Relaxed control for a class of strongly nonlinear impulsive evolution equations*

Pairote Sattayatham
School of Mathematics, Suranaree University of Technology,
Nakhon Ratchasima 30000, Thailand
E-mail address: sattayatham@yahoo.com.

Abstract

Relaxed control for a class of strongly nonlinear impulsive evolution equations are investigated. Existence of solutions of strongly nonlinear impulsive evolution equations is proved and properties of original and relaxed trajectories are discussed. The existence of optimal relaxed control and relaxation results are also presented. For illustration, one example is given.

Keywords: Impulsive system, Banach space, nonlinear monotone operator, evolution triple, relaxation.

1. Introduction

In this paper, we present sufficient conditions of optimality for optimal relaxed control problems arising in systems governed by strongly nonlinear impulsive evolution equations on Banach spaces. The general descriptions of such systems were proposed in [1] as given below.

$$\begin{aligned} \dot{x}(t)+A(t,x(t)) &= g(t,x(t),u(t)) \;\; t \in I \setminus D, \\ x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i)),\; i=1,2,...,n, \end{aligned} \tag{1}$$

where $I \equiv (0,T)$ is a bounded open interval of the real line and let the set $D \equiv \{t_1, t_2, ..., t_n\}$ be a partition on $(0,T)$ such that $0 < t_1 < t_2 < ... < t_n < T$. In general, the operator A is a nonlinear monotone operator, g is a nonlinear nonmonotone perturbation, $\Delta x(t) \equiv x(t_i^+) - x(t_i^-) \equiv x(t_i^+) - x(t_i)$, $i = 1, 2, ..., n$, and $F_i's$ are nonlinear operators. This model includes all the standard models used by many authors in the field (see [2],[3]). The objective functional is given by $J(x,u) = \int_0^T L(t,x(t),u(t))dt$.

*This work was supported by Thailand Research Fund. Grant No. BRG 48 2005.

In a recent paper by the author [1], the existence of optimal control was proved, but sufficient conditions of relaxation for optimality were not addressed. We wish to present just that. Before, we can consider such problems, we need some preparation.The rest of the paper is organized as follows. In section 2, some basic notations and terminologies are presented. Section 3 contains some preparatory results. Relaxed impulsive systems are presented in section 4. Sufficient conditions of relaxation for optimality are disscussed in section 5. In section 6, we presented an example for illustration.

2. System description

Let V be a real reflexive Banach space with topological dual V^* and H be a real separable Hilbert space. Let $V \hookrightarrow H \hookrightarrow V^*$ be an evolution triple and the embedding $V \hookrightarrow H$ be compact.

The system model considered here is based on this evolution triple (see chapter 23 of [4]).

Let $\langle x, y\rangle$ denote the paring of an element $x \in V^*$ and an element $y \in V$. If $x, y \in H$, then $\langle x, y\rangle = (x, y)$, where (x, y) is the scalar product on H. The norm in any Banach space X will be denoted by $\|\cdot\|_X$.

Let $p, q \geq 1$ be such that $2 \leq p < +\infty$ and $1/p + 1/q = 1$ and let $I \equiv (0, T)$. For p, q satisfying the preceeding conditions, it follows from reflexivity of V that both $L_p(I, V)$ and $L_q(I, V^*)$ are reflexive Banach spaces. The pairing between $L_p(I, V)$ and $L_q(I, V^*)$ will be denoted by $\langle\langle \cdot \rangle\rangle$.
Define

$$W_{pq}(I) = \{x : x \in L_p(I, V), \dot{x} \in L_q(I, V^*)\},$$

with the norm

$$\| x \|_{W_{pq}(I)} = \| x \|_{L_p(L,V)} + \| \dot{x} \|_{L_q(L,V^*)}$$

where $\dot{x}$ denoted the derivative of x in the generalized sense. The space $(W_{pq}(I), \|\cdot\|_{W_{pq}(I)})$ becomes a Banach space which is clearly reflexive and separable and the embedding $W_{pq}(I) \hookrightarrow C(\bar{I}, H)$ is continuous. If the embedding $V \hookrightarrow H$ is compact, the embedding $W_{pq}(I) \hookrightarrow L_p(I, H)$ is also compact (see Problem 23.13(b) of [4]). Sometime, we write $W_{pq}(0,T)$ instead of $W_{pq}(I)$. Similarly, we can define $W_{pq}(s,t)$ for $0 \leq s < t \leq T$ and the space $(W_{pq}(s,t), \|\cdot\|_{W_{pq}(s,t)})$ is also a separable reflexive Banach space. Moreover, the embedding $W_{pq}(s,t) \hookrightarrow C([s,t], H)$ is continuous and the embedding $W_{pq}(s,t) \hookrightarrow L_p((s,t), H)$ is also compact. We define the set $PW_{pq}(0,T) = \{x : x\,|_{(t_i, t_{i+1})} \in W_{pq}(t_i, t_{i+1});\ i = 0, 1, 2, ..., n,\ t_0 = 0,\ t_{n+1} = T\}$. For each $x \in PW_{pq}(0,T)$, we define $||x||_{PW_{pq}(0,T)} = \sum_{i=0}^{n} ||x||_{W_{pq}(t_i, t_{i+1})}$. As a result, the space $(PW_{pq}(0,T), \|\cdot\|_{PW_{pq}(0,T)})$ becomes a Banach space. Let $PC([0,T], H) = \{x : x$ is a map from $[0,T]$ into H such that x is continuous at every point $t \neq t_i$, left continuous

