STABILITY OF MATTER IN 2D

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We prove rigorously the stability of matter in 2D.

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1. Introduction

Early investigations of the stability of matter go to the classical work of Dyson and Lenard [1] and more recently to the monumental work of Lieb and Thirring [2, 3]. The Pauli exclusion principle turns out to be not only sufficient for stability but also necessary. In regard to the exclusion principle, or more generally to the spin and statistics connection, it is interesting to quote from the translator's Preface [4] of the classical book by Tomonaga on spin: "*The existence of spin, and the statistics associated with it, is the most subtle and ingenious design of Nature — without it the whole universe would collapse.*" For matter, with the exclusion principle, with Coulomb interactions, the ground-state energy $E_N \sim N$, with N denoting the number of electrons in matter, and matter consisting of $(2N + 2N)$ particles is not favoured over two separate systems brought together, each consisting of $(N + N)$ particles. This is unlike the situation with "matter" without the exclusion principle, for which $E_N \sim N^{\alpha}$ with $\alpha > 1$. A key result in the stability of matter is that for a non-vanishing probability of having the electrons within a sphere of radius R, the latter, necessarily, grows not any slower than $N^{1/3}$ for large N [5], and the infinite density limit does not occur. No wonder why matter occupies so large a volume! Here it is worth recalling the words addressed by Ehrenfest to Pauli in 1931 on the occasion of the Lorentz model (cf. [6]) to this effect: "*We take a piece of metal, or a stone. When we think about it, we are astonished that this quantity of matter should occupy so large a volume.*" He went on by stating that the Pauli exclusion principle is the reason: "*Answer: only the Pauli principle, no two electrons in the same state.*"

There was much interest in recent years in physics in 2D, e.g. [7–10], and in the role of the spin and statistics theorem. It has thus become important to investigate the nature of matter in 2D with the exclusion principle. As a matter of fact, it is an important theoretical question to investigate if the change of the dimensionality of space will change matter from a stable to an unstable or an explosive phase. We show that matter *is* stable in 2D. In 2D we work with the logarithmic potential and, as it turns out, the corresponding Laplace equation is more important for the proof of stability in spite of the fact that the logarithmic potential does not go to zero at infinity and one might expect difficulties for the proof of stability. We do not, however, dwell upon nature for other dimensions here, with the exception of some comments made in the concluding section (Section 6). (Some of the present field theories speculate that at early stages of our universe the dimensionality of space was not necessarily coinciding with three, and by a process which may be referred to as compactification of space, the present three-dimensional character of space arose from the evolution and the cooling down of the universe.)

In Section 2 a detailed study of the Thomas–Fermi (TF) atom is carried out in 2D. Some very preliminary study of this was also carried out in [11]. The TF energy, however, was neither computed in the latter reference nor it was shown that it provides the smallest possible energy value for the TF energy functional. The no-binding theorem [2] is established in Section 3 in 2D from which a lower bound to the electron–electron potential is obtained. A lower bound to the exact-ground-state energy of matter in 2D is derived in Section 4. The expansion of matter is investigated in Section 5 where it is shown that for a non-vanishing probability of having the electrons within a circle of radius R, the latter, necessarily, does not grow any slower than $N^{1/2}$ for large N. Section 6 deals with our conclusion.

2. The TF atom in 2D

The particle density $n(\mathbf{x})$ may be quite elegantly expressed in terms of the Green function $G_{\sigma\sigma'}(\mathbf{x}T; \mathbf{x}0)$ for coincident space points as

$$n(\mathbf{x}) = \sum_{\sigma} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i\epsilon}\, G_{\sigma\sigma}(\mathbf{x}, \hbar\tau; \mathbf{x}, 0), \qquad \epsilon \to +0, \tag{1}$$

where σ, σ' are spin indices, and in the semi-classical approximation

$$G_{\sigma\sigma'}(\mathbf{x}T; \mathbf{x}0) = \delta_{\sigma\sigma'} \int \frac{d^2\mathbf{p}}{(2\pi\hbar)^2}\, \exp\left[-i\left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{x})\right)\tau\right], \tag{2}$$

