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Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. II

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Abstract

The second-order ordinary differential equations can have one, two, three or eight independent symmetries. Sophus Lie showed that the equations with eight symmetries and only these equations can be linearized by a change of variables. Moreover he demonstrated that these equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. We provide a similar description of the equations with three symmetries. There are four different types of such equations. Classes of equations equivalent to one of these equations were studied in [Ibragimov NH, Meleshko SV. Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. Communication in Nonlinear Science and Numerical Simulation, in press], where we presented the candidates for all four types and studied one of these candidates. The present paper is the continuation of the work of Ibragimov and Meleshko and is devoted to other three candidates.

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Keywords: Equations with three symmetries; Candidates; Equivalence test

1. Introduction

According to Lie's classification [2] in the complex domain, any ordinary differential equation of the second order

$$y'' = f(x, y, y'), \tag{1}$$

admitting a three-dimensional Lie algebra belongs to one of four distinctly different types. Each of these four types is obtained by a change of variables from the following canonical representatives (see, e.g. [3], Section 8.4):

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$$y'' + Cy^{-3} = 0, \tag{2}$$

$$y'' + e^{y'} = 0, \tag{3}$$

$$y'' + y'^{(k-2)/(k-1)} = 0, \tag{4}$$

$$y'' + 2\frac{y' + Cy'^{3/2} + y'^2}{x - y} = 0, \tag{5}$$

where k and C are constants such that $k \neq 0, 1/2, 1, 2$ and $C \neq 0$.

Eqs. (2)–(5) admit non-similar three-dimensional Lie algebras L_3 spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \tag{6}$$

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + (y - x)\frac{\partial}{\partial y}, \tag{7}$$

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \tag{8}$$

and

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \tag{9}$$

respectively (see, e.g. [3], Section 8.4).

2. Candidates for equations with three symmetries

Let us subject each of Eqs. (2)–(5) to the arbitrary change of variables

$$t = \varphi(x, y), \qquad u = \psi(x, y), \tag{10}$$

where t is a new independent variable and u is a new dependent variable. Then we obtain from (2)–(5) the equations of the form

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 = 0, \tag{11}$$

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)\exp\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right) = 0, \tag{12}$$

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right)^{(k-2)/(k-1)} = 0, \tag{13}$$

and

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)C\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right)^{3/2} = 0, \tag{14}$$

respectively, where $b_i = b_i(t, u)$, $i = 1, \ldots, 12$. Eqs. (11)–(14) are the candidates for the equations with three symmetries.

All candidates can be encapsulated in the formula

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)f\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right) = 0.$$

Namely, Eqs. (11)–(14) are obtained by letting

$$f(z) = 0, \qquad f(z) = e^z, \qquad f(z) = z^{(k-2)/(k-1)}, \qquad f(z) = z^{3/2}. \tag{15}$$

Using the usual formula for the transformation of derivatives under the change of variables (10), we obtain the following statement.

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Theorem. *Any equation of the form*

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8) f\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right) = 0 \tag{16}$$

is transformed by the change of variables (10) *into an equation of the same form*:

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 + (a_5 y'^3 + 3a_6 y'^2 + 3a_7 y' + a_8) f\left(\frac{a_9 y' + a_{10}}{a_{11} y' + a_{12}}\right) = 0. \tag{17}$$

Here a_i and b_i are functions of x,y and t,u, respectively, and are connected by:

$$\begin{aligned}
a_1 &= \Delta^{-1}[\varphi_y \psi_{yy} - \varphi_{yy}\psi_y + b_4\varphi_y^3 + 3b_3\varphi_y^2\psi_y + 3b_2\varphi_y\psi_y^2 + b_1\psi_y^3],\\
a_2 &= \Delta^{-1}[b_4\varphi_x\varphi_y^2 + b_3\varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x) + b_2\psi_y(\varphi_x\psi_y + 2\varphi_y\psi_x)\\
&\quad + b_1\psi_x\psi_y^2 + (\varphi_x\psi_{yy} - \varphi_{yy}\psi_x - 2\varphi_{xy}\psi_y + 2\varphi_y\psi_{xy})/3],\\
a_3 &= \Delta^{-1}[b_4\varphi_x^2\varphi_y + b_3\varphi_x(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_2\psi_x(2\varphi_x\psi_y + \varphi_y\psi_x)\\
&\quad + b_1\psi_x^2\psi_y + (\varphi_y\psi_{xx} - \varphi_{xx}\psi_y - 2\varphi_{xy}\psi_x + 2\varphi_x\psi_{xy})/3],\\
a_4 &= \Delta^{-1}[b_4\varphi_x^3 + 3b_3\varphi_x^2\psi_x + 3b_2\varphi_x\psi_x^2 + b_1\psi_x^3 - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}],
\end{aligned} \tag{18}$$