Proof: See ([1], Theorem B) for the proof of existence and boundedness. The uniqueness follows from assumption (G)(2). To see this, suppose that system(2) has two solutions x_1, x_2. Then it follows from integration by part formula and monotonicity of $A(t,x)$ that

$$\begin{aligned}
\|x_1(t) - x_2(t)\|_H^2 &= 2\int_0^t \langle \dot{x}_1(s) - \dot{x}_2(s), x_1(s) - x_2(s)\rangle_{V^*-V} ds \\
&= -2\int_0^t \langle A(s, x_1(s)) - A(s, x_2(s)), x_1(s) - x_2(s)\rangle_{V^*-V}\, ds \\
&+ 2\int_0^t \langle g(s, x_1(s)) - g(s, x_2(s)),\, x_1(s) - x_2(s)\rangle_{V^*-V} ds \\
&\le 2\int_0^t \langle g(s, x_1(s)) - g(s, x_2(s)),\, x_1(s) - x_2(s)\rangle_{V^*-V} ds \\
&\le 2\int_0^t \| g(s, x_1(s)) - g(s, x_2(s)) \|_{V^*} \| x_1(s) - x_2(s) \|_V \, ds \\
&\le \; 2L(\rho)\int_0^t \| x_1(s) - x_2(s) \|_H \| x_1(s) - x_2(s) \|_V \, ds \\
&\le 2L(\rho)C_1 \int_0^t \| x_1(s) - x_2(s) \|_H^2 \, ds,
\end{aligned}$$

for some positive constant C_1. By Gronwall's lemma, we get $x_1(t) = x_2(t)$ for all $t \in [0,T]$. Hence $x_1 = x_2$ and this prove the uniqueness of the solution of system (2).

Now, let us consider the corresponding control system. We model the control space by a separable complete metric space Z (i.e., a Polish space). By P_f (P_{fc}),we denote a class of nonempty closed (closed and convex) subset of Z. Let $I = (0,T)$. Recall that a multifunction $\Gamma : I \to P_f(Z)$ is said to be measurable if for each $F \in P_f Z$, $\Gamma^{-1}(F)$ is Lebesque measurable in I. We defined S_Γ to be the set of all measurable selections of $\Gamma(\cdot)$, i.e.,

$$S_\Gamma = \{u : I \to Z \mid u(t) \text{ is measurable and } u(t) \in \Gamma(t),\ \mu\text{-a.e. } t \in I\},$$

where μ is the Lebesgue measure on I. Note that the set $S_\Gamma \neq \phi$ if $\Gamma : I \to P_f(Z)$ is measurable (see [5], Theorem 2.23, p.100). Consider the following control systems

$$\begin{aligned}
\dot{x}(t) + A(t, x(t)) &= g(t, x(t), u(t)) \;\; t \in I, \\
x(0) &= x_0 \in H, \\
\Delta x(t_i) &= F_i(x(t_i)),\; i = 1, 2, ..., n \; (0 < t_1 < t_2 < ... < t_n < T).
\end{aligned} \tag{5}$$

Here, we require the operator A, and $F_i's$ of (5) satisfy hypothesis (A), and (F), respectively as in section 3. We now give some new hypotheses for the remaining data.

(U) $U : I \to P_{fc}(Z)$ is a measurable multifunction satisfying $U(\cdot) \subset F$, where F is a compact subset of Z. For the admissible controls, we choose the set

$U_{ad} = S_U$.

(G1) $g : I \times H \times Z \to V^*$ is an operator such that

(1) $t \mapsto g(t,x,z)$ is measurable, and the map $(x,z) \longmapsto g(t,x,z)$ is continuous on $H \times Z$.

(2) For each fixed z, $g(t,x,z)$ is locally Lipschitz continuous with respect to x and uniformly in t.

(3) There exist constants $a, b > 0$ such that

$$\| g(t,x,z) \|_{V^*} \leq a + b \| x \|_H^{k-1}$$

for all $x \in H$, $t \in (0,T)$, and $z \in Z$, where $2 \leq k < p$.

By assumption (U), the control set S_U is nonempty and is called the class of original control. Now, let us define

$X_0 = \{x \in PW_{pq}(I) \cap PC(\bar{I}, H) \mid x$ is a solution of (5) corresponding to $u\}$.

X_0 is called the class of original trajectories.

$A_0 = \{(x,u) \in PW_{pq}(I) \cap PC(\bar{I}, H) \times S_U \mid x$ is a solution of (5) corresponding to $u\}$.

A_0 is called the class of admissible state-control pairs.

The following theorem guarantee that $X_0 \neq \phi$. Its proof follows immediately from Theorem 3.1 by defining the function $g_u(t,x) = g(t,x,u)$ and noting that g_u satisfies all hypotheses of Theorem 3.1.

Theorem 3.2 Assume that hypotheses (A),(F), (G1) and (U) hold. For every $u \in S_U$, equation (5) has a unique solution $x(u) \in PW_{pq}(I) \cap PC(\bar{I}, H)$. Moreover the set X_0 is bounded in $PW_{pq}(I) \cap PC(\bar{I}, H)$, *i.e.*, $\| x(u) \|_{PW_{pq}(0,T)} \leq M$ and $\| x(u) \|_{PC([0,T],H)} \leq M$ for all $u \in S_U$.

