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EXPONENTIAL STABILITY AND STABILIZATION OF NONLINEAR DYNAMICAL SYSTEMS

Ms. Rattikan Saelim

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Thesis Examining Committee

(Assoc. Prof. Dr. Prapasri Asawakun)
Chairman

(Assoc. Prof. Dr. Pairote Sattayatham)
Thesis Advisor

(Assoc. Prof. Dr. Sarawut Sujitjorn)
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ศึกษาเสถียรภาพชี้กำลังสำหรับชั้นของระบบพลศาสตร์ไม่เชิงเส้นที่มีความไม่แน่นอน โดยเริ่มจากการรบกวนระบบปรกติ (nominal) ที่เสถียรอยู่ก่อนให้สูญเสียเสถียรภาพ แล้วจึงสร้างเสถียรภาพให้แก่ระบบที่ถูกรบกวน จากความมีเสถียรภาพของระบบปรกติ ได้สร้างชั้นของตัวควบคุมย้อนกลับต่อเนื่องที่มีขอบเขตขึ้น ตัวควบคุมนี้เราสามารถประกันความมีเสถียรภาพชี้กำลังของระบบพลศาสตร์ไม่เชิงเส้นที่มีความไม่แน่นอนได้ ทั้งนี้ได้ให้ตัวอย่างที่แสดงการใช้ประโยชน์ ของผลที่ได้จากการวิจัย

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Exponential stability for a class of nonlinear dynamical systems with uncertainties is investigated through destabilization of such system. Based on the stability of the nominal system, a class of bounded continuous feedback controllers is constructed. Those controllers can guarantee exponential stability of uncertain nonlinear dynamical systems. A numerical example is given to demonstrate the use of the main results.

School of Mathematics

Student

Academic Year 2000

Advisor

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Rattikan Saelim

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Chapter I

Introduction

Stability theory is a fundamental topic in mathematics and engineering discipline that consider control essential. The least requirement for a control system is that the system is stable, since only a stable system can be operated no matter whether unknown disturbances or noises are present. There are several kinds of stability concepts such as input-output stability, absolute stability, Lyapunov stability, and stability of periodic solutions. These stability concepts have been studied extensively over a hundred years.

In analyzing and designing a nonlinear control system, Lyapunov's direct method plays a vital role for the, mainly, two reasons. First, Lyapunov's direct method uses an energy-like function, the so called Lyapunov function, to study analytically behaviors of dynamical systems. Second, Lyapunov's direct method is applicable to systems both linear and nonlinear.

Lyapunov stability theory generally includes Lyapunov's first and direct methods. In this thesis, our interest is stability based on Lyapunov's direct method for systems described by nonlinear differential equations. The fact is that virtually *all* physical systems are nonlinear in nature. Sometimes it is possible to describe the operation of a physical system by a linear model, such as a set of ordinary linear differential equations. This is the case, for example, if the model of operation of the physical system does not deviate too much from the *nominal* set of operating conditions (Vidyasagar, 1978). Thus the analysis of linear systems occupies an important place in system theory. But in analyzing the behavior of any physical system, one often encounters situations where the linearized model is inadequate or inaccurate.

There are some different issues between linear systems and nonlinear systems. Firstly, in the case of linear system described by a set of linear ordinary differential equations, it is often possible to derive *closed-form expressions* for the solutions of the system equations (Curtain and Pritchard, 1977;Khalil, 1996). In general, this is not possible in the case of nonlinear systems described by a set of nonlinear ordinary differential equations. As a consequence, it is desirable to be able to make some predictions about the behavior of a nonlinear system even in the absence of closed-form expressions for the solution of the system equations. This type of analysis, called qualitative analysis or approximate analysis, is much less relevant to linear systems. Secondly, the analysis of nonlinear systems make use of a wide variety of approaches and mathematical tools than does the analysis of the linear systems. The main reason for this variety is that no tool or methodology for nonlinear systems analysis is universally applicable. Finally, in general, the level of mathematics needed to master the basic ideas of nonlinear system analysis is higher than that for the linear case.

