + \frac{k^i k^j}{|\mathbf{k}|^2}.
\end{aligned} \tag{6.4.12}$$

In turn, we may define polarization 3×3 tensors by

$$e^{ij}_{\lambda\sigma} = \frac{1}{2}\Big[e^i_\lambda e^{j*}_\sigma + e^{i*}_\sigma e^j_\lambda - \delta_{\lambda\sigma} e^i_\alpha e^{j*}_\alpha \Big], \tag{6.4.13}$$

with $\lambda, \sigma, \alpha = \pm$, and a summation over the repeated index α is assumed, and note that after some algebra, $B^{ij,lm}$ in Eq. (6.3.22) may be rewritten as

$$B^{ij,lm} = \sum_{\lambda,\sigma=\pm} e^{ij}_{\lambda\sigma} e^{*lm}_{\lambda\sigma}. \tag{6.4.14}$$

Using, in the process, Eq. (6.4.13), we note that

$$e^{ij}_{++} = 0, \qquad e^{ij}_{--} = 0, \tag{6.4.15}$$

and

$$e^{ij}_{+-} = e^{i}_{+} e^{j}_{+} \equiv \epsilon^{ij}_{+}, \tag{6.4.16}$$

$$e^{ij}_{-+} = e^{i}_{-} e^{j}_{-} \equiv \epsilon^{ij}_{-}\ , \tag{6.4.17}$$

thus defining the two 3×3 tensors ϵ^{ij}_{+}, ϵ^{ij}_{-}, and rewrite Eq. (6.4.14) as

$$B^{ij,lm} = \sum_{\lambda=\pm} \epsilon^{ij}_{\lambda}\, \epsilon^{lm^*}_{\lambda}. \tag{6.4.18}$$

From Eqs. (6.3.20), (6.3.22), (6.4.18), we conclude that, the probability is less than one as needed,

$$\begin{aligned} \Big|\langle 0_+ \,|\, 0_- \rangle^T\Big|^2 = &\quad \exp\Big[-8\pi G \int \mathrm{d}\omega_{\mathbf{k}} \sum_{\lambda=\pm} \big(T^*_{ij}\epsilon^{ij}_{\lambda}\big)\big(\epsilon^{lm*}_{\lambda} T_{lm}\big)\Big] \\ \leq &\quad 1, \end{aligned} \tag{6.4.19}$$

with equality holding in the limit of vanishing $T_{\mu\nu}$, thus establishing the underlying positivity constraint, as well as the correct spin content of the theory with the graviton having only two polarization states described by $\epsilon^{ij}_{+}, \epsilon^{ij}_{-}$ for a theory with, in general, a *not* necessarily conserved external energy-momentum tensor.

The scalar product in Eq. (6.4.19) may be rewritten from Eq. (6.4.18) as follows

$$\int \mathrm{d}\omega_{\mathbf{k}} \sum_{\lambda=\pm} (T^{*}_{ij}\epsilon^{ij}_{\lambda})(\epsilon^{lm*}_{\lambda} T_{lm}) = \int \mathrm{d}\omega_{\mathbf{k}} T^{*}_{ij} B^{ij,lm} T_{lm}$$

$$= \int (\mathrm{d}x)(\mathrm{d}x') T_{\mu\nu}(x) C^{\mu\nu,\sigma\rho}(x,x') T_{\sigma\rho}(x'), \qquad (6.4.20)$$

where

$$C^{\mu\nu,\sigma\rho}(x,x') = \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}k(x-x')} \pi^{\mu\nu,\sigma\rho}(k), \qquad (6.4.21)$$

$$\pi^{\mu\nu,\sigma\rho}(k) = \frac{1}{2}\left(\beta^{\mu\sigma}\beta^{\nu\rho} + \beta^{\mu\rho}\beta^{\nu\sigma} - \beta^{\mu\nu}\beta^{\sigma\rho}\right), \qquad (6.4.22)$$

$$\beta^{\mu\nu}(k) = \left[\eta^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{(Nk)^2} - \frac{N^{\mu}k^{\nu}}{(Nk)} - \frac{N^{\nu}k^{\mu}}{(Nk)}\right], \qquad (6.4.23)$$

$$Nk = N_{\alpha}k^{\alpha} = -k^0 = -|\mathbf{k}|. \qquad (6.4.24)$$

6.5 Gravitons and Expectation Value Formalism at Finite Temperature

For book-keeping purposes, we use the notation

$$\sqrt{8\pi G}\, \epsilon^{lm*}_{\lambda} T_{lm}(\mathbf{k}) \equiv S(\mathbf{k},\lambda), \qquad (6.5.1)$$

and conveniently introduce a discrete notation (Schwinger,1976; Manoukian, 1986b) for the momentum variable $\mathbf{k}$ by writing, in the process,

$$(\mathbf{k},\lambda) \equiv\ r, \qquad (6.5.2)$$

for these pairs of variables and in turn use the notation S_r for $S(\mathbf{k},\lambda)$. A scalar product as in Eq. (6.4.14) then becomes simply replaced as follows:

$$8\pi G\int \mathrm{d}\omega_{\mathbf{k}}\sum_{\lambda=\pm}\left(T^{*}_{ij}\epsilon^{ij}_{\lambda}\right)\left(\epsilon^{lm*}_{\lambda}T_{lm}\right)\rightarrow\sum_{r}S^{*}_{r}S_{r}. \qquad (6.5.3)$$

With the above notation, and for any two, *a priori*, independent, not necessarily conserved, sources $T^{1}_{\mu\nu}$, $T^{2}_{\mu\nu}$, we introduce the functional

$$\mathscr{F}[T^{1},T^{2}]=\sum_{N}\sum_{N_1+N_2+\ldots=N}\langle 0_-\,|\,N;N_1,N_2,\ldots\rangle^{T^2}\,\langle N;N_1,N_2,\ldots\,|\,0_-\rangle^{T^1}\,, \qquad (6.5.4)$$

where N denotes number of gravitons, N_1 of which have momentum-polarization index r_1, and so on,
with

$$\langle N;N_1,N_2,\ldots\,|\,0_-\rangle^{T^1}\,, \qquad (6.5.5)$$

denoting the amplitude that these N gravitons are emitted by the source T^1,
and is given by

$$\langle N;N_1,N_2,\ldots\,|\,0_-\rangle^{T^1}=\langle 0_+\,|\,0_-\rangle^{T^1}\frac{(\mathrm{i}S^{1}_{r_1})^{N_1}}{\sqrt{N_1!}}\frac{(\mathrm{i}S^{1}_{r_2})^{N_2}}{\sqrt{N_2!}}\ldots\,. \qquad (6.5.6)$$

The expression for the functional $\mathscr{F}[T^1,T^2]$ may be summed exactly by using, in the process, Eq. (6.5.6), to give

$$\mathscr{F}[T^{1},T^{2}]=\left(\langle 0_+\,|\,0_-\rangle^{T^2}\right)^{*}\left(\langle 0_+\,|\,0_-\rangle^{T^1}\right)$$
$$\times\exp\left[8\pi G\int \mathrm{d}\omega_{\mathbf{k}}\sum_{\lambda=\pm}\left(T^{*2}_{ij}\epsilon^{ij}_{\lambda}\right)\left(\epsilon^{lm*}_{\lambda}T^{1}_{lm}\right)\right], \qquad (6.5.7)$$

where we have restored the integration signs. From Eq. (6.5.4), we realize that for the special case that $T^1_{\mu\nu}$ and $T^2_{\mu\nu}$ are equal, we have by unitarity

$$\mathscr{F}[T,T] = \langle 0_- \,|\, 0_- \rangle^T = 1, \tag{6.5.8}$$

which also follows readily from Eq. (6.5.7) and the left-hand side equality in Eq. (6.4.19).

In the expression for $\mathscr{F}[T^1, T^2]$, we write

$$T^1 = T_1 + T_1', \tag{6.5.9}$$

$$T^2 = T_2 + T_2', \tag{6.5.10}$$

where T_1' is switched on after T_1 is switched off, and T_2' is switched on after T_2 is switched off, to obtain from Eqs. (6.5.4) and (6.5.7), *respectively*,

$$\mathscr{F}[T_1 + T_1', T_2 + T_2'] = \sum_{(N)} \langle 0_- \,|\, N; N_1, N_2, \ldots \rangle^{T_2 + T_2'} \langle N; N_1, N_2, \ldots \,|\, 0_- \rangle^{T_1 + T_1'}$$

$$= \sum_{(N),(M)} \langle 0_- \,|\, N; N_1, N_2, \ldots \rangle^{T_2} \langle N; N_1, N_2, \ldots \,|\, M; M_1, M_2, \ldots \rangle^{T_2', T_1'}$$

$$\times \langle M; M_1, M_2, \ldots \,|\, 0_- \rangle^{T_1}, \tag{6.5.11}$$

where

$$\langle N; N_1, N_2, \ldots \,|\, M; M_1, M_2, \ldots \rangle^{T_2', T_1'}$$

$$= \sum_{(L)} \langle N; N_1, N_2, \ldots \,|\, L; L_1, L_2, \ldots \rangle^{T_2'} \times \langle L; L_1, L_2, \ldots \,|\, M; M_1, M_2, \ldots \rangle^{T_1'}, \tag{6.5.12}$$

with $\sum\limits_{(N)}$ denoting a sum over non-negative integers $N, N_1, N_2, \ldots$ such that

$$N_1 + N_2 + \ldots = N, \tag{6.5.13}$$

and similarly for $\sum\limits_{(M)}, \sum\limits_{(L)}$, and

$$\mathscr{F}[T_1 + T_1', T_2 + T_2'] = \mathscr{F}[T_1', T_2'] \exp[S_2^* S_1] \left(\langle 0_+ \,|\, 0_- \rangle^{T_2}\right)^* \left(\langle 0_+ \,|\, 0_- \rangle^{T_1}\right)$$
$$\times \exp[S_2^*(S_1' - S_2')] \exp[-(S_1'^* - S_2'^*) S_1], \tag{6.5.14}$$

where the scalar product $S_2^* S_1$, for example, is defined as on the right-hand side of Eq. (6.5.3) with a sum over r. Upon comparison of the two equivalent expressions for $\mathscr{F}[T_1 + T_1', T_2 + T_2']$ in Eqs. (6.5.11) and (6.5.14), we obtain, in particular, for the diagonal term

$$\langle N; N_1, N_2, \ldots \,|\, N; N_1, N_2, \ldots \rangle^{T^2, T^1}, \tag{6.5.15}$$

valid for *any* two, *a priori*, independent and not necessarily conserved sources $T^1_{\mu\nu}, T^2_{\mu\nu}$, the expression:

$$\langle N; N_1, N_2, \ldots \,|\, N; N_1, N_2, \ldots \rangle^{T^2, T^1} = (N_1! N_2! \cdots) \mathscr{F}[T^1, T^2]$$
$$\times \sum{}^* \prod_i \frac{\left[-(S^{1*}_{r_i} - S^{2*}_{r_i})(S^1_{r_i} - S^2_{r_i})\right]^{N_i - m_i}}{m_i! [(N_i - m_i)!]^2}, \tag{6.5.16}$$

where $\sum^*$ stands for a summation over all non-negative integers $m_1, m_2, \ldots$ such that

$$0 \leq m_i \leq N_i, \qquad i = 1, 2, \ldots. \tag{6.5.17}$$

We now perform a thermal average (Manoukian, 1991) of

$$\langle N; N_1, N_2, \ldots | N; N_1, N_2, \ldots \rangle^{T^2, T^1}, \tag{6.5.18}$$

by multiplying, in the process, the latter by the Boltzmann factor

$$\prod_i (\exp -\beta |\mathbf{k}_i|), \tag{6.5.19}$$

and summing over (N), where

$$\beta = 1/\mathrm{K}\tau, \tag{6.5.20}$$

and we have used the notation K for the Boltzmann constant and τ for temperature in order not to confuse it with the trace T of an energy-momentum tensor.

This gives the statistical thermal average:

$$\mathscr{F}[T^1, T^2; \tau]$$

$$= \mathscr{F}[T^1, T^2; 0] \exp\Big[-8\pi G \int \mathrm{d}\omega_{\mathbf{k}} \sum_{\lambda=\pm} \frac{(T^{1*}_{ij} - T^{2*}_{ij})\epsilon^{ij}_\lambda \epsilon^{lm*}_\lambda (T^1_{lm} - T^2_{lm})}{(\mathrm{e}^{\beta|\mathbf{k}|} - 1)} \Big]. \tag{6.5.21}$$

As mentioned above, we have to average the expression on the left-hand side of Eq. (6.5.16) by the Boltzmann factor. With a normalization constant C introduced in the product $C\prod_i (\exp -\beta|\mathbf{k}_i|)$ determined from the identity (cf. Gradshteyn and Ryzhik, 1985)

$$\sum_{N=0}^{\infty} \sum_{N_1+N_2+\cdots=N} (x_1)^{N_1} (x_2)^{N_2} \cdots = \frac{1}{\prod_{i=1}^{\infty} (1 - x_i)}, \tag{6.5.22}$$

gives $C = \prod_{i=1}^{\infty}(1 - \mathrm{e}^{-\beta k_i^0})$. Therefore the thermal average of Eq. (6.5.16) is

$$\mathscr{F}[T^1, T^2]\; C \sum_{N=0}^{\infty} \sum_{N_1+N_2+\cdots=N} \prod_{i=1}^{\infty} \frac{[\mathrm{e}^{-\beta k_i^0}(\frac{\partial}{\partial a_i} + 1)]^{N_i}}{N_i!}(a_i)^{N_i}, \tag{6.5.23}$$

where $a_i = -|S^1_{r_i} - S^2_{r_i}|^2$.

To carry out the sum in the expression in Eq. (6.5.23), we use the identity

$$\frac{[\mathrm{e}^{-\beta k^0}(\frac{\partial}{\partial a} + 1)]^{N}}{N!}(a)^{N} = \int_{-\infty}^{\infty} \mathrm{d}\rho \int_{-\infty}^{\infty} \frac{\mathrm{d}\gamma}{2\pi} \frac{[\rho \mathrm{e}^{-\beta k^0}(-\mathrm{i}\gamma + 1)]^N}{N!} \mathrm{e}^{\mathrm{i}\gamma(\rho - a)}, \tag{6.5.24}$$

to obtain

$$\begin{aligned}
\mathscr{F}[T^1, T^2; \tau] &= C \sum_{N=0}^{\infty} \sum_{N_1+N_2+\cdots=N} (\prod_{i=1}^{\infty} (\mathrm{e}^{-\beta k_i^0})^{N_i})\, \langle N; N_1, \cdots | \, N, N_1, \cdots \rangle^S \\
&= C\mathscr{F}[T^1, T^2] \prod_{j=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\rho_j \int_{-\infty}^{\infty} \frac{\mathrm{d}\gamma_j}{2\pi} \\
&\quad \times \exp\left[\rho_j \mathrm{e}^{-\beta k_j^0}(-\mathrm{i}\gamma_j + 1) + \mathrm{i}\gamma_j(\rho_j - a_j)\right] \\
&= C\mathscr{F}[T^1, T^2] \prod_{j=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\rho_j \delta(\rho_j - \rho_j \mathrm{e}^{-\beta k_j^0} - a_j) \exp(\rho_j \mathrm{e}^{-\beta k_j^0}) \\
&= \mathscr{F}[T^1, T^2] \prod_{j=1}^{\infty} \exp\left[\frac{\mathrm{e}^{-\beta k_j^0}}{1 - \mathrm{e}^{\beta k_j^0}} a_j\right] \\
&= \mathscr{F}[T^1, T^2] \exp\left[-\int \mathrm{d}\omega_k \sum_{\lambda=1,2} \frac{(T^{1*}_{ij} - T^{2*}_{ij})\epsilon^{ij}_{\lambda}\epsilon^{lm*}_{\lambda}(T^1_{lm} - T^2_{lm})}{\mathrm{e}^{\beta k^0} - 1}\right],
\end{aligned} \tag{6.5.25}$$

this verify Eq. (6.5.21).

In particular, we note from Eqs. (6.5.7), (6.5.8), (6.5.21) that for the special case that $T^1_{\mu\nu}$, $T^2_{\mu\nu}$ are identical, we have the consistent normalization condition

$$\mathscr{F}[T,T;\tau] \equiv 1. \tag{6.5.26}$$

We also verify directly from Eq. (6.5.21) that

$$\mathscr{F}[T^1,T^2;0] = \mathscr{F}[T^1,T^2], \tag{6.5.27}$$

as expected.

As we have not imposed conservation laws on $T^1_{\mu\nu}$, $T^2_{\mu\nu}$, we may vary each of their respective ten components independently to obtain from the quantum dynamical principle (Schwinger, 1951, 1961; Manoukian, Sukkhasena and Siranan, 2007) as applied, respectively, and in the process to

$$\langle L; L_1, \ldots \,|\, M; M_1, \ldots \rangle^{T^1}, \tag{6.5.28}$$

and

$$\langle N; N_1, \ldots \,|\, L; L_1, \ldots \rangle^{T^2}, \tag{6.5.29}$$

in Eq. (6.5.12) with T'_1, T'_2 in it replaced by T^1, T^2, the thermal average $\langle h^{\mu\nu}(x)\rangle^T_\tau$ of the gravitational field

$$\begin{aligned}\langle h^{\mu\nu}(x)\rangle^T_\tau &= (-\mathrm{i})\frac{\delta}{\delta T^1_{\mu\nu}(x)}\mathscr{F}[T^1,T^2;\tau]\bigg|_{T^1=T^2=T} \\ &= (\mathrm{i})\frac{\delta}{\delta T^2_{\mu\nu}(x)}\mathscr{F}[T^1,T^2;\tau]\bigg|_{T^1=T^2=T} \quad ,\end{aligned} \tag{6.5.30}$$

generalizing the expression for

$$\langle 0_- |\, h^{\mu\nu}(x)\, | 0_- \rangle^T, \tag{6.5.31}$$

given by

$$\langle 0_- | h^{\mu\nu}(x) | 0_- \rangle^T = (-\mathrm{i}) \frac{\delta}{\delta T^1_{\mu\nu}(x)} \mathscr{F}[T^1, T^2] \bigg|_{T^1 = T^2 = T}$$

$$= (\mathrm{i}) \frac{\delta}{\delta T^2_{\mu\nu}(x)} \mathscr{F}[T^1, T^2] \bigg|_{T^1 = T^2 = T} , \tag{6.5.32}$$

from zero to finite temperature.

From Eqs. (6.5.21), (6.5.7), (6.4.15), the generating functional $\mathscr{F}[T^1, T^2; \tau]$ may be rewritten as

$$\mathscr{F}[T^1, T^2; \tau] = \bigg(\langle 0_+ | 0_- \rangle^{T^2} \bigg)^* \bigg(\langle 0_+ | 0_- \rangle^{T^1} \bigg)$$
$$\times \exp \bigg[8\pi G \int (\mathrm{d}x)(\mathrm{d}x') T^2_{\mu\nu}(x) C^{\mu\nu,\sigma\rho}(x, x') T^1_{\sigma\rho}(x') \bigg]$$
$$\times \exp \bigg[-8\pi G \int (\mathrm{d}x)(\mathrm{d}x') \Big(T^1_{\mu\nu}(x) - T^2_{\mu\nu}(x) \Big)$$
$$\times D^{\mu\nu,\sigma\rho}(x, x'; \tau) \Big(T^1_{\sigma\rho}(x') - T^2_{\sigma\rho}(x') \Big) \bigg], \tag{6.5.33}$$

where $C^{\mu\nu,\sigma\rho}(x, x')$ is defined in Eq. (6.4.21), and

$$D^{\mu\nu,\sigma\rho}(x, x'; \tau) = \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}k(x-x')} \frac{\pi^{\mu\nu,\sigma\rho}(k)}{(\mathrm{e}^{-\beta(Nk)} - 1)}, \tag{6.5.34}$$

$$Nk = N_\alpha k^\alpha = -k^0 = -|\mathbf{k}|, \tag{6.5.35}$$

where $\pi^{\mu\nu,\sigma\rho}(k)$ is given in Eq. (6.4.22).

We note that the temperature dependence occurs only in the last exponential in Eq. (6.5.33) through $D^{\mu\nu,\sigma\rho}(x, x'; \tau)$. We eventually set

$$T^1_{\mu\nu} = T^2_{\mu\nu}, \tag{6.5.36}$$

after the relevant functional differentiations with respect to these sources are taken. For $\tau \to 0$, the last exponential in Eq. (6.5.33) is equal to one, giving the relation in Eq. (6.5.27).

6.6 Covariance of the Induced Riemann Curvature Tensor

The thermal average $\langle h_{\mu\nu}(x)\rangle_\tau^T$ may be obtained from Eqs. (6.5.30), (6.5.33) to give

$$\begin{aligned}\langle h_{\mu\nu}(x)\rangle_\tau^T &= 8\pi G\mathrm{i}\int(\mathrm{d}x')T^{\sigma\rho}(x')\int \mathrm{d}\omega_{\mathbf{k}}\pi_{\mu\nu,\sigma\rho}(k)\mathrm{e}^{\mathrm{i}k(x-x')}\\ &\quad - 8\pi G\mathrm{i}\int(\mathrm{d}x')T^{\sigma\rho}(x')\int \mathrm{d}\omega_{\mathbf{k}}\pi_{\sigma\rho,\mu\nu}(k)\mathrm{e}^{\mathrm{i}k(x'-x)}\\ &= -16\pi G\int(\mathrm{d}x')T^{\sigma\rho}(x')\int \mathrm{d}\omega_{\mathbf{k}}\sin k(x-x')\pi_{\mu\nu,\sigma\rho}(k)\\ &\equiv \langle 0_-|h_{\mu\nu}(x)|0_-\rangle^T,\end{aligned} \tag{6.6.1}$$

for $x^0 > x'^0$, where after the functional differentiation was carried out with respect to, say, $T^{1\mu\nu}(x)$, we have set

$$T^{2\mu\nu} = T^{1\mu\nu} = T^{\mu\nu}. \tag{6.6.2}$$

We learn that the above expectation value is independent of temperature in the leading linearized theory as a consequence of the fact that the exponent in the last exponential in Eq. (6.5.33) does not contribute if a single functional differentiation with respect to $T^{1\mu\nu}$ is carried out and then by finally setting

$$T^2_{\mu\nu} - T^1_{\mu\nu} = 0. \tag{6.6.3}$$

Radiative corrections and explicit temperature dependence will be discussed in the concluding chapter.

In more detail, we may rewrite Eq. (6.6.1) as:

$$\langle 0_- | h_{\mu\nu}(x) | 0_- \rangle^T = \left\{ 8\pi G \mathrm{i} \int \mathrm{d}\omega_k \mathrm{e}^{\mathrm{i}kx} \left[T_{\mu\nu}(k) - \frac{g_{\mu\nu}}{2} T(k) \right] + c.c. \right\}$$

$$+ \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) + \partial_\mu \partial_\nu \xi(x), \tag{6.6.4}$$

for $x^0 > x'^0$, where

$$\xi_\mu(x) = \left\{ 4\pi G \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}kx} \frac{N_\mu T - 2T_\mu{}^\sigma N_\sigma}{(Nk)} + c.c. \right\}, \tag{6.6.5}$$

$$\xi(x) = \left\{ \frac{4\pi G}{\mathrm{i}} \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}kx} \frac{T + 2T^{\nu\sigma} N_\nu N_\sigma}{(Nk)^2} + c.c. \right\}, \tag{6.6.6}$$

and

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \partial_\mu \partial_\nu \xi, \tag{6.6.7}$$

are the so-called gauge terms (see Eq. (6.2.3)) and are non-covariant depending on the vector N^μ. The induced Riemann curvature tensor in the leading theory is given by

$$\langle 0_- | R_{\mu\nu\sigma\lambda}(x) | 0_- \rangle^T = \langle 0_- | \partial_\mu \partial_\sigma h_{\nu\lambda} + \partial_\nu \partial_\lambda h_{\mu\sigma} - \partial_\mu \partial_\lambda h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\lambda} | 0_- \rangle^T . \tag{6.6.8}$$

By substituting the expression Eq. (6.6.4) in Eq. (6.6.8), we obtain

$$\langle 0_- | R_{\mu\nu\sigma\lambda}(x) | 0_- \rangle^T = \partial_\mu \partial_\sigma h^0_{\nu\lambda} + \langle 0_- | \partial_\mu \partial_\sigma [\partial_\nu \xi_\lambda + \partial_\lambda \xi_\nu + \partial_\nu \partial_\lambda \xi] | 0_- \rangle$$

$$+ \partial_\nu \partial_\lambda h^0_{\mu\sigma} + \langle 0_- | \partial_\nu \partial_\lambda [\partial_\mu \xi_\sigma + \partial_\sigma \xi_\mu + \partial_\mu \partial_\sigma \xi] | 0_- \rangle$$

$$- \partial_\mu \partial_\lambda h^0_{\nu\sigma} - \langle 0_- | \partial_\mu \partial_\lambda [\partial_\nu \xi_\sigma + \partial_\sigma \xi_\nu + \partial_\sigma \partial_\nu \xi] | 0_- \rangle$$

$$- \partial_\nu \partial_\sigma h^0_{\mu\lambda} - \langle 0_- | \partial_\nu \partial_\sigma [\partial_\mu \xi_\lambda + \partial_\lambda \xi_\mu + \partial_\mu \partial_\lambda \xi] | 0_- \rangle$$

$$= \partial_\mu \partial_\sigma h^0_{\nu\lambda} + \partial_\nu \partial_\lambda h^0_{\mu\sigma} - \partial_\mu \partial_\lambda h^0_{\nu\sigma} - \partial_\nu \partial_\sigma h^0_{\mu\lambda}, \qquad (6.6.9)$$

where $h^0_{\mu\nu}(x)$ corresponds to the first term on the right- hand side of Eq. (6.6.4) within the curly brackets.

Therefore, all the terms depending on ξ^μ, ξ *cancel* in the induced Riemann curvature tensor

$$\langle 0_- | R_{\mu\nu\sigma\lambda}(x) | 0_- \rangle^T , \qquad (6.6.10)$$

thus establishing its covariance. This means that one may restrict

$$\langle 0_- | h_{\mu\nu}(x) | 0_- \rangle^T , \qquad (6.6.11)$$

to its covariant gauge-independent part defined by the first term on the left-hand side of Eq. (6.6.4) within the curly brackets, i.e., consider

$$\langle 0_- | h_{\mu\nu}(x) | 0_- \rangle^T = \left\{ 8\pi G \mathrm{i} \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}kx} \Big[T_{\mu\nu}(k) - \frac{\eta_{\mu\nu}}{2} T(k) \Big] + \ c.c. \right\}$$

$$\equiv h^o_{\mu\nu}(x), \qquad (6.6.12)$$

in applications. The expression for the latter may be further simplified to

$$h^o_{\mu\nu}(x) = \left\{ 8\pi G \mathrm{i} \int (\mathrm{d}x') \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}k(x-x')} \Big[T_{\mu\nu}(x') - \frac{\eta_{\mu\nu}}{2} T(x') \Big] + \ c.c. \right\}. \qquad (6.6.13)$$

The $\mathbf{k}$-integration as well as the $x^{0'}$-one may be explicitly carried out leading to

$$h^o_{\mu\nu}(x) = 2G \int \frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \Big[T_{\mu\nu}(x^0 - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}') - \frac{\eta_{\mu\nu}}{2} T(x^0 - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}') \Big]. \qquad (6.6.14)$$

6.7 Induced Geometric, and Induced Correction to the Metric: Application to a Nambu String

The metric of spacetime to the leading contribution in our notation here is defined (Schwinger, 1976) by

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2h^{o}_{\mu\nu}(x), \tag{6.7.1}$$

with the 2 factor, where $h^{o}_{\mu\nu}(x)$ is given in Eq. (6.6.14). The leading contribution to the inverse $g^{\mu\nu}$ is then given by

$$g^{\mu\nu} = \eta^{\mu\nu} - 2h^{o\mu\nu}. \tag{6.7.2}$$

We investigate the contribution to the metric, the induced geometry and corresponding spacetime measurements due to a string. The dynamics of the string is described as follows. The trajectory of the string is described by a vector function $\mathbf{R}(\sigma, t)$, where σ parametrizes the string. The equation of motion of the closed sting considered is taken to be

$$\frac{\partial^2}{\partial t^2}\mathbf{R}(\sigma, t) - \frac{\partial^2}{\partial \sigma^2}\mathbf{R}(\sigma, t) = 0, \tag{6.7.3}$$

with constraints

$$\partial_t \mathbf{R} \cdot \partial_\sigma \mathbf{R} = 0, \tag{6.7.4}$$

$$(\partial_t \mathbf{R})^2 + (\partial_\sigma \mathbf{R})^2 = 1, \tag{6.7.5}$$

$$\mathbf{R}\Big(\sigma + \frac{2\pi}{\omega}, t\Big) = \mathbf{R}(\sigma, t), \tag{6.7.6}$$

for a constant ω. The general solution to Eqs. (6.7.3), (6.7.4) - (6.7.6) is given by

$$\mathbf{R}(\sigma,t)=\frac{1}{2}\Big[\mathbf{\Phi}(\sigma-t)+\mathbf{\Psi}(\sigma+t)\Big], \tag{6.7.7}$$

where $\mathbf{\Phi}$, $\mathbf{\Psi}$, in particular, satisfy the normalization conditions

$$(\partial_\sigma\mathbf{\Phi})^2=(\partial_\sigma\mathbf{\Psi})^2 \ = \ 1. \tag{6.7.8}$$

To verify that any $\phi(\sigma-t)$ is a solution of Eq. (6.7.3) we define $\sigma-t=u$. Therefore

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathbf{\Phi}(\sigma-t) &= \frac{\partial}{\partial t}\left[(\frac{\partial}{\partial t}u)\frac{\partial}{\partial u}\mathbf{\Phi}(u)\right] \\ &= -\frac{\partial}{\partial t}\mathbf{\Phi}'(u) \\ &= (-\frac{\partial}{\partial t}u)\mathbf{\Phi}''(u) \\ &= \mathbf{\Phi}''(u),\end{aligned} \tag{6.7.9}$$

Similarly,

$$\begin{aligned}\frac{\partial^2}{\partial\sigma^2}\mathbf{\Phi}(\sigma-t) &= \frac{\partial}{\partial\sigma}(\frac{\partial u}{\partial\sigma})\mathbf{\Phi}'(u) \\ &= \frac{\partial}{\partial\sigma}\mathbf{\Phi}'(u) \\ &= (\frac{\partial u}{\partial\sigma})\mathbf{\Phi}''(u) \\ &= \mathbf{\Phi}''(u).\end{aligned} \tag{6.7.10}$$

Therefore

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \sigma^2})\mathbf{\Phi}(\sigma - t) = 0, \tag{6.7.11}$$

for any vector function $\mathbf{\Phi}(\sigma - t)$ which is twice differentiable with respect to t and σ

We can also verify that any $\mathbf{\Psi}(\sigma + t)$ is also a solution of Eq. (6.7.3), we set $\sigma + t = v$.

