การศึกษาปฏิกิริยาการสลายตัวของ e^+e^-เป็น $\rho\pi$ และ $\omega\pi$ ในแบบจำลองควาร์ก

นายกฤษดา กิตติมานะพันธ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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INVESTIGATION OF REACTIONS e^+e^- TO $\rho\pi$ AND $\omega\pi$ IN QUARK MODEL

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A Thesis Submitted in Partial Fulfillment of the Requirements

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INVESTIGATION OF REACTIONS e^+e^- TO $\rho\pi$ AND $\omega\pi$ IN QUARK MODEL

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Master of Science.

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วิทยานิพนธ์นี้ได้ทำการศึกษาปฏิกิริยา $e^+e^- \to \omega\pi^0$ และ $e^+e^- \to \rho\pi^0$ ด้วยแบบจำลองควาร์ก $^3P_0$ โดยการจำลองข้อมูลที่ได้ในปฏิกิริยานี้ขึ้นใหม่และเทียบกับผลที่ได้จากการทดลอง จากการศึกษาพบว่าในช่วงของระดับพลังงานของปฏิกิริยานี้ซึ่งเริ่มตั้งแต่พลังงานขีดเริ่มจนถึง 1.5 GeV กระบวนการสองขั้นซึ่งเกิดขึ้นโดยแรกเริ่มคู่ของควาร์กและปฏิควาร์กสร้างตัวเป็นเวกเตอร์เมซอนแล้วจากนั้นจึงสลายตัวให้คู่ของฮาร์ดรอน ส่งผลในกระบวนการเกิดคู่ของฮาร์ดรอนมากกว่ากระบวนการขั้นเดียวซึ่งเกิดขึ้นโดยที่คู่ของควาร์กและปฏิควาร์กถูกจับโดยคู่ของควาร์กและปฏิควาร์กอื่น แล้วสร้างตัวเป็นคู่ของฮาร์ดรอน นอกจากนั้นข้อมูลที่ได้จากการทดลองของปฏิกิริยา $e^+e^- \to \omega\pi^0$ ยืนยันว่า $\rho(1450)$ อยู่ในสถานะ S-wave ในขณะที่ $e^+e^- \to \rho\pi^0$ ยืนยันว่า $\omega(1420)$ อยู่ในสถานะ D-wave

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KRITSADA KITTIMANAPUN : INVESTIGATION OF REACTIONS e^+e^- TO $\rho\pi$ AND $\omega\pi$ IN QUARK MODEL. THESIS ADVISOR : ASSOC. PROF. YUPENG YAN, Ph.D. 64 PP.

$^3P_0$ MODEL/e^+e^- ANNIHILATION

The reactions $e^+e^- \to \omega\pi^0$ and $e^+e^- \to \rho\pi^0$ are investigated in the $^3P_0$ non-relativistic quark model. The experimental data of both the reactions are fairly reproduced in the work. The study suggests that at the energy region from the threshold to 1.5 GeV the two-step process, in which the primary $\overline{q}q$ pair forms first a vector meson and then the meson decays into a hadron pair, is dominant over the one-step process in which the primary $\overline{q}q$ pair is directly dressed by an additional $\overline{q}q$ pair to form a hadron pair. It is found that the experimental data of the reaction $e^+e^- \to \omega\pi^0$ strongly dictate a S-wave $\rho(1450)$ while the data of the reaction $e^+e^- \to \rho\pi^0$ prefer $\omega(1420)$ being a D-wave meson.

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CHAPTER I

INTRODUCTION

The most fundamental unit, atom, expressing the individual chemical properties is at first thought to be the smallest particle. Until the discovery of protons, neutrons, and electrons which are the composition of atoms, only electrons are classified as a type of elementary particles in lepton family. Fortunately, protons and neutrons are indeed consisted of another type of elementary particles called quarks. A bound state of three quarks and a pair of quark-antiquark are respectively baryon such as protons (uud), neutron (ddu), Λ (uds) etc. and meson such as $\pi^{\pm}$ ($u\bar{d}, d\bar{u}$), ω ($\frac{u\bar{u}+d\bar{d}}{2}$) and so on. To reveal details of the tiny structure, the collisions at high energy level of e^+e^- and $p\bar{p}$, for example, have been studied.

The investigation of e^+e^- annihilations has revealed many important and fascinating phenomena in particle physics in a large energy region ranging from a few hundred MeV ($\pi\pi$ threshold) to several hundred GeV (W^-W^+ threshold). The over-decade research on e^-e^+ annihilation reaction in both experimental and theoretical sectors has played a crucial role in confirming the successes of the standard model at high energies and in originating theoretical models at low energies.

The decay modes $\rho\pi$ and $\omega\pi$ are among the most important for the processes of e^+e^- annihilation into hadrons at low energies, giving mainly the 3π and 4π final states, respectively (Akhmetshin et al., 1999; Akhmetshin et al., 2004). These reactions might be used to study the dynamics of light vector mesons, for example, ρ' and ω', which may be formed as the intermediate states and decay then into $\omega\pi$ and $\rho\pi$. The information of mesons except the lightest ones is still rather rare because of the lack of high-quality experimental data and also effective theoretical models. In addition, that ρ' may decay into ω ($\rho' \to \omega\pi$) and ω' to $\rho\pi$ ($\omega' \to \rho\pi$) (Yao et al., 2006) adds more uncertainties to the understanding of the properties of the intermediate states. The analysis of (Achasov et al, 1999) confirms that the uncertainties between the ω-like resonance and ρ-like resonance result to calculations with low accuracy.

Recently, experiments have been set up to study the processes of e^+e^- annihilation at low energies (below 2 GeV). The $e^+e^- \to \pi^0\pi^0\gamma$ process in the SND experiment studied in the energy region $0.6 - 0.97$ GeV gives information of ρ and ω intermediate state mesons (Achasov et al., 2002; Akhmetshin 2005). The reaction of $\omega\pi \to \pi^0\pi^0\gamma$ measured in the center mass energies $0.92 - 1.38$ GeV at CMD-2 shows the interference of $\rho(770)$ meson and $\rho(1450)$ meson, which decays into $\omega\pi^0$ (Akhmetshin et al., 2003). However, the SND experiment with the energy up to 1.4 GeV from the threshold (Achasov et al., 2000) revealed that the experimental cross section can be satisfactorily understood with two excited states included with the masses $m_{\rho'} = 1400$ MeV and $m_{\rho''} = 1600$ MeV in which a contribution of the higher state dominates. However, this result contradicts the theoretical expectation, where ρ' and ρ'' are considered as $2S$ and $1D$ $q\bar{q}$ states respectively and the larger contribution of the lower $2S$ excitation was predicted.

At the energy range up to 1.8 GeV, the cross section measured with $BABAR$ detector is well described by a sum of contributions of four isoscalar resonances ($\omega, \phi, \omega' = \omega(1350)$ and $\omega'' = \omega(1660)$) (Aubert et al., 2004). The similar resonances were also reported by the VEPP-2M collider in the energy region $0.98 - 1.38$ GeV (Achasov, Aulchenko et al., 2002).

The intermediate vector mesons in e^+e^- annihilation reactions at low energies could be simple $\overline{q}q$ states, mixtures of ρ-like and ω-like mesons, or even hybrid states ($\bar{q}q$ plus one or more gluons). The idea of exotic meson (vector hybrid) (Donnachie and Kalashnikova, 1999) has been proposed, but the theoretical results are not in line with the the experimental data. On the other hand, $q\bar{q}$ structured mesons with different radial and orbital excitations have been extensively studied. An earlier work in quark model (Godfrey and Isgur, 1985) predicted a series of excited vector mesons, with $\rho(1450)$ and $\omega(1460)$ being the lowest ρ-type state with the 2^3S_1 excitation which has a large probability to decay into $\omega\pi^0$ and the ω-type state with the 1^3D_1 excitation, in respective. The predictions are consistent with some experimental data but in strong contrast with the observations of CMD-2 (Akhmetshin et al., 1999) and CLEO (Edwards et al., 2000) which support the $a_1(1260)$ dominance in the reaction $e^+e^- \to \omega\pi$.

The prediction in the work (Godfrey and Isgur, 1985) that the meson $\rho(1450)$ has a bigger probability to decay into $\omega\pi$ than the ρ-type mesons with higher masses is not consistent with the results of the SND experiment (Achasov et al., 2000) that the $\rho(1600)$ meson dominates over the $\rho(1400)$ in the reaction $e^+e^- \to \omega\pi$. However, the results of the recent work (Achasov and Kozhevnikov, 1998) do not contradict the assignment of the $\rho(1450)$ and $\omega(1420)$ to the state 2^3S_1.

The properties of the intermediate states in the processes of e^+e^- annihilation into $\rho\pi$ and $\omega\pi$ are still open questions.

In this thesis, we will work in the non-relativistic quark model and the $^3P_0$ model will be employed to study the creation of a light meson pair. The data used to analyze the calculation are mainly form Novosibirsk.

The main objective is to reveal the dominant dynamics of the reactions $e^+e^- \to \omega\pi$, $\rho\pi$ at low energies. The study is also expected to lead us to better understandings of the vector mesons, $\rho(1450)$ and $\omega(1420)$.

This thesis is arranged as follows: In Chapter II, hadrons in non-relativistic quark model is demonstrated. The $^3P_0$ model, the size parameter, and the effective coupling constant are discussed in Chapter III. The transition amplitude and the cross section are shown in Chapter IV and Chapter V consists of discussion and conclusion. The particle data of ρ, ω, and π is displayed in Appendix A. Appendix B is the derivation of spatial wave function in three dimensional harmonic oscillator potential by Shrödinger equation. γ-matrix, trace technology, and reaction of $e^+e^- \rightarrow \mu^+\mu^-$ are discussed in Appendix C and D. In Appendix E is the Wigner 9j symbols used in the calculation of decay process.

CHAPTER II

HADRONS IN NON-RELATIVISTIC QUARK MODEL

Hadrons, the bound state of quarks, can be classified into two groups, a bound state of three quarks called baryon and a bound state of quark-antiquark called mesons. Hadrons have been studied by both non-relativistic and relativistic quark models depending on energy range and mass we are interested. Since we study the collision events in energy level between 1-2 GeV or around the thresholds of ρ and ω resonances, the non-relativistic quark model is totally employed. The important ideas in this model is to construct color, flavor, and spin wave functions by language of group theory. For the spatial wave function, we work out the Schrödinger equation with three dimensional harmonic oscillator potential. Even though in thesis we study only mesons, we will demonstrate the wave functions for both mesons and baryons.

2.1 Meson Wave Function

The states of mesons can be identified by wave functions in four spaces which are color, flavor, spin, and spatial spaces.

2.1.1 Color and Spin-Flavor Wave Function

The color wave function of all particles is observed in only a singlet state

$$|1\rangle = \frac{1}{\sqrt{3}}|q_i\rangle|\bar{q}_i\rangle \tag{2.1}$$

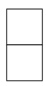
transformed under $SU(3)$ transformation

$$
\begin{aligned}
U|1\rangle &= \frac{1}{\sqrt{3}}|q_i\rangle|\bar{q}_i\rangle \\
&= \frac{1}{\sqrt{3}}\sum_{j,k} U_{ji}|q_j\rangle U^*_{ki}|\bar{q}_k\rangle \\
&= \frac{1}{\sqrt{3}}\sum_{j,k}(UU^\dagger)_{jk}|q_j\rangle|\bar{q}_k\rangle \\
&= \frac{1}{\sqrt{3}}\delta_{jk}|q_j\rangle|\bar{q}_k\rangle \\
&= \frac{1}{\sqrt{3}}|q_i\rangle|\bar{q}_i\rangle. \qquad (2.2)
\end{aligned}
$$

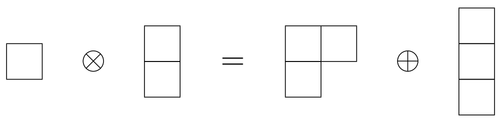
For flavor wave function, method from group theory has been used. The fundamental representation $D(1,0)$, quark flavor, of $SU(3)$ is denoted by the Young tableaux


,

while the conjugation representation $D(0,1)$, referred to antiquark flavor, is depicted by


.

A meson flavor state is represented by the direct product of quark and antiquarks states by which the Young Tableaux for mesons are formed as following:



that is,

$$D(1,0) \otimes D(0,1) = D(1,1) \oplus D(0,0) \qquad (2.3)$$

with the corresponding dimensions begin:

Table 2.1 Meson nonet

Content	Charge	Strangeness	Pseudoscalar	Vector
$u\bar{d}$	+1	0	π^+	ρ^+
$d\bar{u}$	-1	0	π^-	ρ^-
$u\bar{u}$	0	0	π^0	ρ^0
$d\bar{d}$	0	0	η^0	ω^0
$s\bar{s}$	0	0	η'^0	ϕ^0
$u\bar{s}$	+1	+1	K^+	K^{*+}
$d\bar{s}$	0	+1	K^0	K^{*0}
$s\bar{u}$	-1	-1	k^-	K^{*-}
$s\bar{d}$	0	-1	$\bar{K}^0$	$\bar{K}^{*0}$

$$3 \otimes \bar{3} = 8 \oplus 1. \tag{2.4}$$

For spin wave function, each quark and antiquark has two possible spin states, spin up and spin down, namely $S_z = \pm\frac{1}{2}$, transformed under $SU(2)$ group. The representations of mesons in spin space are

$$2 \otimes \bar{2} = 3 \oplus 1, \tag{2.5}$$

where the total spin wave function takes the form of triplet ($S = 1$) and singlet ($S = 0$). The spin-flavor conjugation for mesons are

$$(1_f, 1_s), (1_f, 3_s), (8_f, 1_s), (8_f, 3_s) \tag{2.6}$$

where subscript f and s represent flavor and spin, in respective.

