การควบคุมเหมาะที่สุดแบบเป็นคาบของระบบซึ่งถูกครอบงำด้วย สมการวิวัฒนาการไม่เชิงเส้นในปริภูมิบานาค

Mrs. Wei Wei

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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Periodic Optimal Control of Systems Governed by Nonlinear Evolution Equations in Banach Spaces

Mrs. Wei Wei

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Periodic Optimal Control of Systems Governed by Nonlinear Evolution Equations in Banach Spaces

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วิทยานิพนธ์ฉบับนี้ศึกษาการมีผลเฉลยอย่างเป็นคาบและปฏิคาบของชั้นของสมการวิวัฒนาการไม่เชิงเส้นในปริภูมิบานาค และได้มีการศึกษาปัญหาการควบคุมเหมาะที่สุดที่สมนัยกันด้วย

*ในตอนแรกจะเริ่มต้นด้วยการพิสูจน์เรื่องการมีจริงของผลเฉลยอย่างเป็นคาบของสมการวิวัฒนาการไม่*เชิงเส้น สมการนี้มีตัวดำเนินการทางเดียวไม่เชิงเส้นและเพอร์เทอร์เบชั่นไม่เชิงเส้นรวมอยู่ด้วย เราได้แก้ปัญหาของความไม่เป็นเชิงเส้นอย่างเข้มด้วยการใช้ทฤษฎีของตัวดำเนินการทางเดียวไม่เชิงเส้นและทฤษฎีบทจุดคงที่ของเลอเรย์-เชาว์เดอร์

โดยการใช้วิธีการเช่นเดียวกับข้างต้น เราได้พิสูจน์การมีอยู่จริงของผลเฉลยปฏิคาบของสมการวิวัฒนาการอันดับที่หนึ่ง ต่อจากนั้นด้วยการเลือกการแปลงที่เหมาะสม เราได้พิสูจน์การมีอยู่จริงของผลเฉลยปฏิคาบของสมการวิวัฒนาการอันดับที่สองด้วย

ในตอนต่อมาได้มีการพิจารณาปัญหาการควบคุมเหมาะที่สุดแบบลากรองจ์ เราได้พิสูจน์การมีอยู่จริงของระบบการควบคุมเหมาะที่สุด ซึ่งถูกครอบงำด้วยสมการวิวัฒนาการไม่เชิงเส้นอย่างเป็นคาบบนปริภูมิบานาค

ในท้ายที่สุดได้มีการเสนอตัวอย่างสามข้อซึ่งเกี่ยวข้องกับสมการเชิงอนุพันธ์ย่อยประเภทคล้ายเชิงเส้น ตัวอย่างแรกจะเป็นปัญหาการควบคุมเหมาะที่สุดกำลังสองของระบบสมการซึ่งถูกครอบงำด้วยสมการคล้ายเชิงเส้นพาราโบลาอันดับที่สองพร้อมด้วยเงื่อนไขอย่างเป็นคาบ ตัวอย่างที่สองจะเป็นปัญหาการควบคุมเหมาะที่สุดแบบลากรองจ์ซึ่งถูกครอบงำด้วยสมการคล้ายเชิงเส้นพาราโบลาอันดับที่ 2m ตัวอย่างที่สามจะเป็นปัญหาค่าขอบแบบปฏิคาบของสมการคล้ายเชิงเส้นไฮเพอร์โบลาพร้อมด้วยแรงเสียดทานประเภทไม่เชิงเส้น

สาขาวิชาคณิตศาสตร์ | *Wei Wei* นักศึกษา | อาจารย์ที่ปรึกษา
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WEI WEI: PERIODIC OPTIMAL CONTROL OF SYSTEMS GOVERNED BY NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES THESIS ADVISOR: ASSOC. PROF. DR. PAIROTE SATTAYATHAM, Ph. D. 78 PP. ISBN 974-7359-83-9

PERIODIC SOLUTIONS / ANTI-PERIODIC EQUATIONS / MONOTONE OPERATOR / EVOLUTION EQUATIONS / OPTIMAL CONTROL / EXISTENCE/ PARABOLIC DIFFERENTIAL EQUATIONS / HYPERBOLIC DIFFERENTIAL EQUATIONS /

This thesis systematically studies periodic and anti-periodic solutions for a large class of strongly nonlinear evolution equations in Banach spaces and corresponding optimal control problems.

At first, a new existence result on periodic solutions for first-order nonlinear evolution equations is presented. The equations contain nonlinear monotone operators and nonlinear nonmonotone perturbations. An approach of integrating the theory of nonlinear monotone operators and the Leray-Schauder fixed point theorem was used to successfully overcome some difficulties due to strong nonlinearity.

By virtue of this approach, an existence result of anti-periodic solutions for the first-order nonlinear evolution equations is also obtained. Furthermore, through an appropriate transformation, the existence of anti-periodic solutions for the second order nonlinear evolution equations is verified.

In addition, a corresponding Lagrange optimal control problem is considered. We give an existence result of optimal control of systems governed by periodic nonlinear evolution equations on Banach spaces.

Finally, the results are illustrated by three examples concerning quasi-linear partial differential equations: quadratic optimal control problem of a system governed by a second order quasi-linear parabolic equation with periodic condition; a Lagrange optimal control problem of a system governed by a 2m-order quasi-linear parabolic equation with time periodic condition; an anti-periodic boundary value problem of a quasi-linear hyperbolic equation with nonlinear motion.

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Wei Wei

Contents

Contents (Continued)

Chapter I

Introduction

In this thesis, we systematically study periodic and anti-periodic solutions for a large class of strongly nonlinear evolution equations in Banach spaces and a Lagrange optimal control problem of systems governed by strongly nonlinear periodic evolution equations in Banach spaces.

Optimal control problems are minimizing problems which describe the behavior of systems that can be modified by the action of an operator. Two kinds of variables (or sets of variables) are involved: one of them describes the state of the system and cannot be modified directly by the operator, it is called the *state variable*; the second one, on the contrary, is under the direct control of the operator which may choose a strategy among a given set of admissible ones, it is called the *control variable*. The operator tries to modify the state of the system indirectly, acting on control variables; only these may act on the system, through a link control-state, usually called *state equation*. Finally, the operator, acting directly on controls and indirectly on states through the state equation, must achieve a goal usually written as the minimization of a functional which depends on the control that has been chosen and on the corresponding state: the so-called *cost functional*.

In many technological processes, the state of the system is described by quantities that depend on the position variables and also on time. In the optimal control of such processes, there often appear partial differential equations, functional differential equations, or abstract evolution equations as state equations. In general, these equations are nonlinear. For example, the well-known Navier-Stokes equation arising in hydrodynamic problems is a system of nonlinear diffusion equations. Similarly, in problems of chemical (nuclear) kinetics, involving simultaneously chemical (nuclear) reactions and heat transfer, one has to deal with a system of nonlinear diffusion equations. A fairly large class of such problems can be covered by evolution equations on suitable Banach spaces.

Optimal control problems of distributed parameter systems have been extensively studied since the last century. Most of the work concentrated on linear and semilinear equations is summarized by the books of Lions (1971), Ahmed and Teo (1982), Li and Yong (1995), and Fattorini (1999). Recently, some authors have shifted their attention to nonlinear cases. We refer to the works of Barbu (1992), Papageorgiou (1993), Ahmed and Xiang (1997), etc.

In order to study the optimal control problem, it is very important to analyze the state equations. Since the last decade, some authors have been paying great attention to systems with strong nonlinearities, particularly to nonlinear systems with nonmonotone perturbations. That is, the state equation is

$$\dot{x}(t) + A(t, x(t)) = f(t, x(t)), \quad 0 < t < T$$

with initial condition $x(0) = x_0$. Here, A is a nonlinear monotone operator and f is a nonlinear nonmonotone perturbing operator. There are some works discussing such a class of initial value problems. In 1989, Hirano began to study the influence of nonlinear nonmonotone perturbations. Since then, Ahmed and Xiang (1994), Papageorgiou (1993) have continued to consider this problem. However, in order to obtain the existence of solutions, they imposed some strong restrictions on the perturbation such as restriction of domain or region, growth condition of low order (almost linear), etc.

On the other hand, another important and interesting problem is the nonlinear periodic problem which arises naturally in mathematical modeling of various physical processes, because many processes are cyclic. For example, there are periodic mathematic models in Bonilla and Higuera (1995), Kulshreshtha, Liang, and Muller-Kirsten (1993). There has been a significant amount of research on various periodic problems of nonlinear evolution equations for concrete problems (cf. Fu and Ma (1997), Kolesov (1991), Leung and Ortega (1998), Pao (1999), and Pao (2000)), for semigroup theory (cf. Browder (1965), Prüss (1979), Amann (1978), Xiang and Ahmed (1992), Vrabie (1990) and Papageorgiou (1994), Yong, Fuzhong, Zhenghua, and Wenbin (1999)), and for the theory of monotone operators (cf. Becker Browder (1965), Becker (1981), Zeidler (1990), and Avgerinos (1996)).

In 1990, Zeidler proved the existence of periodic solutions for nonlinear evolution equations only containing a nonlinear monotone operator by using monotone theory. In 1996, Avgerinos continued to consider nonlinear and multivalued systems including a nonlinear monotone operator perturbed by a nonmonotone but regular operator (mapping within the same Hilbert space, thereby excluding differential expressions). In this thesis, we extend the result to the more general case of systems which contain a nonlinear monotone operator and where the perturbing operator admits differential expressions.

Furthermore, many authors pay great attention to the existence of optimal controls of nonlinear evolution systems with initial condition. So far, not much seems to be known for T-time periodic problems. There are only some results for linear periodic evolution equations in Hilbert spaces, see, e.g., Barbu(1991), Barbu (1997), and Li and Yong (1994). In this thesis, we study the optimal control problem of systems governed by nonlinear periodic evolution equations in Banach spaces.

More precisely, the main purpose of this thesis is to present general existence results for the optimal control problem (P): Minimize the functional

$$J(x,u) = \int_0^T L(t, x(t), u(t))\, dt$$

subject to $u \in U_{ad}$ and $x \in W_{pq}$ satisfying the state system

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t)) + B(t)u(t), & 0 < t < T \\ x(0) = x(T) \end{cases}$$

where $A : (0, T) \times V \to V^*$ is a nonlinear monotone operator, $f : (0, T) \times H \to V^*$ is a nonlinear and nonmonotone perturbation, $B \in L_\infty((0, T); \mathcal{L}(E, H))$, E is a reflexive Banach space (control space), $V \hookrightarrow H \hookrightarrow V^*$ is an evolution triple and W_{pq} will be defined in Chapter 2.

Before we discuss the optimal periodic control problem, we study strongly nonlinear periodic evolution equations as follows:

$$\begin{cases} \dot{x}(t) + A(t, x) = f(t, x), & 0 < t < T \\ x(0) = x(T). \end{cases} \tag{1.1}$$

By using nonlinear monotone operator theory and the Leray-Schauder fixed point theorem, we prove successfully a new existence result of problem (1.1) under quite general perturbations. A framework is presented that allows for a weaker the growth condition and $f : (0, T) \times H \to V^*$. The key step lies in choosing suitable work spaces and defining a useful operator.

It is very interesting that our techniques can also be used to solve another problem arising in physics and other natural phenomena, namely, the anti-periodic problem (cf. Okochi(1990), Aizicovici, and Pavel (1992), Aizicovici (1999), and Nakao(1996), etc.). We obtain existence results of anti-periodic solutions under quite reasonable assumptions. In addition, by constructing an appropriate transformation function, we reduce a class of second-order nonlinear equations to first-order nonlinear equations. Moreover, the existence of anti-periodic solutions for second-order nonlinear evolution equations is verified.

Finally, based on the discussion of the existence of periodic solutions for nonlinear evolution equations, we study the periodic optimal control problem (P). Using the monotone operator trick and Balder's (1987) results we prove the existence of optimal controls. The periodic problem is different from the initial value problem, in that we have to show that the limit function is also periodic. We use some techniques of weak convergence to overcome this difficulty.

The thesis is organized as follows. Chapter 2 mainly introduces notation and provides convenient reference to well known facts on abstract evolution equations. Chapter 3 deals with the existence of periodic solutions for a class of strongly nonlinear evolution equations. In chapter 4, we establish analogous results for the anti-periodic solutions of first-order nonlinear evolution equation and give an existence result for second-order nonlinear evolution equations with anti-periodic conditions. In chapter 5, the existence result of optimal controls for periodic nonlinear systems will be obtained. In chapter 6, we give three examples to demonstrate the applicability of our abstract results: a quadratic optimal control problem of a system governed by a second order quasi-linear parabolic equation with periodic condition; a Lagrange optimal control problem of a system governed by a $2m$-order quasi-linear parabolic with time-periodic conditions; and an anti-periodic boundary value problem of a quasi-linear hyperbolic differential equation with nonlinear motion.

Chapter II

Preliminaries

In this chapter, we collect some definitions and propositions which will be used frequently later on.

Before we start, let us introduce some basic notation. Throughout this thesis, E will be a real Banach space and E^* will denote its topological dual. The norm in E will be denoted by $\|\cdot\|_E$. We denote by $\langle x, y\rangle_E$ the pairing of an element $x \in E^*$ with an element $y \in E$. We shall use the symbol lim or $\longrightarrow$ to indicate strong convergence in E and $w-\lim$ or $\xrightarrow{W}$ for weak convergence in E. Let $0 < T < +\infty$ be a constant, $I = (0, T)$ be a fixed interval, $\bar{I} = [0, T]$, and $\mathcal{L}(E, H)$ denote the space of bounded linear operators from E to H.

2.1 Weak convergence

In contrast to finite dimensional Banach spaces, in infinite dimensional Banach spaces there exist bounded sequences, which do not possess convergent subsequences. This is responsible for many difficulties in the calculus of variations and the theory of partial differential equations. In order to overcome this difficulty, weakly convergence is a very important concept.

Definition 1. *A sequence $\{x_n\} \subset E$ is said to converge weakly to $x \in E$ if*

$$\lim_{n\to\infty} \langle v, x_n\rangle_E = \langle v, x\rangle_E \quad \text{for all } v \in E^*.$$

Proposition 1. *(1) If E is a reflexive Banach space, then every bounded sequence $\{u_n\}$ in E has a weakly convergent subsequence.*

If, in addition, each weakly convergent subsequence of $\{u_n\}$ has the same limit u, then

$$u_n \xrightarrow{W} u \quad \text{in } E \quad \text{as } n \to \infty.$$

(2) If

$$v_n \longrightarrow v \text{ in } E^*, \quad u_n \xrightarrow{W} u \text{ in } E \text{ as } n \to \infty,$$

then

$$\langle v_n, u_n\rangle_E \longrightarrow \langle v, u\rangle_E \quad \text{as } n \to \infty.$$

Moreover, if E is a reflexive Banach space, then

$$v_n \xrightarrow{W} v \quad in\ E^*, \qquad u_n \longrightarrow u \quad in \quad E \quad as\ n \to \infty,$$

implies

$$\langle v_n, u_n \rangle_E \longrightarrow \langle v, u \rangle_E \qquad as \quad n \to \infty.$$

Lemma 1. *(Mazur) Let E be a Banach space and K be a convex and (strongly) closed set in E. Then K is weakly closed in E.*

The proofs of these results and more information about weak convergence can be found in Yosida (1980).

2.2 Compact mappings in Banach Spaces

Definition 2. *Let E, F be normed linear spaces. An operator $G : E \to F$ is called compact if it maps every bounded subset of E into a relatively compact subset of F.*

The following the Leray-Schauder fixed point theorem is one of our main approaches in the proof of the existence of solutions for nonlinear evolution equations.

Theorem 1. *Let G be a compact mapping of a Banach space B into itself, and suppose there exists a constant M such that*

$$\|x\|_B < M$$

for all $x \in B$ and $\sigma \in [0, 1]$ satisfying $x = \sigma G x$. Then G has a fixed point.

For the proof of this theorem, we refer to Gilbarg and Trudinger (1977), pp. 231-232.

2.3 The Lebesgue space $L_p(I, E)$

In order to investigate the evolution equation, we need the following more general concepts and propositions for the Lebesgue integral and Lebesgue spaces.

Definition 3. *(1) (Step functions). A function $x : M \subseteq R^N \to E$ is called a step function if x is piecewise constant. To be precise, we suppose that the set M is measurable and that there exist finitely many pairwise disjoint measurable subsets M_i of M such that meas$(M_i) < \infty$ for all i and*

$$x(t) = \begin{cases} a_i & \text{if } t \in M_i, \\ 0 & \text{otherwise.} \end{cases}$$

(2) The integral of a step function is defined to be

$$\int_M x dt = \sum_i (meas\ M_i) a_i.$$

(3) (Measurable functions). A function $x : M \subseteq R^N \to E$ with values in E is called measurable if the following hold:

(i) *The domain of definition M is measurable.*

(ii) *There exists a sequence $\{x_n\}$ of step functions $x_n : M \to E$ such that*

$$\lim_{n\to\infty} x_n(t) = x(t) \qquad \text{for almost all } t \in M.$$

(4) (Integral). A function $x : M \to E$ is called integrable if M is measurable and there exists a sequence $\{x_n\}$ of step functions $x_n : M \to E$ such that

$$\begin{array}{ll} x(t) = \lim_{n\to\infty} x_n & \text{for almost all } t \in M, \\ \int_M \|x_n(t) - x_m(t)\|_E\, dt < \varepsilon & \text{for all } n, m \geq n_0(\varepsilon), \end{array}$$

where n_0 is a constant depending on ε.

We define

$$\int_M x(t) dt = \lim_{n\to\infty} \int_M x_n(t) dt.$$

Proposition 2. *(Majorant convergence principle). We have*

$$\lim_{n\to\infty} \int_M x_n(t) dt = \int_M \lim_{n\to\infty} x_n(t) dt,$$

where all the integrals and limits exist, provided the following conditions are satisfied:

(i) *$\|x_n(t)\| \leq y(t)$ for almost all $t \in M$ and all $n \in \mathbb{N}$, and $\int_M y dt$ exists.*

(ii) *$\lim_{n\to\infty} x_n(t)$ exists for almost all $t \in M$, where $x_n : M \subseteq R^N \to E$ is measurable for all n.*

Definition 4. *(1) The space $C^m\left(\overline{I}, E\right)$ with $m = 0, 1, \cdots$ consists of all continuous functions $x : \overline{I} \to E$ that have continuous derivatives up to order m on $\overline{I}$ with the norm*

$$\| x \|_{C^m(\overline{I},E)} = \sum_{i=0}^{m} \sup \left\{ \| x^{(i)}(t) \|_E, t \in \overline{I} \right\}. \tag{2.1}$$

Here, $x^{(0)}$ means x. We write $C\left(\overline{I}, X\right)$ instead of $C^0\left(\overline{I}, X\right)$.

(2) The space $L_p(I, E)$ with $1 \leq p \leq +\infty$ consists of the equivalence class of strongly measurable functions $x : I \to E$ such that

$$\| x \|_{L_p(I,E)} \equiv \begin{cases} \left(\int_I \| x(t) \|_E^p\, dt\right)^{1/p} < +\infty & \text{for } \ 1 \leq p < \infty, \\ \operatorname{ess\,sup} \{ \| x(t) \|_E,\ t \in I \} < +\infty & \text{for } \ p = \infty. \end{cases} \tag{2.2}$$

where $\operatorname{ess\,sup}$ is the least C such that $\|x(t)\|_E \leq C$ a.e. in I.

(3) From $u_n \longrightarrow u$ *in* $L_p(I,E)$ *as* $n \to \infty$ *it follows that*

$$\int_0^t u_n(s)\,ds \longrightarrow \int_0^t u(s)\,ds \quad \text{in } E \quad \text{as } n \to \infty.$$

(4) From

$$\begin{aligned} u_n &\xrightarrow{S} u \quad \text{in } L_p(I.E) \quad \text{as } n \to \infty, \\ v_n &\xrightarrow{W} v \quad \text{in } L_q(I.E^*) \quad \text{as } n \to \infty, \end{aligned}$$

it follows that

$$\int_0^t \langle v_n(s), u_n(s)\rangle_E\,ds \longrightarrow \int_0^t \langle v(s), u(s)\rangle_E\,ds \quad \text{as } n \to \infty.$$

(5)From

$$\begin{aligned} u_n &\xrightarrow{W} u \quad \text{in } L_p(I,E) \quad \text{as } n \to \infty, \\ v_n &\xrightarrow{S} v \quad \text{in } L_q(I,E^*) \quad \text{as } n \to \infty, \end{aligned}$$

it follows that

$$\int_0^t \langle v_n(s), u_n(s)\rangle_E\,ds \longrightarrow \int_0^t \langle v(s), u(s)\rangle_E\,ds \quad \text{as } n \to \infty.$$

2.4 Evolution triple

The use of several Banach spaces in connection with interpolation inequalities represents an important modern strategy in the theory of nonlinear partial differential equations.

Definition 5. *(Evolution triple) We define an evolution triple*

$$\text{“}V \hookrightarrow H \hookrightarrow V^*\text{”}$$

to be the following:

(1) V *is a real, separable, and reflexive Banach space.*

(2) H *is a real, separable Hilbert space.*

(3) The embedding $V \hookrightarrow H$ *is continuous, that is,*

$$\|v\|_H \le c\,\|v\|_V \quad \text{for all } v \in V$$

with a constant $c > 0$ *and* V *is dense in* H.

In order to simplify the writing of formulas, we agree to use the following abbreviations:

$$\langle x, y\rangle = \langle x, y\rangle_V, \qquad (x, y) = \langle x, y\rangle_H.$$

If $y \in H$ and $x \in V$ then $\langle y, x\rangle = (y, x)$.

For example, if G is a bounded region of R^N with $N \geq 1$, we set $V = W_0^{m,p}(G)$ with $2 \leq p < \infty$ and integer $m \geq 1$, and $H = L_2(G)$ then $V^* = W^{-m,p}(G)$. So, "$V \hookrightarrow H \hookrightarrow V^*$" is an evolution triple and $V \hookrightarrow H$ is compact by Sobolev embedding theorem.

Proposition 7. *Let "$V \hookrightarrow H \hookrightarrow V^*$" be an evolution triple. Then the following hold:*

(1) For each $h \in H$, define a functional $\overline{h} : V \to R$ by

$$\langle \overline{h}, v\rangle = (h, v) \quad \text{for all } v \in V.$$

Then $\overline{h}$ is linear and continuous, i.e., $\overline{h} \in V^*$.