4. Relaxed impulsive systems

We consider the following optimal control problem

$$\text{(P)} \qquad \inf\{J(x,u) = \int_0^T L(t, x(t), u(t))dt\}$$

subject to equation (5).

It is well known that, to solve optimization problem involving (P) and obtain an optimal state-control pair, we need some kind of convexity hypothesis on the orientor field $L(t, x(t), u(t))$. If convexity hypothesis is no longer satisfied. In order to get an optimal admissible pair, we need to pass to a larger systems with measure control (or know as "relaxed control") in which the orientor field have been convexified. For this purpose, we introduce the relaxed control and the corresponding relaxed systems.

Let Z be a separable complete metric space (i.e. a Polish space) and $B(Z)$ be its Borel σ-field. Let (Ω, Σ, μ) be a measure space. We will denote the space of probability measures on the measurable space $(Z, B(Z))$ by $M_+^1(Z)$.

A Caratheodory integrand on $\Omega \times Z$ is a function $f : \Omega \times Z \to R$ such that $f(\cdot, x)$ is Σ-measurable on Ω, $f(\omega, \cdot)$ is continuous on Z for all $\omega \in \Omega$, and $\sup\{| f(\omega, z) |: z \in Z\} \leq \alpha(\omega)$ *a.e.*, for some functions $\alpha(\cdot) \in L_1(\Omega)$. We denote the set of all Caratheodory integrands on $\Omega \times Z$ by $Car(\Omega, Z)$.

By a transition probability, we mean a function $\lambda : \Omega \times B(Z) \to [0,1]$ such that for every $A \in B(Z), \lambda(\cdot, A)$ is Σ-measurable and for every $\omega \in \Omega$, $\lambda(\omega, \cdot) \in M_+^1(Z)$. We use $R(\Omega, Z)$ to denote the set of all transition probability from (Ω, Σ) into $(Z, B(Z))$. Following Balder [6], we can define a topology on $R(\Omega, Z)$ as follows : Let $f \in Car(\Omega, Z)$ and define

$$I_f(\lambda) = \int_\Omega \int_Z f(\omega, z)\lambda(\omega)(dz)d\mu(\omega). \tag{6}$$

The weak topology on $R(\Omega, Z)$ is defined as the weakest topology for which all functionals $I_f : R(\Omega, Z) \to R$, $f \in Car(\Omega, Z)$, are continuous.

Suppose $\Omega = I = [0,T]$ and Z is a compact Polish space, then the space $\mathrm{Car}(I, Z)$ can be identified with the separable Banach space $L_1(I, C(Z))$ where $C(Z)$ is the space of all real valued continuous functions on Z. To see this, we associate to each Caratheodory integrand $\phi(\cdot,\cdot)$ the map $t \longmapsto \phi(t, \cdot) \in C(Z)$. Let $M(Z)$ be the space of all regular bounded countably additive measure defined on $B(Z)$. We note that $M(Z)$ is a Banach space under the total variation norm, i.e., $\| \lambda \|_{M(Z)} = | \lambda | (Z)$. Then by Riesz representation theorem, the dual $[C(Z)]^*$ can be identified algebraically and metrically with $M(Z)$. The duality pair between $M(Z)$ and $C(Z)$ is given by

$$\langle \lambda, f \rangle = \int_Z f(z)\lambda(dz).$$

So $M(Z)$ is a separable (see([7], p.265) dual Banach space and hence has a Radon-Nikodym property. This observation combined with Theorem 1 of Diestel and Uhr ([8], p.98), tell us that

$$L_1(I, C(Z))^* = L_\infty(I, M(Z)). \tag{7}$$

So the weak topology on $R(I, Z)$ coincides with the relative $w^*(L_\infty(I, M(Z))), L_1(I, C(Z))$-topology. The duality pair between $L_\infty(I, M(Z))$ and $L_1(I, C(Z))$ is given by

$$\begin{aligned} \langle\langle \lambda, f \rangle\rangle &= \int_0^T \langle \lambda(t), f(t) \rangle dt \\ &= \int_0^T \int_Z f(t)(z)\lambda(t)(dz)dt \\ &= \int_0^T \int_Z f(t,z)\lambda(t)(dz)dt, \end{aligned} \tag{8}$$

which is the same formula as in (6) with $f(t, z) \equiv f(t)(z)$. This fact will be useful in the study of the relaxed control system where the control functions are transition probability.

Now we introduce some assumptions imposed on the class of relaxed control which will be denoted by S_Σ.

(U1) Z is a compact Polish space, $U : I \to P_{fc}(Z)$ is a measurable multifunction.

Define $\Sigma(t) = \{\lambda \in M_+^1(Z),\ \lambda(U(t)) = 1\}$ and let $S_\Sigma \subseteq R(I, Z)$ be the set of transition probabilities on $I \times B(Z)$ that are measurable selections of $\Sigma(\cdot)$. For any $u \in S_U$, we define the relaxation $\delta_u \in S_\Sigma$ of u by $\delta_u(t) \equiv$ Dirac probability measure at $u(t)$. Then we can identify $S_U \subseteq S_\Sigma$. From now on, we shall consider S_U and S_Σ as a subspace of the topological space $R(I, Z)$ with the weak topology defined above.