Table 2.2 Flavor wave functions of the pseudoscalar and vector meson nonets

Pseudoscalar	Vector	Flavor
π^+	ρ^+	$u\bar{d}$
π^-	ρ^-	$-d\bar{u}$
π^0	ρ^0	$\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$
η_1	ω_1	$\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$
η_8	ω_8	$\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$
K^+	K^{*+}	$u\bar{s}$
k^0	K^{*0}	$d\bar{s}$
k^-	K^{*-}	$-s\bar{u}$
$\bar{k}^0$	$\bar{K}^{*0}$	$s\bar{d}$

2.1.2 Spatial Wave Function in 1S, 2S, and 1D State

The spatial wave function of mesons can be derived from the radial Schrödinger equation in spherical polar coordinates

$$\left[\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left(E - V(r)\right) - \frac{l(l+1)}{r^2}\right] u(r) = 0. \tag{2.7}$$

The potential $V(r)$ should be an interaction which can provide confinement of quarks. In common practice, the three dimensional harmonic oscillator potential is usually employed, taking the form

$$V(r) = \frac{1}{2}\mu\omega^2 r^2. \tag{2.8}$$

The solution or the normalized wave function* is

$$R_{nl}(r) = \left[\frac{2a^3 n!}{\Gamma(n + l + \frac{3}{2})}\right] (ar)^l e^{-\frac{1}{2}a^2r^2} L_n^{l+1/2}(a^2r^2), \tag{2.9}$$

*see Appendix B.

where $L_n^{l+1/2}(\alpha^2 r^2)$ are the associated Laguerre polynomials

$$L(1,\frac{1}{2},a^2p^2) \equiv L_n^{l+1/2}(a^2r^2) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{\Gamma(n+l+\frac{3}{2})}{(n-k)!\Gamma(k+l+\frac{3}{2})} r^{2k}, \tag{2.10}$$

n, l and a are a principle quantum number, an orbital quantum number, and size parameter, in respective. n and l run from $n, l = 0, 1, 2\ldots$ Therefore, the wave function of mesons in momentum space is

$$\psi_{sp}(\vec{p}) = e^{-\frac{1}{2}a^2p^2}(a\vec{p})^l \sqrt{\frac{2a^3 n!}{\Gamma(\frac{3}{2}+l+n)}}\; L(n,\frac{1}{2}+l,a^2p^2) Y_{lm}(\theta,\phi) \tag{2.11}$$

where $\vec{p}$ is the relative momentum $\vec{p} = \frac{1}{2}(\vec{p}_i - \vec{p}_j)$ in which i and j are a quark and an antiquark. The wave function for ground-state or $1S$-state mesons ($n = l = 0$) is

$$\begin{aligned}\psi_{sp} &= e^{-\frac{1}{2}a^2p^2} \sqrt{\frac{2a^3}{\Gamma(\frac{3}{2})}}\; L(0,\frac{1}{2},a^2p^2) Y_{00}(\theta,\phi) \\ &= \frac{\sqrt{a^3}}{\pi^{3/4}} e^{-\frac{1}{2}a^2p^2}. \end{aligned} \tag{2.12}$$

For resonance mesons, the spatial wave functions are considered as excited states, $2S$ and $1D$, for instance. The wave function for $2S$-state mesons ($n = 1, l = 0$) is

$$\begin{aligned}\psi_{sp} &= e^{-\frac{1}{2}a^2p^2} \sqrt{\frac{2a^3}{\Gamma(\frac{5}{2})}}\; L(1,\frac{1}{2},a^2p^2) Y_{00}(\theta,\phi) \\ &= \frac{1}{\pi^{3/4}} \sqrt{\frac{2a^3}{3}} \left(\frac{3}{2} - a^2p^2\right) e^{-\frac{1}{2}a^2p^2}. \end{aligned} \tag{2.13}$$

For $1D$-state mesons ($n = 0, l = 2$), the spatial wave function is

$$\begin{aligned}\psi_{sp} &= e^{-\frac{1}{2}a^2p^2}(ap)^2 \sqrt{\frac{2a^3}{\Gamma(\frac{7}{2})}}\; L(0,\frac{1}{2}+2,a^2p^2) Y_{2m}(\theta,\phi) \\ &= \frac{4}{\pi^{1/4}} \sqrt{\frac{a^3}{15}}\; (ap)^2 e^{-\frac{1}{2}a^2p^2} Y_{2m}(\theta,\phi). \end{aligned} \tag{2.14}$$

The root-mean-square radii for mesons are defined in terms of the size parameters, as follow: For a $1S$-wave meson

$$\langle r^2 \rangle_{1s}^{1/2} = \frac{1}{2}\sqrt{\langle \Phi_{1s}|r^2|\Phi_{1s}\rangle}$$

$$
\begin{aligned}
&= \frac{1}{2}\sqrt{\frac{3}{2}}\; a \;\simeq\; 2.5 \text{ GeV}^{-1} \\
&\simeq\; 0.5 \text{ fm.}
\end{aligned} \tag{2.15}
$$

For a $2S$-wave meson

$$
\begin{aligned}
\langle r^2 \rangle_{2S}^{1/2} &= \frac{1}{2}\sqrt{\langle \Phi_{2s} | r^2 | \Phi_{2s} \rangle} \\
&= \frac{1}{2}\sqrt{\frac{3}{2}}\; a \;\simeq\; 3.83 \text{ GeV}^{-1} \\
&\simeq\; 0.76 \text{ fm.}
\end{aligned} \tag{2.16}
$$

For a $1D$-wave meson

$$
\begin{aligned}
\langle r^2 \rangle_{1D}^{1/2} &= \frac{1}{2}\sqrt{\langle \Phi_{2s} | r^2 | \Phi_{2s} \rangle} \\
&= \frac{1}{2}\sqrt{\frac{3}{2}}\; a \;\simeq\; 7.62 \text{ GeV}^{-1} \\
&\simeq\; 1.5 \text{ fm.}
\end{aligned} \tag{2.17}
$$

where $a = 4.1$ GeV^{-1} and practically fitted to the meson sizes.

2.2 Baryon Wave Function

Since baryons are the bound states of three quarks or three identical fermions in which each quark is $\frac{1}{2}$-spin particle, the total wave function of baryons must be antisymmetric. In the nature, the color wave function of all known and observed particles is singlet, that is, wave function is automatically antisymmetric. Furthermore, Particles are generally considered to occupy a ground-state or S-state giving the spatial wave function to be symmetric. Therefore, the spin-flavor coupling wave function must be symmetric.

The detail of each wave function are discussed in the following sections, starting from color wave function, then spin-flavor wave function, and finally spatial wave function.

2.2.1 Color and Spin-Flavor Wave Function

The color singlet wave function is

$$\psi_c = \frac{1}{\sqrt{6}} \sum_{ijk} \epsilon_{ijk} |q_i\rangle |q_j\rangle |q_k\rangle \tag{2.18}$$

where ϵ_{ijk} is Levi-Civita symbol giving $+1$ for even permutation, -1 for odd permutation and 0 for $i = j, j = k$, or $k = i$. Here is some of its properties

$$\left.\begin{aligned} \epsilon_{ijk}\epsilon_{i'jk} &= 2\delta_{ii'} \\ \epsilon_{ijk}\epsilon_{ijk} &= 6 \\ \epsilon_{ijk}\epsilon_{pqk} &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \end{aligned}\right\} \tag{2.19}$$

where δ_{ij} is Kronecker Delta. The flavor wave function of baryon is constructed by first introducing Young tableaux to be the fundamental representation of $SU(3)$ where three refers to $s, u,$ and d quarks. The product of three quarks system formed by the direct product of three fundamental representations is displayed by direct sum of irreducible representations,

$$\boxed{a} \otimes \boxed{b} \otimes \boxed{c} = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline a \\ \hline b \\ \hline c \\ \hline\end{array} \tag{2.20}$$

with corresponding dimension,

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1. \tag{2.21}$$

The wave functions corresponding to the irreducible representation from the left-hand side are symmetric, mixed symmetric (λ-type, symmetric for the first two particles), mixed antisymmetric (ρ-type, antisymmetric for the first two particles), and antisymmetric, respectively.

The product of fundamental representations of spin states (spin up $\uparrow$ and spin down $\downarrow$) transformed under $SU(2)$ is similarly shown in the direct sum of irreducible representations

$$\boxed{a} \otimes \boxed{b} \otimes \boxed{c} = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1}\end{array} \tag{2.22}$$

with corresponding dimension

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2. \tag{2.23}$$

The antisymmetric representation is vanished because we are dealing with $SU(2)$ group, not $SU(3)$. We now come to the combination between $SU(3)_{flavor}$ and $SU(2)_{spin}$ to be $SU(6)$ multiplet. The table below shows the easily understandable concept of the combination read by

	Symmetric ↓	Mixed Symmetric ↓	Antisymmetric ↓
Symmetric →	S	M	A
Mixed Symmetric →	M	S, M, A	M

Denote symmetric, mixed symmetric, and antisymmetric representations by S, M, and A. The explanation is that the product of symmetric representation with symmetric, mixed symmetric, and antisymmetric representation yields symmetric, mixed symmetric, and antisymmetric representation, respectively. For the mixed symmetric representation, the products with symmetric and antisymmetric are still mixed symmetric but with mixed symmetric the possible products can be all symmetric depending on types of mixed symmetric representation. In case of dimensional picture, we have $10_S, 8_M$ and 1_A in $SU(3)$ and 4_S and 2_M in $SU(2)$. The results can read; flavor 10_S with $\frac{3}{2}$-spin 4_S is totally symmetric, flavor 10_S with $\frac{1}{2}$-spin 2_M is absolutely mixed symmetric, and so on.

Of the total $3^3 \times 2^3 = 216$ states, 56 symmetric, 70 λ-type, 70 ρ-type, and

20 antisymmetric are listed as follow

$$\left.\begin{array}{rr} S: & (10,4)+(8,2)=56 \\ M: & (10,2)+(8,4)+(8,2)+(1,2)=70 \\ A: & (8,2)+(1,4)=20. \end{array}\right\} \tag{2.24}$$

The spin-flavor wave functions of various permutation symmetries, where $\phi^S(\chi^S), \phi^A(\chi^A), \phi^\lambda(\chi^\lambda)$, and $\phi^\rho(\chi^\rho)$ are respectively the flavor(spin) symmetric, antisymmetric, λ-type, and ρ-type symmetric are listed in Table 2.3

Table 2.3 Spin-flavor wave functions of baryons classified by permutation symmetry

Represetation type and number of state	Spin and flavor wave function	
Symmetric, 56	(10,4): $\phi^S\chi^S$	(8,2): $(\phi^\rho\chi^\rho + \phi^\lambda\chi^\lambda)/\sqrt{2}$
Antisymmetric, 20	(1,4): $\phi^A\chi^S$	(8,2): $(\phi^\lambda\chi^\rho - \phi^\rho\chi^\lambda)/\sqrt{2}$
λ-type, 70	(10,2): $\phi^S\chi^\lambda$	(8,4): $\phi^\lambda\chi^S$
	(8,2): $(\phi^\rho\chi^\rho - \phi^\lambda\chi^\lambda)/\sqrt{2}$	(1,2): $\phi^A\chi^\lambda$
ρ-type, 70	(10,2): $\phi^S\chi^\rho$	(8,4): $\phi^\rho\chi^S$
	(8,2): $(\phi^\lambda\chi^\rho + \phi^\rho\chi^\lambda)/\sqrt{2}$	(1,2): $\phi^A\chi^\rho$

The explicit form of the baryon spin-flavor wave function can be derived in the framework of Yamanouchi basis developed in permutation group. What we need is to get the projection operators for Young tableaux of multiplet states, act operators onto general states, then automatically obtain the states with symmetry under permutation group represented by Young tableau. Since we have worked out the representation matrices of permutation group S_3, the projection operators are directly written as

$$\left.\begin{aligned} P^S &= 1 + (12) + (13) + (23) + (123) + (132) \\ P^\lambda &= 1 + (12) - \frac{1}{2}(13) - \frac{1}{2}(23) - \frac{1}{2}(123) - \frac{1}{2}(132) \\ P^\rho &= 1 - (12) + \frac{1}{2}(13) + \frac{1}{2}(23) - \frac{1}{2}(123) - \frac{1}{2}(132) \\ P^A &= 1 - (12) - (13) - (23) + (123) + (132) \end{aligned}\right\} \tag{2.25}$$

where P^S, P^λ, P^ρ, and P^A are the projection operators for symmetric, λ-type symmetric, ρ-type symmetric, and antisymmetric state, respectively. Here is the example of applications. Acting the projection of operators P^λ and P^ρ onto the state udu (with $u \equiv \phi_u$ and $d \equiv \phi_d$), we have

$$\begin{aligned} P^\lambda udu &= udu + duu - \frac{1}{2}udu - \frac{1}{2}uud - \frac{1}{2}duu - \frac{1}{2}uud \\ &= \frac{1}{2}udu + \frac{1}{2}duu - uud \end{aligned} \tag{2.26}$$

$$\begin{aligned} P^\rho udu &= udu - duu + \frac{1}{2}udu + \frac{1}{2}uud - \frac{1}{2}duu - \frac{1}{2}uud \\ &= \frac{3}{2}udu - \frac{3}{2}duu. \end{aligned} \tag{2.27}$$

The normalized forms of $P^\lambda udu$ and $P^\rho udu$ are respectively the flavor wave function for the proton, seen in any text book.

In case of the spin wave function, the spin wave functions with $[3]_S, [21]_\lambda$ and $[21]_\rho$ symmetries can be derived by acting the projection operators in Eq.(2.25) on an arbitrary spin state of quarks, for instance, $\uparrow\downarrow\uparrow$ for spin $s_z = \frac{1}{2}$. We gain

$$\left.\begin{aligned} \chi^S &= \frac{1}{\sqrt{3}}[\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow] \\ \chi^\lambda &= \frac{1}{\sqrt{6}}[2\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow] \\ \chi^S &= \frac{1}{\sqrt{2}}[\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow]. \end{aligned}\right\} \tag{2.28}$$

To determine the normalization coefficients in the spin wave functions, we treat the spin up (down) state similar to the u(d)-quark in the construction of flavor wave functions.

2.2.2 Spatial Wave Function

The baryon spatial wave function worked out by solving the Schrödinger equation with harmonic oscillator interaction

$$V(r) = \frac{1}{2}Kr^2 \tag{2.29}$$

takes in momentum space the form,

$$\psi_{sp} = N_N \; e^{-\frac{1}{2}a^2\left(\frac{\vec{p}_1-\vec{p}_2}{\sqrt{2}}\right)^2} e^{-\frac{1}{2}a^2\left(\frac{\vec{p}_1+\vec{p}_2-2\vec{p}_3}{\sqrt{6}}\right)^2} \tag{2.30}$$

where $N_N = \frac{3^{3/4}a^3}{\pi^{3/2}}$, with $a^2 = \frac{1}{(3Km)^{1/2}}$ where m is the mass of quark.

CHAPTER III

MESON SIZE PARAMETER AND STRENGTH OF $^3P_0$ QUARK MODEL

3.1 $^3P_0$ Quark Model

In this work we study the reactions in a nonperturbative quark model with the $^3P_0$ quark dynamics which describes the quark-antiquark annihilation and creation. The $^3P_0$ model, first introduced by Micu (Micu, 1969), has made considerable successes in understanding low-energies hadron physics (Le Yaouanc et al., 1973, 1974, 1975; Maruyama et al., 1987; Maruyama, Furui et al., 1987; Gutsche et al., 1989; Dover et al., 1992; Muhn et al., 1996; Yan et al., 1997). The $^3P_0$ decay model defines the quantum states of a pair of quark and antiquark destroyed or created from vacuum quantum numbers

$$I^G(J^{PC}) = 0^+(0^{++}) \tag{3.1}$$

to be $J = 0, L = 1, S = 1$ and $T = 0$. The derivation of these quantum numbers is that because of the parity of vacuum, $P = +1$, the quark-antiquark pair (with intrinsic negative parity) must be in an odd relative orbital angular momentum. To obtain the zero total angular momentum, the pair has to be coupled to spin $S = 1$ together with orbital angular momentum $L = 1$ which finally couples to $J = 0$, hence $^3P_0$.

In analogous to a scalar interaction, the vertex in a relativistic approach for Dirac spinors of the annihilating quark and antiquark or fermion-antifermion vacuum interaction can be written in momentum space as

$$W_{ij} = \frac{1}{2}\chi_i^\dagger V_{ij}\chi_j \tag{3.2}$$

with

$$\begin{aligned} V_{ij} &= (\vec{\sigma}_i \cdot \vec{p}_i - \vec{\sigma}_j \cdot \vec{p}_j)\delta(\vec{p}_i + \vec{p}_j) \\ &= \vec{\sigma}_{ij} \cdot (\vec{p}_i - \vec{p}_j)\delta(\vec{p}_i + \vec{p}_j) \end{aligned} \tag{3.3}$$

where $\vec{\sigma}_{ij}$ defined as

$$\vec{\sigma}_{ij} = \frac{\vec{\sigma}_i + \vec{\sigma}_j}{2} \tag{3.4}$$

and having the property

$$\left[\chi_i^\dagger \sigma_{ij}^\mu \chi_j\right]_{SM} = \sqrt{2}\delta_{S,1}\delta_{\mu,-M}(-1)^M \tag{3.5}$$

where

$$\left.\begin{aligned} \sigma_{ij}^1 &= -\frac{1}{\sqrt{2}}(\sigma_{ij}^x + i\sigma_{ij}^y) \\ \sigma_{ij}^0 &= \sigma_{ij}^z \\ \sigma_{ij}^{-1} &= \frac{1}{\sqrt{2}}(\sigma_{ij}^x - i\sigma_{ij}^y). \end{aligned}\right\} \tag{3.6}$$

The interpretation of each Pauli-spinors is as shown; for fermion,

$$\chi(\text{spin up}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi(\text{spin down}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.7}$$

for antifermion,

$$\bar{\chi}(\text{spin up}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \bar{\chi}(\text{spin down}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.8}$$

The operation of $\vec{\sigma}_{ij}$ can be understood as the operation of a quark to an antiquark or the projection of a quark-antiquark pair onto a spin-1 state and it is found that the latter is more convenient.