Therefore

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\mathbf{\Psi}(\sigma + t) &= \frac{\partial}{\partial t}\left[\frac{\partial v}{\partial t}\frac{\partial \mathbf{\Psi}(v)}{\partial v}\right] \\
&= \frac{\partial}{\partial t}\frac{\partial \mathbf{\Psi}(v)}{\partial v} \\
&= \frac{\partial}{\partial v}\left[\frac{\partial}{\partial t}\mathbf{\Psi}(v)\right] \\
&= \frac{\partial}{\partial v}\left[\frac{\partial}{\partial v}\mathbf{\Psi}(v)\right] \\
&= \frac{\partial^2}{\partial v^2}\mathbf{\Psi}(v) \\
&= \mathbf{\Psi}''(v).
\end{aligned} \tag{6.7.12}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2}{\partial \sigma^2}\mathbf{\Psi}(\sigma + t) &= \frac{\partial}{\partial \sigma}\left[\frac{\partial v}{\partial \sigma}\frac{\partial \mathbf{\Psi}(v)}{\partial v}\right] \\
&= \frac{\partial}{\partial \sigma}\frac{\partial \mathbf{\Psi}(v)}{\partial v} \\
&= \frac{\partial}{\partial v}\frac{\partial}{\partial \sigma}\mathbf{\Psi}(v) \\
&= \frac{\partial}{\partial v}\left[\frac{\partial v}{\partial \sigma}\frac{\partial \mathbf{\Psi}(v)}{\partial v}\right]
\end{aligned}$$

$$= \frac{\partial^2}{\partial v^2}\mathbf{\Psi}(v)$$

$$= \mathbf{\Psi}''(v)$$

$$= \mathbf{\Psi}''(\sigma + t). \tag{6.7.13}$$

Therefore

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \sigma^2})\mathbf{\Psi}(\sigma + t) = 0, \tag{6.7.14}$$

for any vector function $\mathbf{\Psi}(\sigma + t)$ of $\sigma + t$ which is twice differentiable with respect to t and σ is also a solution of Eq. (6.7.3).

For the system of Eqs. (6.7.3) - (6.7.6), we consider a solution of the form (Manoukian, 1991, 1992, 1995, 1998):

$$\mathbf{R}(\sigma, t) = \Bigg(\cos\omega\sigma, \sin\omega\sigma, 0 \Bigg) \frac{\sin\omega t}{\omega}, \tag{6.7.15}$$

describing a radially oscillating circular string in a plane.

We verify that $\mathbf{R}(\sigma, t)$ in Eq. (6.7.15) is of the form in Eq. (6.7.7). To this end, we use the elementary trigonometric identities

$$\sin A \sin B = \frac{1}{2}\Bigg[\cos(A - B) - \cos(A + B) \Bigg], \tag{6.7.16}$$

$$\cos A \sin B = \frac{1}{2}\Bigg[\sin(A + B) - \sin(A - B) \Bigg], \tag{6.7.17}$$

Therefore,

$$\mathbf{R}(\sigma, t) = \frac{1}{2\omega}\Bigg(\sin\omega(\sigma + t) - \sin\omega(\sigma - t), \cos\omega(\sigma - t) - \cos\omega(\sigma + t), 0 \Bigg)$$

$$= \frac{1}{2}\left[\frac{1}{\omega}\Big(\sin\omega(\sigma+t), -\cos\omega(\sigma+t), 0\Big) + \frac{1}{\omega}\Big(\cos\omega(\sigma-t), -\sin\omega(\sigma-t), 0\Big)\right], \tag{6.7.18}$$

which is exactly of the form in Eq. (6.7.7).

The general expression for the energy-momentum tensor of the string is given by

$$T^{\mu\nu}(x) = \frac{M\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}\sigma \Big(\partial_t R^\mu \partial_t R^\nu - \partial_\sigma R^\mu \partial_\sigma R^\nu\Big)\delta^3\Big(\mathbf{r} - \mathbf{R}(\sigma, t)\Big), \tag{6.7.19}$$

where

$$R^0 \;=\; t, \qquad \mathbf{r} \;=\; r(\cos\phi, \sin\phi, 0), \tag{6.7.20}$$

and M provides a mass scale.

The various components of the energy-momentum tensor are worked out to be (Manoukian, 1991a,1992, 1995, 1998)

$$T^{00} = \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z), \tag{6.7.21}$$

$$T^{0i} = \frac{M}{2\pi r}(\cos\phi, \sin\phi, 0)\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)\cos\omega t \,\mathbf{sgn}(\sin\omega t), \tag{6.7.22}$$

$$T^{11} = \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)[\cos^2\omega t - \sin^2\phi], \tag{6.7.23}$$

$$T^{12} = \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)\frac{\sin 2\phi}{2}, \tag{6.7.24}$$

$$T^{22} = \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)[\cos^2\omega t - \cos^2\phi], \tag{6.7.25}$$

$$T^{\mu 3} = 0, \tag{6.7.26}$$

where

$$\mathrm{sgn}(\alpha) \;=\; \pm\, 1, \tag{6.7.27}$$

for

$$\alpha \quad \gtrless \quad 0, \tag{6.7.28}$$

is the sign function, $i = 1, 2, 3$, in Eq. (6.7.22).

We note the normalization condition

$$\int \mathrm{d}^3\mathbf{x}\, T^{00}(x) = M. \tag{6.7.29}$$

Also for the trace $T^{\mu}{}_{\mu}(x)$ of the energy-momentum tensor we have

$$T = -\frac{M}{\pi r}\delta\Big(r - \frac{|\sin\omega t|}{\omega}\Big)\delta(z)\sin^2\omega t. \tag{6.7.30}$$

For completeness a Fourier analysis of $T^{\mu\nu}(t, \mathbf{r}, z)$ follows. To this end we perform a multiple Fourier-transform

$$T^{\mu\nu}(t, \mathbf{r}, z) = \int \frac{\mathrm{d}p^0}{(2\pi)} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \int \frac{\mathrm{d}q}{(2\pi)} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}} \mathrm{e}^{\mathrm{i}q\cdot z} \mathrm{e}^{-\mathrm{i}p^0 t} T^{\mu\nu}(p^0, \mathbf{p}, q), \tag{6.7.31}$$

where we have used cylindrical coordinates with $\mathbf{r}$ lying in the plane of the string. Using the periodicity of $T^{\mu\nu}$ in time we can write

$$T^{\mu\nu}(t, \mathbf{r}, z) = \sum_{N=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}NMt} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \int_{-\infty}^{\infty} \mathrm{d}q \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}} \mathrm{e}^{\mathrm{i}qz} B^{\mu\nu}(\mathbf{p}, N), \tag{6.7.32}$$

$$T^{\mu\nu}(p^0, \mathbf{p}, q) \equiv T^{\mu\nu}(p^0, \mathbf{p}) = 2\pi \sum_{N=-\infty}^{\infty} \delta(p^0 - mN) B^{\mu\nu}(\mathbf{p}, N), \tag{6.7.33}$$

where

$$B^{\mu\nu}(\mathbf{p},N)=\frac{m}{2\pi}\int_{\frac{-\pi}{m}}^{\frac{\pi}{m}}\mathrm{e}^{\mathrm{i}Nmt}\int\mathrm{d}^2r\int_{-\infty}^{\infty}\mathrm{d}z\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{r}}\mathrm{e}^{-\mathrm{i}qz}T^{\mu\nu}(\mathbf{r},z,t), \qquad (6.7.34)$$

with

$$\mathbf{p}=p(\cos\phi,\sin\phi,0), \qquad (6.7.35)$$

$$\mathbf{p}\cdot\mathbf{r}=pr\cos(\phi'-\phi), \quad \mathrm{d}^2\mathbf{r}=r\mathrm{d}r\mathrm{d}\phi', \qquad (6.7.36)$$

exhibiting the time translation invariance : $t\to t+\frac{2\pi}{m}$, and the q-independence of the Fourier Transform. We use the following expansion (Gradshteyn and Ryzhik, 1985)

$$\exp\Big[\mathrm{i}\rho\cos(\phi'-\phi)\Big]=\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n(\phi'-\phi)}J_n(\rho), \qquad (6.7.37)$$

in terms of Bessel functions $J_n(\rho)$ of integral order n, and the following basic integrals:

$$\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}NT}J_M(a|\sin T|)\mathrm{d}T=2\pi\cos(\frac{N\pi}{2})J_{\frac{M+N}{2}}(\frac{a}{2}), \qquad (6.7.38)$$

$$\int_{\pi}^{\pi}\mathrm{e}^{\mathrm{i}Nt}\mathrm{sgn}(\sin T)J_M(a|\sin T|)\mathrm{d}T=2\mathrm{i}\pi\sin(\frac{N\pi}{2})J_{\frac{M-N}{2}}(\frac{a}{2})J_{\frac{M+N}{2}}(\frac{a}{2}), \qquad (6.7.39)$$

to obtain after a very lengthy process the following expression for $B^{\mu\nu}=B^{\nu\mu}$:

$$B^{00}=\beta_nJ_n^2(x), \qquad (6.7.40)$$

$$B^{0a}=\beta_n\frac{p^0p^a}{\mathbf{p}^2}J_n^2(x), \quad a=1,2, \qquad (6.7.41)$$

$$B^{ab} = \beta_n A_n \delta^{ab} + \beta_n E_n \frac{p^a p^b}{\mathbf{p}^2}, \quad a,\ b = 1, 2, \tag{6.7.42}$$

$$B^{\mu 3} = 0, \tag{6.7.43}$$

where

$$A_n = \frac{1}{4}\left[J_{n+1}^2(x) + J_{n-1}^2(x) - 2J_{n-1}(x)J_{n+1}(x)\right], \tag{6.7.44}$$

$$E_n = J_{n+1}(x)J_{n-1}(x), \tag{6.7.45}$$

$$x = \frac{|\mathbf{p}|}{2m}, \tag{6.7.46}$$

$$n = \frac{N}{2}, \tag{6.7.47}$$

$$\beta_n = m(-1)^n \cos(n\pi), \tag{6.7.48}$$

and $\mathbf{p}$ is defined in Eq. (6.7.35).

Now we verify the explicit equalities Eqs. (6.7.21) - (6.7.26), (6.7.29), (6.7.30). For example

$$T^{00}(x) = \frac{M\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}\sigma (1-0)\delta^3\Big(\mathbf{r} - \mathbf{R}(\boldsymbol{\sigma}, t)\Big), \tag{6.7.49}$$

$$T^{\mu\nu} = \frac{\omega^2}{2\pi}\int_0^{\frac{2\pi}{\omega}} \mathrm{d}\sigma \Big(\partial_t R^\mu \partial_t R^\nu - \partial_\sigma R^\mu \partial_\sigma R^\nu\Big)\delta^3\Big(\mathbf{r} - \mathbf{R}(\sigma, t)\Big), \tag{6.7.50}$$

$$\mathbf{R}(\sigma,t) = \frac{1}{\omega}(\cos\omega\sigma, \sin\omega\sigma, 0)\sin\omega\sigma, \qquad \mathbf{r} = r(\cos\phi', \sin\phi', 0)$$

$$= \frac{|\sin\omega t|}{\omega}\Bigg(\underbrace{\cos\omega\sigma\,\mathbf{sgn}(\sin\omega t)}_{\cos\phi}, \underbrace{\sin\omega\sigma\,\mathbf{sgn}(\sin\omega t)}_{\sin\phi}, 0\Bigg), \tag{6.7.51}$$

$$\delta^3\Bigg(\mathbf{r} - \mathbf{R}(\sigma,t)\Bigg) = \frac{r - \dfrac{|\sin\omega t|}{\omega}}{r}\delta(\phi-\phi')\delta(z), \qquad \phi = \phi(\sigma'), \tag{6.7.52}$$

$$T^{\mu\nu} = \frac{\omega}{2\pi r}\int_0^{2\pi} \mathrm{d}\sigma'\Bigg(\partial_t R^\mu \partial_t R^\nu - \partial_\sigma R^\mu \partial_\sigma R^\nu\Bigg)\delta\Bigg(r - \frac{|\sin\omega t|}{\omega}\Bigg)\delta(\phi-\phi')\delta(z), \tag{6.7.53}$$

$$\begin{cases} \cos\phi = \cos\sigma'\mathbf{sgn}(\sin\omega t), \\ \sin\phi = \sin\sigma'\mathbf{sgn}(\sin\omega t), \end{cases} \tag{6.7.54}$$

$$\begin{cases} \mathbf{sgn}(\sin\omega t) > 0 \quad \phi = \sigma', \\ \mathbf{sgn}(\sin\omega t) < 0 \quad \phi = \sigma' + \pi, \end{cases} \tag{6.7.55}$$

$$\phi = (\sigma' + \frac{(1 - \mathbf{sgn}(\sin\omega t))}{2}\pi), \tag{6.7.56}$$

$$\begin{cases} \text{sgn}(\sin\omega t) = +1 & \to \phi = \sigma', \\ \text{sgn}(\sin\omega t) = -1 & \to \phi = \sigma' + \pi \to \cos\phi = -\sin\phi' \text{ , and } \sin\phi = -\sin\sigma', \end{cases} \tag{6.7.57}$$

thus

$$\phi = \delta\Bigg(\sigma' + (\frac{1 - \text{sgn}(\sin\omega t)}{2})\pi - \phi'\Bigg), \tag{6.7.58}$$

$$T^{\mu\nu} = \frac{\omega}{2\pi r}\int_0^{2\pi} \mathrm{d}\sigma' \Bigg(\partial_t R^\mu \partial_t R^\nu - \partial_\sigma R^\mu \partial_\sigma R^\nu\Bigg)\delta\Bigg(r - \frac{|\sin\omega t|}{\omega}\Bigg)$$
$$\times\,\delta\Bigg(\sigma' - \Bigg[\phi' - (\frac{1 - \text{sgn}(\sin\omega t)}{2})\pi\Bigg]\Bigg), \tag{6.7.59}$$

$$T^{00} = \frac{\omega}{2\pi r}\delta\Bigg(r - \frac{|\sin\omega t|}{\omega}\Bigg)\delta(z), \tag{6.7.60}$$

$$T^{0i} = \frac{\omega}{2\pi r}\int_0^{2\pi} \mathrm{d}\sigma'(\cos\omega t)(\cos\sigma', \sin\sigma', 0)\delta\Bigg(r - \frac{|\sin\omega t|}{m}\Bigg)$$
$$\times\,\delta\Bigg(\sigma' - [\phi' - (\frac{1 - \text{sgn}(\sin\omega t)}{2})\pi]\Bigg)\delta(z), \tag{6.7.61}$$

$$T^{0i} = \frac{\omega}{2\pi r}(\cos\omega t)\delta\Bigg(r - \frac{|\sin\omega t|}{\omega}\Bigg)\delta(z)(\cos\phi', \sin\sigma', 0)\text{sgn}(\sin\omega t)\sin(\omega t), \tag{6.7.62}$$

Finally

$$\begin{aligned} T &= T^{ii} - T^{00} \\ &= \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)\left[\cos^2\omega t - \sin^2\phi + \cos^2\omega t - \cos^2\phi + 0 - 1\right], \end{aligned} \tag{6.7.63}$$

or

$$T = \frac{M}{\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)(\cos^2\omega t - 1), \tag{6.7.64}$$

$$T = -\frac{M}{\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)\sin^2\omega t, \tag{6.7.65}$$

thus checking Eq. (6.7.30) as well.

The explicit demonstration of the conservation of the energy-momentum tensor associated with the string is worked out through the following steps.

$$T^{0i} = \frac{\omega}{2\pi r_0^2}(x, y, 0)\,\delta\left(\sqrt{x^2 + y^2} - r_0\right)\delta(z)\cos\omega t \text{ sgn}(\sin\omega t), \tag{6.7.66}$$

$$\partial_x T^{0x} = \frac{\omega}{2\pi r_0^2}\,\delta(r - r_0)\left\{1 - \frac{1}{(r - r_0)}\frac{x^2}{r}\right\}(\cdots), \tag{6.7.67}$$

$$\partial_y T^{0y} = \frac{\omega}{2\pi r_0^2}\,\delta(r - r_0)\left\{1 - \frac{y^2}{(r - r_0)r}\right\}(\cdots), \tag{6.7.68}$$

$$\begin{aligned} \partial_i T^{0i} &= \frac{\omega}{2\pi r_0^2}\,\delta(r - r_0)\left\{1 - \frac{r}{r - r_0}\right\}(\cdots) \\ &= \frac{\omega}{2\pi, r_0^2}\,\delta(r - r_0)\frac{(r - 2r_0)}{(r - r_0)}(\cdots), \end{aligned} \tag{6.7.69}$$

$$\partial_i T^{0i} = -\frac{\omega}{2\pi r_0}\frac{\delta(r - r_0)}{(r - r_0)}(\cdots), \tag{6.7.70}$$

$$\partial_i T^{0i} = -\frac{\omega}{2\pi r_0}\frac{\delta(r-r_0)}{(r-r_0)}\cos\omega t\;\mathrm{sgn}(\sin\omega t), \tag{6.7.71}$$

$$\partial_0 T^{00} = -\frac{\omega}{2\pi r_0}\frac{\delta(r-r_0)}{(r-r_0)}\begin{cases}(-)\cos\omega t & , \quad \mathrm{sgn}(\sin\omega t) > 0\\ (+)\cos\omega t & , \quad \mathrm{sng}(\sin\omega t) < 0,\end{cases} \tag{6.7.72}$$

$$\partial_0 T^{00} = \frac{\omega}{2\pi r_0}\frac{\delta(r-r_0)}{(r-r_0)}\cos\omega t\;\mathrm{sgn}(\sin\omega t), \tag{6.7.73}$$

$$\partial_\mu T^{0\mu} = 0, \tag{6.7.74}$$

A similar proof may be carried out for $\partial_\mu T^{i\mu} = 0$. It is most interesting to consider spacetime measurements along the most symmetrical direction in the problem, that is, along the z–(x^3–) axis perpendicular to the plane of oscillations. Before doing so, we note that in the plane of oscillations of the string, $g_{\phi\phi}$ cannot be a function of ϕ by symmetry. Also no cross term $g_{r\phi}$ can occur in this plane, i.e., $g_{r\phi} = 0$. The metric contributions $h_{rr},\ h_{00}$, in the plane of oscillations, are readily obtained. To this end Eqs. (6.6.14), (6.7.21) - (6.7.26), (6.7.30) lead for

$$r \gg 1/\omega, \tag{6.7.75}$$

$$\begin{aligned}2h_{11}(x) &\simeq \frac{4G}{r}\int \mathrm{d}^3\mathbf{x}'\Big[T_{11}(x^0-r,\mathbf{x}') - \frac{T(x^0-r,\mathbf{x}')}{2}\Big]\\ &= \frac{2GM}{r}\\ &\simeq 2h_{22}(x),\end{aligned} \tag{6.7.76}$$

$$h_{12} \simeq 0, \tag{6.7.77}$$

where $1/\omega$ is the maximum radial extension of the string, and where the factor 2 multiplying $h_{11}(x)$ is due to the definition Eq. (6.7.2).

In detail, the result in Eq. (6.7.76) follows from the explicit integrations in

$$2h_{11}(x) = 4G \int \frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \left[T_{11}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') - \frac{1}{2} T(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') \right], \tag{6.7.78}$$

where

$$T_{11}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') = \frac{M}{2\pi r'}\, \delta\left(r' - \frac{|\sin\omega(t-|\mathbf{x}-\mathbf{x}'|)|}{\omega} \right)$$

$$\times\, \delta(z')(\cos^2\omega(t-|\mathbf{x}-\mathbf{x}'|) - \sin^2\phi'), \tag{6.7.79}$$

$$T(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') = -\frac{M}{\pi r'}\, \delta\left(r' - \frac{|\sin\omega(t-|\mathbf{x}-\mathbf{x}'|)|}{\omega} \right)$$

$$\times\, \delta(z') \sin^2\omega(t-|\mathbf{x}-\mathbf{x}'|), \tag{6.7.80}$$

and

$$\mathrm{d}^3\mathbf{x}' = r'\,\mathrm{d}r'\,\mathrm{d}\phi'\,\mathrm{d}z'. \tag{6.7.81}$$

Therefore

$$2h_{11}(x) = \frac{2MG}{\pi} \int_0^\infty r'\,\mathrm{d}r' \int_0^{2\pi} \mathrm{d}\phi' \int_{-\infty}^{\infty} \mathrm{d}z' \frac{M}{2\pi r'}\, \delta\left(r' - \frac{|\sin\omega(t-|\mathbf{x}-\mathbf{x}'|)|}{\omega} \right)$$

$$\times\, \delta(z') \frac{1}{|\mathbf{x}-\mathbf{x}'|} \left[\cos^2\omega(t-|\mathbf{x}-\mathbf{x}'|) - \sin^2\phi' + \sin^2\omega(t-|\mathbf{x}-\mathbf{x}'|) \right]$$

$$=\frac{2MG}{\pi}\int_0^\infty \mathrm{d}r' \int_0^{2\pi} \mathrm{d}\phi'\delta\left(r' - \frac{|\sin\omega(t-|\mathbf{x}-\mathbf{x}'|)|}{\omega}\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|}[1-\sin^2\phi'], \tag{6.7.82}$$

$r' = |\mathbf{x}'| \leq 1/\omega$, hence for points of observation $r \gg 1/\omega$, $|\mathbf{x}-\mathbf{x}'| \simeq r$, i.e.,

$$\begin{aligned} 2h_{11}(x) \simeq &\frac{2MG}{\pi r}\int_0^\infty \mathrm{d}r'\delta\left(r' - \frac{|\sin\omega(t-r)|}{\omega}\right)\int_0^{2\pi}\mathrm{d}\phi'[\cos^2\phi'] \\ = &\frac{2MG}{r}. \end{aligned} \tag{6.7.83}$$

Using the identity which relates polar coordinates to the Cartesian ones:

$$h_{rr} = \cos^2\phi\, h_{11} + \sin^2\phi\, h_{22} + \sin 2\phi\, h_{12}, \tag{6.7.84}$$

we conclude that the above leads to the following expression for g_{rr}:

$$g_{rr} \simeq \Big(1 + \frac{2GM}{r}\Big). \tag{6.7.85}$$

On the other hand,

$$\begin{aligned} 2h_{00}(x) &\simeq \frac{4GM}{r}\int \mathrm{d}^3\mathbf{x}'\Big[T_{00}(x^0 - r, \mathbf{x}') + \frac{T(x^0 - r, \mathbf{x}')}{2}\Big] \\ &= \frac{4GM}{r}\cos^2\omega(t-r), \end{aligned} \tag{6.7.86}$$

This finally leads to

$$g_{00}(x) \simeq -\Big(1 - \frac{4GM}{r}\cos^2\omega(t-r)\Big), \tag{6.7.87}$$

where we recall that the Minkowski metric is taken to be $[\eta_{\mu\nu}] = \mathrm{diag}[-1,1,1,1]$.

For an observer at a fixed r satisfying

$$r \gg 1/\omega, \tag{6.7.88}$$

in the plane of oscillations of the string, then we infer that time slows down by a factor

$$\frac{1}{(T_2 - T_1)} \int_{T_1}^{T_2} \sqrt{-g_{00}}\mathrm{d}t$$
$$= 1 - \frac{GM}{r} \left\{ 1 + \cos\omega(T_1 + T_2 - 2r) \frac{\sin\omega(T_2 - T_1)}{\omega(T_2 - T_1)} \right\}, \tag{6.7.89}$$

relative to a time lapsed of length $(T_2 - T_1)$ in empty space.

For spacetime measurements along the z-axis, we have explicitly

$$2h^o_{33}(x) = 4GM \int_0^\infty \frac{\mathrm{d}r'}{\sqrt{r'^2 + z^2}} \delta \left(r' - \frac{\sin\omega(t - \sqrt{r'^2 + z^2})}{\omega} \right) r'^2 \omega^2. \tag{6.7.90}$$

Again, since r' does not exceed $1/\omega$, we have for an observer at

$$|z| \gg 1/\omega, \tag{6.7.91}$$

$$g_{33}(x) \simeq 1 + \frac{4GM}{|z|} \sin^2\omega(t - |z|), \tag{6.7.92}$$

showing an interesting oscillatory behaviour in the space metric with a relative expansion of length. Here

$$\mathbf{x} = (0, 0, z), \tag{6.7.93}$$

$$\mathbf{x}' = (r' \cos\phi', r' \sin\phi', 0), \tag{6.7.94}$$

therefore

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{\mathbf{x}^2+\mathbf{x}'^2-2\mathbf{x}\cdot\mathbf{x}'}, \tag{6.7.95}$$

But $\mathbf{x}\cdot\mathbf{x}'=0$, hence

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{z^2+r'^2}. \tag{6.7.96}$$

Since the maximum radial extension of the string is $1/\omega$, therefore for $|z| \gg 1/\omega$,

$$|\mathbf{x}-\mathbf{x}'| \simeq |z|. \tag{6.7.97}$$

Thus the integral expression for $2h_{00}$ becomes:

$$2h_{00} \simeq \frac{4GM}{|z|}\int \mathrm{d}^3\mathbf{r}'\Big[T_{00}(x^0-|z|,\mathbf{r}') + \frac{T(x^0-|z|,\mathbf{r}')}{2}\Big]. \tag{6.7.98}$$

Thus

$$g_{00} = \eta_{00} + 2h_{00}, \tag{6.7.99}$$

and from Eq. (6.7.98) leads to

$$\begin{aligned}
g_{00} &= -1 + 4G\int \frac{\mathrm{d}^3\mathbf{r}'}{|\mathbf{x}-\mathbf{x}'|}[T_{00}(x^0-|\mathbf{x}-\mathbf{x}'|,\mathbf{r}') - \frac{\eta_{00}}{2}T(x^0-|\mathbf{x}-\mathbf{x}'|,\mathbf{r}')] \\
&= -1 + \frac{4G}{|z|}\int \mathrm{d}^3\mathbf{r}'\Big[T_{00}(x^0-|z|,\mathbf{r}') + \frac{T(x^0-|z|,\mathbf{r}')}{2}\Big] \\
&= -1 + \frac{4G}{|z|}\int \mathrm{d}^3\mathbf{r}'\Big[\frac{M}{2\pi r'}\delta(r' - \frac{|\sin\omega(t-|z|)|}{\omega})\delta(z) \\
&\qquad - \frac{M}{2\pi r'}\delta(r' - \frac{|\sin\omega(t-|z|)|}{\omega})\delta(z)\sin^2(\omega(t-|z|))\Big]
\end{aligned}$$

$$
\begin{aligned}
&= -1 + \frac{4GM}{|z|}\int_0^\infty r'\mathrm{d}r'\mathrm{d}\phi'\mathrm{d}z\Big[\frac{1}{2\pi r'}\delta(r' - \frac{|\sin\omega(t-|z|)|}{\omega})\delta(z) \\
&\qquad - \frac{1}{2\pi r'}\delta(r' - \frac{|\sin\omega(t-|z|)|}{\omega})\delta(z)\sin^2(\omega(t-|z|)\Big] \\
&= -1 + \frac{4GM}{|z|}\cos^2\omega(t-|z|). \qquad (6.7.100)
\end{aligned}
$$

Finally, we obtain

$$
g_{00}(x) \simeq -\left(1 - \frac{4GM}{|z|}\cos^2\omega(t-|z|)\right), \qquad (6.7.101)
$$

and time slows by a factor

$$
\begin{aligned}
&\frac{1}{(T_2-T_1)}\int_{T_1}^{T_2}\sqrt{-g_{00}}\,\mathrm{d}t \\
&\qquad = 1 - \frac{GM}{|z|}\left[1 + \cos\omega(T_1+T_2-2|z|)\frac{\sin\omega(T_2-T_1)}{\omega(T_2-T_1)}\right], \qquad (6.7.102)
\end{aligned}
$$

relative to a time lapsed of length $(T_2 - T_1)$ in the absence of the string.