$\tau = T/\hbar$, for a spin independent potential $V(\mathbf{x})$. This gives

$$n(\mathbf{x}) = -\frac{qm}{2\pi\hbar^2}\, V(\mathbf{x}), \tag{3}$$

where q is the spin multiplicity. For the kinetic energy T, we similarly have

$$T = \sum_{\sigma} \int d^2\mathbf{x}\, \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i\epsilon}\left[i\frac{\partial}{\partial\tau} - V(\mathbf{x})\right] G_{\sigma\sigma}(\mathbf{x}, \hbar\tau; \mathbf{x}, 0), \tag{4}$$

$\epsilon \to +0$, giving

$$T = \frac{\pi\hbar^2}{qm} \int d^2\mathbf{x}\, [n(\mathbf{x})]^2 . \tag{5}$$

Of particular importance in 2D is the Poisson equation

$$\nabla^2 \ln\left(\frac{r}{C}\right) = 2\pi\delta^2(\mathbf{r}) \tag{6}$$

for any dimensional scale factor C.

We define the TF energy functional $F[n]$,

$$F[n] = \frac{\pi\hbar^2}{qm}\int d^2\mathbf{x}\,[n(\mathbf{x})]^2 + 2Ze^2\int d^2\mathbf{x}\,\ln\left(\frac{|\mathbf{x}|}{2r_0}\right)n(\mathbf{x}) - e^2\int d^2\mathbf{x}\,d^2\mathbf{x}'\,n(\mathbf{x})\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)n(\mathbf{x}'), \tag{7}$$

where $r_0 = \left(\hbar^2/2qme^2\right)^{1/2}$, and the correctness of the dimensional scale factor $2r_0$ will be established below.

The TF density $n_{\mathrm{TF}}(\mathbf{x})$ and potential energy $V_{\mathrm{TF}}(\mathbf{x})$ may be obtained by functionally differentiating $F[n]$ with respect to $n(\mathbf{x})$ and setting the resulting expression equal to zero. This gives

$$n_{\mathrm{TF}}(\mathbf{x}) = -\frac{Ze^2qm}{\pi\hbar^2}\ln\left(\frac{|\mathbf{x}|}{2r_0}\right) + \frac{qme^2}{\pi\hbar^2}\int d^2\mathbf{x}'\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)n_{\mathrm{TF}}(\mathbf{x}'), \tag{8}$$

and from (3) we have

$$V_{\mathrm{TF}}(\mathbf{x}) = 2Ze^2\ln\left(\frac{|\mathbf{x}|}{2r_0}\right) + \frac{qme^2}{\pi\hbar^2}\int d^2\mathbf{x}'\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)V_{\mathrm{TF}}(\mathbf{x}'). \tag{9}$$

Eq. (8) leads from (6) to

$$\nabla^2 n_{\mathrm{TF}}(\mathbf{x}) = -\frac{2Ze^2qm}{\hbar^2}\,\delta^2(\mathbf{x}) + \frac{2qme^2}{\hbar^2}\,n_{\mathrm{TF}}(\mathbf{x}) \tag{10}$$

which, with spherical symmetry, may be rewritten as

$$\left[r^2\frac{d^2}{dr^2} + r\frac{d}{dr} - \frac{2qme^2}{\hbar^2}r^2\right]n_{\mathrm{TF}}(r) = -\frac{2Ze^2qm}{\hbar^2}\,r\,\delta(r)\,\delta(\theta). \tag{11}$$

The properly normalized solution of (11) is given by

$$n_{\mathrm{TF}}(r) = \frac{qmZe^2}{\pi\hbar^2}\,\mathrm{K}_0\left(\frac{r}{r_0}\right), \tag{12}$$

where K_0 is a modified Bessel function of order zero, and we have abandoned the function $\mathrm{I}_0(r/r_0)$ which increases exponentially in r for $r \to \infty$, $r_0 = (\hbar^2/2qme^2)^{1/2}$.