$$\begin{aligned}
a_5 &= \Delta^{-1}[b_8\varphi_y^3 + 3b_7\varphi_y^2\psi_y + 3b_6\varphi_y\psi_y^2 + b_5\psi_y^3],\\
a_6 &= \Delta^{-1}[b_8\varphi_x\varphi_y^2 + b_7\varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x) + b_6\psi_y(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_5\psi_x\psi_y^2],\\
a_7 &= \Delta^{-1}[b_8\varphi_x^2\varphi_y + b_7\varphi_x(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_6\psi_x(2\varphi_x\psi_y + \varphi_y\psi_x) + b_5\psi_x^2\psi_y],\\
a_8 &= \Delta^{-1}[b_8\varphi_x^3 + 3b_7\varphi_x^2\psi_x + 3b_6\varphi_x\psi_x^2 + b_5\psi_x^3],
\end{aligned} \tag{19}$$

$$a_9 = b_{10}\varphi_y + b_9\psi_y,\ a_{10} = b_{10}\varphi_x + b_9\psi_x,\ a_{11} = b_{12}\varphi_y + b_{11}\psi_y,\ a_{12} = b_{12}\varphi_x + b_{11}\psi_x, \tag{20}$$

where

$$\Delta = (\varphi_x\psi_y - \varphi_y\psi_x) \neq 0$$

is the Jacobian of the change of variables (10).

It follows from Eqs. (20) that

$$a_9a_{12} - a_{10}a_{11} = \Delta(b_9b_{12} - b_{10}b_{11}).$$

Hence the equation

$$a_9a_{12} - a_{10}a_{11} = 0 \tag{21}$$

is invariant under the change of variables (10). If $a_9a_{12} - a_{10}a_{11} = 0$, and hence $b_9b_{12} - b_{10}b_{11} = 0$, the function f disappears in both Eqs. (16) and (17). This leads to the equations equivalent to Eq. (2), i.e. to the case considered in [1]. Therefore, we assume in what follows that

$$b_9b_{12} - b_{10}b_{11} \neq 0, \qquad a_9a_{12} - a_{10}a_{11} \neq 0.$$

3. Equations equivalent to Eqs. (3) and (4)

The test for equivalence to both Eqs. (3) and (4) have the same form. The only difference is that Eqs. (3) and (4) have the candidates (12) and (13), respectively, with the different functions f.

Eqs. (3) and (4) have the form (17) with

$$\begin{aligned}
&a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0, \quad a_6 = 0,\\
&a_7 = 0, \quad a_8 - 1 = 0, \quad a_9 - a_{12} = 0, \quad a_{10} = 0, \quad a_{11} = 0,
\end{aligned} \tag{22}$$

whereas the function f has the form $f(z) = e^z$ for (3) and $f(z) = z^{(k-2)/(k-1)}$ for (4). Furthermore, the change of variables (10) leaves invariant each candidate. Hence, the equations which are equivalent to (3) and (4) belong to equations of the form (16):

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8) f\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right) = 0.$$

Thus for the functions (10) $\varphi(x,y)$ and $\psi(x,y)$ one obtains the overdetermined system of equations which consists of Eqs. (22), where the coefficients a_i, $(i=1,2,\ldots,12)$ are defined by the relations (18)–(20).

Analysis of compatibility of the overdetermined system depends on the value of b_{12}. If the argument of the function f in (16) is a linear function with respect to the derivative u', then without loss of generality one can assume that $b_{11}=0$ and $b_{12}=1$. If the argument of the function f in (16) is a rational function with respect to the derivative u', then without loss of generality one can assume that $b_{11}=1$.

