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Physica B 269 (1999) 115–122

PHYSICA B

Čerenkov radiation in discontinuous media: a quantum view-point

E.B. Manoukian*, P. Bantitadawit

School of Physics, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand

Received 12 November 1998; accepted 6 January 1999

Abstract

The photon propagator in the Čerenkov radiation off a charged particle moving in a dielectric slab immersed within another dielectric medium is derived. From the vacuum-to-vacuum transition amplitude, an explicit expression is obtained for the photon number density of given frequency of photons radiated per unit path length of the particle. In particular, it is shown that near threshold, the density behaves like $\sin\theta_c$ rather than of the well known behavior of $\sin^2\theta_c$ for uniformly extended media and may be of interest experimentally, where θ_c is the Čerenkov-cone half-angle. The derived expression is applied to the visible region. The analysis is given from a field-theory view-point. © 1999 Published by Elsevier Science B.V. All rights reserved.

PACS: 41.60.Bq; 12.20.Ds; 11.10. − z; 03.50. − z

Keywords: Čherenkov radiation; Quantum electrodynamics; Field theory

1. Introduction

There has been much interest recently [1–4,14–16] in the classic Čerenkov radiation [5,6] and its experimental verification has been repeated many times over the years (cf. [7–9,17,18]). Its numerous applications in particle detection, in nuclear and cosmic ray physics, in high-energy physics and tests of properties of materials have been well documented (cf. [10,19,20]). The purpose of this work is to carry out a systematic analysis of Čerenkov radiation off a charged particle moving in a plane homogeneous isotropic dielectric slab of permitivity κ_1 with the latter immersed in another homogeneous isotropic dielectric medium of permitivity $\kappa_2 > \kappa_1$ via the Green's function (propagator) of the photon. With the physical problem in mind, a direct derivation is given for the latter and then we use the vacuum-to-vacuum transition amplitude (e.g., [1,11,14]) to obtain an explicit expression for the number [1,12,14,21,22] density of photons radiated within a given frequency range as the particle transverses a unit distance parallel to the surface of the slab and, for simplicity, at midpoint. Of particular interest in our results are that near threshold for emission, the density behaves like $\sin\theta_c$ rather than of the well known behavior of

* Corresponding author.

PII: S0921-4526(99)00101-5

$\sin^2\theta_c$ for uniformly extended media and may be of interest experimentally, where θ_c is the Čerenkov-cone half-angle. This may provide an overall enhancing factor for emission. Secondly, near threshold, the density is seen to be damped out at higher frequencies unlike the well known constant behavior for uniformly extended media. This also may be of interest experimentally. An application is given to the visible region and the number of photons emitted is estimated. The permitivities are allowed, in general, to be frequency dependent in the formalism.

The paper is organized as follows. In Section 2, the boundary conditions for the photon propagator are spelled out and the expression for the latter is derived with the physical problem in mind. In Section 3, we obtain an explicit expression for the photon number density for emission within a frequency range $(\omega, \omega + \mathrm{d}\omega)$ as the particle traverses a given path length L, and the important positivity of the density is established. Section 4 deals with the threshold behavior of the number density.

2. The photon propagator

We work in the celebrated temporal gauge $A^0 = 0$ (cf. [1,13,14]) for the propagator. The permitivity varies along the z-axis as

$$\kappa(z) = \kappa_1, \quad -a/2 < z < a/2,$$
$$\kappa(z) = \kappa_2, \quad z > a/2, \quad z < -a/2, \tag{1}$$

where a denotes the thickness of the slab of permitivity κ_1. The Green's function will be denoted by $D^{ij}(x', x)$, $x = (x^0, \boldsymbol{x}_{||}, z)$, where $\boldsymbol{x}_{||}$ lies parallel to the surfaces of the slab, $x^0 = ct$; $i, j = 1, 2, 3$. Once and for all, we set $-a/2 < z < a/2$, where the motion of the charged particle takes place, and eventually set $z = 0$ corresponding to its trajectory (Section 3). The photon propagator for the problem at hand reduces in projecting out the $i = j = 1$ component of $D^{ij}(x', x)$.