CHAPTER VII

CONCLUSION

The present thesis was involved with a systematic analysis of the so-called closed-time path, also known as the expectation value, formalism of quantum physics and in more details of quantum field theory in the functional *differential* treatment via the application of the celebrated and the powerful tool referred to as the *quantum dynamical principle*. The formalism allows one to obtain expectation values as well as of probabilities directly by functional differentiations of a generating functional with respect to external sources coupled to dynamical variables, such as fields, without the necessity of deriving first transition amplitudes associated with a given theory. The emphasis of the application of the general underlying theory in quantum physics was on the role of the environment and its coupling to quantum systems, while in quantum field theory the emphasis was on the far more complex problem dealing with gravitons as the particle excitations of the gravitational field and the graviton propagator which plays a central role in mediating the gravitational interaction between all particles in nature in quantum gravity, as well as of the induced geometry of an external agent (source) in spacetime in a quantum setting. We summarize our most pertinent results and point out additional technical details related to the work carried out here.

A general expression was obtained in Eqs. (4.2.11), (4.2.15) for the transition probability, as a closed-time process, from a given state $|a;0\rangle$ to a state $|b;t\rangle$ in time t responding, in the process to the environment, given by

$$\text{Prob}[(a;0) \to (b;t)]_E = \mathcal{O}(\mathcal{O}')^* \, \langle b;t\,|\,a;0\rangle^{F_1,S_1} \, (\langle b;t\,|\,a;0\rangle^{F_1',S_1'})^* \\ \mathscr{F}[F_2,S_2;F_2',S_2']\Big|, \qquad (7.1)$$

where

$$\mathcal{O} = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau H_I\big(-\mathrm{i}\hbar\frac{\delta}{\delta F_1(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta S_1(\tau)}, -\mathrm{i}\hbar\frac{\delta}{\delta F_2(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta S_2(\tau)}, \tau\big)\right), \quad (7.2)$$

with $\mathcal{O}'$ defined similarly with

$$F_1, S_1, F_2, S_2, \quad (7.3)$$

replaced by

$$F_1', S_1', F_2', S_2', \quad (7.4)$$

respectively, and the presence of the letter E attached to the probability on the left-hand side of Eq. (7.1) is to emphasize the coupling of the environment to the physical system as the latter evolves in time. The functional $\mathscr{F}$ is given by

$$\mathscr{F}[F_2, S_2; F_2', S_2'] = \sum_n \langle B_n; t\,|\,A; 0\rangle^{F_2,S_2}\left(\langle B_n; t\,|\,A; 0\rangle^{F_2',S_2'}\right)^*, \quad (7.5)$$

where the closed-time path concept is emphasized in the above expression, *where* for time 0 to t we have sources F_2, S_2, which in the reversed path from t to 0, we have *a priori* different set of sources F_2' and S_2'. At the end of all manipulations F_2' will be set equal to F_2 and S_2' will be set equal to S_2 which will all taken to be zero. Eq. (7.2) involves functional differentiations, with respect to classical sources, using functional calculus techniques. It is important to note that as the functional $\mathscr{F}$ in Eq. (7.5) cannot be written as the product of two terms one involving the sources F_1, S_1, F_2, S_2, and one involving the sources F_1', S_1', F_2', S_2', one necessarily has to deal directly with transition probabilities of the physical system as it evolves in time in response to the environment rather than amplitudes. In case the amplitudes $\langle b; t\,|\,a; 0\rangle^{F_i,S_i}$ in Eq. (7.1) are not explicitly given for the decoupled physical system from the environment, one may use the integral expression in Eq. (4.2.2) to carry out various approximations suitable for the system in consideration. The main analysis shows the power of the functional differential treatment, involving functional differentiations with respect to classical, thus commuting, functions.

We have subsequently derived a novel expression for the graviton propagator, from Lagrangian field theory, valid for the case when the external source $T_{\mu\nu}$ coupled to the gravitational field is not *a priori* necessarily conserved, by working in a gauge where only two polarization physical states of the graviton arise to ensure positivity in the quantum treatment thus avoiding non-physical states. That such a conservation should *a priori* not to be imposed is a necessary mathematical requirement so that all the ten components of the external source $T_{\mu\nu}$ may be varied independently in order to generate interactions of the gravitational field with matter and produce non-linearities of the gravitational field itself in the functional procedure. The latter requirement arises by noting that such interactions are generated by the application (Manoukian, 1986a; Limboonsong and Manoukian, 2006) of some functional $F[-\mathrm{i}\delta/\delta T_{\mu\nu}]$ to $\langle 0_+ \,|\, 0_- \rangle^T$, where $\langle 0_+ \,|\, 0_- \rangle$ corresponding to other particles, as well as functional derivatives of their corresponding sources in F, have been suppressed to simplify the notation. Accordingly, to vary the ten components of $T_{\mu\nu}$ independently, no conservation may *a priori* be imposed. The $1/\mathbf{k^2}$ terms in Eqs. (5.2.149) - (5.2.152) are apparent singularities due to the sufficient powers in k in the corresponding denominators and the three-dimensional character of space, in the same way that this happens for the photon propagator in the Coulomb gauge in quantum electrodynamics, and give rise to static $1/r$ type interactions complicated by the tensorial character of a spin two object. It is important to note that for a conserved $T_{\mu\nu}$, i.e., for $\partial^\mu T_{\mu\nu} = 0$, all the terms in the propagators in Eq. (5.2.149), with the exception of the terms $(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\sigma\lambda})/2$, do not contribute in Eq. (5.2.153) since *all* the other terms in Eqs. (5.2.151), (5.2.152) involve derivatives of $T_{\mu\nu}$ and the graviton propagator $\Delta_+^{\mu\nu;\sigma\lambda}(x, x')$ effectively *goes over* to the well documented expression

$$\frac{1}{(-\Box - \mathrm{i}\epsilon)} \frac{(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\sigma\lambda})}{2}, \tag{7.6}$$

which has been known for years (Schwinger, 1970, 1976; Manoukian, 1990, 1997). This is unlike the corresponding time-ordered product which does not go over to the result in Eq. (7.6) for $\partial^\mu T_{\mu\nu} = 0$. This may be shown by solving for the time-ordered product in Eq. (5.1.6) in terms of the propagator and carrying out explicitly, say, the functional derivatives $\delta h^{0i}/\delta T_{\mu\nu}$, $\delta h^{00}/\delta T_{\mu\nu}$, as arising on the right-hand side of Eq. (5.1.6), by using, in the process, Eqs. (5.2.85), (5.2.86). In any case, it is the propagator $\Delta_+^{\mu\nu;\sigma\lambda}$, as given in Eq. (5.2.149), is the one that appears in the theory and not the time-ordered product as is often naïvely assumed. After all the functional derivatives with respect to $T_{\mu\nu}$ are carried out in the theory, one may impose a conservation law on $T_{\mu\nu}$ or even set $T_{\mu\nu}$ equal to zero if required on physical grounds.

It is unlike the expression in Eq. (7.6) involving only 3 terms as expected naïvely by imposing, rather incorrectly, first a conservation law $\partial_\mu T^{\mu\nu} = 0$ and then carrying out functional differentiations of $\langle 0_+ \,|\, 0_- \rangle$ with respect to $T^{\mu\nu}$. In this respect see also, for example, Eqs. (6.1.1) - (6.1.3) where a contradictory result is obtained by the incorrect procedure just described. That the *additional terms* obtained for our graviton propagator *contribute* is easily seen by explicit functional differentiations of $\langle 0_+ \,|\, 0_- \rangle$ given in Eq. (5.2.153) in defining the correlation functions:

$$(-\mathrm{i})\frac{\delta}{\delta T_{\mu_1\nu_1}(x_1)} \cdots (-\mathrm{i})\frac{\delta}{\delta T_{\mu_n\nu_n}(x_n)} \langle 0_+ \,|\, 0_- \rangle \Big|_{\partial_\mu T^{\mu\nu}=0} \tag{7.7}$$

where the conservation law $\partial_\mu T^{\mu\nu} = 0$ is imposed *after* all the functional differentiations $(-\mathrm{i})\delta/\delta T_{\mu_1\nu_1}(x_1), \cdots, (-\mathrm{i})\delta/\delta T_{\mu_n\nu_n}(x_n)$ are carried out. Such methods have led to the discovery (Manoukian, 1986; Limboonsong and Manoukian, 2006), in the functional differential treatment via the quantum dynamical principle differential approach, of Faddeev–Popov (FP) factors, and of their *generalizations*, in non-abelian gauge theories such as in QCD and in other theories. Re-iterating the discussion above, the relevance of the analysis and the explicit expression derived for the graviton propagator for, *a priori*, not conserved external source $T_{\mu\nu} : \partial^{\mu} T_{\mu\nu} \neq 0$ is immediate. If, in contrast, a conservation law is *a priori*, imposed then variations with respect to one of the components of $T_{\mu\nu}$ would automatically imply, via such a conservation law, variations with respect some of its *other* components as well. Also, as mentioned above, the present method, based on the functional differential treatment, as applied to non-abelian gauge theories such as QCD (Manoukian, 1986a; Limboonsong and Manoukian, 2006) leads automatically to the presence of the FP determinant modifying naïve Feynman rules. The *physical* relevance of such a factor is important as its omission would lead to a violation of unitarity. The novel expression for the graviton propagator derived involved 30 terms as just mentioned and is given in detail in Eqs. (5.2.149) - (5.2.152) and is too complicated to be reproduced here. It is interesting to extend the analyses in (Manoukian, 1986a; Limboonsong and Manoukian, 2006), as well as of gauge transformations (Manoukian, 1986a), and covariance (Manoukian, 1987b), to theories involving gravity. This would be exponentially much harder to do and will be attempted in future investigations. In this regard, our ultimate interest in the future is in aspects of renormalizability (Manoukian, 1983) and rules for additional physical applications that would follow from our, *a priori*, systematic analysis carried out at the outset, in a quantum setting with the newly modified propagator, by a functional *differential* treatment, in the presence of external sources, to generate non-linearities in gravitation and interactions with matter.

The positivity constraint as well as the spin content of the theory of gravitons interacting with *a priori* non-conserved external energy-momentum tensor was established. As emphasized throughout, relaxing this conservation law is necessary so that variations of the ten components of the energy-momentum tensor may be varied independently which goes to the heart of the functional differential formalism of quantum field theory. The expectation value formalism of the theory within the above context was derived at finite temperature for gravitons in chapter VI and is given by

$$\begin{aligned}\mathscr{F}[T^1, T^2; \tau] = &(\langle 0_+ \,|\, 0_- \rangle^{T^2})^* (\langle 0_+ \,|\, 0_- \rangle^{T^1}) \\ &\times \exp\left[8\pi G \int (\mathrm{d}x)(\mathrm{d}x') T^2_{\mu\nu}(x) C^{\mu\nu,\sigma\rho}(x,x') T^1_{\sigma\rho}(x') \right] \\ &\times \exp\left[-8\pi G \int (\mathrm{d}x)(\mathrm{d}x') (T^1_{\mu\nu}(x) - T^2_{\mu\nu}(x)) \right. \\ &\left. \times D^{\mu\nu,\sigma\rho}(x,x';\tau)(T^1_{\sigma\rho}(x') - T^2_{\sigma\rho}(x')) \right], \qquad (7.8)\end{aligned}$$

where $C^{\mu\nu,\sigma\rho}(x,x')$ is defined by

$$C^{\mu\nu,\sigma\rho}(x,x') = \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}k(x-x')} \pi^{\mu\nu,\sigma\rho}(k), \qquad (7.9)$$

and

$$D^{\mu\nu,\sigma\rho}(x,x';\tau) = \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}k(x-x')} \frac{\pi^{\mu\nu,\sigma\rho}(k)}{(\mathrm{e}^{-\beta(Nk)} - 1)}, \qquad (7.10)$$

$$Nk = N_\alpha k^\alpha = -k^0 = -|\mathbf{k}|, \qquad (7.11)$$

where $\pi^{\mu\nu,\sigma\rho}(k)$ is given by

$$\pi^{\mu\nu,\sigma\rho}(k) = \frac{1}{2}\left(\beta^{\mu\sigma}\beta^{\nu\rho} + \beta^{\mu\rho}\beta^{\nu\sigma} - \beta^{\mu\nu}\beta^{\sigma\rho}\right). \qquad (7.12)$$

and

$$\beta^{\mu\nu}(k) = \left[\eta^{\mu\nu} - \frac{k^\mu k^\nu}{(Nk)^2} - \frac{N^\mu k^\nu}{(Nk)} - \frac{N^\nu k^\mu}{(Nk)} \right], \qquad (7.13)$$

We note that the temperature dependence occurs only in the last exponential in

Eq. (7.8) through $D^{\mu\nu,\sigma\rho}(x,x';\tau)$. We eventually set

$$T^1_{\mu\nu} = T^2_{\mu\nu}, \tag{7.14}$$

in applications.

Thermal averages of the generated gravitational field and their correlations may be then obtained by functional differentiations of the resulting generating functional at finite temperature which coincide with the corresponding expectation values $\langle 0_- | \cdot | 0_- \rangle$ at zero temperature. The covariance of the induced Riemann curvature tensor was established in spite of the gauge constraint which ensures only two polarization states of the graviton. An application was carried out to determine the induced correction to the Minkowski metric resulting from a closed string arising from the Nambu action as a solution of a circularly oscillating string.

In addition to several results derived in this context, we have explicitly obtained an expression for the time slowing factor for an observer situated at a point z along the vertical axis situated perpendicular to the plane of oscillations of the string above the center origin for $|z| \gg 1/\omega$, where ω is the frequency of oscillations of the string, and in the present units $1/\omega$ denotes the maximum radial extension of the string. This time slowing factor is given by

$$\frac{1}{(T_2 - T_1)} \int_{T_1}^{T_2} \sqrt{-g_{00}}\mathrm{d}t = 1 - \frac{GM}{|z|}\left[1 + \cos\omega(T_1 + T_2 - 2|z|)\frac{\sin\omega(T_2 - T_1)}{\omega(T_2 - T_1)}\right], \tag{7.15}$$

relative to a time lapsed of length $(T_2 - T_1)$ in the absence of the string. Radiative corrections play an important role as the induced geometry may, in general, depend on temperature. Technically, this may be seen as follows. The multiplicative factor in the generating functional $\mathscr{F}[T^1, T^2; \tau]$ in Eq. (7.8) depending on temperature is given by

$$\exp\left[-8\pi G \int (\mathrm{d}x)(\mathrm{d}x')\Big(T^1_{\mu\nu}(x) - T^2_{\mu\nu}(x)\Big) D^{\mu\nu,\sigma\rho}(x,x';\tau)\Big(T^1_{\sigma\rho}(x') - T^2_{\sigma\rho}(x')\Big)\right], \tag{7.16}$$

where $D^{\mu\nu,\sigma\rho}(x,x';\tau)$ is defined in Eqs. (7.10), (7.11) - (7.13). Consider a familiar correction to the leading order in the Lagrangian density given by $h^{\mu\nu}(x)\left(\tau_{\mu\nu} + T^{(m)}_{\mu\nu}\right)$, where $\tau_{\mu\nu}$, $T^{(m)}_{\mu\nu}$ are energy-momentum tensors of the gravitational field and matter, respectively. For example, if $T^{(m)}_{\mu\nu}$ corresponds to a real scalar field coupled in turn to an external source $K(x)$, then the multiplicative factor in the corresponding generating functional of the scalar field depending on temperature is clearly given by

$$\exp\left[-\int (\mathrm{d}x)(\mathrm{d}x')\Big(K^1(x) - K^2(x)\Big)\Delta^{(+)}(x,x';\tau)\Big(K^1(x') - K^2(x')\Big)\right], \tag{7.17}$$

where

$$\Delta^{(+)}(x,x';\tau) = \int \frac{\mathrm{d}^3\mathbf{k}\,\mathrm{e}^{\mathrm{i}k(x-x')}}{(2\pi)^3 2\sqrt{\mathbf{k}^2+m^2}}\left(\mathrm{e}^{\beta\sqrt{\mathbf{k}^2+m^2}} - 1\right)^{-1}, \tag{7.18}$$

$k^0 = +\sqrt{\mathbf{k}^2+m^2}$, and m is the mass of the scalar field. Now both $\tau_{\mu\nu}$ and $T^{(m)}_{\mu\nu}$ are *quadratic* in their respective fields. To generate the term $h^{\mu\nu}\tau_{\mu\nu}$, we then need to functionally differentiate Eq. (7.16), say, with the external source $T^1_{\mu\nu}$ *three* times, also additively with respect to $T^2_{\mu\nu}$ according to the quantum dynamical principle (Schwinger, 1961; Manoukian, 1987c, 1988b, 1988c, 1991b). On the other, hand to generate $T^{(m)}_{\mu\nu}$, we have to functionally differentiate Eq. (7.17) twice with respect to the external sources $K^{1,2}$ of the scalar field. Finally to generate the thermal average of $h_{\mu\nu}$, we have to functionally differentiate once more w.r.t. $T^1_{\mu\nu}$ and then set $T^1_{\mu\nu} = T^2_{\mu\nu} \equiv T_{\mu\nu}$, and $K^1 = K^2 \equiv K$. That is, all in all, we have an *even* number of functional differentiations with respect to the corresponding external sources to generate the thermal average $\langle h_{\mu\nu}\rangle^T_\tau$ before setting the equality of the sources just mentioned and thus generate a temperature dependence in $\langle h_{\mu\nu} \rangle_{\tau}^{T}$. This is unlike the situation in the leading order in which we have to differentiate only once w.r.t. $T^1_{\mu\nu}$ to generate $\langle h_{\mu\nu} \rangle_{\tau}^{T}$ before setting $T^1_{\mu\nu} - T^2_{\mu\nu} = 0$, resulting no temperature dependence in the former expression as seen in Eq. (6.6.1). The study of higher orders, however, requires a detailed analysis of Faddeev-Popov-like factors of the type discovered in (Limboonsong and Manoukian, 2006; Manoukian and Sukkhasena, 2007b), as generated in the functional differential treatment (see Sect.3 in Manoukian and Sukkhasena, 2007b; Manoukian, 1986a, 1987a; Limboonsong and Manoukian, 2006; Manoukian et al., 2007) which would in turn lead to extra vertices coming from the second term on the right-hand side of Eq. (6.1.6) and its generalizations and complicates matter quite a bit in gravitation. This formidable problem as well as convergence aspects (Manoukian, 1983) will be hopefully investigated in the future. Physically, temperature dependence of the underlying induced geometry is also clear. When we perform a thermal average, we introduce in the process, a *background* of gravitons, and in general other particles depending on the matter fields considered. These particles in turn would then act as additional *sources* of gravitation contributing to the net induced gravitational field and this happens *only* when non-linearities as field *interactions* are considered, and corresponding radiative corrections are taken into account.

We hope that our contribution to the expectation value formalism at any temperature for gravitons in the functional differential formalism of quantum field theory will be important for further developments of the very challenging and long standing problems of quantum gravity as the progress of the underlying theory has been slow due to the well known non-renormalizability of Einstein's theory of gravitation.

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APPENDIX

PUBLICATIONS

Progress of Theoretical Physics, Vol. 116, No. 5, November 2006

Functional Treatment of Quantum Scattering via the Dynamical Principle

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A careful functional treatment of quantum scattering is given using Schwinger's dynamical principle which involves a functional differentiation operation applied to a generating functional written in closed form. For long range interactions, such as for the Coulomb one, it is shown that this expression may be used to obtain explicitly the asymptotic "free" modified Green function near the energy shell.

§1. Introduction

The purpose of this communication is to use Schwinger's[6),10)–13)] most elegant quantum dynamical principle to provide a careful functional treatment of quantum scattering. We derive rigorously an expression for the scattering amplitude involving a functional differentiation operation applied to a functional, depending on the potential, written in closed form. The main result of this paper is given in Eq. (2·28). In particular, it provides a systematic starting point for studies of deviations from so-called straight-line "trajectories" of particles, with small deviation angles, by mere functional differentiations. An investigation of a time limit of a function related to this expression shows that the latter may be also used to obtain the asymptotic "free" modified Green functions for theories with long range potentials such as for the Coulomb potential with the latter defining the transitional potential between short and long range potentials. Functional methods have been also introduced earlier in the literature[1)–5),9),14)–17)] in quantum scattering dealing with path integrals or variational optimization methods which, however, are not in the spirit of the present paper based on the dynamical principle. The present study is an adaptation of quantum field theory methods[7)] to quantum potential scattering.

§2. Functional treatment of scattering

Given a Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}) \tag{2·1}$$

for a particle of mass m interacting with a potential $V(\boldsymbol{x})$, we introduce a Hamiltonian $H'(\lambda, \tau)$ involving external sources $\boldsymbol{F}(\tau)$, $\boldsymbol{S}(\tau)$ coupled linearly to $\boldsymbol{x}$ and $\boldsymbol{p}$ as

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follows:

$$H'(\lambda,\tau)=\frac{\boldsymbol{p}^2}{2m}+\lambda V(\boldsymbol{x})-\boldsymbol{x}\cdot\boldsymbol{F}(\tau)+\boldsymbol{p}\cdot\boldsymbol{S}(\tau), \tag{2·2}$$

where λ is an arbitrary parameter which will be eventually set equal to one. Schwinger's[10)–13),6)] dynamical principle states, that the variation of the transformation function $\langle \boldsymbol{x}t\,|\,\boldsymbol{p}t'\rangle$ with respect to the parameter λ for the theory governed by the Hamiltonian $H'(\lambda,\tau)$ is given by

$$\delta\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle=\left(-\frac{i}{\hbar}\right)\int_{t'}^{t}d\tau\,\delta\left(\lambda V\left(-i\hbar\frac{\delta}{\delta\boldsymbol{F}(\tau)}\right)\right)\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle. \tag{2·3}$$

Here $V(-i\hbar\delta/\delta\boldsymbol{F}(\tau))$ denotes $V(\boldsymbol{x})$ with $\boldsymbol{x}$ in it replaced by $-i\hbar\delta/\delta\boldsymbol{F}(\tau)$. Equation (2·3) may be readily integrated for $\lambda=1$, $\boldsymbol{F}(\tau)$, $\boldsymbol{S}(\tau)$ set equal to zero, that is for the theory governed by the Hamiltonian H in (2·1), to obtain

$$\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle=\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}d\tau\,V\left(-i\hbar\frac{\delta}{\delta\boldsymbol{F}(\tau)}\right)\right]\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle^{(0)}\Big|_{\boldsymbol{F}=0,\boldsymbol{S}=0}. \tag{2·4}$$

The transformation function $\langle \boldsymbol{x}t\,|\,\boldsymbol{p}t'\rangle^{(0)}$ corresponds to a theory developing in time via the Hamiltonian

$$H'(0,\tau)=\frac{\boldsymbol{p}^2}{2m}-\boldsymbol{x}\cdot\boldsymbol{F}(\tau)+\boldsymbol{p}\cdot\boldsymbol{S}(\tau), \tag{2·5}$$

to which we now pay special attention.

With $\boldsymbol{p}$ replaced by $i\hbar\delta/\delta\boldsymbol{S}(\tau)$, the dynamical principle, exactly as in (2·4), gives

$$\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle^{(0)}=\exp\left[-\frac{i}{2m\hbar}\int_{t'}^{t}d\tau\left(i\hbar\frac{\delta}{\delta\boldsymbol{S}(\tau)}\right)^2\right]\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle_0, \tag{2·6}$$

where the transformation function $\langle \boldsymbol{x}t\,|\,\boldsymbol{p}t'\rangle_0$ is governed by the "Hamiltonian"

$$\hat{H}(\tau)=-\boldsymbol{x}\cdot\boldsymbol{F}(\tau)+\boldsymbol{p}\cdot\boldsymbol{S}(\tau), \tag{2·7}$$

involving no kinetic energy term.

The Heisenberg equations corresponding to $\hat{H}(\tau)$ give the equations

$$\boldsymbol{x}(\tau)=\boldsymbol{x}(t)-\int_{t'}^{t}d\tau'\,\Theta(\tau'-\tau)\boldsymbol{S}(\tau'), \tag{2·8}$$

$$\boldsymbol{p}(\tau)=\boldsymbol{p}(t')+\int_{t'}^{t}d\tau'\,\Theta(\tau-\tau')\boldsymbol{F}(\tau'), \tag{2·9}$$

where $\Theta(\tau)$ is the step function $\Theta(\tau)=1$ for $\tau>0$ and $=0$ for $\tau<0$. Using the relations

$${}_0\langle \boldsymbol{x}t|\,\boldsymbol{x}(t)=\boldsymbol{x}\,\langle \boldsymbol{x}t|\,, \tag{2·10}$$

$$\boldsymbol{p}(t')\left|\boldsymbol{p}t'\right\rangle_0=\left|\boldsymbol{p}t'\right\rangle\boldsymbol{p}, \tag{2·11}$$

and the dynamical principle, we obtain from taking the matrix elements of $\boldsymbol{x}(\tau)$, $\boldsymbol{p}(\tau)$ in (2·8) and (2·9) between the states ${}_0\langle \boldsymbol{x}t|$, $|\boldsymbol{p}t'\rangle_0$, the functional differential equations

$$-i\hbar\frac{\delta}{\delta \boldsymbol{F}(\tau)}\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle_0 = \left[\boldsymbol{x} - \int_{t'}^{t} d\tau'\,\Theta(\tau'-\tau)\boldsymbol{S}(\tau')\right]\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle_0 , \tag{2·12}$$

$$i\hbar\frac{\delta}{\delta \boldsymbol{S}(\tau)}\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle_0 = \left[\boldsymbol{p} + \int_{t'}^{t} d\tau'\,\Theta(\tau-\tau')\boldsymbol{F}(\tau')\right]\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle_0 . \tag{2·13}$$

These equations may be integrated to yield

$$\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle_0 = \exp\left[\frac{i}{\hbar}\boldsymbol{x}\cdot\left(\boldsymbol{p}+\int_{t'}^{t} d\tau\,\boldsymbol{F}(\tau)\right)\right]\exp\left[-\frac{i}{\hbar}\boldsymbol{p}\cdot\int_{t'}^{t} d\tau\,\boldsymbol{S}(\tau)\right]$$
$$\times\exp\left[-\frac{i}{\hbar}\int_{t'}^{t} d\tau\int_{t'}^{t} d\tau'\,\boldsymbol{S}(\tau)\cdot\boldsymbol{F}(\tau')\Theta(\tau-\tau')\right], \tag{2·14}$$

satisfying the familiar boundary condition $\exp(i\boldsymbol{x}\cdot\boldsymbol{p}/\hbar)$ for $\boldsymbol{F}$, $\boldsymbol{S}$ set equal to zero.

