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On the Compatibility of Overdetermined Systems of Double Waves

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Abstract. Obtaining equations for double waves in the case of a general quasilinear system of partial differential equations poses some difficulties. They are connected with the complexity and awkwardness of the study of overdetermined systems, describing solutions of this class. However, there are general statements about double waves of autonomous quasilinear systems of equations. This article is devoted to the classification of irreducible double waves of autonomous nonhomogeneous systems.

Keywords: Partially invariant solutions, degenerate hodograph, multiple waves, double waves.

1. Introduction

A solution $u_i = u_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$, of the autonomous quasilinear system of equations

$$\sum_{\alpha=1}^{n} A_\alpha(u)\frac{\partial u}{\partial x_\alpha} = f(u) \tag{1}$$

is called a multiple wave of rank r if a rank of the Jacobi matrix $\partial(u_1, u_2, \ldots, u_m)/\partial(x_1, x_2, \ldots, x_n)$ is equal to r in a domain G of the independent variables $x_1, x_2, \ldots, x_n$. Here A_α are rectangular $N \times m$ matrices with elements $a_{ij}^{\alpha}(u)$ and $f = (f_1(u), \ldots, f_N(u))$.

Depending on the value of r, a multiple wave is called a simple $(r = 1)$, double $(r = 2)$ or triple $(r = 3)$ wave. The value $r = 0$ corresponds to uniform flow with constant u_i, $(i = 1, 2, \ldots, m)$, and $r = n$ corresponds to the general case of nondegenerate solutions. Multiple waves of all ranks compose a class of degenerate hodograph solutions.

The singularity of the Jacobi matrix means that the functions $u_i(x)$ $(i = 1, 2, \ldots, m)$ are functionally dependent (hodograph is degenerate), with $m-r$ number of functional constraints

$$u_i = \Phi_i(\lambda^1, \lambda^2, \ldots, \lambda^r), \quad (i = 1, 2, \ldots, m). \tag{2}$$

The variables $\lambda^1(x), \lambda^2(x), \ldots, \lambda^r(x)$ are called parameters of the wave. The solutions with a degenerate hodograph are a generalization of travelling waves: the wave parameters of the travelling waves are linear forms of independent variables. To find the r-multiple wave, it is necessary to substitute the representation (2) into system (1). We get an overdetermined

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system of differential equations for the wave parameters $\lambda^i(x)$ $(i = 1, 2, \ldots, r)$, which should be studied for compatibility. A review of applications of multiple waves in gas dynamics can be found in [1].

The main problem of the theory of solutions with a degenerate hodograph is getting a closed system of equations in the space of dependent variables (hodograph), establishing the arbitrariness of the general solution and determining flow in the physical space.

An arbitrary nonhomogeneous system (1) does not change under the transformations

$$x_i' = x_i + b_i, \quad (i = 1, 2, \ldots, n),$$

that compose a group G^n. For homogeneous systems (1) $(f = 0)$, there is one more scale transformation[1] $x_i' = ax_i$ $(i = 1, 2, \ldots, n)$. From the group analysis point of view, an r-multiple wave is a partially invariant solution with respect to G^n (or G^{n+1}) [2]. A class of partially invariant solutions of some group H is characterized by rank σ and defect δ: class $H(\sigma, \delta)$-solutions. If some class $H(\sigma, \delta)$-solutions are class $H_1(\sigma, \delta_1)$-solutions with fewer defects $\delta_1 < \delta$, then it is said that the class $H(\sigma, \delta)$-solutions are reduced to having fewer defects. For example, if $\delta_1 = 0$, then such a solution is reducible to an invariant solution with respect to the subgroup H_1.

A study of partially invariant solutions shows that classes of solutions of a given rank with fewer defects are easier to obtain. This is connected with the idea that the analysis of compatibility for the solutions with greater defects is more difficult. Therefore, it is useful to a priori clarify the structure properties of the overdetermined system.

There are only a few sufficient conditions of the reducibility [2] that allow us to predict a reduction on the basis of the structure properties of an overdetermined system. One of these conditions is a restriction on the ability to define all first derivatives of a solution (otherwise the solution is reduced to an invariant solution). Others are concerned with double waves. If in the process of obtaining compatibility conditions for the wave parameters of a double wave, we obtain $N = 2n - 1$ homogeneous equations of type (1), then this double wave is an invariant solution. In particular, plane nonisobaric double waves with the general state equation which has a defect of invariance $\delta = 2$ are isoentropic [2]. Another application of these conditions to double waves of gas dynamics equations leads to the result [3] that the class of irreducible to invariant solutions of plane isoentropic irrotational double waves is described by the flows obtained in [4]. For homogeneous systems of type (1) with $N = 2n - 2$ and $n = 3$, a full classification of double waves with the additional assumption about having functional arbitrariness of the solution was carried out in [5].

This article is devoted to the study of nonhomogeneous systems of type (1) with $N = 2n-1$ equations, the solutions of which are not reducible to invariant.

2. Nonhomogeneous Systems ($N = 2n - 1$)

Let a system of $N = 2n - 1$ independent autonomous quasilinear equations on the wave parameters λ and μ of a double wave be of type (1). It can be obtained as a result of substitution of the representation of a double wave:

$$u_i = u_i(\lambda, \mu), \ (i = 1, 2 \ldots, m)$$

[1] The full Lie group admissible by system (1) can be wider than G^n (or G^{n+1}).

into the initial system and some analysis of compatibility.[2] Without loss of generality equations, for the wave parameters can be rewritten as

$$\begin{aligned} \lambda_i &= p_i(\lambda,\mu)\lambda_1 + f_i(\lambda,\mu), \\ \mu_j &= q_j(\lambda,\mu)\lambda_1 + g_j(\lambda,\mu), \quad (i=1,\dots,n;\ j=1,\dots,n). \end{aligned} \tag{3}$$

Here $\lambda_i = \partial\lambda/\partial x_i$, $\mu_j = \partial\mu/\partial x_j$ and, for the sake of simplicity, we set $p_1 \equiv 1$, $f_1 \equiv 0$.

The problem is to classify systems of type (3), the solutions of which are irreducible to invariant solutions.