Introducing a one-rank tensor operator $\hat{O}_1$, we have, according to the Wigner Eckart theorem,

$$\begin{aligned}\langle 0,0|\hat{O}_{1\mu}|J,M\rangle &= \langle J1,M\mu|0,0\rangle\langle 0||\hat{O}_1||J\rangle \\ &= \frac{(-1)^{1-M}}{\sqrt{3}}\delta_{J,1}\delta_{M,-\mu}\langle 0|\hat{O}_1|J\rangle. \end{aligned} \tag{3.9}$$

Let

$$\langle 0|\hat{O}_1|J\rangle = -\sqrt{6}, \tag{3.10}$$

then

$$\langle 0,0|\hat{O}_{1\mu}|J,M\rangle = (-1)^M\sqrt{2}\delta_{J,1}\delta_{M,-\mu}. \tag{3.11}$$

For the flavor, we may introduce a unit operator

$$\hat{O}^F = 1^F \tag{3.12}$$

with the property

$$\langle 0,0|\hat{O}^F|T,T_z\rangle = \sqrt{2}\delta_{T,0}\delta_{T_z,0}. \tag{3.13}$$

Consequently, the $^3P_0$ operator is read in the form

$$\begin{aligned} V_{ij} = \frac{1}{\sqrt{3}}\sum_{\mu}(-1)^{1+\mu}\langle 0,0|_{ij}^{S}\langle 0,0|_{ij}^{F}\hat{O}^{S}_{-\mu,ij}O^{F}_{ij} \\ \cdot Y_{1\mu}(\vec{p}_i-\vec{p}_j)\delta(\vec{p}_i+\vec{p}_j). \end{aligned} \tag{3.14}$$

In case of the decay of a spin-1 state to the quark-antiquark pair, $Y^*_{1\mu}(\vec{p}_i-\vec{p}_j)$ is used in place of $Y_{1\mu}(\vec{p}_i-\vec{p}_j)$ by the relation

$$Y^*_{l,m} = (-1)^m Y_{l,-m}. \tag{3.15}$$

Here, we have used the formula

$$\vec{A}\cdot\vec{B} = \sum_{\mu}(-1)^{\mu}A_{1,-\mu}B_{1\mu}$$

$$= \sum_{\mu} (-1)^{\mu} Y_{1,-\mu}(\vec{A}) Y_{1\mu}(\vec{B}). \quad (3.16)$$

Mixed up with other ideas, ${}^3P_0$ quark model has recently successfully given creditable results to the two-step process calculation of e^+e^- annihilation to $\pi^+\pi^-$ and intermediate $\pi\pi$ scattering (Suebka, 2005; Yan et al., 2005).

The reaction $\rho' \rightarrow \pi\omega$ and $\omega' \rightarrow \pi\rho$ are similarly investigated, so the following calculation in the ${}^3P_0$ quark model displays only the former one. The transition amplitude then takes the form

$$T_v = \langle \pi\omega | V_{q\bar{q}} | \rho' \rangle \quad (3.17)$$

where $V_{q\bar{q}}$ is the effective vertex for the creation and destruction of a quark-antiquark pair in quark model, as shown in the form of spin part, flavor part, color part and a constant of effective strength parameter λ referred to how large of frequency of the reaction occurs. The effective vertex is alternatively in the form

$$V_{ij} = \lambda \vec{\sigma}_{ij} \cdot (\vec{p}_i - \vec{p}_j) \hat{F}_{ij} \hat{C}_{ij} \delta(\vec{p}_i + \vec{p}_j). \quad (3.18)$$

Sandwiched (3.18) by the state of $\pi\omega$ and ρ' and applied the ${}^3P_0$ model, the effective parameter turns to, as shown in three compositions,

$$\langle 0,0 | \hat{F}_{ij} | T, T_z \rangle = \sqrt{2} \delta_{T,0} \delta_{T_z,0} \quad (3.19)$$

$$\langle 0,0 | \hat{C}_{ij} | q^i_\alpha \bar{q}^j_\beta \rangle = \delta_{\alpha\beta} \quad (3.20)$$

$$\langle 0,0 | \sigma^\mu_{ij} | \left[\bar{\chi}_i \otimes \chi_j \right]_{JM} \rangle = (-1)^M \sqrt{2} \delta_{J,1} \delta_{M,-\mu} \quad (3.21)$$

where the first, the second, the last represent the flavor, the color and the spin state in vacuum, respectively. Since the isospin of the vacuum state is zero, (3.19) is given in this form. For the color state, α and δ are color indices and $\bar{\sigma}$ is the Pauli matrix. For simplicity, we consider only S-wave mesons that means mesons involved have the orbital angular momentum equal to zero or ground state.

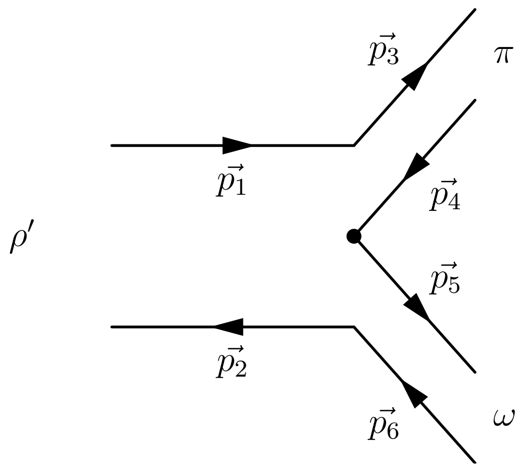


Figure 3.1 $\rho' \to \omega\pi$ in the $^3P_0$ non relativistic quark model.

To compute the transition amplitude, we have Ψ_f and Ψ_i to be the final and initial state wave functions, in respective. For ρ, the state of one meson is

$$|\Psi_i\rangle = \Psi_{spatial} \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right]_{S_i,M_i} \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right]_{T,T_z} \tag{3.22}$$

where a spatial wave function ($\Psi_{spatial}$) is different in each radial and orbital excitation. From the particle data book, we know that, for ρ meson, spin $S_i = 1$, isospin $T_i = 1$, and isospin projection $T_z = 0$. The final state $|\Psi_f\rangle$ formed by coupling of two final S-wave mesons is, as shown,

$$\begin{aligned} |\Psi_f\rangle = N_1 N_2 e^{-\frac{1}{8}a^2(\vec{p}_3-\vec{p}_4)^2} e^{-\frac{1}{8}a^2(\vec{p}_5-\vec{p}_6)^2} & \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)} \right]_{S_1} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right]_{S_2} \right]_{S_f,M_f} \\ \times & \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)} \right]_{T_1} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right]_{T_2} \right]_{T,T_z} \end{aligned} \tag{3.23}$$

where a is the size parameter related to the size of meson.

3.2 Size Parameter Determination

The size parameter a in the meson wave function in Eq. (2.11) can be determined by studying the process $\rho^0 \to e^+e^-$. The transition amplitude of the reaction is derived as

$$
\begin{aligned}
T_{\rho\to e^+e^-} &= \langle e^+e^-|T|q\bar{q}\rangle\langle q\bar{q}|\rho\rangle \\
&= \sum_{\alpha}\sum_{m_q m_{\bar{q}}}\sum_{t_q t_{\bar{q}}} \frac{1}{\sqrt{3}}\; C(\frac{1}{2}\frac{1}{2}S, m_q m_{\bar{q}} S_z) C(\frac{1}{2}\frac{1}{2}I, t_q t_{\bar{q}} I_z) \\
&\quad \int \frac{d\vec{p}_1}{(2\pi)^{3/2} 2E_q}\; \psi_\rho(p_1)\; T_{q\bar{q}\to e^+e^-} \qquad (3.24)
\end{aligned}
$$

where $T_{q\bar{q}\to e^+e^-} \equiv \langle e^+e^-|T|q\bar{q}\rangle$, the transition amplitude* of a quark-antiquark pair to an electron-positron pair, can be evaluated by the standard method in Quantum Field Theory taking the form

$$
\begin{aligned}
\langle e^+e^-|T|q\bar{q}\rangle &= -\frac{e_q e}{s}\bar{u}_e(p_{e^-}, m_{e^-})\gamma^\mu v_e(p_{e^+}, m_{e^+}) \\
&\quad \times \bar{v}_q(p_{\bar{q}}, m_{\bar{q}})\gamma_\mu u_q(p_q, m_q) \qquad (3.25)
\end{aligned}
$$

where $s = (p_q + p_{\bar{q}})^2$, e_q is the charge of quarks, and the Dirac spinors are normalized according to $\bar{u}u = \bar{v}v = 2m$. The decay width is generally in the form

$$
\Gamma = \frac{1}{(4\pi)^2}\frac{p_f}{E_f}\int d\Omega |T_{\rho\to e^+e^-}|^2. \qquad (3.26)
$$

In the small quark momentum approximation, the decay width for the transition of a vector meson to an electron-positron pair can be easily calculated,

$$
\Gamma_{\rho^0\to e^+e^-} = \frac{16\pi\alpha^2 Q^2}{M_V^2}|\psi(0)|^2 \qquad (3.27)
$$

where Q is the squared sum of the charges of the quarks in the meson, with $Q^2 = 1/2$ for ρ, $1/18$ for ω, and $1/9$ for ϕ; and $\psi(0) = 1/(\pi a^2)^{3/4}$ is the coordinate space wave function of the vector meson at the origin. Using as an input $M_\rho = 0.7758$ GeV, $\alpha = 1/137$, and the experimental value of $\Gamma_{\rho^0\to e^+e^-} = 7.02 \pm 0.11$ keV, we obtain $a = 3.847$ GeV^{-1} for the size parameter of ρ meson. The size parameter may slightly be different from meson to meson.

*See Appendix C

3.3 Effective Coupling Constant Determination

We use the reaction $\rho \to \pi^+\pi^-$ to determine the effective coupling constant λ in the quark-antiquark $^3P_0$ vertex,

$$\begin{aligned} V_{ij} &= \lambda \vec{\sigma}_{ij} \cdot (\vec{p}_i - \vec{p}_j)\hat{F}_{ij}\hat{C}_{ij}\delta(\vec{p}_i + \vec{p}_j) \\ &= \lambda \sum_{\mu} \sqrt{\frac{4\pi}{3}} (-1)^{\mu} \sigma^{\mu}_{ij} y_{1\mu}(\vec{p}_i - \vec{p}_j)\hat{F}_{ij}\hat{C}_{ij}\delta(\vec{p}_i + \vec{p}_j) \end{aligned} \tag{3.28}$$

where $y_{1\mu}(\vec{q}) = |\vec{q}|\ Y_{1\mu}(\hat{q}),\ \vec{\sigma}_{ij} = (\vec{\sigma}_i + \vec{\sigma}_j)/2$. $\vec{p}_i$ and $\vec{p}_j$ are the momenta of quark and antiquark created form vacuum. $\hat{F}_{ij}$ and $\hat{C}_{ij}$ are the flavor and color operators projecting a quark-antiquark pair to the respective vacuum quantum number. The decay width of the reaction is

$$\Gamma_{\rho^0 \to \pi^+\pi^-} = \frac{\pi}{4} M_\rho^2 \sqrt{1 - \frac{4M_\pi^2}{M_\rho^2}} |T_{\rho^0 \to \pi^+\pi^-}|^2 \tag{3.29}$$

where $T_{\rho^0 \to \pi^+\pi^-}$ is the transition amplitude in the center of mass system. We can consequently determine the effective coupling constant by substituting the previously calculated transition amplitude to the decay width and comparing then to the experimental data. The transition amplitude in Eq. (3.29) is demonstrated as followed,

$$\begin{aligned} T_{\rho^0 \to \pi^+\pi^-} &= \langle \pi^+\pi^- | V_{ij} | \rho^0 \rangle \\ &= \lambda \sqrt{\frac{4\pi}{3}} \sum_{\mu} T^{\mathrm{Sp}}_{\mu} T^{\mathrm{S}}_{\mu} T_{\mathrm{F}} T_{\mathrm{C}} \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} T^{\mathrm{Sp}}_{\mu} &= \int d\vec{Q} \left(\frac{2a^3}{\Gamma(\frac{3}{2})4\pi} \right)^{3/2} e^{-\frac{3}{2}a^2Q^2 - \frac{1}{12}a^2k^2} \left(\frac{4k}{3} Y^*_{1\mu}(\hat{k}) - 2QY^*_{1\mu}(\hat{Q})\right) \\ &= \left(\frac{2a^3}{\Gamma(\frac{3}{2})4\pi} \right)^{3/2} 4\pi e^{-\frac{1}{12}a^2k^2} \int dQ\ Q^2 e^{-\frac{3}{2}a^2Q^2} \frac{4k}{3} Y^*_{1\mu}(\hat{k}) \\ &= \frac{8\sqrt{2}\ a^{3/2} e^{-\frac{1}{12}a^2k^2} k}{9\sqrt{3}\ \pi^{3/4}} \end{aligned} \tag{3.31}$$

$$
\begin{aligned}
T_\mu^{\mathrm{S}} &= -\sqrt{2}\langle(\frac{1}{2}\frac{1}{2})1,(\frac{1}{2}\frac{1}{2})1,0|(\frac{1}{2}\frac{1}{2})0,(\frac{1}{2}\frac{1}{2})0,\rangle\ C\left(1S_z^{12};1,-\mu;0\right) \\
&= -\sqrt{\frac{3}{2}}\ C(1S_z;1,-\mu;0) \qquad (3.32) \\
T_{\mathrm{C}} &= \frac{1}{\sqrt{3}}\langle q_\alpha^{(3)}|\langle\bar{q}_\alpha^{(4)}|\frac{1}{\sqrt{3}}\langle q_\gamma^{(5)}|\langle\bar{q}_\gamma^{(6)}|\hat{C}_{ij}\frac{1}{\sqrt{3}}|q_\beta\rangle|\bar{q}_\beta\rangle \\
&= \frac{1}{\sqrt{3}} \qquad (3.33) \\
T_{\mathrm{F}} &= \sqrt{2}\langle(\frac{1}{2}\frac{1}{2})1,(\frac{1}{2}\frac{1}{2})0,1|(\frac{1}{2}\frac{1}{2})1,(\frac{1}{2}\frac{1}{2})1,1\rangle C\left(1T_z^{12},00;1T_z\right) \\
&= 1. \qquad (3.34)
\end{aligned}
$$

The general detailed calculation is shown in the next chapter. Finally, the transition amplitude of $\rho^0 \to \pi^+\pi^-$ is

$$
\begin{aligned}
T_{\rho^0\to\pi^+\pi^-} &= -\frac{16\lambda\ a^{3/2}e^{-\frac{1}{12}a^2k^2}k}{27\ \pi^{1/4}}\sum_\mu C(1S_z,;1,-\mu;0) \\
&= -\frac{16\lambda\ a^{3/2}e^{-\frac{1}{12}a^2k^2}k}{27\sqrt{3}\ \pi^{1/4}} \qquad (3.35)
\end{aligned}
$$

where $\lambda = 0.92$ in the non-relativistic approximation and $\lambda = 2.5$ in the minimum relativity approximation (Machleidt, Holinde, and Elster, 1987) by which the transition amplitude takes the form

$$
T_{\text{minimum}} = \sqrt{\frac{m_{\pi^+}}{E_{\pi^+}}}\sqrt{\frac{m_{\pi^-}}{E_{\pi^-}}}\ T_{\rho^0\to\pi^+\pi^-}. \qquad (3.36)
$$

CHAPTER IV

STUDY OF REACTIONS $e^+e^- \to \rho\pi, \omega\pi$

The reactions $e^+e^- \to \rho\pi, \omega\pi$ may stem from two possible processes, namely, the one-step process where the e^+e^- pair annihilates into a virtual time-like photon, the virtual photon decays into a $\overline{q}q$ pair, and finally the $\overline{q}q$ pair is dressed directly by an additional quark-antiquark pair pumped out of the vacuum to form the $\pi\omega$ final state, and the two-step process where the e^+e^- pair annihilates into a virtual time-like photon, the virtual photon decays into a $\overline{q}q$ pair, the $\overline{q}q$ pair first form a vector meson, and finally the vector meson decay into the $\pi\omega$ final state.