Let us consider the first case

$$b_{11}=1, \qquad b_{12}=0.$$

In this case the result of compatibility analysis gives that $b_5b_{10}\neq 0$ and

$$\begin{aligned}
&b_4=0, \qquad b_6=0, \quad b_7=0, \qquad b_8=0,\\
&b_{5t}-3b_5b_3=0, \qquad b_{10}b_{5u}-3b_5(2b_{10}b_2+b_{9t})=0, \qquad b_{9u}+b_{10}b_1=0,\\
&b_{10t}+3b_{10}b_3=0, \qquad b_{10u}+3b_{10}b_2+b_{9t}=0.
\end{aligned}$$

The functions $\varphi(x,y)$ and $\psi(x,y)$ are found from the compatible system of equations

$$\varphi_x=\frac{b_9}{b_{10}^2b_5}, \qquad \varphi_y=-\frac{1}{b_{10}^2b_5}, \qquad \psi_x=-\frac{1}{b_{10}b_5}, \qquad \psi_y=0.$$

The generators corresponding to (7) are

$$\begin{aligned}
&X_1=(b_{10}^2b_5)^{-1}\left[b_9\frac{\partial}{\partial t}-b_{10}\frac{\partial}{\partial u}\right], \qquad X_2=(b_{10}^2b_5)^{-1}\frac{\partial}{\partial t},\\
&X_3=(b_{10}^2b_5)^{-1}\left[((b_9+1)x-y)\frac{\partial}{\partial t}-b_{10}x\frac{\partial}{\partial u}\right].
\end{aligned}$$

The generators corresponding to (8) are

$$\begin{aligned}
&X_1=(b_{10}^2b_5)^{-1}\left[b_9\frac{\partial}{\partial t}-b_{10}\frac{\partial}{\partial u}\right], \qquad X_2=(b_{10}^2b_5)^{-1}\frac{\partial}{\partial t},\\
&X_3=(b_{10}^2b_5)^{-1}\left[(b_9x-ky)\frac{\partial}{\partial t}-b_{10}x\frac{\partial}{\partial u}\right].
\end{aligned}$$

In the second case $b_{12}=1$ one obtains that $b_8\neq 0$ and

$$\begin{aligned}
&b_{11t}b_{11}-b_{11u}+b_{11}^3b_4-3b_{11}^2b_3+3b_{11}b_2-b_1=0,\\
&3b_2(b_{11}b_{10}-b_9)b_8+(b_{11}b_{10}-b_9)b_{8u}+3b_{11}b_8b_{10u}=0,\\
&6b_3(b_{11}b_{10}-b_9)b_8+(b_{11}b_{10}-b_9)b_{8t}+3b_{11}b_8b_{10t}+3b_8b_{10u}=0,\\
&b_4(b_{11}b_{10}-b_9)+b_{10t}=0, \qquad b_5=b_{11}^3b_8, \qquad b_6=b_{11}^2b_8, \qquad b_7=b_{11}b_8,\\
&b_8(2b_{11t}b_{10}+b_{10t}b_{11}+b_{10u}-2b_{9t})+b_{8t}(b_{11}b_{10}-b_9)=0,\\
&(b_{11}b_{10}-b_9)(2b_{8u}-2b_{11t}b_8-b_{8t}b_{11})+b_8(2b_{11u}b_{10}-b_{10t}b_{11}^2+3b_{10u}b_{11}-2b_{9u})=0.
\end{aligned}$$

The functions $\varphi(x,y)$ and $\psi(x,y)$ are found from the compatible system of equations

$$\begin{aligned}
&\varphi_x=\frac{b_9}{b_8(b_{11}b_{10}-b_9)^2}, \qquad \varphi_y=-\frac{b_{11}}{b_8(b_{11}b_{10}-b_9)^2},\\
&\psi_x=-\frac{b_{10}}{b_8(b_{11}b_{10}-b_9)^2}, \qquad \psi_y=\frac{1}{b_8(b_{11}b_{10}-b_9)^2}.
\end{aligned}$$