A classification is derived with respect to equivalence transformations, admitted by system (3):

(a) linear nondegenerate replacement of independent variables;
(b) replacement of wave parameters: $\lambda' = L(\lambda,\mu)$, $\mu' = M(\lambda,\mu)$.

In the last case, the coefficients p_i, q_i and the functions f_i, g_i are transformed by formulae:

$$p_1' = 1, \quad p_i' = \frac{p_i L_\lambda + q_i L_\mu}{L_\lambda + q_1 L_\mu}, \quad q_j' = \frac{p_j M_\lambda + q_j M_\mu}{L_\lambda + q_1 L_\mu},$$

$$f_1' = 0, \quad f_i' = f_i L_\lambda + g_i L_\mu - g_1 L_\mu p_i', \quad g_j' = f_j M_\lambda + g_j M_\mu - g_1 L_\mu q_j',$$

$$(i = 2,\dots,n;\ j = 1,\dots,n).$$

As a result of such transformations (as in the homogeneous case [2]), it is possible to let $q_1 = 0$. For this purpose, it is enough to choose a function $L(\lambda,\mu)$, which satisfies the equation $L_\lambda + q_1 L_\mu = 0$.

If $\sum_i q_i^2 \neq 0$, then the coefficients of system (3) can be transformed to

$$q_1 = 0, \quad q_2 = 1. \tag{4}$$

Simultaneous to the equalities $q_1 = 0$, $q_2 = 1$ under replacement of the wave parameters, iff

$$M_\lambda = 0, \quad L_\lambda = M_\mu,$$

results in

$$L = \lambda M'(\mu) + \omega(\mu), \quad M = M(\mu). \tag{5}$$

Another case corresponds to system (3) with

$$q_i = 0 \quad (i = 1, 2, \dots, n). \tag{6}$$

There is no case (6) for homogeneous systems, because conditions (6) contradict the definition of a double wave for such a kind of systems: rank of the Jacobi matrix is less than two.

A study of the compatibility of system (3) consists of the following. As a result of a reduction of the overdetermined system (3) to an involutive system, we get equations with a structure of nonhomogeneous quadratic forms with respect to the derivative λ_1. If at least

[2] A case of homogeneous $N = 2n - 1$ equations was studied by Ovsiannikov [2].

one of the coefficients of these forms is not equal to zero, then it means that a solution of the system satisfies the overdetermined system of equations from which all first derivatives can be found. By virtue of the reduction theorem [2], it gives the reduction of this solution to an invariant solution. Therefore, these forms are decomposed on subsystems on functions p_i, q_j, f_i, g_j: quadratic, linear and 'zero' terms with respect to power of the derivative λ_1. Further simplifications are connected with more the detailed study of the compatibility conditions of systems of types (4) and (6).

3. Systems of Type (4)

The value of $\lambda_{11} = a\lambda_1 + b$ can be defined from the expression $D_1(\mu_2 - \lambda_1 - g_2) - D_2(\mu_1 - g_1) = 0$, where D_i is a total derivative with respect to x_i, $a = p_2 g_{1\lambda} + g_{1\mu} - g_{2\lambda}$, $b = f_2 g_{1\lambda} + g_2 g_{1\mu} - g_1 g_{2\mu}$. It can be noted that all second derivatives λ_{ij} and μ_{ij} can be found. Therefore arbitrariness of the general solution of system of type (4) is only constant. For example, the derivatives

$$\lambda_{i1} = p_{i\lambda}\lambda_1^2 + \lambda_1(ap_i + f_{i\lambda} + g_1 p_{i\mu}) + bp_i + g_1 f_{i\mu}, \quad (i = 2, 3, \ldots, n)$$

can be found from the expressions $D_1(\lambda_i - p_i\lambda_1 - f_i) = 0$. After substituting them into $F_i \equiv D_1\mu_i - D_i\mu_1 = 0$, $(i = 2, 3, \ldots, n)$, we obtain nonhomogeneous quadratic forms with respect to the derivative λ_1. By virtue of the prohibition of reduction of the solution of system (3) to an invariant, the coefficients of these quadratic forms F_i have to be equal to zero:

$$q_{i\lambda} = 0, \tag{7}$$

$$q_i(p_2 g_{1\lambda} - g_{2\lambda}) + g_1 g_{i\mu} + g_{i\lambda} - p_i g_{1\lambda} = 0, \tag{8}$$

$$q_i b + g_1 g_{i\mu} - f_i g_{1\lambda} - g_i g_{1\mu} = 0, \quad (i = 2, 3, \ldots, n). \tag{9}$$

In the same way from the quadratic forms $D_i\lambda_j - D_j\lambda_i = 0$, we get

$$q_j p_{i\mu} = q_i p_{j\mu}, \tag{10}$$

$$\begin{aligned} &f_j p_{i\lambda} + g_j p_{i\mu} + q_j f_{i\mu} + p_i g_1 p_{j\mu} = f_i p_{j\lambda} + g_i p_{j\mu} + q_i f_{j\mu} + p_j g_1 p_{i\mu}, \\ &f_j f_{i\lambda} + g_j f_{i\mu} + p_i g_1 f_{j\mu} = f_i f_{j\lambda} + g_i f_{j\mu} + p_j g_1 f_{i\mu}, \quad (i, j = 2, 3, \ldots, n;\ i \neq j). \end{aligned} \tag{11}$$

And from the equalities $D_i\mu_j - D_j\mu_i = 0$, we find

$$q_j(p_{i\lambda} - q_{j\mu}) = q_i(p_{j\lambda} - q_{j\mu}), \tag{12}$$

$$\begin{aligned} &g_j q_{i\mu} + q_i(p_j a + f_{j\lambda} + g_1 p_{j\mu}) + p_j g_{i\lambda} + q_j g_{i\mu} \\ &\quad = g_i q_{j\mu} + q_j(p_i a + f_{i\lambda} + g_1 p_{i\mu}) + p_i g_{j\lambda} + q_i g_{j\mu}, \end{aligned} \tag{13}$$

$$\begin{aligned} &q_i(p_j b + g_1 f_{j\mu}) + f_j g_{i\lambda} + g_j g_{i\mu} \\ &\quad = q_j(p_i b + g_1 f_{i\mu}) + f_i g_{j\lambda} + g_i g_{j\mu}, \quad (i, j = 2, 3, \ldots, n;\ i \neq j). \end{aligned} \tag{14}$$