At very high energies the reactions $e^+e^- \to \rho\pi, \omega\pi$ are likely dominated by the one-step process while in the low-energy region, especially close to the threshold, the reactions are expected to be dominated by the two-step process. It is found that the reactions $e^+e^- \to \pi\pi, \overline{N}N$ at low energies are dominated by the two-step process (Suebka, 2005; Yan et al., 2005).

For one-step process shown in Fig. 4.1, the transition amplitude of the reactions $e^+e^- \to \rho\pi, \omega\pi$ might be expressed formally as

$$T_1 = \langle \pi\omega | V_{\overline{q}q} | \overline{q}q \rangle \langle \overline{q}q | T | e^+e^- \rangle \tag{4.1}$$

where $\langle \overline{q}q | T | e^+e^- \rangle$ is simply the transition amplitude of e^+e^- to a primary quark pair while $\langle \pi\omega | V_{\overline{q}q} | \overline{q}q \rangle$ denotes the amplitude of the process of a $\overline{q}q$ pair to the $\pi\omega$ final state. $V_{\overline{q}q}$ is the effective vertex for creation and destruction of a quark-antiquark pair in quark models. In this work we will employ the $^3P_0$ quark-antiquark dynamics which has been proven the most successful non-relativistic quark model.

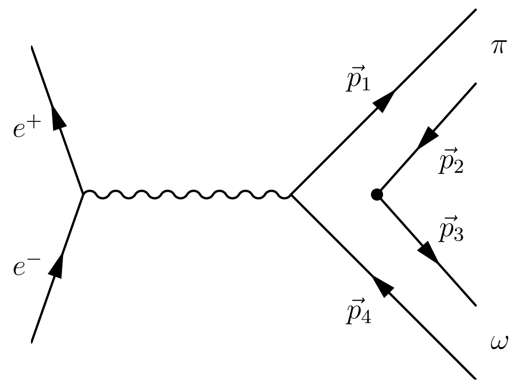


Figure 4.1 Reaction $e^+e^- \to \omega(\rho)\pi$ in one-step process.

For two-step process shown in Fig. 4.2, the transition amplitude of the reactions $e^+e^- \to \rho\pi, \omega\pi$ for the two step process takes the form

$$T_2 = \langle \pi\omega | V_{\overline{q}q} | \rho' \rangle \langle \rho' | G | \rho' \rangle \langle \rho' | \overline{q}q \rangle \langle \overline{q}q | T | e^+e^- \rangle \tag{4.2}$$

where $\langle \rho' | \overline{q}q \rangle$ is simply the wave function of the intermediate meson ρ', $\langle \rho' | G | \rho' \rangle$ the Green function describing the propagation of the intermediate meson, and $\langle \pi\omega | V_{\overline{q}q} | \rho' \rangle$ the transition amplitude of ρ' annihilation into the $\pi\omega$ pair in non-relativistic quark models.

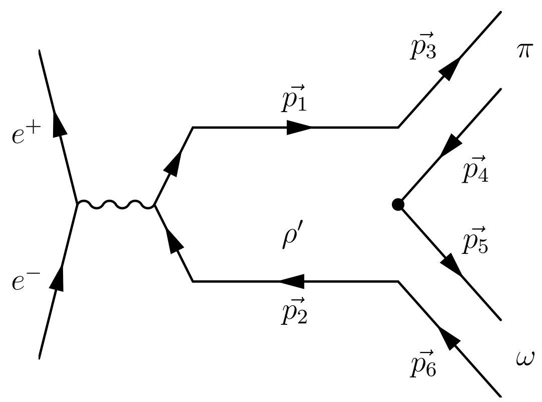


Figure 4.2 Reaction $e^+e^- \to \omega(\rho)\pi$ in two-step process.

The difficult part in working out the transition amplitude Eq. (4.2) is to calculate the decay of an intermediate particle by means of $^3P_0$ model. Here, we demonstrate the general detailed calculation of an intermediate meson decaying into two mesons. The wave function of a meson and two mesons are, respectively,

$$
\begin{aligned}
|m\rangle &= \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right]_{TT_z} \left[\left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right]_{S'^{12}S_z'^{12}} \otimes Y_{lm}\right]_{S^{12}S_z^{12}} R_{nl}^{12} \\
&= C(S'^{12}S_z'^{12}, l^{12}m^{12}; S^{12}S_z^{12}) \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right]_{T^{12}T_z^{12}} \\
&\quad \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right]_{S'^{12}S_z'^{12}} R_{nl}^{12} Y_{lm}^{12}
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
|m_1m_2\rangle &= \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{T'T_z'} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{T''T_z''}\right]_{TT_z} \\
&\quad \left[\left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{S'S_z'} \otimes Y_{l'm'}\right]_{J'} \otimes \left[\left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{S''S_z''} \otimes Y_{l''m''}\right]_{J''}\right]_{JJ_z} \\
&\quad R_{n'l'}R_{n''l''} \\
&= \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{T'T_z'} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{T''T_z''}\right]_{TT_z} \\
&\quad \sum_{sl} \langle (S'S'')_S, (l'l'')_l, J | (S'l')_{J'}, (S''l'')_{J''}, J \rangle \\
&\quad \left[\left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{S'S_z'} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{S''S_z''}\right]_{SS_z} \otimes [l'm' \otimes l''m'']_{lm}\right]_{JJ_z} \\
&\quad R_{n'l'}R_{n''l''} \\
&= \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{T'T_z'} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{T''T_z''}\right]_{TT_z} \\
&\quad \langle (S'S'')_S, (l'l'')_l, J | (S'l')_{J'}, (S''l'')_{J''}, J \rangle \\
&\quad C(SS_z, lm; JJ_z) \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{S'S_z'} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{S''S_z''}\right]_{SS_z} \\
&\quad [l'm' \otimes l''m'']_{lm} R_{n'l'}R_{n''l''} \\
&= \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{T'T_z'} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{T''T_z''}\right]_{TT_z}
\end{aligned}
$$

$$
\begin{aligned}
&\langle (S'S'')_S, (l'l'')_l, J | (S'l')_{J'}, (S''l'')_{J''}, J\rangle \\
&C(SS_z, lm; JJ_z)\left[\left[\frac{1}{2}^{(3)}\otimes\frac{1}{2}^{(4)}\right]_{S'S'_z}\otimes\left[\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right]_{S''S''_z}\right]_{SS_z} \\
&C(l'm', l''m''; lm)R_{n'l'}Y_{l'm'}R_{n''l''}Y_{l''m''}.
\end{aligned}
\tag{4.4}
$$

The transition amplitude of the decay of any intermediate meson is

$$
T = \langle m_1 m_2 | O | m\rangle \tag{4.5}
$$

where $|m_1m_2\rangle, |m\rangle$ and O are a final state wave function, an initial state wave function and an interaction operator, respectively. We use the interaction operator from $^3P_0$ model,

$$
\begin{aligned}
V_{ij} &= \lambda\vec{\sigma}_{ij}\cdot(\vec{p}_i - \vec{p}_j)\hat{F}_{ij}\hat{C}_{ij}\delta(\vec{p}_i + \vec{p}_j) \\
&= \lambda\sum_{\mu}\sqrt{\frac{4\pi}{3}}(-1)^{1+\mu}\sigma^{\mu}_{ij}Y_{1\mu}(\vec{p}_i - \vec{p}_j)\hat{F}_{ij}\hat{C}_{ij}\delta(\vec{p}_i + \vec{p}_j).
\end{aligned}
\tag{4.6}
$$

Then, the transition amplitude is

$$
\begin{aligned}
T &= \langle m_1 m_2 | O | m\rangle \\
&= \sum_{\mu} A\ T^{\mathrm{Sp}}_{\mu}T^{\mathrm{S}}_{\mu}T_C T_F
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
A &= \lambda\sqrt{\frac{4\pi}{3}}\ C(S'^{12}S'^{12}_z, l^{12}m^{12}; S^{12}S^{12}_z)C(SS_z, lm; JJ_z)C(l'm', l''m''; lm) \\
&\quad \langle (S'S'')_S, (l'l'')_l, J | (S'l')_{J'}, (S''l'')_{J''}, J\rangle \\
T^{Sp}_{\mu} &= \int dQ\mathcal{L}_1\mathcal{L}_2\mathcal{L}\ e^{-\frac{3}{2}a^2Q^2 - \frac{1}{12}a^2k^2}(\frac{4k}{3}Y^*_{1\mu}(\hat{k}) - 2QY^*_{1\mu}(\hat{Q})) \\
&\quad a^{l'_1+l'_2+l}Y^*_{l'_1m'_1}(2\vec{Q} - \frac{\vec{k}}{3})Y^*_{l'_2m'_2}(2\vec{Q} - \frac{\vec{k}}{3})Y_{l^{12}m^{12}}(2\vec{Q} + \frac{2\vec{k}}{3}) \\
T^S_{\mu} &= -\sqrt{2}\langle (\frac{1}{2}\frac{1}{2})S^{12}, (\frac{1}{2}\frac{1}{2})1, S | (\frac{1}{2}\frac{1}{2})S', (\frac{1}{2}\frac{1}{2})S'', S\rangle C\left(S^{12}S^{12}_z; 1, -\mu; SS_z\right) \\
T_C &= \frac{1}{\sqrt{3}}
\end{aligned}
$$

$$T_F = \sqrt{2}\langle(\frac{1}{2}\frac{1}{2})T^{12}, (\frac{1}{2}\frac{1}{2})0, T|(\frac{1}{2}\frac{1}{2})T', (\frac{1}{2}\frac{1}{2})T'', T\rangle C\left(T^{12}T_z^{12}, 00; TT_z\right) \tag{4.8}$$

and $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}$ which are functions of $n_1', l_1', n_2', l_2', n$, and l take the form,

$$\mathcal{L}(n,l,m) = \sqrt{\frac{2a^3 n!}{\Gamma(\frac{3}{2}+l+n)}} L(n, \frac{1}{2}+l, a^2(2\vec{Q}+\frac{2\vec{k}}{3})^2). \tag{4.9}$$

Next, the detailed calculation of color, flavor, spin and spatial space is displayed.

4.1 Color, Flavor, Spin, and Spatial Transition Amplitude

The transition amplitude of color part is

$$\begin{aligned} T_C &= \langle m_1 m_2 | O_C | m\rangle \\ &= \frac{1}{\sqrt{3}}\langle q_\alpha^{(3)}|\langle \bar{q}_\alpha^{(4)}|\frac{1}{\sqrt{3}}\langle q_\gamma^{(5)}|\langle \bar{q}_\gamma^{(6)}|\hat{C}_{ij}\frac{1}{\sqrt{3}}|q_\beta\rangle|\bar{q}_\beta\rangle \\ &= \frac{1}{3\sqrt{3}}\delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{\gamma\beta} \\ &= \frac{1}{3\sqrt{3}}\delta_{\alpha\alpha} \\ &= \frac{1}{\sqrt{3}} \end{aligned} \tag{4.10}$$

where

$$\langle 0,0|\hat{C}_{ij}|q_\alpha^i \bar{q}_\beta^j\rangle = \delta_{\alpha\beta} \tag{4.11}$$

and α and β are color indices. The transition amplitude of spin part is

$$\begin{aligned} T_\mu^S &= \left[\left\langle \frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right|_{S'S_z'} \otimes \left\langle \frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right|_{S''S_z''}\right]_{SS_z} O_S \left|\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right\rangle_{S^{12}S_z^{12}} \\ &= \langle(\frac{1}{2}\frac{1}{2})S^{36}, (\frac{1}{2}\frac{1}{2})S^{45}, S|(\frac{1}{2}\frac{1}{2})S', (\frac{1}{2}\frac{1}{2})S'', S\rangle \\ &\quad \left[\left\langle \frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(6)}\right|_{S^{36}S_z^{36}} \otimes \left\langle \frac{1}{2}^{(4)} \otimes \frac{1}{2}^{(5)}\right|_{S^{45}S_z^{45}}\right]_{SS_z} (-1)^{1+\mu}\sigma_{ij}^\mu \left|\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right\rangle_{S^{12}S_z^{12}} \\ &= \langle(\frac{1}{2}\frac{1}{2})S^{36}, (\frac{1}{2}\frac{1}{2})S^{45}, S|(\frac{1}{2}\frac{1}{2})S', (\frac{1}{2}\frac{1}{2})S'', S\rangle C\left(S^{36}S_z^{36}, S^{45}S_z^{45}; SS_z\right) \\ &\quad \left[\left\langle \frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(6)}\right|_{S^{36}S_z^{36}} \left\langle \frac{1}{2}^{(4)} \otimes \frac{1}{2}^{(5)}\right|_{S^{45}S_z^{45}}\right]_{SS_z} (-1)^{1+\mu}\sigma_{ij}^\mu \left|\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right\rangle_{S^{12}S_z^{12}} \end{aligned}$$

$$= \langle (\frac{1}{2}\frac{1}{2})S^{36}, (\frac{1}{2}\frac{1}{2})S^{45}, S|(\frac{1}{2}\frac{1}{2})S', (\frac{1}{2}\frac{1}{2})S'', S\rangle C\left(S^{36}S_z^{36}, S^{45}S_z^{45}; SS_z\right)$$
$$\delta_{S^{36},S^{12}}\delta_{S_z^{36},S_z^{12}}(-1)^{1+\mu}(-1)^{S_z^{45}}\sqrt{2}\delta_{S^{45},1}\delta_{S_z^{45},-\mu}$$
$$= -\sqrt{2}\langle (\frac{1}{2}\frac{1}{2})S^{12}, (\frac{1}{2}\frac{1}{2})1, S|(\frac{1}{2}\frac{1}{2})S', (\frac{1}{2}\frac{1}{2})S'', S\rangle C\left(S^{12}S_z^{12}; 1, -\mu; SS_z\right) \quad (4.12)$$

where

$$\langle 0,0|\sigma_{ij}^{\mu}|[\bar{\chi}_i \otimes \chi_j]_{JM}\rangle = (-1)^M\sqrt{2}\delta_{J,1}\delta_{M,-\mu}. \quad (4.13)$$