Using the integral

$$\int_0^\infty x\,dx\,\mathrm{K}_0(x) = 1 \tag{13}$$

we explicitly verify the correct normalization condition

$$\int d^2\mathbf{x}\, n_{\mathrm{TF}}(\mathbf{x}) = Z. \tag{14}$$

Other useful integrals involving $\mathrm{K}_0(x)$ are

$$\int_0^\infty x\,dx\ \ln(x)\,\mathrm{K}_0(x) = -\gamma + \ln 2, \tag{15}$$

where $\gamma = 0.57724$ is Euler's constant, and

$$\int_0^\infty x\,dx \left[\mathrm{K}_0(x)\right]^2 = \frac{1}{2}. \tag{16}$$

For $x \to 0$, we may use the asymptotic expression

$$\mathrm{K}_0(x) \to -\left[\ln\left(\frac{x}{2}\right) + \gamma\right]. \tag{17}$$

Accordingly, from (12), (17) and (8), we have for $r \to 0$,

$$\begin{aligned} -\frac{qmZe^2}{\pi\hbar^2}\left[\ln\left(\frac{r}{2r_0}\right) + \gamma\right] = &-\frac{qmZe^2}{\pi\hbar^2}\ln\left(\frac{r}{2r_0}\right) \\ &+\frac{qmZe^2}{\pi\hbar^2}\int_0^\infty x'\,dx'\ \ln x'\,\mathrm{K}_0(x') \\ &-\frac{qmZe^2}{\pi\hbar^2}\ln(2)\int_0^\infty x'\,dx'\ \mathrm{K}_0(x') \end{aligned} \tag{18}$$

and using the integrals (13) and (15) in (18) we, in particular, verify the correctness of the dimensional scale factor $2r_0$ in (7).

The evaluation of $F[n_{\mathrm{TF}}]$, with n_{TF} given in (12), is now straightforward thanks to the integrals in (13), (15) and (16). In particular,

$$\frac{\pi\hbar^2}{qm}\int d^2\mathbf{x}\, n_{\mathrm{TF}}^2(\mathbf{x}) = \frac{Z^2e^2}{2}, \tag{19}$$

$$2Ze^2\int d^2\mathbf{x}\ \ln\left(\frac{|\mathbf{x}|}{2r_0}\right) n_{\mathrm{TF}}(\mathbf{x}) = -\,2Z^2e^2\gamma, \tag{20}$$

and

$$\begin{aligned} -e^2\int d^2\mathbf{x}\,d^2\mathbf{x}'\ n_{\mathrm{TF}}(\mathbf{x})\ \ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right) n_{\mathrm{TF}}(\mathbf{x}') &= Z^2e^2(\ln 2 - 0.61592) \\ &= 0.07723\, Z^2e^2, \end{aligned} \tag{21}$$

where in evaluating the last integral we have used the fact that

$$\int_0^{2\pi} d\theta\ \ln\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\theta} = \pi\,\ln(\rho_>^2), \tag{22}$$

where $\rho_>$ is the largest of ρ, ρ'.

The TF energy $E_{\mathrm{TF}} = F[n_{\mathrm{TF}}]$ is then given by

$$E_{\mathrm{TF}} = -0.5773\, Z^2 e^2. \tag{23}$$

For the TF potential energy $V_{\mathrm{TF}}(\mathbf{x})$, we have from (9) and (6),

$$\nabla^2 V_{\mathrm{TF}}(\mathbf{x}) = 4\pi\, Ze^2\, \delta^2(\mathbf{x}) - 4\pi\, e^2 n_{\mathrm{TF}}(\mathbf{x}), \tag{24}$$

with the first term corresponding to the nucleus at the origin, while the second term corresponds to the electron density. Upon integration over $\mathbf{x}$, and using (14), we obtain

$$\int d^2\mathbf{x}\, \nabla^2 V(\mathbf{x}) = 0 \tag{25}$$

verifying the neutrality of the TF atom.