Since we are interested in (2·4), in particular, for the case when $\boldsymbol{S}$ is set equal to zero, the functional differentiation in (2·6) may be easily carried out giving

$$\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle^{(0)}\Big|_{\boldsymbol{S}=\boldsymbol{0}} = \exp\left[\frac{i}{\hbar}\left(\boldsymbol{x}\cdot\boldsymbol{p}-\frac{\boldsymbol{p}^2}{2m}(t-t')\right)\right]$$
$$\times\exp\left[\frac{i}{\hbar}\int_{t'}^{t} d\tau\,\boldsymbol{F}(\tau)\cdot\left(\boldsymbol{x}-\frac{\boldsymbol{p}}{m}(t-\tau)\right)\right]$$
$$\times\exp\left[-\frac{i}{2m\hbar}\int_{t'}^{t} d\tau\int_{t'}^{t} d\tau'\,\boldsymbol{F}(\tau)\cdot\boldsymbol{F}(\tau')(t-\tau_>)\right], \tag{2·15}$$

where $\tau_>$ is the largest of τ and $\tau' : \tau = \max(\tau,\tau')$.

We recall from (2·4) that we eventually set $\boldsymbol{F}(\tau)$ equal to zero. This allows us to interchange the exponential factor in (2·4) involving the $V(-i\hbar\delta/\delta\boldsymbol{F}(\tau))$ term and the last two exponential factors in (2·15). This gives for $\langle \boldsymbol{x}t|\boldsymbol{p}t'\rangle$ in (2·4) the expression

$$\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{p}t'\right\rangle = \exp\left[\frac{i}{\hbar}\left(\boldsymbol{x}\cdot\boldsymbol{p}-\frac{\boldsymbol{p}^2}{2m}(t-t')\right)\right]$$
$$\times\exp\left[\frac{i\hbar}{2m}\int_{t'}^{t} d\tau\int_{t'}^{t} d\tau'\,[t-\tau_>]\,\frac{\delta}{\delta\boldsymbol{F}(\tau)}\cdot\frac{\delta}{\delta\boldsymbol{F}(\tau')}\right]$$
$$\times\exp\left[-\frac{i}{\hbar}\int_{t'}^{t} d\tau\,V\left(\boldsymbol{x}-\frac{\boldsymbol{p}}{m}(t-\tau)+\boldsymbol{F}(\tau)\right)\right]\Bigg|_{\boldsymbol{F}=\boldsymbol{0}}. \tag{2·16}$$

Since we finally set $\boldsymbol{F}=\boldsymbol{0}$ in (2·16), the theory becomes translational invariant in time and $\langle \boldsymbol{x}t|\boldsymbol{p}t'\rangle$ is a function of $t-t'\equiv T$.

For $t>t'$, we have the definition of the Green function

$$\left\langle \boldsymbol{x}t\,\middle|\,\boldsymbol{x}'t'\right\rangle = G_+(\boldsymbol{x}t,\boldsymbol{x}'t'), \tag{2·17}$$

with $G_+(\boldsymbol{x}t,\boldsymbol{x}'t')$=0 for $t < t'$, and

$$\begin{aligned}\langle \boldsymbol{x}t\,|\,\boldsymbol{p}t'\rangle &= G_+(\boldsymbol{x}t,\boldsymbol{p}t') \\ &= \int d^3\boldsymbol{x}'\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}'/\hbar} G_+(\boldsymbol{x}t,\boldsymbol{x}'t'). \end{aligned} \tag{2·18}$$

We may now introduce the Fourier transform defined by

$$G_+(\boldsymbol{p},\boldsymbol{p}';p^0) = -\frac{i}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty dT\, e^{i(p^0+i\epsilon)T/\hbar}\int d^3\boldsymbol{x}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\,\langle \boldsymbol{x}T\,|\,\boldsymbol{p}'0\rangle, \tag{2·19}$$

for $\epsilon \to +0$, where $\langle \boldsymbol{x}T\,|\,\boldsymbol{p}0\rangle$ is given in (2·16) with $t - t' \equiv T$. From (2·19) and (2·16), we may rewrite $G_+(\boldsymbol{p},\boldsymbol{p}';p^0)$ as

$$G_+(\boldsymbol{p},\boldsymbol{p}';p^0) = -\frac{i}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty d\alpha\, e^{i[p^0-E(\boldsymbol{p}')+i\epsilon]\alpha/\hbar}\int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')/\hbar}K(\boldsymbol{x},\boldsymbol{p}';\alpha), \tag{2·20}$$

where $E(\boldsymbol{p}) = \boldsymbol{p}^2/2m$,

$$\begin{aligned} K(\boldsymbol{x},\boldsymbol{p}';\alpha) = \exp&\left[\frac{i\hbar}{2m}\int_{t'}^{t} d\tau\int_{t'}^{t} d\tau'[t-\tau_>]\frac{\delta}{\delta\boldsymbol{F}(\tau)}\cdot\frac{\delta}{\delta\boldsymbol{F}(\tau')}\right] \\ \times\exp&\left[-\frac{i}{\hbar}\int_{t'}^{t} d\tau\, V\left(\boldsymbol{x}-\frac{\boldsymbol{p}'}{m}(t-\tau)+\boldsymbol{F}(\tau)\right)\right]\Bigg|_{\boldsymbol{F}=\boldsymbol{0}} \end{aligned} \tag{2·21}$$

with $t - t' \equiv \alpha$ playing the role of time − a notation used for it quite often in field theory.

In the α-integrand in the exponential in (2·20), we recognize $[p^0 - E(\boldsymbol{p}) + i\epsilon]$ as the inverse of the free Green function in the energy-momentum representation.

The scattering amplitude $f(\boldsymbol{p},\boldsymbol{p}')$ for scattering of the particle with initial and final momenta $\boldsymbol{p}'$, $\boldsymbol{p}$, respectively, is defined by

$$f(\boldsymbol{p},\boldsymbol{p}') = -\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{p}''\, V(\boldsymbol{p}-\boldsymbol{p}'')G_+(\boldsymbol{p}'',\boldsymbol{p}';p^0)[p^0-E(\boldsymbol{p}')]\Big|_{p^0=E(\boldsymbol{p}')}, \tag{2·22}$$

where $V(\boldsymbol{p}) = \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot\boldsymbol{p}/\hbar}V(\boldsymbol{x})$. This suggests to multiply (2·20) by $[p^0 - E(\boldsymbol{p}')]$ giving

$$\begin{aligned} G_+(\boldsymbol{p},\boldsymbol{p}';p^0)[p^0-E(\boldsymbol{p}')] = &-\frac{1}{(2\pi\hbar)^3}\int_0^\infty d\alpha\left(\frac{\partial}{\partial\alpha}e^{i\alpha[p^0-E(\boldsymbol{p}')+i\epsilon]/\hbar}\right) \\ &\times\int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')/\hbar}K(\boldsymbol{x},\boldsymbol{p}';\alpha). \end{aligned} \tag{2·23}$$

From the fact that $\langle \boldsymbol{x}\,|\,\boldsymbol{p}\rangle = \exp(i\boldsymbol{x}\cdot\boldsymbol{p}/\hbar)$ and the definition of $K(\boldsymbol{x},\boldsymbol{p}';\alpha)$ in (2·21), we have

$$K(\boldsymbol{x},\boldsymbol{p}';0) = 1. \tag{2·24}$$

We now consider the cases for which

$$\lim_{\alpha\to\infty}\int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')/\hbar}K(\boldsymbol{x},\boldsymbol{p}';\alpha), \tag{2·25}$$

exists. This, in particular, implies that ($\epsilon > 0$)

$$\lim_{\alpha\to\infty} e^{-\epsilon\alpha} \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')/\hbar} K(\boldsymbol{x},\boldsymbol{p}';\alpha) = 0. \tag{2·26}$$

We may then integrate over α in (2·23) to obtain simply

$$G_+(\boldsymbol{p},\boldsymbol{p}';p^0)[p^0 - E(\boldsymbol{p}')]\Big|_{p^0=E(\boldsymbol{p}')} = \lim_{\alpha\to\infty} \frac{1}{(2\pi\hbar)^3} \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')/\hbar} K(\boldsymbol{x},\boldsymbol{p}';\alpha), \tag{2·27}$$

on the energy shell $p^0 = E(\boldsymbol{p})$, and for the scattering amplitude, in (2·22), *after* integrating over $\boldsymbol{p}''$, the expression

$$f(\boldsymbol{p},\boldsymbol{p}') = -\frac{m}{2\pi\hbar^2} \lim_{\alpha\to\infty} \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')/\hbar} V(\boldsymbol{x}) K(\boldsymbol{x},\boldsymbol{p}';\alpha), \tag{2·28}$$

with $K(\boldsymbol{x},\boldsymbol{p}';\alpha)$ defined in (2·21). Here we recognize that the formal replacement of $K(\boldsymbol{x},\boldsymbol{p}';\alpha)$ by one gives the celebrated Born approximation. On the other hand, part of the argument $[\boldsymbol{x}-\boldsymbol{p}'(t-\tau)/m]$ of $V(\boldsymbol{x}-\boldsymbol{p}'(t-\tau)/m+\boldsymbol{F}(\tau))$ in (2·21), represents a "straight line trajectory" of a particle, with the functional differentiations with respect to $\boldsymbol{F}(\tau)$, as defined in (2·21), leading to deviations of the dynamics from such a straight line trajectory. With a straight line approximation, ignoring all of the functional differentiations, with respect to $\boldsymbol{F}(\tau)$ and setting the latter equal to zero, gives the following explicit expression for the scattering amplitude $f(\boldsymbol{p},\boldsymbol{p}')$ in (2·28):

$$f(\boldsymbol{p},\boldsymbol{p}') = -\frac{m}{2\pi\hbar^2} \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')/\hbar} V(\boldsymbol{x}) \exp\left[-\frac{i}{\hbar}\int_0^\infty d\alpha\, V\left(\boldsymbol{x}-\frac{\boldsymbol{p}'}{m}\alpha\right)\right]. \tag{2·29}$$

This modifies the Born approximation by the presence of an additional phase factor in the integrand in (2·29), depending on the potential, accumulated during the scattering process. Here one recognizes the expression which leads to scattering with small deflections at high energies (the so-called eikonal approximation) obtained from the straight line trajectory approximation discussed above. Deviations from this approximation may be then systematically obtained by carrying out a functional power series expansion of $V(\boldsymbol{x}-\boldsymbol{p}'(t-\tau)/m+\boldsymbol{F}(\tau))$ in $\boldsymbol{F}(\tau)$ and performing the functional differential operation as dictated by the first exponential in (2·21) and finally setting $\boldsymbol{F}(\tau)$ equal to zero.

We note that formally that the τ-integral, involving the potential V, in (2·21) increases with no bound for $\alpha \to \infty$ for the Coulomb potential and for potentials of longer range with the former potential defining the transitional potential between long and short range potentials. And in case that the limit in (2·25) does not exist, as encountered for the Coulomb potential, (2·23) cannot be integrated by parts. This is discussed in the next section.

§3. Asymptotic "free" Green function

In case the $\alpha \to \infty$ limit in (2·25) does not exist, one may study the behaviour of $G_+(\boldsymbol{p}, \boldsymbol{p}'; p^0)$ near the energy shell $p^0 \simeq \boldsymbol{p}'^2/2m$ directly from (2·20). To this end, we introduce the integration variable

$$z = \frac{\alpha}{\hbar}\left[p^0 - E(\boldsymbol{p}')\right], \tag{3·1}$$

in (2·20), to obtain

$$G_+(\boldsymbol{p}, \boldsymbol{p}'; p^0)[p^0 - E(\boldsymbol{p}')] = -\frac{i}{(2\pi\hbar)^3}\int_0^\infty dz\, e^{iz(1+i\epsilon)} \times \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')/\hbar} K\left(\boldsymbol{x}, \boldsymbol{p}'; \frac{z\hbar}{p^0 - E(\boldsymbol{p}')}\right), \tag{3·2}$$

for $\epsilon \to +0$. For $p^0 - E(\boldsymbol{p}') \gtrsim 0$, i.e., near the energy shell, we may substitute

$$K\left(\boldsymbol{x}, \boldsymbol{p}'; z\hbar/(p^0 - E(\boldsymbol{p}'))\right) \simeq \exp\left[-\frac{i}{\hbar}\int_0^{z\hbar/(p^0-E(\boldsymbol{p}'))} d\alpha\, V\left(\boldsymbol{x} - \frac{\boldsymbol{p}'}{m}\alpha\right)\right], \tag{3·3}$$

in (3·2) to obtain for the following integral

$$\int d^3\boldsymbol{p}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar} G_+(\boldsymbol{p}, \boldsymbol{p}'; p^0) \simeq \frac{-ie^{i\boldsymbol{x}\cdot\boldsymbol{p}'/\hbar}}{[p^0 - E(\boldsymbol{p}') + i\epsilon]}\int_0^\infty dz\, e^{iz(1+i\epsilon)} \times \exp\left[-\frac{i}{\hbar}\int_0^{z\hbar/(p^0-E(\boldsymbol{p}'))} d\alpha\, V\left(\boldsymbol{x} - \frac{\boldsymbol{p}'}{m}\alpha\right)\right], \tag{3·4}$$

For the Coulomb potential $V(\boldsymbol{x}) = \lambda/|\boldsymbol{x}|$,

$$\int_0^{z\hbar/(p^0-E(\boldsymbol{p}'))} d\alpha\, V\left(\boldsymbol{x} - \frac{\boldsymbol{p}'}{m}\alpha\right) \simeq \frac{\lambda m}{|\boldsymbol{p}'|}\ln\left(\frac{2|\boldsymbol{p}'|z\hbar}{m(p^0 - E(\boldsymbol{p}'))|\boldsymbol{x}|(1-\cos\theta)}\right), \tag{3·5}$$

where $\cos\theta = \boldsymbol{p}'\cdot\boldsymbol{x}/|\boldsymbol{p}'||\boldsymbol{x}|$. Hence

$$\exp -\frac{i}{\hbar}\int_0^{z\hbar/(p^0-E(\boldsymbol{p}'))} d\alpha\, V\left(\boldsymbol{x} - \frac{\boldsymbol{p}'}{m}\alpha\right) \simeq \frac{1}{[p^0 - E(\boldsymbol{p}') + i\epsilon]^{-i\gamma}} \times \exp -i\gamma\ln\left(\frac{2p'^2 z\hbar}{m(p'x - \boldsymbol{p}'\cdot\boldsymbol{x})}\right), \tag{3·6}$$

where $\gamma = \lambda m/\hbar p'$. Finally using the integral

$$\int_0^\infty dz\, e^{iz(1+i\epsilon)}(z)^{-i\gamma} = ie^{\pi\gamma/2}\Gamma(1 - i\gamma), \tag{3·7}$$

for $\epsilon \to +0$, where Γ is the gamma function, we obtain from (3·4)

$$\int d^3\boldsymbol{p}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar} G_+(\boldsymbol{p}, \boldsymbol{p}'; p^0) \simeq e^{i\boldsymbol{x}\cdot\boldsymbol{p}'/\hbar}\frac{e^{-i\gamma\ln(2p'^2/m)}}{[p^0 - E(\boldsymbol{p}') + i\epsilon]^{1-i\gamma}} \times \exp i\gamma\ln\left(\frac{p'x - \boldsymbol{p}'\cdot\boldsymbol{x}}{\hbar}\right) e^{\pi\gamma/2}\Gamma(1 - i\gamma), \tag{3·8}$$

to be compared with earlier results (e.g., Ref. 8)), and for the asymptotic "free" Green function, in the energy-momentum representation, the expression:

$$G^0_+(\boldsymbol{p}) = \frac{e^{-i\gamma \ln(2p^2/m)}}{[p^0 - E(\boldsymbol{p}) + i\epsilon]^{1-i\gamma}} e^{\pi\gamma/2} \Gamma(1 - i\gamma), \tag{3·9}$$

showing on obvious modification from the Fourier transform of the free Green function $[p^0 - E(\boldsymbol{p}) + i\epsilon]^{-1}$.

§4. Conclusion

The expression (2·28) provides a functional expression for the scattering amplitude with $K(\boldsymbol{x}, \boldsymbol{p}'; \alpha)$ defined in (2·21) and the latter is obtained by the functional differential operation carried out on the functional, involving the potential V, of argument $\boldsymbol{x} - \boldsymbol{p}'(t-\tau)/m + \boldsymbol{F}(\tau)$, for all $t' \leq \tau \leq t$, represented by the first exponential in (2·21). The "straight line trajectory" approximation of a particle consisting of retaining $\boldsymbol{x} - \boldsymbol{p}'(t-\tau)/m$ only in the argument of V and neglecting the functional differentiations with respect to $\boldsymbol{F}(\tau)$, with the latter operation leading systematically to modifications from this linear "trajectory", gives rise to the familiar eikonal approximation. The existence of the time limit in (2·25) distinguishes between so-called potentials of short and long ranges with the Coulomb potential providing the transitional potential between these two general classes of potentials and belonging to the latter class. In case the time limit in (2·25) does not exist, corresponding to potentials of long ranges, (2·20) may be used to obtain the asymptotic "free" modified Green function near the energy shell as seen in §3. In a subsequent report, our functional expression in (2·28) for the scattering amplitude will be generalized for long range potentials as well.

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Variational derivatives of transformation functions in quantum field theory

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Abstract
A systematic explicit derivation is given for variational derivatives of transformation functions in field theory with respect to parameters variations, also known as the quantum dynamical principle (QDP), by introducing, in the process, two unitary time-dependent operators which in turn allow an otherwise non-trivial interchange of the orders of the parameters variations of transformation functions with specific time-dependent ones. Special emphasis is put on dependent fields, as appearing, particularly, in gauge theories, and on the Lagrangian formalism. The importance of the QDP and its practicality as a powerful tool in field theory are spelled out, which cannot be overemphasized, and a complete derivation of it is certainly lacking in the literature. The derivation applies to gauge theories as well.

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1. Introduction

The purpose of this study is to *derive* systematically variational derivatives of transformation functions, also referred to as the quantum dynamical principle (QDP), with respect to parameters occurring in the theory and with respect to external sources, coupled to the underlying fields, in quantum field theory. The very elegant QDP [1–14] is undisputably recognized as a very powerful tool for carrying out explicit computations in quantum field theory, and in the quantization problem, in general. How these applications and constructions are carried out using variational derivatives of transformation functions, derived below, and will be spelled out for the convenience of the reader in the concluding section. In particular, the QDP has been used to quantize gauge theories [10–13] in constructing the vacuum-to-vacuum transition amplitude and the direct generation and *derivation* [10–13] of Faddeev–Popov (FP) [15] factors, encountered in non-abelian gauge theories and their further generalizations [13] with not much effort and without making an appeal to path integrals or to commutation rules and without [10–13] even going into the well known complicated structure of the Hamiltonian [16]. In particular, it has been shown [13] that the so-called FP factor needs to be modified in more general cases of gauge theories and that a gauge invariant theory does not necessarily imply the familiar FP factor for proper quantization as may be otherwise naïvely expected based on symmetry arguments. As the QDP provides the variations of transformation functions with respect to external parameters, such as coupling constants and external sources coupled to the quantum fields, upon integrations of the amplitudes over these parameters yield the expression for the latter (see e.g. [8–14]). To derive variational derivatives of transformation functions, we introduce in the process, two unitary time-dependent operators which in turn allow an otherwise non-trivial interchange of the orders of parameters variations with specific time-dependent ones. This procedure answers the otherwise rather mysterious question as to why the variation of a transformation function, with respect to given parameters, is restricted solely to the variation of the Lagrangian in question with the states defining the transformation function, which may depend on these parameters, kept non-varied! The answer is based, mostly on equations (6) and (7) below and a key identity derived in (9) written in terms of the two unitary time-dependent operators mentioned above. The derivation is an extension of the corresponding one in quantum mechanics [9] to the more complicated case of quantum field theory, where now emphasis is also put on dependent fields, as occurring, particularly, in gauge theories and on the Lagrangian formalism. There has been renewed interest recently in Schwinger's action principle (see e.g. [17–20]) emphasizing generally operator aspects of a theory, as deriving, for example, various commutation relations, rather than dealing with the computational aspects directly related to transformation functions and transition amplitudes through their variational derivatives as done here, and most importantly, to be derived in this work. In the concluding section, we spell out how variational derivatives of transformation functions are used in various aspects of the theory, emphasizing the underlying method as a powerful tool in quantum field theory.

2. The QDP

Consider a Hamiltonian of the general form

$$H(t,\lambda) = H_1(t) + H_2(t,\lambda), \tag{1}$$

where $H_1(t)$, $H_2(t,\lambda)$ may be time-dependent but $H_2(t,\lambda)$ may, in addition, depend on some parameters denoted by λ. Typically, in quantum field theory, $H_1(t)$ may stand for the free Hamiltonian written in terms of the physically observed masses referred to renormalized masses and $H_1(t)$ will be time-independent. In this latter case, $H_2(t,\lambda)$ will denote the remaining part of the Hamiltonian which, in particular, depends on renormalization constants, coupling constants and so-called external sources coupled to the quantum fields. The coupling constants and the external sources will be then collectively denoted by λ. A derivative of a transformation function with respect to λ with the latter denoting an external source will then represent a functional derivative (see e.g. [10]).

The time evolution operator $U(t,\lambda)$, corresponding to the Hamiltonian $H(t,\lambda)$, satisfies the equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}U(t,\lambda) = H(t,\lambda)U(t,\lambda). \tag{2}$$

For the theory given in a specific description, we have

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle at| = \langle at|H(t,\lambda), \tag{3}$$

where the states $\langle at|$ will depend on the parameters λ. Typically, the states $\langle a|$, assumed independent of λ, may represent multi-particle states of free particles associated with a given self-adjoint operator such as the momentum operator, with the single particle energies written in terms of the observed masses, or may represent the vacuum-state. One may also introduce the time evolution operator $U_1(t)$, corresponding to $H_1(t)$, satisfying the equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_1(t) = H_1(t)U_1(t), \tag{4}$$

and the states ${}_1\langle at|$ which are independent of the parameters λ, satisfy

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,{}_1\langle at| = {}_1\langle at|H_1(t). \tag{5}$$

The states $\langle at|$ of interest are related to the states ${}_1\langle at|$ by the equation

$$\langle at| = {}_1\langle at|V(t,\lambda), \tag{6}$$

where

$$V(t,\lambda) = U_1^{\dagger}(t)U(t,\lambda), \tag{7}$$

with the latter satisfying

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}V(t,\lambda) = U_1^{\dagger}(t)H_2(t,\lambda)U(t,\lambda). \tag{8}$$

The QDP is involved with the study of the variation of a transformation function $\langle at_2|bt_1\rangle$, with respect to the parameters λ.

For $\tau \neq t_2$, $\tau \neq t_1$ and $\lambda \neq \lambda'$, we have the following useful *key* identity in the entire analysis

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}\left[V(t_2,\lambda)V^{\dagger}(\tau,\lambda)V(\tau,\lambda')V^{\dagger}(t_1,\lambda')\right]$$
$$= V(t_2,\lambda)\left[U^{\dagger}(\tau,\lambda)(H(\tau,\lambda') - H(\tau,\lambda))U(\tau,\lambda')\right]V^{\dagger}(t_1,\lambda'), \tag{9}$$

which will be subsequently used.

The independent quantum fields of the theory will be denoted by $\chi(x)$ and their canonical conjugate momenta by $\pi(x)$, suppressing all obvious indices. The dependent fields will be denoted by $\eta(x)$ whose canonical conjugate momenta vanish, by definition. Here $x = (t,\ \mathbf{x})$. The Hamiltonian $H(t,\lambda)$ may be then written as

$$H(t,\lambda) = H(\chi,\pi,\lambda,t), \tag{10}$$

which, in particular, is a function of $\chi(\mathbf{x})$, $\pi(\mathbf{x})$ with the latter defined in the so-called Schrödinger representation at $t = 0$, which are independent of λ. In the Heisenberg representation we have

$$\chi(x) = U^{\dagger}(t,\lambda)\chi(\mathbf{x})U(t,\lambda), \tag{11}$$

$$\pi(x) = U^{\dagger}(t,\lambda)\pi(\mathbf{x})U(t,\lambda) \tag{12}$$

having non-trivial dependence on the parameters λ.

Now we integrate the relation in (9) over τ from t_1 to t_2 to obtain

$$\left[V(t_2,\lambda')V^{\dagger}(t_1,\lambda') - V(t_2,\lambda)V^{\dagger}(t_1,\lambda)\right] = -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)$$
$$\times\left[\int_{t1}^{t_2}\mathrm{d}\tau\, U^{\dagger}(\tau,\lambda)(H(\tau,\lambda') - H(\tau,\lambda))U(\tau,\lambda')\right]V(t_1,\lambda'). \tag{13}$$

By setting $\lambda' = \lambda + \delta\lambda$, one obtains the variational form of the above equation

$$\delta[V(t_2,\lambda)V^{\dagger}(t_1,\lambda)]$$
$$= -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\left[\int_{t_1}^{t_2}\mathrm{d}\tau\, U^{\dagger}(\tau,\lambda)\delta H(\tau,\lambda)U(\tau,\lambda)\right]V^{\dagger}(t_1,\lambda). \tag{14}$$

Upon defining the Heisenberg representation of $H(\tau,\lambda)$ at time τ, by

$$I\!H(\tau,\lambda) = U^{\dagger}(\tau,\lambda)H(\chi,\pi,\tau,\lambda)U(\tau,\lambda), \tag{15}$$

we may rewrite (14), as

$$\delta[V(t_2,\lambda)V^{\dagger}(t_1,\lambda)] = -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\left[\int_{t_1}^{t_2}\mathrm{d}\tau\,\delta I\!H(\tau,\lambda)\right]V^{\dagger}(t_1,\lambda) \tag{16}$$

provided the variations of $I\!H$ with respect to λ in (16) are carried out *by keeping* $\chi(x)$, $\pi(x)$, given in (11) and (12), fixed.

We take the matrix elements of (16) with respect to ${}_1\langle at_2|, |bt_1\rangle_1$ (see (5)), use (6), *and* note the λ independence of ${}_1\langle at_2|, |bt_1\rangle_1$, to obtain

$$\delta\,\langle at_2|bt_1\rangle = -\frac{\mathrm{i}}{\hbar}\Big\langle at_2\,\Big|\int_{t_1}^{t_2}\mathrm{d}\tau\;\delta I\!H(\tau,\lambda)\Big|\,bt_1\Big\rangle, \tag{17}$$

with the variation in $I\!H$, with respect to λ, carried out with the independent fields $\chi(x)$ and their canonical conjugate momenta $\pi(x)$ kept *fixed*.

The Hamiltonian $I\!H$ in the Heisenberg representation in (15) may be rewritten as

$$I\!H(t,\lambda) = H(\chi(t),\pi(t),\lambda,t), \tag{18}$$

as obtained from the Hamiltonian $H(t,\lambda)$ in (10) at t, by carrying out the explicit operation given in (15). Equation (18) is, in particular, written in terms of the independent (Heisenberg) fields at time t and their canonical conjugate momenta. The effective Lagrangian L_* of the system is related to $I\!H$ by the equation

$$L_*(\chi(t),\dot\chi(t),\lambda,t) = \int \mathrm{d}^3\mathbf{x}\,\pi(x)\dot\chi(x) - H(\chi(t),\pi(t),\lambda,t), \tag{19}$$

with a summation over the fields understood.

The canonical conjugate momenta $\pi(x)$ of the fields are defined through the equation

$$L_*(\chi(t),\dot\chi(t)+\delta\dot\chi(t),\lambda,t) - L_*(\chi(t),\dot\chi(t),\lambda,t) = \int \mathrm{d}^3\mathbf{x}\;\pi(x)\delta\dot\chi(x). \tag{20}$$

Equations (19) and (20) allow us to consider the variation of $H(\chi(\tau),\pi(\tau),\lambda,\tau)$, with respect to λ, by keeping χ,π fixed as required in (17), in relationship to the variation of L_*. From (19) and (20), we then obtain, with χ,π kept fixed, that

$$\delta L_*(\chi(\tau),\dot\chi(\tau),\lambda,\tau) = -\delta H(\chi(\tau),\pi(\tau),\lambda,\tau), \tag{21}$$

upon *cancellation* of the term on the right-hand side of (20), where, now the variation of L_* in (21) is carried out with respect to λ by keeping $\chi(\tau)$ and $\dot\chi(\tau)$ fixed.