We note that the expressions $D_1\lambda_{i1} - D_i\lambda_{11} = 0$ are cubic polynomials with respect to the derivative λ_1: $p_{i\lambda\lambda}\lambda_1^3 + \cdots = 0$. Therefore,

$$p_{i\lambda\lambda} = 0, \quad (i = 2, 3, \ldots, n).$$

With the help of equivalence transformations (5) that leave the conditions $q_1 = 0$, $q_2 = 1$ unchanged, because of the choice of functions $\omega(\mu)$ and $\psi(\mu)$, we can assume that $p_2 = 0$. Then from (6), (10), (12), we get

$$q_{i\lambda} = 0, \quad p_{i\mu} = 0, \quad p_{i\lambda} = q_{i\mu}, \quad (i = 2, 3, \ldots, n). \tag{15}$$

By using (15) in the expressions $D_1\lambda_{i1} - D_i\lambda_{11} = 0$ $(i = 2, 3, \ldots, n)$, we find

$$q_i a_\mu = 2ap_{i\lambda} + f_{i\lambda\lambda}, \tag{16}$$

$$f_i a_\lambda + g_i a_\mu + q_i b_\mu = 3bp_{i\lambda} + g_1(p_i a_\mu + 2f_{i\lambda\mu}) + g_{1\lambda} f_{i\mu}, \tag{17}$$

$$ag_1 f_{i\mu} + b_\lambda f_i + g_i b_\mu = bf_{i\lambda} + g_1(p_i b_\mu + g_1 f_{i\mu\mu} + g_{1\mu} f_{i\mu}), \tag{18}$$

The functions p_i, q_j, f_i, g_j must satisfy (8), (9), (11), (14), (13), (15–18) for the irreducibility of solutions of system (3) to invariant solutions.

We note that

$$p_i = \lambda A_i + B_i, \quad q_j = \mu A_i + C_i, \quad (i = 2, 3, \ldots, n),$$

are the general solutions of Equations (15), where

$$A_1 = 0, \quad B_1 = 1, \quad C_1 = 0, \quad A_2 = 0, \quad B_2 = 0, \quad C_2 = 1,$$

and A_i, B_i, C_i $(i = 3, \ldots, n)$ are arbitrary constants. Further simplifications of equations of system (3) are connected with an application of equivalence transformations, which correspond to a replacement of the independent variables. By means of the replacement

$$x_1' = B_\alpha x_\alpha, \quad x_2' = C_\alpha x_\alpha, \quad x_i' = x_i, \quad (i = 3, 4, \ldots, n)$$

we can obtain $B_i = 0, C_i = 0$, $(i = 3, 4, \ldots, n)$.

Further, we have to consider two cases: (a) all $A_i = 0$ $(i = 3, 4, \ldots, n)$ and (b) $\sum_i A_i^2 \neq 0$.

In the first case (a), system (3) has the form

$$\begin{aligned} \lambda_2 &= f_2, \quad \lambda_i = f_i, \\ \mu_1 &= g_1, \quad \mu_2 = \lambda_1 + g_2, \quad \mu_i = g_i, \quad i \geq 3. \end{aligned} \tag{19}$$

In the second case (b), without loss of generality, we can regard $A_3 \neq 0$. Then as a result of one more linear transformation of the independent variables

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = A_\alpha x_\alpha, \quad x_i' = x_i, \quad (i = 4, 5, \ldots, n),$$

system (3) becomes

$$\begin{aligned} \lambda_2 &= f_2, \quad \lambda_3 = \lambda\lambda_1 + f_3, \quad \lambda_i = f_i, \\ \mu_1 &= g_1, \quad \mu_2 = \lambda_1 + g_2, \quad \mu_3 = \mu\mu_2 + g_3, \quad \mu_i = g_i, \quad i \geq 4. \end{aligned} \tag{20}$$

Further successive simplifications of systems (19) and (20) are connected with the analysis of the constants C_i.

3.1. System (19)

In this case, Equations (8), (9), (11), (14) are reduced to

$$g_i = C_i\mu + K_i, \quad f_i = C_i\lambda + R_i,$$
$$C_i(\lambda g_{1\lambda} + \mu g_{1\mu} - g_1) + R_i g_{1\lambda} + K_i g_{1\mu} = 0,$$
$$C_i(\lambda g_{2\lambda} + \mu g_{2\mu} - g_2) + R_i g_{2\lambda} + K_i g_{2\mu} = 0,$$
$$C_i(\lambda f_{2\lambda} + \mu f_{2\mu} - f_2) + R_i f_{2\lambda} + K_i f_{2\mu} = 0,$$
$$C_i R_j = C_j R_i, \quad C_i K_j = C_j K_i, \quad (i, j = 3, 4, \dots, n), \tag{21}$$

where C_i, R_i, K_i are arbitrary constants.

3.1.1. *Case* $C_3 \neq 0$

If at least one of the constants C_i is not equal to zero (without loss of generality, we can take $C_3 \neq 0$), then with the help of transformations

$$\lambda' = \lambda + \frac{R_3}{C_3}, \quad \mu' = \mu + \frac{K_3}{C_3},$$

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = \sum_{\alpha=3}^{n} C_\alpha x_\alpha, \quad x_i' = x_i, \quad (i = 4, \dots, n),$$

system (19) becomes

$$\lambda_3 = \lambda, \quad \mu_3 = \mu, \quad \lambda_i = 0, \quad \mu_i = 0, \quad (i = 4, 5, \dots, n),$$
$$\lambda_2 = \lambda F(\mu/\lambda), \quad \mu_1 = \lambda\Psi_1(\mu/\lambda), \quad \mu_2 = \lambda_1 + \lambda\Psi_2(\mu/\lambda). \tag{22}$$

The functions F, Ψ_1, Ψ_2 must satisfy a system of three ordinary differential equations of the second order. This system is obtained after substitution of

$$f_2 = \lambda F(\mu/\lambda), \quad g_1 = \lambda\Psi_1(\mu/\lambda), \quad g_2 = \lambda\Psi_2(\mu/\lambda),$$

into Equations (16–18):

$$\Psi_1'' + y\Psi_2'' - y^2 F'' = 0,$$
$$(y^2 F - y\Psi_2 - \Psi_1)F'' = 0, \quad (y^2 F - y\Psi_2 - \Psi_1)\Psi_2'' = 0,$$

where $y \equiv \mu/\lambda$.