(2) The mapping $h \mapsto \overline{h}$ from H into V^* is linear, injective, and continuous.

By Proposition 7, we may identify $\overline{h}$ with h. In this sense,

$$H \hookrightarrow V^*.$$

Henceforth, we shall write h instead of $\overline{h}$. Then the following are valid:

$$\begin{aligned} \langle h, v\rangle &= (h, v) \quad \text{for all } h \in H, \quad v \in V, \\ \|h\|_{V^*} &\leq c\,\|h\|_H \quad \text{for all } h \in H. \end{aligned}$$

In the following, the relation

$$V \hookrightarrow H \hookrightarrow V^*$$

for evolution triples is to be understood in the sense of this sense. And we have the following proposition:

Proposition 8. *Let "$V \hookrightarrow H \hookrightarrow V^*$" be an evolution triple. Then*

(1) The embedding $H \hookrightarrow V^*$ is continuous.

(2) H is dense in V^*.

(3) V^* is a reflexive and separable Banach space.

2.5 Generalized derivatives

The following definition is basic for understanding evolution equations.

Definition 6. *Let Y and Z be Banach spaces. Let $u \in L_1(I, Y)$ and $w \in L_1(I, Z)$. Then, the function w is said to be the nth generalized derivative of the function $u \in I$ if*

$$\int_0^T \psi^{(n)}(t)\, u(t)\, dt = (-1)^n \int_0^T \psi(t)\, w(t)\, dt \quad \text{for all } \psi \in C_0^\infty(I). \qquad (2.4)$$

We write $w = u^{(n)}$. Equation (2.4) means the integrals appearing on the right and left member belong both to $Y \cap Z$.

Remark 1. *(Generalized Derivatives and Distributions). Let* $u \in L_1(I,Y)$. *One can assign to the function* u *a distribution* U *via*

$$U(\phi) = \int_0^T \phi(t)\, u(t)\, dt \quad \text{for all } \phi \in C_0^\infty(0,T).$$

For each n, *this distribution has nth distributive derivative* $U^{(n)}$ *which is defined by*

$$U^{(n)}(\phi) = (-1)^n U\left(\phi^{(n)}\right) \quad \text{for all } \phi \in C_0^\infty(0,T). \tag{2.5}$$

If (2.4) holds, then $U^{(n)}$ *can be obviously represented in the form*

$$U^{(n)}(\phi) = \int_0^T \phi(t)\, u^{(n)}(t)\, dt \quad \text{for all } \phi \in C_0^\infty(0,T). \tag{2.6}$$

The advantage of the distribution conception consists in that each function $u_1 \in L_1(I,Y)$ *possesses derivatives of every order in the distributional sense. The definition singles out the cases in which by (2.6) the nth distributional derivative of* u *can be represented by a function* $w \in L_1(I,Z)$. *In this case, we set* $u^{(n)} = w$ *and we write briefly*

$$u \in L_1(I,Y) \quad , u^{(n)} \in L_1(I,Z).$$

Proposition 9. *(Uniqueness of Generalized Derivatives). Let* Y *and* Z *be Banach spaces. Moreover, suppose that* $u \in L_1(I,Y)$ *and* $v, w \in L_1(I,Z)$. *If*

$$u^{(n)} = v \quad \text{and} \quad u^{(n)} = w$$

in the sense of generalized derivatives, then we obtain

$$v(t) = w(t) \quad \text{almost everywhere on } (0,T),$$

that is $v = w$ *in* $L_1(I,Z)$.

Proposition 10. *(Generalized Derivatives and Weak Convergence). Let* Y *and* Z *be Banach spaces with the continuous embedding* $Y \subseteq Z$. *Then it follows from*

$$u_k^{(n)} = v_k \quad \text{on } I \quad \text{for all } k \quad \text{and} \quad \text{fixed } n \geq 1$$

and

$$u_k \xrightarrow{W} u \quad \text{in } L_p(I,Y) \ ,$$

$$v_k \xrightarrow{W} v \quad \text{in } L_q(I,Z),$$

as $k \to \infty, 1 \leq p,q < \infty$ *that*

$$u^{(n)} = v \quad \text{on } (0,T).$$

Proposition 11. *(Existence of $u^{(n)}$). Let "$V \hookrightarrow H \hookrightarrow V^*$" be an evolution triple and $1 \le p, q \le +\infty$. Then the following are valid.*

(1) Existence. Let $u \in L_p(I, V)$. Then there exists the generalized derivative

$$u^{(n)} \in L_q(I, V^*)$$

if there is a function $w \in L_q(I, V^*)$ such that

$$\int_0^T (u(t), v)\, \psi^{(n)}(t)\, dt = (-1)^n \int_0^T \langle w(t), v\rangle_V\, \psi(t)\, dt$$

for all $v \in V$ and $\psi \in C_0^\infty(\overline{I})$.

Then $u^{(n)} = w$ and

$$\frac{d^n}{dt^n}(u(t), v) = \left\langle u^{(n)}(t), v\right\rangle_V$$

holds for all $v \in V$ and almost all $t \in I$. Here, d^n/dt^n means the nth generalized derivative of real functions on I.

(2) Uniqueness. For $u \in L_p(I, V)$, the generalized derivative $u^{(n)}$ is unique as an element of $L_q(I, V^*)$, that is, $t \mapsto u^{(n)}(t)$ can be modified only on a subset of I of measure zero.

2.6 Sobolev space W_{pq} and embedding theorems

The following proposition, in particular the integration by parts formula (2.9) below, will play a central role in the treatment of evolution equations.

Proposition 12. *(Extension principle) Suppose that Y and Z are Banach spaces, and that the linear operator $A : D(A) \subseteq Y \to Z$ satisfies the inequality*

$$\|Au\|_Z \le c\,\|u\|_Y \tag{2.7}$$

for all $u \in D(A)$, where c is a constant and the set $D(A)$ is a linear subspace of Y which dense in Y. Then:

(a) The operator A can be uniquely extended to a linear continuous operator $A : Y \to Z$ with (2.7) for all $u \in Y$.

(b) If $A : D(A) \subseteq Y \to Z$ is compact, then so is $A : Y \to Z$.

(See Zeidler(1990), p.71).

Let "$V \hookrightarrow H \hookrightarrow V^*$" be an evolution triple. Define $X = L_p(I, V)$ and $X^* = L_q(I, V^*)$.

For all $u \in X^*$ and $v \in X$, we have that

$$\| u \|_{X^*}^q = \int_0^T \| u(t) \|_{V^*}^q\, dt, \quad \| v \|_X^p = \int_0^T \| v(t) \|_V^p\, dt,$$

and

$$\langle u,\ v\rangle_X = \int_0^T \langle u\,(t)\,,\ v\,(t)\rangle dt.$$

Definition 7. *Define the space*

$$W_{pq} = \{x : x \in X, \quad \dot{x} \in X^*\}\,,$$

with the norm

$$\| x \|_{W_{pq}} = \| x \|_X + \| \dot{x} \|_X\ .$$

where the derivative in the definition should be understood in the sense of generalized derivative.

The space $(W_{pq}, \| \cdot \|)$ becomes a Banach space, which is clearly reflexive and separable. $C^1(\overline{I}, H)$ is dense in W_{pq}. (cf. Zeidler (1990), p. 422 and p. 446).

We have the following two important embedding theorems which are frequently used in this thesis.

Proposition 13. *The embedding $W_{pq} \hookrightarrow C\left(\overline{I}, H\right)$ is continuous.*

Proof. Let $u, v \in C^1\left(\overline{I}, H\right)$, then

$$\frac{d}{dt}\left(u\,(t)\,, v\,(t)\right) = (\dot{u}(t), v(t)) + (u(t), \dot{v}(t)).$$

Hence

$$(u(t), v(t)) - (u(s), v(s)) = \int_s^t (\dot{u}(\tau), v(\tau)) + (u(\tau), \dot{v}(\tau))d\tau,$$

for all $0 \le s \le t \le T$. By the property of evolution triple, we have

$$(u, v) = \langle u, v\rangle, \qquad \text{for all } u, v \in V.$$

Then for $u, v \in C^1\left(\overline{I}, V\right)$, this implies

$$(u(t), v(t)) - (u(s), v(s)) = \int_s^t \langle \dot{u}(\tau), v(\tau)\rangle + \langle u(\tau), \dot{v}(\tau)\rangle d\tau.$$

Now we choose a test function $\phi \in C^1(R)$ with $\phi(s) = 0$ and $\phi(t) = 1$. Moreover, let $|\phi| + |\dot{\phi}| \le 1$ on $\overline{I}$ and $v = \phi u$, then we get

$$\begin{aligned} & (u(t), \phi(t)u(t)) - (u(s), \phi(s)u(s)) \\ = & \int_s^t \langle \dot{u}(\tau), \phi(\tau)u(\tau)\rangle + \langle \dot{\phi}(\tau)u(\tau) + \phi(\tau)\dot{u}(\tau), u(\tau)\rangle d\tau \\ = & \ 2\int_s^t \phi(\tau)\langle \dot{u}(\tau), u(\tau)\rangle d\tau + \int_s^t \dot{\phi}(\tau)\langle u(\tau), u(\tau)\rangle d\tau. \end{aligned}$$

From the Holder inequality and continuity of the embedding $V \hookrightarrow V^*$, we get

$$\begin{aligned}
\| u(t) \|_H^2 &\leq 2\int_0^T \langle u(\tau), \dot{u}(\tau)\rangle d\tau + \int_0^T \langle u(\tau), u(\tau)\rangle d\tau \\
&\leq C_1 (\int_0^T \| u(\tau) \|_V^p \, d\tau)^{1/p} (\int_0^T \| \dot{u}(\tau) \|_{V^*}^q \, d\tau)^{1/q} \\
&\quad + C_2 (\int_0^T \| u(\tau) \|_V^p \, d\tau)^{1/p} (\int_0^T \| u(\tau) \|_V^q \, d\tau)^{1/q} \\
&\leq C \left((\int_0^T \| u(\tau) \|_V^p \, d\tau)^{1/p} + (\int_0^T \| \dot{u}(\tau) \|_{V^*}^q \, d\tau)^{1/q} \right)^2 \\
&= C \| u \|_{W_{pq}}^2,
\end{aligned}$$

for any $u \in C^1\left(\overline{I}, V\right)$ with some positive constants C_1, C_2, and C. So

$$\| u \|_{C\left(\overline{I},H\right)} \leq C \| u \|_{W_{pq}}, \text{ for all } u \in C^1\left(I, V\right).$$

Since the set $C^1\left(\overline{I}, V\right)$ is dense in W_{pq}, by the extension principle (Proposition 12) the embedding operator $j : C^1\left(\overline{I}, V\right) \subset W_{pq} \to C\left(\overline{I}, H\right)$ has a unique continuous extension $j : W_{pq} \to C\left(\overline{I}, H\right)$. In this sense, the embedding

$$W_{pq} \hookrightarrow C\left(\overline{I}, H\right)$$

is continuous. □

Since the embedding of W_{pq} into $C\left(\overline{I}, H\right)$ is continuous, so every element in W_{pq} has a representative in $C(\overline{I}, H)$. Therefore it makes sense to speak of $u(t)$ for each $t \in \overline{I}$.

Proposition 14. *(1) (Dense subset). The set of all polynomials $w : I \to V$, that is*

$$w(t) = \sum_i t^i a_i \quad \text{with } a_i \in V \quad \text{for all } i,$$

is dense in the spaces W_{pq}, $L_p(I, V)$, and $L_p(I, H)$.

(2) (Integration by parts). For all functions $u, v \in W_{pq}$ and $t, s \in \overline{I}$ with $s < t$ the following generalized integration by parts formula holds:

$$\left(u(t), v(t)\right)_H - \left(u(s), v(s)\right)_H = \int_s^t \langle \dot{u}(\tau),\ v(\tau)\rangle_V + \langle \dot{v}(\tau),\ u(\tau)\rangle_V d\tau.$$

(See, problem 23.10e of Zeidler, 1990).

Proposition 15. *The embedding $W_{pq} \hookrightarrow L_p(I, H)$ is compact if the embedding $V \hookrightarrow H$ is compact.*

Proof. First, we claim that for any $\delta > 0$, there exists a $C_\delta > 0$, such that

$$\|x\|_H \leq \delta \|x\|_V + C_\delta \|x\|_{V^*} \quad \text{for all } x \in V. \tag{2.8}$$

If not, then there exists a $\delta > 0$ and a weakly convergent sequence $\{x_n\} \subset V$, $\|x_n\|_V = 1$, such that

$$\|x_n\|_H \geq \delta + n \|x_n\|_{V^*} \quad \text{for } n \geq 1. \tag{2.9}$$

By the compactness of the embedding $V \hookrightarrow H$, there exists a subsequence of $\{x_n\}$, denoted by $\{x_n\}$ again, such that

$$x_n \longrightarrow \overline{x} \quad \text{in } H \quad \text{as } n \to \infty.$$

Then (2.9) implies that

$$x_n \longrightarrow 0 \quad \text{in } H \quad \text{as } n \to \infty.$$

Thus, $\overline{x} = 0$. This means,

$$x_n \longrightarrow 0 \quad \text{in } H \quad \text{as } n \to \infty.$$

This is contradiction to (2.9). Hence our claim holds.

Now, let $\{h_n\} \subset W_{pq}$ be a bounded sequence. Because $1 < p, q < +\infty$ and V is reflexive, we have that $L_p(I, V)$ and $L_q(I, V^*)$ are reflexive (See Proposition 4). So

$$\begin{aligned} h_n &\xrightarrow{W} h \quad \text{in } L_p(I, V) \\ \dot{h}_n &\xrightarrow{W} \dot{h} \quad \text{in } L_q(I, V^*) . \end{aligned}$$

Without loss generality, we may assume that $h = 0$.

In addition, for any $s \in [0, T)$, we have

$$h_n(s) = h_n(t) - \int_s^t \dot{h}(\tau)\, d\tau, \quad t \in [0, T].$$

Integrate it over $(s, s+\sigma)$ $(\sigma \in (0, T-s])$

$$\begin{aligned} h_n(s) &= \frac{1}{\sigma}\left[\int_s^{s+\sigma} h_n(t)\, dt - \int_s^{s+\sigma}\int_s^t \dot{h}_n(\tau)\, d\tau dt\right] \\ &= \frac{1}{\sigma}\int_s^{s+\sigma} h_n(t)\, dt - \frac{1}{\sigma}\int_s^{s+\sigma} (s+\sigma-\tau)\, \dot{h}_n(\tau)\, d\tau \\ &\equiv a_n + b_n. \end{aligned}$$

We observe that

$$\begin{aligned} \|b_n\|_{V^*} &\leq \frac{1}{\sigma}\int_s^{s+\sigma} (s+\sigma-\tau) \left\|\dot{h}_n(\tau)\right\|_{V^*} d\tau \\ &\leq \frac{1}{\sigma}\int_s^{s+\sigma} (s+\sigma-s) \left\|\dot{h}_n(\tau)\right\|_{V^*} d\tau \\ &\leq \left(\int_s^{s+\sigma} \left\|\dot{h}_n(\tau)\right\|_{V^*}^q d\tau\right)^{1/q} \left(\int_s^{s+\sigma} 1^p d\tau\right)^{1/p} \\ &= \left\|\dot{h}_n\right\|_{L_q(I,V^*)} \cdot \sigma^{1/p} \leq C\sigma^{1-1/q}. \end{aligned} \tag{2.10}$$

On the other hand, since $h_n \xrightarrow{W} 0$ in $L_p(I,V)$ as $n \to \infty$, for any fixed $\sigma > 0$, it follows from Proposition 6 that

$$\begin{aligned} \langle v, a_n \rangle_V &= \left\langle v, \frac{1}{\sigma} \int_s^{s+\sigma} h_n(t)\, dt \right\rangle_V \\ &= \frac{1}{\sigma} \int_s^{s+\sigma} \langle v, h_n(t) \rangle_V \, dt \\ &\longrightarrow \frac{1}{\sigma} \int_s^{s+\sigma} \langle v, 0 \rangle_V \, dt = 0 \qquad \text{for all } v \in V^* \end{aligned} \tag{2.11}$$

as $n \to \infty$. That is

$$a_n \xrightarrow{W} 0 \quad \text{in } V \text{ as } n \to \infty.$$

Since the embedding $V \hookrightarrow H$ is compact,

$$a_n \longrightarrow 0 \quad \text{in } H \quad \text{as } n \to \infty. \tag{2.12}$$

Combining (2.10) and (2.12), we get

$$h_n(s) \longrightarrow 0 \quad \text{in } V^*.$$

In addition, $W_{pq} \hookrightarrow C(I,H)$ is continuous and h_n is bounded in W_{pq}, we have the boundedness of h_n in $C(\bar{I}, H)$, moreover for any $t \in I$, $h_n(t)$ is bounded in V^*. Hence, by the Majorized Convergence Theorem, we get

$$\lim_{n \to \infty} \|h_n\|_{L_p(I,V^*)} = 0. \tag{2.13}$$

Finally, for any $\delta > 0$, by (2.13), we get

$$\|h_n\|_{L_p(I,H)} \le \delta \|h_n\|_{L_p(I,V)} + C_\delta \|x\|_{L_p(I,V^*)} \quad \text{for } n = 1, 2, \cdots.$$

For any $\varepsilon > 0$, we can take $\delta > 0$ small enough such that

$$\delta \|h_n\|_{L_p(I,V)} < \frac{\varepsilon}{2} \quad \text{for } n = 1, 2, \cdots$$

and then for this $\delta > 0$, by (2.12) we can take N big enough such that

$$C_\delta \|h_n\|_{L_p(I,V^*)} < \frac{\varepsilon}{2} \quad \text{for all } n \ge N.$$

Therefore

$$\|h_n\|_{L_p(I,H)} < \varepsilon \quad \text{for all } n \ge N.$$

That is

$$h_n \longrightarrow 0 \quad \text{in } L_p(I,H) \quad \text{as } n \to \infty.$$

□

2.7 The Nemyckii operator

In order to discuss nonlinear problems and apply the abstract form to partial differential equations, we require the properties of the Nemyckii operator F is defined by

$$(Fx)(z) = f(z, x_1(z), x_2(z), \cdots, x_n(z))$$

with $x = (x_1, x_2, \cdots, x_n)$. Thus, F results when one replaces all the variables x_j by $x_j(z)$ in $f(z, x_1(z), \cdots, x_n(z))$.

Assume:

(H1) Carathéodory condition. Let $f : G \times R^n \to R$ be a given function, where G is a nonempty measurable set in R^N and $N \geq 1$. Moreover, the following hold:

$z \mapsto f(z, x)$ is measurable on G for all $x \in R^n$;

$x \mapsto f(z, x)$ is continuous on R^n for almost all $z \in G$.

(H2) Growth condition. For all $(z, x) \in G \times R^n$,

$$|f(z, x)| \leq a(z) + b \sum_{i=1}^{n} |x_i|^{p_i/q}.$$

Here, $b > 0$ is a constant, the function $a \in L_q(G)$ is nonnegative, and $1 \leq q, p_i < \infty$ for all i.

Proposition 16. *If (H1) and (H2) hold, then the Nemyckii operator*

$$F : \prod_{i=1}^{n} L_{p_i}(G) \to L_q(G)$$

is continuous and bounded with

$$\|Fx\|_{L_q(G)} \leq C \left(\|a\|_{L_q(G)} + \sum_{i=1}^{n} \|x_i\|_{L_{p_i}(G)}^{p_i/q} \right)$$

for all $x \in \prod_{i=1}^{n} L_{p_i}(G)$, where $C > 0$ is a constant.

For the proof of this proposition, we can refer to Zeidler (1990), pp. 561-564.

In this thesis, the system model considered is based on the evolution triple $V \hookrightarrow H \hookrightarrow V^*$ and the Sobolev space W_{pq}. In order to study the Nemyckii operator in Banach spaces, we have to consider the measurability of functions with values in Banach spaces. The following theorem tells us the relationship between the measurability of functions with values in Banach spaces and the measurability of real functions. We will frequently use it later.

Theorem 2. *(Pettis) Let Y be a real separable Banach space and let $f : M \subset R^N \to Y$ be a measurable function. Then the following two statements are equivalent:*

(1) The function f is measurable.

(2) The real functions $x \mapsto \langle g, f(x)\rangle_Y$ are measurable on M for all functional $g \in Y^*$.

Proposition 17. *(Measurable functions via substitution). We set*

$$F(z) = f(z, x(z)).$$

If the function $x : M \subseteq R^N \to E$ is measurable, then the function $F : M \to Y$ is also measurable provided the following assumptions are satisfied:

(1) The set M is measurable and the Banach spaces E and Y are separable.

(2) The function $f : M \times E \to Y$ satisfies the Caratheodory condition, i.e.,

$$\begin{aligned} z &\mapsto f(z,x) \text{ is measurable on } M \text{ for all } x \in E, \\ x &\mapsto f(z,x) \text{ is continuous on } E \text{ for almost all } z \in M. \end{aligned}$$

Suppose that an operator $A : I \times V \to V^*$ satisfies:

(H3) The function $t \mapsto A(t)$ is weakly measurable, i.e., the function

$$t \mapsto \langle A(t)\, x, y\rangle_V$$

is measurable on I, for all $x, y \in V$ and $A(t)\, x_n \xrightarrow{W} A(t)\, x$ in V^* whenever $x_n \longrightarrow x$ in V.

(H4) There exist a nonnegative function $c_1 \in L_q(I)$ and a constant $c_2 > 0$ such that

$$\|A(t)\, x\|_{V^*} \leq c_1(t) + c_2 \|x\|_V^{p-1} \quad \text{for all } x \in V, \quad t \in I.$$

We also require the properties of the Nemyckii operator $A : L_p(I, V) \to L_q(I, V^*)$ is defined by

$$(Ax)(t) = A(t, x(t)).$$

Proposition 18. *If hypotheses (H3) and (H4) hold, then for each $x \in X$, $t \longmapsto A(t, x(t))$ is measurable from $(0,T)$ to V^*.*

The idea of proof is definition of measurable via substitution (see Proposition 17).

Proof. Let $w \in V$ be fixed. We set

$$g(t, v) = \langle A(t, v), w\rangle_V \quad \text{for all } t \in I \quad \text{and } v \in V.$$

and we know the interval $(0, T)$ is measurable and the Banach space V and R are real and separable.

By assumption $(H3)$, for each $v \in V$, the function $t \longmapsto g(t, v)$ is measurable on $(0, T)$ and for each $t \in (0, T)$, the function $v \longmapsto g(t, v)$ is continuous on V. That is, the function $g : (0, T) \times V \to V^*$ satisfies the Caratheodory condition.