The transition amplitude of the flavor part is

$$T_F = \left[\left\langle \frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right|_{T'T_z'} \otimes \left\langle \frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right|_{T''T_z''}\right]_{TT_z} O_F \left|\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right\rangle_{T^{12}T_z^{12}}$$
$$= \langle (\frac{1}{2}\frac{1}{2})T^{36}, (\frac{1}{2}\frac{1}{2})T^{45}, T|(\frac{1}{2}\frac{1}{2})T', (\frac{1}{2}\frac{1}{2})T'', T\rangle$$
$$\left[\left\langle \frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(6)}\right|_{T^{36}T_z^{36}} \otimes \left\langle \frac{1}{2}^{(4)} \otimes \frac{1}{2}^{(5)}\right|_{T^{45}T_z^{45}}\right]_{TT_z} \hat{F}_{ij} \left|\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right\rangle_{T^{12}T_z^{12}}$$
$$= \langle (\frac{1}{2}\frac{1}{2})T^{36}, (\frac{1}{2}\frac{1}{2})T^{45}, T|(\frac{1}{2}\frac{1}{2})T', (\frac{1}{2}\frac{1}{2})T'', T\rangle C\left(T^{36}T_z^{36}, T^{45}T_z^{45}; TT_z\right)$$
$$\left[\left\langle \frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(6)}\right|_{T^{36}T_z^{36}} \left\langle \frac{1}{2}^{(4)} \otimes \frac{1}{2}^{(5)}\right|_{T^{45}T_z^{45}}\right]_{TT_z} O_F \left|\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right\rangle_{T^{12}T_z^{12}}$$
$$= \langle (\frac{1}{2}\frac{1}{2})T^{36}, (\frac{1}{2}\frac{1}{2})T^{45}, T|(\frac{1}{2}\frac{1}{2})T', (\frac{1}{2}\frac{1}{2})T'', T\rangle C\left(T^{36}T_z^{36}, T^{45}T_z^{45}; TT_z\right)$$
$$\delta_{T^{36},T^{12}}\delta_{T_z^{36},T_z^{12}}\sqrt{2}\delta_{T^{45},0}\delta_{T_z^{45}}$$
$$= \sqrt{2}\langle (\frac{1}{2}\frac{1}{2})T^{12}, (\frac{1}{2}\frac{1}{2})0, T|(\frac{1}{2}\frac{1}{2})T', (\frac{1}{2}\frac{1}{2})T'', T\rangle C\left(T^{12}T_z^{12}, 00; TT_z\right) \quad (4.14)$$

where

$$\langle 0,0|\hat{F}_{ij}|T,T_z\rangle = \sqrt{2}\delta_{T,0}\delta_{T_z,0}. \quad (4.15)$$

The normalized spatial wave function is

$$\psi_{sp} = e^{-\frac{1}{2}a^2p^2}(ap)^l\sqrt{\frac{2a^3n!}{\Gamma(\frac{3}{2}+l+n)}}\,L(n,\frac{1}{2}+l,a^2p^2)Y_{lm}(\theta,\phi) \quad (4.16)$$

where $\vec{p}$ is the relative momentum $\vec{p} = \frac{\vec{p}_i - \vec{p}_j}{2}$. Let

$$\mathcal{L}(n,l,m) = \sqrt{\frac{2a^3n!}{\Gamma(\frac{3}{2}+l+n)}}\,L(n,\frac{1}{2}+l,a^2p^2) \quad (4.17)$$

where, $L(n, \frac{1}{2}+l, a^2p^2)$ is the generalized Laguerre polynomial. The special transition amplitude is

$$T_\mu^{Sp} = \int d\vec{p}_1 d\vec{p}_2 d\vec{p}_3 d\vec{p}_4 d\vec{p}_5 d\vec{p}_6 (\mathcal{L}_1^* e^{-\frac{1}{8}a^2(\vec{p}_3-\vec{p}_4)^2} a^{l_1} Y_{l_1m_1}^*(\vec{p}_3-\vec{p}_4))$$
$$(\mathcal{L}_2^* e^{-\frac{1}{8}a^2(\vec{p}_5-\vec{p}_6)^2} a^{l_2} Y_{l_2m_2}^*(\vec{p}_5-\vec{p}_6)) Y_{1\mu}^*(\vec{p}_4-\vec{p}_5)\delta(\vec{p}_4+\vec{p}_5)\delta(\vec{p}_1-\vec{p}_3)$$
$$(\mathcal{L}\ e^{-\frac{1}{8}a^2(\vec{p}_1-\vec{p}_2)^2} a^{l} Y_{lm}(\vec{p}_1-\vec{p}_2))\delta(\vec{p}_2-\vec{p}_6)\delta(\vec{p}_1+\vec{p}_2)\delta(\vec{p}_3+\vec{p}_4-\vec{k}) \quad (4.18)$$

where

$$\left.\begin{aligned} Y_{l_1m_1}^*(\vec{p}_3-\vec{p}_4) &\equiv (\vec{p}_3-\vec{p}_4)^{l_1} Y_{l_1m_1}^*(\hat{p}_{34}) \\ Y_{l_2m_2}^*(\vec{p}_5-\vec{p}_6) &\equiv (\vec{p}_5-\vec{p}_6)^{l_2} Y_{l_2m_2}^*(\hat{p}_{56}) \\ Y_{lm}(\vec{p}_1-\vec{p}_2) &\equiv (\vec{p}_1-\vec{p}_2)^{l} Y_{lm}(\hat{p}_{12}) \end{aligned}\right\} \quad (4.19)$$

and $\hat{p}_{ij} = \vec{p}_{ij}/|\vec{p}_{ij}|$. The transition amplitude then turns to

$$T_\mu^{Sp} = \int d\vec{p}_1 (\mathcal{L}_1 e^{-\frac{1}{8}a^2(2\vec{p}_1-\vec{k})^2} a^{l_1} Y_{l_1m_1}^*(2\vec{p}_1-\vec{k}))(\mathcal{L}_2 e^{-\frac{1}{8}a^2(2\vec{p}_1-\vec{k})^2} a^{l_2} Y_{l_2m_2}^*(2\vec{p}_1-\vec{k}))$$
$$Y_{1\mu}^*(2\vec{p}_4)(\mathcal{L}\ e^{-\frac{1}{8}a^2(2\vec{p}_1)^2} a^{l_1} Y_{lm}(2\vec{p}_1))$$
$$= \int d\vec{p}_1 \mathcal{L}_1\mathcal{L}_2\mathcal{L}\ e^{-\frac{3}{2}a^2[(\vec{p}_1-\frac{\vec{k}}{3})^2+\frac{k^2}{18}]} Y_{1\mu}^*(2\vec{k}-2\vec{p}_1) a^{l_1+l_2+l} Y_{l_1m_1}^*(2\vec{p}_1-\vec{k})$$
$$Y_{l_2m_2}^*(2\vec{p}_1-\vec{k}) Y_{lm}(2\vec{p}_1). \quad (4.20)$$

Let $\vec{Q} = \vec{p}_1 - \frac{\vec{k}}{3}$, the transition amplitude becomes

$$T_\mu^{Sp} = \int d\vec{Q} \mathcal{L}_1\mathcal{L}_2\mathcal{L}\ e^{-\frac{3}{2}a^2Q^2-\frac{1}{12}a^2k^2} (\frac{4k}{3} Y_{1\mu}^*(\hat{k}) - 2Q Y_{1\mu}^*(\hat{Q}))$$
$$a^{l_1+l_2+l} Y_{l_1m_1}^*(2\vec{Q}-\frac{\vec{k}}{3}) Y_{l_2m_2}^*(2\vec{Q}-\frac{\vec{k}}{3}) Y_{lm}(2\vec{Q}+\frac{2\vec{k}}{3}) \quad (4.21)$$

However,the spatial transition amplitude seems more complicated than other types of amplitude because it depends on which state we are interested, that is, 2S and 1D states.

4.2 Spatial Transition Amplitude in 2S Wave

The general form of spatial transition amplitude in previous section is

$$T_{sp} = \int d\vec{Q}\mathcal{L}_1\mathcal{L}_2\mathcal{L}\ e^{-\frac{3}{2}a^2Q^2-\frac{1}{12}a^2k^2}(\frac{4k}{3}Y^*_{1\mu}(\hat{k}) - 2QY^*_{1\mu}(\hat{Q}))a^{l_1+l_2+l}$$
$$Y^*_{l_1m_1}(2\vec{Q} - \frac{\vec{k}}{3})Y^*_{l_2m_2}(2\vec{Q} - \frac{\vec{k}}{3})Y_{lm}(2\vec{Q} + \frac{2\vec{k}}{3}) \tag{4.22}$$

In case of e^+e^- annihilation into 2 mesons with $n'_1 = n'_2 = l'_1 = l'_2 = 0$ and $n = 1, l = 0$, the amplitude then turns to

$$T_{sp} = \int d\vec{Q}\frac{2a^3}{\Gamma(\frac{3}{2})}\sqrt{\frac{2a^3}{\Gamma(\frac{5}{2})}}\ e^{-\frac{3}{2}a^2Q^2-\frac{1}{12}a^2k^2}(\frac{4k}{3}Y^*_{1\mu}(\hat{k}) - 2QY^*_{1\mu}(\hat{Q}))$$
$$\frac{1}{4\pi\sqrt{4\pi}}\ L(1, \frac{1}{2}, a^2(\vec{Q} + \frac{\vec{k}}{3})^2) \tag{4.23}$$

Consider a term

$$\begin{aligned} L(1, \frac{1}{2}, a^2(\vec{Q} + \frac{\vec{k}}{3})^2) &= \frac{3}{2} - a^2(\vec{Q} + \frac{\vec{k}}{3})^2 \\ &= \frac{3}{2} - a^2(Q^2 + \frac{k^2}{9} + \frac{2}{3}\sum_\nu(-1)^\nu Q_{1\nu}k_{1-\nu}) \\ &= \frac{3}{2} - a^2Q^2 - \frac{a^2k^2}{9} - \frac{2a^2}{3}\sum_\nu(-1)^\nu Y_{1\nu}(\vec{Q})Y_{1-\nu}(\vec{k}) \end{aligned} \tag{4.24}$$

where $\vec{Q}\cdot\vec{k} = \sum_\nu(-1)^\nu Q_{1\nu}k_{1-\nu}$, $Q_{1\nu} = Y_{1\nu}(\vec{Q}) = QY_{1\nu}(\hat{Q})$ and $k_{1-\nu} = Y_{1-\nu}(\vec{k}) = kY_{1-\nu}(\hat{k})$. Substitute (4.24) into (4.23),

$$T_{sp} = \int d\vec{Q}\frac{2a^3}{\Gamma(\frac{3}{2})}\sqrt{\frac{2a^3}{\Gamma(\frac{5}{2})}}\ e^{-\frac{3}{2}a^2Q^2-\frac{1}{12}a^2k^2}(\frac{4k}{3}Y^*_{1\mu}(\hat{k}) - 2QY^*_{1\mu}(\hat{Q}))$$
$$\frac{1}{4\pi\sqrt{4\pi}}(\frac{3}{2} - a^2Q^2 - \frac{a^2k^2}{9} - \frac{2a^2}{3}Qk\sum_\nu(-1)^\nu Y_{1\nu}(\hat{Q})Y_{1-\nu}(\hat{k})))$$
$$= \frac{\sqrt{2}a^{9/2}}{\sqrt{3}\pi^{9/4}}e^{-\frac{1}{12}a^2k^2}\int d\vec{Q}\ e^{-\frac{3}{2}a^2Q^2}\Big\{2kY^*_{1\mu}(\hat{k}) - \frac{4a^2k}{3}Q^2Y^*_{1\mu}(\hat{k}) - \frac{4a^2k^3}{27}Y^*_{1\mu}(\hat{k})$$
$$-\frac{8a^2k^2}{9}Q\sum_\nu(-1)^\nu Y_{1\nu}(\hat{Q})Y_{1-\nu}(\hat{k})Y^*_{1\mu}(\hat{k}) - 3QY^*_{1\mu}(\hat{Q}) + 2a^2Q^3Y^*_{1\mu}(\hat{Q})$$
$$+\frac{2a^2k^2}{9}QY^*_{1\mu}(\hat{Q}) + \frac{4a^2}{3}Q^2k\sum_\nu(-1)^\nu Y_{1\nu}(\hat{Q})Y_{1-\nu}(\hat{k})Y^*_{1\mu}(\hat{Q})\Big\}. \tag{4.25}$$

We have found that some terms in (4.25) vanished because

$$\int d\Omega\ Y_{lm}Y^*_{l'm'} = \delta_{ll'}\delta_{mm'} \tag{4.26}$$

$$\int d\Omega\ Y_{lm} = \int d\Omega\ Y^*_{l'm'} = 0 \tag{4.27}$$

then (4.25) turns to

$$\begin{aligned}
T_{sp} &= \frac{\sqrt{2}a^{9/2}}{\sqrt{3}\pi^{9/4}}e^{-\frac{1}{12}a^2k^2}\int d\vec{Q}\ e^{-\frac{3}{2}a^2Q^2}(2kY^*_{1\mu}(\hat{k}) - \frac{4a^2k}{3}Q^2Y^*_{1\mu}(\hat{k}) - \frac{4a^2k^3}{27}Y^*_{1\mu}(\hat{k}) \\
&+ \frac{4a^2}{3}Q^2k\sum_\nu(-1)^\nu Y_{1\nu}(\hat{Q})Y_{1-\nu}(\hat{k})Y^*_{1\mu}(\hat{Q})) \\
&= \frac{\sqrt{2}a^{9/2}}{\sqrt{3}\pi^{9/4}}e^{-\frac{1}{12}a^2k^2}(\frac{4\sqrt{2}\pi^{3/2}k}{3\sqrt{3}a^3}Y^*_{1\mu}(\hat{k}) - \frac{8\sqrt{2}\pi^{3/2}k}{9\sqrt{3}a^3}Y^*_{1\mu}(\hat{k}) - \frac{8\sqrt{2}\pi^{3/2}k^3}{81\sqrt{3}a}Y^*_{1\mu}(\hat{k}) \\
&+ \frac{4a^2}{3}\int d\vec{Q}\ e^{-\frac{3}{2}a^2Q^2}Q^2k\sum_\nu(-1)^\nu Y_{1\nu}(\hat{Q})Y_{1-\nu}(\hat{k})Y^*_{1\mu}(\hat{Q})) \\
&= \frac{\sqrt{2}a^{9/2}}{\sqrt{3}\pi^{9/4}}e^{-\frac{1}{12}a^2k^2}[(\frac{4\sqrt{2}k\pi^{3/2}}{9\sqrt{3}a^3} - \frac{8\sqrt{2}k^3\pi^{3/2}}{81\sqrt{3}a})Y^*_{1\mu}(\hat{k}) \\
&+ \frac{4a^2}{3}\int dQ\ e^{-\frac{3}{2}a^2Q^2}Q^4k\sum_\nu(-1)^\nu Y_{1-\nu}(\hat{k})]\delta_{\mu\nu} \\
&= \frac{\sqrt{2}a^{9/2}}{\sqrt{3}\pi^{9/4}}e^{-\frac{1}{12}a^2k^2}(\frac{4\sqrt{2}k\pi^{3/2}}{9\sqrt{3}a^3} - \frac{8\sqrt{2}k^3\pi^{3/2}}{81\sqrt{3}a} + \frac{2k\sqrt{2\pi}}{9\sqrt{3}a^3})Y^*_{1\mu}(\hat{k}) \\
&= BY^*_{1\mu}(\hat{k})
\end{aligned} \tag{4.28}$$

where

$$\begin{aligned}
B &= \frac{\sqrt{2}a^{9/2}}{\sqrt{3}\pi^{9/4}}e^{-\frac{1}{12}a^2k^2}(\frac{4\sqrt{2}k\pi^{3/2}}{9\sqrt{3}a^3} - \frac{8\sqrt{2}k^3\pi^{3/2}}{81\sqrt{3}a} + \frac{2k\sqrt{2\pi}}{9\sqrt{3}a^3}) \\
&= \alpha k(1-\beta k^2)
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
\alpha &= \frac{8(a\pi)^{3/2}(1+\frac{1}{2\pi})e^{-\frac{1}{12}a^2k^2}}{27\pi^{9/4}} \\
\beta &= \frac{4\pi a^2}{9(2\pi+1)}.
\end{aligned} \tag{4.30}$$

4.3 Spatial Transition Amplitude in 1D Wave

For 1D wave calculation, we similarly start from the general form of spatial transition amplitude which is

$$T_{sp} = \int d\vec{Q}\mathcal{L}_1\mathcal{L}_2\mathcal{L}\ e^{-\frac{3}{2}a^2Q^2-\frac{1}{12}a^2k^2}(\frac{4k}{3}Y^*_{1\mu}(\hat{k}) - 2QY^*_{1\mu}(\hat{Q}))a^{l_1+l_2+l}$$
$$Y^*_{l_1m_1}(2\vec{Q} - \frac{\vec{k}}{3})Y^*_{l_2m_2}(2\vec{Q} - \frac{\vec{k}}{3})Y_{lm}(2\vec{Q} + \frac{2\vec{k}}{3}) \quad (4.31)$$