It can be noted that system (22) is invariant with respect to the transformation: $\lambda' = -\lambda$, $\mu' = -\mu$. Therefore, we can consider that $\lambda > 0$. It allows one more simplification by transformation:

$$\lambda' = \frac{\mu}{\lambda}, \quad \mu' = \ln(\lambda), \quad x_1' = x_2, \quad x_2' = x_1, \quad x_i = x_i, \quad (i = 3, 4, \dots, n).$$

System (22) is reduced to

$$\lambda_2 + \lambda\lambda_1 = \hat{\Psi}_1(\lambda), \quad \lambda_i = 0, \quad (i = 3, 4, \ldots, n),$$
$$\mu_1 = F(\lambda), \quad \mu_2 = \lambda_1 + \hat{\Psi}_2(\lambda), \quad \mu_3 = 1, \quad \mu_i = 0, \quad (i = 4, \ldots, n). \tag{23}$$

Here $\hat{\Psi}_1(\lambda) = \Psi_1(\lambda) + \lambda\Psi_2(\lambda) - \lambda^2 F(\lambda)$, $\hat{\Psi}_2(\lambda) = -\Psi_2(\lambda) + \lambda F(\lambda)$.

Let us make some remarks about solutions of system (23). A solution of (23) has the form

$$\lambda = \Lambda(x_1, x_2), \quad \mu = x_3 + G(x_1, x_2),$$

where the function $G(x_1, x_2)$ can be found from the totally integrable compatible system of differential equations. These solutions are invariant solutions of Equations (23) with respect to algebra with generators:

$$\partial_{x_3} + \partial_\mu, \quad \partial_{x_i}, \quad (i = 4, \ldots, n). \tag{24}$$

Assume that the functions $\Lambda(x_1, x_2)$ and $G((x_1, x_2)$ are functionally dependent, then the Jacobian

$$W(x_1, x_2) = \frac{\partial(\lambda, \mu)}{\partial(x_1, x_2)} = \lambda_1^2 + \lambda_1(\hat{\Psi}_2 + \lambda F) - F\hat{\Psi}_1 = 0.$$

This equation supplies the sufficient conditions for the reducibility of the solution of system (23) to an invariant solution with respect to $H \subset G^n$. Therefore, for irreducible solutions, the functions $\Lambda(x_1, x_2)$ and $G((x_1, x_2)$ are functionally independent or $W(x_1, x_2) \neq 0$.

We note that if $\hat{\Psi}_1 \neq 0$, then functions F, Ψ_1, Ψ_2 are linear: $F = k_1\lambda + k_2$, $\Psi_2 = k_3\lambda + k_4$, $\Psi_2 = k_5\lambda + k_6$ with arbitrary constants k_i $(i = 1, 2, \ldots, 6)$. If $\hat{\Psi}_1 = 0$, then $\hat{\Psi}_2'(\lambda) + \lambda F'(\lambda) = 0$ and $\Lambda = x_1/x_2$ up to shifts of the independent variables and because of $W = x_2^{-2}(1 + x_2\hat{\Psi}_2 + x_1 F) \neq 0$, then the solution is not reducible to an invariant solution of $H \subset G^n$.

3.1.2. *Case $C_i = 0$ ($i = 3, 4, \ldots, n$)*

Let us consider the case with all constants zero, $C_i = 0$.

Firstly, assume that at least one of the constants K_i is not equal to zero (without loss of generality, we can consider that $K_3 \neq 0$). Then from (21) we get

$$g_1 = g_1(\lambda - R\mu), \quad g_2 = g_2(\lambda - R\mu), \quad f_2 = f_2(\lambda - R\mu),$$

where $R = R_3/K_3$. If $g_1' = g_2' = f_2' = 0$, then the solution of system (23) is linear with respect to the independent variables, i.e. it is invariant with respect to some subgroup $H \subset G^n$. Therefore a prohibition of reducibility to an invariant solution leads to conditions $(g_1')^2 + (g_2')^2 + (f_2')^2 \neq 0$ or from (21) we have $R_i = RK_i$. After the transformation

$$x_3' = \sum_{i=3}^{n} K_i x_i, \quad x_i' = x_i, \quad i \neq 3$$

we obtain $f_3 = R$, $g_3 = 1$, $g_i = 0$, $f_i = 0$, $(i = 4, 5, \ldots, n)$. In addition we can reckon that $R = 0$. Really, if it is not so, then after one more transformation

$$\lambda' = \lambda - R\mu, \quad \mu' = R\mu,$$
$$x_1' = R^{-1}x_1 - x_2, \quad x_2' = x_2, \quad x_3' = Rx_3,$$

the same system can be obtained, but with $R = 0$. Irreducibility conditions (16–18) in this case become

$$f_2 = k_1\lambda + k_2, \quad g_1'' f_2 = 0, \quad g_2'' f_2 = 0$$

with arbitrary constants k_1, k_2. We note that if $f_2 = 0$ ($k_1 = 0, k_2 = 0$), then a solution of (19) is $\lambda = \varphi(x_1)$, $\mu = x_3 + cx_2 + \psi(x_1)$, which is invariant with respect to some subalgebra $H \subset G^n$. Here c is a constant. Therefore, for systems irreducible to invariant solutions, we have to consider only the case when $f_2 \neq 0$. In this case, functions g_1 and g_2 are linear $g_1 = k_3\lambda + k_4$, $g_2 = k_5\lambda + k_6$ and system (19) is

$$\begin{aligned} \lambda_2 &= k_1\lambda + k_2, \quad \lambda_i = 0, \quad (i = 3, 4, \ldots, n), \\ \mu_1 &= k_3\lambda + k_4, \quad \mu_2 = \lambda_1 + k_5\lambda + k_6, \quad \mu_3 = 1, \quad \mu_j = 0, \quad (j = 4, 5, \ldots, n). \end{aligned} \tag{25}$$