2.1. Case $-a/2 < z' < a/2$

$D^{ij}(x', x)$ consists of two parts:

$$D^{ij}(x', x) = D_1^{ij}(x', x) + D_2^{ij}(x', x), \tag{2}$$

where $D_1^{ij}(x', x)$ is the particular solution of

$$[(-\partial'^2 + \kappa_1 \partial'^{0^2})\delta^{im} + \partial'^i \partial'^m] D_1^{mj}(x', x) = \delta^{ij}\delta^4(x', x), \tag{3}$$

and $D_2^{ij}(x', x)$ is the homogeneous solution of Eq. (3) with the right-hand side of the latter replaced by zero. For $x'^0 > x^0$, the solution to Eq. (3) is well known (cf. [1,14]):

$$D_1^{ij}(x', x) = i\int \sqrt{\kappa_1}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})}\mathrm{e}^{-\mathrm{i}k(x'^0-x^0)}}{2k} \times \left(\delta^{ij} - \frac{k^i k^j}{k^2}\right), \tag{4}$$

$k \equiv |\boldsymbol{k}|$. For $D_2^{ij}(x', x)$ we have

$$[(-\partial'^2 + \kappa_1 \partial'^{0^2})\delta^{im} + \partial'^i \partial'^m] D_2^{mj}(x', x) = 0; \tag{5}$$

or equivalently

$$(-\partial'^2 + \kappa_1 \partial'^{0^2}) D_2^{ij}(x', x) = 0, \tag{6}$$

$$(-\partial^2 + \kappa_1 \partial^{0^2}) D_2^{ij}(x', x) = 0, \tag{7}$$

$$\partial'^i D^{ij}(x', x) = 0, \qquad \partial^j D^{ij}(x', x) = 0. \tag{8}$$

Eqs. (6) and (7) together imply that, in particular, for $a, b = 1,2$, quite generally we may write

$$D_2^{ab}(x', x) = i\int \sqrt{\kappa_1}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}\boldsymbol{K}\cdot(\boldsymbol{x}'_{||}-\boldsymbol{x}_{||})}\mathrm{e}^{-\mathrm{i}k(x'^0-x^0)}}{2k} \times [\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}q(z'-z)}A^{ab}(\boldsymbol{k}) + \mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}q(z'+z)}B^{ab}(\boldsymbol{k})], \tag{9}$$

where $\boldsymbol{k} = (\boldsymbol{K}, q)$, $k = |\boldsymbol{k}|$, and A^{ab}, B^{ab} are unknown. For $z = 0$, the above in Eq. (9) must be an even function in z' by symmetry. That is

$$D_2^{ab}(x', x) = i\int \sqrt{\kappa_1}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}\boldsymbol{K}\cdot(\boldsymbol{x}'_{||}-\boldsymbol{x}_{||})}\mathrm{e}^{-\mathrm{i}k(x'^0-x^0)}}{2k} \times \cos(\sqrt{\kappa_1}qz')G_+^{ab}(\boldsymbol{k}), \tag{10}$$

where we have set $z = 0$ and $G_+^{ab}(k)$ is yet to be determined. Upon combining Eqs. (4) and (10) we are led $(x'^0 > x^0,\ z = 0)$ to

$$D^{ab}(x', x) = \mathrm{i}\int \sqrt{\kappa_1}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}\boldsymbol{K}\cdot(\boldsymbol{x}'_{||}-\boldsymbol{x}_{||})}\mathrm{e}^{-\mathrm{i}k(x'^0-x^0)}}{2k} \times \left[\left(\delta^{ab} - \frac{K^a K^b}{k^2}\right)\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}qz'} + \cos(\sqrt{\kappa_1}qz')G_+^{ab}(\boldsymbol{k})\right]. \tag{11}$$

We will also need to find the general structure of $D^{3b}(x', x)$. To this end we may infer from the first relation Eq. (8) that ($x'^0 > x^0$, $z = 0$)

$$D^{3b}(x', x) = \mathrm{i}\int \sqrt{\kappa_1}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}\boldsymbol{K}\cdot(\boldsymbol{x}'_{\|}-\boldsymbol{x}_{\|})}\mathrm{e}^{-\mathrm{i}k(x'^0-x^0)}}{2k} \times\left[\frac{-qK^b}{k^2}\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}qz'} - \mathrm{i}\frac{\sin(\sqrt{\kappa_1}qz')}{q}K^aG_+^{ab}\right]. \quad (12)$$

Quite generally G_+^{ab} is of the following form

$$G_+^{ab} = \left(\delta^{ab}A_+(\boldsymbol{k}) - \frac{K^aK^b}{k^2}B_+(\boldsymbol{k})\right), \quad (13)$$

where A_+ and B_+ are to be determined.