We list two lemmas which will be useful in discussing relaxation problem. The proofs can be found in J. Warga ([7], Theorem IV 2.1) and Balder ([6], Corollary 3) respectively.

Lemma 4.1 Suppose Z is a compact Polish space. Then S_Σ is convex, compact, and sequentially compact.

Lemma 4.2 S_U is dense in S_Σ.

Theorem 4.3 Let $h : I \times H \times Z \to R$ be such that

1. $t \longmapsto (t, x, z)$ is measurable and $(x, z) \longmapsto h(t, x, z)$ is continuous.
2. $| h(t, x, z) | \le \psi(t) \in L_1(I)$ for all $(x, z) \in H \times Z$.

   If $x_k \to x \in C([0, T], H)$ then

$$\bar{h}_k(\cdot, \cdot) \to \bar{h}(\cdot, \cdot) \text{ in } L_1(I, C(Z))$$

as $k \to \infty$, where $\bar{h}_k(t, z) = h(t, x_k(t), z)$ and $\bar{h}(t, z) = h(t, x(t), z)$.

Proof : The proof is similar to Lemma 3.3 in [3].

Next, let us consider this new larger system know as "relaxed impulsive system"

$$\begin{aligned} \dot{x}(t) + A(t, x(t)) &= \textstyle\int_Z g(t, x(t), z)\lambda(t)(dz),\ 0 \le t \le T, \\ x(0) &= x_0, \\ \Delta x(t_i) &= F_i(x(t_i)), \quad i = 1, 2, ..., n. \end{aligned} \tag{9}$$

We will denote the set of trajectories of (9) by X_r, i.e.,
$X_r = \{x \in PW_{pq}(I) \cap PC(\bar{I}, H) \mid x \text{ is a solution of (9) corresponding to } \lambda \in S_\Sigma\}$.
Moreover, the set of admissible state-control pairs of (9) will be denoted by
$A_r = \{(x, \lambda) \in PW_{pq}(I) \cap PC(\bar{I}, H) \times S_\Sigma \mid x$ is a solution of (9) corresponding to $\lambda \in S_\Sigma\}$.
Note that $X_0 \subseteq X_r$, since $S_U \subseteq S_\Sigma$ and if the hypotheses of Theorem 3.2 are satisfied, $X_0 \ne \phi \Rightarrow X_r \ne \phi$. To see this, given any relaxed control $\lambda \in S_\Sigma$, if we set $\bar{g}(t, x(t), \lambda(t)) = \int_Z g(t, x(t), z)\lambda(t)(dz)$ then, working as in the proof of Theorem 3.2, one can show that there exists a relaxed admissible trajectory $x(\lambda)$ corresponding to λ. We summarize the above discussion into a theorem.

Theorem 4.4 Assume that hypotheses (A), (F), (G1) and (U1) hold. For every $\lambda \in S_\Sigma$, equation (9) has a unique solution $x(\lambda) \in PW_{pq}(I) \cap PC(\bar{I}, H)$. Moreover the set X_r is bounded in $PW_{pq}(I) \cap PC(\bar{I}, H)$, *i.e.*, $\| x(\lambda) \|_{PW_{pq}(0,T)} \leq M$ and $\| x(\lambda) \|_{PC([0,T],H)} \leq M$ for all $\lambda \in S_\Sigma$.

The next theorem give us a useful relation betweeen X_0 and X_r.

Theorem 4.5 If assumption (A), (F), (G1) and (U1) hold, then $X_r = \overline{X_0}$ (closure is taken in $PC(\bar{I}, H)$).

Before proving this theorem, we need a lemma.

Lemma 4.6 If assumption (A), (F), (G1) and (U1) hold and $\lambda_k \to \lambda$ in $R(I, Z)$. Suppose that $\{x_k, x\}$ is the solution of (9) corresponding to $\{\lambda_k, \lambda\}$, by working with a subsequence if necessary, $x_k \to x$ in $PC(\bar{I}, H)$ as $k \to \infty$.

Proof. Suppose that $\lambda_k \to \lambda$ in $R(I, Z)$ as $k \to \infty$ and $\{x_k, x\}$ is the solution of (9) corresponding to $\{\lambda_k, \lambda\}$. Since $(x_k, \lambda_k) \in A_r$ for each positive integer k, then (x_k, u_k) must satisfy the equation

$$\begin{aligned}
&\dot{x}_k(t) + A(t, x_k(t)) = \textstyle\int_Z g(t, x_k(t), z)\lambda_k(t)(dz), \\
&x_k(0) = x_0 \in H, \\
&\Delta x_k(t_i) = F_i(x_k(t_i)), \ i = 1, 2, ..., n \ (0 < t_1 < t_2 < ... < t_n < T).
\end{aligned} \tag{10}$$

To finish the proof, we try to choose $y \in X_r$ such that y is a solution of (9) correspond to this λ and $x_k \to y$ in $PC(\bar{I}, H)$ as $k \to \infty$. The uniqueness property of the solution of (9) implies $x = y$ and hence $x_k \to x$ in $PC(\bar{I}, H)$. This prove that $x \in \overline{X_0}$ and we are done. We shall do this by considering in each case separately.