If $k_1 \neq 0$, then by equivalence transformations we can consider that $k_1 = 1$, $k_2 = 0$. In this case

$$\lambda = \varphi(x_1)e^{x_2}, \quad \mu = (\varphi' + k_5\varphi)e^{x_2} + k_6x_2 + x_3,$$

where the function $\varphi = \varphi(x_1)$ satisfies the homogeneous linear ordinary differential equation

$$\varphi'' - k_3\varphi' + k_5\varphi = 0.$$

If $k_1 = 0$, but $k_2 \neq 0$, then, as in previous case, via equivalence transformations we can put $k_1 = 0$, $k_2 = 1$. And then

$$\lambda = x_2 + \varphi(x_1), \quad \mu = x_3 + x_2\left(\varphi' + \frac{k_5}{2}x_2 + k_5\varphi + k_6\right) + \psi,$$

where the functions $\varphi = \varphi(x_1)$ and $\psi = \psi(x_1)$ satisfy the ordinary differential equations

$$\varphi'' + k_5\varphi' - k_3 = 0, \quad \psi' = k_3\varphi + k_4.$$

Now let all constants $K_i = 0$. If at least one of the constants R_i is not equal to zero (without loss of generality, we can account that $R_3 \neq 0$), then by transformation

$$\lambda' = \mu, \quad \mu' = \lambda, \quad x_1' = x_2, \quad x_2' = x_1, \quad x_i = x_i, \quad (i = 3, 4, \ldots, n),$$

the same system is obtained as was considered in the previous case. If all $R_i = 0$, then for such a solution

$$\lambda = \Lambda(x_1, x_2), \quad \mu = G(x_1, x_2)$$

and it is invariant with respect to the subalgebra $H \subset G^n$, which corresponds to the subalgebra $\{\partial_{x_3}, \partial_{x_4}, \ldots, \partial_{x_n}\}$.

3.2. SYSTEM (20)

A study of compatibility of system (20) is more cumbersome. In this case, Equations (8), (9), (11), (14), (16–18) can be reduced to

$$
\begin{aligned}
&g_{3\lambda} = \lambda g_{1\lambda} + \mu g_{2\lambda} - g_1,\\
&s_2 \equiv \mu b + g_1 g_{3\mu} - f_3 g_{1\lambda} - g_3 g_{1\mu} = 0,\\
&f_{3\mu} = \mu f_{2\mu} - f_2,\\
&f_2 f_{3\lambda} + g_2 f_{3\mu} + \lambda g_1 f_{2\mu} = f_3 f_{2\lambda} + g_3 f_{2\mu},\\
&g_2 + \mu f_{2\lambda} + g_{3\mu} = \lambda g_{2\lambda} + \mu g_{2\mu} + f_{3\lambda},\\
&s_6 \equiv \mu g_1 f_{2\mu} + f_2 g_{3\lambda} + g_2 g_{3\mu} - (f_3 g_{2\lambda} + g_3 g_{2\mu} + \lambda b + g_1 f_{3\mu}),\\
&f_i = 0, \quad g_i = 0, \quad (i = 4, 5, \ldots, n),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&a_\mu = f_{2\lambda\lambda},\\
&\mu a_\mu = 2a + f_{3\lambda\lambda},\\
&f_2 a_\lambda + g_2 a_\mu + b_\mu = g_1(2f_{2\lambda\mu}) + g_{1\lambda} f_{2\mu},\\
&f_3 a_\lambda + g_3 a_\mu + \mu b_\mu = 3b + g_1(\lambda a_\mu + 2f_{3\lambda\mu}) + g_{1\lambda} f_{3\mu},\\
&a g_1 f_{2\mu} + b_\lambda f_2 + g_2 b_\mu = b f_{2\lambda} + g_1(g_1 f_{2\mu\mu} + g_{1\mu} f_{2\mu}),\\
&a g_1 f_{3\mu} + b_\lambda f_3 + g_3 b_\mu = b f_{3\lambda} + g_1(\lambda b_\mu + g_1 f_{3\mu\mu} + g_{1\mu} f_{3\mu}).
\end{aligned}
\tag{27}
$$

The problem is to find a general solution (up to equivalence transformation) of system (26), (27). Because Equations (26) and (27) are not sufficient for irreducibility of a solution of system (20) to invariant solution, then the next problem is to try to analyze a solution of (20) with the found functions f_i, g_j and coefficients p_i, q_j.

All further intermediate calculations in the study of the compatibility of system (26) were made on a computer using the system REDUCE [6]. Here we give the method of computations and final results.

Let us input the new function $G_3 = g_3 - \mu g_2$ instead of g_3. From $(26)_1$ and $(26)_5$, we find $G_{3\lambda}$, $G_{3\mu}$ and from $(27)_1$: $f_{2\lambda\lambda}$ and $f_{3\lambda\lambda}$. After substitution of the found expressions into $\partial G_{3\lambda}/\partial\mu - \partial G_{3\mu}/\partial\lambda = 0$, we get the equation $(\lambda(g_{1\mu} - g_{2\lambda}))_\lambda = 0$. Without loss of generality, the last equation can be integrated:

$$
g_1 = \varphi_\lambda, \quad g_2 = \varphi_\mu + \psi_1 \log\lambda,
\tag{28}
$$

where $\varphi = \varphi(\lambda, \mu)$ and $\psi_1 = \psi_1(\mu)$ are arbitrary functions. After substitution of (28) into expressions for $f_{2\lambda\lambda}$ and $f_{3\lambda\lambda}$, we get

$$
f_{2\lambda\lambda} = -\frac{\psi_1'}{\lambda}, \quad f_{3\lambda\lambda} = \frac{2\psi_1 - \mu\psi_1'}{\lambda}.
$$