2.2. Case $z' > a/2$ (or $z' < -a/2$)

For $z' > a/2$, we use the notation $D_>^{ij}(x', x)$ for the Green function. The equations for the latter are

$$(-\partial^2 + \kappa_1\partial^{0^2})D_>^{ij}(x', x) = 0, \quad (14)$$

$$(-\partial'^2 + \kappa_2\partial'^{0^2})D_>^{ij}(x', x) = 0, \quad (15)$$

$$\partial'^iD_>^{ij}(x', x) = 0, \qquad \partial^jD_>^{ij}(x', x) = 0. \quad (16)$$

Quite generally, $D_>^{ab}(x', x)$, $a, b = 1, 2$, has the structure ($x'^0 > x^0$):

$$D_>^{ab}(x', x) = \int\frac{\mathrm{d}^2\boldsymbol{K}}{(2\pi)^2}\frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}q'}{2\pi}\frac{\mathrm{d}k^0}{2\pi} \times\mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{x}'_{\|}-\boldsymbol{x}_{\|})}\mathrm{e}^{\mathrm{i}q'z'}\mathrm{e}^{-\mathrm{i}qz}\mathrm{e}^{-\mathrm{i}k^0(x'^0-x^0)}\,\mathrm{e}^{-\mathrm{i}k^0(x'^0-x^0)\tilde{D}_>^{ab}}. \quad (17)$$

Eqs. (14) and (15) give, respectively,

$$(\boldsymbol{K}^2 + q^2 - \kappa_1k^{0^2})\tilde{D}_>^{ab} = 0, \quad (18)$$

$$(\boldsymbol{K}^2 + q'^2 - \kappa_2k^{0^2})\tilde{D}_>^{ab} = 0, \quad (19)$$

from which we infer, in particular, that with

$$q' = \left(\frac{\kappa_2}{\kappa_1}\boldsymbol{k}^2 - \boldsymbol{K}^2\right)^{1/2}\operatorname{sgn}q \equiv Q, \quad (20)$$

$k = (\boldsymbol{K}^2 + q^2)^{1/2} \equiv |\boldsymbol{k}|$, where $\operatorname{sgn}q = |q|$ for $q > 0$, $\operatorname{sgn}q = -|q|$ for $q < 0$, we may write

$$D_>^{ab}(x', x) = \mathrm{i}\int\sqrt{\kappa_1}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\uparrow\sqrt{\kappa_1}\boldsymbol{K}\cdot(\boldsymbol{x}'_{\|}-\boldsymbol{x}_{\|})}\mathrm{e}^{-\mathrm{i}k(x'^0-x^0)}}{2k} \times\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}Qz'}(\mathrm{e}^{-\mathrm{i}\sqrt{\kappa_1}qz}G_1^{ab}(\boldsymbol{k}) + \mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}qz}G_2^{ab}(\boldsymbol{k})) \quad (21)$$

$\boldsymbol{k} = (\boldsymbol{K}, q)$, which may be rewritten as $z = 0$ as

$$D_>^{ab}(x', x) = \mathrm{i}\int\sqrt{\kappa_1}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}\boldsymbol{K}\cdot(\boldsymbol{x}'_{\|}-\boldsymbol{x}_{\|})}\mathrm{e}^{-\mathrm{i}k(x'^0-x^0)}}{2k} \times\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}Qz'}G_>^{ab}(\boldsymbol{k}) \quad (22)$$

in terms of a new function G^{ab}. By using Eq. (16) we also have ($x'^0 > x^0$, $z = 0$)

$$D_>^{3b}(x', x) = \mathrm{i}\int\sqrt{\kappa_1}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}\boldsymbol{K}\cdot(\boldsymbol{x}'_{\|}-\boldsymbol{x}_{\|})}\mathrm{e}^{-\mathrm{i}k(x'^0-x^0)}}{2k} \times\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}Qz'}(-K^aG_>^{ab}(\boldsymbol{k})/Q). \quad (23)$$

Similar expressions may be given for $z' < -a/2$ which, however, due to symmetry, are not essential here.