The generators corresponding to (7) are

$$X_1 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[b_9\frac{\partial}{\partial t} - b_{10}\frac{\partial}{\partial u}\right],$$
$$X_2 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[b_{11}\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right],$$
$$X_3 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[(b_{11}(x-y) + b_9x)\frac{\partial}{\partial t} + (y - (1+b_{10})x)\frac{\partial}{\partial u}\right].$$

The generators corresponding to (8) are

$$X_1 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[b_9\frac{\partial}{\partial t} - b_{10}\frac{\partial}{\partial u}\right],$$
$$X_2 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[b_{11}\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right],$$
$$X_3 = b_8^{-1}(b_{11}b_{10} - b_9)^{-2}\left[(b_9x - b_{11}ky)\frac{\partial}{\partial t} + (ky - b_{10}x)\frac{\partial}{\partial u}\right].$$

4. Equations equivalent to Eq. (5)

Eq. (5) has the form (17) with

$$\begin{array}{l} a_1 = 0, \quad 3a_2 - 2/(x-y) = 0, \quad 3a_3 - 2/(x-y) = 0, \\ a_4 = 0, \quad a_5 = 0, \quad a_6 = 0, \quad a_7 = 0, \quad a_8 - C(x-y)^{-1} = 0, \\ a_9 - a_{12} = 0, \quad a_{10} = 0, \quad a_{11} = 0, \end{array} \tag{23}$$

and f has the form $f(z) = Cz^{3/2}$. The equations that are equivalent to (5) belong to equations of the form (16):

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + \left(b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8\right)f\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right) = 0.$$

Thus for the functions (10) $\varphi(x,y)$ and $\psi(x,y)$ one obtains the overdetermined system of equations which consists of Eqs. (23), where the coefficients a_i $(i = 1,2,\ldots,12)$ are defined by the relations (18)–(20).

Analysis of compatibility of the overdetermined system depends on the value of b_{12}. If the argument of the function f in (16) is a linear function with respect to the derivative u', then without loss of generality one can assume that $b_{11} = 0$ and $b_{12} = 1$. If the argument of the function f in (16) is a rational function with respect to the derivative u', then without loss of generality one can assume that $b_{11} = 1$.

Let us consider the first case

$$b_{11} = 1, \qquad b_{12} = 0.$$

In this case the result of compatibility analysis gives that $b_5b_{10} \neq 0$ and

$$b_4 = 0, \quad b_6 = 0, \quad b_7 = 0, \quad b_8 = 0,$$
$$b_{10t}C - b_{10}(2b_{10}^2b_5 - 3b_3C) = 0,$$
$$4b_{10}^2b_9b_5 + 2b_{10}^2b_5 - 3b_{10}b_2C - b_{9t}C - b_{10u}C = 0,$$
$$3b_5(-b_{10}^2b_5 + b_3C) - b_{5t}C = 0,$$
$$3b_5(-3b_{10}^2b_9b_5 - b_{10}^2b_5 + 2b_{10}b_2C + b_{9t}C) - b_{5u}b_{10}C = 0,$$
$$b_{10}(2b_9^2b_5 + 2b_9b_5 - b_1C) - b_{9u}C = 0.$$

The functions $\varphi(x, y)$ and $\psi(x, y)$ are found from the compatible system of equations

$$\varphi_x = \frac{b_9 C}{b_{10}^2 b_5 (x-y)}, \qquad \varphi_y = -\frac{C}{b_{10}^2 b_5 (x-y)}, \qquad \psi_x = -\frac{C}{b_{10} b_5 (x-y)}, \qquad \psi_y = 0.$$

The generators corresponding to (9) are

$$X_1 = (b_{10}^2 b_5 (x-y))^{-1} \left[(b_9 - 1)\frac{\partial}{\partial t} - b_{10}\frac{\partial}{\partial u} \right],$$
$$X_2 = (b_{10}^2 b_5 (x-y))^{-1} \left[(b_9 x - y)\frac{\partial}{\partial t} - b_{10} x\frac{\partial}{\partial u} \right],$$
$$X_3 = (b_{10}^2 b_5 (x-y))^{-1} \left[(b_9 x^2 - y^2)\frac{\partial}{\partial t} - b_{10} x^2\frac{\partial}{\partial u} \right].$$