Integration of the last expressions allows us to find the functions

$$
f_2 = \lambda\psi_1'(1 - \log\lambda) + \lambda\psi_2 + \psi_3, \quad f_3 = \lambda(\mu\psi_1' - 2\psi_1)(1 - \log\lambda) + \lambda\psi_4 + \psi_5
$$

with arbitrary functions $\psi_i = \psi_i(\mu)$ $(i = 2, 3, 4, 5)$. From $(26)_3$, we have

$$\lambda(\psi_2 + \psi_4' - \mu\psi_2') + \psi_3 + \psi_5' - \mu\psi_3' = 0.$$

After splitting with respect to λ, we get

$$\psi_4' = \mu\psi_2' - \psi_2, \quad \psi_5' = \mu\psi_3' - \psi_3$$

or, if we input a new function $\psi_6 = \psi_6(\mu)$ by $\psi_4 = \psi_6' + \mu\psi_2 - \psi_1$, then $\psi_2 = (\psi_1' - \psi_6'')/2$. In this case,

$$\frac{\partial G_3}{\partial \lambda} = -\varphi_\lambda + \lambda\varphi_{\lambda\lambda}, \quad \frac{\partial G_3}{\partial \mu} = -2\varphi_\lambda + \lambda\varphi_{\lambda\mu} + \psi_6'$$

which can be integrated as $G_3 = -2\varphi + \lambda\varphi_\lambda + \psi_6$.

A composition of differentiating $(26)_6$ with respect to λ and subtracting it by differentiating $(26)_2$ with respect to μ and adding it to $(27)_3$ is

$$\psi_1\varphi_{\lambda\mu} - \psi_1'\varphi_\lambda + \frac{\psi_1}{\lambda} = 0.$$

If $\psi_1 \neq 0$, then we can get a contradiction. Really, let $\psi_1 \neq 0$, then the last equation can be integrated

$$\varphi = \psi_1(G - \mu \log \lambda) + \psi_7,$$

where $G = G(\lambda)$ and $\psi_7 = \psi_7(\mu)$ are arbitrary functions. In this case, Equation $(26)_4$ has the form

$$G(a_1\lambda \log \lambda + a_2\lambda + a_3) + a_4\lambda \log^2 \lambda + a_5\lambda \log \lambda + a_6\lambda + a_7 \log \lambda + a_8 = 0, \tag{29}$$

where a_i, $(i = 1, 2, \ldots, 8)$ are polynomials of functions $\psi_1, \psi_3, \psi_5, \psi_6, \psi_7$ and their derivatives. It can be shown that (29) is possible only if $\psi_1 = 0$. But it contradicts the original assumption about ψ_1. Therefore, we have to consider $\psi_1 = 0$.

Further consideration is based on the analysis of the compatibility of Equations $(26)_4$ and $\partial s_2/\partial\mu - \partial s_6/\partial\lambda = 0$, which have the forms:

$$\varphi_\mu h - 2\varphi h' + \psi_6 h' - \psi_3(\mu\psi_6'' - 2\psi_6') + \psi_5\psi_6'' = 0, \tag{30}$$

$$-3\varphi_\lambda\varphi_{\mu\mu} + \varphi_\lambda\psi_6'' + 3\varphi_\mu\varphi_{\lambda\mu} - \varphi_{\lambda\lambda}h = 0, \tag{31}$$

where $h = \lambda\psi_6'' - 2\psi_3$.

Assume that $h = 0$, so $\psi_3 = 0$, $\psi_6 = c_1\mu + c_2$, where c_1 and c_2 are constants. We note that in this case $\psi_5' = 0$. Analysis of (31) requires that we need to study two cases: (a) $\varphi_\mu = 0$ and (b) $\varphi_\mu \neq 0$.

Let $\varphi_\mu = 0$, then from (31) we get

$$(c_1\lambda + \psi_5)\varphi_{\lambda\lambda} - c_1\varphi_\lambda = 0.$$

If $c_1 \neq 0$, then without loss of generality, system (20) can be written as

$$\begin{aligned} \lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1 + \lambda, \quad \lambda_i = 0, \\ \mu_1 &= 2c\lambda, \quad \mu_2 = \lambda_1, \quad \mu_3 = \mu\lambda_1 + \mu + c_2, \quad \mu_i = 0, \quad i \geq 4. \end{aligned} \tag{32}$$

A solution of this system is

$$\lambda = -x_1\phi(x_3), \quad \mu = (cx_1^2 + x_2 + c_2e^{x_3})\phi(x_3),$$

where $\phi(x_3) = \mathrm{e}^{x_3}/(\mathrm{e}^{x_3} - 1)$.

If $c_1 = 0$ and $\psi_5 \neq 0$, then without loss of generality, system (20) can be written as

$$\begin{aligned} \lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1 + 1, \quad \lambda_i = 0, \\ \mu_1 &= c, \quad \mu_2 = \lambda_1, \quad \mu_3 = \mu\lambda_1 - c\lambda + c_2, \quad \mu_i = 0, \quad i \geq 4. \end{aligned} \tag{33}$$

A solution of this system is

$$\lambda = -\frac{x_1}{x_3} + \frac{x_3}{2}, \quad \mu = c\left(x_1 - \frac{x_3^2}{6}\right) - \frac{x_2}{x_3},$$

where c is an arbitrary constant.

If $c_1 = 0$ and $\psi_5 = 0$, then without loss of generality, system (20) can be written as

$$\begin{aligned} \lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1, \quad \lambda_i = 0, \\ \mu_1 &= \varphi', \quad \mu_2 = \lambda_1, \quad \mu_3 = \mu\lambda_1 + \lambda\varphi' - 2\varphi, \quad \mu_i = 0, \quad i \geq 4, \end{aligned} \tag{34}$$

where $\varphi = \varphi(\lambda)$ is an arbitrary function of λ. A solution of this system is

$$\lambda = -\frac{x_1}{x_3}, \quad \mu = -\frac{x_2}{x_3} - x_3\varphi(\lambda).$$

Let $\varphi_\mu \neq 0$, then from (31) we get $\varphi = F(\xi)$, where $\xi = \mu + \psi(\lambda)$. The functions $\psi(\lambda)$ and $F(\xi)$ are functions of one argument ($F' \neq 0$), which have to satisfy the equations

$$\psi''(c_1\lambda + \psi_5) = 0, \quad F''(2F - c_1\xi - c_3) + c_1F' - (F')^2 = 0.$$

Here, by virtue of the first equation, $c_3 \equiv \psi'(c_1\lambda + \psi_5) - c_1\psi$ is a constant.