In the other hand, let $x \in X$ be given, then the function $t \longmapsto x(t)$ is measurable on $(0, T)$. By the substitution theorem (Proposition 16), we get that the function $t \to g(t, x(t))$ is measurable on $(0, T)$ for all $w \in V$. From the Pettis Theorem, we get that for each $x \in X$, the function $t \longmapsto A(t, x(t))$ is measurable from $(0, T)$ to V^*. $\square$

The system model considered in this thesis is based on the evolution triple "$V \hookrightarrow H \hookrightarrow V^*$", the compact embedding $V \hookrightarrow H$, and $2 \leq p < +\infty$.

Chapter III

Periodic Solutions of Evolution Equations

In this chapter, we will study the existence of periodic solutions for nonlinear evolution equations. The first section contains some monotone and maximal monotone theories. In the second section, an existence and uniqueness results of periodic solutions for a class of nonlinear evolution equations with nonlinear uniform monotone operator will be presented. In the third section, we will discuss the existence of periodic solutions for a class of strongly nonlinear evolution equations including a nonlinear monotone operator and a nonlinear nonmonotone perturbation.

3.1 Monotone operator theory

The theory of nonlinear monotone operators generalizes the following elementary result. We consider the real equation

$$F(x) = b \qquad x \in R \tag{3.1}$$

and assume that:

(1) The function $F : R \to R$ is monotone;

(2) F is continuous;

(3) $F(x) \to \pm\infty$ as $x \to \pm\infty$.

Then, for each $b \in R$, equation (3.1) has a solution. If F is strictly monotone, then the solution is unique.

Now we want to generalize the result above to monotone operator equations of the form

$$Ax = b \qquad x \in X. \tag{3.2}$$

where $A : X \to X^*$ is an operator on the real reflexive Banach space X. A natural problem arises when there exists a solution of the operator equation (3.2) for each $b \in X^*$.

The following definitions are basic.

Definition 8. *Let Y be real Banach space and let $A : Y \to Y^*$ be an operator. Then:*

(1) A is called monotone *if*

$$\langle Ax - Ay,\ x - y\rangle_Y \geq 0 \quad \textit{for all } x,\ y \in Y.$$

(2) A is called strictly monotone *if*

$$\langle Ax - Ay,\ x - y\rangle_Y > 0 \quad \textit{for all } x, y \in Y \quad \textit{with } x \neq y.$$

(3) A is called strongly monotone *if there is a constant* $c > 0$ *such that*

$$\langle Ax - Ay,\ x - y\rangle_Y \geq c\,\|x - y\|_Y^2 \quad \textit{for all } x, y \in Y.$$

(4) A is called uniformly monotone *if*

$$\langle Ax - Ay,\ x - y\rangle_Y \geq a\left(\|x - y\|_Y\right)\|x - y\|_Y \quad \textit{for all } x, y \in Y,$$

where the continuous function $a : R_+ \to R_+$ *is strictly monotone increasing with* $a(0) = 0$ *and* $a(t) \to +\infty$ *as* $t \to +\infty$.

For example, we may choose $a(t) = c\,|t|^{p-1}$ with $p > 1$ and $c > 0$. In this case, we obtain

$$\langle Ax - Ay,\ x - y\rangle_Y \geq c\,\|x - y\|_Y^p \quad \text{for all } x, y \in Y.$$

(5) If Y *is a reflexive Banach space and* $A : Y \to Y^*$ *is an operator, A is called* pseudomonotone *if* $x_n \overset{W}{\to} x$ *as* $n \to \infty$ *and*

$$\overline{\lim_{n\to\infty}}\,\langle Ax_n,\ x_n - x\rangle_Y \leq 0$$

implies

$$\langle Ax,\ x - w\rangle_Y \leq \underline{\lim}_{n\to\infty}\langle Ax_n, x_n - w\rangle_Y \quad \textit{for all } w \in Y.$$

(6) A is called coercive *if*

$$\lim_{\|x\|\to\infty} \frac{\langle Ax, x\rangle_Y}{\|x\|_Y} = +\infty.$$

Obviously, we have the following implications:

A is strongly monotone $\Rightarrow$ A is uniformly monotone $\Rightarrow$ A is strictly monotone $\Rightarrow$ A is monotone.

Definition 9. *Let* $A : Y \to Y^*$ *be an operator on the Banach space* Y.

(1) A is called demicontinuous *if*

$$x_n \to x \quad \textit{as} \quad n \to +\infty$$

implies $Ax_n \overset{W}{\to} Ax$ *as* $n \to \infty$.

(2) A is called hemicontinuous *if the real function*

$$t \mapsto \langle A(x + ty), w\rangle_Y$$

is continuous on $[0, 1]$ *for all* $x, y, w \in Y$.

Definition 10. *Let Y be a reflexive Banach space. The operator $A : Y \to Y^*$ satisfies the* condition *(M) if*

$$x_n \xrightarrow{W} x, \quad Ax_n \xrightarrow{W} b, \quad \overline{\lim_{n\to\infty}} \langle Ax_n, x_n \rangle_Y \leq \langle b, x \rangle_Y$$

as $n \to +\infty$ implies

$$Ax = b.$$

Proposition 19. *Let $A, B : Y \to Y^*$ be operators on the real reflexive Banach space Y. Then*

(1) If A is monotone and hemicontinuous, then A is pseudomonotone.

(2) If A is monotone and hemicontinuous, then A is demicontinuous.

(3) If A is monotone and hemicontinuous, then A satisfies condition (M).

(4) If A is pseudomonotone and locally bounded, then A is demicontinuous.

. *(5) If A and B are pseudomonotone, then $A + B$ is pseudomonotone.*

For more information, one can refer to Chapter 25, Chapter 16, and Chapter 27 of Zeidler(1990).

The notion of a maximal monotone operator is the most important concept in the theory of monotone operators. Each monotone operator possesses a maximal monotone extension. We first consider some basic notions for multivalued mappings.

Definition 11. *Let $A : M \to 2^Y$ be a multivalued mapping, i.e., A assigns to each point $x \in M$ a subset Ax of Y.*

(1) The set

$$D(A) = \{x \in M : Ax \neq \varnothing\}$$

is called the effective domain of A.

(2) The set

$$R(A) = \bigcup_{x \in M} Ax$$

is called the range of A.

(3) The set

$$G(A) = \{(x, y) \in M \times Y : x \in D(A), y \in Ax\}$$

is called the graph of A.

Each single-valued map

$$A : D(A) \subseteq M \to Y$$

can be identified with a multivalued map

$$\bar{A} : M \to 2^Y$$

by setting

$$\bar{A}x = \begin{cases} \{Ax\} & \text{if } x \in D(A), \\ \emptyset & \text{otherwise.} \end{cases}$$

Instead of $\bar{A}$ we will briefly write A.

The map $B : M \to 2^Y$ is called an extension of $A : M \to 2^Y$ if $G(A) \subseteq G(B)$.

Definition 12. *We consider the multivalued map map*

$$A : M \to 2^{Y^*}$$

where M is a subset of the real Banach space Y.

(1) A subset S of $M \times Y^*$ is called monotone *if*

$$\langle x^* - y^*, x - y\rangle_Y \geq 0 \quad \textit{for all } (x, x^*), (y, y^*) \in S.$$

(2) A subset S of $M \times Y^*$ is called maximal monotone *if it is monotone and there is no proper monotone extension in $M \times Y^*$.*

(3) The map A is called monotone *if the graph $G(A)$ is monotone set in $M \times Y^*$.*

(4) The map A is called maximal monotone *if the graph $G(A)$ is a maximal monotone set in $M \times Y^*$.*

An operator

$$A : D(A) \subseteq Y \to Y^*$$

is to be understood as a multivalued map $A : Y \to 2^{Y^*}$. Thus A is called *maximal monotone* if A is monotone and it follows from

$$(x, x^*) \in Y \times Y^* \quad \text{and} \quad \langle x^* - Ay, x - y\rangle_Y \geq 0 \quad \text{for all } y \in D(A)$$

that $x \in D(A)$ and $x^* = Ax$.

Proposition 20. *Let $A : Y \to Y^*$ be a monotone and hemicontinuous on the reflexive Banach space Y. Then*

(1) A is maximal monotone.

(2) A satisfies condition (M).

(3) It follows from either

$$x_n \longrightarrow x \quad \textit{in } Y \quad \textit{and} \quad Ax_n \xrightarrow{W} b \quad \textit{in } Y^* \quad \textit{as } n \to \infty,$$

or

$$x_n \xrightarrow{W} x \quad \textit{in } Y \quad \textit{and} \quad Ax_n \longrightarrow b \quad \textit{in } Y^* \quad \textit{as } n \to \infty,$$

that

$$Ax = b.$$

In 1968, Browder built a fundamental result in the theory of monotone operators. He considered the basic equation

$$b \in Lx + Ax, \qquad x \in C. \tag{3.3}$$

Assume:

(H5) C is a nonempty closed convex set in the reflexive Banach space Y;

(H6) $L : C \to 2^{Y^*}$ is maximal monotone;

(H7) $A : C \to Y^*$ is pseudomonotone, bounded and demicontinuous;

(H8) If the set C is unbounded, then the operator A is coercive with respect to the fixed element $b \in Y^*$, that is, there exists a point $x_0 \in C \cap D(L)$ and a number $r > 0$ such that

$$\langle Ax, x - x_0\rangle_Y > \langle b, x - x_0\rangle_Y$$

for all $x \in C$ with $\|x\| > r$.

Theorem 3. *(Browder (1968)). Let $b \in Y^*$ be given and assume (H5) through (H8). Then the original problem (3.3) has a solution.*

3.2 Evolution equations with nonlinear monotone operator

Browder's theorem will be helpful to assert the existence of solutions of some initial value problems and periodic problems in differential equations. In this thesis, we are interested in periodic problems. Let us consider the periodic problem

$$\begin{cases} \dot{x}(t) + Ax(t) = b(t), & 0 < t < T \\ x(0) = x(T). \end{cases} \tag{3.4}$$

By a solution x of problem (3.4), we mean a function $x \in \{x \in W_{pq} : x(0) = x(T)\}$ that satisfies

$$\langle \dot{x}(t), v\rangle_V + \langle Ax(t), v\rangle_V = \langle b(t), v\rangle_V$$

for all $v \in V$ and almost all $t \in I$. $I = (0, T)$.

Recall that

$$X = L_p(I, V), \quad X^* = L_q(I, V^*)$$

where "$V \hookrightarrow H \hookrightarrow V^*$" is an evolution triple, $p \geq 1$, and $p^{-1} + q^{-1} = 1$.

Lemma 2. *Define the operator*

$$Lx = \dot{x}, \quad D(L) = \{x \in W_{pq} : x(0) = x(T)\}.$$

Then the linear operator $L : D(L) \subseteq X \to X^*$ is maximal monotone.

Proof. (1) L is linear and monotone. It is obvious L is linear. And, from the integration by parts formula

$$\langle Lx, x\rangle_X = \int_0^T \langle \dot{x}(t), x(t)\rangle dt = \frac{1}{2}(\|x(T)\|_H^2 - \|x(0)\|_H^2) = 0$$

for any $x \in D(L)$. Hence L is monotone.

(2) L is maximal monotone. Suppose that $(y,\ y^*) \in X \times X^*$ and

$$0 \le \langle y^* - Lx, y - x\rangle_X \quad \text{for any } x \in D(L). \tag{3.5}$$

We have to show that $y \in D(L)$ and $y^* = Ly$. We choose $x = \psi z$ where $\psi \in C_0^\infty(0, T)$ and $z \in V$, then from (3.5) and $\langle Lx, x\rangle_X = 0$, it follows that

$$0 \le \langle y^*, y\rangle_X - \int_0^T \left\langle \psi(t)\, y^*(t) + \dot{\psi}(t)\, y(t), z \right\rangle_V dt$$

for all $z \in V$. The above inequality implies

$$\int_0^T \psi(t) y^*(t) + \dot{\psi}(t) y(t) dt = 0 \quad \text{for all } \psi \in C_0^\infty(0, T).$$

Hence

$$\dot{y} = y^* \quad \text{and} \quad y \in W_{pq},$$

because $y^* \in X^*$. It remains to show that $y \in D(L)$. Since

$$0 \le \langle \dot{y} - \dot{x}, y - x\rangle_X = \frac{1}{2}\left(\|y(T) - x(T)\|_H^2 - \|y(0) - x(0)\|_H^2\right)$$

for any $x \in D(L)$, we get

$$\|y(T)\|_H^2 - \|y(0)\|_H^2 + 2\,(y(0) - y(T), x(0)) \ge 0$$

for any $x \in D(L)$. In particular, we can choose $x(t) \equiv a$ for arbitrary $a \in V$. That is,

$$\|y(T)\|_H^2 - \|y(0)\|_H^2 + 2\,(y(0) - y(T), a) \ge 0 \tag{3.6}$$

for any $a \in V$ and therefore

$$\|y(T)\|_H^2 - \|y(0)\|_H^2 \ge 0$$

by taking $a = 0$. Set $\varepsilon = \|y(T)\|_H^2 - \|y(0)\|_H^2$ and substitute $na \in V$ and $-na \in V$, $n = 1, 2, \cdots$ into (3.6), one can get

$$-\frac{\varepsilon}{n} \leq (y(0) - y(T), a) \leq \frac{\varepsilon}{n}, \quad \text{for } n = 1, 2, \cdots \text{ and any } a \in V.$$

Letting $n \to +\infty$,

$$(y(0) - y(T), a) = 0 \quad \text{for any } a \in V.$$

Note that V is dense in H, so $y(0) = y(T)$ in H and $y \in D(L)$. Therefore, L is maximal monotone. □

The problem (3.4) can be reformulated in the following operator equation

$$Lx + Ax = b \qquad x \in D(L).$$

Theorem 4. *Suppose that operator $A : X \to X^*$ is pseudomonotone, coercive, and bounded. Then, for each $b \in X^*$, problem (3.4) has a solution. If, in addition, A is strictly monotone, then the solution is unique.*

Proof. 1. Existence.

We identify L with a multivalued map

$$\bar{L} : X \to 2^{X^*}$$

by setting

$$\bar{L}x = \begin{cases} \{Lx\} & \text{if } x \in D(L), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $D(\bar{L}) = D(L)$ and $R(\bar{L}) = R(L)$. By Lemma 2, the operator $L : D(L) \subseteq X \to X^*$ is maximal monotone, we get that $\bar{L} : X \to 2^{X^*}$ is also maximal monotone. The existence of a solution of the problem (3.4) follows from Browder's Theorem (Theorem 3) with $C = X$ and $x_0 = 0$.

2. Uniqueness. Let x_1, x_2 be two solutions of problem (3.4). It follows from

$$Lx_i + Ax_i = b, \qquad i = 1, 2$$

and as L is monotone, that

$$\begin{aligned} 0 &= \langle L(x_1 - x_2), x_1 - x_2 \rangle_X + \langle Ax_1 - Ax_2, x_1 - x_2 \rangle_X \\ &= \langle Ax_1 - Ax_2, x_1 - x_2 \rangle_X . \end{aligned}$$

Since A is strictly monotone, $x_1 = x_2$. □

3.3 Evolution equations with nonlinear monotone and nonmonotone operators

In the proceeding section, we presented an existence result for periodic solutions of evolution equations in which the nonlinear operator is uniformly monotone. In this section, we will extend this result to more general nonlinear evolution equations including nonlinear nonmonotone perturbations. That is, we consider the existence of periodic solutions for the following nonlinear equation

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t)), & t \in I \\ x(0) = x(T) \end{cases} \tag{3.7}$$

with nonlinear monotone operator A and nonlinear nonmonotone perturbation f.

Let "$V \hookrightarrow H \hookrightarrow V^*$" be an evolution triple and suppose that the embedding $V \hookrightarrow H$ is compact. Let $X = L_p(I, V)$ and $p \geq 2$.

We need the following hypotheses on the data problem of (3.7).

(A1) $A : I \times V \longrightarrow V^*$ is an operator such that

1. $t \longmapsto A(t, x)$ is measurable ;

2. For each $t \in I$, the operator $A(t) : V \longrightarrow V^*$ is uniformly monotone and hemicontinuous, that is, there exists a constant $C_1 \geq 0$ such that

$$\langle A(t, x_1) - A(t, x_2), x_1 - x_2 \rangle \geq C_1 \| x_1 - x_2 \|_V^p \quad \text{for all } x_1, x_2 \in V,$$

and the map $s \longmapsto \langle A(t, x + sz), y \rangle$ is continuous on $[0, 1]$ for all $x, y, z \in V$;

3. Growth condition. There exist a constant $C_2 > 0$ and a nonnegative function $a_1(\cdot) \in L_q(I)$ such that

$$\| A(t, x) \|_{V^*} \leq a_1(t) + C_2 \| x \|_V^{p-1} \quad \text{for all } x \in V, \text{ a.e. on } I;$$

(F1) $f : I \times H \longrightarrow V^*$ is an operator such that

1. $t \longmapsto f(t, x)$ is measurable;

2. $x \longmapsto f(t, x)$ is continuous and $f(t, x)$ is Hölder continuous respect x with exponent $0 < \alpha \leq 1$ in H uniformly in t. That is, there is a constant L such that

$$\| f(t, x_1) - f(t, x_2) \|_{V^*} \leq L \| x_1 - x_2 \|_H^\alpha \quad \text{for all } x_1, x_2 \in H, \ t \in I.$$

3. There exist a nonnegative function $h_1(\cdot) \in L_q(I)$ and a constant $C_3 > 0$ such that

$$\| f(t, x) \|_{V^*} \leq h_1(t) + C_3 \| x \|_H^{k-1} \quad \text{for all } x \in V, t \in I,$$

where $1 \leq k < p$ is a constant.

We assume without loss of generality that $A(t,0) = 0$ for all $t \in \overline{I}$.

At first, we reformulate the equation (3.7) into an operator equation in some appropriate spaces. It is often convenient to write system (3.7) as an operator equation in

$$W_{pq}(T) = \{x \in W_{pq} : x(0) = x(T)\}.$$

For $x \in X$, we set

$$A(x)(t) = A(t, x(t)), \quad F(x)(t) = f(t, x(t)), \quad \text{for all } t \in I.$$

Then, we obtain that the original problem (3.7) is equivalent to the following operator equation:

$$\begin{cases} \dot{x} + A(x) = F(x), \\ x \in W_{pq}(T). \end{cases} \tag{3.8}$$

The operator A and the operator F have some properties which we will be stated below.

Lemma 3. *If hypotheses (A1) and (F1) hold, then for each $x \in X$, $t \longmapsto A(t, x(t))$ is measurable from $(0,T)$ to V^* and for each $x \in L_p(I,H)$, the function $t \longmapsto f(t, x(t))$ is measurable from $(0,T)$ to V^*.*

Lemma 4. *If hypothesis (A1) holds, then the operator $A : X \to X^*$ is uniformly monotone, hemicontinuous, coercive, and bounded. That is,*

$$\begin{aligned} \langle Ax_1 - Ax_2, x_1 - x_2\rangle_X &\geq C_1 \|x_1 - x_2\|_X^p \quad \text{for all } x_1, x_2 \in X, \\ \lim_{\lambda \to \mu} \langle A(x + \lambda y), z\rangle_X &= \langle A(x + \mu y), z\rangle_X \text{ for all } x, y, z \in X \text{ and } \lambda, \mu \in [0,1], \\ \langle Ax, x\rangle_X &\geq C_1 \|x\|_X^p \quad \text{for all } x \in X, \\ \|Ax\|_{X^*} &\leq M_1 + M_2 \|x\|_X^{p-1} \quad \text{for all } x \in X. \end{aligned}$$

Proof. 1. Boundedness. We show that the operator $A : X \to X^*$ is bounded. Let $x \in X$, it follows from the growth condition $(A1)(3)$ and $p/q = p - 1$ that

$$\begin{aligned} &\| \; A(t, x(t)) \|_{V^*}^q \leq \left(a_1(t) + C_2 \| x(t) \|_V^{p-1}\right)^q \\ &\leq \alpha_1 \left(| a_1(t) |^q + \| x(t) \|_V^p\right) \end{aligned} \tag{3.9}$$

for some constant $\alpha_1 > 0$. By Proposition 18 in Section 2.7, the function $A(t, x(t))$ is measurable from $(0,T)$ to V^*. By the property that linear combinations of measurable functions, norm functions of measurable functions are also measurable. We obtain that the real function $t \mapsto \| A(t, x(t)) \|_{V^*}^q$ is measurable on $(0,T)$. Since $a_1 \in L_q(0,T)$ and $x \in L_p(I,V)$, the function on the right-hand side of (3.9) is integrable over $(0,T)$. By integration, we obtain from (3.9) that

$$\| Ax \|_{X^*} \leq \alpha_1 \left(\| a_1 \|_{L_q(I)} + \| x \|_X^{p/q} \right) \quad \text{for all } x \in X.$$

Let $M_1 = \alpha_1 \cdot \|a_1\|_{L_q(I)}$ and $M_2 = \alpha_1$, we get the boundedness of A.

2. Monotonicity. We show that $A : X \to X^*$ is monotone. Let $x_1,\ x_2 \in X$, since $Ax_1, Ax_2 \in X^*$, then proposition 17 of Section 2.7 implies that the real function
$$t \mapsto \langle A(t, x_1(t)), x_2(t)\rangle$$
is integrable over $(0,T)$. By the monotonicity of $A(t) : V \to V^*$ for each $t \in (0,T)$, it implies that
$$\begin{aligned} & \langle A(x_1) - A(x_2),\ x_1 - x_2\rangle_X \\ = & \int_0^T \langle A(t,\ x_1(t)) - A(t,\ x_2(t)),\ x_1(t) - x_2(t)\rangle_V dt \\ \geq & \int_0^T C_1 \parallel x_1(t) - x_2(t) \parallel_V^p dt \\ = & C_1 \parallel x_1 - x_2 \parallel_X^p , \end{aligned}$$
for all $x_1,\ x_2 \in X$.

3. Coerciveness. The operator $A : X \to X^*$ is coercive, since
$$\begin{aligned} \langle A(x), x\rangle_X &= \int_0^T \langle A(t, x(t)) - A(t,0), x(t) - 0\rangle_V dt \\ &\geq \int_0^T C_1 \parallel x(t) \parallel_V^p dt = C_1 \parallel x \parallel_X^p , \end{aligned}$$
for all $x \in X$.