2.3. Applications of the boundary conditions

The boundary conditions for $D^{ib}(x', x)$ are given by

$$D^{ab}(x', x), \quad (24)$$

$$\kappa(z')D^{3b}(x', x), \quad (25)$$

$$[\partial'^aD^{3b}(x', x) - \partial'^3D^{ab}(x', x)], \quad (26)$$

$a, b = 1, 2$, are continuous at $z' = \pm a/2$.

The application of these boundary conditions at $z' = +a/2$, for example, give from Eqs. (11)–(13), (22) and (23), respectively:

$$\left(\delta^{ab} - \frac{K^aK^b}{k^2}\right)\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}qa/2} + \cos(\sqrt{\kappa_1}qa/2)G_+^{ab} = \mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}Qa/2}G_>^{ab}, \quad (27)$$

$$\kappa_1\left[\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}qa/2}\left(-\frac{qK^b}{k^2}\right) - \mathrm{i}\frac{\sin(\sqrt{\kappa_1}qa/2)}{q}K^aG_+^{ab}\right] = \kappa_2\mathrm{e}^{\mathrm{i}\sqrt{\kappa_1}Qa/2}G_>^{3b}, \quad (28)$$

$$K^a\left[e^{i\sqrt{\kappa_1}qa/2}\left(-\frac{qK^b}{k^2}\right)-i\frac{\sin(\sqrt{\kappa_1}qa/2)}{q}K^b\right.$$
$$\left.\times\left(A_+-\frac{\boldsymbol{K}^2}{k^2}B_+\right)\right]-q\left[e^{i\sqrt{\kappa_1}qa/2}\left(\delta^{ab}-\frac{K^aK^b}{k^2}\right)\right.$$
$$\left.+\,i\sin(\sqrt{\kappa_1}qa/2)G_+^{ab}\right]=[K^aG_>^{3b}-QG_>^{ab}]e^{i\sqrt{\kappa_1}Qa/2},\tag{29}$$

where in writing Eq. (29), we have used Eq. (13). A tedious analysis of the B. C. in Eqs. (27)–(29) determines the functions A_+, B_+ relevant to the problem at hand:

$$A_+=\frac{(q-Q)e^{i\sqrt{\kappa_1}qa/2}}{Q\cos(\sqrt{\kappa_1}qa/2)-iq\sin(\sqrt{\kappa_1}qa/2)},\tag{30}$$

$$\frac{B_+}{k^2}=\frac{1}{Qk^2}\frac{e^{i\sqrt{\kappa_1}qa/2}}{[i\kappa_1Q\sin(\sqrt{\kappa_1}qa/2)-q\kappa_2\cos(\sqrt{\kappa_1}qa/2)]}$$
$$\times\left[(\kappa_2Q-\kappa_1q)q+i\frac{(\kappa_2-\kappa_1)k^2(q-Q)\sin(\sqrt{\kappa_1}qa/2)}{[Q\cos(\sqrt{\kappa_1}qa/2)-iq\sin(\sqrt{\kappa_1}qa/2)]}\right].\tag{31}$$

2.4. The photon propagator

The photon propagator for the physical problem at hand corresponds to $a=1=b$, $z=0=z'$, $x_2'=0=x_2$ in Eq. (9), where the charged particle moves in the x_1 direction $\boldsymbol{x}=(x_1, x_2, z)$. That is, from Eqs. (9) and (12) ($x'^0>x^0$):

$$D^{11}(x', x)=i\int\sqrt{\kappa_1}\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i\sqrt{\kappa_1}K_1(x_1'-x_1)}e^{-ik(x'^0-x^0)}}{2k}$$
$$\times\left[(1+A_+)-\frac{(K_1)^2}{k^2}(1+B_+)\right],\tag{32}$$

where A_+, B_+ are given by Eqs. (30), (31), respectively. Here $\boldsymbol{k}=(K_1, K_2, q)$, $k=|\boldsymbol{k}|$, and from Eq. (19), Q may be rewritten as

$$Q=\left(\left(\frac{\kappa_2}{\kappa_1}-1\right)k^2+q^2\right)^{1/2}\operatorname{sgn}q.\tag{33}$$