The dependent fields will be denoted by $\eta(x)$ and their canonical conjugate momenta vanish, by definition. The Lagrangian of the underlying field theory may be written as $L(\chi(t),\dot\chi(t),\eta(t),\lambda,t)$, which upon the elimination of $\eta(t)$ in favour of $\chi(t)$, $\dot\chi(t)$ and λ generating the Hamiltonian under study as well as the effective Lagrangian L_*. We consider the variation of L, with respect to λ, by keeping $\chi(t),\dot\chi(t)$ fixed. Now since $\eta(t)$ will, in general, *depend* on λ, we have

$$\delta L = E_\eta\frac{\partial\eta}{\partial\lambda}\delta\lambda + \delta L\Big|_{\chi,\dot\chi,\eta}, \tag{22}$$

where we note that the Lagrangian does not contain terms depending on $\dot\eta$, by definition. The first term on the right-hand side defined as an integral in abbreviated form, E_η in it corresponds to the Euler–Lagrange equation of η, which vanishes, and the second term on the right-hand denotes the variation of L, with respect to λ, by keeping χ, $\dot\chi$ and η fixed. The latter property was first noted in [7]. The Lagrangian density $\mathscr{L} = \mathscr{L}(x) = \mathscr{L}(x,\lambda)$ of the system is related to the Lagrangian L through

$$L(\chi(t),\dot\chi(t),\eta(t),\lambda,t) = \int \mathrm{d}^3\mathbf{x}\,\mathscr{L}(x,\lambda). \tag{23}$$

From (21), (22) and (23), we obtain the celebrated QDP or the Schwinger dynamical (action) principle

$$\delta\langle at_2|bt_1\rangle = \frac{\mathrm{i}}{\hbar}\Big\langle at_2\,\Big|\int_{t_1}^{t_2}(\mathrm{d}x)\,\delta\mathscr{L}(x,\lambda)\Big|\,bt_1\Big\rangle, \tag{24}$$

where $(\mathrm{d}x) = \mathrm{d}t\,\mathrm{d}^3\mathbf{x}$, and the variation $\delta\mathscr{L}(x,\lambda)$, with respect to λ, is carried out with the fields, independent *and* dependent, and their derivatives $\partial_\mu\chi$, $\nabla\eta$, all kept *fixed*. The interesting thing to note is that although the states $|at_2\rangle$, $|bt_1\rangle$ depend on λ, in the variation of the transformation function $\langle at_2|bt_1\rangle$, the same (non-varied) states appear on the right-hand side of (24) with the entire variation being applied to the Lagrangian density $\mathscr{L}(x,\lambda)$ with the fields and their canonical conjugate momenta kept fixed. This is thanks to the U and V operators elaborated upon in (2)–(8), the independence of the states${}_1\langle at_2|,{}_1\langle bt_1|$ of λ, and the key identify given in (9). In practice the limits $t_2\to+\infty$, $t_1\to-\infty$ are taken in (24) in scattering processes.

Now consider an arbitrary function

$$B(\chi(x),\pi(x),\lambda,t) \equiv I\!B(t,\lambda), \tag{25}$$

of the variables indicated, with $\chi(x),\pi(x)$ in the Heisenberg representation in (11) and (12). We may write

$$I\!B(t,\lambda) = U^\dagger(t,\lambda)B(\chi(\mathbf{x}),\pi(\mathbf{x}),\lambda,t)U(t,\lambda), \tag{26}$$

with $\chi(\mathbf{x}),\pi(\mathbf{x})$ on the right-hand side in the Schrödinger representation at time $t=0$. We note the identity

$$\begin{aligned}&V(t_2,\lambda)I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)\\ &\quad= V(t_2,\lambda)V^\dagger(\tau,\lambda)U_1^\dagger(\tau)B(\chi(\mathbf{x}),\pi(\mathbf{x}),\lambda,\tau)\\ &\qquad\times U_1(\tau)V(\tau,\lambda)V^\dagger(t_1,\lambda).\end{aligned} \tag{27}$$

Hence (14) and (27) give for the following variation with respect to λ ($t_1<\tau<\tau_2$)

$$\begin{aligned}&\delta[V(t_2,\lambda)I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)]\\ &\quad= -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_\tau^{t_2}\mathrm{d}\tau'\,\delta H(\tau',\lambda)I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)\\ &\qquad+V(t_2,\lambda)\delta I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)\\ &\qquad-\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_{t_1}^{\tau}\mathrm{d}\tau'\,I\!B(\tau,\lambda)\delta H(\tau',\lambda)V^\dagger(t_1,\lambda),\end{aligned} \tag{28}$$

where according to (28), the variation in $\delta I\!B(\tau,\lambda) = \delta B(\chi(\mathbf{x},\tau),\pi(\mathbf{x},\tau),\lambda,\tau)$, with respect to λ, is carried out by keeping the (Heisenberg) fields $\chi(\mathbf{x},\tau),\pi(\mathbf{x},\tau)$ *fixed*.

We may use the definition of the chronological time ordering product to rewrite (28) in the more compact form

$$\begin{aligned}&\delta[V(t_2,\lambda)I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)]\\ &\quad= -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_{t_1}^{t_2}\mathrm{d}\tau'(I\!B(\tau,\lambda)\delta H(\tau',\lambda))_+V^\dagger(t_1,\lambda)\\ &\qquad+V(t_2,\lambda)\delta I\!B(\tau,\lambda)V^\dagger(t_1,\lambda).\end{aligned} \tag{29}$$

Upon taking the matrix element of (29) with respect to ${}_1\langle at_2|$, $|bt_1\rangle_1$, and using (6), (15) and (24) we have for $t_1 < \tau < t_2$

$$\delta\langle at_2|\mathbb{B}(\tau,\lambda)|bt_1\rangle = \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2}(\mathrm{d}x')\big\langle at_2\,\big|(\mathbb{B}(\tau,\lambda)\delta\mathscr{L}(x',\lambda))_+\big|\,bt_1\big\rangle + \langle at_2\,|\delta\mathbb{B}(\tau,\lambda)|\,bt_1\rangle\,, \quad (30)$$

where in the variation $\delta\mathscr{L}(x',\lambda)$, with respect to λ, all the fields and their derivatives $\partial_\mu\chi$, $\nabla\eta$ are kept fixed, while in $\delta\mathbb{B}(\tau,\lambda)$, *expressed* in terms of $\chi(\mathbf{x},\tau)$, $\pi(\mathbf{x},\tau)$, the latter are kept fixed, and an extra λ-dependence may arise from the elimination of η in favour of χ,π. To our knowledge Equation (30) appears first in [7]. The second term in (30) is *responsible* for the generation of the FP factor and its generalizations in gauge theories (see [10, 12, 13]).

3. Conclusion

The importance of the QDP as a powerful tool in field theory cannot be overemphasized and a detailed derivation of it was given by introducing, in the process, two unitary time-dependent operators. The latter in turn allowed the interchange of variations of transformation functions with respect to given parameters with specific time-dependent operations so crucial for the validity of the QDP. A key identity has been derived in (9) which was essential for the entire derivation. For the convenience of the reader we spell out how variational derivatives of transformation functions are used in some aspects of an underlying theory. (i) The integration of (24) for the QDP over λ is carried out by introducing, in the process, external sources coupled to the fields, where the external sources (currents) are necessarily taken initially to be non-conserved so that variations of all of their components may be varied independently (see [10, 13]). From the expression of the vacuum-to-vacuum transition amplitude, for example, thus obtained, transition amplitudes of *all* processes may be extracted by factoring out amplitudes for the emission and absorption of the underlying particles by the external sources. By functional differentiation of the vacuum-to-vacuum transition amplitude with respect to the external sources, integral equations, such as Schwinger–Dyson equations, relating various Green's functions may be derived. We also recall that the path integral expressions may be derived, for example, directly from the application of the QDP principle (see e.g. [8, 9]). It is also far simpler to carry out (functional) differentiations than to deal with infinite dimensional continual integrals. (ii) In the presence of dependent fields, with no time derivatives of them occurring in their respective field equations, these dependent fields will, in general, be functions of independent fields (and their conjugate momenta) *and* external sources. With the rules set up in (24) and in (30), additional terms will then occur coming from the second term on the right-hand side of (30) by taking functional derivatives of matrix elements of such dependent fields in (30) with respect to external sources by keeping the independent fields (and their conjugate momenta) fixed. Such terms lead precisely to FP factors and their generalizations, for example, in gauge theories, from the applications of (30), as just mentioned, in the present formalism (see [8, 10–14]). For such intricate and additional details, the reader may refer to the just given references as well as to some of the earlier ones such as [21, 22].

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Coupling of Quantum Systems to the Environment: Functional Differential Treatment

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Abstract

A functional differential treatment, via the quantum dynamical principle, is given for the coupling of quantum systems to the environment. As one is involved in taking the trace over the dynamical variables of the environment, the analysis necessarily deals with transition probabilities rather than with amplitudes. It is shown that the functional differential treatment is quite suitable for such a study as it involves in carrying out functional differentiations, with respect to classical sources, on functionals describing decoupled physical systems from the environment.

Mathematics Subject Classification: 35Q40, 49S05, 81Q15, 58D25

Keywords: Functional differential equations and quantum mechanics, variational methods in function spaces, coupling to the environment and quantum decoherence

1 Introduction

The functional differential treatment [2–9, 11–13], via the quantum dynamical principle, has been a very powerful tool for investigating properties of quantum systems and for carrying out explicit computations. In this regard, it has been quite successful in gauge theories and of the generation of essential modifications [2–6] needed for their proper quantization with no much effort. For a pedagogical treatment of the theory and for several applications of the functional differential method, via the quantum dynamical principle, in quantum mechanics, the reader may wish to refer to [7, Ch. 11]. The purpose of this work is to carry out an analysis, using the functional differential approach, of the *coupling* of quantum mechanical systems to the environment, understood

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to be surrounding a physical system, as the former systems, in the real world, are never in isolation from the latter. The incorporation of the environment in quantum mechanical systems has led to much physical insights into such fundamental problems as quantum decoherence, Schrödinger's cat and in measurement theory, in general [7, §8.7, §8.9, §12.7], [1,10,14,15]. We will see, that the functional differential approach is quite suitable for studying the coupling of quantum mechanical systems to the environment. It involves in carrying out functional differentiations, with respect to classical sources, of a functional describing "decoupled" systems from the environment. As one is involved in taking the *trace* over the dynamical variables of the environment in studying the response of physical systems to it, the analysis necessarily involves in dealing directly with transition probabilities rather than amplitudes. This is a basic departure from the far simpler case of studying quantum mechanical systems in isolation. In dealing with probabilities and in taking traces, it turns out that two different sets of classical sources, coupled to the dynamical variables of the theory, should, *a priori*, be introduced. The physically relevant probabilities are then recovered in the limit as the two sets of sources coincide and are eventually set equal to zero. The general expression for transition probabilities of quantum mechanical systems, coupled to the environment, is given in Eq.(5) involving functional differentiations with respect to these two sets of classical sources. The method used in this work generalizes to quantum field theory and will be studied in a forthcoming report.

2 Transition Probabilities and the Role of the Environment

Typically a quantum mechanical system may be described by a Hamiltonian

$$H_1(t) = H(q,p,t) - qF(t) + pS(t) \tag{1}$$

written in terms of dynamical variables in the (q,p) language, where $F(t)$ and $S(t)$ are classical source functions introduced to generate functions of q and p, respectively. For simplicity of the notation, we have in Eq.(1), suppressed indices in q and p reflecting the dimensionality of space and of the number of particles involved in the theory. In most applications, the classical sources $F(t)$, $S(t)$ are set equal to zero *after* all relevant functional differentiations in the theory, with respect to them, are carried out, with $H(q,p;t)$ finally emerging as the Hamiltonian describing the actual physical system into consideration.

The quantum dynamical principle states [cf. 7, Ch. 11], that a transition amplitude $\langle at_2 \,|\, bt_1 \rangle$ for the system governed by (1), from time t_1 to time t_2, is given by

$$\langle at_2 \,|\, bt_1 \rangle = \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t_1}^{t_2} \mathrm{d}\tau H \left[-\mathrm{i}\hbar \frac{\delta}{\delta F(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta S(\tau)}, \tau \right] \right) \langle at_2 \,|\, bt_1 \rangle_0 \tag{2}$$

where H in (2) is obtained from $H(q,p,\tau)$ by simply replacing q and p in the latter by the operators of functional differentiations $-\mathrm{i}\hbar\delta/\delta F(\tau)$, $\mathrm{i}\hbar\delta/\delta S(\tau)$, respectively, and $\langle at_2 \,|\, bt_1 \rangle_0$ denotes the transition amplitude governed by the simple "Hamiltonian" $[-qF(t) + pS(t)]$ only.

To investigate the role of the environment on the quantum mechanical system, governed initially by the Hamiltonian $H(t)$ in (1), one modifies the latter Hamiltonian by including, in the Hamiltonian, the contribution of the environment and of its interaction with the physical system at hand. Of particular interest is in the response of the physical system to the environment. Accordingly, one takes a trace over the dynamical variables of the environment in the manner to be spelled out below. The Hamiltonian of the combined system is taken to be of the form

$$\begin{aligned} H(t) = H_1(q_1,p_1,t) - q_1F_1(t) + p_1S_1(t) + H_2(q_2,p_2,t) - q_2F_2(t) + p_2S_2(t) \\ + H_I(q_1,p_1,q_2,p_2,t), \end{aligned} \tag{3}$$

where the indices 1, 2 correspond, respectively, to the physical system and the environment, and H_I specifies the interaction term between them.

The transition amplitude for the combined system to evolve from a state, say, $|a,A;0\rangle$, initially at time $t=0$, to a state, say, $|b,B;t\rangle$, at time $t>0$, is then given by

$$\begin{aligned} \langle b,B;t \,|\, a,A;0 \rangle = \exp\Big(-\frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}\tau H_I \big(-\mathrm{i}\hbar \frac{\delta}{\delta F_1(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta S_1(\tau)}, \\ -\mathrm{i}\hbar \frac{\delta}{\delta F_2(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta S_2(\tau)}, t \big) \Big) \langle b;t \,|\, a;0 \rangle^{F_1,S_1} \langle B;t \,|\, A;0 \rangle^{F_2,S_2} \end{aligned} \tag{4}$$

as in (2), where $\langle b;t \,|\, a;0 \rangle^{F_1,S_1}$, $\langle B;t \,|\, A;0 \rangle^{F_2,S_2}$ are the transition amplitudes of the *decoupled* subsystems in the presence of their respective classical sources.

To find the response of the physical system, described by the Hamiltonian $H_1(q_1,p_1,t)$ in (3), to the environment, it is necessary to work with transition *probabilities*, corresponding to the process associated with the expression in (4), rather than with amplitudes as done in the latter equation, and "trace out" over the environment. To this end, let $\{|B_n;t\rangle\}$ denote a complete set of states pertaining to the environment, then the probability for the physical system to make a transition from an initial state $|a;0\rangle$ to a state $|b;t\rangle$ in time t, responding in the process to the environment, emerges as

$$\begin{aligned} \mathrm{Prob}[(a;0) \to (b;t)]_E = \mathcal{O}(\mathcal{O}')^* \langle b;t \,|\, a;0 \rangle^{F_1,S_1} (\langle b;t \,|\, a;0 \rangle^{F_1',S_1'})^* \\ \mathcal{F}[F_2,S_2;F_2',S_2'] \Big| \end{aligned} \tag{5}$$

where

$$\mathcal{O} = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau H_I\big(- \mathrm{i}\hbar\frac{\delta}{\delta F_1(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta S_1(\tau)}, -\mathrm{i}\hbar\frac{\delta}{\delta F_2(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta S_2(\tau)}, \tau\big)\right) \quad (6)$$

with $\mathcal{O}'$ defined similarly with F_1, S_1, F_2, S_2 replaced by F_1', S_1', F_2', S_2', respectively, and the presence of the letter E attached to the probability on the left-hand side of (6) is to emphasize the coupling of the environment to the physical system as the latter evolves in time. The functional $\mathcal{F}$ is given by

$$\mathcal{F}[F_2, S_2; F_2', S_2'] = \sum_n \langle B_n; t\,|\,A; 0\rangle^{F_2,S_2}\left(\langle B_n; t\,|\,A; 0\rangle^{F_2',S_2'}\right)^* . \quad (7)$$

We note that (7) reduces to the trace over the environment in the special case for which F_2' is set equal to F_2, and S_2' to S_2. One cannot, *a priori*, set such equalities until the functional differentiations, with respect to these sources, as accomplished by the operators $\mathcal{O}, (\mathcal{O}')^*$, are independently carried out. The bar sign on the right-hand side of (5) refers to the fact that finally one is to set $F = F' = 0$, $S = S' = 0$, after all the operations of functional differentiations have been done.

Eq.(5) gives the general expression for the transition probability of a physical system, as it evolves in time, in response to the environment.

Of significance importance is for systems written in terms of creation and annihilation operators, which most conveniently describe processes of transitions between their allowed states. Such a typical example is given by the Hamiltonian

$$H(t) = H_1(t) + H_2(t) + H_{12}(t) \quad (8)$$

with

$$H_1(t) = \hbar\omega a^\dagger a - a^\dagger F(t) - F^*(t)a \quad (9)$$

$$H_2(t) = \sum_k \hbar\omega_k b_k^\dagger b_k - \sum_k \left(K_k(t) b_k^\dagger + b_k K_k^*(t)\right) \quad (10)$$

$$H_{12}(t) = a^\dagger \sum_k \lambda_k b_k + a \sum_k \lambda_k^* b_k^\dagger \quad (11)$$

and $(a, a^\dagger)$, $(b_k, b_k^\dagger)$, pertaining to the physical system in consideration and the environment, respectively, $[a, a^\dagger] = 1$, $[b_k, b_{k'}^\dagger] = \delta_{kk'}$ for the corresponding commutators.

Suppose that the environment is initially in the ground-state $|0;0\rangle_2$. Let $U_2(t)$ denote the time evolution unitary operator describing the time evolution of the environment in the absence of the physical system. The so-called Heisenberg operator $b_k(t)$ associated with b_k is given by

$$b_k(t) = U_2^\dagger(t) b_k U_2(t) \tag{12}$$

which works out to be

$$b_k(t) = b_k \mathrm{e}^{-\mathrm{i}\omega_k t} + \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau\, K_k(\tau)\mathrm{e}^{-\mathrm{i}\omega_k(t-\tau)} \tag{13}$$

The quantum dynamical principle [2–9, 11–13] for the vacuum-to-vacuum transition amplitude $\langle 0;t\,|\,0;0\rangle_2^K$ gives

$$-\mathrm{i}\hbar \frac{\delta}{\delta K_k^*(t')}\,\langle 0;t\,|\,0;0\rangle_2^K = \langle 0;t\,|b_k(t')|0;0\rangle_2^K \tag{14}$$

for $0 < t' < t$. From the expression in (13), Eq.(14) simplifies to

$$-\mathrm{i}\hbar \frac{\delta}{\delta K_k^*(t')}\,\langle 0;t\,|\,0;0\rangle_2^K = \frac{\mathrm{i}}{\hbar}\,\langle 0;t\,|\,0;0\rangle_2^K \int_0^{t'} \mathrm{d}\tau\, K_k(\tau)\mathrm{e}^{-\mathrm{i}\omega_k(t'-\tau)} \tag{15}$$

which integrates out to

$$\langle 0;t\,|\,0;0\rangle_2^K = \exp\left(-\frac{1}{\hbar^2}\sum_k \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \mathrm{e}^{-\mathrm{i}\omega_k(\tau-\tau')} K_k^*(\tau)\Theta(\tau-\tau')K_k(\tau')\right) \tag{16}$$

where $\Theta(\tau-\tau')$ is the step function.

The functional $\mathcal{F}[K,K']$, corresponding to the one in (7), may be worked out in closed form. To this end, set $K(t) = K_1(t) + K_2(t)$, with $K_1(t), K_2(t)$ localized in time between $(0,t)$, such that $K_2(t)$ is "switched on" after the source $K_1(t)$ is "switched off". That is, in particular, $K_2(t)$ and $K_1(t)$ do not overlap in time.

From (16) we may then write

$$\begin{aligned}\langle 0;t\,|\,0;0\rangle_2^{K_1+K_2} = \langle 0;t\,|\,0;0\rangle_2^{K_2} \exp&\left[\sum_k \left(\int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega_k\tau}\frac{\mathrm{i}}{\hbar}K_2^*(\tau)\right)\right.\\ &\times\left.\left(\int_{-\infty}^{\infty} \mathrm{d}\tau'\, \mathrm{e}^{\mathrm{i}\omega_k\tau}\frac{\mathrm{i}}{\hbar}K_1(\tau)\right)\right]\langle 0;t\,|\,0;0\rangle_2^{K_1}\end{aligned} \tag{17}$$

where due to the fact that $K_1(\tau), K_2(\tau')$ are localized in time, we have extended the time integrations in the middle exponential from $-\infty$ to ∞.

Let $|n; n_{k_1}, n_{k_2}, \ldots\rangle_2$ denote a state of n excitations, n_{k_1} of which in the state k_1, n_{k_2} of which in state k_2, and so on, i.e., such that $n = n_{k_1} + n_{k_2} + \ldots$. Then upon introducing the unitarity completeness property

$$\langle 0; t \,|\, 0; 0\rangle_2^{K_1+K_2} = \sum_{n=0}^{\infty} \sum_{(n_{k_1}+n_{k_2}+\ldots=n)} \langle 0; t \,|\, n; n_{k_1}, n_{k_2}, \ldots\rangle_2^{K_2} \langle n; n_{k_1}, n_{k_2}, \ldots \,|\, 0\rangle_2^{K_1} \tag{18}$$

where the intermediate states are evaluated at any time after the switching off of source K_1 and before the switching on of source K_2, and the Fourier transform

$$K_k(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega_k}{2\pi}\, K_k(\omega) \mathrm{e}^{-\mathrm{i}\omega_k t} \tag{19}$$

we obtain by expanding the middle exponential in powers of the source functions $K_{1k}(\omega_k)$, $K_{2k}(\omega_k)$ the expression

$$\langle n; n_{k_1}, n_{k_2}, \ldots; t \,|\, 0; 0\rangle_2^{K} = \langle 0; t \,|\, 0; 0\rangle_2^{K} \frac{\left(\frac{\mathrm{i}}{\hbar} K_{k_1}(\omega_{k_1})\right)^{n_{k1}}}{\sqrt{n_{k_1}!}} \frac{\left(\frac{\mathrm{i}}{\hbar} K_{k_2}(\omega_{k_2})\right)^{n_{k2}}}{\sqrt{n_{k_2}!}} \cdots \tag{20}$$

for a given source $K(t)$. The functional $\mathcal{F}[K, K']$, corresponding to the one in (7), is then given by

$$\mathcal{F}[K, K'] = \sum_{n=0}^{\infty} \sum_{(n_{k_1}+n_{k_2}+\ldots=n)} \langle n; n_{k_1}, n_{k_2}, \ldots; t \,|\, 0; 0\rangle_2^{K} \times \left(\langle n; n_{k_1}, n_{k_2}, \ldots; t \,|\, 0; 0\rangle_2^{K'}\right)^* \tag{21}$$

and may be summed exactly over n giving

$$\mathcal{F}[K, K'] = \langle 0; t \,|\, 0; 0\rangle_2^{K} \exp\left[\frac{1}{\hbar^2} \sum_k \left(\int_0^t \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega_k \tau} K_k(\tau)\right) \times \left(\int_0^t \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}\omega_k \tau'} K'^{*}_k(\tau')\right)\right] \left(\langle 0; t \,|\, 0; 0\rangle_2^{K'}\right)^* \tag{22}$$

which cannot be expressed as the product of two functionals one depending on K and the other on K', as expected. Formally one checks the *unitarity condition* : $\mathcal{F}[K, K] = 1$ directly from (22).

Suppose that the physical system is initially in the ground-state, i.e., the vacuum-state $|0; 0\rangle$. The vacuum persistence amplitude of the physical system, in isolation from the environment, but in the presence of the external sources $F(t), F^*(t)$ in (9), may be then inferred from (16) to be

$$\langle 0;t\,|\,0;0\rangle_1^F = \exp\left(-\frac{1}{\hbar^2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \mathrm{e}^{-\mathrm{i}\omega(\tau-\tau')} F^*(\tau)\Theta(\tau-\tau')F(\tau')\right) \tag{23}$$

From our general expression in (5), we then obtain for the vacuum persistence probability of the physical system, in response to the environment,

$$\mathrm{Prob}[(0;0)\to(0;t)]_E = \mathcal{O}\,(\mathcal{O}')^*\,\langle 0;t\,|\,0;0\rangle_1^F \left(\langle 0;t\,|\,0;0\rangle_1^{F'}\right)^* \mathcal{F}[K,K']\Big| \tag{24}$$

where

$$\mathcal{O} = \exp -\frac{\mathrm{i}}{\hbar}\sum_k \int_0^t \mathrm{d}\tau \left[\lambda_k \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta F(\tau)}\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta K_k^*(\tau)} + \lambda_k^* \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta F^*(\tau)}\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta K_k(\tau)}\right] \tag{25}$$

and $\mathcal{O}'$ similarly defined with F, F^*, K_k, K_k^* replaced, respectively, by F', F'^*, K'_k, K'^*_k, and $\mathcal{F}[K,K']$ given by the explicit expression in (22).

To evaluate the expression on the right-hand side of (24), we use, in the process, the identity

$$\mathrm{e}^A\mathrm{e}^B = \exp\left(\mathrm{e}^A B \mathrm{e}^{-A}\right)\mathrm{e}^A \tag{26}$$

for two operators A, B. We note that $\delta/\delta F(\tau)$, $\delta/\delta F^*(\tau)$, in (25), give rise to translation operators, via $\mathcal{O}$, to functionals of F and F^* as given, for example, in (23), and similarly for $\delta/\delta K(\tau)$, $\delta/\delta K^*(\tau)$. The functional differentiations operations in (24) are then readily carried for a physical system weakly coupled to the environment, and after setting the classical sources equal to zero, we obtain for the survival probability the expression

$$\mathrm{Prob}[(0;0)\to(0;t)]_E = \exp\left(-2\sum_k \frac{|\lambda_k|^2}{\hbar^2}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \cos[(\omega-\omega_k)(\tau'-\tau)]\right) \tag{27}$$

For the environment described by an infinite set of degrees of freedom, we replace the sum over k by an integral over the frequency $\omega_k \to \omega$, and in turn introduce a frequency density $n(\omega')$ to rewrite (27) as

$$\mathrm{Prob}[(0;0)\to(0;t)]_E = \exp\left(-\frac{1}{\hbar^2}\int_0^\infty \mathrm{d}\omega' |\lambda(\omega')|^2\, n(\omega') \frac{\sin^2(\omega'-\omega)\dfrac{t}{2}}{(\omega'-\omega)^2/4}\right) \tag{28}$$

Upon introducing the integration variable $x = (\omega' - \omega)t/2$, one may rewrite the integral in (28) as

$$2t \int_{-\omega t/2}^{\infty} \mathrm{d}x \left|\lambda(\omega(1 + \frac{x}{\omega t}))\right|^2 n(\omega(1 + \frac{x}{\omega t}))\frac{\sin^2 x}{x^2} \tag{29}$$

If one makes the Markov approximation by assuming that $|\lambda(\omega')|^2 n(\omega')$ is slowly varying around the point $\omega' = \omega$, and hence for $\omega t \gg \pi$, it may be taken outside the integral evaluated at ω, one gets for the integral in (29)

$$2t|\lambda(\omega)|^2 n(\omega) \int_{-\omega t/2}^{\infty} \mathrm{d}x \, \frac{\sin^2 x}{x^2} \tag{30}$$

with increasing accuracy for $\omega t \gg \pi$. And for $\omega t \gg \pi$, we obtain from (28), (30) the familiar exponential law

$$\mathrm{Prob}[(0;0) \to (0;t)]_E = \mathrm{e}^{-\gamma t} \tag{31}$$

where γ is the decay constant $2\pi|\lambda(\omega)|^2 n(\omega)/\hbar^2$. This expression is strictly valid for $\pi/\omega \ll t \ll 1/\gamma$ consistent with the property of the decay of quantum systems and the Paley-Wiener Theorem [cf. 7, §3.5], that the exponential law may be valid for intermediate values of t and not in the truly asymptotic limit $t \to \infty$.