4. Hemicontinuity. Let $x, y, z \in X$ and $0 \leq \lambda, \mu \leq 1$. For all $t \in (0,T)$, using the inequality (3.9)
$$\begin{aligned} & \mid \langle A(t, x(t) + \lambda y(t)), z(t)\rangle \mid \leq \parallel A(t, x(t) + \lambda y(t)) \parallel_{V^*} \parallel z(t) \parallel_V \\ \leq & M_1 \left(\mid a_1(t) \mid + \parallel x(t) + \lambda y(t) \parallel_V^{p/q} \right) \parallel z(t) \parallel_V \leq k(t), \end{aligned}$$
where
$$k(t) = M_1 \left(\mid a_1(t) \mid + \parallel x(t) \parallel_V^{p/q} + \parallel y(t) \parallel_V^{p/q} \right) \parallel z(t) \parallel_V .$$
Since $a_1(\cdot) \in L_q(I)$, then $a_1(\cdot) \in L_1(I)$, and because of $x, y, z \in X$ it follows that
$$\parallel x(\cdot) \parallel_V^{p/q}, \parallel y(\cdot) \parallel_V^{p/q} \in L_q(I) \quad \text{and} \quad \parallel z(\cdot) \parallel_V \in L_p(I) .$$
By the Hölder inequality, the majorant function $k(\cdot)$ belongs to $L_1(0,T)$. Therefore, it follows from the principle of Majorized convergence (see Proposition 2 in Section 2.3) that
$$\begin{aligned} \lim_{\lambda\to\mu} \langle A(x + \lambda y),\ z\rangle_X &= \lim_{\lambda\to\mu} \int_0^T \langle A(t,\ x(t) + \lambda y(t)),\ z(t)\rangle_V dt \\ &= \int_0^T \lim_{\lambda\to\mu} \langle A(t,\ x(t) + \lambda y(t)),\ z(t)\rangle_V dt \\ &= \langle A(x + \mu y),\ z\rangle_X. \end{aligned}$$
That is, $A : X \to X^*$ is hemicontinuous. □

Lemma 5. *Under assumption $(F1)$, the operator $F : L_p(I,H) \to X^*$ satisfies*

1. *Hölder continuous with exponent α, $0 < \alpha \le 1$, i.e.,*

$$\|F(x_1) - F(x_2)\|_{X^*} \le L_1 \|x_1 - x_2\|^{\alpha}_{L_p(I,H)} \qquad \textit{for all } x_1, x_2 \in L_p(I,H)$$

   with some constant $L_1 > 0$;

2. *Bounded, i.e.,*

$$\| F(x) \|_{X^*} \le M_3 + M_4 \| x \|^{(k-1)}_{L_p(I,H)} \qquad \textit{for all } x \in L_p(I,H).$$

   where M_3, M_4 are positive constants.

3. *$F(x_n) \longrightarrow F(x)$ in X^* whenever $x_n \xrightarrow{W} x$ in W_{pq} as $n \to \infty$.*

Proof. 1. Boundedness. We show that $F : L_p(I,H) \to X$ is also bounded. Let $x \in L_p(I,H)$, from the hypothesis of $(F1)(3)$, we get

$$\begin{aligned} \| \ f(t, x(t)) \|^q_{V^*} &\le \left(h_1(t) \ + \ C_3 \| x \|^{k-1}_H\right)^q \\ &\le \ \alpha_2 \left(| h_1(t) |^q + \| x(t) \|^{(k-1)q}_H \right) \end{aligned} \tag{3.10}$$

for some constant $\alpha_2 > 0$. Following an argument similar to proposition 18 in Section 2.7, the function $f(t, x(t))$ is measurable from $(0,T)$ to V^* for any $x \in L_p(I,H)$, so that the real function $t \longmapsto \|f(t,x(t))\|^q_{V^*}$ is measurable on $(0,T)$. By integration, we obtain that

$$\| F(x) \|_{X^*} \le M_3 + M_4 \| x \|^{(k-1)}_{L_p(I,H)} \qquad \text{for all } x \in L_p(I,H).$$

2. For any $x_1, \ x_2 \in L_p(I,H)$, by hypothesis $(F1)$ and Hölder inequality, we get

$$\begin{aligned} \| \ & F(x_1) - F(x_2) \|_{X^*} \\ = \ & \left(\int_0^T \| \ f(t, x_1(t)) - f(t, x_2(t)) \|^q_{V^*} \, dt \right)^{1/q} \\ \le \ & L \left(\int_0^T \| \ x_1(t) - x_2(t) \|^{q\alpha}_H \, dt \right)^{1/q} \\ \le \ & L \left(\int_0^T \| \ x_1(t) - x_2(t) \|^p_H \, dt \right)^{\alpha/p} \left(\int_0^T 1^{1-\frac{p}{q\alpha}} dt \right)^{\frac{\alpha}{q\alpha - 1}} \\ \le \ & L_1 \| \ x_1 - x_2 \|^{\alpha}_{L_p(I,H)} . \end{aligned}$$

with some constant L_1. This proves that F is Hölder continuous with exponent α in $L_p(I,H)$. Hence F is continuous on $L_p(I,H)$.

3. Since the embedding $V \hookrightarrow H$ is compact, the embedding $W_{pq} \hookrightarrow L_p(I,H)$ is compact. That is

$$x_n \longrightarrow x \ \text{ in } L_p(I,H) \qquad \text{whenever} \qquad x_n \overset{W}{\longrightarrow} x \ \text{ in } W_{pq}.$$

By using the above relation and the continuity of F, we have

$$F(x_n) \longrightarrow F(x) \ \text{ in } X^* \ \text{ whenever } \ x_n \overset{W}{\longrightarrow} x \ \text{ in } W_{pq}.$$

□

Lemma 6. *Assume (A1) and (F1) are satisfied , then the set*

$$S1 = \{x \in W_{pq}(T) \mid \dot{x} + A(x) = \sigma F(x), \ \text{for some } \sigma \in [0,1]\} \tag{3.11}$$

is bounded in W_{pq}. Moreover, there exists positive constants M such that

$$\| A(x) \|_{X^*} \leq M, \ \ \text{and} \ \ \max_{t \in I} \| x(t) \|_H \leq M$$

for all $x \in S1$.

Proof. Let $x \in S1$, then

$$\langle \dot{x}, x \rangle_X + \langle A(x), x \rangle_X = \langle \sigma F(x), x \rangle_X.$$

From Lemma 4, Lemma 5, $x(0) = x(T)$, and the continuous embedding $X \hookrightarrow L_p(I,H)$, using integration by parts we get

$$\begin{aligned} C_1 \|x\|_X^p &\leq \|F(x)\|_{X^*} \cdot \|x\|_X \\ &\leq \left(M_3 + M_4 \|x\|_{L_p(I,H)}^{k-1}\right) \cdot \|x\|_X \\ &\leq \left(M_3 + \alpha_3 \|x\|_X^{k-1}\right) \cdot \|x\|_X \end{aligned} \tag{3.12}$$

for some constants $M_3 > 0$ and $\alpha_3 > 0$. Now the real function

$$g(\xi) = C_1 |\xi|^{p-1} - \alpha_3 |\xi|^{k-1} - M_3$$

goes to $+\infty$ as $\xi \to +\infty$ when $1 \leq k < p$. Thus, by virtue of the inequality (3.12), we can find a constant $\alpha_4 > 0$ such that

$$\| x \|_X \leq \alpha_4 \tag{3.13}$$

for all $x \in S1$.

Now, let $\phi(\cdot)$ be an arbitrary element of X and suppose $x \in S1$, then we have

$$\langle \dot{x}, \phi \rangle_X + \langle A(x), \phi \rangle_X = \langle \sigma F(x), \phi \rangle_X.$$

Apply Hölder inequality, Lemma 4, and Lemma 5 again, we get

$$\begin{aligned} | \dot{x}(\phi) | &\leq \|A(x)\|_{X^*} \|\phi\|_X + |\sigma| \|F(x)\|_{X^*} \|\phi\|_X \\ &\leq \left(M_1 + M_2 \| x \|_X^{p-1} + M_3 + M_4 \| x \|_{L_p(I,H)}^{k-1} \right) \| \phi \|_X \end{aligned} \tag{3.14}$$

By using the continuous embedding $L_p(I,V) \hookrightarrow L_p(I,H)$ and (3.13), then (3.14) implies

$$\| \dot{x} \|_{X^*} \leq \alpha_5 \tag{3.15}$$

for some positive constant α_5 and for all $x \in S1$. It follows from (3.13) and (3.15) that

$$\| x \|_{W_{pq}} \leq \alpha_4 + \alpha_5.$$

Hence, $S1$ is bounded subset of W_{pq}.

It follows from boundedness of A (Lemma 4) and (3.13) that

$$\| A(x) \|_{X^*} \leq \alpha_6$$

for some $\alpha_6 > 0$ and for all $x \in S1$.

Finally, we note that as $W_{pq} \hookrightarrow C(\overline{I}, H)$ is continuous, then

$$\max_{t \in I} \| x(t) \|_H \leq \alpha \| x \|_{W_{pq}} \leq \alpha_7$$

for some positive constants α , α_7, and for all $x \in S1$.

Taking $M = \max(\alpha_4 + \alpha_5, \alpha_6, \alpha_7)$, we get the assertion. □

In order to prove the existence of a solution of problem (3.8), we introduce a map $G : L_p(I,H) \times [0,1] \to L_p(I,H)$ defined by

$$G(x,\sigma) = y$$

where y is the unique solution of the following problem

$$\begin{cases} \dot{y} + A(y) = \sigma F(x) \\ y(0) = y(T). \end{cases} \tag{3.16}$$

Since A is uniformly monotone, then A is strictly monotone. By Theorem 4, for any $x \in S$, problem (3.16) has a unique solution $y \in W_{pq} \subset L_p(I,H)$. So G is well defined. In the following, we will show that G has a fixed point in $L_p(I,H)$ by verifying that G satisfies the hypotheses of the Leray-Schauder fixed point theorem (Theorem 1 in Section 2.2).

Theorem 5. *Under assumptions (A1) and (F1), the equation (3.8) has a solution $x \in W_{pq}$.*

Proof. 1. The map $G : L_p(I, H) \times [0,1] \to L_p(I, H)$ is compact.

Firstly, we show that the mapping $G : L_p(I,H) \times [0,1] \to L_p(I,H)$ is continuous. In fact, for any sequence $(x_n, \sigma_n) \subset L_p(I,H) \times [0,1]$ such that

$$(x_n, \sigma_n) \longrightarrow (x, \sigma) \quad \text{in} \quad L_p(I,H) \times [0,1],$$

let y_n be a solution of the problem

$$\begin{cases} \dot{y}_n + A(y_n) = \sigma_n F(x_n) \\ y_n(0) = y_n(T), \end{cases}$$

and y be a solution of the problem

$$\begin{cases} \dot{y} + A(y) = \sigma F(x) \\ y(0) = y(T). \end{cases}$$

So,

$$\begin{aligned} &\langle \dot{y}_n - \dot{y}, y_n - y\rangle_X + \langle A(y_n) - A(y), y_n - y\rangle_X \\ = \ &\langle \sigma_n F(x_n) - \sigma F(x), y_n - y\rangle_X. \end{aligned} \tag{3.17}$$

Using integration by parts and the monotonicity of the operator A, we obtain from (3.17) that

$$\begin{aligned} &\frac{1}{2}\left(\| y_n(T) - y(T)\|_H^2 - \| y_n(0) - y(0)\|_H^2\right) + C_1 \| y_n - y\|_X^p \\ \leq \ &\langle \sigma_n F(x_n) - \sigma F(x), y_n - y\rangle_X \\ \leq \ &\frac{\varepsilon}{p} \| y_n - y\|_X^p + \frac{\varepsilon^{-q/p}}{q} \| \sigma_n F(x_n) - \sigma F(x)\|_{X^*}^q. \end{aligned} \tag{3.18}$$

for all $\varepsilon > 0$. By choosing ε in (3.18) small enough and from Lemma 4, Lemma 5, we get

$$\begin{aligned} &\beta_1 \| y_n - y\|_X^p \leq \beta_2 \| \sigma_n F(x_n) - \sigma F(x)\|_{X^*}^q \\ = \ &\beta_2 \| \sigma_n F(x_n) - \sigma_n F(x) + \sigma_n F(x) - \sigma F(x)\|_{X^*}^q \\ \leq \ &\beta_3 (\| F(x_n) - F(x)\|_{X^*}^q + |\sigma_n - \sigma|^q \| F(x)\|_{X^*}^q) \\ \leq \ &\beta_4 \| x_n - x\|_{L_p(I,H)}^{q\alpha} + \beta_5 |\sigma_n - \sigma|^q \left(1 + \| x\|_{L_p(I,H)}^{(k-1)}\right)^q. \end{aligned}$$

for some positive constants $\beta_1, \beta_2, \beta_3, \beta_4$, and β_5. Noting that the embedding $L_p(I,V) \hookrightarrow L_p(I,H)$ is continuous, we have

$$\| y_n - y\|_{L_p(I,H)} \leq \beta_6 (\| x_n - x\|_{L_p(I,H)}^{\frac{q\alpha}{p}} + |\sigma_n - \sigma|^{q/p} (1 + \| x\|_{L_p(I,H)}^{(k-1)})^{q/p})$$

for some constant $\beta_6 > 0$. Hence, $G : L_p(I,H) \times [0,1] \to L_p(I,H)$ is continuous.

Moreover, we will show that G maps every bounded set in $L_p(I,H) \times [0,1]$ to a relative compact set in $L_p(I,H)$.

Let y be solution of problem (3.16) with $\| x \|_{L_p(I,H)} < b_1$ and some $\sigma \in [0,1]$, where $b_1 > 0$ is a constant. Similar to the arguments in proof of Lemma 6, one can show that there exists constants $b_2 > 0$,

$$\| y \|_{W_{pq}} \leq b_2.$$

G maps bounded sets in $L_p(I,H) \times [0,1]$ into bounded sets in W_{pq}. As the embedding $W_{pq} \hookrightarrow L_p(I,H)$ is compact.
$G: \ L_p(I,H) \times [0,1] \longrightarrow L_p(I,H)$ is compact operator.

2. A priori estimates.

   Let

$$X_\sigma = \{x \in L_p(I,H) \mid x = G(x,\sigma) \text{ for some } 0 \leq \sigma \leq 1\}.$$

   Assume $x \in L_p(I,H)$ and $x = G(x, \sigma)$, then $x \in W_{pq}$ and satisfies the problem

$$\begin{cases} \dot{x} + A(x) = \sigma F(x), \\ x(0) = x(T). \end{cases}$$

   By lemma 6, we get

$$\| x \|_{W_{pq}} \leq M \quad \text{for all } x \in X_\sigma.$$

   Again, as the embedding $W_{pq} \hookrightarrow L_p(I,H)$ is compact, we get

$$\| x \|_{L_p(I,H)} \leq \beta_7 \ \text{ for all } x \in X_\sigma$$

   with some constant $\beta_7 > 0$.

3. We show that $G(x, 0) = 0$ for any $x \in L_p(I,H)$.

   For any $x \in L_p(I,H)$, set $G(x, 0) = y_0$, where y_0 is the solution of

$$\begin{cases} \dot{y}_0 + A(y_0) = 0, \\ y_0(0) = y_0(T). \end{cases} \tag{3.19}$$

   By uniqueness of the solution and as $A(t,0) = 0$ for all $t \in I$, we get

$$y_0 = 0 \quad \text{in } W_{pq}.$$

   But the embedding $W_{pq} \hookrightarrow L_p(I,H)$ is continuous, so we get

$$y_0 = 0 \qquad \text{in} L_p(I,H),$$

   that is,

$$G(x, 0) = 0, \quad \text{for any } x \in L_p(I,H).$$

Applying the Leray-Schauder fixed point theorem in the space $L_p(I,H)$, there exists a fixed point $y^* \in L_p(I,H) \cap W_{pq}$ such that

$$v^* = G(v^*, 1).$$

$y^* \in W_{pq}$ is just the periodic solution of (3.8). By the equivalence of problems (3.7) and (3.8), there exists a periodic solution for the nonlinear evolution equation (3.7). □

Chapter IV

Anti-periodic Solutions of Evolution Equations

It is very interesting that our method can be used to discuss the existence of anti-periodic solutions which arise in many physical models. In this chapter, we prove results analogous to those in Chapter 3 for anti-periodic solutions of nonlinear evolution equation. Moreover, we will present an existence result for anti-periodic solutions of second-order nonlinear evolution equations.

As before, let "$V \hookrightarrow H \hookrightarrow V^*$", $X = L_p(I, V)$, and $X^* = L_q(I, V^*)$, where $2 \leq p < +\infty$, $I = (0, T)$, and $0 < T < +\infty$.

4.1 Evolution equations with nonlinear monotone operator

Similar to the preceding chapter, we start with an existence theorem for the following anti-periodic problem

$$\begin{cases} \dot{x}(t) + Ax(t) = b, & 0 < t < T \\ x(0) = -x(T). \end{cases} \tag{4.1}$$

Theorem 6. *Suppose that the operator*

$$A : X \to X^*$$

is pseudomonotone, coercive, and bounded. Then, for each $b \in X^*$, problem (4.1) has a solution. If, in addition, A is strictly monotone, then the solution is unique.

Proof. (1) The anti-periodic operator $\Lambda : D(\Lambda) \subseteq X \to X^*$ is a maximal monotone operator.

We set

$$\Lambda x = \dot{x}, \quad D(\Lambda) = \{x \in W_{pq} : x(0) = -x(T)\} \tag{4.2}$$

It is obvious that Λ is linear. From the integration by parts formula

$$\int_0^T \langle \dot{x}(t), x(t) \rangle \, dt = \frac{1}{2} \left(\|x(T)\|_H^2 - \|x(0)\|_H^2 \right)$$

for all $x \in W_{pq}$, we get

$$\langle \Lambda x, x \rangle_X = \int_0^T \langle \dot{x}(t), x(t) \rangle \, dt = 0, \qquad \text{for all } x \in D(\Lambda). \tag{4.3}$$

Combining (4.3) with linearity of Λ, we get that Λ is monotone.

Furthermore, Λ is maximal monotone. To prove this, suppose that $(y, y^*) \in X \times X^*$ and

$$0 \leq \langle y^* - \Lambda x, y - x \rangle_X \qquad \text{for all } x \in D(\Lambda). \tag{4.4}$$

We have to show that $y \in D(\Lambda)$ and $y^* = \Lambda y$. Choose $x = \psi z$, where $\psi \in C_0^\infty(0, T)$ and $z \in V$. Then $\dot{x} = \dot{\psi} z$ and $x \in D(\Lambda)$. It follows from (4.3) and (4.4), that

$$0 \leq \langle y^*, y \rangle_X - \int_0^T \left\langle \dot{\psi}(t) y(t) + \psi(t) y^*(t), z \right\rangle_V dt$$

for all $z \in V$. Therefore,

$$\int_0^T \left(\dot{\psi}(t) y(t) + \psi(t) y^*(t) \right) dt = 0$$

for all $\psi \in C_0^\infty(0, T)$. Since $y^* \in X^*$, we get

$$\Lambda y = \dot{y} = y^* \quad \text{and} \quad y \in W_{pq}.$$

Again, from

$$\begin{aligned} 0 &\leq \langle \dot{y} - \dot{x}, y - x \rangle_X = \frac{1}{2} \left(\|y(T) - x(T)\|_H^2 - \|y(0) - x(0)\|_H^2 \right) \\ &= \frac{1}{2} \left(y(T) - x(T), y(T) - x(T) \right) - \frac{1}{2} \left(y(0) - x(0), y(0) - x(0) \right), \end{aligned}$$

we obtain

$$\|y(T)\|_H^2 - \|y(0)\|_H^2 + 2 \left(y(0) + y(T), x(0) \right) \geq 0$$

for all $x \in D(\Lambda)$. In particular, we choose $x(t) = 0$ for all $t \in [0, T]$, then

$$\|y(T)\|_H^2 - \|y(0)\|_H^2 \geq 0.$$

We choose $x(t) = \cos \frac{\pi t}{T} \cdot a$ for arbitrary $a \in V$, then $x \in D(\Lambda)$ and $x(0) = a$. Denoting $l = \|y(T)\|_H^2 - \|y(0)\|_H^2 \geq 0$, we get

$$-\frac{l}{2n} \leq \left(y(T) + y(0), a \right) \leq \frac{l}{2n}$$

because V is a Banach space, $a \in V$ implies $-a \in V$, $na \in V$, and $-na \in V$, for $n = 1, 2, \cdots$. Letting $n \to +\infty$, we get

$$\left(y(T) + y(0), a \right) = 0$$

for any $a \in V$. By the density of V in H, we get $y(0) = -y(T)$. That is, $v \in D(\Lambda)$.

(2) Existence and uniqueness of anti-periodic solutions.

By the definition of the operator Λ, problem (4.1) is replaced by an operator equation as follows:

$$\Lambda x + Ax = b, \quad x \in D(\Lambda).$$

We identify Λ with a multivalued map

$$\overline{\Lambda} : X \to 2^{X^*}$$

by setting

$$\overline{\Lambda} x = \begin{cases} \{\Lambda x\} & \text{if } x \in D(\Lambda), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $D(\overline{\Lambda}) = D(\Lambda)$ and $R(\overline{\Lambda}) = R(\Lambda)$.

It follows from maximal monotonicity of the operator $\Lambda : D(\Lambda) \subseteq X \to X^*$ that $\overline{\Lambda} : X \to 2^{X^*}$ is also maximal monotone. Since $A : X \to X^*$ is pseudomonotone, bounded, and coercive, we obtain that for each $b \in X^*$, problem (4.1) has a solution by using Browder's Theorem (Theorem 3) with $C = X$ and $x_0 = 0$.