The integrand in Eq. (32) is an even function of K_2. The q integral over q may be rewritten over the region $0<q<\infty$, by replacing $q\to-q$ over the $-\infty<q<0$ region. A tedious analysis from Eqs. (30)–(33), then shows that we may, equivalently to Eq. (32), write

$$D^{11}(x', x)=4i\int_{-\infty}^{\infty}\frac{dK_1}{2\pi}\int_0^{\infty}\frac{dK_2}{2\pi}\int_0^{\infty}\frac{dq}{2\pi}$$
$$\times\frac{e^{i\sqrt{\kappa_1}K_1(x_1'-x_1)}e^{-ik(x'^0-x^0)}}{2k}A\left[1-\frac{\kappa_1(K_1)^2}{k^2}B\right],\tag{34}$$

where

$$A=\frac{qQ}{Q^2\cos^2(\sqrt{\kappa_1}qa/2)+q^2\sin^2(\sqrt{\kappa_1}qa/2)},\tag{35}$$

$$B=\frac{(\kappa_2-\kappa_1)k^2\sin^2(\sqrt{\kappa_1}qa/2)+\kappa_2q^2}{\kappa_1^2Q^2\sin^2(\sqrt{\kappa_1}qa/2)+\kappa_2^2q^2\cos^2(\sqrt{\kappa_1}qa/2)}.\tag{36}$$

A and B are real.

We insert the unit operator

$$\int_0^{\infty}d\omega\,\delta(\omega-kc)=1\tag{37}$$

in Eq. (34). Also the K_2-q integrals suggest to integrate for the latter in polar coordinates: $q=R\cos\theta$, $K_2=R\sin\theta$ with $0\leqslant R<\infty$, $0\leqslant\theta\leqslant\pi/2$. With the help of the delta function in Eq. (37) we may then explicitly integrate over R to obtain

$$D^{11}(x', x)=\frac{i}{2\pi c}\int_0^{\infty}d\omega\int_{-\infty}^{\infty}\frac{dK}{2\pi^2}\,\Theta\left(\frac{\omega^2}{c^2}-\frac{K^2}{\kappa_1}\right)$$
$$\times e^{iK(x_1'-x_1)}e^{-i\omega(x'^0-x^0)/c}$$
$$\times\int_0^{\pi/2}d\theta\,A_1\left(1-\frac{c^2K^2}{\omega^2}B_1\right),\tag{38}$$

where we have finally made the change of variable $\sqrt{\kappa_1}K_1=K$, $A_1=A$ given in Eq. (35),

$$B_1=\frac{(\kappa_2-\kappa_1)\sin^2(Z\cos\theta)+\kappa_2q^2}{\kappa_1^2Q^2\sin^2(Z\cos\theta)+\kappa_2^2q^2\cos^2(Z\cos\theta)}\tag{39}$$

and *now*

$$q = \left(1 - \frac{1}{\kappa_1}\left(\frac{cK}{\omega}\right)^2\right)^{1/2} \cos\theta\,, \tag{40}$$

$$Q = \left(\frac{\kappa_2}{\kappa_1} - 1 + \left(1 - \frac{1}{\kappa_1}\left(\frac{cK}{\omega}\right)^2\right)\cos^2\theta\right)^{1/2}, \tag{41}$$

$$Z = \sqrt{\kappa_1}\,\frac{\omega a}{2c}\left(1 - \frac{1}{\kappa_1}\left(\frac{cK}{\omega}\right)^2\right)^{1/2}. \tag{42}$$