It remains to show that n_{TF} provides the smallest possible value for $F[n]$ in (7), that is

$$F[n] \geq F[n_{\mathrm{TF}}]. \tag{26}$$

Let

$$\begin{aligned} n(\mathbf{x}) &= tn_1(\mathbf{x}) + (1-t)n_2(\mathbf{x}) \equiv tn_1 + (1-t)n_2, \\ n(\mathbf{x}') &= tn_1(\mathbf{x}') + (1-t)n_2(\mathbf{x}') \equiv tn_1' + (1-t)n_2', \qquad 0 \leq t \leq 1. \end{aligned}$$

From convexity or directly we have

$$[tn_1 + (1-t)n_2]^2 \leq tn_1^2 + (1-t)n_2^2 \tag{27}$$

and

$$[tn_1+(1-t)n_2][tn_1'+(1-t)n_2'] = tn_1n_1'+(1-t)n_2n_2'-t(1-t)(n_1-n_2)(n_1'-n_2'). \tag{28}$$

Using the fact that

$$\int d^2\mathbf{x}\, d^2\mathbf{x}'\, (n_1 - n_2) \ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)(n_1' - n_2')$$

is negative, for example by Fourier transform, we may then infer that

$$F[tn_1 + (1-t)n_2] \leq t\, F[n_1] + (1-t)\, F[n_2]. \tag{29}$$

It is also easily seen that for $n_2 \equiv n_{\mathrm{TF}}$, we have from (8),

$$\left.\frac{d}{dt} F[tn_1 + (1-t)n_2]\right|_{t=0} = 0. \tag{30}$$

From (29) and (30), the inequality in (26) readily follows by replacing n_1 by n.

3. No-binding theorem

We introduce a TF energy-like functional

$$\mathcal{F}[\rho] = \frac{\pi\hbar^2}{qm\beta}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}) + 2\sum_{j=1}^{k} Z_j\,e^2\int d^2\mathbf{x}\,\rho(\mathbf{x})\,\ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right) - 2e^2\sum_{i<j}^{k} Z_i\,Z_j\,\ln\left(\frac{|\mathbf{R}_i-\mathbf{R}_j|}{2r_0}\right) - e^2\int d^2\mathbf{x}\,d^2\mathbf{x}'\,\rho(\mathbf{x})\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho(\mathbf{x}'), \tag{31}$$

where $\beta > 0$ is arbitrary. Here the $\mathbf{R}_i$ denote the positions of the nuclei. Taking the functional derivative of $\mathcal{F}[\rho]$ with respect to $\rho(\mathbf{x})$ and setting the resulting expression equal to zero, we have

$$\frac{\pi\hbar^2}{qm\beta}\rho_0(\mathbf{x}) = -\sum_{j=1}^{k} Z_j\,e^2\ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right) + e^2\int d^2\mathbf{x}'\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho_0(\mathbf{x}') \tag{32}$$

for the corresponding solution $\rho_0(\mathbf{x})$,

$$\int d^2\mathbf{x}\,\rho_0(\mathbf{x}) = \sum_{j=1}^{k} Z_j. \tag{33}$$

We introduce the functionals with some of the Z_j scaled by a positive parameter λ as

$$\mathcal{F}[\rho;\lambda Z_1,\ldots,\lambda Z_l, Z_{l+1},\ldots,Z_k;\mathbf{R}_1,\ldots,\mathbf{R}_k] \tag{34}$$

and

$$\mathcal{F}[\rho;\lambda Z_1,\ldots,\lambda Z_l;\mathbf{R}_1,\ldots,\mathbf{R}_l] \tag{35}$$

for $l < k$, with the corresponding solutions to ρ_0 in (32) denoted by $\rho_1(\mathbf{x})$, $\rho_2(\mathbf{x})$, respectively. That is,

$$\frac{\pi\hbar^2}{qm\beta}\rho_1(\mathbf{x}) = -\lambda\sum_{j=1}^{l} Z_j\,e^2\ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right) - \sum_{j=l+1}^{k} Z_j\,e^2\ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right) + e^2\int d^2\mathbf{x}'\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho_1(\mathbf{x}'), \tag{36}$$

$$\frac{\pi\hbar^2}{qm\beta}\rho_2(\mathbf{x}) = -\lambda\sum_{j=1}^{l} Z_j\,e^2\ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right) + e^2\int d^2\mathbf{x}'\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho_2(\mathbf{x}'). \tag{37}$$