For uniqueness, let x_k $(k = 1, 2)$ be solutions of problem (4.1). It follows from

$$\Lambda x_k + Ax_k = b, \quad k = 1, 2$$

and monotonicity of Λ that

$$\begin{aligned} 0 &= \langle \Lambda(x_1 - x_2), x_1 - x_2 \rangle_X + \langle Ax_1 - Ax_2, x_1 - x_2 \rangle_X \\ &\geq \langle Ax_1 - Ax_2, x_1 - x_2 \rangle_X . \end{aligned}$$

Since A is strictly monotone, $x_1 = x_2$. □

4.2 Evolution equations with nonlinear monotone and nonlinear nonmonotone operators

We study the existence of anti-periodic solutions of the following nonlinear evolution:

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t)), & t \in I \\ x(0) = -x(T). \end{cases} \tag{4.5}$$

We impose the same hypotheses (A1) and (F1) on the operators A and f as in Section 3.3. By a solution x of problem (4.5), we mean a function

$$x \in \{x \in W_{pq} : x(0) = -x(T)\}$$

that satisfies

$$\langle \dot{x}(t), v\rangle_V + \langle A(t, x(t)), v\rangle_V = \langle f(t, x(t)), v\rangle_V$$

for all $v \in V$ and almost all $t \in I$. We may write system (4.5) as an operator equation:

$$\begin{cases} \dot{x} + A(x) = F(x), \\ x \in \overline{W}_{pq}(T). \end{cases} \tag{4.6}$$

where

$$\overline{W}_{pq}(T) = \{x \in W_{pq} : \ x(0) = -x(T)\}.$$

Recall that

$$X = L_p(I, V), \ X^* = L_q(I, V^*).$$

The Nemyckii operator $A : X \to X^*$ is defined by

$$A(x)(t) = A(t, x(t)) \quad \text{for all } t \in I,$$

and the Nemyckii operator $F : L_p(I, H) \to X^*$ is defined by

$$F(x)(t) = f(t, x(t)) \quad \text{for all } t \in I.$$

A and F have the same properties which were stated in Lemma 4 and Lemma 5 of Section 3.3.

Before, we turn to proof the existence of solutions for (4.6), we note that a result similar to Lemma 6 is true for anti-periodic problems with basically same proof.

Lemma 7. *Assume that (A1) and (F1) are satisfied. Then the set*

$$\overline{S1} = \{x \in \overline{W}_{pq}(T) \mid \ \dot{x} + A(x) = \sigma F(x) \quad \textit{for some } \sigma \in [0, 1]\} \tag{4.7}$$

is bounded in W_{pq}. Moreover, there exists a positive constant M such that

$$\| A(x) \|_{X^*} \leq M \quad \textit{and} \quad \max_{t \in I} \| x(t) \|_H \leq M$$

for all $x \in \overline{S1}$.

We shall prove the following existence result for the anti-periodic problem.

Theorem 7. *Under assumptions (A1) and (F1), the problem (4.5) has a solution $x \in W_{pq}$.*

Proof. The proof is analogous to that of Theorem 5 in Section 3.3. We use the Leray-Schauder fixed point theorem to prove it.

1. Define an operator $\overline{G} : L_p(I,H) \times [0,1] \to L_p(I,H)$

$$\overline{G}(x,\sigma) = y$$

where y is the solution of

$$\begin{cases} \dot{y} + Ay = \sigma F x \\ y(0) = -y(T). \end{cases} \tag{4.8}$$

This operator is well defined by Theorem 6.

2. $\overline{G} : L_p(I,H) \times [0,1] \to L_p(I,H)$ is compact.

   Using Lemma 4, Lemma 5 in Section 3.3, for any sequence $\{(x_n,\sigma_n)\} \subset L_p(I,H) \times [0,1]$ such that

$$\begin{array}{lll} x_n & \longrightarrow & x \quad \text{in } L_p(I,H), \\ \sigma_n & \longrightarrow & \sigma \quad \text{in } R \end{array}$$

as $n \to \infty$, one can show that

$$y_n \longrightarrow y \quad \text{in } L_p(I,H)$$

as $n \to \infty$, where y_n is a solution of

$$\begin{cases} \dot{y}_n + A(y_n) = \sigma_n F(x_n) \\ y_n(0) = -y_n(T), \end{cases}$$

and y is a solution of

$$\begin{cases} \dot{y} + A(y) = \sigma F(x) \\ y(0) = -y(T). \end{cases}$$

That is,

$$\overline{G}(x_n,\sigma_n) \longrightarrow \overline{G}(x,\sigma) \quad \text{in } L_p(I,H)$$

as $n \to \infty$.

By virtue of the compactness of the embedding $W_{pq} \hookrightarrow L_p(I,H)$ and using similar arguments as in proof of lemma 7, we obtain that $\overline{G}$ maps bounded sets in $L_p(I,H) \times [0,1]$ into bounded sets in W_{pq}, and thus into compact sets in $L_p(I,H)$.

Therefore, $\overline{G} : L_p(I,H) \times [0,1] \to L_p(I,H)$ is compact.

3. It follows from Lemma 7 that if $x = \overline{G}(x,\sigma)$ then

$$\| x \|_{L_p(I,H)} \leq M$$

for some positive constant $M > 0$.

4. It is obvious that

$$\overline{G}(x,0) = 0,$$

since $A(t,0) = 0$ for all $t \in I$.

Applying the Leray-Schauder fixed point theorem (Theorem 1 in Section 2.2) in the space $L_p(I,H)$, we see that there exists a fixed point $y^* \in L_p(I,H) \cap W_{pq}$ with $\overline{G}(x,1) = y$. Therefore, y^* satisfies

$$\begin{cases} \dot{y}^* + Ay^* = Fy^* \\ y^*(0) = -y^*(T) \end{cases}$$

and $y^* \in W_{pq}$. Thus y^* is a solution of problem (4.5) from the equivalence of the problem with equation (4.8). □

4.3 Second order nonlinear evolution equations

As an application of results of Section 4.2, we can obtain the existence of anti-periodic solutions for a class of second order nonlinear evolution equations by reducing second order evolution equations to first order evolution equations. We consider the following second order anti-periodic problem

$$\begin{cases} \ddot{x}(t) + A(t,\dot{x}(t)) + Nx(t) = f(t,x(t)) & 0 < t < T, \\ x(0) = -x(T), \quad \dot{x}(0) = -\dot{x}(T), \\ x \in C(\overline{I},V), \qquad \dot{x} \in W_{pq}. \end{cases} \tag{4.9}$$

where $A : I \times V \to V^*$ and $f : I \times H \to V^*$ satisfying hypotheses (A1) and (F1). In addition, we assume:

(N1) The operator $N : V \to V^*$ is linear, monotone, and symmetric, i.e.,

$$\langle Nv, w\rangle = \langle Nw, v\rangle \quad \text{for all } v, w \in V.$$

For $x \in X$, we set $(Nx)(t) = Nx(t)$, then problem (4.9) can be reduced to the following equation:

$$\begin{cases} \ddot{x} + A(\dot{x}) + N(x) = F(x), \\ x(0) = -x(T), \quad \dot{x}(0) = -\dot{x}(T), \\ x \in C(\overline{I},V), \quad \dot{x} \in W_{pq}. \end{cases} \tag{4.10}$$

Now we define an operator $S : L_p(I,V) \to C(\overline{I},V)$ by

$$(Sy)(t) = \frac{1}{2}\left(\int_0^t y(s)ds - \int_t^T y(s)ds\right).$$

Letting $x = Sy$, problem (4.9) is reduced to the first order evolution equation:

$$\begin{cases} \dot{y}(t) + A(t, y(t)) + NSy(t) = f(t, Sy(t)), \\ y(0) = -y(T), \quad y \in W_{pq}. \end{cases} \tag{4.11}$$

As before, for $y \in X$, we set

$$A(y)(t) = A(t, y(t)), \quad (NS)y(t) = N(Sy(t)) \quad \text{for all } t \in I.$$

It follows from Lemma 4 in Chapter 3 that the operator $A : X \to X^*$ is bounded, uniformly monotone, hemicontinuous and coercive. Furthermore, $NS : X \to X^*$. For $y \in L_p(I, H)$, we set

$$F_1(y)(t) = f(t, Sy(t)) \quad \text{for all } t \in I.$$

The equation (4.10) can be reformulated as the following equation:

$$\begin{cases} \dot{y} + A(y) + NSy = F_1(y), \\ y(0) = -y(T), \quad y \in W_{pq}. \end{cases} \tag{4.12}$$

Lemma 8. *Under assumption (F1), the operator $F_1 : L_p(I, H) \to X^*$ is*

1. *Hölder continuous with exponent α, $0 < \alpha \leq 1$, i.e.,*

$$\| F_1(y_1) - F_1(y_2) \|_{X^*} \leq L_{11} \| y_1 - y_2 \|^{\alpha}_{L_p(I,H)} \quad \text{for all } y_1, y_2 \in L_p(I, H)$$

with some constant $L_{11} > 0$;

2. *bounded, i.e.,*

$$\| F_1(y) \|_{X^*} \leq M_{13} + M_{14} \| y \|^{(k-1)}_{L_p(I,H)} \quad \text{for all } x \in L_p(I, H),$$

where M_{13}, M_{14} are positive constants.

Proof. If $y \in L_p(I, H)$, we get $Sy \in C(\overline{I}, H) \subseteq L_p(I, H)$. Thus F_1 is well defined.

1. $F_1 : L_p(I, H) \to X^*$ is Hölder continuous with exponent $0 < \alpha \leq 1$.

Since

$$\begin{aligned} \| F_1(y_1) - F_2(y_2) \|^q_{X^*} &= \int_0^T \| f(t, Sy_1(t)) - f(t, Sy_2(t)) \|^q_{V^*} \, dt \\ \leq & \int_0^T L \| Sy_1(t) - Sy_2(t) \|^{q\alpha}_H \, dt \\ = & \; L \int_0^T \| \int_0^t (y_1(s) - y_2(s)) ds - \int_t^T (y_1(s) - y_2(s)) ds \|^{q\alpha}_H \, dt \\ \leq & \; 2LTC \int_0^T \| y_1(t) - y_2(t) \|^{q\alpha}_H \, dt \\ \leq & \; \beta \| y_1 - y_2 \|^{q\alpha}_{L_p(I,H)} \quad \text{for any } y_1, y_2 \in L_p(I, H). \end{aligned}$$

We obtain that

$$\| F_1(y_1) - F_1(y_2) \|_{X^*} \leq L_{11} \| y_1 - y_2 \|^{\alpha}_{L_p(I,H)}$$

for any $y_1, y_2 \in L_p(I, H)$, where L_{11} is a constant.

2. Boundedness. For any $y \in L_p(I, H)$,

$$\begin{aligned}
\| F_1(y) \|_{X^*}^q &= \int_0^T \| f(t, Sy(t)) \|_{V^*}^q \, dt \\
&\leq C \int_0^T (|h_1(t)|^q + \| Sy(t) \|_H^{(k-1)q}) dt \\
&\leq C \| h_1 \|_{Lq(I)}^q + \int_0^T \| \int_0^t y(s)ds - \int_t^T y(s)ds \|_H^{(k-1)q} \, dt \\
&\leq C \| h_1 \|_{Lq}^q + C_2 \| y \|_{L_p(I,H)}^{(k-1)q}
\end{aligned}$$

where C and C_2 are constants. So

$$\| F_1(y) \|_{X^*} \leq M_{13} + M_{14} \| y \|_{L_p(I,H)}^{k-1} \qquad \text{for all } y \in L_p(I, H),$$

where $M_{13} = C^{1/q} \| h_1 \|$ and $M_{14} = C_2^{1/q}$ are constants.

Theorem 8. *Under assumptions (A1), (F1), and (N1), problem (4.9) has a solution.*

Proof. Step 1: Equivalence of (4.10) and (4.12).
Let x be a solution of (4.10), we set

$$y = \dot{x}.$$

Then, $y \in W_{pq}$, $y(0) = -y(T)$, and

$$\begin{aligned}
\dot{y} + A(y) + NSy - F_1(y) &= \ddot{x} + A(\dot{x}) + NSy - F_1(y) \\
&= F(x) - F_1(y) - Nx + NSy \\
&= F(x) - F_1(y) - N(x - Sy),
\end{aligned}$$

but

$$\begin{aligned}
(x - Sy)(t) &= x(t) - \frac{1}{2}\left(\int_0^t \dot{x}(s)ds - \int_t^T \dot{x}(s)ds\right) \\
&= 0 \qquad \text{for all } t \in I. \qquad (4.13)
\end{aligned}$$

Then $x - Sy = 0$ in $C(\overline{I}, H)$. Therefore,

$$N(x - Sy) = N(0) = 0,$$

and by assumption (F1) and (4.13), we get

$$\begin{aligned}
\| F(x) - F_1(y) \|_{X^*}^q &= \int_0^T \| f(t, x(t)) - f(t, Sy(t)) \|_{V^*}^q \, dt \\
&\leq L^q \int_0^T \| x(t) - Sy(t) \|_H^{q\alpha} \, dt = 0
\end{aligned}$$

That is, y is a solution of (4.12).

Conversely, let y be a solution of (4.12), then $y \in C(I, H)$. We set $x = Sy$, thus $\dot{x} = y$. Since $y \in L_p(I, V)$, we obtain $x \in C(I, V)$,

$$\ddot{x} + A(\dot{x}) + Nx - F(x) = \dot{y} + A(y) + NSy - F_1(y) = 0,$$

and

$$x(0) = -x(T) \qquad \dot{x}(0) = -\dot{x}(T).$$

Step 2: We show that the equation (4.12) has a solution.

For this purpose, we show that $NS : X \to X^*$ is linear, continuous, and monotone.

In fact, it is obvious that NS is linear since N and S are linear. The Hölder inequality implies

$$\begin{aligned} \| Sy \|_X^p &= \frac{1}{2^p} \int_0^T \| \int_0^t y(s)ds - \int_t^T y(s)ds \|_V^p \, dt \\ &\leq \frac{\beta_1}{2^p} \int_0^T (\| \int_0^t y(s)ds \|_V^p + \| \int_t^T y(s)ds \|_V^p) dt \\ &\leq \frac{\beta_1}{2^{p-1}} \int_0^T \int_0^T \| y(s) \|_V^p \, ds \cdot T^{p/q} dt \\ &= \beta_2 \| y \|_X^p, \end{aligned}$$

for every $y \in X$, where β_1 is a constant and $\beta_2 = \beta_1 T^{p/q+1}/2^{p-1}$. That is, $S : X \to X$ is continuous. By assumption (N1), the operator $N : V \to V^*$ is linear and monotone, we get $N : V \to V^*$ is continuous (see Zeidler (1990), Proposition 26.4). Thus, we obtain

$$\begin{aligned} \| Ny \|_{X^*}^q &= \int_0^T \| Ny(s) \|_{V^*}^q \, ds \\ &\leq \beta_3 \int_0^T \| y(s) \|_V^q \, ds \\ &\leq \beta_4 (\int_0^T \| y(s) \|_V^p \, ds)^{p/q} = \beta_4 \| y \|_X^q, \end{aligned}$$

for some constant $\beta_4 > 0$ since $p \geq 2$, $1 < q \leq p$. Hence $NS : X \to X^*$ is linear and continuous.

The operator $NS : X \to X^*$ is monotone. To prove this, let P denote the set of all polynomials $p : [0, T] \to V$ with coefficients of p in V. Since P is dense in X, it is sufficient to show that

$$\langle NSp, p \rangle_X \geq 0 \quad \text{for all } p \in P.$$

Let $q \in P$. As $N : V \to V^*$ is symmetric, we get

$$\begin{aligned} \frac{d}{dt} \langle Nq(t), q(t) \rangle_V &= \langle N\dot{q}(t), q(t) \rangle_V + \langle Nq(t), \dot{q}(t) \rangle_V \\ &= 2 \langle Nq(t), \dot{q}(t) \rangle_V. \end{aligned}$$

so

$$2\int_0^T \langle Nq(s), \dot{q}(s)\rangle_V ds = \langle Nq(T), q(T)\rangle_V - \langle Nq(0), q(0)\rangle_V. \tag{4.14}$$

Substituting $q = Sp$ into (4.14), and noting that

$$q(0) = \frac{1}{2}\left(\int_0^T p(s)ds - \int_0^T p(s)ds\right) = -\frac{1}{2}\int_0^T p(s)ds = -q(T)$$

for all $p \in P$, we obtain

$$\begin{aligned} \langle NSp, p\rangle_X &= \int_0^0 \langle Nq(s), \dot{q}(s)\rangle_V ds \\ &= \frac{1}{2}(\langle Nq(T), q(T)\rangle_V - \langle Nq(0), q(0)\rangle_V). \\ &= 0 \quad \text{for all } p \in P. \end{aligned}$$

Therefore, $NS : X \to X^*$ is linear, continuous and monotone. And from assumption of (A1), we get that the operator $A + NS : X \to X^*$ is monotone, hemicontinuous, coercive and bounded.

From Lemma 8, the operator $F_1 : L_p(I, H) \to X^*$ is bounded and Hölder continuous, so there exists a solution of (4.12) by theorem 7 in section 4.2. Hence, equation (4.11) has a solution.

Combining step 1 and step 2, the assertion of theorem is valid. □

Chapter V

Optimal Periodic Control

In this chapter, we study the existence of optimal solutions for a Lagrange optimal control problem governed by a class strongly nonlinear evolution equations with periodic condition which we stated in chapter 3. For the periodic setting, we only need to consider the optimal control problem in a periodic interval.

Definition 13. *Let C be a Banach space and $L : S \subseteq C \to [-\infty, +\infty]$ be given. The function L is said to be sequentially lower semicontinuous at a point $x \in S$ if*

$$L(x) \leq \underline{\lim}_{n\to\infty} L(x_n)$$

holds for each sequence $(x_n) \subset S$ such that

$$x_n \longrightarrow x \quad \text{as } n \to \infty.$$

L is said to be sequentially lower semicontinuous on S when L is sequentially lower semicontinuous for all $x \in S$. We briefly write that L is sequentially l.s.c in S.

Let E be a Banach space and suppose the control policies $u(t), t \in I$, take their values in E. We denote the collection of nonempty, closed, convex subsets of E by $P(E)$. A multifunction $U : I \to 2^E$ is called measurable if

$$Gr(U) = \{(t, v) \in I \times E : v \in U(t)\} \in \mathfrak{B}(I) \times \mathfrak{B}(E),$$

where $\mathfrak{B}(I)$ and $\mathfrak{B}(E)$ are the Borel σ-fields of I and E respectively.

Definition 14. *Let $U : I \to 2^E$ be a multifunction. A function $u : I \to E$ is called a selection of $U(\cdot)$ if*

$$u(t) \in U(t) \quad \text{a.e. } t \in I.$$

If such a u is measurable, then u is called a measurable selection of $U(\cdot)$.

Theorem 9. *Let $U : I \to 2^E$ be measurable taking closed set values. Then U admits a measurable selection.*

For the proof of this theorem, we refer to Li and Yong (1994), pp. 100-101.

Obviously, the existence of measurable selections is ensured if $U(t)$ take closed set values for almost all $t \in I$.

We assume:

(U1) E is a reflexive separable Banach space, and

$$U : I \longrightarrow P(E)$$

is a measurable multifunction such that

$$t \longrightarrow |U(t)| = \sup\{\| u \|_E : u \in U(t)\} \quad \text{belongs to } L_r(I).$$

As admissible controls, we choose the set of all selections of $U(\cdot)$ that belong to space $L_r(I, E), 1 < q \leq r < +\infty$; that is,

$$U_{ad} = \{u \in L_r(I, E) : u(t) \in U(t) \quad \text{a.e. on } [0, T]\}.$$

Any element in U_{ad} is called a control.

(B1) $B \in L_\infty(I, \mathcal{L}(E, H))$.

(L1) $L : I \times V \times E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is such that

(1) $(t, x, u) \longrightarrow L(t, x, u)$ is measurable. That is, L is a given $\mathfrak{B}(I) \times \mathfrak{B}(V \times E)-$ measurable function;

(2) $(x, u) \longrightarrow L(t, x, u)$ is sequentially l.s.c.;

(3) $u \rightarrow L(t, x, u)$ is convex;

(4) There exist a nonnegative bounded measurable function $\phi(\cdot) \in L_1(0, T)$ and a nonnegative constant C_6 such that

$$L(t, x, u) \geq \phi(t) - C_6(\| x \|_V + \| u \|_E)$$

for almost all $t \in I$, all $x \in V$, and all $u \in E$.

Note that if hypothesis (U1) is satisfied, then by Theorem 9,

$$U_{ad} \neq \emptyset.$$

The evolution system we are considering is the following

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t)) + B(t)u(t) & \text{a.e. } t \in I \\ x(0) = x(T), & u \in U_{ad}. \end{cases} \tag{5.1}$$

Any solution x of (5.1) is referred to as a state trajectory of the evolution system corresponding to the control $u \in U_{ad}$. Let us introduce some notions.

Definition 15. *A pair (x, u) is said to be admissible if x is a solution of (5.1) corresponding to $u \in U_{ad}$. We call x, u, and (x, u) an admissible trajectory, an admissible control, and an admissible pair, respectively.*

We let

$$\begin{aligned} A_{ad} &= \{(x,u) \in W_{pq} \times U_{ad} | (x,u) \text{ is admissible}\}, \\ X_{ad} &= \{x \in W_{pq} | \exists u \in U_{ad}, \text{ suct that } (x,u) \in A_{ad}\}. \end{aligned}$$

The cost functional considered is:

$$J(x,u) = \int_0^T L(t, x(t), u(t))dt \quad \forall (x,u) \in A_{ad}.$$

Then our optimal control problem can be stated as follows.

Problem (P).

Find $(x_0, u_0) \in A_{ad}$, such that

$$J(x_0, u_0) = \min_{(x,u)\in A_{ad}} J(x,u) = m.$$

If such a pair (x_0, u_0) exists, then we call (x_0, u_0) an optimal control pair.

5.1 Existence of admissible trajectories

In order to prove the existence of optimal control problem (P), at first, we should insure the existence of admissible trajectories of equation (5.1).

Theorem 10. *Assume that hypotheses (A1), (F1), (B1), and (U1) hold. Then the admissible pair set A_{ad} is nonempty and X_{ad} is bounded in $W_{pq} \cap C(I, H)$.*

Proof. (1) $A_{ad} \neq \emptyset$.

In fact, hypothesis (U1) implies

$$U_{ad} \neq \emptyset.$$

For every $u \in U_{ad}$, we define

$$f_u(t,x) = f(t, x(t)) + B(t)u(t).$$

Clearly, f_u satisfies hypothesis (F1)(1) and

$$\|f_u(t,x_1) - f_u(t,x_2)\|_{V^*} \leq L \|x_1 - x_2\|_H^\alpha$$

for all $x_1, x_2 \in H$ and almost all $t \in I$. That is, f_u is Hölder continuous respect x in H uniformly in t. Furthermore,

$$\begin{aligned} \| f_u(t,x) \|_{V^*} &\leq h_1(t) + C_4 \| x \|_H^{k-1} + \| B(t)u(t) \|_{V^*} \\ &\leq (h_1(t) + \| B \|_{L_\infty} \| u(t) \|_E) + C_4 \| x \|_H^{k-1} \end{aligned}$$

for all $x \in H$ and almost all $t \in I$. By assumption, set

$$\bar{h}_1(t) = h_1(t) + \|B\|_{L_\infty(I)} \|u(t)\|_E,$$

then $\bar{h}_1(t) \in L_1(I)$ and is positive. Hence f_u satisfies hypothesis (F1). By Theorem 4 , the assertion holds.