If $c_1 \neq 0$, then as a result of equivalence transformations, we can set $c_1 = 1$, $\psi_5 = 0$, $\psi = 0$, and system (20) can be written as

$$\begin{aligned} \lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1 + \lambda, \quad \lambda_i = 0, \\ \mu_1 &= 0, \quad \mu_2 = \lambda_1 + F', \quad \mu_3 = \mu\lambda_1 + \mu + \mu F' - 2F, \quad \mu_i = 0, \quad i \geq 4, \end{aligned} \tag{35}$$

where the function $F = F(\mu)$ satisfies

$$(\mu - 2F)F'' = F'(1 - F'), \quad (F' \neq 0).$$

A solution of this system is

$$\lambda = \frac{x_1\,\mathrm{e}^{x_3}}{1 - \mathrm{e}^{x_3}}, \quad \mu = \mu(x_2, x_3),$$

where the function $\mu(x_2, x_3)$ satisfies a compatible overdetermined system of equations.

If $c_1 = 0$ and $\psi_5 \neq 0$, then without loss of generality and because of equivalence transformations, system (20) can be written as

$$\begin{aligned} \lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1 + 1, \quad \lambda_i = 0, \\ \mu_1 &= 0, \quad \mu_2 = \lambda_1 + 2c\mu, \quad \mu_3 = \mu\lambda_1, \quad \mu_i = 0, \quad i \geq 4, \end{aligned} \tag{36}$$

where $c \neq 0$ is a constant. The solution of this system (up to scaling x_1, x_2, x_3 and μ) is

$$\lambda = -\frac{x_1}{x_3} + x_3, \quad \mu = \frac{1}{x_3}(\gamma\, e^{x_2} + 1),$$

where $\gamma = 0$ or $\gamma = 1$. If $\gamma = 0$, then the solution is invariant with respect to the subalgebra $\partial_{x_2}, \partial_{x_i}, (i = 4, 5, \ldots, n)$.

If $c_1 = 0$ and $\psi_5 = 0$, then without loss of generality, system (20) can be written as

$$\begin{aligned}
\lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1, \quad \lambda_i = 0,\\
\mu_1 &= \psi' F', \quad \mu_2 = \lambda_1 + F',\\
\mu_3 &= \mu\lambda_1 + (\mu + \psi'\lambda)F' - 2F, \quad \mu_i = 0, \quad i \geq 4,
\end{aligned} \tag{37}$$

where $\psi = \psi(\lambda)$ is an arbitrary function, $F = c(\xi + c_3)^2$, $\xi = \mu + \psi(\lambda)$ and c, c_3 are constants ($c \neq 0$). With the help of equivalence transformation, this system can be simplified to

$$\begin{aligned}
\lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1, \quad \lambda_i = 0,\\
\mu_1 &= \psi'(\mu + \lambda_1), \quad \mu_2 = \lambda_1 + \mu,\\
\mu_3 &= \mu\lambda_1 + (\lambda\psi' - \psi)(\mu + \lambda_1), \quad \mu_i = 0, \quad i \geq 4,
\end{aligned} \tag{38}$$

The general solution of this system is (up to equivalence transformation)

$$\lambda = -\frac{x_1}{x_3}, \quad \mu = \frac{1}{x_3}(\gamma\, e^{x_2 - x_3\psi} + 1),$$

where $\gamma = 0$ or $\gamma = 1$. If $\gamma = 0$, then the solution is invariant with respect to the subalgebra $\partial_{x_2}, \partial_{x_i}, (i = 4, 5, \ldots, n)$.

Now we consider the case $h \equiv \lambda\psi_6'' - 2\psi_3 \neq 0$.

Let $\psi_6'' \neq 0$, then system (30), (31) is compatible (up to equivalence transformations) only if system (20) has the form

$$\begin{aligned}
\lambda_2 &= (\lambda + \alpha)\mu, \quad \lambda_3 = \lambda\lambda_1, \quad \lambda_i = 0,\\
\mu_1 &= 0, \quad \mu_2 = \lambda_1 + \mu(\mu + \beta), \quad \mu_3 = \mu\lambda_1, \quad \mu_i = 0, \quad i \geq 4,
\end{aligned} \tag{39}$$

where α, β are constants. A solution of this system depends on β.

If $\beta \neq 0$, then the solution is (up to equivalence transformation)

$$\lambda = \frac{x_1 - \alpha\gamma\, e^{x_2}}{\gamma\, e^{x_2} - x_3}, \quad \mu = -\frac{1 + \beta^2\gamma\, e^{x_2}}{\gamma\, e^{x_2} - x_3},$$

where $\gamma = 0$ or $\gamma = 1$. If $\gamma = 0$, then the solution is invariant with respect to the subalgebra $\partial_{x_2}, \partial_{x_i}, (i = 4, 5, \ldots, n)$.

If $\beta = 0$, then the solution is (up to equivalence transformation)

$$\lambda = -\frac{x_1 + \alpha x_2^2}{x_3 + x_2^2}, \quad \mu = -\frac{x_2}{x_3 + x_2^2}.$$

Let $\psi_6'' = 0$ or $\psi_6 = c_1\mu + c_2$ and $\psi_3 \neq 0$. Changing the function φ to $Q(\lambda, \mu) = (\varphi - \psi_6/2)/h^2$ simplifies Equations (30) and $(27)_3$, further. Equation $(27)_3$ can be integrated:

$$\frac{\partial Q}{\partial \lambda} = 6Q^2 \frac{\psi_3\psi_3'' - (\psi_3')^2}{\psi_3} - 3Q\frac{c_1\psi_3'}{2\psi_3^2} + \psi_8,$$

where $\psi_8 = \psi_8(\mu)$. Then from these two equations by cross-differentiating, we get

$$AQ^2 + BQ + C = 0,$$

where $A = 6\psi_3^2(\psi_3^2\psi_3''' - 2\psi_3\psi_3'\psi_3'' + (\psi_3')^3)$, $B = 3c_1\psi_3(\psi_3')^2/2$, $C = \psi_8'\psi_3^4 - 3c_1^2\psi_3'/16$.