A fundamental problem in science is that of obtaining a mathematical model of a physical or process. Usually scientific theory developed by the theoretician postulating a mathematical model, based perhaps on some physical laws, and then this model is validated against experimental evidence until the model reasonably represents the process under consideration. It is then possible to make predictions from the model, or design controls for the process so that it functions in some desirable manners. By far largest class of mathematical models are those given in terms of differential equations, and in this thesis nonlinear differential equation is considered. A commonly used model for a nonlinear system is

$$\dot{x}(t) = f(t, x(t), u(t)), \quad \forall t \geq 0, \tag{1.1}$$

where t denotes time; $x(t)$ denotes the value of the function $x(\cdot)$ at time t and is an n-dimensional vector; $u(t)$ is similarly defined and is an m-dimensional vector; and the function f associates, with each value of t, $x(t)$, and $u(t)$, a corresponding n-dimensional vector. The system (1.1), a first-order vector differential equation, represents a continuous-time system. The quantity $x(t)$ is generally referred to the state of the system at time t, while $u(t)$ is called the input or the control function. The discrete-time analogue of the continuous-time system (1.1) is

$$x(k+1) = f(k, x(k), u(k)), \ k = 0, 1, 2, ... \tag{1.2}$$

which is a first-order vector difference equation. A physical phenomenon can be described by the continuous-time differential equation (1.1) or the discrete-time difference equation (1.2). Our objective is to consider the continuous-time differential equation (1.1). In studying the system (1.1), one can make a distinction between two aspects, generally referred to as analysis and synthesis, respectively. Suppose the input function $u(\cdot)$ in (1.1) is specified and one wishes to study the behavior of the corresponding function $x(\cdot)$; this is usually referred to as *analysis*. Now suppose the system description (1.1) is given, as well as the desired behavior of the function $x(\cdot)$, and the problem is to find a suitable input function $u(\cdot)$ that would cause $x(\cdot)$ to behave in this desired fashion; this is usually referred to as *synthesis*. The system (1.1) is said to be *forced*, or to have an input; in contrast, a system described by an equation of the form

$$\dot{x}(t) = f(t, x(t)), \quad \forall t \geq 0, \tag{1.3}$$

is said to be *unforced*. In recent years much attention has been focusing on controlling the behavior of partially known systems, also called *uncertain systems*. Systems under consideration of analysis and control design are often not perfectly known because their modeling calls for many assumptions. An accurate model of the system should contain or can be separated into two parts: the identified model as the known part of its dynamics and an uncertain part. The uncertain part of system dynamics, called *uncertainty*, will be of limited magnitude. That is, the uncertainty is not known, nor is it completely unknown. The use of such partial known models to describe a physical system not only reflects the reality but also makes it possible for us to study the way of designing controls that compensate for the unknown and achieve better performance of modern sophisticated systems.

Consider the system

$$\dot{x} = f(t,x) + g(t,x) \tag{1.4}$$

where $f : [0,\infty) \times D \to \mathbb{R}^n$ and $g : [0,\infty) \times D \to \mathbb{R}^n$ are piecewise continuous in t and locally Lipschitz in x on $[0,\infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. We think of this system as a perturbation of the nominal system (1.3). The perturbation term $g(t,x)$ could result from modeling error, aging, or uncertainties and disturbances which exist in any realistic problems. In a typical situation, we do not know $g(t,x)$ but we know some information about it, like knowing an upper bound on $\|g(t,x)\|$. Here, we represent the perturbation as an additional term on the right-hand side of the state equation. Uncertainties which do not change the system order can be represented in this form. Suppose the nominal system (1.3) has a uniformly asymptotically stable equilibrium point at the origin. One is interested in what we can say about the stability behavior of the perturbed system (1.4). A natural approach to address this question is to use a Lyapunov function for the nominal system (1.3) as a Lyapunov function candidate for the perturbed system (1.4). In section 4.3, we restrict our attention to the case when the nominal system has a stable equilibrium point at the origin. The exponential stability is of interest since it implies various types of stability such as stability, itself, uniform stability, and asymptotic stability. In control theory one is interested in the question of stabilization problem, i.e., how the input of the system should be chosen to assure that the corresponding output has the desired properties. Throughout this thesis, we are working on both analysis and synthesis. Firstly, we try to obtain sufficient conditions for exponential stability of the unforced system (1.3) using Lyapunov's second method (analysis). After that, we apply the stability result to the stabilization problem of the forced system (1.1) (synthesis). Finally, we give an example illustrating the use of our main result. The followings are our precise objectives. Consider the nonlinear differential equation (1.3). Assume there exist a sufficiently smooth function $V(t,x)$, positive constants $\lambda_1, \lambda_2, \lambda_3, p$,and q such that