(2) Let $x(\cdot) \in X_{ad}$, then there exists $u : I \to E$ measurable, $u(t) \in U(t)$, a.e. on I such that

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t)) + B(t)u(t) \quad \text{a.e. on } I \\ x(0) = x(T). \end{cases} \tag{5.2}$$

Therefore,

$$\langle \dot{x}, x\rangle_X + \langle A(x), x\rangle_X = \langle F(x), x\rangle_X + \langle Bu, x\rangle_X. \tag{5.3}$$

Now, we observe the term $\langle Bu, x\rangle_X$. By assumption $B(t)u(t) \in H$ and using Cauchy's inequality with $\varepsilon > 0$, we get

$$\begin{aligned} \langle Bu, x\rangle_X &= \int_0^T \langle B(t)u(t), x(t)\rangle dt \\ &= \int_0^T (B(t)u(t), x(t)) dt \\ &\le \frac{1}{\varepsilon^q q} \int_0^T \|B(t)u(t)\|_H^q \, dt + \frac{\varepsilon^p}{p} \int_0^T \|x(t)\|_H^p \, dt \\ &\le \frac{1}{\varepsilon^q q} \|B\|_{L_\infty(I)}^q \, \||U|\|_{L_r(I)}^q + \frac{\varepsilon^p}{p} \|x\|_{L_p(I,H)}^p . \end{aligned}$$

Since the embedding $V \hookrightarrow H$ is continuous and $1 < q \le r < +\infty$ implies that the embedding $L_r(I) \hookrightarrow L_q(I)$ is also continuous, there exist constants $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\langle Bu, x\rangle_X \le \frac{\beta_1}{\varepsilon^q q} \|B\|_{L_\infty(I)}^q \, \||U|\|_{L_q(I)}^q + \frac{\varepsilon^p \beta_2}{p} \|x\|_X^p \qquad \text{for all } \varepsilon > 0.$$

Let $\varepsilon = \left(\frac{pC_1}{2\beta_2}\right)^{1/p}$, where C_1 is chosen as in hypothesis (A1), then

$$\langle Bu, x\rangle_X \le \beta_3 + \frac{C_1}{2} \|x\|_X^p \tag{5.4}$$

where $\beta_3 = \frac{\beta_1}{(pC_1/2\beta_2)^{q/p} \cdot q} \|B\|_{L_\infty(I)}^q \cdot \||U|\|_{L_q(I)}^q$ is independent of u and x.

From the coerciveness of A (see Lemma 4 in Section 3.3), boundedness of F (see Lemma 5 in Section 3.3), (5.4), and (5.3), we obtain

$$\frac{1}{2} C_1 \|x\|_X^p \le \beta_3 + \beta_4 \|x\|_X + \beta_5 \|x\|_X^k . \tag{5.5}$$

where $C_1, \beta_3, \beta_4, \beta_5$ are positive constants.

Now we consider the real function

$$g(\zeta) = \frac{C_1}{2}\zeta^p - \beta_5 \zeta^k - \beta_4 \zeta - \beta_3,$$

the function $g(\zeta) \to +\infty$, as $\zeta \to +\infty$ since $k < p$, but (5.5) implies

$$g(\|x\|_X) \le 0 \qquad \text{for all } x \in X_{ad}.$$

Hence, there exists a constant $\beta_6 > 0$ such that

$$\|x\|_X \le \beta_6 \qquad \text{for all } x \in X_{ad}. \tag{5.6}$$

Moreover it follows from the continuity of the embedding $X \hookrightarrow L_p(I, H)$ that there is a constant $\beta_7 > 0$ such that

$$\|x\|_{L_p(I,H)} \le \beta_7 \quad \text{for all } x \in X_{ad}. \tag{5.7}$$

In addition, given $p(\cdot) \in X = L_p(I, V) = (L_q(I, V^*))^*$, we get from (5.2) that

$$\langle \dot{x}, p\rangle_X + \langle A(x), p\rangle_X = \langle F(x), p\rangle_X + \langle Bu, p\rangle_X.$$

This implies

$$\begin{aligned}
\langle \dot{x}, p\rangle_X &\le \|A(x)\|_{X^*} \|p\|_X + \|F(x)\|_{X^*} \|p\|_X + \|Bu\|_{X^*} \|p\|_X \\
&\le \left(M_1 + M_2 \|x\|_X^{p-1} + M_3 + M_4 \|x\|_{L_p(I,H)}^{k-1}\right) \|p\|_X \\
&\quad + \left(\|B\|_{L_\infty(I)} \cdot \||U|\|_{L_q(I)}\right) \|p\|_X .
\end{aligned} \tag{5.8}$$

Substitute (5.6) and (5.7) into (5.8), we get

$$\langle \dot{x}, p\rangle_X \le \beta_8 \|p\|_X$$

where $\beta_8 > 0$ is a constant independent x and p. Since $p \in X$ was arbitrary, we deduce that

$$\|\dot{x}\|_{X^*} \le \beta_8 \quad \text{for all } x \in X_{ad}.$$

Hence,

$$\|x\|_{W_{pq}} = \|x\|_X + \|\dot{x}\|_{X^*} \le \beta_6 + \beta_8 \quad \text{for all } x \in X_{ad}.$$

Furthermore, we note that as the embedding $W_{pq} \hookrightarrow C(\bar{I}, H)$ is continuous,

$$\max_{t \in \bar{I}} \|x(t)\|_H \le \beta_9 \quad \text{for all } x \in X_{ad}.$$

We have accomplished the proof of the theorem. □

5.2 Existence of optimal control pairs

In order to prove the existence of optimal control pairs, we need the following result given by Balder (1986).

Theorem 11. *The following three conditions*

(1) $L(t,\cdot,\cdot)$ *is sequentially l.s.c. on* $V \times E$ *a.e. on* I,

(2) $L(t,x,\cdot)$ *is convex on* E *for every* $x \in X$ *a.e. on* I,

(3) there exist $M > 0$ *and* $\psi \in L_1(\mathbb{R})$ *such that*

$$L(t,x,u) \geq \psi(t) - M(\|x\|_V + \|u\|_E) \quad \text{for all } x \in V, u \in E \text{ a.e. on } I$$

are sufficient for sequential strong-weak lower semicontinuity of J *on* $L_1(I,V) \times L_1(I,E)$. *Moreover, they are also necessary, provided that* $J(\overline{x},\overline{u}) < +\infty$ *for some* $\overline{x} \in L_1(I,V)$, $\overline{u} \in L_1(I,E)$.

Theorem 12. *If hypotheses (A1), (F1), (U1), (B1), and (L1) hold, there exists an admissible control pair* (x,u) *such that* $J(x,u) = m$.

Proof. By Theorem 10, we get $A_{ad} \neq \emptyset$. Let $(x_n,\ u_n)$ be a minimizing sequence, that is, $\{(x_n,u_n)\} \subseteq A_{ad}$ and $\lim_{n\to+\infty} J(x_n,u_n) = m$.

From $(x_n,u_n) \in A_{ad}$, we get

$$\begin{cases} \dot{x}_n + A(x_n) = F(x_n) + Bu_n \\ x_n(0) = x_n(T), \quad x_n \in W_{pq}, \quad u_n \in U_{ad}. \end{cases} \tag{5.9}$$

Since E is a reflexive Banach space, $L_q(I,E)$ is a reflexive Banach space. In addition, for $u \in U_{ad}$ we get

$$\begin{aligned} \|u\|_{L_r(I,E)} &= \left(\int_0^T \|u(t)\|_E^r\, dt\right)^{1/r} \\ &\leq \left(\int_0^T \||U(t)\||_E^r\, dt\right)^{1/r} \\ &= \||U\||_{L_r(I)}, \end{aligned}$$

i.e., $\{u_n\}$ is bounded in $L_r(I,E)$. Hence, there exists subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that

$$u_n \xrightarrow{W} u \quad \text{in } L_r(I,E) \quad \text{as } n \to \infty$$

Moreover, as $U(t)$ is a closed convex subset of E, it is obvious that U_{ad} is a closed convex subset of the reflexive Banach space $L_r(I,E)$, so U_{ad} is a weakly closed subset of $L_r(I,E)$ from Marzur's Lemma (Lemma 1 in Chapter 2). Hence $u \in U_{ad}$.

On the other hand, since $\{x_n\} \subset X_{ad}$, it follows from Theorem 10 and reflexivity of W_{pq} that there exist a subsequence of $\{x_n\}$, again denoted by $\{x_n\}$, an element $x \in W_{pq}$ such that

$$x_n \xrightarrow{W} x \quad \text{in } W_{pq} \quad \text{as } n \to \infty. \tag{5.10}$$

In the following , we shall show that $(x, u) \in A_{ad}$, that is,

$$\begin{cases} \dot{x} + A(x) = F(x) + Bu \\ x(0) = x(T). \end{cases}$$

First, we claim $x(0) = x(T)$. In fact, from the continuous embedding $W_{pq} \hookrightarrow C(\overline{I}, H)$ and (5.10), we get

$$x_n \xrightarrow{W} x \quad \text{in } C(\overline{I}, H) \quad \text{as } n \to \infty.$$

For given $y_0 \in H$ and $t_0 \in \overline{I}$, we define

$$g(x) = (x(t_0), y_0) \qquad \text{for any } x \in C(\overline{I}, H).$$

Then $g(x)$ is linear and

$$|g(x)| = |(x(t_0), y_0)| \le \| x(t_0) \|_H \| y_0 \| \le \| x \|_{C(\overline{I},H)} \| y_0 \|.$$

That is $g \in \left(C(\overline{I}, H)\right)^*$. Hence

$$g(x_n) \longrightarrow g(x) \quad \text{as } n \to +\infty.$$

That is,

$$(x_n(t_0), y_0) \longrightarrow (x(t_0), y_0) \quad \text{as } n \to +\infty.$$

Since $y_0 \in H$ is arbitrary and H is Hilbert space, this implies for any $t_0 \in \overline{I}$,

$$x_n(t_0) \xrightarrow{W} x(t_0) \quad \text{in } H \quad \text{as } n \to +\infty.$$

So,

$$x_n(0) \xrightarrow{W} x(0) \quad \text{in } H$$

and

$$x_n(T) \xrightarrow{W} x(T) \quad \text{in } H$$

as $n \to \infty$. It follows from $x_n(0) = x_n(T)$ that

$$x(0) = x(T) \quad \text{in } H.$$

Secondly, since embedding $W_{pq} \hookrightarrow X$ is continuous, the embedding $W_{pq} \hookrightarrow L_p(I, H)$ is compact, and the operator $A : X \to X^*$ maps bounded sets to bounded sets, it follows from (5.10) that there is a subsequence of $\{x_n\}$, again denoted $\{x_n\}$, such that

$$\begin{aligned} &x_n \xrightarrow{W} x \quad \text{in } X, \\ &\dot{x}_n \xrightarrow{W} \dot{x} \quad \text{in } X^*, \\ &x_n \xrightarrow{S} x \quad \text{in } L_p(I,H) \\ &Ax_n \xrightarrow{W} w \quad \text{in } X^*, \end{aligned} \tag{5.11}$$

as $n \to +\infty$ where $\dot{x} \in X^*$ is the generalized derivative of x and $w \in X^*$. Thanks to Lemma 5 in Section 3.3, we get

$$F(x_n) \longrightarrow F(x) \quad \text{in } X^*. \tag{5.12}$$

Hence

$$\langle F(x_n), x_n \rangle_X \longrightarrow \langle F(x), x \rangle_X \quad \text{as } n \to \infty. \tag{5.13}$$

Note that $B \in L_\infty(I, \mathcal{L}(E,H))$ implies $B : L_r(I,E) \to L_q(I,H)$ is linear and continuous, which implies

$$Bu_n \xrightarrow{W} Bu \quad \text{in } L_q(I,H) \tag{5.14}$$

as $n \to \infty$ as $u_n \xrightarrow{W} u$ in $L_r(I,E)$. In addition, since

$$x_n \longrightarrow x \quad \text{in } L_p(I,H)$$

as $n \to \infty$, we get

$$\begin{aligned} \langle Bu_n,\ x_n \rangle_X &\longrightarrow \langle Bu,\ x \rangle_{L_p(I,H),L_q(I,H)} \\ &= \langle Bu,\ x \rangle_X \quad \text{as } n \to +\infty. \end{aligned} \tag{5.15}$$

Again it follows from (5.9) that

$$\begin{aligned} &\langle \dot{x}_n, x_n - x \rangle_X + \langle A(x_n), x_n - x \rangle_X \\ = &\langle F(x_n), x_n - x \rangle_X + \langle Bu_n, x_n - x \rangle_X \quad \text{for } n = 1, 2, \cdots \end{aligned} \tag{5.16}$$

From the integration by parts formula, we have

$$\begin{aligned} \langle \dot{x}_n, x_n - x \rangle_X &= \langle \dot{x}, x_n - x \rangle_X + \frac{1}{2}\left(\|x_n(T) - x(T)\|_H^2 - \|x_n(0) - x(0)\|_H^2 \right) \\ &= \langle \dot{x}, x_n - x \rangle_X \\ &\longrightarrow 0 \end{aligned} \tag{5.17}$$

as $n \to \infty$. Letting $n \to \infty$ in (5.16) and noting (5.12), (5.15), and (5.17), we obtain

$$\lim_{n\to\infty} \langle A(x_n), x_n - x \rangle_X = 0.$$

But $A : X \to X^*$ is hemicontinuous, monotone, and so A satisfies condition (M) (see Definition 10 and Proposition 19 in Section 3.1). We deduce that

$$w = A(x),$$

that is

$$A(x_n) \xrightarrow{W} A(x) \quad \text{in } X^*$$

as $n \to \infty$.

Thus, for every $\phi \in X$,

$$\langle \dot{x}_n, \phi \rangle_X + \langle A(x_n), \phi \rangle_X = \langle F(x_n), \phi \rangle_X + \langle Bu_n, \phi \rangle_X,$$

letting $n \to +\infty$, then

$$\langle \dot{x}, \phi \rangle_X + \langle A(x), \phi \rangle_X = \langle F(x), \phi \rangle_X + \langle Bu, \phi \rangle_X.$$

Therefore, $(x, u) \in A_{ad}$.

Again, it follows from $(x_n, u_n) \in A_{ad}$ and $(x, u) \in A_{ad}$ that

$$\begin{aligned} & \langle \dot{x}_n - \dot{x}, x_n - x \rangle_X + \langle A(x_n) - A(x), x_n - x \rangle_X \\ = \; & \langle F(x_n) - F(x), x_n - x \rangle_X + \langle Bu_n - Bu, x_n - x \rangle_X \quad \text{for all } n = 1, 2, \cdots. \end{aligned}$$

From integration by parts formula, we get

$$\begin{aligned} C_1 \|x_n - x\|_X^p & \leq \langle Ax_n - Ax, x - x_n \rangle_X \\ & = \langle F(x_n) - F(x), x_n - x \rangle_X + \langle B(u_n) - B(u), x_n - x \rangle_X \\ & \longrightarrow 0 \qquad \text{as } n \to \infty. \end{aligned}$$

That is,

$$x_n \longrightarrow x \quad \text{in } X \quad \text{as } n \to \infty.$$

Note that the embeddings $L_p(I, V) \hookrightarrow L_1(I, V)$ and $L_q(I, E) \hookrightarrow L_1(I, E)$ are continuous, then

$$x_n \longrightarrow x \qquad \text{in } L_1(I, V)$$

and

$$u_n \xrightarrow{W} u \qquad \text{in } L_1(I, E)$$

as $n \to +\infty$. From hypothesis (L1) and Theorem 11, we obtain

$$\begin{aligned} J(x, u) & = \int_0^T L(t, x(t), u(t)) dt \\ & \leq \underline{\lim} \int_0^T L(t, x_n(t), u_n(t)) dt \\ & = \underline{\lim} J(x_n, u_n) = m. \end{aligned}$$

This shows that (x, u) is the desired optimal pair. □

Chapter VI

Applications

In this chapter, to illustrate the applicability of our work, we apply Theorem 5 in Section 3.3 and Theorem 12 in Section 5.2 to prove the existence of optimal control of systems governed by second-order periodic quasi-linear parabolic differential equations. More general, we obtain the existence of periodic solutions for quasi-linear parabolic differential equation of order $2m$ and the existence corresponding to Langrage optimal control. Moreover, we present an existence result of anti-periodic solutions for quasi-linear hyperbolic differential equations by using Theorem 8 in Section 4.3.

In doing so we will use the following notations. $z = (z_1, z_2, \cdots, z_n)$ is a variable point in the n -dimensional Euclidean space R^n. An n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is called a multiindex and we define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \quad \text{for } z = (z_1, z_2, \cdots, z_n).$$

Denoting $D_k = \partial/\partial z_k$ and $D = (D_1, D_2, \cdots, D_n)$ we have

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}.$$

Let Ω be a bounded domain in R^n with $n \geq 1$ and piecewise smooth boundary $\partial\Omega$, i.e., $\partial\Omega \in C^{0,1}$. Recall that $\partial\Omega \in C^{0,1}$ if for each point $z \in \partial\Omega$ there is a ball B with center at z such that $\partial\Omega \cap B$ can be represented in the form $z_i = \phi(z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n)$ for some i with $\phi \in C^{0,1}$. $Q_T = (0, T) \times \Omega$, $0 < T < \infty$ is fixed. Suppose $p \geq 2$ and $q = p/(p-1)$, $W^{m,p}(\Omega)$ denotes the standard Sobolev space with the usual norm:

$$\|\varphi\|_{m,p} = \Big(\sum_{|\alpha| \leq m} \| D^\alpha \varphi \|_{L^p(\Omega)}^p \Big)^{1/p}, \quad m = 0, 1, 2, \cdots.$$

Then $W^{m,p}(\Omega)$ is separable, uniformly convex and hence reflexive.

Set

$$W_0^{m,p}(\Omega) = \{\varphi \in W^{m,p} \mid \quad D^\beta \varphi|_{\partial\Omega} = 0, \quad |\beta| \le m-1\}.$$

Since $\partial\Omega$ is smooth, $C_0^\infty(\Omega)$ is dense in $W_0^{m,p}(\Omega)$ and $L_2(\Omega)$, hence $W_0^{m,p}(\Omega)$ is dense in $L_2(\Omega)$. From Sobolev's embedding theorem, we have that the embeddings $C_0^\infty(\Omega) \hookrightarrow W_0^{m,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-m,p}(\Omega)$ are continuous and the embedding $W_0^{m,p}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Denote $V \equiv W_0^{m,p}(\Omega), H \equiv L_2(\Omega)$, then $V^* \equiv W^{-m,q}(\Omega)$ and $V \hookrightarrow H \hookrightarrow V^*$ is an evolution triple.

6.1 Optimal control of a system governed by a second order quasi-linear parabolic equation with time periodic condition

Example 1. *For $2 \le p < +\infty$ and $\theta \ge 0$, the optimal control problem under consideration is the following:*

(P1) :

$$\begin{aligned} J(x,u) &= \int_0^T \int_\Omega \frac{1}{2} |x(t,z) - y_0(z)|^2 \, dz dt + \frac{\theta}{2} \int_0^T \int_\Omega |u(t,z)|^2 \, dz dt \\ &\to \inf = m, \\ y_0(\cdot) &\in L^2(\Omega) \text{ is a target state} \end{aligned}$$

such that

$$\begin{cases} \frac{\partial}{\partial t} x(t,z) - \sum\limits_{i=1}^n D_i(|D_i x(t,z)|^{p-2} D_i x(t,z)) \\ = \sum_{i=1}^n D_i f_i(t,z,x(t,z)) + f_0(t,z,x(t,z)) + b(t)u(t) \quad \text{on } Q_T, \\ x(t,z) = 0 \quad \text{on } [0,T] \times \partial\Omega, \\ x(0,z) = x(T,z) \quad \text{on } \Omega, \quad |u(t,z)| \le r(t,z) \quad a.e. \end{cases} \tag{6.1}$$

Note that, when $p = 2$, the partial differential operator in the dynamical equation is the Laplacian. So, this example incorporates nonlinear and non-monotone perturbations of the Laplace equation. The differential operator in divergence form $-\sum_{i=1}^n D_i(|D_i x|^{p-2} D_i x)$, which generates A, is usually called the pseudo-Laplacian.

We set

$$V = W_0^{1,p}(\Omega), \qquad H = L_2(\Omega)$$

then $V^* = W^{-1,p}(\Omega)$ and V is dense in H since $C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega) \subset L_2(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and $L_2(\Omega)$. It follows from Sobolev's embedding theorem that the embedding V into H is compact.

We impose the following hypotheses.

(F) $f_i : I \times \Omega \times R \to R \quad (i = 0, 1, \cdots, N)$ is a function such that

(1) $(t, z) \to f_i(t, z, x)$ is measurable for all $x \in R$;

(2) $|f_i(t, z, x)| \le a(t, z) + \gamma_1 |x|^{k-1}$ for all $t \in I$ and $z \in \Omega$ with $a(\cdot, \cdot) \in L_q(Q_T)$, a constant $\gamma_1 \ge 0$, and a constant $1 \le k < p$.

(3) $f_i(t, z, x)$ satisfies the Lipschiz condition

$$|f_i(t, z, x_1) - f_i(t, z, x_2)| \le L|x_1 - x_2|$$

for all $x_1, x_2 \in R$ and $(t, z) \in Q_T$ with some constant $L > 0$.

(B) $b \in L_\infty(I)$.

(R) $r \in L_q^+(I \times \Omega)$.