In case of e^+e^- annihilation into 2 mesons with $n'_1 = n'_2 = l'_1 = l'_2 = 0$ and $n = 0, l = 2$, the amplitude then turns to

$$T_{sp} = \int d\vec{Q}\frac{2a^3}{\Gamma(\frac{3}{2})}\sqrt{\frac{2a^3}{\Gamma(\frac{7}{2})}}\ e^{-\frac{3}{2}a^2Q^2-\frac{1}{12}a^2k^2}$$
$$(\frac{4k}{3}Y^*_{1\mu}(\hat{k}) - 2QY^*_{1\mu}(\hat{Q}))a^2\frac{1}{4\pi}Y_{2m}(2\vec{Q} + \frac{2\vec{k}}{3}) \quad (4.32)$$

where $L(0, \frac{5}{2}, a^2(\vec{Q} + \frac{\vec{k}}{3})^2) = 1$. Consider a term

$$Y_{2m}(2\vec{Q} + \frac{2}{3}\vec{k}) = Y_{2m}(\vec{p}) \quad (4.33)$$

From

$$Y_{lm}(\vec{r}) \equiv r^lY_{lm}(\hat{r}) = (\alpha\vec{\lambda} + \beta\vec{\rho})^lY_{lm}(\hat{r})$$
$$= \sqrt{4\pi}\sum_{l_\lambda l_\rho}^{l}\sum_{m_\lambda m_\rho}\delta_{l_\lambda+l_\rho,l}\left[\frac{(2l+1)!}{(2l_\lambda+1)!(2l_\rho+1)!}\right]^{1/2}(\alpha\lambda)^{l_\lambda}(\beta\rho)^{l_\rho}$$
$$\langle lm|l_\lambda m_\lambda l_\rho m_\rho\rangle Y_{l_\lambda m_\lambda}(\hat{\lambda})Y_{l_\rho m_\rho}(\hat{\rho}), \quad (4.34)$$

we have

$$Y_{2m}(2\vec{Q} + \frac{2}{3}\vec{k}) = \sqrt{4\pi}\sum_{l_Q l_k}^{2}\sum_{m_Q m_k}\delta_{l_Q+l_k,2}\left[\frac{5!}{(2l_Q+1)!(2l_k+1)!}\right]^{\frac{1}{2}}(2Q)^{l_Q}(\frac{2}{3}k)^{l_k}$$
$$\langle 2m|l_Q m_Q l_k m_k\rangle Y_{l_Q m_Q}(\hat{Q})Y_{l_k m_k}(\hat{k})$$
$$= (\frac{5!}{5!1!})^{\frac{1}{2}}(2Q)^2\delta_{mm_Q}Y_{2m_Q}(\hat{Q}) + (\frac{5!}{1!5!})^{\frac{1}{2}}(\frac{2}{3}k)^2\delta_{mm_k}Y_{2m_k}(\hat{k})$$

$$+\sum_{m_Q m_k}\sqrt{4\pi}(\frac{5!}{3!3!})^{\frac{1}{2}}\frac{4}{3}Qk\langle 2m|1m_Q 1m_k\rangle Y_{1m_Q}(\hat{Q})Y_{1m_k}(\hat{k})$$

$$=\sum_{m_Q m_k}\frac{4\sqrt{10}}{3\sqrt{3}}Qk\sqrt{4\pi}\langle 2m|1m_Q 1m_k\rangle Y_{1m_Q}(\hat{Q})Y_{1m_k}(\hat{k})$$

$$+4Q^2Y_{2m}(\hat{Q})+\frac{4}{9}k^2Y_{2m}(\hat{k}). \tag{4.35}$$

Insert (4.35) into (4.32),

$$T_{sp}=\int d\vec{Q}\frac{2a^3}{\Gamma(\frac{3}{2})}\sqrt{\frac{2a^3}{\Gamma(\frac{7}{2})}}\ e^{-\frac{3}{2}a^2Q^2-\frac{1}{12}a^2k^2}(\frac{4k}{3}Y^*_{1\mu}(\hat{k})-2QY^*_{1\mu}(\hat{Q}))$$

$$\frac{a^2}{4\pi}\Big[4Q^2Y_{2m}(\hat{Q})+\frac{4}{9}k^2Y_{2m}(\hat{k})$$

$$+\sum_{m_Q m_k}\frac{4\sqrt{10}}{3\sqrt{3}}Qk\sqrt{4\pi}\langle 2m|1m_Q 1m_k\rangle Y_{1m_Q}(\hat{Q})Y_{1m_k}(\hat{k})\Big]$$

$$=\frac{4a^{13/2}}{\sqrt{15}\pi^{7/4}}\ e^{-\frac{1}{12}a^2k^2}\int d\vec{Q}\ e^{-\frac{3}{2}a^2Q^2}(\frac{16k}{3}Q^2Y_{2m}(\hat{Q})Y^*_{1\mu}(\hat{k})$$

$$+\frac{16k^3}{27}Y_{2m}(\hat{k})Y^*_{1\mu}(\hat{k})-8Q^3Y_{2m}(\hat{Q})Y^*_{1\mu}(\hat{Q})-\frac{8}{9}Qk^2Y_{2m}(\hat{k})Y^*_{1\mu}(\hat{Q})$$

$$+\sum_{m_Q m_k}\frac{16\sqrt{40\pi}k^2}{9\sqrt{3}}Q\langle 2m|1m_Q 1m_k\rangle Y_{1m_Q}(\hat{Q})Y_{1m_k}(\hat{k})Y^*_{1\mu}(\hat{k})$$

$$-\sum_{m_Q m_k}\frac{8\sqrt{10}}{3\sqrt{3}}kQ^2\sqrt{4\pi}\langle 2m|1m_Q 1m_k\rangle Y_{1m_Q}(\hat{Q})Y_{1m_k}(\hat{k})Y^*_{1\mu}(\hat{Q})). \tag{4.36}$$

We have found that some terms in (4.36) are vanished because

$$\int d\Omega\ Y_{lm}Y^*_{l'm'}=\delta_{ll'}\delta_{mm'} \tag{4.37}$$

$$\int d\Omega\ Y_{lm}=\int d\Omega\ Y^*_{l'm'}=0. \tag{4.38}$$

(4.36) turns to

$$T_{sp}=\frac{4a^{13/2}}{\sqrt{15}\pi^{7/4}}\ e^{-\frac{1}{12}a^2k^2}\int d\vec{Q}\ e^{-\frac{3}{2}a^2Q^2}\Big[\frac{16k^3}{27}Y_{2m}(\hat{k})Y^*_{1\mu}(\hat{k})$$

$$-\sum_{m_Q m_k}\frac{8\sqrt{10}}{3\sqrt{3}}kQ^2\sqrt{4\pi}\langle 2m|1m_Q 1m_k\rangle Y_{1m_Q}(\hat{Q})Y_{1m_k}(\hat{k})Y^*_{1\mu}(\hat{Q})\Big]$$

$$=\frac{4a^{13/2}}{\sqrt{15}\pi^{7/4}}\ e^{-\frac{1}{12}a^2k^2}\Big[\frac{32\sqrt{2}k^3\pi^{3/2}}{81\sqrt{3}a^3}Y_{2m}(\hat{k})Y^*_{1\mu}(\hat{k})$$

$$-\int d\vec{Q}\ e^{-\frac{3}{2}a^2Q^2}\sum_{m_Q}\frac{16\sqrt{10\pi}}{3\sqrt{3}}kQ^2\langle 2m|1m_Q,1m-m_Q\rangle$$

$$
\begin{aligned}
& Y_{1m_Q}(\hat{Q})Y_{1,\ m-m_Q}(\hat{k})Y^*_{1\mu}(\hat{Q})\Big] \\
= & \frac{4a^{13/2}}{\sqrt{15}\pi^{7/4}}\ e^{-\frac{1}{12}a^2k^2}\Big[\frac{32\sqrt{2}k^3\pi^{3/2}}{81\sqrt{3}a^3}Y_{2m}(\hat{k})Y^*_{1\mu}(\hat{k}) - \frac{16\sqrt{10\pi}}{3\sqrt{3}}k\int dQ\ Q^4 e^{-\frac{3}{2}a^2Q^2} \\
& \sum_{m_Q}\langle 2m|1m_Q, 1m-m_Q\rangle\delta_{m_Q,\mu}Y_{1,\ m-m_Q}(\hat{k})\Big] \\
= & \frac{4a^{13/2}}{\sqrt{15}\pi^{7/4}}\ e^{-\frac{1}{12}a^2k^2}\ \Big[\frac{32\sqrt{2}k^3\pi^{3/2}}{81\sqrt{3}a^3}Y_{2m}(\hat{k})Y^*_{1\mu}(\hat{k}) - \frac{16\sqrt{5}k\pi}{27a^5} \\
& \langle 2m|1\mu; 1, m-\mu\rangle Y_{1,\ m-\mu}(\hat{k})\Big] \\
= & A_1 Y_{2m}(\hat{k})Y^*_{1\mu}(\hat{k}) + A_2 Y_{1,\ m-\mu}(\hat{k})
\end{aligned} \tag{4.39}
$$

where

$$
\begin{aligned}
A_1 &= \frac{128\sqrt{2}a^{7/2}k^3}{243\sqrt{5}\pi^{1/4}}\ e^{-\frac{1}{12}a^2k^2} \\
A_2 &= -\frac{64a^{3/2}k}{27\sqrt{3}\pi^{3/4}}\langle 2m|1\mu; 1, m-\mu\rangle e^{-\frac{1}{12}a^2k^2}
\end{aligned} \tag{4.40}
$$

We can write the transition amplitude in the form of summation of orbital angular momentum,

$$
T = \sum_{l'm'} T_{l'm'}Y^*_{l'm'}(\hat{k}) \tag{4.41}
$$

where the complex conjugate stands for the outgoing wave. Factors $T_{l'm'}$ are worked out from

$$
T_{l'm'} = \int d\Omega\ T\ Y_{l'm'}(\hat{k}). \tag{4.42}
$$

For $l' = 1$, we have

$$
T_{1,m'} = \int d\Omega\ (A_1 Y^*_{1,\mu}(\hat{k})Y_{2m}(\hat{k}) + A_2 Y_{1,m-\mu}(\hat{k}))Y_{1m'}(\hat{k}). \tag{4.43}
$$

By means of

$$
\int d\Omega Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{4.44}
$$

$$
Y^*_{1\mu}(\hat{k}) = (-1)^{\mu}Y_{l,-\mu}(\hat{k}) \tag{4.45}
$$

and

$$\int \sin\theta d\theta d\phi \; Y^*_{l_3,m_3}(\theta,\phi)Y_{l_1m_1}(\theta,\phi)Y_{l_2m_2}(\theta,\phi)$$
$$= \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}}\; C(l_1l_2l_3,000)C(l_1l_2l_3,m_1m_2m_3), \tag{4.46}$$

the first term of Eq. (4.43) becomes

$$\begin{aligned}\int d\Omega\; A_1Y^*_{1,\mu}(\hat{k})Y_{2m}(\hat{k})Y_{1m'}(\hat{k}) &= A_1\sqrt{\frac{5}{4\pi}}\; C(121,000)C(121;m',m,\mu) \\ &= -A_1\sqrt{\frac{5}{4\pi}}\sqrt{\frac{2}{5}}\; C(121;\mu-m,m,\mu) \\ &= -A_1\sqrt{\frac{1}{2\pi}}\; C(121;\mu-m,m,\mu) \end{aligned} \tag{4.47}$$

and the latter turns to

$$\begin{aligned}\int d\Omega\; A_2Y_{1,m-\mu}(\hat{k})Y_{1m'}(\hat{k}) &= \int d\Omega\; A_2(-1)^{m'}Y_{1,m-\mu}(\hat{k})Y^*_{1,-m'}(\hat{k}) \\ &= A_2(-1)^{m'}\delta_{m-\mu,-m'} \\ &= a_2(-1)^{\mu-m}\langle 2,m|1\mu;1,m-\mu\rangle. \end{aligned} \tag{4.48}$$

where

$$a_2 = -\frac{64a^{3/2}k}{27\sqrt{3}\pi^{3/4}}e^{-\frac{1}{12}a^2k^2}.$$

Then, (4.43) is as shown

$$T_{1,m'} = -A_1\sqrt{\frac{1}{2\pi}}\; C(121;\mu-m,m,\mu) + a_2(-1)^{\mu-m}\langle 2,m|1\mu;1,m-\mu\rangle. \tag{4.49}$$

For $l'=2$, we have

$$T_{2m'} = \int d\Omega\; (A_1Y^*_{1\mu}(\hat{k})Y_{2m}(\hat{k}) + A_2Y_{1,m-\mu}(\hat{k}))Y_{2m'}(\hat{k}). \tag{4.50}$$

By Eqs. (4.44), (4.45), and (4.46), we obtain

$$T_{2m'} = 0 \tag{4.51}$$

where $C(221,000) = 0$. For $l' = 3$, we have

$$\begin{aligned}
T_{3m'} &= \int d\Omega\ (A_1 Y^*_{1,\mu}(\hat{k}) Y_{2m}(\hat{k}) + A_2 Y_{1,m-\mu}(\hat{k})) Y_{3m'}(\hat{k}) \\
&= \int d\Omega\ A_1 Y^*_{1,\mu}(\hat{k}) Y_{2m}(\hat{k}) Y_{3m'}(\hat{k}) \\
&= A_1 \sqrt{\frac{5\cdot 7}{4\pi \cdot 3}}\ C(321,000) C(321; m', m, \mu) \\
&= A_1 \sqrt{\frac{3}{4\pi}}\ C(321; \mu - m, m, \mu). \qquad (4.52)
\end{aligned}$$

Hence, the spatial transition amplitude is

$$\begin{aligned}
T_{\text{spatial}} &= \sum_{l'm'} T_{l'm'} Y^*_{l'm'}(\hat{k}) \\
&= T_{1m'} Y^*_{1m'}(\hat{k}) + T_{3m'} Y^*_{3m'}(\hat{k}) \\
&= (-A_1 \sqrt{\frac{1}{2\pi}}\ C(121; \mu - m, m, \mu) + a_2 (-1)^{\mu - m} \langle 2, m | 1\mu; 1, m - \mu \rangle) \\
&\quad \cdot Y^*_{1,\mu-m}(\hat{k}) + A_1 \sqrt{\frac{3}{4\pi}}\ C(321; \mu - m, m, \mu) Y^*_{3,\mu-m}(\hat{k}) \qquad (4.53)
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \frac{128\sqrt{2} a^{7/2} k^3}{243\sqrt{5}\pi^{1/4}}\ e^{-\frac{1}{12}a^2k^2} \\
a_2 &= -\frac{64 a^{3/2} k}{27\sqrt{3}\pi^{3/4}} e^{-\frac{1}{12}a^2k^2}.
\end{aligned}$$

4.4 Total Cross Section

The total transition amplitude form Eq. (4.2) is

$$\begin{aligned}
T_{e^+e^- \to m_1 m_2} &= \langle \pi\omega | V_{\bar{q}q} | \rho' \rangle \langle \rho' | G | \rho' \rangle \langle \rho' | \bar{q}q \rangle \langle \bar{q}q | T | e^+ e^- \rangle \\
&= T_{m \to m_1 m_2} \langle \rho' | G | \rho' \rangle T_{e^+e^- \to m} \qquad (4.54)
\end{aligned}$$

where $T_{m \to m_1 m_2}$ is the transition amplitude of the intermediate meson decay into two mesons as shown in the previous section, $T_{e^+e^- \to m}$ is the transition amplitude of the annihilation of an electron and a positron into the intermediate meson, and $\langle\rho'|G|\rho'\rangle$ is the Green function describing the propagation of the intermediate meson and taking the form

$$\langle\rho'|G|\rho'\rangle = \frac{1}{E_{cm} - M_m - \frac{i\Gamma}{2}}, \tag{4.55}$$

where M_m is the mass of an intermediate meson, and Γ is the decay width of the intermediate meson. The cross section can be obtained form

$$\sigma = \frac{2\pi E_1 E_2 q}{4p\sqrt{s}E_{cm}} \int |T_{e^+e^-\to m_1m_2}|^2 d\Omega \tag{4.56}$$

where $E_1E_2, p, \sqrt{s}, q$ are the product of the energy of each meson, the energy of electron, the center of mass energy, and the momentum of each outgoing meson, respectively.