Let

$$Q_i(\mathbf{x}) = \frac{\pi\hbar^2}{qm\beta}\rho_i(\mathbf{x}), \qquad i = 1, 2. \tag{38}$$

Then

$$Q_1(\mathbf{x}) - Q_2(\mathbf{x}) = -\sum_{j=l+1}^{k} Z_j\, e^2 \ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right) + \frac{qm\beta e^2}{\pi\hbar^2}\int d^2\mathbf{x}' \ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)[Q_1(\mathbf{x}') - Q_2(\mathbf{x}')] \tag{39}$$

and, in particular,

$$\nabla^2\left[Q_1(\mathbf{x}) - Q_2(\mathbf{x})\right] = -2\pi \sum_{j=l+1}^{k} Z_j\, e^2\, \delta^2(\mathbf{x}-\mathbf{R}_j) + \frac{2qm\beta}{\hbar^2}\, e^2[Q_1(\mathbf{x}) - Q_2(\mathbf{x})]. \tag{40}$$

By an elementary application of Green's theorem, we may infer, as in the 3D case, that $Q_1(\mathbf{x}) - Q_2(\mathbf{x}) \geq 0$. Now in reference to the functional

$$\mathcal{F}[\rho; Z_{l+1}, \ldots, Z_k; \mathbf{R}_{l+1}, \ldots, \mathbf{R}_k], \tag{41}$$

let $\rho_3(\mathbf{x})$ satisfy

$$\frac{\pi\hbar^2}{qm\beta}\, \rho_3(\mathbf{x}) = -\sum_{j=l+1}^{k} Z_j\, e^2 \ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right) + e^2 \int d^2\mathbf{x}' \ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho_3(\mathbf{x}'). \tag{42}$$

An almost identical analysis as in the 3D case then shows that

$$\mathcal{F}[\rho_1; Z_1, \ldots, Z_k; \mathbf{R}_1, \ldots, \mathbf{R}_k] \geq \mathcal{F}[\rho_2; Z_1, \ldots, Z_l; \mathbf{R}_1, \ldots, \mathbf{R}_l] + \mathcal{F}[\rho_3; Z_{l+1}, \ldots, Z_k; \mathbf{R}_{l+1}, \ldots, \mathbf{R}_k], \tag{43}$$

and more generally

$$\mathcal{F}[\rho_0; Z_1, \ldots, Z_k; \mathbf{R}_1, \ldots, \mathbf{R}_k] \geq \sum_{i=1}^{k} \mathcal{F}[\rho^i_{\mathrm{TF}}; Z_i, \mathbf{R}_i], \tag{44}$$

where

$$\frac{\pi\hbar^2}{qm\beta}\, \rho^i_{\mathrm{TF}}(\mathbf{x}) = -Z_i\, e^2 \ln\left(\frac{|\mathbf{x}-\mathbf{R}_i|}{2r_0}\right) + e^2 \int d^2\mathbf{x}' \ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right) \rho^i_{\mathrm{TF}}(\mathbf{x}). \tag{45}$$

Replacing $\mathbf{x}$ by $\mathbf{x}+\mathbf{R}_i$, and setting

$$\rho^i_{\mathrm{TF}}(\mathbf{x}+\mathbf{R}_i) = n_{\mathrm{TF}}(\mathbf{x})\big|_{m\to m\beta,\ Z\to Z_i}, \tag{46}$$

where $n_{\mathrm{TF}}(\mathbf{x})$ satisfies (8) and (12), we obtain

$$\mathcal{F}[\rho; Z_1, \ldots, Z_k; \mathbf{R}_1, \ldots, \mathbf{R}_k] \geq -0.5773\, e^2 \sum_{i=1}^{k} Z_i^2 \tag{47}$$

and we have used (23). Here $\rho \geq 0$ is arbitrary since, as in (26), ρ_0 provides the smallest value for $\mathcal{F}[\rho]$. Eq. (44) embodies a no-binding theorem.