3 Conclusion

A general expression was obtained in (5) for the transition probability of quantum systems when coupled to the environment, and in response to it, involving functional differentiations, with respect to classical sources, using functional calculus techniques. It is important to note that as the functional $\mathcal{F}$ in (7) cannot be written as the product of two terms one involving the sources F_1, S_1, F_2, S_2, and one involving the sources F_1', S_1', F_2', S_2', one necessarily has to deal directly with transition probabilities of the physical system as it evolves in time in response to the environment rather than amplitudes. In case the amplitudes $\langle b; t \,|\, a; 0\rangle^{F_i, S_i}$ in (5) are not explicitly given for the decoupled physical system from the environment, one may use the integral expression in (2) to carry out various approximations suitable for the system in consideration. The main analysis shows the power of the functional differential treatment, involving functional differentiations with respect to classical, thus commuting, functions. The method developed in this work will be extended to quantum field theories, including gauge theories, in a subsequent report.

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The graviton propagator with a non-conserved external generating source

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A novel general expression is obtained for the graviton propagator from Lagrangian field theory by taking into account the necessary fact that in the functional differential approach of quantum field theory, in order to generate non-linearities in gravitation and interactions with matter, the external source $T_{\mu\nu}$, coupled to the gravitational field, should *a priori* not be conserved $\partial^{\mu} T_{\mu\nu} = 0$, so variations with respect to its ten components may be varied *independently*. The resulting propagator is the one which arises in the functional approach and does *not* coincide with the corresponding time-ordered product of two fields and it includes so-called Schwinger terms. The quantization is carried out in a gauge corresponding to physical states with two polarization states to ensure positivity in quantum applications.

1 Introduction

A basic ingredient in quantum gravity computations is the graviton propagator (cf. [1–5]. The latter mediates the gravitational interaction between all particles to the leading order in the gravitational coupling constant. In the so-called functional differential treatment [6–9, 11] of quantum field theory, referred as the quantum dynamical principle approach, based on functional derivative techniques with respect to external sources coupled to the underlying fields in a theory, functional derivatives are taken of the so-called vacuum-to-vacuum transition amplitude. The latter generates *n*-point functions by functional differentiations leading finally to transition amplitudes for various physical processes. For higher spin fields such as the electromagnetic vector potential A^{μ}, the gluon field A^{μ}_{a}, and certainly the gravitational field $h^{\mu\nu}$, the respective external sources J_{μ}, J^{a}_{μ}, $T_{\mu\nu}$, coupled to these fields, cannot *a priori* taken to be conserved so that their respective components may be varied *independently*. The consequences of relaxing the conservation of these external sources are highly non-trivial. For one thing the corresponding field propagators become modified. Also they have led to the rediscovery [6, 7] of Faddeev-Popov (FP) [12] factors in non-abelian gauge theories and the discovery [7] of even more generalized such factors, directly from the functional *differential* treatment, via the application of the quantum dynamical principle, in the presence of external sources, without making an appeal to path integrals, without using commutation rules, and without even going to the well known complicated structures of the underlying Hamiltonians. A brief account of this is given in the concluding section for the convenience of the reader.

For higher spin fields, the propagator and the time-ordered product of two fields do *not* coincide as the former includes so-called Schwinger terms which, in general, lead to a simplification of the expression for the propagator over the time-ordered one. This is well known for spin 1, and, as shown below, is also true

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for the graviton propagator. Let $h^{\mu\nu}$ denote the gravitational field (see Sect. 2). We work in a gauge

$$\partial_i h^{i\nu} = 0\,, \tag{1}$$

where $i = 1, 2, 3;\ \nu = 0, 1, 2, 3$, which guarantees that only two states of polarization occur with the massless particle and ensures positivity in quantum applications avoiding non-physical states. Let $T_{\mu\nu}$ denote an external source coupled to the gravitational field $h^{\mu\nu}$ (see Sect. 2), and let $\langle 0_+ \,|\, 0_- \rangle^T$ denote the vacuum-to-vacuum transition amplitude in the presence of the external source. The propagator of the gravitational field is then defined by

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \mathrm{i}\left((-\mathrm{i})\frac{\delta}{\delta T_{\mu\nu(x)}}(-\mathrm{i})\frac{\delta}{\delta T_{\sigma\lambda(x')}}\,\langle 0_+ \,|\, 0_- \rangle^T \right) \Big/ \,\langle 0_+ \,|\, 0_- \rangle^T\,, \tag{2}$$

in the limit of the vanishing of the external source $T_{\mu\nu}$. In more detail we may rewrite (2) as

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \mathrm{i}\frac{\left\langle 0_+ \left| \left(h^{\mu\nu}(x) h^{\sigma\lambda}(x') \right)_+ \right| 0_- \right\rangle^T}{\langle 0_+ \,|\, 0_- \rangle^T} + \frac{\left\langle 0_+ \left| \dfrac{\delta}{\delta T_{\mu\nu(x)}} h^{\sigma\lambda}(x') \right| 0_- \right\rangle^T}{\langle 0_+ \,|\, 0_- \rangle^T} \tag{3}$$

in the limit of vanishing $T_{\mu\nu}$, where the first term on the right-hand side, up to the i factor, denotes the time-ordered product. In the second term, the functional derivative with respect to the external source $T_{\mu\nu}(x)$ is taken by keeping the independent field components of $h^{\sigma\lambda}(x')$ fixed. The dependent field components depend on the external source and lead to extra terms on the right-hand side of (3) in addition to the time-ordered product and may be referred to as Schwinger terms. For a detailed derivation of the general identity in (3) see [10] (see also [11]). These additional terms lead to a simplification of the expression for the propagator over the time-ordered product. Accordingly, the propagator and the time-ordered product do *not* coincide and it is the propagator $\Delta_+^{\mu\nu;\sigma\lambda}$ that appears in the functional approach and not the time-ordered product. The derivation of the explicit expression for $\Delta_+^{\mu\nu;\sigma\lambda}(x,x')$ follows by relaxing the conservation of $T_{\mu\nu}$ and it includes 30 terms in contrast to the well known case involving only 3 terms when a conservation law of $T_{\mu\nu}$ is imposed. It is important to emphasize that our interest here is in the propagator, the basic component which appears in the theory, and not the time-ordered product. In the concluding section, some additional pertinent comments are made regarding our expression for the propagator. A brief account on how FP factors arise *directly* in gauge theories in the functional *differential* formalism, in the presence of external sources, is also given. Our notation for the Minkowski metric is $g^{\mu\nu} = \mathrm{diag}[-1, 1, 1, 1]$, also quite generally we set $i, j, k, l = 1, 2, 3$, $a, b = 1, 2$, while $\mu, \nu, \sigma, \lambda = 0, 1, 2, 3$.

2 The graviton propagator

For the Lagrangian density of the gravitational field $h^{\mu\nu}$ coupled to an external source $T_{\mu\nu}$, we take

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} + \tfrac{1}{2}\partial^\alpha h^\sigma{}_\sigma \partial_\alpha h^\beta{}_\beta - \partial^\alpha h_{\alpha\mu}\partial^\mu h^\sigma{}_\sigma \\ &+ \tfrac{1}{2}\partial_\alpha h^{\alpha\nu}\partial^\beta h_{\beta\nu} + \tfrac{1}{2}\partial_\alpha h^\mu{}_\nu \partial_\mu h^{\alpha\nu} + h^{\mu\nu}T_{\mu\nu},\end{aligned} \tag{4}$$

where $h^{\mu\nu} = h^{\nu\mu}$, and as a result $T_{\mu\nu}$ is chosen to be symmetric. We consider the ten components of $T_{\mu\nu}$ to be independent by, *a priori*, not imposing a conservation law for $T_{\mu\nu}$. The action corresponding to the Lagrangian density in (4), in the absence of the external source $T_{\mu\nu}$, is invariant under the gauge transformation $h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu + \partial^\mu\partial^\nu \xi$. The gauge constraint in (1) allows us to solve, say, $h_{3\nu}$, in terms of other components:

$$\begin{aligned} &h_{30} = -(\partial_3)^{-1}\partial_a h_{a0},\ h_{3a} = -(\partial_3)^{-1}\partial_b h_{ba}\,, \\ &h_{33} = -(\partial_3)^{-1}\partial_a h_{3a} = (\partial_3)^{-2}\partial_a\partial_b h_{ab}\,, \end{aligned} \tag{5}$$

where $a, b = 1, 2$. Upon substituting the expressions for $h_{3\nu}$ in (4), and varying h_{ab}, we obtain

$$(\Box h_{ab} + T_{ab}) - \frac{\partial_b}{\partial_3}(\Box h_{a3} + T_{a3}) - \frac{\partial_a}{\partial_3}(\Box h_{b3} + T_{b3}) + \frac{\partial_a \partial_b}{(\partial_3)^2}(\Box h_{33} + T_{33}) + \left[\delta_{ab} + \frac{\partial_a \partial_b}{(\partial_3)^2}\right](\boldsymbol{\partial}^2 h_{00} - \Box h_{ii}) = 0\,, \tag{6}$$

$a, b = 1, 2$. Upon multiplying (6) by $(\delta_{ab} - \partial_a \partial_b / \boldsymbol{\partial}^2)$, where $\boldsymbol{\partial}^2 = \partial^i \partial_i$, $i = 1, 2, 3$, some tedious algebra leads to

$$-\boldsymbol{\partial}^2 h_{00} = -\frac{1}{2}\Box h_{ii} + \frac{1}{2}\left(\delta_{ij} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2}\right) T_{ij}\,. \tag{7}$$

On the other hand, with the expressions for $h_{3\nu}$ in (5) replaced in (4), variations with respect to h_{00}, h_{0a}, $a = 1, 2$, give, respectively,

$$-\boldsymbol{\partial}^2 h_{ii} = T_{00}\,, \tag{8}$$

$$-\boldsymbol{\partial}^2 h_{0a} + \frac{\partial_a}{\partial_3}\boldsymbol{\partial}^2 h_{03} = \left(T_{0a} - \frac{\partial_a}{\partial_3} T_{03}\right)\,. \tag{9}$$

We note that (9) is valid if we formally replace a by 3 since this simply gives $0 = 0$. Accordingly, we may rewrite (9) as

$$-\boldsymbol{\partial}^2 h_{0i} + \frac{\partial_i}{\partial_3}\boldsymbol{\partial}^2 h_{03} = T_{0i} - \frac{\partial_i}{\partial_3} T_{03}\,, \tag{10}$$

where $i = 1, 2, 3$. Upon taking the divergence ∂^i of (10) and using (1), we obtain

$$\frac{\partial_i}{\partial_3}\boldsymbol{\partial}^2 h_{03} = \frac{\partial_i}{\boldsymbol{\partial}^2}\left(\partial_j T_{0j} - \frac{\boldsymbol{\partial}^2}{\partial_3} T_{03}\right)\,, \tag{11}$$

which upon substitution in (10) gives

$$-\boldsymbol{\partial}^2 h_{0i} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\boldsymbol{\partial}^2}\right) T_{0j}. \tag{12}$$

Also upon substitution (8) in (7), and using the fact that $\Box = \boldsymbol{\partial}^2 - \partial_0^2$, we obtain for (7)

$$-\boldsymbol{\partial}^2 h_{00} = T_{00} + \frac{T}{2} - \frac{1}{2\boldsymbol{\partial}^2}\left(\partial^0 \partial^0 T_{00} + \partial_i \partial_j T_{ij}\right)\,, \tag{13}$$

where $T = g^{\mu\nu} T_{\mu\nu} = T^\nu{}_\nu$.

Eqs. (8), (12), (13) are not equations of motion as they involve no time derivatives of the corresponding fields and they yield to constraints which together the gauge condition in (1) give rise to two degrees of freedom corresponding to two polarization states for the graviton as it should be.

We now substitute the expression for $-\boldsymbol{\partial}^2 h_{00}$, as given in (13), in (6) and use (8) to obtain an equation involving h_{ij}, $i, j = 1, 2, 3$. Upon multiplying the resulting equation from (6) by $\partial_a \partial_b$ and using the expressions for h_{33} in (5) we obtain after some very tedious algebra

$$(\Box h_{33}+T_{33})-\frac{1}{2}\left(1-\frac{(\partial_3)^2}{\boldsymbol{\partial}^2}\right)T+\frac{1}{2\boldsymbol{\partial}^2}\left(-\partial^0\partial^0T_{00}+\partial^i\partial^jT_{ij}\right)$$
$$-\frac{2}{\boldsymbol{\partial}^2}\partial^i\partial_3T_{i3}+\frac{(\partial_3)^2}{2\boldsymbol{\partial}^2}\left(\frac{\partial^i\partial^j}{\boldsymbol{\partial}^2}T_{ij}+\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}\right)=0. \tag{14}$$

Similarly, upon multiplying (6) by ∂_a and using the expression for h_{b3} in (5), we obtain

$$(\Box h_{b3}+T_{b3})-\frac{1}{\boldsymbol{\partial}^2}\left[\partial_3\partial^iT_{ib}+\partial_b\partial^iT_{i3}-\frac{\partial_b\partial_3}{2}\left(\frac{\partial^i\partial^j}{\boldsymbol{\partial}^2}T_{ij}+\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}+T\right)\right]=0\,. \tag{15}$$

To obtain the equation for h_{ab}, we substitute (14), (15) in (6), to obtain after some lengthy algebra

$$(\Box h_{ab}+T_{ab})-\frac{1}{2}\left(\delta_{ab}-\frac{\partial_a\partial_b}{\boldsymbol{\partial}^2}\right)T-\frac{1}{\boldsymbol{\partial}^2}(\partial_a\partial^iT_{ib}+\partial_b\partial^iT_{ia})$$
$$+\frac{\delta_{ab}}{2\boldsymbol{\partial}^2}\left(-\partial^0\partial^0T_{00}+\partial^i\partial^jT_{ij}\right)+\frac{\partial_a\partial_b}{2(\boldsymbol{\partial}^2)^2}\left(\partial^i\partial^jT_{ij}+\partial^0\partial^0T_{00}\right)=0. \tag{16}$$

Eqs. (14)–(16) may be now combined in the form

$$-\Box h_{ij}=T_{ij}-\frac{1}{2}\left(\delta_{ij}-\frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}\right)\left(T+\frac{\partial^0\partial^0}{\boldsymbol{\partial}^2}T_{00}\right)$$
$$-\frac{1}{\boldsymbol{\partial}^2}\left[\partial_i\partial^kT_{kj}+\partial_j\partial^kT_{ki}-\frac{1}{2}\left(\delta_{ij}+\frac{\partial_i\partial_j}{\boldsymbol{\partial}^2}\right)\partial^k\partial^lT_{kl}\right], \tag{17}$$

where $i,j,k,l=1,2,3$.

Eqs. (17), (13), (12) give the equations for the various components of $h_{\mu\nu}$. To obtain the unifying equation for $h_{\mu\nu}$, we note that we may write

$$h^{\mu\nu}=g^{\mu i}h_{ij}g^{j\nu}+g^{\mu i}h_{i0}g^{0\nu}+g^{\mu 0}h_{0j}g^{j\nu}+g^{\mu 0}h_{00}g^{0\nu}, \tag{18}$$

with $i,j=1,2,3$; $\mu,\ \nu=0,1,2,3$, and use in the process the identity

$$g^{\mu i}\partial_i=(\partial^\mu+N^\mu\partial_0), \tag{19}$$

where N^μ is the unit time-like vector ($N^\mu N_\mu=-1$)

$$(N^\mu)=(g^\mu{}_0)=(1,0,0,0). \tag{20}$$

Finally, we use the identity relating a tensor $A_{\lambda\sigma}$, e.g., to the components A_{ij} as follows:

$$g^{\mu i}A_{ij}g^{j\nu}=\left[g^{\mu\lambda}g^{\nu\sigma}+N^\mu N^\lambda g^{\sigma\nu}+N^\nu N^\sigma g^{\lambda\mu}+N^\mu N^\nu N^\lambda N^\sigma\right]A_{\lambda\sigma}, \tag{21}$$

and the fact that $\Box=\boldsymbol{\partial}^2-\partial^{0^2}$. A lengthy analysis from (12), (13), (17) then gives the following explicit expression for $h^{\mu\nu}$:

$$h^{\mu\nu}=\frac{1}{(-\Box-\mathrm{i}\epsilon)}\left\{\frac{g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}-g^{\mu\nu}g^{\sigma\lambda}}{2}\right.$$
$$+\frac{1}{2\boldsymbol{\partial}^2}\left[g^{\mu\nu}\partial^\sigma\partial^\lambda+g^{\sigma\lambda}\partial^\mu\partial^\nu-g^{\nu\sigma}\partial^\mu\partial^\lambda-g^{\nu\lambda}\partial^\mu\partial^\sigma-g^{\mu\sigma}\partial^\nu\partial^\lambda-g^{\mu\lambda}\partial^\nu\partial^\sigma+\frac{\partial^\mu\partial^\nu\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2}\right]$$

$$+\frac{1}{2}\left(g^{\mu\nu}+\frac{\partial^\mu\partial^\nu}{\boldsymbol{\partial}^2}\right)\left(\frac{N^\sigma\partial^\lambda+N^\lambda\partial^\sigma}{\boldsymbol{\partial}^2}\right)\partial_0+\frac{1}{2}\left(g^{\sigma\lambda}+\frac{\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2}\right)\left(\frac{N^\nu\partial^\mu+N^\mu\partial^\nu}{\boldsymbol{\partial}^2}\right)\partial_0$$

$$-\frac{1}{2}\Big[g^{\nu\sigma}(N^\mu\partial^\lambda+N^\lambda\partial^\mu)+g^{\nu\lambda}(N^\mu\partial^\sigma+N^\sigma\partial^\mu)$$

$$+g^{\mu\sigma}(N^\nu\partial^\lambda+N^\lambda\partial^\nu)+g^{\mu\lambda}(N^\nu\partial^\sigma+N^\sigma\partial^\nu)\Big]\frac{\partial_0}{\boldsymbol{\partial}^2}$$

$$+\frac{\partial^\mu\partial^\nu}{\boldsymbol{\partial}^2}N^\sigma N^\lambda+\frac{\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2}N^\mu N^\nu\Bigg\}T_{\sigma\lambda}$$

$$+\frac{1}{\boldsymbol{\partial}^2}\left\{\frac{\partial^\mu\partial^\nu}{\boldsymbol{\partial}^2}N^\sigma N^\lambda+\frac{\partial^\sigma\partial^\lambda}{\boldsymbol{\partial}^2}N^\mu N^\nu\right\}T_{\sigma\lambda}\,, \tag{22}$$

$\epsilon\to+0$.

From (22) the explicit expression for the graviton propagator $\Delta_+^{\mu\nu;\sigma\lambda}(x,x')$ emerges as:

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x')=\int\frac{(\mathrm{d}k)}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-x')}\left[\frac{\Delta_1^{\mu\nu;\sigma\lambda}(k)}{k^2-\mathrm{i}\epsilon}+\frac{\Delta_2^{\mu\nu;\sigma\lambda}(k)}{\mathbf{k}^2}\right], \tag{23}$$

$\epsilon\to+0$, where $(\mathrm{d}k)=\mathrm{d}k^0\mathrm{d}k^1\mathrm{d}k^2\mathrm{d}k^3$, $k^2=\mathbf{k}^2-k^{0^2}$, and

$$\Delta_1^{\mu\nu;\lambda\sigma}(k)=\frac{(g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}-g^{\mu\nu}g^{\sigma\lambda})}{2}$$

$$+\frac{1}{2\mathbf{k}^2}\Bigg[g^{\mu\nu}k^\sigma k^\lambda+g^{\sigma\lambda}k^\mu k^\nu-g^{\nu\sigma}k^\mu k^\lambda-g^{\nu\lambda}k^\mu k^\sigma-g^{\mu\sigma}k^\nu k^\lambda-g^{\mu\lambda}k^\nu k^\sigma$$

$$+\frac{k^\mu k^\nu k^\sigma k^\lambda}{\mathbf{k}^2}\Bigg]$$

$$-\frac{1}{2}\left(g^{\mu\nu}+\frac{k^\mu k^\nu}{\mathbf{k}^2}\right)\left(\frac{N^\sigma k^\lambda+N^\lambda k^\sigma}{\mathbf{k}^2}\right)k^0-\frac{1}{2}\left(g^{\sigma\lambda}+\frac{k^\sigma k^\lambda}{\mathbf{k}^2}\right)\left(\frac{N^\nu k^\mu+N^\mu k^\nu}{\mathbf{k}^2}\right)k^0$$

$$+\frac{1}{2}\Big[g^{\nu\sigma}(N^\mu k^\lambda+N^\lambda k^\mu)+g^{\nu\lambda}(N^\mu k^\sigma+N^\sigma k^\mu)$$

$$+g^{\mu\sigma}(N^\nu k^\lambda+N^\lambda k^\nu)+g^{\mu\lambda}(N^\nu k^\sigma+N^\sigma k^\nu)\Big]\frac{k^0}{\mathbf{k}^2}$$

$$+\frac{k^\mu k^\nu}{\mathbf{k}^2}N^\sigma N^\lambda+\frac{k^\sigma k^\lambda}{\mathbf{k}^2}N^\mu N^\nu\,, \tag{24}$$

$$\Delta_2^{\mu\nu;\lambda\sigma}(k)=\frac{k^\mu k^\nu}{\mathbf{k}^2}N^\sigma N^\lambda+\frac{k^\sigma k^\lambda}{\mathbf{k}^2}N^\mu N^\nu\,. \tag{25}$$

The vacuum-to-vacuum transition amplitude for the gravitational field coupled to an external source is then given by

$$\langle 0_+\,|\,0_-\rangle^T=\exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')T_{\mu\nu}(x)\Delta_+^{\mu\nu;\sigma\lambda}(x,x')T_{\sigma\lambda}(x')\right] \tag{26}$$

with the graviton propagator given by the explicit expression in (23)–(25). Now we are ready to make pertinent comments concerning the graviton propagator thus obtained.

3 Conclusion

We have derived a novel expression for the graviton propagator, from Lagrangian field theory, valid for the case when the external source $T_{\mu\nu}$ coupled to the gravitational field is not necessarily conserved, by working in a gauge where only two polarization physical states of the graviton arise to ensure positivity in the quantum treatment thus avoiding non-physical states. That a conservation should *a priori* not to be imposed is a necessary mathematical requirement so that all the ten components of the external source $T_{\mu\nu}$ may be varied independently in order to generate interactions of the gravitational field with matter and produce non-linearity of the gravitational field itself in the functional procedure. The latter requirement arises by noting that such interactions are generated by the application (cf. [6,7]) of some functional $F[-\mathrm{i}\delta/\delta T_{\mu\nu}]$ to $\langle 0_+ | 0_- \rangle^T$, where $\langle 0_+ | 0_- \rangle$ corresponding to other particles, as well as functional derivatives of their corresponding sources in F, have been suppressed to simplify the notation. Accordingly, to vary the ten components of $T_{\mu\nu}$ independently, no conservation may *a priori* be imposed. The $1/\mathbf{k}^2$ terms in (23)–(25) are apparent singularities due to the sufficient powers in k in the corresponding denominators and the three-dimensional character of space, in the same way that this happens for the photon propagator in the Coulomb gauge in quantum electrodynamics, and give rise to static $1/r$ type interactions complicated by the tensorial character of a spin two object. It is important to note that for a conserved $T_{\mu\nu}$, i.e., for $\partial^\mu T_{\mu\nu} = 0$, all the terms in the propagator in (23), with the exception of the terms $(g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\sigma\lambda})/2$ in (24), do not contribute in (26) since *all* the other terms in (24), (25) involve derivatives of $T_{\mu\nu}$ and the graviton propagator $\Delta_+^{\mu\nu;\sigma\lambda}(x,x')$ effectively *goes over* to the well documented expression

$$\frac{1}{(-\Box - \mathrm{i}\epsilon)} \frac{(g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\sigma\lambda})}{2}, \tag{27}$$

which has been known for years (cf. [1,2]). This is unlike the corresponding time-ordered product which does not go over to the result in (27) for $\partial^\mu T_{\mu\nu} = 0$. This may be shown by solving for the time-ordered product in (3) in terms of the propagator and carrying out explicitly, say, the functional derivatives $\delta h^{0i}/\delta T_{\mu\nu}$, $\delta h^{00}/\delta T_{\mu\nu}$, as arising on the right-hand side of (3), by using, in the process, Eqs. (12), (13). In any case, it is the propagator $\Delta_+^{\mu\nu;\sigma\lambda}$, as given in (23), is the one that appears in the theory and not the time-ordered product as is often naïvely assumed. After all the functional derivatives with respect to $T_{\mu\nu}$ are carried out in the theory, one may impose a conservation law on $T_{\mu\nu}$ or even set $T_{\mu\nu}$ equal to zero if required on physical grounds. Such methods have led to the discovery [6,7], in the functional quantum dynamical principle differential approach, of Faddeev-Popov (FP) factors, and of their generalizations, in non-abelian gauge theories such as in QCD and in other theories.

Re-iterating the discussion above, the relevance of the analysis and the explicit expression derived for the graviton propagator for, *a priori*, not conserved external source $T_{\mu\nu} : \partial^\mu T_{\mu\nu} \neq 0$ is immediate. If, in contrast, a conservation law is, *a priori*, imposed then variations with respect to one of the components of $T_{\mu\nu}$ would automatically imply, via such a conservation law, variations with respect some of its *other* components as well. A problem that may arise otherwise, may be readily seen from a simple example. The functional derivative of an expression like $[a_{\mu\nu}(x)+b(x)\partial_\mu\partial_\nu]T^{\mu\nu}(x)$, with respect to a component $T^{\sigma\lambda}(x')$ is $(1/2)[a_{\mu\nu}(x) + b(x)\partial_\mu\partial_\nu](\delta_\sigma{}^\mu\delta_\lambda{}^\nu + \delta_\lambda{}^\mu\delta_\sigma{}^\nu)\delta^4(x,x')$, where $a_{\mu\nu}(x),\ b(x)$, for example, depend x, and not $(1/2)a_{\mu\nu}(x)(\delta_\sigma{}^\mu\delta_\lambda{}^\nu + \delta_\lambda{}^\mu\delta_\sigma{}^\nu)\delta^4(x,x')$ as one may naïvely assume by, *a priori*, imposing a conservation law. Also, as mentioned above, the present method, based on the functional differential treatment, as applied to non-abelian gauge theories such as QCD [6,7] leads automatically to the presence of the FP determinant modifying naïve Feynman rules. The *physical* relevance of such a factor is important as its omission would lead to a violation of unitarity. For the convenience of the reader we briefly review, before closing the concluding section, on how the FP determinant arises in the functional differential treatment [6,7].

Consider, for simplicity of the demonstration, the non-abelian gauge theory with Lagrangian density

$$\mathscr{L} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + J^\mu_a A^a_\mu\,, \tag{28}$$

where J_a^μ is an external source taken, *a priori*, not to be conserved. Here

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_o f^{abc} A^b_\mu A^c_\nu \,. \tag{29}$$

We work in the Coulomb gauge. The gauge field propagator, in analogy to the graviton one in (24), (25), is given by

$$D^{\mu\nu}_{ab} = \delta_{ab}[g^{\mu\nu} - \frac{(\partial^\mu\partial^\nu + N^\mu\partial^\nu\partial_0 + N^\nu\partial^\mu\partial_0)}{\boldsymbol{\partial}^2}]\frac{1}{(-\Box - \mathrm{i}\varepsilon)} \tag{30}$$

with $k = 1, 2, 3$.

The quantum dynamical principle states that

$$\frac{\partial}{\partial g_o}\langle 0_+ \,|\, 0_- \rangle = \mathrm{i}\,\langle 0_+ | \int (dx) \frac{\partial}{\partial g_o}\mathscr{L}(x)\,|0_-\rangle \;, \tag{31}$$

where, with $k = 1, 2, 3$,

$$\frac{\partial}{\partial g_o}\mathscr{L}(x) = -f^{abc} A^b_k (A^c_0 G^{k0}_a + \frac{1}{2} A^c_l G^{kl}_a) \tag{32}$$

and G^{kl}_a may be expressed in terms of independent fields, that is, for which the canonical conjugate momenta do not vanish. On the other hand, G^{k0}_a depends on the dependent field A^0_a. By using the identity

$$(-\mathrm{i})\frac{\delta}{\delta J^\mu_a(x')}\langle 0_+ |\, \mathscr{O}(x)\,|0_-\rangle = \langle 0_+ |\,(A^a_\mu(x')\mathscr{O}(x))_+\,|0_-\rangle - \mathrm{i}\,\langle 0_+ |\,\frac{\delta}{\delta J^\mu_a(x')}\mathscr{O}(x)\,|0_-\rangle \tag{33}$$

for an operator $\mathscr{O}(x)$, where $(\ldots)_+$ denotes the time-ordered product, and the functional derivative $\delta\mathscr{O}(x)/\delta J^\mu_a(x')$ in the second term on the right-hand side of (33) is taken by keeping the independent fields and their canonical conjugate kept fixed in $\mathscr{O}(x)$, after the latter is expressed in terms of these fields, together, possibly, in terms of the dependent fields and the external current [7,10].