Case1. Find y on the interval $(0, t_1)$. For notational convenience, we let $I_1 = (0, t_1)$, $X_1 = L_p(I_1, V)$ and $X_1^* = L_q(I_1, V^*)$. We note that $X_1 = L_p(I_1, V)$ can be consider as a closed subspace of $X = L_p(I, V)$. Let x_k^1 and λ_k^1 be the restriction of the functions x_k, λ_k on the interval I_1 $(k = 1, 2, ...)$. Hence, by Theorem 4.4, $\{x_k^1\}$ is bounded in $W_{pq}(I_1)$. By reflexivity of $W_{pq}(I_1)$ there is a subsequence of $\{x_k^1\}$, again denoted by $\{x_k^1\}$, such that

$$x_k^1 \xrightarrow{w} x^1 \ \text{ in } \ W_{pq}(I_1) \ \text{ as } k \to \infty. \tag{11}$$

Since the embedding $W_{pq}(I_1) \hookrightarrow X_1$ is continuous, the embedding $W_{pq}(I_1) \hookrightarrow L_p(I_1, H)$ is compact and the operator $A : X_1 \to X_1^*$ maps bounded sets to bounded sets, it follows from (11) that there exists a subsequence of $\{x_k^1\}$, again denoted by $\{x_k^1\}$ such that

$$\begin{aligned}
&x_k^1 \xrightarrow{w} x^1 \ in \ X_1, \ \ \dot{x}_k^1 \xrightarrow{w} \dot{x}^1 \ in \ X_1^*, \\
&Ax_k^1 \xrightarrow{w} \xi \ in \ X_1^*, \\
&x_k^1 \xrightarrow{s} x^1 \ in \ L_p(I_1, H), \ and, \text{ by ([3], theorem 3.B.)}, \ x_k^1 \xrightarrow{s} x^1 \ in \ C([0, t_1], H),
\end{aligned} \tag{12}$$

as $k \to \infty$ and for some $\xi \in X_1^*$. Consider the following equation,

$$\begin{aligned} \dot{x}_k^1(t) + A(t, x_k^1(t)) &= \textstyle\int_Z g(t, x_k^1(t), z)\lambda_k^1(t)(dz); 0 \le t < t_1 \\ x_k^1(0) &= x_0. \end{aligned} \tag{13}$$

Define an operator $G_k : I_1 \to V^*$ and $G : I_1 \longrightarrow V^*$ as follow

$$G_k(t) = \textstyle\int_Z g(t, x_k^1(t), z)\lambda_k^1(t)(dz) \qquad (k = 1, 2, 3, ...)$$

$$G(t) \ = \textstyle\int_Z g(t, x^1(t), z)\lambda^1(t)(dz).$$

It follows from assumption (G1) that G and $G_k \in L_q(I_1, V^*)$

With this new notation, equation (13) can be rewritten into an equivalent operator equation of the form

$$\begin{aligned} \dot{x}_k^1 + A(x_k^1) &= G_k \qquad ; 0 < t < t_1 \\ x_k^1(0) &= x_0. \end{aligned} \tag{14}$$

For each fixed $v \in V$, define

$$\bar{g}_k(t, z) = \langle g(t, x_k^1(t), z), v \rangle_{V^*-V}$$

$$\bar{g}(t, z) = \langle g(t, x^1(t), z), v \rangle_{V^*-V}$$

It follow from assumption (G1) that, for each fixed $t \in I_1$,

$\bar{g}_k(t, \cdot)$, and $\bar{g}(t, \cdot) \in C(Z)$

and furthermore

$\bar{g}_k(\cdot, \cdot)$, and $\bar{g}(\cdot, \cdot) \in L_1(I_1, C(Z))$.

Since $x_k^1 \xrightarrow{s} x^1$ in $C([0, t_1], H)$ (see equation (12)), then Theorem 4.3 gives

$\bar{g}_k(\cdot, \cdot) \longrightarrow \bar{g}(\cdot, \cdot)$ in $L_1(I_1, C(Z))$ as $k \to \infty$.

Since $\lambda_k^1 \to \lambda^1$ in $R(I_1, Z)$, by equation (7), we have $\lambda_k^1 \xrightarrow{w^*} \lambda^1$ in $(L_1(I_1, C(Z)))^*$ as $k \to \infty$. Hence, it follows from Proposition 21.6(e) of Ziedler ([4],page 216) that

$\langle\langle \lambda_k^1, \bar{g}_k \rangle\rangle \to \langle\langle \lambda^1, \bar{g} \rangle\rangle$ as $k \to \infty$.

This means that

$$\int_0^{t_1}\int_Z\langle g(t,x_k^1(t),z),v\rangle\lambda_k^1(t)(dz)dt \to \int_0^{t_1}\int_Z\langle g(t,x^1(t),z),v\rangle\lambda^1(t)(dz)dt \qquad (15)$$

as $k\to\infty$. The convergence in (15) is true for all $v\in V$ then we get

$$G_k \xrightarrow{w} G \text{ as } k\to\infty \text{ in } L_q(I_1,V^*).$$

By equation (12), $x_k^1 \xrightarrow{s} x^1$ in $C([0,t_1],H)$ and this implies $x_k^1(0)\to x^1(0)$ in H as $k\to\infty$. Referring to the initial condition ,we have $x_k^1(0)=x_0\in H$ for all $k=1,2,3,...$ Thus $x^1(0)=x_0$.