From (31) and (47), we then obtain the following basic inequality:

$$
\begin{aligned}
-2e^2\sum_{i<j}^{k} Z_i\, Z_j\, \ln\left(\frac{|\mathbf{R}_i-\mathbf{R}_j|}{2r_0}\right) \geq & -\frac{\pi\hbar^2}{qm\beta}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}) \\
& -2\sum_{j=1}^{k} Z_j\, e^2 \int d^2\mathbf{x}\,\rho(\mathbf{x})\,\ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right) \\
& + e^2\int d^2\mathbf{x}\, d^2\mathbf{x}'\,\rho(\mathbf{x})\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho(\mathbf{x}') \\
& -0.5773\, e^2 \sum_{i=1}^{k} Z_i^2. \qquad (48)
\end{aligned}
$$

The latter also implies that

$$
\begin{aligned}
-2e^2\sum_{i<j}^{N} \ln\left(\frac{|\mathbf{x}_i-\mathbf{x}_j|}{2r_0}\right) \geq & -\frac{\pi\hbar^2}{qm\beta}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}) \\
& -2\sum_{i=1}^{N} e^2 \int d^2\mathbf{x}\,\rho(\mathbf{x})\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}_i|}{2r_0}\right) \\
& + e^2\int d^2\mathbf{x}\, d^2\mathbf{x}'\,\rho(\mathbf{x})\,\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho(\mathbf{x}') \\
& -0.5773\, e^2 N. \qquad (49)
\end{aligned}
$$

4. Lower bound for the exact ground-state energy of matter in 2D

Let

$$\rho(\mathbf{x}) = N\sum_{\sigma_1,\ldots,\sigma_N}\int d^2\mathbf{x}_2\cdots d^2\mathbf{x}_N\,\left|\psi(\mathbf{x}\sigma_1,\mathbf{x}_2\sigma_2,\ldots,\mathbf{x}_N\sigma_N)\right|^2 \qquad (50)$$

satisfy the normalization condition

$$\int d^2\mathbf{x}\,\rho(\mathbf{x}) = N. \qquad (51)$$

To derive a lower bound to the ground-state energy of matter, we need a lower bound to the expectation value of the kinetic energy $\langle\psi|\sum_{i=1}^{N}\mathbf{p}_i^2/2m\,|\psi\rangle = T$. To this end, set

$$f(\mathbf{x}) = 2\,\frac{\rho(\mathbf{x})}{\int d^2\mathbf{x}\,\rho^2(\mathbf{x})}\,T \qquad (52)$$

then

$$\left\langle\psi\middle|\sum_{i=1}^{N}\left[\frac{\mathbf{p}_i^2}{2m}-f(\mathbf{x}_i)\right]\middle|\psi\right\rangle = -T. \qquad (53)$$

An adaptation of the Schwinger bound [12] then leads to

$$\frac{\hbar^2}{3qm}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}) \leq T \tag{54}$$

by making use, in the process, of the exclusion principle for the effective Hamiltonian $\sum_{i=1}^{N}\left[\mathbf{p}_i^2/2m - f(\mathbf{x}_i)\right]$.

We define the total Hamiltonian of the system

$$\begin{aligned} H = {} & \sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m} + 2e^2\sum_{i=1}^{N}\sum_{j=1}^{k} Z_j \ln\left(\frac{|\mathbf{x}_i-\mathbf{R}_j|}{2r_0}\right) \\ & - 2e^2\sum_{i<j}^{N}\ln\left(\frac{|\mathbf{x}_i-\mathbf{x}_j|}{2r_0}\right) - 2e^2\sum_{i<j}^{k} Z_i\,Z_j \ln\left(\frac{|\mathbf{R}_i-\mathbf{R}_j|}{2r_0}\right). \end{aligned} \tag{55}$$

From (49) we have the following bound:

$$\begin{aligned} & \left\langle\psi\left| -2e^2\sum_{i<j}\ln\left(\frac{|\mathbf{x}_i-\mathbf{x}_j|}{2r_0}\right)\right|\psi\right\rangle \\ & \quad \geq -\frac{\pi\hbar^2}{qm\beta}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}) - e^2\int d^2\mathbf{x}\,d^2\mathbf{x}'\,\rho(\mathbf{x})\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho(\mathbf{x}') \\ & \quad\quad - 0.5773\,e^2N, \end{aligned} \tag{56}$$

and explicitly

$$\left\langle\psi\left|2e^2\sum_{i=1}^{N}\sum_{j=1}^{k} Z_j \ln\left(\frac{|\mathbf{x}_i-\mathbf{R}_j|}{2r_0}\right)\right|\psi\right\rangle = 2e^2\sum_{j=1}^{k} Z_j\int d^2\mathbf{x}\,\ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right)\rho(\mathbf{x}). \tag{57}$$