In case of two intermediate ρ mesons, the total transition amplitude used to calculate the cross section is in the form of the summation of each transition amplitude

$$T_{e^+e^-\to m_1m_2} = T^{\rho'}_{e^+e^-\to m_1m_2} + T^{\rho}_{e^+e^-\to m_1m_2}. \tag{4.57}$$

CHAPTER V

RESULTS AND DISCUSSIONS

Shown in Fig. 5.1 are the theoretical predictions and experimental data for the cross section of the reaction $e^+e^- \to \omega\pi^0$. Note that for comparing with the experimental data of the reaction $e^+e^- \to \omega\pi^0 \to \pi^0\pi^0\gamma$, we have multiplied our theoretical predictions by the decay branch ratio (0.087) of the $\omega(780)$ to $\pi^0\gamma$. In the theoretical calculation we have included both $\rho(1450)$ and $\rho(770)$ mesons as the intermediate meson states. There is no free parameter in the study. The relevant masses and widths of mesons are taken from the particle table*, and the effective coupling constant of the $^3P_0$ quark vertex is fixed to be 2.5 by the reaction $\rho \to \pi\pi$ and the meson length parameter is fixed to be 3.847 by the reaction $\rho \to e^+e^-$.

It is found in Fig. 5.1 that the theoretical prediction, with the $\rho(1450)$ meson being in the $2S$-state, is well close to the experimental data while the result from a $1D$-state $\rho(1450)$ is too small. Therefore, one may comfortably concluded that the $\rho(1450)$ meson is in the $2S$-state.

Although both $\rho(1450)$ and $\rho(770)$ mesons, as the intermediate states, contribute to the reaction $e^+e^- \to \omega\pi^0$, the theoretical result reveals that the $\rho(1450)$ meson dominate over the $\rho(770)$. The broad peak in Fig. 5.1 stem mainly from the occurrence of $\rho(1450)$ meson.

That the prediction for the cross section of the reaction $e^+e^- \to \omega\pi^0$ in the two-step process with the $\rho(1450)$ as the intermediate state reproduces well the experimental data at energies below 1.4 GeV leaves no room for the one-step

*See Appendix A

process to contribute to the reaction at a sizable scale at this energy region. For higher energies one may have to include more mesons into the intermediate states and/or to consider the contribution of the one-step process.

The reaction $e^+e^- \to \rho\pi^0$ is also studied in the work though data are very scarce. ρ is a broad meson with a width of $\Gamma = 150$ MeV, and hence can not be observed directly in experiments. It is usually detected via the coincident measurement of its 2π decay products. Since there are a large number of particles with masses around 1 GeV decaying into 2π, it is very difficult to distinguish ρ from other mesons in the reaction $e^+e^- \to \pi^+\pi^-\pi^0$. In our knowledge, there exists only one set of experimental data from the work (Antonelli et al., 1992). There are apparently two resonances in the experimental data, with the first probably from the decay of $\omega(1420)$ meson while the second from the $\omega(1650)$.

Presented in Fig. 5.2 are the theoretical predictions for the cross section of the reaction $e^+e^- \to \rho\pi^0$ in the two-step process where the $\omega(1420)$ and $\omega(780)$ mesons are included into the intermediate meson states. Again, there is no free parameter in the study. The relevant masses and widths of mesons are taken from the particle table, and the effective coupling constant of the $^3P_0$ quark vertex and the meson length parameter are the same as for the reaction $e^+e^- \to \omega\pi^0$.

It is found in Fig. 5.2 that the theoretical prediction, with the $\omega(1420)$ meson being a $2S$-state, is much larger than the experimental data. Therefore, one may rule out that the $\omega(1420)$ is a $2S$ meson. With the $\omega(1420)$ meson as a $1D$ meson, the calculated cross section is more or less consistent with the data, hence one may suggest that the $\omega(1420)$ meson is likely to be dominated by the $1D$-wave.

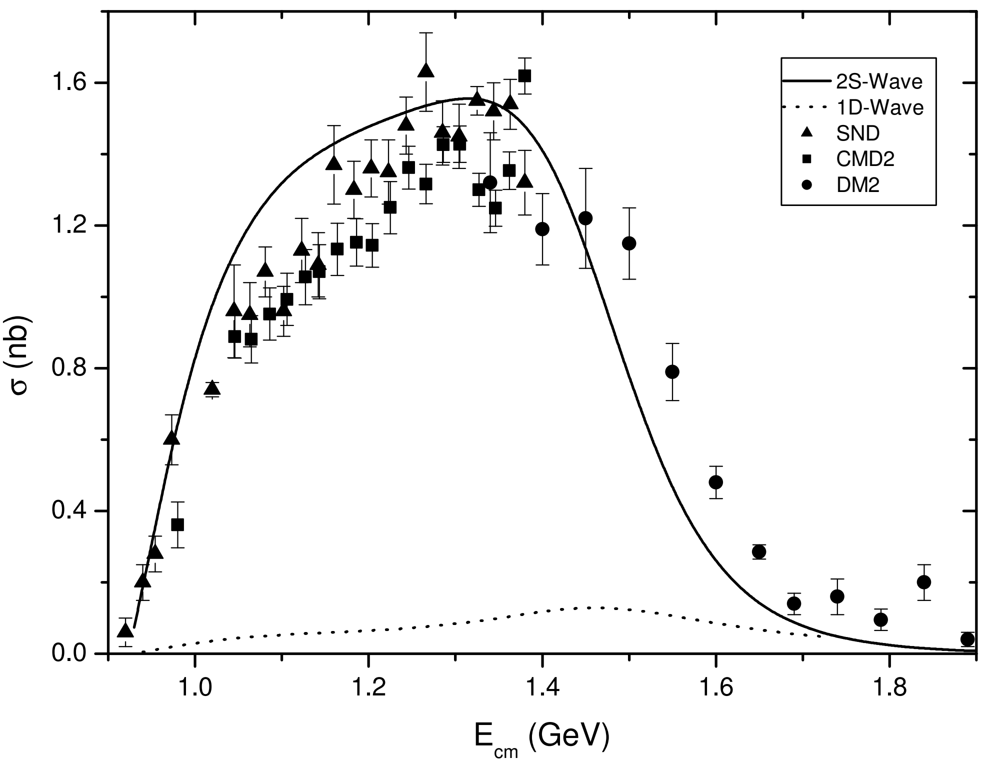


Figure 5.1 Theoretical prediction for the cross section of reaction $e^+e^- \to \omega\pi^0 \to \pi^0\pi^0\gamma$ in two-step process with ρ' in both 2S (solid line)and 1D (dotted line) waves. Experimental data taken from the SND (Achasov et al., 2000), DM2 (Bisello et al., 1991), CMD2 (Akhmetshin et al., 1999).

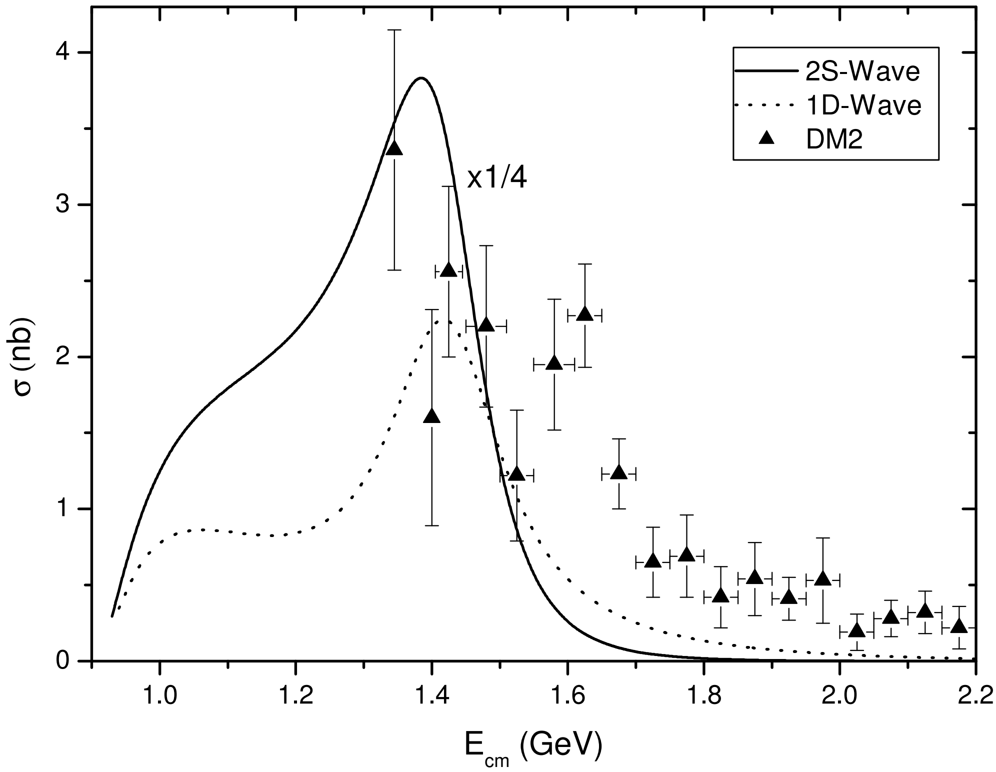


Figure 5.2 Theoretical prediction for the cross section of reaction $e^+e^- \rightarrow \rho\pi^0$ in two-step process with ω' in both 2S (solid line) and 1D (dotted line) waves. Experimental data taken from the DM2 (Antonelli et al., 1992). The cross section simulated from ω in 2S-wave is multiplied by a factor 1/4.

In conclusion, the reaction $e^+e^- \to \omega\pi^0$ and $e^+e^- \to \rho\pi^0$ are investigated in the $^3P_0$ non-relativistic quark model without any free parameter. The experimental data of both reactions are fairly reproduced in the work. The study suggests that at the energy region from the threshold to 1.4 GeV the two-step process is dominant over the one-step one. The experimental data of the reaction $e^+e^- \to \omega\pi^0$ strongly dictate a $2S$-wave $\rho(1450)$ while the data of the reaction $e^+e^- \to \rho\pi^0$ prefer $\omega(1420)$ being a $1D$-wave meson.

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APPENDICES

APPENDIX A

PARTICLE DATA

Data are from J. Phys. G **33**, (2006).

Table A.1 Pseudoscalar meson (spin = 0)

Meson	Quark content	Charge	Mass (MeV)	Lifetime (s)	Principal decays
$\pi^{\pm}$	$u\bar{d}, d\bar{u}$	$+1, -1$	139.570	2.60×10^{-8}	$\mu\nu_{\mu}$
π^{0}	$(u\bar{u} - d\bar{d})/\sqrt{2}$	0	134.977	8.4×10^{-17}	$\gamma\gamma$

Table A.2 Vector meson (spin = 1)

Meson	Quark content	Charge	Mass (MeV)	Full width Γ (MeV)	Principal decays
ρ	$u\bar{d}, d\bar{u}, (u\bar{u} - d\bar{d})/\sqrt{2}$	± 1, 0	775.5 ± 0.4	149.4 ± 1.0	$\pi\pi$
ω	$(u\bar{u} + d\bar{d})/\sqrt{2}$	0	782.6 ± 0.1	8.49 ± 0.08	$\pi^{+}\pi^{-}\pi^{0}$

APPENDIX B

THREE DIMENSIONAL HARMONIC OSCILLATOR

The potential of harmonic oscillator is widely employed to study the interaction in quark-antiquark system of mesons and three quarks system of baryon. The main character of potential considered to be harmonic is

$$V(r) \propto r^2. \tag{B.1}$$

In the spherical coordinate, the radial schrödinger equation is

$$\left[\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}(E - \frac{1}{2}\mu\omega^2 r^2) - \frac{l(l+1)}{r^2}\right] u(r) = 0 \tag{B.2}$$

where

$$u(r) = rR(r). \tag{B.3}$$

Eq. (B.2) can be contracted to

$$\left[\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} + \lambda - \rho^2\right] u(\rho) = 0 \tag{B.4}$$

by introducing the dimensionless variable

$$\begin{aligned} \rho &= \alpha r \\ \lambda &= \frac{2E}{\hbar\omega} \end{aligned} \tag{B.5}$$

where

$$\alpha = \left(\frac{\mu\omega}{\hbar}\right)^{1/2}. \tag{B.6}$$

The study of an asymptotic behavior of $u(\rho)$ leads to, when $\rho \to 0$,

$$u(\rho) \sim \rho^{l+1} \tag{B.7}$$

and, when $\rho \to \infty$,

$$\left[\frac{d^2}{d\rho^2} - \rho^2\right] u(\rho) = 0. \tag{B.8}$$

The solution of asymptotic equation is

$$u(\rho) \sim e^{-\rho^2/2}. \tag{B.9}$$

According to the asymptotic behaviors in Eqs. (B.7) and (B.9), the solution of $u(\rho)$ in (B.4) is assumed as

$$u(\rho) = e^{-\rho^2/2}\rho^{l+1}g(\rho). \tag{B.10}$$

Introduced $y = \rho^2$ and inserted (B.10) into (B.4), equation of $g(\rho)$ becomes

$$y\frac{d^2g(y)}{dy^2} + \left[(l+\frac{3}{2}) - y\right]\frac{dg(y)}{dy} - \left[\frac{1}{2}(l+\frac{3}{2}) - \frac{\lambda}{4}\right]g(y) = 0. \tag{B.11}$$

This is the Kummer-Laplace differential equation whose solution, regular at the origin, is

$$g(y) = CF\left(\frac{l}{2} + \frac{3}{4} - \frac{\lambda}{4}, l + \frac{3}{2}, y\right) \tag{B.12}$$

where C is a constant and F is the confluent hypergeometric function,

$$\begin{aligned} F(\alpha,\gamma,\rho) &= 1 + \frac{\alpha}{\gamma}\frac{\rho}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{\rho^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\gamma)_k}\frac{\rho^k}{k!}. \end{aligned} \tag{B.13}$$

The spherical wave function or the simultaneous eigenfunction of the observables (H, L^2, L_z) reads

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi) \tag{B.14}$$

where $R_{nl}(r)$ is the radial wave functions and $Y_{lm}(\theta,\phi)$ is the spherical harmonics. From (B.12), the radial wave function behaves

$$R_{nl}(r) \sim (\alpha r)^l e^{-\frac{1}{2}\alpha^2 r^2} F(-n, l+\frac{3}{2}, \alpha^2 r^2). \tag{B.15}$$

The normalized wave function reads

$$R_{nl}(r) = \alpha^{3/2}\left[\frac{2^{l+2-n}(2l+2n+1)!!}{\sqrt{\pi}n![(2l+1)!!]^2}\right](\alpha r)^l e^{-\frac{1}{2}\alpha^2 r^2} F(-n, l+\frac{3}{2}, \alpha^2 r^2). \tag{B.16}$$

It is more often and convenient to write the above equation in terms of Lagurre polynomials,

$$R_{nl}(r) = \left[\frac{2\alpha^3 n!}{\Gamma(n+l+\frac{3}{2})}\right](\alpha r)^l e^{-\frac{1}{2}\alpha^2 r^2} L_n^{l+1/2}(\alpha^2 r^2) \tag{B.17}$$

where $L_n^{l+1/2}(\alpha^2 r^2)$ are the associated Laguerre polynomials

$$L_n^{l+1/2}(\alpha^2 r^2) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{\Gamma(n+l+\frac{3}{2})}{(n-k)!\Gamma(k+l+\frac{3}{2})} r^{2k}. \tag{B.18}$$

The radial wave functions have the orthogonal property

$$\int_0^\infty r^2 dr R_{nl}(r) R_{n'l}(r) = \delta_{nn'}. \tag{B.19}$$

By the Fourier transformation, the analytical wave function of a harmonic oscillator in momentum space is shown

$$\psi_{nlm}(\vec{p}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int d\vec{r}\psi_{nlm}(\vec{r}) e^{-i\vec{p}\cdot\vec{r}} = (-i)^{2n+l} R_{nl}(p) Y_{lm}(\vec{p}) \tag{B.20}$$

where

$$R_{nl}(r) = \left[\frac{2\beta^3 n!}{\Gamma(n+l+\frac{3}{2})}\right](\beta r)^l e^{-\frac{1}{2}\beta^2 r^2} L_n^{l+1/2}(\beta^2 r^2) \tag{B.21}$$

and

$$\beta = \frac{1}{\alpha\hbar} \tag{B.22}$$

In our calculation, the spatial wave functions in momentum space are always used and β is interpreted as a size parameter in unit of GeV^{-1}. For mesons (quark-antiquark boundstates), $\vec{p}$ is the momentum of the center of mass

$$\vec{p} = \frac{\vec{p}_1 - \vec{p}_2}{2} \tag{B.23}$$

where $\vec{p}_1$ and $\vec{p}_2$ are momentums of quark and antiquark, respectively.