Up to this point, we can conclude that x^1satisfies the following equation

$$\begin{aligned}\dot{x}^1(t)+\xi(t) &= \int_Z g(t,x^1(t),z)\lambda(t)(dz)\\ x^1(0) &= x_0\in H\end{aligned}$$

Next we aim to prove that $\xi = Ax^1$ in X_1^*.
To prove this we note from equation(14) that

$$\begin{aligned}\langle\langle A(x_k^1),x_k^1\rangle\rangle_{X_1^*-X_1} &= \langle\langle A(x_k^1),x^1\rangle\rangle_{X_1^*-X_1} - \langle\langle \dot{x}_k^1, x_k^1-x^1\rangle\rangle_{X_1^*-X_1}\\ &\quad +\langle\langle G_k, x_k^1-x^1\rangle\rangle_{X_1^*-X_1} \qquad (16)\end{aligned}$$

From integration by part formula, we have

$$\begin{aligned}\langle\langle \dot{x}_k^1, x_k^1-x^1\rangle\rangle_{X_1^*-X_1} &= \langle\langle \dot{x}^1, x_k^1-x^1\rangle\rangle_{X_1^*-X_1} + \tfrac{1}{2}(\| x_k^1(t_1)-x^1(t_1)\|_H^2\\ &\quad -\| x_k^1(0)-x^1(0)\|_H^2) \qquad (17)\end{aligned}$$

Substitute (17) into (16) and noting that the second term on the right hand side of (17) is always nonnegative, then we get

$$\begin{aligned}\langle\langle A(x_k^1),x_k^1\rangle\rangle_{X_1^*-X_1} &\le \langle\langle A(x_k^1),x^1\rangle\rangle_{X_1^*-X_1} - \langle\langle \dot{x}^1, x_k^1-x^1\rangle\rangle_{X_1^*-X_1}\\ &\quad +\| x_k^1(0)-x^1(0)\|_H^2 + \langle\langle G_k, x_k^1-x^1\rangle\rangle_{X_1^*-X_1}\end{aligned}$$

Therefore

$$\lim_{k\to\infty}\langle\langle A(x_k^1),x_k^1\rangle\rangle_{X_1^*-X_1} \le \langle\langle \xi, x^1\rangle\rangle_{X_1^*-X_1},$$

and hence A satisfies condition (M) (see [4] page 474). Then we have

$$A(x^1)=\xi.$$

Now we can say that x^1 is the solution of the following equation

$$\dot{x}^1(t) + A(t, x^1(t)) = \int_Z g(t, x^1(t), z)\lambda(t)(dz)$$
$$x^1(0) = x_0 \in H$$

This proves that x^1 satisfies (9) on the interval $(0, t_1)$ and x^1 is the required y on $(0, t_1)$.

Case 2 : Find y on the interval (t_1, t_2).

The proof is similar to case 1. Here, let $I_2 = (t_1, t_2)$, $X_2 = L_p(I, V)$ and $X_2^* = L_q(I, V^*)$. Let x_k^2, u_k^2 be the restriction of the functions x_k and u_k on the interval I_2 respectively $(k = 1, 2, ...)$. It follows from equation (10) that (x_k^2, u_k^2) satisfies the operator equation

$$\dot{x}_k^2 + A(x_k^2) = G_k \qquad ; t_1 < t < t_2$$
$$x_k^2(t_1^+) = x_k^2(t_1^-) + F_1(x_k^2(t_1)) \tag{18}$$

where $x_k^2(t_1^-) = x_k^2(t_1) = x_k^1(t_1)$ $(k = 1, 2, 3, ...)$. By using the same proof as in case 1, we get that

$$x_k^2 \xrightarrow{w} x^2 \text{ in } W_{pq}(t_1, t_2) \text{ and } x_k^2 \xrightarrow{s} x^2 \text{ in } C([t_1, t_2], H) \text{ as } k \to \infty$$

which implies that $x_k^2(t_1^+) \to x^2(t_1^+)$ in H as $k \to \infty$ and, moreover, x_2 is also satisfied the operator equation

$$\dot{x}^2 + A(x^2) = G \qquad ; t_1 < t < t_2.$$

We are left to verify the initial condition at t_1.To see this, we note that the expression on the right hand side of (18) converges to $x^1(t_1) + F_1(x^1(t_1))$ as $k \to \infty$ (see hypothesis (F)). On the other hand, the left hand side $x_k^2(t_1^+) \to x^2(t_1^+)$ in H as $k \to \infty$. Hence, $x^2(t_1^+) = x^1(t_1) + F_1(x^1(t_1)) \equiv x^2(t_1^-) + F_1(x^2(t_1))$. This proves that x^2 satisfies (9) on the interval (t_1, t_2) and x^2 is the required y on (t_1, t_2). Continue this process we can find y on the interval (t_j, t_{j+1}) ,$j = 0, 1, ..., n$. By piecing them together from $j = 1, 2, ..., n$ and taking into account the impact of jump, we obtain y which is the solution of (9) corresponding to the relaxed control λ satisfying $x_k \to y$ in $PC(\bar{I}, H)$ as $k \to \infty$. Since $x = y$, $x_k \to x$ in $PC(\bar{I}, H)$ as $k \to \infty$. The proof of Lemma 4.6 is now complete.