In order to study the existence of solutions for optimal control problem, first we consider the existence of periodic solutions for the quasi-linear parabolic equation:

$$\begin{cases} \frac{\partial}{\partial t} x(t, z) - \sum_{i=1}^{n} D_i \left(|D_i x(t, z)|^{p-2} D_i x(t, z) \right) \\ = \sum_{i=1}^{n} D_i f_i(t, z, x(t, z)) + f_0(t, z, x(t, z)) \quad \text{on } Q_T, \\ x(t, z) = 0 \quad \text{on } [0, T] \times \partial\Omega, \\ x(0, z) = x(T, z) \quad \text{on } \Omega. \end{cases} \tag{6.2}$$

Definition 16. *The generalized problem associated with (6.2) reads as follows. We seek $x \in W_{pq}$ such that, for all $v \in V$ and almost all $t \in (0, T)$,*

$$\begin{cases} \frac{d}{dt}(x(t), v) + a(x, v) = \widehat{f}(t; x(t), v) \\ x(0) = x(T). \end{cases} \tag{6.3}$$

where

$$\begin{aligned} a(x, y) &= \int_\Omega \sum_{i=1}^{n} |D_i x|^{p-2} (D_i x)(D_i y) dz, \\ \widehat{f}(t, x, y) &= \int_\Omega \sum_{i=1}^{n} f_i(t, z, x) D_i y dz + \int_\Omega f_0(t, z, x) y dz. \end{aligned}$$

In (6.3), $\frac{d}{dt}$ denotes the generalized derivative on I.

Lemma 9. *There exists an operator $A : V \to V^*$ such that*

$$\langle A(x), y \rangle_V = a(x, y) \quad \textit{for all } x, y \in V$$

and the assumptions regarding A in (A1) of Section 3.3 are fulfilled.

Proof. (I) From Hölder's inequality, we get

$$\begin{aligned} |a(x,y)| &\leq \sum_{i=1}^{N} \left(\int_\Omega \left| \frac{\partial x}{\partial z_i} \right|^p dz \right)^{1/q} \left(\int_\Omega \left| \frac{\partial y}{\partial z_i} \right|^p dz \right)^{1/p} \\ &\leq C \|x\|_V^{p/q} \cdot \|y\|_V = C \|x\|_V^{p-1} \|y\|_V \end{aligned}$$

with some constant $C > 0$ for all $x, y \in V$. That is, $y \to a(x,y)$ is linear and bounded. So, there exists an operator $A : V \to V^*$ with

$$a(x,y) = \langle Ax, y \rangle_{V,V^*}$$

and

$$\|Ax\|_{V^*} \leq C \|x\|_V^{p-1}.$$

That is, $A(\cdot)$ is bounded.

(II) A key inequality.
From Tartar's inequality

$$(|\lambda|^{p-2}\lambda - |\mu|^{p-2}\mu)(\lambda - \mu) \geq C_1 |\lambda - \mu|^p$$

for all $\lambda, \mu \in R$ with some constant $C_1 > 0$, we get that there exists a constant $C_2 > 0$ such that

$$a(x, x-y) - a(y, x-y) \geq C_2 |x-y|_V^p$$

for all $x, y \in V$. Hence

$$\langle Ax - Ay, x - y \rangle_V \geq C_2 |x-y|_V^p$$

for all $x, y \in V$. That is, A is uniformly monotone.
Since $A(0) = 0$, we get that

$$\langle Ax, x \rangle_V \geq C_2 |x|_V^p.$$

That is, A is coercive.

(III) Continuity of A.
Let

$$x_n \longrightarrow x \quad \text{in } V \quad \text{as } n \to +\infty.$$

This implies

$$\frac{\partial x_n}{\partial z_i} \longrightarrow \frac{\partial x}{\partial z_i} \quad \text{in } L_p(\Omega) \quad \text{as } n \to +\infty$$

according to the definition of convergence in $V = W_0^{1,p}(\Omega)$. Set

$$G(x) = |x|^{p-2} x \quad \text{for all } x \in R,$$

then

$$|G(x)| \leq |x|^{p-1} \quad \text{for all } x \in R$$

and it follows from Proposition 16 in Chapter 2 that the Nemyckii operator

$$G : L_p(\Omega) \to L_q(\Omega)$$

is continuous. Therefore,

$$\frac{\partial x_n}{\partial z_i} \longrightarrow \frac{\partial x}{\partial z_i} \quad \text{in } L_p(\Omega) \quad \text{as } n \to +\infty$$

implies

$$G(\frac{\partial x_n}{\partial z_i}) \longrightarrow G(\frac{\partial x}{\partial z_i}) \quad \text{in } L_q(\Omega) \quad \text{as } n \to +\infty.$$

By the Hölder inequality, for all $y \in V$, we obtain that

$$\begin{aligned} |\langle Ax_n - Ax, y\rangle_V| &= |\int_\Omega \sum_{i=1}^{N}(G(\frac{\partial x_n}{\partial z_i}) - G(\frac{\partial x}{\partial z_i}))\frac{\partial y}{\partial z_i}| \\ &\leq \sum_{i=1}^{N} \left\| G(\frac{\partial x_n}{\partial z_i}) - G(\frac{\partial x}{\partial z_i}) \right\|_{L_q(\Omega)} \cdot \left\| \frac{\partial y}{\partial z_i} \right\|_{L_p(\Omega)} \\ &\leq \sum_{i=1}^{N} \left\| G(\frac{\partial x_n}{\partial z_i}) - G(\frac{\partial x}{\partial z_i}) \right\|_{L_q(\Omega)} \cdot \|y\|_V . \end{aligned}$$

Hence

$$\|Ax_n - Ax\|_{V^*} \longrightarrow 0 \qquad \text{as} \quad n \to +\infty.$$

That is, $A(\cdot)$ is demicontinuous, hence hemicontinuous.

Therefore, $A(\cdot)$ is defined above satisfies hypothesis (A1) in Chapter 3. □

Lemma 10. *Under hypothesis (F), for every $t \in I$, there exists an operator $f(t) : H \to V^*$ such that*

$$\langle f(t)x, y\rangle_V = \widehat{f}(t, x, y) \quad \text{for all } x \in H \quad , y \in V$$

and $f(t)$ satisfies all hypothesis (F1) in Section 3.3.

Proof. Since $\widehat{f} : I \times H \times V \to R$ is defined by

$$\widehat{f}(t, x, y) = \int_\Omega \sum_{i=1}^{N} f_i(t, z, x(z)) \frac{\partial y(z)}{\partial z_i} dz + \int_\Omega f_0(t, z, x) y(z) dz,$$

then from hypothesis (F)(2), we get

$$|f_i(t, z, x)|^q \leq C_4(|a_2(t, z)|^q + |x|^{(k-1)q}).$$

It follows from $x \in W_0^{1,p}(\Omega)$, $a_2(t,\cdot) \in L_q(\Omega)$, and the majorant criterion that

$$|f_i(t,z,x)| \in L_q(\Omega), \quad i = 0, 1, \cdots, N.$$

By Hölder's inequality, we get

$$\begin{aligned} |\widehat{f}(t,x,y)| &\le \sum_{i=1}^{N} (\int_\Omega |f_i(t,z,x)|^q dz)^{1/q} \cdot (\int_\Omega |\frac{\partial y}{\partial z_i}|^p dz)^{1/p} \\ &\quad + (\int_\Omega |f_0(t,z,x)|^q dz)^{1/q} \cdot (\int_\Omega |y|^p dz)^{1/p} \\ &\le C_5 (\sum_{i=0}^{N} \int_\Omega |f_i(t,z,x)|^q dz)^{1/q} \cdot (\sum_{i=1}^{N} \int_\Omega |\frac{\partial y}{\partial z_i}|^p dz + \int_\Omega |y|^p dz)^{1/p} \\ &\le C_6 (C_7 + \|x\|_H^{k-1}) \cdot \|y\|_V \end{aligned} \tag{6.4}$$

for some constants $C_5 > 0, C_6 > 0$, and $C_7 > 0$. That is, for each $x \in H$ and $t \in I$, $y \to \widehat{f}(t,x,y)$ is a continuous linear from on V. Hence there exists an operator $f : I \times H \to V^*$ such that

$$\langle f(t,x), y \rangle_V = \widehat{f}(t,x,y).$$

And from (6.4), we have

$$|f(t,x)|_{V^*} \le M_1 + M_2 \|x\|_H^{k-1} \quad \text{for all } t \in I \text{ and } x \in H$$

where $M_1 > 0$ and $M_2 > 0$ are constants. Next from hypothesis (F)(4), we get

$$\begin{aligned} & |\langle F(t,x_1) - F(t,x_2), y \rangle_V| \\ = & \left| \widehat{f}(t,x_1,y) - \widehat{f}(t,x_2,y) \right| \\ = & \left| \int_\Omega \left(\sum_{i=1}^{N} (f_i(t,z,x_1) - f_i(t,z,x_2)) \frac{\partial y}{\partial z_i} + (f_0(t,z,x_1) - f_0(t,z,x_2)) y \right) dz \right| \\ \le & \int_\Omega \sum_{i=1}^{N} |f_i(t,z,x_1) - f_i(t,z,x_2)| \left| \frac{\partial y}{\partial z_i} \right| dz + \int_\Omega |f_0(t,z,x_1) - f_0(t,z,x_2)| \, |y| \, dx \\ \le & \int_\Omega \sum_{i=1}^{N} L_i |x_1 - x_2| \cdot \left| \frac{\partial y}{\partial z_i} \right| dz + \int_\Omega L_0 |x_1 - x_2| \, |y| \, dz \\ \le & L' \int_\Omega |x_1 - x_2| \left(\sum_{i=1}^{N} \left| \frac{\partial y}{\partial z_i} \right| + |y| \right) dz \\ \le & L \left(\int_\Omega |x_1 - x_2|^q \, dz \right)^{1/q} \cdot \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial y}{\partial z_i} \right|^p dz + \int_\Omega |y|^p \, dz \right)^{1/p} \\ = & L \|x_1 - x_2\|_{Lq(\Omega)} \cdot \|y\|_V , \end{aligned}$$

for all $x_1, x_2 \in H$ and $t \in I$, where $L' = \max_{0 \le i \le N} (L_i)$ and $L > 0$ is a constant. Since the embedding $L_2(\Omega) \hookrightarrow L_q(\Omega)$ is continuous for $1 \le q \le 2$, this implies $|x_1 - x_2| \in L_q(\Omega)$.

From the above discussion, we obtain that the operator $F : I \times H \to V^*$ satisfies hypothesis (F1) of Section 3.1. $\square$

Let $E = L_q(\Omega)$, $r : I \times \Omega \to R^+$ and $r \in L_q(I \times \Omega)$. Let $U : I \to P(E)$ be defined by

$$U(t) \equiv \{v \in L_q(\Omega) = E : \quad \| v \|_q \leq \| r(t,\cdot) \|_{L_q(\Omega)} = \widehat{r}(t)\}.$$

Note that

$$GrU = \{(t,v) \in T \times E : r(t) - \| v \|_q \geq 0\}$$

By Fubini's theorem, $t \to \widehat{r}(t)$ is measurable, that is, for fixed $v \in E$, $t \to \widehat{r}(t) - \| v \|_q$ is measurable, and $v \to \widehat{r}(t) - \| v \|_q$ is continuous. Hence $(t,v) \to \widehat{r}(t) - \| v \|_q$ is a Caratheodory function. $GrU \in B(T) \times B(V)$, that is $U(\cdot)$ is measurable and $|U(t)| \leq \widehat{r}(t)$ a.e $t \in I$ and $\widehat{r}(\cdot) \in L_q^+(I)$. Thus hypothesis (U1) is satisfied, The set of admissible controls U_{ad} is chosen as

$$U_{ad} \equiv \{u \in L_q(I,E), \quad u(t) \in U(t) \quad a.e.\}.$$

If $b \in L_\infty(0,T)$, we can define $B : I \to \mathcal{L}(E,H)$ by

$$B(t)u(\cdot) = b(t)u(\cdot).$$

then $B \in L_\infty(I, \mathcal{L}(E,H))$.

Let $L : I \times V \times E \to R$ be defined

$$\begin{aligned} L(t,x,u) &= \frac{1}{2}\int_\Omega |x(t,z) - y_0(t,z)|^2 dz + \frac{\theta}{2}\int_\Omega |u(t,z)|^2 dz, \\ u(t) &= \{u \in L_q(\Omega) : \|v\|_{L_q(\Omega)} \leq \widehat{r}(t) = \|r(t,\cdot)\|_{L_q(\Omega)}\}, \end{aligned}$$

where $r \in L_\infty(I \times \Omega)$. Then, it is easy to see that, hypotheses (A1), (F1), (U1), and (L1) are satisfied.

Our problem can be taken to the following abstract form:

$$\inf \quad J(x,u) = \int_0^T L(t,x(t),u(t))dt$$

such that

$$\begin{cases} \dot{x}(t) + Ax(t) = F(t,x(t)) + B(t)u(t) \\ x(0) = x(T) \\ u(t) \in U(t) \quad \text{a.e. } u(\cdot) \text{ is measurable.} \end{cases}$$

This problem is equivalent to problem (P). Applying Theorem 11 in Section 5.2, we get

Theorem 13. *If hypothesis (F) holds and* $r \in L_\infty(I \times \Omega)$, $b \in L_\infty(I)$, *then problem (P1) admits a solution*

$$(x,u) \in (L_p(I, W_0^{1,p}(\Omega)) \bigcap C(\bar{I}, L_2(\Omega))) \times L_q(I \times \Omega)$$

with $\partial x/\partial t \in L_q(I, W^{-1,q}(\Omega))$.

Corollary 1. *(Properties of solution x.) If x is a solution of (6.2), then*

$$\lim_{t \to s} \int_\Omega (x(t,z) - x(s,z))^2 dz = 0 \quad \textit{for all } s \in [0,T].$$

Therefore, the function $t \mapsto x(t,z)$ *is continuous on* $[0,T]$ *in the mean.*

6.2 Optimal control of a system governed by a $2m$-order quasi-linear parabolic equation with time periodic condition

Example 2. *The distributed parameter, periodic parabolic optimal control problem under consideration is the following:*

$$(P2) \qquad inf\ J(x,u) = \inf \int_0^T \int_\Omega L(t,z,\eta(x(t,z)), u(t,z))dzdt$$

subject to the time-periodic $2m$- *order quasi-linear parabolic equation:*

$$\begin{cases} \frac{\partial}{\partial t}x(t,z) + \sum\limits_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,z,\eta(x)(t,z)) \\ = \sum\limits_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha f_\alpha(t,z,x(t,z)) + b(t)u(t) \quad \text{on } Q_T, \\ D^\beta x(t,z) = 0 \quad \text{on } [0,T] \times \partial\Omega \qquad \text{for all } \beta : |\beta| \le m-1, \\ x(0,z) = x(T,z) \qquad \text{on } \Omega. \\ |u(t,z)| \le r(t,z) \qquad \text{a.e. } t \in I \end{cases} \tag{6.5}$$

where $\eta(x) \equiv \{(D^\gamma x), |\gamma| \le m\}$, *i.e.,* $\eta(x)$ *denotes the tuple of all partial derivatives with respect z up to order m including x.*

The boundary condition in (6.5) means that all partial derivatives with respect z up to order $m-1$ *should vanish on* $\partial\Omega$ *for all* $t \in I$. *Set* $M = \frac{(n+m)!}{n!m!}$.

This kind of problem occurrs frequently. Examples are a controlled chemical reaction process with diffusion or controlled single species population dynamics with diffusion. The process is desired to be periodic (with period T).

We will need the following hypotheses on the data of problem (P2).

(A') All the functions $A_\alpha(|\alpha| \leq m) : Q_T \times R^M \to R$ are functions such that

(1) $(t,z) \to A_\alpha(t,z,\eta)$ are measurable on Q_T for all $\eta \in R^M$, $\eta \to A_\alpha(t,z,\eta)$ are continuous on R^M a.e. $(t,z) \in Q_T$;

For each fixed $t \in I$, the following assumptions are satisfied:

(2) Growth condition

$$|A_\alpha(t,z,\eta)| \leq a_1(t,z) + c_1(z) \sum_{|\gamma| \leq m} |\eta_\gamma|^{p-1}$$

with $a_1(\cdot,\cdot) \in L_q(Q_T)$ and $c_1(\cdot) \in L^\infty(\Omega)$.

(3) Uniform monotonicity

$$\sum_{|\alpha| \leq m} \left(A_\alpha\left(t,z,\eta\right) - A_\alpha\left(t,z,\widetilde{\eta}\right)\right)\left(\eta_\alpha - \widetilde{\eta_\alpha}\right) \geq C_1 \sum_{|\gamma| \leq m} |\eta_\gamma - \widetilde{\eta_\gamma}|^p ,$$

with C_1 is a positive constant.

(4) $A_\alpha(t,z,0) = 0$ for all $(t,z) \in Q_T$.

(F') $f_\alpha : Q_T \times R \to R$ are functions such that

(1) $(t,z) \to f_\alpha(t,z,x)$ is measurable on Q_T for all $x \in R$,

$x \to f_\alpha(t,z,x)$ is continuous on R for a.e. $(t,z) \in Q_T$;

(2) For each fixed $t \in I$,

$$|f_\alpha\left(t,z,x\right)| \leq a_2\left(t,z\right) + c_4 \left|x\right|^{k-1}$$

with $a_2\left(\cdot,\cdot\right) \in L_q\left(Q_T\right)$, $c_4 > 0$ and $1 \leq k < p$ are constants.

(3) $f_\alpha\left(t,z,x\right)$ is Hölder continuous with respect to x and exponent $0 < \alpha \leq 1$, that is, there is a constant L

$$|f_\alpha\left(t,z,x_1\right) - f_\alpha\left(t,z,x_2\right)| \leq L\left|x_1 - x_2\right|$$

for any $x_1,\ x_2 \in R$, $(t,z) \in Q_T$.

(B') $b \in L_\infty\left(I\right)$.

(R') $r \in L_q^+\left(I \times \Omega\right)$.

(L') $L : I \times \Omega \times R^M \times R \to \overline{R} = R \cup \{+\infty\}$ is an integrand such that

(1) $(t,z,\eta,u) \to L\left(t,z,\eta,u\right)$ is measurable;

(2) $(\eta,u) \to L\left(t,z,\eta,u\right)$ is l.s.c.;

(3) $u \to L\left(t,z,\eta,u\right)$ is convex;

(4) $\phi\left(t,z\right) - l\left(z\right)\left(\|\eta\|_V + \|u\|_E\right) \leq L\left(t,z,\eta,u\right)$ a.e. with $\phi \in L_1(I \times \Omega)$ *andl* $\in L_\infty\left(\Omega\right)$ is a positive function.

In order to study the existence of solutions to the optimal control problem, first we consider the existence periodic solutions of the $2m$ order quasi-linear parabolic equation:

$$\begin{cases} \frac{\partial}{\partial t}x(t,z) + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,\ z,\ \eta(x)(t\ ,z)) \\ = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha(t,\ z,\ x(t,\ z)) \text{ on } Q_T, \\ D^\beta x(t,\ z) = 0 \quad \text{on } [0,T]\times\partial\Omega \quad \text{for all } \beta : |\beta| \le m-1, \\ x(0,\ z) = x(T,\ z), \quad \text{on } \Omega. \end{cases} \tag{6.6}$$

Definition 17. *The generalized problem associated with (6.6) reads as follows. We seek a function $x \in W_{pq}$ such that, for all $y \in V$ and almost all $t \in I$,*

$$\begin{cases} \frac{d}{dt}(x(t), y) + a(t; x(t), y) = f(t; x(t), y) \\ x(0) = x(T) \end{cases} \tag{6.7}$$

where

$$a(t,x,y) = \int_\Omega \sum_{|\alpha|\le m} A_\alpha(t,z,\eta(x)(t,z))\, D^\alpha y dz$$

and

$$f(t,x,y) = \int_\Omega \sum_{|\beta|\le m} f_\beta(t,z,x(t,z))\, D^\beta y(z)\, dz.$$

In (6.7) $\frac{d}{dt}$ denotes the generalized derivative on I.

Remark 2. *One obtains the generalized problem (6.7) by multiplying the original problem (6.6) by the function $y \in C_0^\infty(\Omega)$ and integrating by parts with respect to the spatial variable.*

Proposition 21. *If hypotheses (A') and (F') hold, then the generalized problem (6.7) corresponding to the original problem (6.6) is equivalent to equation (3.7), and the hypotheses (A1) and (F1) are fulfilled.*

Proof. (1) We define another norm which is equivalent the usually norm on V by

$$\|\phi\|_{m,p,0} = \left(\sum_{|\alpha|=m} \|D^\alpha \phi\|^p_{L_p(\Omega)}\right)^{1/p}.$$

One can show that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \|\phi\|_{m,p} \le \|\phi\|_{m,p,0} \le c_2 \|\phi\|_{m,p} \quad \text{for all } \phi \in V.$$

(2) The Nemyckii operator. For each $t \in I$ fixed, we set

$$\left(\widehat{A}_\alpha (t) x\right)(z) = A_\alpha (t, z, \eta (x)) \quad \text{for all } z \in \Omega.$$

Obviously, we have

$$D^\beta x \in L_p (\Omega) \quad \text{if } x \in V \text{ and } |\beta| \le m.$$

It follows from $(A')(1), (2)$ and Proposition 16 in Chapter 2 that the Nemyckii operator

$$\widehat{A}_\alpha (t) : V \to L_q (\Omega)$$

is continuous and

$$\left\| \widehat{A}_\alpha (t) x \right\|_{L_q(\Omega)} \le M_1 \left(\|a_1 (t)\|_{L_q(\Omega)} + \|x\|_V^{p/q} \right) \quad \text{for all } x \in V$$

where $M_1 > 0$ is a constant.

(3) Boundedness. By Hölder's inequality, we get

$$\begin{aligned} |a (t; x, y)| &\le \sum_{|\alpha| \le m} \left\| \widehat{A}_\alpha (t) x \right\|_{L_q(\Omega)} \|D^\alpha y\|_{L_p(\Omega)} \\ &\le M_2 \left(\|a_1 (t)\|_{L_q(\Omega)} + \|x\|_V^{p/q} \right) \|y\|_V \quad \text{for all } x, y \in V \end{aligned}$$

where $M_2 > 0$ is a constant.

Hence, for each $x \in V$, $y \to a (t; x, y)$ is a linear continuous form on V. By the Riesz Representation Theorem, there exists an operator $A : I \times V \to V^*$ such that

$$\langle A(t, x),\ y \rangle_V = a (t; x, y)$$

and

$$\|A (t, x)\|_{V^*} \le M_2 \left(\|a_1 (t)\|_{L_q(\Omega)} + \|x\|_V^{p-1} \right) \quad \text{for all } x \in V \text{ and } t \in I,$$

i.e., $A(t, x)$ is bounded.