From the Lagrangian density in (28), the following relation follows

$$G^{k0}_a = \pi^k_a - \partial^k D_{ab} J^0_b \tag{34}$$

as a matrix equation, where π^k_a denotes the canonical conjugate momentum of A^k_a, and D_{ab} is the Green operator satisfying

$$\left[\delta^{ac}\boldsymbol{\partial}^2 + g_o f^{abc} A^b_k \partial^k\right] D^{cd}(x, x'; g_o) = \delta^4(x, x')\delta^{ad} \,. \tag{35}$$

Accordingly, with, *a priori*, non-conserved $J^\mu_a(x')$, we may vary each of its components independently to obtain from (34)

$$\frac{\delta}{\delta J^\mu_a(x')} G^{k0}_a(x) = -\delta_\mu{}^0 \partial^k D_{ac}(x, x'; g_o) \,. \tag{36}$$

Hence from (32), (33), and (36), we may write

$$\langle 0_+ |\, \frac{\partial}{\partial g_o}\mathscr{L}(x)\,|0_-\rangle = \left[\left(\frac{\partial}{\partial g_o}\mathscr{L}\right)' + \mathrm{i} f^{bca} A'^b_k \partial^k D'^{ac}(x, x; g_o)\right]\langle 0_+ \,|\, 0_-\rangle \;, \tag{37}$$

where the primes mean to replace $A^c_\mu(x)$ in the corresponding expressions by the functional differential operator $(-\mathrm{i})\delta/\delta J^\mu_c(x)$.

Clearly, upon an elementary integration over g_o in (31) by using, in the process, (37) and the equation for D^{ac} *in* (35), we *obtain* the FP determinant

$$\exp \operatorname{Tr} \ln \left[1 - \mathrm{i} g_o \frac{1}{\partial^2} A'_k \partial^k \right] \tag{38}$$

as a multiplicative modifying differential operating factor in $\langle 0_+ \,|\, 0_- \rangle$. For additional related details see [6,7] and also for further generalizations of the occurrence of such factors in field theory.

It is interesting to extend such analyses [6,7], as well as of gauge transformations [6], and covariance [13], to theories involving gravity. This would be exponentially much harder to do and will be attempted in further investigations. In this regard, our ultimate interest is in aspects of renormalizability [14] and rules for physical applications that would follow from our, *a priori*, systematic analysis carried out at the outset, in a quantum setting with the newly modified propagator, by a functional *differential* treatment, in the presence of external sources, to generate non-linearities in gravitation and interactions with matter.

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Gravitons, induced geometry and expectation value formalism at finite temperature

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After establishing the positivity constraint and spin content of the theory for gravitons interacting with a necessarily, and *a priori*, *non*-conserved external energy-momentum tensor, the expectation value formalism of the theory is developed at *finite* temperature in the functional *differential* treatment of quantum field theory. The necessity of having, *a priori*, a non-conserved external energy-momentum tensor is an obvious technical requirement so that its respective ten components may be varied *independently* in order to generate expectation values and non-linearities in the theory. The covariance of the *induced* Riemann curvature tensor, in the initial vacuum, is established even for the quantization in a gauge corresponding only to two physical states of the gravitons as established above. As an application, the *induced* correction to the metric and the underlying geometry is investigated due to a closed string arising from the Nambu action as a solution of a circularly oscillating string as, perhaps, the simplest generalization of a limiting point-like object. Finally it is discussed on why the geometry of spacetime may, in general, depend on temperature due to radiative corrections and its physical significance is emphasized.

1 Introduction

The graviton propagator [1–6] plays a central role in the quantum field theory treatment of gravitation. It mediates the gravitational interaction between all particles to the leading order in the gravitational coupling constant. It is well known that in the functional *differential* formalism of quantum field theory, pioneered by Schwinger [7], functional derivatives (e.g., [7–12]) are taken of the so-called vacuum-to-vacuum transition amplitude $\langle 0_+ | 0_- \rangle$ with respect to external sources, via the application, in the process, of the quantum dynamical (action) principle (e.g., [8, 11, 12]) to generate non-linearities (interactions) in the theory and n-point functions leading finally to transition amplitudes for various physical processes. [For a recent modern and a detailed derivation of the quantum dynamical principle see [12].] For higher spin fields such as the electromagnetic vector potential A^μ, the gluon field A^μ_a, and, of course, the gravitational field $h^{\mu\nu}$, the respective external sources J_μ, J^a_μ, $T_{\mu\nu}$, coupled to these fields, cannot *a priori* taken to be conserved so that their respective components may be varied *independently* in the functional differentiations process. A problem that may arise otherwise, may be readily seen from a simple example given in [1]: The functional derivative of an expression like $[a_{\mu\nu}(x) + b(x)\partial_\mu\partial_\nu]T^{\mu\nu}(x)$, with respect to $T^{\sigma\lambda}(x')$ is $(1/2)[a_{\mu\nu}(x) + b(x)\partial_\mu\partial_\nu](\delta_\sigma{}^\mu\delta_\lambda{}^\nu + \delta_\lambda{}^\mu\delta_\sigma{}^\nu)\delta^4(x, x')$, where $a_{\mu\nu}(x), b(x)$, for example, depend on x, and *not* $(1/2)a_{\mu\nu}(x)(\delta_\sigma{}^\mu\delta_\lambda{}^\nu + \delta_\lambda{}^\mu\delta_\sigma{}^\nu)\delta^4(x, x')$ as one may naïvely assume by, *a priori*,

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imposing a conservation law on $T^{\mu\nu}(x)$ *prior* to functional differentiation. The consequences of relaxing the conservation of the such external sources are highly non-trivial. For one thing the corresponding field propagators become modified. Also they have led to the rediscovery [8,9] of Faddeev-Popov (FP) [13]-like factors in non-abelian gauge theories [8,9] and the discovery of even further *generalizations* [9] of such factors, directly from the functional *differential* treatment, via the application of the quantum dynamical principle [12], in the presence of external sources, without making an appeal to path integrals, without using symmetry arguments which may be broken, and without even going into the well known complicated structures of the underlying Hamiltonians. An account of this procedure, which is also pedagogical, was given in the concluding section of [1] for the convenience of the reader and needs not to be repeated.

For higher spin fields, the propagator and time-ordered product of two fields do not, in general, coincide as the former includes so-called Schwinger terms which, in general, lead to a simplification for the propagator over the time-ordered one. This is well known for spin 1 and is also true for the graviton propagator [1]. Let $h^{\mu\nu}$ denote the gravitational field. We work in a gauge

$$\partial_i h^{i\nu} = 0, \tag{1.1}$$

where $i = 1,2,3;\ \nu = 0,1,2,3$, which, as established in Sect.3, guarantees that only two states of polarization occur for the graviton even with a non-conserved external source $T_{\mu\nu}$ in the theory.

If we denote the vacuum-to-vacuum transition amplitude for the interaction of gravitons with the external source $T_{\mu\nu}$ by $\langle 0_+ \,|\, 0_-\rangle^T$, then the propagator of the gravitational field is defined by

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \mathrm{i}\left((-\mathrm{i})\frac{\delta}{\delta T_{\mu\nu}(x)}(-\mathrm{i})\frac{\delta}{\delta T_{\sigma\lambda}(x')}\,\langle 0_+ \,|\, 0_-\rangle^T\right) \Big/ \langle 0_+ \,|\, 0_-\rangle^T\,, \tag{1.2}$$

in the limit of the vanishing of the external source $T_{\mu\nu}$. In more detail we may rewrite (1.2) as

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \mathrm{i}\,\frac{\left\langle 0_+ \left| \left(h^{\mu\nu}(x)h^{\sigma\lambda}(x')\right)_+ \right| 0_- \right\rangle^T}{\langle 0_+ \,|\, 0_-\rangle^T} + \frac{\left\langle 0_+ \left| \dfrac{\delta}{\delta T_{\mu\nu}(x)}\, h^{\sigma\lambda}(x') \right| 0_- \right\rangle^T}{\langle 0_+ \,|\, 0_-\rangle^T} \tag{1.3}$$

in the limit of vanishing $T_{\mu\nu}$, where the first term on the right-hand side, up to the i factor, denotes the time-ordered product. In the second term, the functional derivative with respect to $T_{\mu\nu}(x)$ is taken by keeping the independent field components of $h^{\sigma\lambda}(x')$ fixed. The dependent field components depend on the external source and lead to extra terms on the right-hand side of (1.3) in addition to the time-ordered product and may be referred to as Schwinger terms. For a detailed derivation of the general identity in (1.3) is given in [12] (see also [11]). It is the propagator $\Delta_+^{\mu\nu;\sigma\lambda}$ that appears in this formalism and not the time-ordered product. The propagator $\Delta_+^{\mu\nu;\sigma\lambda}(x,x')$ has been derived in [1] and will be elaborated upon in Sect. 2. It includes 30 terms in contrast to the well known one involving only 3 terms when a conservation law of $T_{\mu\nu}$ is imposed. The positivity constraint of the vacuum persistence probability $|\,\langle 0_+ \,|\, 0_-\rangle\,|^2 \leq 1$, as well as the correct spin content of the theory is established in Sect.3 for, *a priori*, *non*-conserved external energy-momentum tensor.

The expectation value formalism, pioneered by Schwinger [14], also known as the closed-time path formalism, in quantum field theory has been a useful tool in performing expectation values without first evaluating transition amplitudes. For a partial list of studies of the expectation value formalism, the reader may refer to [15,16] in the functional differential formalism. See also related work in [17–20] emphasizing on non-equilibrium phenomenae and [21–23] emphasizing Feynman path integrals.

In order to study gravitational effects such as the induced geometry due to external sources and even due to fluctuating quantum fields, the expectation value formalism turns out to be of practical value. In Sect. 4, we develop the expectation value formalism for gravitons interacting with an external energy-momentum tensor $T_{\mu\nu}$ at *finite* temperature with *a priori* not conserved $T_{\mu\nu}$, so that variations with respect to its ten components may be varied independently in order to generate expectation values. *After* all the relevant functional differentiations with respect to $T_{\mu\nu}$ are carried out, the conservation law on $T_{\mu\nu}$ may be then imposed. We establish the covariance of the *induced* Riemann curvature tensor, in the initial vacuum, due to the external source, in spite of the quantization carried out in a gauge which ensures only two polarization states for the graviton. As an application, we investigate the *induced* correction to the metric and the underlying geometry due a closed string arising from the Nambu action (e.g., [24–26]) as a solution of a circularly oscillating string [27–30] as, perhaps, the simplest generalization of a limiting point-like object. Finally, it is discussed on why the geometry of spacetime may, in general, depend on temperature due to radiative corrections and its physical significance is emphasized. The Minkowski metric is denoted by $[\eta_{\mu\nu}]$=diag[-1,1,1,1], and we use units such that $\hbar = 1, c = 1$.

2 Graviton propagator and vacuum-to-vacuum transition amplitude

The action for the gravitational field $h^{\mu\nu}$ coupled to an external energy-momentum tensor source $T_{\mu\nu}$ is taken to be

$$A = \frac{1}{8\pi G}\int (\mathrm{d}x)\mathscr{L}(x) + \int (\mathrm{d}x) h^{\mu\nu}(x) T_{\mu\nu}(x), \tag{2.1}$$

with

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h^\sigma{}_\sigma \partial_\alpha h^\beta{}_\beta - \partial^\alpha h_{\alpha\mu}\partial^\mu h^\sigma{}_\sigma \\ &+ \frac{1}{2}\partial_\alpha h^{\alpha\nu}\partial^\beta h_{\beta\nu} + \frac{1}{2}\partial_\alpha h^\mu{}_\nu \partial_\mu h^{\alpha\nu} \ ,\end{aligned} \tag{2.2}$$

and G is Newton's gravitational constant. The action part $\int(\mathrm{d}x)\mathscr{L}$ is invariant under gauge transformations

$$h^{\mu\nu}(x) \to h^{\mu\nu}(x) + \partial^\mu \xi^\nu(x) + \partial^\nu \xi^\mu(x) + \partial^\mu\partial^\nu \xi(x), \tag{2.3}$$

As mentioned above the external energy-momentum tensor $T_{\mu\nu}$ is, *a priori*, taken to be *not* conserved so that variations of its respective ten components may be varied independently - a necessary *technical* requirement. Details on dependent fields due to the gauge constraints are spelled out in [12] as well as in [1].

The vacuum-to-vacuum transition amplitude is then given by [1]

$$\langle 0_+ \,|\, 0_- \rangle^T = \exp\left[4\pi G \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x') T_{\mu\nu}(x) \Delta_+^{\mu\nu;\sigma\lambda}(x,x') T_{\sigma\lambda}(x')\right], \tag{2.4}$$

$$(\mathrm{d}x) = \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3. \tag{2.5}$$

Here we note that the exponent is scaled by the factor $8\pi G$ to satisfy the boundary condition that the gravitational attraction of two widely separated static sources is given by Newton's law [2]. The graviton propagator $\Delta_+^{\mu\nu;\sigma\lambda}(x,x')$ contains 30 terms and *not* only just the first 3 terms as may be naïvely expected, and is given by

$$\Delta_+^{\mu\nu;\sigma\lambda}(x,x') = \int \frac{(\mathrm{d}k)}{(2\pi)^4} \mathrm{e}^{\mathrm{i}k(x-x')} \left[\frac{\Delta_1^{\mu\nu;\sigma\lambda}(k)}{k^2 - \mathrm{i}\epsilon} + \frac{\Delta_2^{\mu\nu;\sigma\lambda}(k)}{\mathbf{k}^2}\right], \tag{2.6}$$

$\epsilon \to +0$, where $(\mathrm{d}k) = \mathrm{d}k^0 \mathrm{d}k^1 \mathrm{d}k^2 \mathrm{d}k^3$, $k^2 = \mathbf{k}^2 - k^{0^2}$, and

$$\begin{aligned}
\Delta_1^{\mu\nu;\lambda\sigma}(k) = & \frac{(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\sigma\lambda})}{2} \\
& + \frac{1}{2\mathbf{k}^2}\Big[\eta^{\mu\nu}k^\sigma k^\lambda + \eta^{\sigma\lambda}k^\mu k^\nu - \eta^{\nu\sigma}k^\mu k^\lambda - \eta^{\nu\lambda}k^\mu k^\sigma \\
& - \eta^{\mu\sigma}k^\nu k^\lambda - \eta^{\mu\lambda}k^\nu k^\sigma + \frac{k^\mu k^\nu k^\sigma k^\lambda}{\mathbf{k}^2}\Big] \\
& - \frac{1}{2}\left(\eta^{\mu\nu} + \frac{k^\mu k^\nu}{\mathbf{k}^2}\right)\left(\frac{N^\sigma k^\lambda + N^\lambda k^\sigma}{\mathbf{k}^2}\right)k^0 \\
& - \frac{1}{2}\left(\eta^{\sigma\lambda} + \frac{k^\sigma k^\lambda}{\mathbf{k}^2}\right)\left(\frac{N^\nu k^\mu + N^\mu k^\nu}{\mathbf{k}^2}\right)k^0 \\
& + \frac{1}{2}\Big[\eta^{\nu\sigma}(N^\mu k^\lambda + N^\lambda k^\mu) + \eta^{\nu\lambda}(N^\mu k^\sigma + N^\sigma k^\mu) \\
& + \eta^{\mu\sigma}(N^\nu k^\lambda + N^\lambda k^\nu) + \eta^{\mu\lambda}(N^\nu k^\sigma + N^\sigma k^\nu)\Big]\frac{k^0}{\mathbf{k}^2} \\
& + \frac{k^\mu k^\nu}{\mathbf{k}^2}N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{\mathbf{k}^2}N^\mu N^\nu \,,
\end{aligned} \tag{2.7}$$

$$\Delta_2^{\mu\nu;\lambda\sigma}(k) = \frac{k^\mu k^\nu}{\mathbf{k}^2}N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{\mathbf{k}^2}N^\mu N^\nu. \tag{2.8}$$

Here $(N^\mu) = (\eta^\mu{}_0) = (1,0,0,0)$. The $\mathrm{i}\epsilon$ factor in (2.6) corresponds to the Schwinger-Feynman boundary condition.

It is far from obvious that with a *non-conserved* energy-momentum tensor, the vacuum-to-vacuum amplitude $\langle 0_+ \,|\, 0_- \rangle$ in (2.4) satisfies the positivity constraint $|\,\langle 0_+ \,|\, 0_- \rangle\,|^2 \le 1$. This together with the correct spin content of the theory is established in the next section.

3 Positivity constraint and spin content

We rewrite the vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle$ in (2.4) as

$$\langle 0_+ \,|\, 0_- \rangle^T = \exp\left[4\pi G\mathrm{i}\int (\mathrm{d}x) T_{\mu\nu}(x) H^{\mu\nu}(x)\right], \tag{3.1}$$

with

$$T_{\mu\nu}H^{\mu\nu} = T_{00}H^{00} + 2T_{0i}H^{0i} + T_{ij}H^{ij}, \tag{3.2}$$

$i, j = 1, 2, 3$, and we may infer from Eq.(13) in [1] that

$$H^{00} = -\frac{1}{\boldsymbol{\partial}^2}\left[T^{00} + \frac{T}{2} - \frac{1}{2\boldsymbol{\partial}^2}\left(\partial^0\partial^0 T_{00} + \partial^i\partial^j T_{ij}\right)\right], \tag{3.3}$$

$T = T_{ii} - T_{00}$, and H^{00} is *real*. Also from Eq.(12) in [1], we may infer that

$$H^{0i} = -\frac{1}{\boldsymbol{\partial}^2}\left[\delta^{ij} - \frac{\partial^i\partial^j}{\boldsymbol{\partial}^2}\right]T_{0j}, \tag{3.4}$$

which is again *real*. That is,

$$\exp\left[4\pi G\mathrm{i}\int(\mathrm{d}x)\left(T_{00}(x)H^{00}(x)+2T_{0i}(x)H^{0i}(x)\right)\right] \tag{3.5}$$

is a phase factor.

On the other hand, we may infer from Eq.(17) in [1] that

$$H^{ij}=\frac{1}{(-\Box-\mathrm{i}\epsilon)}A^{ij,lm}T_{lm}-\frac{1}{2}\frac{1}{\boldsymbol{\partial}^2}\left(\delta^{ij}-\frac{\partial^i\partial^j}{\boldsymbol{\partial}^2}\right)T_{00}, \tag{3.6}$$

and the second term above involving T_{00} is *real*, while $A^{ij,lm}$ is given by

$$A^{ij,lm}=\frac{(\delta^{il}\delta^{jm}+\delta^{im}\delta^{jl}-\delta^{ij}\delta^{lm})}{2}-\frac{1}{2\boldsymbol{\partial}^2}\Big[\partial^i\partial^l\delta^{jm}+\partial^i\partial^m\delta^{jl}+\partial^j\partial^l\delta^{im}+\partial^j\partial^m\delta^{il}$$
$$+\partial^i\partial^j\delta^{lm}-\delta^{ij}\partial^l\partial^m-\frac{\partial^i\partial^j\partial^l\partial^m}{\boldsymbol{\partial}^2}\Big], \tag{3.7}$$

where $i,j,l,m=1,2,3$.

Accordingly, from (3.1), (3.5)-(3.7), we may rewrite

$$\langle 0_+\,|\,0_-\rangle^T=\mathrm{e}^{\mathrm{i}G[T]}\exp\left[4\pi G\mathrm{i}\int(\mathrm{d}x)T_{ij}(x)\frac{1}{(-\Box-\mathrm{i}\epsilon)}A^{ij,lm}T_{lm}(x)\right], \tag{3.8}$$

where $\exp\mathrm{i}G[T]$ is a phase factor.

By using the facts that the reality of $T_{ij}(x)$ implies that $T_{ij}(k)^*=T_{ij}(-k)$, where $(k^\mu)=(k^0,\mathbf{k})$, and the identity

$$\frac{\mathrm{i}}{2}\left(\frac{1}{k^2-\mathrm{i}\epsilon}-\frac{1}{k^2+\mathrm{i}\epsilon}\right)=-\pi\delta(k^2)=-\frac{\pi}{|\mathbf{k}|}\left[\delta(k^0-|\mathbf{k}|)+\delta(k^0+|\mathbf{k}|)\right] \tag{3.9}$$

for $\epsilon\to+0$, in the sense of distributions, we obtain that

$$\left|\langle 0_+\,|\,0_-\rangle^T\right|^2=\exp\left[-8\pi G\int\mathrm{d}\omega_{\mathbf{k}}T^*_{ij}(k)B^{ij,lm}(k)T_{lm}(k)\right], \tag{3.10}$$

where now $k^0=+|\mathbf{k}|$, $\mathrm{d}\omega_{\mathbf{k}}=\mathrm{d}^3\mathbf{k}/(2\pi)^3 2|\mathbf{k}|$, and

$$B^{ij,lm}(k)=\frac{1}{2}\Big[(\delta^{il}-\frac{k^ik^l}{\mathbf{k}^2})(\delta^{jm}-\frac{k^jk^m}{\mathbf{k}^2})+(\delta^{im}-\frac{k^ik^m}{\mathbf{k}^2})(\delta^{jl}-\frac{k^jk^l}{\mathbf{k}^2})$$
$$-(\delta^{ij}-\frac{k^ik^j}{\mathbf{k}^2})(\delta^{lm}-\frac{k^lk^m}{\mathbf{k}^2})\Big], \tag{3.11}$$

with $i,j,l,m=1,2,3$ as before.

For a given 3-vector $\mathbf{k}$, we introduce two orthonormal complex 3-vectors $\mathbf{e}_+,\mathbf{e}_-$,

$$\mathbf{e}_+\cdot\mathbf{e}_+^*=1=\mathbf{e}_-\cdot\mathbf{e}_-^*,\qquad \mathbf{e}_+\cdot\mathbf{e}_-^*=0 \tag{3.12}$$

such that $\mathbf{k}/|\mathbf{k}|,\mathbf{e}_+,\mathbf{e}_-$ constitute three mutually orthonormal vectors. That is, in addition to the conditions in (3.12),

$$\mathbf{k}\cdot\mathbf{e}_+=0,\qquad \mathbf{k}\cdot\mathbf{e}_-=0 \tag{3.13}$$

Upon writing

$$\mathbf{k} = |\mathbf{k}|(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \tag{3.14}$$

we may set

$$\mathbf{e}_+ = \frac{1}{\sqrt{2}}(\cos\phi\cos\theta - \mathrm{i}\sin\phi, \sin\phi\cos\theta + \mathrm{i}\cos\phi, -\sin\theta), \tag{3.15}$$

$$\mathbf{e}_- = \frac{1}{\sqrt{2}}(\cos\phi\cos\theta + \mathrm{i}\sin\phi, \sin\phi\cos\theta - \mathrm{i}\cos\phi, -\sin\theta), \tag{3.16}$$

and note that

$$\mathbf{e}_- = \mathbf{e}_+^* \quad . \tag{3.17}$$

The above allows us to introduce the completeness relation

$$\begin{aligned}\delta^{ij} &= \sum_{\lambda=\pm} e^i_\lambda e^{*j}_\lambda + \frac{k^i k^j}{|\mathbf{k}|^2} \\ &= \sum_{\lambda=\pm} e^{i*}_\lambda e^j_\lambda + \frac{k^i k^j}{|\mathbf{k}|^2} .\end{aligned} \tag{3.18}$$

In turn, we may define polarization 3x3 tensors by

$$e^{ij}_{\lambda\sigma} = \frac{1}{2}\left[e^i_\lambda e^{j*}_\sigma + e^{i*}_\sigma e^j_\lambda - \delta_{\lambda\sigma} e^i_\alpha e^{j*}_\alpha\right] \tag{3.19}$$

with $\lambda, \sigma, \alpha = \pm$, and a summation over the repeated index α is assumed, and note that after some algebra, $B^{ij,lm}$ in (3.11) may be rewritten as

$$B^{ij,lm} = \sum_{\lambda,\sigma=\pm} e^{ij}_{\lambda\sigma} e^{*lm}_{\lambda\sigma} \, , . \tag{3.20}$$

Using, in the process, (3.19), we note that

$$e^{ij}_{++} = 0, \qquad e^{ij}_{--} = 0, \tag{3.21}$$

and

$$e^{ij}_{+-} = e^i_+ e^j_+ \equiv \epsilon^{ij}_+, \tag{3.22}$$

$$e^{ij}_{-+} = e^i_- e^j_- \equiv \epsilon^{ij}_- \tag{3.23}$$

thus defining the two 3x3 tensors ϵ^{ij}_+, ϵ^{ij}_-, and rewrite (3.20) as

$$B^{ij,lm} = \sum_{\lambda=\pm} \epsilon^{ij}_\lambda \, \epsilon^{lm^*}_\lambda \, . \tag{3.24}$$

From (3.10), (3.11), (3.24), we conclude that

$$\left|\langle 0_+ \,|\, 0_- \rangle^T\right|^2 = \exp\left[-8\pi G \int \mathrm{d}\omega_{\mathbf{k}} \sum_{\lambda=\pm} \left(T^*_{ij}\epsilon^{ij}_\lambda\right)\left(\epsilon^{lm*}_\lambda T_{lm}\right)\right] \leq 1 \tag{3.25}$$

with equality holding in the limit of vanishing $T_{\mu\nu}$, thus establishing the underlying positivity constraint, as well as the correct spin content of the theory with the graviton having only two polarization states described by $\epsilon_+^{ij}, \epsilon_-^{ij}$ for a theory with, in general, a *not* necessarily conserved external energy-momentum tensor.

The scalar product in (3.25) may be rewritten from (3.24) as follows

$$\int \mathrm{d}\omega_{\mathbf{k}} \sum_{\lambda=\pm} (T_{ij}^* \epsilon_\lambda^{ij})(\epsilon_\lambda^{lm*} T_{lm}) = \int \mathrm{d}\omega_{\mathbf{k}} T_{ij}^* B^{ij,lm} T_{lm}$$

$$= \int (\mathrm{d}x)(\mathrm{d}x') T_{\mu\nu}(x) C^{\mu\nu,\sigma\rho}(x,x') T_{\sigma\rho}(x'), \tag{3.26}$$

where

$$C^{\mu\nu,\sigma\rho}(x,x') = \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}k(x-x')} \pi^{\mu\nu,\sigma\rho}(k), \tag{3.27}$$

$$\pi^{\mu\nu,\sigma\rho}(k) = \frac{1}{2} \left(\beta^{\mu\sigma}\beta^{\nu\rho} + \beta^{\mu\rho}\beta^{\nu\sigma} - \beta^{\mu\nu}\beta^{\sigma\rho} \right), \tag{3.28}$$

$$\beta^{\mu\nu}(k) = \left[\eta^{\mu\nu} - \frac{k^\mu k^\nu}{(Nk)^2} - \frac{N^\mu k^\nu}{(Nk)} - \frac{N^\nu k^\mu}{(Nk)} \right], \tag{3.29}$$

$$Nk = N_\alpha k^\alpha = -k^0 = -|\mathbf{k}|. \tag{3.30}$$

4 Gravitons and expectation value formalism at finite temperature

For book-keeping purposes, we use the notation

$$\sqrt{8\pi G}\, \epsilon_\lambda^{lm*} T_{lm}(\mathbf{k}) \equiv S(\mathbf{k}, \lambda), \tag{4.1}$$

and conveniently introduce a discrete notation [2, 31] for the momentum variable $\mathbf{k}$ by writing, in the process, $(\mathbf{k}, \lambda) \equiv r$ for these pairs of variables and in turn use the notation S_r for $S(\mathbf{k}, \lambda)$. A scalar product as in (3.25) then becomes simply replaced as follows:

$$8\pi G \int \mathrm{d}\omega_{\mathbf{k}} \sum_{\lambda=\pm} \left(T_{ij}^* \epsilon_\lambda^{ij} \right) \left(\epsilon_\lambda^{lm*} T_{lm} \right) \to \sum_r S_r^* S_r \,. \tag{4.2}$$

With the above notation, and for any two, *a priori*, independent, not necessarily conserved, sources $T_{\mu\nu}^1$, $T_{\mu\nu}^2$, we introduce the functional

$$\mathscr{F}[T^1, T^2] = \sum_N \sum_{N_1+N_2+\ldots=N} \langle 0_- \,|\, N; N_1, N_2, \ldots \rangle^{T^2} \langle N; N_1, N_2, \ldots \,|\, 0_- \rangle^{T^1} , \tag{4.3}$$

where N denotes number of gravitons, N_1 of which have momentum-polarization index r_1, and so on, with $\langle N; N_1, N_2, \ldots \,|\, 0_- \rangle^{T^1}$ denoting the amplitude that these N gravitons are emitted by the source T^1, and is given by

$$\langle N; N_1, N_2, \ldots \,|\, 0_- \rangle^{T^1} = \langle 0_+ \,|\, 0_- \rangle^{T^1} \frac{(\mathrm{i}S_{r_1}^1)^{N_1}}{\sqrt{N_1!}} \frac{(\mathrm{i}S_{r_2}^1)^{N_2}}{\sqrt{N_2!}} \ldots . \tag{4.4}$$

The expression for the functional $\mathscr{F}[T^1, T^2]$ may be summed exactly by using, in the precess, (4.4), to give

$$\mathscr{F}[T^1, T^2] = \left(\langle 0_+ \,|\, 0_- \rangle^{T^2} \right)^* \left(\langle 0_+ \,|\, 0_- \rangle^{T^1} \right) \exp \left[8\pi G \int \mathrm{d}\omega_{\mathbf{k}} \sum_{\lambda=\pm} \left(T_{ij}^{*2} \epsilon_\lambda^{ij} \right) \times \left(\epsilon_\lambda^{lm*} T_{lm}^1 \right) \right], \tag{4.5}$$

where we have restored the integration signs. From (4.3), we realize that for the special case that $T^1_{\mu\nu}$ and $T^2_{\mu\nu}$ are equal, we have by unitarity

$$\mathscr{F}[T,T] = \langle 0_- \,|\, 0_- \rangle^T = 1\,, \tag{4.6}$$

which also follows readily from (4.5) and the left-hand side equality in (3.25).