Proof of Theorem 4.5 Firstly, we shall show that $X_r \subseteq \overline{X_0}$. Let $x \in X_r$, then there exists $\lambda \in S_\Sigma$ such that $(x, \lambda) \in A_r$. By virtue of density result Lemma 4.2, there exists a sequence $\{u_k\} \in S_U$ such that $\delta_{u_k} \to \lambda$ in $R(I, Z)$. Let x_k be the solution of (9) corresponding to u_k. Then we have a sequence $\{(x_k, u_k)\} \subset A_0$. Since $(x_k, u_k) \in A_0$ for each positive integer k, then (x_k, u_k) must satisfy the equation

$$\begin{aligned}\dot{x}_k(t) + A(t, x_k(t)) &= \int_Z g(t, x_k(t), z)\delta_{u_k}(t)(dz)\\ x_k(0) &= x_0 \in H\\ \Delta x_k(t_i) &= F_i(x_k(t_i)),\ i = 1, 2, ..., n,\ k = 1, 2, 3, ...,\\ &(0 < t_1 < t_2 < ... < t_n < T).\end{aligned}$$

Apply Lemma 4.6, we get $x_k \to x$ in $PC(\bar{I}, H)$. This prove that $x \in \overline{X_0}$ and hence $X_r \subseteq \overline{X_0}$. Finally, we will show that X_r is closed in $PC(\bar{I}, H)$. Let $\{x_k\}$ be a sequence of points in X_r such that $x_k \to x$ in $PC(\bar{I}, H)$ as $k \to \infty$. By definition of X_r, there is a sequence $\{\lambda_k\}$ of points in S_Σ such that $(x_k, \lambda_k) \in A_r$, $k = 1, 2, 3, ...$ By Lemma 4.1, S_Σ is compact in $R(I, Z)$ under the weak topology. Moreover, $R(I, Z)-$topology coincides with the relative $w^*(L_\infty(I, M(Z)), L_1(I, C(Z)))$-topology which is metrizable (see [2] page 276). Then, by passing to a subsequence if necessary, we may assume that $\lambda_k \to \lambda$ in $R(I, Z)$. Apply Lemma 4.6, there is $x \in X_r$ such that $x_k \to x$ in $PC(\bar{I}, H)$ as $k \to \infty$. Hence X_r is closed in $PC(\bar{I}, H)$ and, consequently, $\overline{X_0} \subseteq \overline{X_r} = X_r$. The proof of Theorem 4.5 is now complete.

The following corollary is an immediate consequence of Lemma 4.6

Corollary 4.7 Under assumption of Theorem 4.5, the function $\lambda \longmapsto x(\lambda)$ is continuous from $S_\Sigma \subseteq R(I, Z)$ into $PC(\bar{I}, H)$.

5. Existence of optimal controls

Consider the following Lagrange optimal control problem (P_r): Find a control policy $\bar{\lambda} \in S_\Sigma$, such that it imparts a minimum to the cost functional J given by

$$(P_r) \qquad J(\lambda) \equiv J(x^\lambda, \lambda) \equiv \int_I \int_Z l(t, x^\lambda(t)), z)\lambda(t)(dz)dt$$

where x^λ is the solution of the system (9) corresponding to the control $\lambda \in S_\Sigma$. We make the following hypothesis concerning the integrand $l(.,.,.)$.

(L) $l : I \times H \times Z \to R \cup \{+\infty\}$ is Borel measurable satisfying the following conditions:

(1) $(\xi, z) \longmapsto l(t, \xi, z)$ is lower semicontinuous on $H \times Z$ for each fixed t.
(2) $\psi(t) \leq l(t, \xi, z)$ almost everywhere with $\psi(t) \in L_1(I)$.

Let $m_r = \inf\{J(\lambda) : \lambda \in S_\Sigma\}$. We have the following theorem on the existence of optimal impulsive control.

Theorem 5.1 Suppose assumptions (A), (F), (G1), (U1), (L) hold and Z is compact Polish space, then there exists $(\bar{x}, \bar{\lambda}) \in A_r$ such that $J(\bar{x}, \bar{\lambda}) = m_r$.

Proof. If $J(\lambda) = +\infty$ for all $\lambda \in S_\Sigma$, then every control is admissible. Assume $\inf\{J(\lambda) : \lambda \in S_\Sigma\} = m_r < +\infty$. By assumption (L), we have $m_r > -\infty$. Hence m_r is finite. Let $\{\lambda_k\}$ be a minimizing sequence so that $\lim_{k\to\infty} J(\lambda_k) = m_r$ By Lemma 4.1, S_Σ is compact in the topology $R(I, Z)$. Hence, by passing to a subsequence if neccesary, we may assume that $\lambda_k \to \bar{\lambda}$ in $R(I, Z)$ as $k \to \infty$. This means that $\lambda_k \overset{w^*}{\to} \bar{\lambda}$ in $L_\infty(I, M(Z))$ as $k \to \infty$. Let $\{x_k, \bar{x}\}$ be the solution of (9) correspond to $\{\lambda_k, \bar{\lambda}\}$. By Lemma 4.6, we get $x_k \to \bar{x}$ in $PC(I, H)$ and $(\bar{x}, \bar{\lambda}) \in A_r$. Next, we shall prove that $(\bar{x}, \bar{\lambda})$ is an optimal pair.