APPENDIX C

γ-MATRICES AND TRACE TECHNOLOGY

Four dimensional γ-matrices are defined by the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\gamma^{\mu\nu} + \mathbf{1}_{n\times n}. \tag{C.1}$$

Definitions base on "An Introduction to Quantum Field Theory"* with priority. Specific Weyl or chiral representations are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{C.2}$$

where σ^i is Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.3}$$

To easily attack QED problems, the trace techniques produced by R. P. Feynman has been a very important tools. Here are some proves and properties. The prove of trace of one γ matrix is

$$\left.\begin{aligned} \mathrm{tr}(\gamma^\mu) &= \mathrm{tr}(\gamma^5\gamma^5\gamma^\mu) && \text{since } (\gamma^5)^2 = 1 \\ &= -\mathrm{tr}(\gamma^5\gamma^\mu\gamma^5) && \text{since } \{\gamma^\mu, \gamma^5\} = 0 \\ &= -\mathrm{tr}(\gamma^5\gamma^5\gamma^\mu) && \text{using cyclic properties of trace} \\ &= -\mathrm{tr}(\gamma^\mu) \end{aligned}\right\} \tag{C.4}$$

where $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Any parameter equal to minus itself must be vanished. The result is also applied to trace of odd number of γ matrix. For the trace of two

*Michael E. Peskin and Daniel V. Schroeder, 1995

γ matrices, we use the anticommutation property and the cyclic property of trace,

$$\begin{aligned}\mathrm{tr}(\gamma^\mu\gamma^\nu) &= \mathrm{tr}(2g^{\mu\nu}\cdot 1 - \gamma^\mu\gamma^\nu) && \text{(anticommutation)}\\ &= 8g^{\mu\nu} - \mathrm{tr}(\gamma^\mu\gamma^\nu) && \text{(cyclicity).}\end{aligned} \tag{C.5}$$

Hence $\mathrm{tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$. The trace of any even number of γ matrices are evaluated in the same way by anticommuting the first γ matrix all the way to right, then cycle it back to the left. For the trace of four γ matrices, we have

$$\begin{aligned}\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{tr}(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma)\\ &= \mathrm{tr}(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2g^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho 2g^{\mu\sigma} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu).\end{aligned} \tag{C.6}$$

Using the cyclic property on the last term and moving it to the left hand side, we obtain

$$\begin{aligned}\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= g^{\mu\nu}\mathrm{tr}(\gamma^\rho\gamma^\sigma) - g^{\mu\rho}\mathrm{tr}(\gamma^\mu\gamma^\sigma) + g^{\mu\sigma}\mathrm{tr}(\gamma^\nu\gamma^\rho)\\ &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).\end{aligned} \tag{C.7}$$

For $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, the trace of γ^5 and any odd number of other matrices is vanish. The trace of γ^5 itself, however, is also zero,

$$\mathrm{tr}(\gamma^5) = \mathrm{tr}(\gamma^0\gamma^0\gamma^5) = -\mathrm{tr}(\gamma^0\gamma^5\gamma^0) = -\mathrm{tr}(\gamma^0\gamma^0\gamma^5) = -\mathrm{tr}(\gamma^5) = 0. \tag{C.8}$$

These are summary of trace theorems;

$$\left.\begin{aligned}\mathrm{tr}(1) &= 4\\ \mathrm{tr}(\text{any odd number of }\gamma^s) &= 0\\ \mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu}\\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\\ \mathrm{tr}(\gamma^5) &= 0\\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5) &= 0\\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) &= -4i\epsilon^{\mu\nu\rho\sigma}.\end{aligned}\right\} \tag{C.9}$$

The last formula can be simplified by

$$\left.\begin{aligned} \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} &= -24 \\ \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} &= -6\delta^{\mu}_{\nu} \\ \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\rho\sigma} &= -2(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}). \end{aligned}\right\} \tag{C.10}$$

The order of all γ matrices can be reversed,

$$\mathrm{tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\ldots) = \mathrm{tr}(\ldots\gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}). \tag{C.11}$$

Two γ matrices with similar indices dotted together can be reduced by

$$\gamma^{\mu}\gamma_{\mu} = g_{\mu\nu}\gamma^{\mu}\gamma^{\nu} = \frac{1}{2}g_{\mu\nu}\{\gamma^{\mu},\gamma^{\nu}\} = g_{\mu\nu}g^{\mu\nu} = 4. \tag{C.12}$$

In addition, several γ matrices dotted together and having the following form can be reduced by contraction identities, easily proved by using the anticommutation relations,

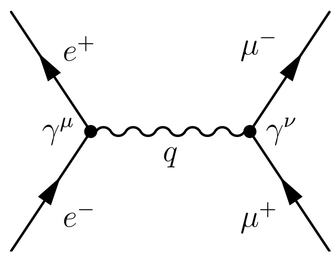
$$\left.\begin{aligned} \gamma^{\mu}\gamma^{\nu}\gamma_{\mu} &= -2\gamma^{\nu} \\ \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma_{\mu} &= -2g^{\nu\rho} \\ \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma_{\mu} &= -2\gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}. \end{aligned}\right\} \tag{C.13}$$

All these properties are important in the QED calculation of differential cross section.

APPENDIX D

REACTION OF $e^+e^- \rightarrow \mu^+\mu^-$

The $e^+e^- \rightarrow \mu^+\mu^-$ reaction has been the primary process to study a pair of quark-antiquark bound state. In this case, the calculation corresponding to charges is only treated by quantum electrodynamics. However, the final state of muon reaction is crucially different from the quark-antiquark final state in which there are not free quark observed so far. The transition amplitude is modified to avoid free a quark problem by coupling both quark and antiquark to its bound state, mesons. We want to show the whole calculation by starting form $e^+e^- \rightarrow \mu^+\mu^-$. A Feynman diagram of the reaction is



The transition amplitude is, from the feynman diagram,

$$
\begin{aligned}
T &= \bar{v}(p')(-ie\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k)(-ie\gamma^\nu)v(k') \\
&= \frac{ie^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big). \qquad \text{(D.1)}
\end{aligned}
$$

The factor e^2 will be modified later to Q^2 where Q is charges of quark. To find the differential cross section, we need to find a complex conjugate of T leading to $|T|^2$. The complex conjugate of T is

$$
T^* = \frac{ie^2}{q^2}\Big(\bar{u}(p)\gamma^\mu v(p')\Big)\Big(\bar{v}(k')\gamma_\mu u(k)\Big) \qquad \text{(D.2)}
$$

and

$$|T|^2 = \frac{e^4}{q^4}\Big(\bar{v}(p')\gamma^\mu u(p)\bar{u}(k)\gamma_\mu v(k')\Big)\Big(\bar{u}(p)\gamma^\mu v(p')\bar{v}(k')\gamma_\mu u(k)\Big) \tag{D.3}$$

where

$$(\bar{v}\gamma^\mu u)^* = u^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger v = u^\dagger(\gamma^\mu)^\dagger\gamma^0 v = u^\dagger\gamma^0\gamma^\mu v = \bar{u}\gamma^\mu v. \tag{D.4}$$

Since the beam of electron and position is unpolarized and the detector cannot distinguish the polarization, $|T|^2$ is averaged over spins of electron and positron and summed over muon spins, that is,

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'} |T(s,s'\to r,r')|^2. \tag{D.5}$$

By trace technology, (D.3) becomes

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|T|^2 &= \frac{16e^4}{q^4}\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\right)\\ &\quad\cdot\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_\mu^2)\right)\\ &= \frac{8e^4}{q^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\Big]\end{aligned} \tag{D.6}$$

where $m_e = 0$ because of high energy approximation or electron mass much fewer than muon mass. Let the reaction occur in the center of mass frame and all momenta, energies and mass be

$$\begin{aligned} p &= (E, E\hat{z}); & p' &= (E, -E\hat{z})\\ k &= (E, k); & k' &= (E, -k)\\ |k| &= \sqrt{E^2 - m_\mu^2}; & k\cdot\hat{z} &= |k|\cos\theta\\ q^2 &= (p+p')^2 = 4E^2; & p\cdot p' &= 2E^2\\ p\cdot k &= p'\cdot k' = E^2 - E|k|\cos\theta; & p\cdot k' &= p'\cdot k = E^2 + E|k|\cos\theta. \end{aligned}$$

Eq. (D.6) turns to

$$\frac{1}{4}\sum_{\text{spins}}|T|^2 = \frac{8e^4}{16E^4}\Big[E^2(E - |k|\cos\theta)^2 + E^2(E + |k|\cos\theta)^2 + 2m_\mu^2 E^2\Big]$$

$$= e^4 \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right]. \tag{D.7}$$

The differential cross section of two final states is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_a 2E_b |v_a - v_b|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{cm}} |T(p_a, p_b \to p_1 p_2)|^2 \tag{D.8}$$

where $E, |v_a - v_b|$, $\mathbf{p}$ are the energy of each initial particle, the relative velocity of the beams as viewed from the laboratory frame, and the momentum of final particle, respectively. In the case of $e^+e^- \to \mu^+\mu^-$, $E_a = E_b = \frac{E_{cm}}{2}$ and $|v_a - v_b| = 2c = 2, c = 1$, and $\mathbf{p}_1 = k$. The cross section of the reaction is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{2E_c m^2} \frac{|\mathbf{k}|}{16\pi^2 E_{cm}} \cdot \frac{1}{4} \sum_{\text{spins}} |T|^2 \\ &= \frac{\alpha^2}{4E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right] \end{aligned} \tag{D.9}$$

where $\alpha = \frac{e^2}{4\pi}$. Integrated over angular part, the cross section is

$$\sigma_{\text{total}} = \frac{4\pi\alpha^2}{3E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{m_\mu^2}{2E^2}\right). \tag{D.10}$$

APPENDIX E

WIGNER'S 9J SYMBOLS

In the calculation of four quarks system or two mesons, the selection of coupled pair of quarks corresponding to initial state has played a role. To help couple independently, Wigner's 9j symbols mainly used in the coupling of four angular momenta is employed. Suppose there are four angular momenta $\vec{J_i}$ with $i = 1, 2, 3, 4$ in different spaces, the eigenstates of the operators (J_i^2, J_{zi}) are $|l_i, m_i\rangle$. The direct product states

$$|j_1 j_2 j_3 j_4; m_1 m_2 m_3 m_4\rangle \equiv |j_1 m_1\rangle|j_2 m_2\rangle|j_3 m_3\rangle|j_4 m_4\rangle \tag{E.1}$$

are the eigenstates of operators (J_i^2, J_{zi}) and form a complete basis in the direct product space of dimension $(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)$ with the transformation according to the direct production representation

$$D(\vec{\lambda}) = D^{j_1}(\vec{\lambda}) \otimes D^{j_2}(\vec{\lambda}) \otimes D^{j_3}(\vec{\lambda}) \otimes D^{j_4}(\vec{\lambda}). \tag{E.2}$$

The operator operating on the eigenstates is

$$\vec{J} = \vec{J_1} + \vec{J_2} + \vec{J_3} + \vec{J_4}. \tag{E.3}$$

The eigenstate of the total angular momentum can be formed by different ways. We can couple the first and the second momenta to j_{12} and the third and the fourth momenta to j_{34}, then couple both together to the total angular momentum j,

$$|(j_1 \otimes j_2)_{j_{12}} \otimes (j_3 \otimes j_4)_{j_{34}}; jm\rangle. \tag{E.4}$$

Another ways is to couple the first and the third momenta to j_{13} and the second and the fourth momenta to j_{24}, and then combine together to the total angular momentum j,

$$|(j_1 \otimes j_3)_{j_{13}} \otimes (j_2 \otimes j_4)_{j_{24}}; jm\rangle. \tag{E.5}$$

The relation between above bases is

$$\begin{aligned} &|(j_1 \otimes j_2)_{j_{12}} \otimes (j_3 \otimes j_4)_{j_{34}}; jm\rangle \\ &\quad = \sum_{j_{13}j_{24}} \langle (j_1 j_3)_{j_{13}} (j_2 j_4)_{j_{24}}; jm | (j_1 j_2)_{j_{12}} (j_3 j_4)_{j_{34}}; jm\rangle \\ &\quad \cdot |(j_1 \otimes j_3)_{j_{13}} \otimes (j_2 \otimes j_4)_{j_{24}}; jm\rangle \end{aligned} \tag{E.6}$$

with

$$\begin{aligned} &\langle (j_1 j_3)_{j_{13}} (j_2 j_4)_{j_{24}}; jm | (j_1 j_2)_{j_{12}} (j_3 j_4)_{j_{34}}; jm\rangle \\ &= \sqrt{(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)} \cdot \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{array} \right\} \end{aligned} \tag{E.7}$$

where

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{array} \right\} \tag{E.8}$$

is called Wigner's $9j$ symbols. Here are some properties of the Wigner's $9j$ symbols

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{array} \right\} = (-1)^R \left\{ \begin{array}{ccc} j_3 & j_4 & j_{34} \\ j_1 & j_2 & j_{12} \\ j_{13} & j_{24} & j \end{array} \right\} = (-1)^R \left\{ \begin{array}{ccc} j_2 & j_1 & j_{12} \\ j_4 & j_3 & j_{34} \\ j_{24} & j_{13} & j \end{array} \right\} \tag{E.9}$$

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{array} \right\} = \sum_k (-1)^{2k} (2k+1)$$

$$\left\{\begin{matrix} j_1 & j_3 & j_{13} \\ j_{24} & j & k \end{matrix}\right\}\left\{\begin{matrix} j_2 & j_4 & j_{24} \\ j_3 & k & j_{34} \end{matrix}\right\}\left\{\begin{matrix} j_{12} & j_{34} & j \\ k & j_1 & j_2 \end{matrix}\right\} \quad \text{(E.10)}$$

$$\left\{\begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & 0 \end{matrix}\right\} = \delta j_{12} j_{34} \delta j_{13} j_{24} \frac{(-1)^{j_2+j_3+j_{12}+j_{13}}}{\sqrt{(2j_{12}+1)(2j_{13}+1)}}$$

$$\begin{pmatrix} j_1 & j_2 & j_{12} \\ j_4 & j_3 & j_{13} \end{pmatrix} \quad \text{(E.11)}$$

where $R = j_1 + j_2 + j_3 + j_4 + j_{12} + j_{34} + j_{13} + j_{24} + j$.

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