In the second case $b_{12} = 1$ one obtains that $b_8 \neq 0$ and

$$\begin{aligned}
& b_5 = b_{11}^3 b_8, \qquad b_6 = b_{11}^2 b_8, \qquad b_7 = b_{11} b_8, \\
& - b_{10t} C + (b_{11} b_{10} - b_9)(2 b_{10} b_8 (b_{10} + 1) - b_4 C) = 0, \\
& b_{11t} b_{11} - b_{11u} + b_{11}^3 b_4 - 3 b_{11}^2 b_3 + 3 b_{11} b_2 - b_1 = 0, \\
& - 3 b_{10u} b_{11} b_8 C - b_{8u} C (b_{11} b_{10} - b_9) + 3 b_8^2 (b_{11} + b_9)(b_{11}^2 b_{10}^2 - b_9^2) - 3 (b_{11} b_{10} - b_9) b_8 b_2 C = 0, \\
& - 3 b_{10t} b_{11} b_8 C - 3 b_{10u} b_8 C + b_{8t} C (-b_{11} b_{10} + b_9) + 3 b_8^2 (b_{11}^2 b_{10}^3 + 3 b_{11}^2 b_{10}^2 + 2 b_{11} b_{10}^2 b_9 - 2 b_{11} b_{10} b_9 \\
& \quad - 3 b_{10} b_9^2 - b_9^2) - 6 (b_{11} b_{10} - b_9) b_3 b_8 C = 0, \\
& 2 b_{11t} b_{10} b_8 C + b_{10t} b_{11} b_8 C + b_{10u} b_8 C - 2 b_{9t} b_8 C + b_{8t} C (b_{11} b_{10} - b_9) \\
& \quad + b_8^2 (b_{11}^2 b_{10}^3 - b_{11}^2 b_{10}^2 - 2 b_{11} b_{10}^2 b_9 + 2 b_{11} b_{10} b_9 + b_{10} b_9^2 - b_9^2) = 0, \\
& 2 b_{11t} b_8 C (-b_{11} b_{10} + b_9) + 2 b_{11u} b_{10} b_8 C - b_{10t} b_{11}^2 b_8 C + 3 b_{10u} b_{11} b_8 C + b_{8t} b_{11} C (-b_{11} b_{10} + b_9) \\
& \quad + 2 b_{8u} C (b_{11} b_{10} - b_9) + b_8^2 (-b_{11}^3 b_{10}^3 - b_{11}^3 b_{10}^2 + 4 b_{11}^2 b_{10}^2 b_9 + 2 b_{11}^2 b_{10} b_9 - 5 b_{11} b_{10} b_9^2 \\
& \quad - b_{11} b_9^2 + 2 b_9^3) - 2 b_{9u} b_8 C = 0
\end{aligned}$$

The functions $\varphi(x,y)$ and $\psi(x,y)$ are found from the compatible system of equations

$$\varphi_x = \frac{b_9 C}{(b_{11} b_{10} - b_9)^2 (x-y) b_8}, \qquad \varphi_y = -\frac{b_{11} C}{(b_{11} b_{10} - b_9)^2 (x-y) b_8},$$
$$\psi_x = -\frac{b_{10} C}{(b_{11} b_{10} - b_9)^2 (x-y) b_8}, \qquad \psi_y = \frac{C}{(b_{11} b_{10} - b_9)^2 (x-y) b_8}.$$

The generators corresponding to (9) are

$$X_1 = \left((b_{11} b_{10} - b_9)^2 (x-y) b_8 \right)^{-1} \left[(b_9 - b_{11})\frac{\partial}{\partial t} + (1 - b_{10})\frac{\partial}{\partial u} \right],$$
$$X_2 = \left((b_{11} b_{10} - b_9)^2 (x-y) b_8 \right)^{-1} \left[(b_9 x - b_{11} y)\frac{\partial}{\partial t} + (y - b_{10} x)\frac{\partial}{\partial u} \right],$$
$$X_3 = \left((b_{11} b_{10} - b_9)^2 (x-y) b_8) \right)^{-1} \left[(b_9 x^2 - b_{11} y^2)\frac{\partial}{\partial t} + (y^2 - b_{10} x^2)\frac{\partial}{\partial u} \right].$$

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