From (54)–(57), we obtain

$$\begin{aligned} \left\langle\psi\left|H\right|\psi\right\rangle \geq {} & \frac{\pi\hbar^2}{qm\beta'}\int d^2\mathbf{x}\,\rho^2(\mathbf{x}) + 2e^2\sum_{j=1}^{k} Z_j\int d^2\mathbf{x}\,\ln\left(\frac{|\mathbf{x}-\mathbf{R}_j|}{2r_0}\right)\rho(\mathbf{x}) \\ & - e^2\int d^2\mathbf{x}\,d^2\mathbf{x}'\,\rho(\mathbf{x})\ln\left(\frac{|\mathbf{x}-\mathbf{x}'|}{2r_0}\right)\rho(\mathbf{x}') \\ & - 2e^2\sum_{i<j}^{k} Z_i\,Z_j\ln\left(\frac{|\mathbf{R}_i-\mathbf{R}_j|}{2r_0}\right) - 0.5773\,e^2N, \end{aligned} \tag{58}$$

where $1/\beta' = (1/3\pi) - (1/\beta)$, with $\beta > 3\pi$ chosen as such for consistency.

With β replaced by β' in (48), the latter in conjunction with (58) then gives

$$\langle\psi|H|\psi\rangle \geq -0.5773\,e^2\left[N + \sum_{j=1}^{k} Z_j^2\right]. \tag{59}$$

Finally, using the bound

$$\sum_{j=1}^{k} Z_j^2 \leq Z_{\max} \sum_{j=1}^{k} Z_j = Z_{\max} N \tag{60}$$

in Eq. (59), we obtain

$$\langle \psi | H | \psi \rangle \geq -0.5773\, e^2 N [1 + Z_{\max}], \tag{61}$$

where $Z_{\max}$ corresponds to the nucleus with largest charge in units of $|e|$.

5. Inflation of matter

Let $\mathbf{x}$ denote the position of an electron relative, for example, to the center of mass of the nuclei. We define the set function

$$\chi_R(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \text{ lies within a circle of radius } R; \\ 0, & \text{otherwise.} \end{cases} \tag{62}$$

Accordingly, for the probability of finding the electrons within a circle of radius R,

$$\begin{aligned} \mathtt{Prob}[|\mathbf{x}_1| \leq R, \ldots, |\mathbf{x}_N| \leq R] &\leq \mathtt{Prob}[|\mathbf{x}_1| \leq R] \\ &= \frac{1}{N} \int d^2\mathbf{x}\, \rho(\mathbf{x})\, \chi_R(\mathbf{x}) \\ &\leq \frac{1}{N} \left[\int d^2\mathbf{x}\, \rho^2(\mathbf{x}) \right]^{1/2} A_R^{1/2}, \end{aligned} \tag{63}$$

where in the last inequality we have used the Schwarz inequality. Here A_R denotes the area in which the electrons are confined, i.e. $A_R = \pi R^2$.

Let V denote the interaction part in (55), i.e. $H = \sum_{i=1}^{N} \mathbf{p}_i^2/2m + V$. Then for a strictly negative energy state of matter $|\psi(m)\rangle$, i.e. $\langle \psi(m)|H|\psi(m)\rangle < 0$, we have

$$\langle \psi(m)| \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} |\psi(m)\rangle < -\langle \psi(m)| \left(\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} + V \right) |\psi(m)\rangle. \tag{64}$$