Further analysis depends on the value of Q_λ. There are only two possibilities: (a) $A = 0$, $B = 0$, $C = 0$ and (b) $Q_\lambda = 0$.

In case (a), because $B = 0$, we need to consider two cases. In the first case $\psi_3' = 0$, and then, without loss of generality, system (20) can be reduced to

$$\begin{aligned} \lambda_2 &= 1, \quad \lambda_3 = \lambda(\lambda_1 + c_1) - \mu + c_2, \\ \mu_1 &= k, \quad \mu_2 = \lambda_1 + c_1, \quad \mu_3 = \mu\lambda_1 - k\lambda + k_1, \end{aligned} \tag{40}$$

where k and k_1 are constants and c_1 attains two values: either $c_1 = 1$ or $c_1 = 0$. In the second case, $c_1 = 0$, and without loss of generality, the system (20) can be reduced to

$$\begin{aligned} \lambda_2 &= -\frac{1}{2}(\mu - k)^2, \quad \lambda_3 = \lambda\lambda_1 - \frac{1}{6}(\mu + 2k)(\mu - k)^2, \\ \mu_1 &= \frac{(\mu - k)^4}{6(\lambda - k_1)^2}, \quad \mu_2 = \lambda_1 - \frac{2(\mu - k)^3}{3(\lambda - k_1)}, \\ \mu_3 &= \mu\lambda_1 - \frac{(\mu - k)^2(\lambda\mu + 3k\lambda - 2k_1\mu - 2kk_1)}{6(\lambda - k_1)^2}, \end{aligned} \tag{41}$$

where k and k_1 are constants.

Let us now consider case (b) $Q_\lambda = 0$. From $s_6 = 0$ we get $Q\psi_3'' = 0$. If $c_1 = 0$, then system (20) can be reduced to

$$\begin{aligned} \lambda_2 &= \psi_3, \quad \lambda_3 = \lambda\lambda_1 + \psi_5, \\ \mu_1 &= 0, \quad \mu_2 = \lambda_1 + k\psi_3\psi_3', \quad \mu_3 = \mu\lambda_1 + k\psi_3\psi_5', \end{aligned} \tag{42}$$

where k is a constant and ψ_3 is an arbitrary function of one argument and the function ψ_5 is connected with ψ_3 by: $\psi_5' = \mu\psi_3' - \psi_3$. If $c_1 \neq 0$, then system (20) can be reduced to

$$\begin{aligned} \lambda_2 &= 1, \quad \lambda_3 = \lambda(\lambda_1 + 1) - \mu + k_1, \\ \mu_1 &= 0, \quad \mu_2 = \lambda_1 + 1, \quad \mu_3 = \mu\lambda_1 + k, \end{aligned} \tag{43}$$

where k and k_1 are constants.

We can thus formulate the following theorem:

THEOREM. *System (19) can have solutions irreducible to invariant solutions only if it is equivalent to one of the systems: (23), (25), (32–36), (37) (or (38)).*

4. Systems of Type (6)

Systems of the type (6) have the form

$$\lambda_i = p_i(\lambda,\mu)\lambda_1 + f_i(\lambda,\mu), \quad \mu_j = g_j(\lambda,\mu), \quad (i = 1,\dots,n;\ j = 1,\dots,n). \tag{44}$$

As with systems of type (4), we can obtain the necessary irreducibility conditions from expressions $D_i\mu_j - D_j\mu_i = 0$:

$$g_{i\lambda} = p_i g_{1\lambda}, \quad g_{i\mu}g_1 = f_i g_{1\lambda} + g_i g_{1\mu}, \quad (p_j f_i - p_i f_j)g_{1\lambda} + g_i g_{j\mu} - g_j g_{i\mu} = 0, \tag{45}$$

and

$$\begin{aligned}&(p_i p_{j\mu} - p_j p_{i\mu})g_1 + p_{i\lambda} f_j + p_{i\mu} g_j - p_{j\lambda} f_i - p_{j\mu} g_i = 0,\\ &(p_i f_{j\mu} - p_j f_{i\mu})g_1 + f_{i\lambda} f_j + f_{i\mu} g_j - f_{j\lambda} f_i - f_{j\mu} g_i = 0,\end{aligned} \tag{46}$$

from expressions $D_i\lambda_j - D_j\lambda_i = 0$. Here $i, j = 2, 3, \dots, n$.

Assume that $g_1 \neq 0$. If $g_{1\lambda} = 0$, then without loss of generality, we can consider $g_1 = 1$. In this case, from (45) we can conclude that g_i, $(i, j = 2, 3, \dots, n)$ are constants, even up to equivalence transformations we can regard them as $g_i = 0$, $(i, j = 2, 3, \dots, n)$. Solution of such a system is $\mu = x_1$, which is partially invariant with defect $\delta \leq 1$. It is possible to obtain a further simplification of system (44).

If $g_{1\lambda} \neq 0$, then without loss of generality we can consider $g_1 = \lambda$. Because in this case, from (45) we have

$$p_i = g_{i\lambda}, \quad f_i = \lambda g_{i\mu}, \quad (i = 2, 3, \dots, n).$$

It gives that the first $n - 1$ equations $\lambda_i = p_i\lambda_1 + f_i$, 0, $(i, j = 2, 3, \dots, n)$ are consequences of the other equations. But we have assumed that the equations of system (44) are not dependent.

If $g_1 = 0$, then without loss of generality we can consider that $g_2 = 1$. From (45) and changing the independent variables, we can obtain $g_j = 0$, $(j = 3, 4, \dots, n)$. The solution of such a system is $\mu = x_2$, which is partially invariant with defect $\delta \leq 1$. As before, it is possible for a further simplification of system (44).

5. Conclusion

In this paper, the classification of systems of type (3) with $N = 2n - 1$ for double waves of nonhomogeneous quasilinear equations is performed.

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