3. The photon number density and its positivity

For the charged particle, we specialize in a current density $\boldsymbol{J} = (J_1, 0, 0)$ in the form

$$J_1(x) = f(x_1, t)\delta(x_2)\delta(z)\,, \tag{43}$$

$x = (ct, x_1, x_2, z)$. The vaccum-to-vacuum transition amplitude in the presence of a current density $J_1(x)$ is given by (cf. [1,11,12,14,22])

$$\langle 0_+ | 0_- \rangle = \times \exp\left[\frac{\mathrm{i}}{2\hbar c^3}\int (\mathrm{d}x')(\mathrm{d}x)\, J_1(x') D^{11}(x',x) J_1(x)\right] \tag{44}$$

and hence for the average number of photons emitted by J_1 we have [1,12,14,22] from Eqs. (38) and (43)

$$\langle N \rangle = \frac{1}{8\pi^2\hbar c}\int_0^\infty \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}K\, \Theta\left(\frac{\omega^2}{c^2} - \frac{K^2}{\kappa_1}\right) \times |\tilde{f}(K,\omega)|^2 G(K,\omega)\,, \tag{45}$$

where

$$G(K,\omega) = \frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}\,\theta\, A_1\left(1 - \frac{c^2K^2}{\omega^2}B_1\right), \tag{46}$$

where A_1, B_1, q, Q, Z defined through Eqs. (38)–(42).

For a charged particle of charge e we write

$$f(x_1, t) = ev\delta(x_1 - vt) \tag{47}$$

to obtain formally by a double Fourier transform

$$|\tilde{f}(K,\omega)|^2 = 2\pi e^2 \delta\left(K - \frac{\omega}{v}\right)\int_0^L \mathrm{d}x_1\,, \tag{48}$$

where L denotes the distance travelled by the particle. The average number density of photons *emitted with* (angular) frequency within the interval $(\omega, \omega + \mathrm{d}\omega)$ as the particle traverses a distance L is then

$$\langle N(\omega, L) \rangle = \frac{\alpha L}{c} G\left(\frac{\omega}{v}, \omega\right) \tag{49}$$

with the threshold condition obtained from Eq. (48) and the step function in Eq. (45) to be $\sqrt{\kappa_1}\beta > 1$, where $\beta = v/c$, and $\alpha = e^2/(4\pi\hbar c)$ is the fine-structure constant. In detail

$$G\left(\frac{\omega}{v}, \omega\right) = \frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}\theta\, qQ(Q^2\cos^2(Z\cos\theta) + q^2\sin^2(Z\cos\theta))^{-1} \left[1 - \frac{1}{\beta^2}\frac{(\kappa_2 - \kappa_1)\sin^2(Z\cos\theta) + \kappa_2 q^2}{[\kappa_1^2 Q^2 \sin^2(Z\cos\theta) + \kappa_2^2 q^2 \cos^2(Z\cos\theta)]}\right], \tag{50}$$

where now

$$q = (1 - 1/\kappa_1\beta^2)^{1/2}\cos\theta,$$

$$Q = (\kappa_2/\kappa_1 - 1 + (1 - 1/\kappa_1\beta^2)\cos^2\theta)^{1/2}\,, \tag{51}$$

$$Z = \sqrt{\kappa_1}\,\frac{\omega a}{2c}(1 - 1/\kappa_1\beta^2)^{1/2}\,. \tag{52}$$

3.1. Positivity of $\langle N(\omega,L) \rangle$

To establish the very essential positivity of $\langle N(\omega, L) \rangle$ given in Eq. (49), we derive a chain of some inequalities. To this end we note that with the condition $\kappa_2 > \kappa_1$ set, the second term in the square brackets in the integrand in Eq. (50) without the minus sign

$$\frac{(\kappa_2 - \kappa_1)\sin^2(Z\cos\theta) + \kappa_2 q^2}{\beta^2\kappa_1^2 Q^2 \sin^2(Z\cos\theta) + \beta^2\kappa_2^2\cos^2(Z\cos\theta)} \quad (\geqslant 0) \tag{53}$$

as indicated is positive. To establish the positivity of $G(\omega/v, \omega)$ in Eq. (50) we have to show that the expression in Eq. (53) does not exceed one since $qQ \geqslant 0$ in Eq. (50). To this end, the denominator in