(4) Uniformly monotonicity. By $(A')(3)$, for all $x_1, x_2 \in V$, and $t \in I$

$$\begin{aligned} \langle A (t, x_1) - A (t, x_2), x_1 - x_2 \rangle_V &= a (t; x_1, x_1 - x_2) - a (t; x_2, x_1 - x_2) \\ &\ge C_1 \|x_1 - x_2\|_V^p, \end{aligned}$$

i.e., $A (t, x)$ is uniformly monotone.

(5)Lastly, we show that for each $t \in I$, $A (t) : V \to V^*$ is continuous. Indeed, let

$$x_n \to x \quad \text{in } V \quad \text{as } n \to +\infty.$$

Since $\widehat{A}_\alpha(t) : V \to L_q(\Omega)$ is continuous,

$$\widehat{A}_\alpha(t)\, x_n \longrightarrow \widehat{A}_\alpha(t)\, x \quad \text{in } L_q(\Omega) \quad \text{as } n \to \infty.$$

By Hölder's inequality,

$$|a(t; x_n, y) - a(t; x, y)| \le \sum_{|\alpha| \le m} \left\| \widehat{A}_\alpha(t)\, x_n - \widehat{A}_\alpha(t)\, x \right\|_{L_q(\Omega)} \|y\|_V ,$$

for all $y \in V$. This implies

$$\|A(t, x_n) - A(t, x)\|_{V^*} \le \sum_{|\alpha| \le m} \left\| \widehat{A}_\alpha(t)\, x_n - \widehat{A}_\alpha(t)\, x \right\|_{L_q(\Omega)} .$$

Hence

$$\|A(t, x_n) - A(t, x)\|_{V^*} \longrightarrow 0 \quad \text{as } n \to +\infty.$$

That is, the operator $A(t) : V \to V^*$ is continuous.

Therefore, the operator $A : I \times V \to V^*$ satisfies hypothesis (A1).

By a similar method, one can show that there exists an operator $F : I \times H \to V^*$ such that

$$\langle F(t, x), y \rangle_V = f(t; x, y)$$

and the operator F satisfies the hypothesis $(F1)$. We have accomplished the proof of proposition. □

Using the operators A and F as defined above, Eq. (6.6) can be written in abstract form:

$$\begin{cases} \dot{x} + A(t, x) = F(t, x), \\ x(0) = x(T). \end{cases} \tag{6.8}$$

Applying Theorem 5 in Section 3.3, we get the following theorem.

Theorem 14. *If hypotheses (A') and (F') hold, then there exists a periodic solution $x \in L_p(I, W_0^{m,p}(\Omega))$, $\frac{\partial x}{\partial t} \in L_q(I, W^{-m,q}(\Omega))$ of equation (6.6).*

From the continuity of the embedding $W_{pq} \hookrightarrow C(I, H)$ and the construction of W_{pq}, we get

Corollary 2. *(Properties of solution x) If x is a solution of problem (6.6), then (1)*

$$\lim_{t \to s} \int_\Omega (x(t, z) - x(s, z))^2\, dz = 0 \quad \textit{for all } s \in [0, T].$$

Therefore, the function $t \mapsto x(t, z)$ is continuous on $[0, T]$ in the mean.

(2) The function $z \mapsto x(t, z)$ belongs to $W_0^{m,p}(\Omega)$ for almost all $t \in (0, T)$. That is, this function has generalized derivatives up to order m with respect to the spatial variable x.

Next, we consider the optimal control problem (P2).

Let $U : I \to L_q(\Omega)$ be defined by

$$U(t) = \left\{ v \in L_q(\Omega) : \|v\|_{L_q(\Omega)} \le \|r(t,\cdot)\|_{L_q(\Omega)} \equiv \widehat{r}(t) \right\}.$$

Then the range of $U(t)$ is nonempty since $r(t,\cdot)$ belongs to the range of $U(t)$, closed and convex because $L_q(\Omega)$ is closed and uniform convex space.

Note that the graph of U is

$$GrU = \left\{ (t,v) \in I \times E : 0 \le \widehat{r}(t) - \|v\|_{L_q(\Omega)} \right\}.$$

But $t \to \widehat{r}(t)$ is measurable by Fubini's theorem, and $v \to \widehat{r}(t) - \|v\|_{L_q(\Omega)}$ is continuous. Hence the graph is measurable, which implies that $GrU \in B(I) \times B(E)$; i.e., $U(\cdot)$ is measurable. Furthermore,

$$|U(t)| \le \widehat{r}(t) \qquad \text{a.e.}$$

and $\widehat{r}(\cdot) \in L_q^1(I)$ by hypothesis (R').

Therefore, we have the hypothesis (U1) is satisfied.

Let $\widehat{L} : I \times V \times E \to \overline{R}$ be defined by

$$\widehat{L}(t,y,u) = \int_\Omega L(t,z,\eta(y(z)),u(z))\,dz.$$

One can find Caratheodory integrand $L_k : I \times \Omega \times R^M \times R \to R$ (i.e., $(t,z) \to L_k(t,z,y,u)$ is measurable and $(y,u) \to L_k(t,z,y,u)$ is continuous, $k \ge 1$) such that $L_k \uparrow L$ as $k \to +\infty$ and

$$\phi(t,z) - l(z)\left(\|\eta\|_V + \|u\|_E\right) \le L(t,z,\eta,u) \le k \text{ almost every } t \in I.$$

Set

$$\widehat{L}_k(t,y,u) = \int_\Omega L_k(t,z,y(z),u(z))\,dz.$$

It is easy to see that, for every $k \ge 1$, $t \to \widehat{L}_k(t,y,u)$ is measurable and $(y,u) \to \widehat{L}_k(t,y,u)$ is continuous. Hence, for every $k \ge 1$, $(t,y,u) \to \widehat{L}_k(t,y,u)$ is jointly measurable. Furthermore, from the monotone convergence theorem, we have that $\widehat{L}_k \uparrow \widehat{L}$ as $k \to +\infty$. Hence $\widehat{L}$ is measurable. Also using hypotheses $(L')(2)$, $(L')(3)$ and Fatou's lemma, we can see that $\widehat{L}(t,\cdot,\cdot)$ is l.s.c. and $\widehat{L}(t,x,\cdot)$ is convex. Finally, it follows from hypotheses $(L')(4)$ that

$$\widehat{\phi}(t) - \widehat{l}\left(\|y\|_V + \|u\|_E\right) \le \widehat{L}(t,y,u),$$

with

$$\widehat{\phi}(t) = \|\phi(t,\cdot)\|_{L_1(\Omega)}, \quad \widehat{l} = \|l(\cdot)\|_{L_\infty(\Omega)}.$$

So hypothesis $(L1)$ is satisfied.

Finally, problem (P2) admits the following equivalent abstract formulation:

$$\inf \widehat{J}(x,u) = \int_0^T \widehat{L}(t, x(t), u(t))\, dt = m$$

subject to

$$\begin{cases} \dot{x}(t) + A(t,x(t)) = F(t,x(t)) + b(t)u(t) & \text{a.e. } I \\ x(0) = x(T), \\ u(t) \in U(t) \quad \text{a.e.}, \quad u(\cdot)\text{=measurable.} \end{cases} \tag{6.9}$$

Problem (P2) is the same as problem (P) in Chapter 5. So, applying Theorem 11 in Chapter 5, we get the following theorem.

Theorem 15. *Under hypotheses (A′), (F′), (B′), (R′), and (L′), the problem (P2) has an optimal pair* $(x,u) \in L_p(I, W_0^{m,p}(\Omega) \bigcap C(\bar{I}, L^2(\Omega))) \times L_q(Q_T)$.

6.3 Anti-periodic boundary value problem of quasi-linear hyperbolic differential equations

Example 3. *We consider the following anti-periodic boundary problem corresponding to a wave equation with nonlinear motion:*

$$\begin{cases} x_{tt}(t,z) - \Delta x(t,z) - \sum_{i=1}^n D_i(|D_i x_t(t,z)|^{p-2} D_i x_t(t,z)) \\ = \sum_{i=1}^n D_i f_i(t,z,x(t,z)) + f_0(t,z,x(t,z)) \quad on \quad Q_T \\ x(t,z) = 0 \quad on\ [0,T] \times \partial\Omega \\ x(0,z) = -x(T,z), \quad x_t(0,z) = -x_t(T,z) \quad on\ \Omega. \end{cases} \tag{6.10}$$

We assume that f_i $(i = 0, 1, \cdots, n)$ satisfies the hypothesis (F) in example 1.

Let $V = W_0^{1,p}(\Omega)$ and $H = L_2(\Omega)$. Then $V^* = W^{-1,q}(\Omega)$. Here $p \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Hence "$V \hookrightarrow H \hookrightarrow V^*$" is an evolution triple, and the embedding $V \hookrightarrow H$ is compact.

Definition 18. *The generalized problem corresponding to (6.10) reads as follows. We seek a function* $t \longmapsto x(t)$ *such that, for all* $y \in V$ *and almost all* $t \in (0,T)$,

$$\begin{cases} \frac{d^2}{dt^2}(x(t),y)_H + a(\dot{x}(t),y) + c(x(t),y) = f(x(t),y), \\ x(0) = -x(T), \qquad \dot{x}(0) = -\dot{x}(T), \\ x \in C([0,T],V), \qquad \dot{x} \in W_{pq}. \end{cases} \tag{6.11}$$

where we set

$$\begin{aligned}
a(x,y) &= \int_\Omega \sum_{i=1}^{n} |D_i x|^{p-2}(D_i x)(D_i y)dz, \\
c(x,y) &= \int_\Omega \sum_{i=1}^{n} D_i x D_i y dz, \\
\widehat{f}(t,x,y) &= \int_\Omega \sum_{i=1}^{N} f_i(t,z,x(z))D_i y dz + \int_\Omega f_0(t,z,x)y(z)dz, \\
(x,y)_H &= \int_\Omega xy dx
\end{aligned}$$

for all $x, y \in V$ and all $t \in I$. Here, d^2/dt^2 is to be understood as a second generalized derivative on $(0,T)$.

From example 1, we can get the following lemma.

Lemma 11. *Suppose that (F) in example 1 holds. Then for all $t \in I$, there exists an operator $A(t) : V \to V^*$ such that*

$$\langle A(t)x, y\rangle_V = a(x,y) \quad \textit{for all } x, y \in V,$$

and an operator $f(t) : H \to V^*$ such that

$$\langle f(t)x, y\rangle_V = \widehat{f}(t,x,y) \quad \textit{for all } x \in H \textit{ and } y \in V,$$

The operator $A : I \times V \to V^*$ satisfies the hypothesis (A1) in Section 3.3, and the operator $F : I \times H \to V^*$ satisfies the hypothesis (F1).

Lemma 12. *There exists an operator $N : V \to V^*$ such that*

$$\langle Nx, y\rangle_V = c(x,y) \quad \textit{for all } x, y \in V$$

and N is linear, symmetric, and uniformly monotone.

Proof. Obviously, the map $c : V \times V \to R$ is bilinear and symmetric. By Hölder's inequality,

$$\begin{aligned}
|c(x,y)| &\leq \sum_i \int_\Omega |D_i x D_i y| \, dz \\
&\leq \sum_i \left(\int_\Omega |D_i x|^q \, dz\right)^{1/q} \left(\int_\Omega |D_i y|^p \, dz\right)^{1/p} \\
&\leq C \|x\|_V^{p/q} \cdot \|y\|_V,
\end{aligned}$$

with some constant $C > 0$ for all $x, y \in V$. That is, $y \longmapsto c(x,y)$ is linear and bounded. Hence, there is an linear operator $N : V \to V^*$ such that

$$\langle Nx, y\rangle_V = c(x,y),$$

$$\|Nx\|_{V^*} \leq C \|x\|_V^{p-1},$$

and N is symmetric.

In addition,

$$\begin{aligned} c(x, x-y) - c(y, x-y) &= c(x-y, x-y) \\ &= \int_\Omega \sum_{i=1}^n (D_i(x-y))^2 dz \\ &= \|x-y\|^2_{W^{1,2}(\Omega)}. \end{aligned}$$

Since $p \geq 2$, the embedding $W^{1,p}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ is continuous. So, there exists a constant $\alpha_1 > 0$ such that

$$\begin{aligned} \langle Nx - Ny, x-y \rangle_V &= c(x-y, x-y) \\ &\geq \alpha_1 \|x-y\|_V^p. \end{aligned}$$

That is, $N : V \to V^*$ is uniformly monotone. □

Lemma 10 and Lemma 11 imply that hypotheses (A1), (F1), and (N1) are satisfied. Hence,

Theorem 16. *If (F) holds, then the generalized problem (6.11) corresponding to the original problem (6.10) is equivalent to the operator equation*

$$\begin{cases} \ddot{x}(t) + A(t)\dot{x}(t) + Nx(t) = f(t, x(t)) \quad 0 < t < T, \\ x(0) = -x(T), \quad \dot{x}(0) = -\dot{x}(T), \\ x \in C([0,T], V), \qquad \dot{x} \in W_{pq}, \end{cases}$$

and all the hypotheses of Theorem 8 are fulfilled. Therefore, problem (6.11) has a solution. i.e., there exists a periodic solution $x \in C\left(\overline{I}, W_0^{1,p}(\Omega)\right)$, $\frac{\partial x}{\partial t} \in L_p(I, W_0^{1,p}(\Omega)) \bigcap C\left(\overline{I}, L_2(\Omega)\right)$, $\frac{\partial^2 x}{\partial t^2} \in L_q\left(I, W^{-1,q}(\Omega)\right)$ *of equation (6.10).*

Chapter VII

Conclusion

7.1 Thesis summary

In this thesis, we have studied the existence of periodic and anti-periodic solutions for a large class of strongly nonlinear evolution equations in Banach spaces and sufficient conditions for the existence of a corresponding optimal periodic control.

7.1.1 Problems

The system model considered is based on an evolution triple "$V \hookrightarrow H \hookrightarrow V^*$" and the compact embedding $V \hookrightarrow H$. Let T be a positive number, $2 \leq p < +\infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. The Banach space W_{pq} is defined by

$$W_{pq} = \{x : x \in L_p(0,T;V), \quad \dot{x} \in L_q(0,T;V^*)\}$$

with the norm

$$\| x \|_{W_{pq}} = \| x \|_{L_p(0,T;V)} + \| \dot{x} \|_{L_q(0,T;V^*)} \, .$$

where the derivative in the definition should be understood in the sense of the generalized derivative.

This thesis has considered the following problems:

1. Existence of periodic solutions for the nonlinear evolution equation

$$\begin{cases} \dot{x}(t) + A(t,x(t)) = f(t,x(t)), & t \in (0,T), \\ x(0) = x(T). \end{cases} \tag{7.1}$$

where $A : (0,T) \times V \to V^*$ is a nonlinear monotone operator and $f : (0,T) \times H \to V^*$ is a nonlinear nonmonotone perturbation.

2. Existence of anti-periodic solutions for the first order nonlinear evolution equation

$$\begin{cases} \dot{x}(t) + A(t,x(t)) = f(t,x(t)), & t \in (0,T), \\ x(0) = -x(T). \end{cases} \tag{7.2}$$

where $A : (0,T) \times V \to V^*$ is a nonlinear monotone operator and $f : (0,T) \times H \to V^*$ is a nonlinear nonmonotone perturbation.

3. Existence of anti-periodic solutions for the second order nonlinear evolution equation

$$\begin{cases} \ddot{x}(t) + A(t, \dot{x}(t)) + Nx(t) = f(t, x(t)), & 0 < t < T, \\ x(0) = -x(T), \quad \dot{x}(0) = -\dot{x}(T), \\ x \in C([0,T], V), \quad \dot{x} \in W_{pq}. \end{cases} \tag{7.3}$$

where $A : (0,T) \times V \to V^*$ is a nonlinear monotone operator, $f : (0,T) \times H \to V^*$ is a nonlinear nonmonotone perturbation, and $N : V \to V^*$ is a linear, monotone, and symmetric operator.

4. Sufficient conditions for the optimal control problem (P).

Let E be a control space and the control systems are given as follows:

$$\begin{cases} \dot{x}(t) + A(t, x(t)) = f(t, x(t)) + B(t)u(t), & 0 < t < T, \\ x(0) = x(T), \quad u \in U_{ad}. \end{cases} \tag{7.4}$$

where U_{ad} is the admissible control set.

We denote

$$\begin{aligned} A_{ad} &= \{(x,u) \in W_{pq} \times U_{ad} | \quad (x,u) \text{ satisfies } (7.4)\}. \\ X_{ad} &= \{x \in W_{pq} | \quad \exists\, u \in U_{ad}, \text{ such that } (x,u) \in A_{ad}\}. \end{aligned}$$

Problem (P). Find $(x_0, u_0) \in A_{ad}$, such that

$$J(x_0, u_0) = \min_{(x,u) \in A_{ad}} J(x,u) = m.$$

7.1.2 Hypotheses

(A1) $A : I \times V \longrightarrow V^*$ is an operator such that

1. $t \longmapsto A(t,x)$ is measurable ;

2. For each $t \in I$, the operator $A(t) : V \longrightarrow V^*$ is uniformly monotone and hemicontinuous, that is, there exists a constant $C_1 \geq 0$ such that

$$\langle A(t,x_1) - A(t,x_2), x_1 - x_2 \rangle \geq C_1 \| x_1 - x_2 \|_V^p, \text{ for all } x_1, x_2 \in V,$$

and the map $s \longmapsto \langle A(t, x+sz), y \rangle$ is continuous on $[0,1]$ for all $x, y, z \in V$;

3. Growth condition. There exist a constant $C_2 > 0$ and a nonnegative function $a_1(\cdot) \in L_q(I)$ such that

$$\| A(t,x) \|_{V^*} \leq a_1(t) + C_2 \| x \|_V^{p-1}, \text{ for all } x \in V, \text{ a.e. on } I.$$

We assume that $A(t,0) = 0$ for all $t \in \bar{I}$ without loss of generality.

(F1) $f: I \times H \longrightarrow V^*$ is an operator such that

1. $t \longmapsto f(t,x)$ is measurable;

2. $x \longmapsto f(t,x)$ is continuous and $f(t,x)$ is Hölder continuous with respect to x with exponent $0 < \alpha \le 1$ in H uniformly in t. That is, there exists a constant L such that

$$\| f(t,x_1) - f(t,x_2) \|_{V^*} \le L \| x_1 - x_2 \|_H^\alpha, \quad \text{for all } x_1,\ x_2 \in H,\ t \in I.$$

3. There exist a nonnegative function $h_1(\cdot) \in L_q(I)$ and a constant $C_3 > 0$ such that

$$\| f(t,x) \|_{V^*} \le h_1(t) + C_3 \| x \|_H^{k-1}, \text{ for all } x \in V, t \in I,$$

where $1 \le k < p$ is a constant.

(N1) The operator $N : V \to V^*$ is linear, monotone, and symmetric, i.e.,

$$\langle Nv, w \rangle = \langle Nw, v \rangle \quad \text{for all } v, w \in V.$$

(U1) E is a reflexive separable Banach space.

$$U : I \longrightarrow P(E) := \text{the class of nonempty, closed, convex subsets of E}$$

is a measurable multifunction such that

$$t \longrightarrow |U(t)| = \sup \{\| u \|_E : u \in U(t)\} \quad \text{belongs to } L_r(I)$$

where $1 < q \le r < +\infty$.

For the admissible controls, we choose the set of all selectors of $U(\cdot)$ that belong to Lebesgue-Bochner space, that is,

$$U_{ad} = \{u \in L_r(I,E) : u(t) \in U(t) \text{ a.e. on } [0,T]\}.$$

(B1) $B \in L^\infty(I, \ell(E,H))$.

(L1) $L : I \times V \times E \longrightarrow R \cup \{+\infty\}$ such that

(1) $(t,x,u) \longrightarrow L(t,x,u)$ is measurable;

(2) $(x,u) \longrightarrow L(t,x,u)$ is l.s.c.;

(3) $u \longrightarrow L(t,x,u)$ is convex;

(4) There exist a nonnegative bounded measurable function $\phi(\cdot) \in L_1(0,T)$ and a nonnegative constant C_6 such that

$$L(t,x,u) \ge \phi(t) - C_6 (\| x \|_V + \| u \|_E)$$

for almost all $t \in I$, all $x \in V$, and all $u \in E$.

7.1.3 Results

1. Under hypotheses (A1) and (F1), problem (7.1) has a solution $x \in W_{pq}$.
2. Under hypotheses (A1) and (F1), problem (7.2) also has a solution $x \in W_{pq}$.
3. Under hypotheses (A1), (F1), and (N1), problem (7.3) has a solution.
4. Under hypotheses (A1), (F1), (B1), and (U1), A_{ad} is nonempty and X_{ad} is bounded in $W_{pq} \cap C(I, H)$.
5. Under hypotheses (A1), (F1), (B1), (U1), and (L1), there exists a pair $(x_0, u_0) \in A_{ad}$ such that $J(x_0, u_0) = \min_{(x,u)\in A_{ad}} J(x, u)$.

7.2 Limitations

1. For the existence of a solution of the nonlinear evolution equation, we have required that the nonmonotone perturbation satisfies some growth condition and is Hölder continuous.
2. For the optimal periodic control problem, the control part appears linearly in the control system.

7.3 Applications

All results in the abstract framework of this thesis can be applied to quasi-linear partial differential equations. Three examples concerning quasi-linear partial differential equations and the corresponding optimal control problems have been presented. These are a quadratic optimal control problem of system governed by second order quasi-linear parabolic equation with time periodic condition, Lagrange optimal control problem of a system governed by $2m$-order quasi-linear parabolic equation with time periodic condition, and an anti-periodic boundary value problem of a quasi-linear hyperbolic equation with nonlinear motion.

7.4 Suggestion for further work

We should observe that further problems can be considered. For instance, how to deal with optimal periodic or anti-periodic control problems in which controls appear nonlinear? What form of Cesari conditions can guarantee existence of optimal periodic or anti-periodic control? Discuss the relaxed optimal periodic control problem without convexity condition. Extend our results to strongly nonlinear differential inclusions. We will continue to study in this field.

Remark 3. *In this thesis, the existence of periodic solutions is obtained by virtue of the Leray-Schauder fixed point theorem. This theorem guarantees only the existence but not the uniqueness. The usual approach to verify uniqueness of solutions for an initial value problem is to use integration by parts and Gronwall's Lemma. However, this approach fails in the case of a periodic problem. The uniqueness of periodic solutions and the number of periodic solutions are still open problems even for ordinary differential equations. For more detail some of the investigations to the uniqueness of periodic solutions carried out by author, one may consult on the internal report.*

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