As before, we identify the space of Caratheodory integrand $Car(I, Z)$ with the separable Banach space $L_1(I, C(Z))$. We note that every semicontinuous measurable integrand $l : I \times H \times Z \to R \cup \{+\infty\}$ is the limit of an increasing sequence of Caratheodory integrand $\{l_j\} \in L_1(I, C(Z))$ for each fixed $h \in H$. Thus, there exists an increasing sequence of Caratheodory integrand $\{l_j\} \in L_1(I, C(Z))$ such that

$$l_j(t, \bar{x}(t), z) \uparrow l(t, \bar{x}(t), z) \text{ as } j \to \infty \quad \text{for all } t \in I, z \in Z.$$

Since $x_k \to \bar{x}$ in $PC(\bar{I}, H)$, by applying Theorem 4.3 on each subinterval of $[0, T]$, $l_j(t, x_k(t), z) \to l_j(t, x(t), z)$ as $k \to \infty$ for almost all $t \in I$ and all $z \in Z$. We note that since $\lambda_k \overset{w^*}{\to} \bar{\lambda}$ in $L_\infty(I, M(Z))$ as $k \to \infty$, then

$$\begin{aligned} J(\bar{x}, \bar{\lambda}) = \langle\langle \bar{\lambda}, l \rangle\rangle &= \int_I \int_Z l(t, \bar{x}(t), z)\bar{\lambda}(t)(dz)dt \\ &= \lim_{j\to\infty} \int_I \int_Z l_j(t, \bar{x}(t), z)\bar{\lambda}(t)(dz)dt \\ &= \lim_{j\to\infty} \lim_{k\to\infty} \int_I \int_Z l_j(t, x_k(t), z)\lambda_k(t)(dz)dt \\ &\le \lim_{k\to\infty} \lim_{j\to\infty} \int_I \int_Z l_j(t, x_k(t), z)\lambda_k(t)(dz)dt \ = m_r. \end{aligned}$$

However, by definition of m_r, it is obvious that $J(\bar{x}, \bar{\lambda}) \ge m_r$. Hence $J(\bar{x}, \bar{\lambda}) = m_r$. This implies that $(\bar{x}, \bar{\lambda})$ is an optimal pair.

Remark. If $J_0(x, u) = \int_I l(t, x(t), u(t))dt$ is the cost functional for the original problem and $m = \inf\{J_0(x, u) : u \in U_{ad}\}$. In general we have $m_r \le m$. it is desirable that $m_r = m$, i.e., our relaxation is resonable. With some stronger conditions on l, i.e., the map $(\xi, \eta, z) \mapsto l(t, \xi, z)$ is continuous and $| l(t, \xi, z) | \le \theta_R(t)$ for all most all $t \in I$ and $\theta_R \in L_1(I)$, one can show that $m_r = m$. The proof is similar to Theorem 4.B. in [3].

6. Example

In this section we present an example of strongly nonlinear impulsive system for which our general theory can be applied. Let $I = (0, T)$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^1 boundary $\partial\Omega$. For $p \ge 2$ and $\theta \ge 0$, we consider the following quasi-linear parabolic control problem

It is clear that $U : I \to P_{fc}(Z)$ is measurable. The set of admissible controls U_{ad} is chosen as $U_{ad} \equiv S_U \equiv \{u : I \to R^N \mid u$ is measurable and $u(t) \in U(t)$ a.e. $t \in [0,T]\}$. Hence the multifunction U satisfies (U1).

Next, For $t \in I, \phi \in H, w \in Z$ define a function $b^w : I \times H \times V \to R$ by $b^w(t,\phi,\psi) = \int_\Omega f(t,z,\phi,w)\psi(z)dz$. Then, the map $\psi \longmapsto b^w(t,\phi,\psi)$ is bounded on V and hence is a continuous linear form on V. Thus there exists an operator $g : I \times H \times Z \to V^*$ such that

$$b^w(t,\phi,\psi) = \langle g(t,\phi,w),\psi\rangle_{V^*-V} . \tag{21}$$

By using hypothesis (G'), we obtain that g satisfies hypothesis (G1) of Section 3.

Using the operator A and g as defined in equation (20) and (21), one can rewrite system (19) in an abstract form as in (9). So, apply Theorem 4.4, system (19) has a solution.

Acknowledgements

I would like to thank prof. X.L. Xiang for her valuable suggestions and comments.

References

[1] P. Sattayatham, Strongly nonlinear impulsive evolution equations and optimal control, Nonlinear Analysis, 52(2004) 1005-1020.

[2] N.S. Papageorgiou, Properties of the relaxed trajectories of evolution equations and optimal control, SIAM J. control Optim., 27(1989) 267-288.

[3] X. Xiang, P. Sattayatham, and Wei Wei, Relaxed controls for a class of strongly nonlinear delay evolution equations, Nonlinear Analysis 52(2003) 703-723.

[4] E. Zeidler, Nonlinear Functional Analysis and its applications, Vol I,II, Springer, New York 1990.

[5] X.Li and J.Yong, Optimal Control Theory for Infinite Dimensional Systems, Birkhauser, Boston, 1995.

[6] E. Balder, A general denseness result for relaxed control theory, Bull. Austral. Math. Soc. 20(1984) 463-475.

[7] J.Warga, Optimal control of differential and functional equations, Academic press, New York, 1972.

[8] J. Diestel and J.J. Uhr, Vector measures, American Mathematical Society, 1997.