Eq. (53) may be rewritten as

$$\beta^2\kappa_1^2(Q^2-q^2)\sin^2(Z\cos\theta)+\beta^2(\kappa_1^2\sin^2(Z\cos\theta)$$
$$+\kappa^2\cos^2(Z\cos\theta))q^2=\beta^2\kappa_1(\kappa_2-\kappa_1)\sin^2$$
$$\times(Z\cos\theta)+\beta^2(\kappa_1^2\sin^2(Z\cos\theta)$$
$$+\kappa_2^2\cos^2(Z\cos\theta))q^2$$
$$=(\kappa_2-\kappa_1)\sin^2(Z\cos\theta)$$
$$+\beta^2\kappa_1^2\left[\left(\frac{\kappa_2}{\kappa_1}-1\right)\frac{\sin^2(Z\cos\theta)}{\cos^2\theta}\right.$$
$$\left.+\sin^2(Z\cos\theta)+\frac{\kappa_2^2}{\kappa_1^2}\cos^2(Z\cos\theta)\right]q^2, \quad (54)$$

where we have used the definitions in Eq. (51). We have the following chain of inequalities:

$$\beta^2\kappa_1^2\left[\left(\frac{\kappa_2}{\kappa_1}-1\right)\frac{\sin^2(Z\cos\theta)}{\cos^2\theta}+\sin^2(Z\cos\theta)\right.$$
$$\left.+\frac{\kappa_2^2}{\kappa_1^2}\cos^2(Z\cos\theta)\right]$$
$$\geqslant\beta^2\kappa_1\kappa_2\left[\sin^2(Z\cos\theta)+\frac{\kappa_2}{\kappa_1}\cos^2(Z\cos\theta)\right]$$
$$=\beta^2\kappa_1\kappa_2\left[1+\left(\frac{\kappa_2}{\kappa_1}-1\right)\cos^2(Z\cos\theta)\right]$$
$$\geqslant\beta^2\kappa_1\kappa_2\geqslant\kappa_2, \quad (55)$$

thus establishing from (54) that the expression in Eq. (53) does not exceed one, and hence of the positivity of $\langle N(\omega, L)\rangle$.

3.2. *Expression for* $\langle N(\omega, L)\rangle$

Upon using the definitions (51), (52) in Eqs. (49) and (50) we obtain the final expression

$$\langle N(\omega, L)\rangle=\frac{\alpha L}{c}\frac{2}{\pi}\int_0^{\pi/2}\mathrm{d}\theta f_1(\cos\theta,\omega)f_2(\cos\theta,\omega), \quad (56)$$

where

$$f_1(\cos\theta,\omega)=$$
$$\frac{\sin^3\theta_c((b-1)+\sin^2\theta_c\cos^2\theta)^{1/2}}{[(b-1)\cos^2(Z\cos\theta)+\sin^2\theta_c\cos^2\theta]}\cos\theta, \quad (57)$$

$$f_2(\cos\theta,\omega)=\frac{(b-1)\sin^2(Z\cos\theta)+F\cos^2\theta}{(b-1)\sin^2(Z\cos\theta)+G\sin^2\theta_c\cos^2\theta},$$
$$F=b(\sin^2\theta_c-1)+\sin^2(Z\cos\theta)+b^2\cos^2(Z\cos\theta),$$
$$G=\sin^2(Z\cos\theta)+b^2\cos^2(Z\cos\theta), \quad (58)$$

where

$$b=\kappa_2/\kappa_1, \quad (59)$$

$$\sin\theta_c=(1-1/\kappa_1\beta^2)^{1/2}, \quad (60)$$

$$Z=\sqrt{\kappa_1}\frac{\omega a}{zc}\sin\theta_c. \quad (61)$$

In particular for $\kappa_2\to\kappa_1\equiv\kappa$, $a\to 0$,

$$\frac{f_1(\cos\theta,\omega)}{\sin^2\theta_c}\to 1,\quad f_2(\cos\theta,\omega)\to 1,$$

and we recover the classic result for uniformly extended media

$$\langle N(\omega, L)\rangle_0=\frac{\alpha L}{c}\sin^2\theta_c. \quad (62)$$

4. Threshold behavior

Of particular interest is the threshold behavior $\sin\theta_c\sim 0$ of $\langle N(\omega, L)\rangle$ in Eqs. (56)–(58). To this end

$$f_1(\cos\theta,\omega)\simeq\frac{\sin^3\theta_c\cos\theta}{(b-1)^{1/2}}, \quad (63)$$

$$f_2(\cos\theta,\omega)\simeq\frac{b(b-1)}{(b^2+(b-1)\omega^2\kappa_1a^2/4c^2)}\frac{1}{\sin^2\theta_c} \quad (64)$$

and the θ-integral in Eq. (56) is readily carried out to give

$$\langle N(\omega, L)\rangle\simeq\frac{4\alpha L}{\pi a\sqrt{\kappa_1}}\left(\frac{\omega_0}{\omega^2+\omega_0^2}\right)\sin\theta_c, \quad (65)$$

where

$$\omega_0=\frac{2c(\kappa_2/\kappa_1)}{a(\kappa_2-\kappa_1)^{1/2}}. \quad (66)$$