Let $-\mathcal{E}_N$ denote the ground-state energy of matter, i.e. from (61),

$$-\mathcal{E}_N \geq -0.5773\, e^2 N [1 + Z_{\max}]. \tag{65}$$

Since the state $|\psi(m/2)\rangle$ cannot lead for $\langle \psi(m/2)|H|\psi(m/2)\rangle$ to a numerical value lower than $-\mathcal{E}_N$, we have

$$-\mathcal{E}_N \leq \langle \psi(m/2)|H|\psi(m/2)\rangle. \tag{66}$$

From Eq. (64), we then obtain

$$\langle \psi(m)| \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} |\psi(m)\rangle \leq 2\, \mathcal{E}_N. \tag{67}$$

This inequality in conjunction with the one in (54) then gives

$$\int d^2\mathbf{x}\, \rho^2(\mathbf{x}) \leq 3.464\, \frac{qme^2}{\hbar^2}\, N[1 + Z_{\max}] \tag{68}$$

where we have finally made use of (65).

Now we invoke the inequality (63) to infer that

$$\mathtt{Prob}[|\mathbf{x}_1| \leq R, \ldots, |\mathbf{x}_N| \leq R]\left(\frac{N}{A_R}\right)^{1/2} \leq 1.861\sqrt{\frac{qme^2}{\hbar^2}[1 + Z_{\max}]}. \tag{69}$$

This leads to an inescapable fact that necessarily, for a non-vanishing probability of having the electrons within a circle of radius R, the corresponding area A_R grows not any slower than the single power of N for $N \to \infty$, since otherwise the left-hand side of (69) would go to infinity in this limit and would be in contradiction with the finite upper bound in (69). Upon multiplying (69) by $(A_R/N)^{1/2}$, we also note that the infinite density limit $N/A_R \to \infty$ does not occur, as the probability on the left-hand side of (69) would go to zero in this limit.

Finally, we note that for the expectation value

$$\begin{aligned}
\left\langle \sum_{i=1}^{N} \frac{|\mathbf{x}_i|}{N} \right\rangle &= \sum_{\sigma_1,\ldots,\sigma_N} \int d^2\mathbf{x}_1 \cdots d^2\mathbf{x}_N \left(\sum_{i=1}^{N} \frac{|\mathbf{x}_i|}{N}\right) |\psi(\mathbf{x}_1\sigma_1, \ldots, \mathbf{x}_N\sigma_N)|^2 \\
&= \frac{1}{N} \int d^2\mathbf{x}\, |\mathbf{x}|\, \rho(\mathbf{x}) \\
&\geq \frac{R}{N} \int_{|\mathbf{x}|>R} d^2\mathbf{x}\, \rho(\mathbf{x}) \\
&= R\,(1 - \mathtt{Prob}[|\mathbf{x}| \leq R]) \\
&\geq R\left(1 - \left(\frac{\pi R^2}{N}\right)^{1/2} 1.861\sqrt{\frac{qme^2}{\hbar^2}[1 + Z_{\max}]}\right).
\end{aligned} \tag{70}$$

Optimizing the extreme right-hand side of (70) over R, we obtain

$$R = 0.2687\left(\frac{N}{\pi}\,\frac{\hbar^2}{qme^2}\,\frac{1}{[1 + Z_{\max}]}\right)^{1/2} \tag{71}$$

leading from (70) to the explicit non-vanishing lower bound

$$\left\langle \sum_{i=1}^{N} \frac{|\mathbf{x}_i|}{N} \right\rangle \geq 0.1343\, N^{1/2}\left(\frac{\hbar^2}{qme^2\pi[1 + Z_{\max}]}\right)^{1/2}. \tag{72}$$

6. Conclusion

Two-dimensional matter is physically relevant, see e.g. [7–10], and our analysis shows that matter is stable in 2D. It is an important theoretical question to investigate if the change of the dimensionality of space will alter these properties, or that stability is a characteristic of the space dimension. We do not wish to speculate on this until a detailed rigorous study of this is carried out in dimensions higher than three. A preliminary study of this shows that the TF density is too singular at the origin leading to serious problems with normalizability conditions in conformity with earlier studies [11]. The situation becomes more extreme in 1D as the potential rises linearly as opposed to the 2D case which is only logarithmic. These problems will be considered in a forthcoming report.

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