In the expression for $\mathscr{F}[T^1, T^2]$, we write $T^1 = T_1 + T_1'$, $T^2 = T_2 + T_2'$, where T_1' is switched on after T_1 is switched off, and T_2' is switched on after T_2 is switched off, to obtain from (4.3) and (4.5), *respectively*,

$$\begin{aligned}\mathscr{F}[T_1 + T_1', T_2 + T_2'] &= \sum_{(N)} \langle 0_- \,|\, N; N_1, N_2, \ldots \rangle^{T_2 + T_2'} \langle N; N_1, N_2, \ldots \,|\, 0_- \rangle^{T_1 + T_1'} \\ &= \sum_{(N),(M)} \langle 0_- \,|\, N; N_1, N_2, \ldots \rangle^{T_2} \langle N; N_1, N_2, \ldots \,|\, M; M_1, M_2, \ldots \rangle^{T_2', T_1'} \\ &\qquad \times \langle M; M_1, M_2, \ldots \,|\, 0_- \rangle^{T_1}\ ,\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}\langle N; N_1, N_2, \ldots \,|\, M; M_1, M_2, \ldots \rangle^{T_2', T_1'} &= \sum_{(L)} \langle N; N_1, N_2, \ldots \,|\, L; L_1, L_2, \ldots \rangle^{T_2'} \\ &\qquad \times \langle L; L_1, L_2, \ldots \,|\, M; M_1, M_2, \ldots \rangle^{T_1'}\,,\end{aligned} \tag{4.8}$$

with $\sum_{(N)}$ denoting a sum over non-negative integers $N, N_1, N_2, \ldots$ such that $N_1 + N_2 + \ldots = N$, and similarly for $\sum_{(M)}$, $\sum_{(L)}$, *and*

$$\begin{aligned}\mathscr{F}[T_1 + T_1', T_2 + T_2'] &= \mathscr{F}[T_1', T_2'] \exp[S_2^* S_1] \left(\langle 0_+ \,|\, 0_- \rangle^{T_2}\right)^* \left(\langle 0_+ \,|\, 0_- \rangle^{T_1}\right) \\ &\qquad \times \exp[S_2^*(S_1' - S_2')] \exp[-(S_1^{'*} - S_2^{'*}) S_1]\,,\end{aligned} \tag{4.9}$$

where the scalar product $S_2^* S_1$, for example, is defined as on the right-hand side of (4.2) with a sum over r. Upon comparison of the two equivalent expressions for $\mathscr{F}[T_1 + T_1', T_2 + T_2']$ in (4.7) and (4.9), we obtain, in particular, for the diagonal term $\langle N; N_1, N_2, \ldots \,|\, N; N_1, N_2, \ldots \rangle^{T^2, T^1}$, valid for *any* two, *a priori*, independent and not necessarily conserved sources $T^1_{\mu\nu}, T^2_{\mu\nu}$, the expression:

$$\begin{aligned}\langle N; N_1, N_2, \ldots \,|\, N; N_1, N_2, \ldots \rangle^{T^2, T^1} &= (N_1! N_2! \cdots) \mathscr{F}[T^1, T^2] \\ &\qquad \times \sum{}^* \prod_i \frac{\left[-(S^{1*}_{r_i} - S^{2*}_{r_i})(S^1_{r_i} - S^2_{r_i})\right]^{N_i - m_i}}{m_i! [(N_i - m_i)!]^2}\,.\end{aligned} \tag{4.10}$$

where $\sum^*$ stands for a summation over all non-negative integers $m_1, m_2, \ldots$ such that $0 \leq m_i \leq N_i$, $i = 1, 2, \ldots$.

We now perform a thermal average [16] of $\langle N; N_1, N_2, \ldots \,|\, N; N_1, N_2, \ldots \rangle^{T^2, T^1}$ by multiplying, in the process, the latter by the Boltzmann factor $\prod_i (\exp -\beta|\mathbf{k}_i|)$ and summing over (N), where $\beta = 1/\mathrm{K}\tau$, and we have used the notation K for the Boltzmann constant and τ for temperature in order not to confuse it with the trace T of an energy-momentum tensor. This gives the statistical thermal average:

$$\mathscr{F}[T^1, T^2; \tau] = \mathscr{F}[T^1, T^2; 0] \exp\left[-8\pi G \int \mathrm{d}\omega_{\mathbf{k}} \sum_{\lambda = \pm} \frac{(T^{1*}_{ij} - T^{2*}_{ij}) \epsilon^{ij}_\lambda \epsilon^{lm*}_\lambda (T^1_{lm} - T^2_{lm})}{(\mathrm{e}^{\beta|\mathbf{k}|} - 1)}\right]\,. \tag{4.11}$$

In particular, we note from (4.5), (4.6), (4.11) that for the special case that $T^1_{\mu\nu}$, $T^2_{\mu\nu}$ are identical, we have the consistent normalization condition

$$\mathscr{F}[T, T; \tau] \equiv 1. \tag{4.12}$$

We also verify directly from (4.11) that

$$\mathscr{F}[T^1, T^2; 0] = \mathscr{F}[T^1, T^2], \tag{4.13}$$

as expected.

As we have not imposed conservation laws on $T^1_{\mu\nu}$, $T^2_{\mu\nu}$, we may vary each of their respective ten components independently to obtain from the quantum dynamical principle [7, 12, 14] as applied, respectively, and in the process to $\langle L; L_1, \ldots | M; M_1, \ldots \rangle^{T^1}$ and $\langle N; N_1, \ldots | L; L_1, \ldots \rangle^{T^2}$ in (4.8) with T'_1, T'_2 in it replaced by T^1, T^2, the thermal average $\langle h^{\mu\nu}(x)\rangle^T_\tau$ of the gravitational field

$$\begin{aligned}\langle h^{\mu\nu}(x)\rangle^T_\tau &= (-\mathrm{i})\,\frac{\delta}{\delta T^1_{\mu\nu}(x)}\,\mathscr{F}[T^1, T^2; \tau]\bigg|_{T^1=T^2=T} \\ &= (\mathrm{i})\,\frac{\delta}{\delta T^2_{\mu\nu}(x)}\,\mathscr{F}[T^1, T^2; \tau]\bigg|_{T^1=T^2=T}\quad ,\end{aligned} \tag{4.14}$$

generalizing the expression for $\langle 0_- |\, h^{\mu\nu}(x)\, | 0_- \rangle^T$ given by

$$\begin{aligned}\langle 0_- | h^{\mu\nu}(x) | 0_- \rangle^T &= (-\mathrm{i})\,\frac{\delta}{\delta T^1_{\mu\nu}(x)}\,\mathscr{F}[T^1, T^2]\bigg|_{T^1=T^2=T} \\ &= (\mathrm{i})\,\frac{\delta}{\delta T^2_{\mu\nu}(x)}\,\mathscr{F}[T^1, T^2]\bigg|_{T^1=T^2=T}\quad ,\end{aligned} \tag{4.15}$$

from zero to finite temperature.

From (4.11), (4.5), (3.26), the generating functional $\mathscr{F}[T^1, T^2; \tau]$ may be rewritten as

$$\begin{aligned}&\mathscr{F}[T^1, T^2; \tau] = (\langle 0_+ | 0_- \rangle^{T^2})^* (\langle 0_+ | 0_- \rangle^{T^1}) \\ &\times \exp\left[8\pi G \int (\mathrm{d}x)(\mathrm{d}x') T^2_{\mu\nu}(x) C^{\mu\nu,\sigma\rho}(x, x') T^1_{\sigma\rho}(x')\right] \\ &\times \exp\left[-8\pi G \int (\mathrm{d}x)(\mathrm{d}x') (T^1_{\mu\nu}(x) - T^2_{\mu\nu}(x)) D^{\mu\nu,\sigma\rho}(x, x'; \tau) (T^1_{\sigma\rho}(x') - T^2_{\sigma\rho}(x'))\right],\end{aligned} \tag{4.16}$$

where $C^{\mu\nu,\sigma\rho}(x, x')$ is defined in (3.27), and

$$D^{\mu\nu,\sigma\rho}(x, x'; \tau) = \int \mathrm{d}\omega_{\mathbf{k}} \mathrm{e}^{\mathrm{i}k(x-x')}\,\frac{\pi^{\mu\nu,\sigma\rho}(k)}{(\mathrm{e}^{-\beta(Nk)} - 1)}, \tag{4.17}$$

$Nk = N_\alpha k^\alpha = -k^0 = -|\mathbf{k}|$, where $\pi^{\mu\nu,\sigma\rho}(k)$ is given in (3.28).

We note that the temperature dependence occurs only in the last exponential in (4.16) through $D^{\mu\nu,\sigma\rho}(x, x'; \tau)$. We eventually set $T^1_{\mu\nu} = T^2_{\mu\nu}$ *after* the relevant functional differentiations with respect to these sources are taken. For $\tau \to 0$, the last exponential in (4.16) is equal to one, giving the relation in (4.13).

5 Covariance of the induced Riemann curvature tensor

The thermal average $\left\langle h_{\mu\nu}(x)\right\rangle_{\tau}^{T}$ may be obtained from (4.14), (4.16) to give

$$\begin{aligned}\left\langle h_{\mu\nu}(x)\right\rangle_{\tau}^{T} &= 8\pi G\mathrm{i}\int(\mathrm{d}x')T^{\sigma\rho}(x')\int \mathrm{d}\omega_{\mathbf{k}}\pi_{\mu\nu,\sigma\rho}(k)\mathrm{e}^{\mathrm{i}k(x-x')} \\ &\quad - 8\pi G\mathrm{i}\int(\mathrm{d}x')T^{\sigma\rho}(x')\int \mathrm{d}\omega_{\mathbf{k}}\pi_{\sigma\rho,\mu\nu}(k)\mathrm{e}^{\mathrm{i}k(x'-x)} \\ &= -16\pi G\int(\mathrm{d}x')T^{\sigma\rho}(x')\int \mathrm{d}\omega_{\mathbf{k}}\sin k(x-x')\pi_{\mu\nu,\sigma\rho}(k) \\ &\equiv \left\langle 0_{-}\left|h_{\mu\nu}(x)\right|0_{-}\right\rangle^{T}\end{aligned} \tag{5.1}$$

for $x^0 > x'^0$, where after the functional differentiation was carried out with respect to, say, $T^{1\mu\nu}$, we have set $T^{2\mu\nu} = T^{1\mu\nu} = T^{\mu\nu}$. We learn that the above expectation value is independent of temperature in the leading linearized theory as a consequence of the fact that the exponent in the last exponential in (4.16) does not contribute if a single functional differentiation w.r.t. $T^{1\mu\nu}$ is carried out and then by finally setting $T^2_{\mu\nu} - T^1_{\mu\nu} = 0$. Radiative corrections and explicit temperature dependence will be discussed in Sect. 7.

In more detail, we may rewrite (5.1) as:

$$\begin{aligned}\left\langle 0_{-}\left|h_{\mu\nu}(x)\right|0_{-}\right\rangle^{T} &= \left\{8\pi G\mathrm{i}\int \mathrm{d}\omega_{k}\mathrm{e}^{\mathrm{i}kx}\left[T_{\mu\nu}(k) - \frac{\eta_{\mu\nu}}{2}T(k)\right] + \text{c.c.}\right\} \\ &\quad + \partial_\mu\xi_\nu(x) + \partial_\nu\xi_\mu(x) + \partial_\mu\partial_\nu\xi(x)\end{aligned} \tag{5.2}$$

$$\xi_\mu(x) = \left\{4\pi G\int \mathrm{d}\omega_{\mathbf{k}}\mathrm{e}^{\mathrm{i}kx}\frac{N_\mu T - 2T_\mu{}^\sigma N_\sigma}{(Nk)} + \text{c.c.}\right\} \tag{5.3}$$

$$\xi(x) = \left\{\frac{4\pi G}{\mathrm{i}}\int \mathrm{d}\omega_{\mathbf{k}}\mathrm{e}^{\mathrm{i}kx}\frac{T + 2T^{\nu\sigma}N_\nu N_\sigma}{(Nk)^2} + \text{c.c.}\right\} \tag{5.4}$$

and $\partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \partial_\mu\partial_\nu\xi$ are the so-called gauge terms (see (2.3)) and are non-covariant depending on the vector N^μ. Also in this section, since we are not carrying out further functional differentiations with respect to the sources we have finally imposed the conservation law $\partial_\mu T^{\mu\nu} = 0$ in (5.2).

The induced Riemann curvature tensor in the leading theory is given by

$$\left\langle 0_{-}\left|R_{\mu\nu\sigma\lambda}(x)\right|0_{-}\right\rangle^{T} = \left\langle 0_{-}\left|\partial_\mu\partial_\sigma h_{\nu\lambda} + \partial_\nu\partial_\lambda h_{\mu\sigma} - \partial_\mu\partial_\lambda h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\lambda}\right|0_{-}\right\rangle^{T} \tag{5.5}$$

By substituting the expression (5.2) in (5.5), we see that all the terms depending on ξ^μ, ξ *cancel* in the induced Riemann curvature tensor $\left\langle 0_{-}\left|R_{\mu\nu\sigma\lambda}(x)\right|0_{-}\right\rangle^{T}$ thus establishing its covariance. This means that one may restrict $\left\langle 0_{-}\left|h_{\mu\nu}(x)\right|0_{-}\right\rangle^{T}$ to its covariant gauge-independent part

$$\left\langle 0_{-}\left|h_{\mu\nu}(x)\right|0_{-}\right\rangle^{T} = \left\{8\pi G\mathrm{i}\int \mathrm{d}\omega_{\mathbf{k}}\mathrm{e}^{\mathrm{i}kx}[T_{\mu\nu}(k) - \frac{\eta_{\mu\nu}}{2}T(k)] + \text{ c.c.}\right\} \equiv h^{o}_{\mu\nu}(x) \tag{5.6}$$

in applications. The expression for the latter may be further simplified to

$$h^{o}_{\mu\nu}(x) = \left\{8\pi G\mathrm{i}\int(\mathrm{d}x')\int \mathrm{d}\omega_{\mathbf{k}}\mathrm{e}^{\mathrm{i}k(x-x')}[T_{\mu\nu}(x') - \frac{\eta_{\mu\nu}}{2}T(x')] + \text{ c.c.}\right\} \tag{5.7}$$

The $\mathbf{k}$-integration as well as the $x^{0'}$-one may be explicitly carried out leading to

$$h^{o}_{\mu\nu}(x) = 2G\int \frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}[T_{\mu\nu}(x^0 - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}') - \frac{\eta_{\mu\nu}}{2}T(x^0 - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')]. \tag{5.8}$$

6 The induced correction to the metric: Application to a Nambu string

The metric of spacetime to the leading contribution in our notation here is defined [2] by

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2h^{o}_{\mu\nu}(x), \tag{6.1}$$

with the 2 factor, where $h^{o}_{\mu\nu}(x)$ is given in (5.8). The leading contribution to the inverse $g^{\mu\nu}$ is then given by $g^{\mu\nu} = \eta^{\mu\nu} - 2h^{o\mu\nu}$.

We investigate the contribution to the metric, the induced geometry and corresponding spacetime measurements due to a string. The dynamics of the string is described as follows. The trajectory of the string is described by a vector function $\mathbf{R}(\sigma, t)$, where σ parametrizes the string. The equation of motion of the closed sting considered is taken to be

$$\frac{\partial^2}{\partial t^2}\mathbf{R}(\sigma, t) - \frac{\partial^2}{\partial \sigma^2}\mathbf{R}(\sigma, t) = 0, \tag{6.2}$$

with constraints

$$\partial_t \mathbf{R} \cdot \partial_\sigma \mathbf{R} = 0, \quad (\partial_t \mathbf{R})^2 + (\partial_\sigma \mathbf{R})^2 = 1, \quad \mathbf{R}(\sigma + \frac{2\pi}{\omega}, t) = \mathbf{R}(\sigma, t), \tag{6.3}$$

for a constant ω. The general solution to (6.2), (6.3) is given by

$$\mathbf{R}(\sigma, t) = \frac{1}{2}[\boldsymbol{\Phi}(\sigma - t) + \boldsymbol{\Psi}(\sigma + t)], \tag{6.4}$$

where $\boldsymbol{\Phi}, \boldsymbol{\Psi}$, in particular, satisfy the normalization conditions $(\partial_\sigma \boldsymbol{\Phi})^2 = (\partial_\sigma \boldsymbol{\Psi})^2 = 1$. For the system (6.2)-(6.4), we consider a solution of the form [27–30]

$$\mathbf{R}(\sigma, t) = (\cos\omega\sigma, \sin\omega\sigma, 0)\frac{\sin\omega t}{\omega} \tag{6.5}$$

describing a radially oscillating circular string in a plane. The general expression for the energy-momentum tensor of the string is given by

$$T^{\mu\nu}(x) = \frac{M\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}\sigma (\partial_t R^\mu \partial_t R^\nu - \partial_\sigma R^\mu \partial_\sigma R^\nu)\delta^3(\mathbf{r} - \mathbf{R}(\sigma, t)), \tag{6.6}$$

where $R^0 = t$, $\mathbf{r} = r(\cos\phi, \sin\phi, 0)$, and M provides a mass scale. The various components of the energy-momentum tensor are worked out to be [27–30]

$$T^{00} = \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z), \tag{6.7}$$

$$T^{0i} = \frac{M}{2\pi r}(\cos\phi, \sin\phi, 0)\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)\cos\omega t \,\mathrm{sgn}(\sin\omega t), \tag{6.8}$$

$$T^{11} = \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)[\cos^2\omega t - \sin^2\phi], \tag{6.9}$$

$$T^{12} = \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)\frac{\sin 2\phi}{2}, \tag{6.10}$$

$$T^{22} = \frac{M}{2\pi r}\delta\left(r - \frac{|\sin\omega t|}{\omega}\right)\delta(z)[\cos^2\omega t - \cos^2\phi], \tag{6.11}$$

$$T^{\mu 3} = 0\,, \tag{6.12}$$

where $\mathrm{sgn}(\alpha) = \pm 1$ for $\alpha \gtrless 0$ is the sign function, $i = 1, 2, 3$.

We note the normalization condition

$$\int \mathrm{d}^3\mathbf{x} T^{00}(x) = M. \tag{6.13}$$

Also for the trace $T^{\mu}{}_{\mu}(x)$ of the energy-momentum tensor we have

$$T = -\frac{M}{\pi r}\delta\left(r - \frac{|\sin \omega t|}{\omega}\right)\delta(z)\sin^2 \omega t. \tag{6.14}$$

It is most interesting to consider spacetime measurements along the most symmetrical direction in the problem, that is, along the $z - (x^3-)$ axis perpendicular to the plane of oscillations. Before doing so, we note that in the plane of oscillations of the string, $g_{\phi\phi}$ cannot be a function of ϕ by symmetry. Also no cross term $g_{r\phi}$ can occur in this plane, i.e., $g_{r\phi} = 0$. The metric contributions h_{rr}, h_{00}, in the plane of oscillations, are readily obtained. To this end (5.8), (6.7)-(6.12), (6.14) lead for $r \gg 1/\omega$

$$\begin{aligned} 2h_{11}(x) &\simeq \frac{4G}{r}\int \mathrm{d}^3\mathbf{x}'\left[T_{11}(x^0 - r, \mathbf{x}') - \frac{T(x^0 - r, \mathbf{x}')}{2}\right] = \frac{2GM}{r} \\ &\simeq 2h_{22}(x), \qquad h_{12} \simeq 0\,, \end{aligned} \tag{6.15}$$

where $1/\omega$ is the maximum radial extension of the string. Using the identity $h_{rr} = \cos^2\phi\, h_{11} + \sin^2\phi\, h_{22} + \sin 2\phi\, h_{12}$, it leads to

$$g_{rr} \simeq \left(1 + \frac{2GM}{r}\right). \tag{6.16}$$

On the other hand,

$$\begin{aligned} 2h_{00}(x) &\simeq \frac{4GM}{r}\int \mathrm{d}^3\mathbf{x}'\left[T_{00}(x^0 - r, \mathbf{x}') + \frac{T(x^0 - r, \mathbf{x}')}{2}\right] \\ &= \frac{4GM}{r}\cos^2 \omega(t - r) \end{aligned} \tag{6.17}$$

or

$$g_{00}(x) \simeq -\left(1 - \frac{4GM}{r}\cos^2\omega(t - r)\right), \tag{6.18}$$

where we recall that the Minkowski metric is taken to be $[\eta_{\mu\nu}]$=diag[-1,1,1,1].

For an observer at a fixed $r \gg 1/\omega$ in the plane of oscillations of the string, then time slows down by a factor

$$\frac{1}{(T_2 - T_1)}\int_{T_1}^{T_2}\sqrt{-g_{00}}\mathrm{d}t = 1 - \frac{GM}{r}\left\{1 + \cos\omega(T_1 + T_2 - 2r)\frac{\sin\omega(T_2 - T_1)}{\omega(T_2 - T_1)}\right\} \tag{6.19}$$

relative to a time lapsed of length $(T_2 - T_1)$ in empty space.

For spacetime measurements along the z-axis, we have explicitly

$$2h^{o}_{33}(x) = 4GM\int_0^{\infty}\frac{\mathrm{d}r'}{\sqrt{r'^2 + z^2}}\delta\left(r' - \frac{|\sin\omega(t - \sqrt{r'^2 + z^2})|}{\omega}\right)r'^2\omega^2. \tag{6.20}$$

Again, since r' does not exceed $1/\omega$, we have for an observer at $|z| \gg 1/\omega$

$$g_{33}(x) \simeq 1 + \frac{4GM}{|z|} \sin^2 \omega(t - |z|), \tag{6.21}$$

showing an interesting oscillatory behaviour in the space metric with a relative expansion of length.

Similarly, we obtain

$$g_{00}(x) \simeq -\left(1 - \frac{4GM}{|z|} \cos^2 \omega(t - |z|)\right). \tag{6.22}$$

7 Conclusion

The positivity constraint as well as the spin content of the theory of gravitons interacting with *a priori* non-conserved external energy-momentum tensor was established. As emphasized throughout, relaxing this conservation law is necessary so that variations of the ten components of the energy-momentum tensor may be varied independently which goes to the heart of the functional differential formalism of quantum field theory. The expectation value formalism of the theory within the above context was derived at finite temperature for gravitons. Thermal averages of the generated gravitational field and their correlations may be then obtained by functional differentiations of the resulting generating functional at finite temperature which coincide with the corresponding expectation values $\langle 0_- | \cdot | 0_- \rangle$ at zero temperature. The covariance of the induced Riemann curvature tensor was established in spite of the gauge constraint which ensures only two polarization states of the graviton. An application was carried out to determine the induced correction to the Minkowski metric resulting from a closed string arising from the Nambu action as a solution of a circularly oscillating string. Radiative corrections play an important role as the induced geometry may, in general, depend on temperature. Technically, this may be seen as follows. The multiplicative factor in the generating functional $\mathscr{F}[T^1, T^2; \tau]$ in (4.16) depending on temperature is given by

$$\exp\left[-8\pi G \int (\mathrm{d}x)(\mathrm{d}x')(T^1_{\mu\nu}(x) - T^2_{\mu\nu}(x)) D^{\mu\nu,\sigma\rho}(x, x'; \tau)(T^1_{\sigma\rho}(x') - T^2_{\sigma\rho}(x'))\right], \tag{7.1}$$

where $D^{\mu\nu,\sigma\rho}(x, x'; \tau)$ is defined in (4.17), (3.28)-(3.30). Consider a familiar correction to the leading order in the Lagrangian density given by $h^{\mu\nu}(x)\left(\tau_{\mu\nu} + T^{(m)}_{\mu\nu}\right)$, where $\tau_{\mu\nu}$, $T^{(m)}_{\mu\nu}$ are energy-momentum tensors of the gravitational field and matter, respectively. For example, if $T^{(m)}_{\mu\nu}$ corresponds to a real scalar field coupled in turn to an external source $K(x)$, then the multiplicative factor in the corresponding generating functional of the scalar field depending on temperature is clearly given by

$$\exp\left[-\int (\mathrm{d}x)(\mathrm{d}x')(K^1(x) - K^2(x))\Delta^+(x, x'; \tau)(K^1(x') - K^2(x'))\right], \tag{7.2}$$

where

$$\Delta^+(x, x'; \tau) = \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{e}^{\mathrm{i}k(x-x')}}{(2\pi)^3 2\sqrt{\mathbf{k}^2 + m^2}} (\mathrm{e}^{\beta\sqrt{\mathbf{k}^2+m^2}} - 1)^{-1}, \tag{7.3}$$

$k^0 = +\sqrt{\mathbf{k}^2 + m^2}$, and m is the mass of the scalar field. Now both $\tau_{\mu\nu}$ and $T^{(m)}_{\mu\nu}$ are *quadratic* in their respective fields. To generate the term $h^{\mu\nu}\tau_{\mu\nu}$, we then need to functionally differentiate (7.1), say, with the external source $T^1_{\mu\nu}$ *three* times, also additively w.r.t. $T^2_{\mu\nu}$ according to the quantum dynamical principle [14–16]. On the other, hand to generate $T^{(m)}_{\mu\nu}$, we have to functionally differentiate (7.2) twice with respect to the external sources $K^{1,2}$ of the scalar field. Finally to generate the thermal average of

$h_{\mu\nu}$, we have to functionally differentiate once more w.r.t. $T^1_{\mu\nu}$ and then set $T^1_{\mu\nu} = T^2_{\mu\nu} \equiv T_{\mu\nu}$, and $K^1 = K^2 \equiv K$. That is, all in all, we have an *even* number of functional differentiations w.r.t. the corresponding external sources to generate the thermal average $\langle h_{\mu\nu} \rangle^T_\tau$ before setting the equality of the sources just mentioned and thus generate a temperature dependence in $\langle h_{\mu\nu} \rangle^T_\tau$. This is unlike the situation in the leading order in which we have to differentiate only once w.r.t. $T^1_{\mu\nu}$ to generate $\langle h_{\mu\nu} \rangle^T_\tau$ before setting $T^1_{\mu\nu} - T^2_{\mu\nu} = 0$, resulting no temperature dependence in the former expression as seen in (5.1). The study of higher orders, however, requires a detailed analysis of Faddeev-Popov-like factors of the type discovered in [1, 9], as generated in the functional differential treatment (see Sect.3 in [1, 8, 9], [12]) which would in turn lead to extra vertices coming from the second term on the right-hand side of (1.3) and its generalizations and complicates matter quite a bit in gravitation. This formidable problem as well as convergence aspects [32] will be investigated in a future report. Physically, temperature dependence of the underlying induced geometry is also clear. When we perform a thermal average, we introduce in the process, a *background* of gravitons, and in general other particles depending on the matter fields considered. These particles in turn would then act as additional *sources* of gravitation contributing to the net induced gravitational field and this happens *only* when non-linearities as field *interactions* are considered, and corresponding radiative corrections are taken into account.

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