The $\sin\theta_c$ behavior in Eq. (65), rather than $\sin^2\theta_c$ as in Eq. (62), is to be noted, as well as the frequency dependence in the denominator.

We consider the visible region, and we assume, approximate constancy of the permitivities in this region to obtain

$$N(\omega_2, \omega_1) = \int_{\omega_2}^{\omega_1} \mathrm{d}\omega \langle N(\omega, L)\rangle \simeq \frac{4\alpha}{\pi\sqrt{\kappa_1}}\left(\frac{L}{a}\right) \times \sin\theta_c \tan^{-1}\left(\frac{(\omega_1-\omega_2)/\omega_0}{1+\omega_1\omega_2/\omega_0^2}\right), \quad (67)$$

where we take $\omega_1/c \simeq 1.653\times10^5$/cm, $\omega_2/c \simeq 8.380\times10^4$/cm. For example consider a layer of hydrogen gas $\kappa_1 = \kappa(H_2) = 1.00026$, of thickness a, trapped long enough in air, $\kappa_2 = \kappa(\text{air}) = 1.00060$, so that the particle traverses a distance L. Then

$$\frac{\omega_0}{c} = 1.085\times10^2/a. \quad (68)$$

Since $(\omega_1\omega_2)^{1/2}/c \sim 10^5$/cm, we may consider the following three cases.

I. $a \sim 10^{-3}$cm.

This corresponds to the case $\omega_0^2 \sim \omega_1\omega_2$. For definiteness, let $a = 10^{-3}$ cm, then with $\alpha = (137)^{-1}$, Eq. (67) gives

$$N(\omega_2, \omega_1)/L \simeq 3\sin\theta_c \text{ photons/cm.} \quad (69)$$

II. $a \gg 10^{-3}$cm.

Of course, a cannot be taken arbitrarily large here because of the approximation made in Eqs. (63) and (64). This case corresponds to $\omega_1\omega_2 \gg \omega_0^2$ and hence

$$N(\omega_2, \omega_1)/L \simeq \frac{9.29\times10^{-3}}{a}\sin\theta_c\omega_0\left(\frac{\omega_1-\omega_2}{\omega_1\omega_2}\right) \simeq \frac{6\times10^{-6}}{a}\sin\theta_c \quad (70)$$

and for $a \simeq 1$ cm,

$$N(\omega_2, \omega_1)/L \simeq 6\times10^{-6}\sin\theta_c \text{ photons/cm.} \quad (71)$$

Needless to say that Eq. (71) is the number of photons radiated per charged particle and the result may be significant when there is a large number of radiating particles.

III. $a \ll 10^{-3}$cm.

In this case the a dependence in Eq. (67) essentially disappears since $\omega_1\omega_2 \ll \omega_0^2$, and $(\omega_0 a)$ is independent of a, and we obtain

$$N(\omega_2, \omega_1)/L \simeq 7\sin\theta_c \text{ photons/cm.} \quad (72)$$

Eqs. (69), (71) and (72) are to be compared with $N_0(\omega_2, \omega_1)/L$ as obtained from Eq. (62) for uniformly extended media for the same frequency range. The latter is given by

$$N_0(\omega_2, \omega_1)/L = \alpha\sin^2\theta_c(\omega_1-\omega_2)/c \simeq 595\sin^2\theta_c \text{ photons/cm.} \quad (73)$$

It is interesting to note that the problem sets a scale $a \sim 10^{-3}$ cm at which the number of photons may be arbitrarily enhanced over the uniformly extended media when a particle moves arbitrarily close to its threshold value. An enhancement that may occur in Eq. (69) or a suppression that may occur in Eq. (71) over the uniformly extended media may be a clear indication of an inhomogeneity of the permitivity with a given extension around the charged particle and may be of experimental interest.

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