$$\lambda_1 \|x\|^p \leq V(t,x) \leq \lambda_2 \|x\|^q \tag{1.5}$$

and the derivative of V along the solution of (1.3) satisfies

$$\dot{V}(t,x) \leq -\lambda_3 \|x\|^q . \tag{1.6}$$

We try to prove that under the conditions (1.5) and (1.6) the equilibrium point of the system (1.3) is exponentially stable. After that we apply the stability result to the stabilizability problem of the forced system (1.1), i.e., we need to synthesise the feedback control $u(t) = h(t, x(t))$ such that the system (1.1) is exponentially stable.

This thesis is organized as follows. In Chapter 2, we give such general background on differential equations as the existence and uniqueness, local and global, theorem, the interval of definition, the maximum interval, the maximal solution, and so on. In Chapter 3, we give the general ideas of Lyapunov stability and control theory, for example, various types of stability, methods used to investigate the stability of an equilibrium point of nonlinear dynamical systems, terms used in control problems, and so on. In Chapter 4, section 4.2, we first derive the sufficient conditions for the exponential stability of the nonlinear dynamical system. After that, in section 4.3, we apply the stability result (from 4.2) to the stabilization problem for the control system (1.1). Then, in section 4.4, we give a numerical example to illustrate the use of theorem in section 4.3. Finally, in Chapter 5, the conclusion and some suggestions for one who may be interested in exploring further on stability and control problems.

Chapter II

Nonlinear Differential Equations

2.1 Preliminary Notations and Definitions

Suppose we are given the function f mapping a subset of $\mathbb{R}^m$, Euclidean m-dimensional space, into $\mathbb{R}^n$, Euclidean n-dimensional space. If

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

is in the domain of f, and we denote its image under f by

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

then we may write

$$y = f(t,x) = \begin{pmatrix} f_1(t,x) \\ \vdots \\ f_n(t,x) \end{pmatrix},$$

where we define

$$y_i = f_i(t,x) = f_i(t, x_1, ..., x_m), i = 1, ..., n.$$

Note in particular, that the transpose of the $n \times 1$ column vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is the $1 \times n$ row vector

$$x^T = (x_1, ..., x_n).$$

Moreover, for a matrix $A \in \mathbb{R}^{n \times m}$, set of all real n by m matrices, the transpose of the $n \times m$ matrix, $A = (a_{ij}), i = 1, ..., n, j = 1, ..., m$, is the $m \times n$ matrix

$$A^T = (a_{ji}),$$

where $j = 1, ..., m, i = 1, ..., n$.

We will say that f is continuous in x if each f_i is continuous in x. Furthermore, we define the vector of partial derivatives as

$$\frac{\partial f}{\partial x_j} = \begin{pmatrix} \frac{\partial f_1}{\partial x_j} \\ \vdots \\ \frac{\partial f_n}{\partial x_j} \end{pmatrix},$$

where $1 \leq j \leq m$. If $f(t, x)$ is a smooth function, the gradient of $f(t, x)$ is defined by

$$\begin{aligned} \nabla_x f(t,x) &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n} \right)^T f(t,x) \\ &= \frac{\partial}{\partial x_1} f_1(t,x) \frac{dx_1}{dt} + \frac{\partial}{\partial x_2} f_2(t,x) \frac{dx_2}{dt} + ... + \frac{\partial}{\partial x_n} f_n(t,x) \frac{dx_n}{dt}. \end{aligned}$$

Given the n-dimensional vector $x(t) = (x_1(t), ..., x_n(t))^T$, where t is a real variable and each $x_i(t)$ is real-valued, we say $x(t)$ is continuous at $t = t_0$ if each $x_i(t)$ is continuous at $t = t_0$, and it is differentiable if each $x_i(t)$ is differentiable. We then may express the derivative vector as

$$\dot{x}(t) = \frac{dx}{dt} = (\dot{x_1}(t), ..., \dot{x_n}(t))^T,$$

and successive derivatives will be denoted by $\ddot{x}(t), x^{(3)}(t), ..., x^{(k)}(t)$. If $x(t)$ is given as above, we denote the norm of $x(t)$, Euclidean norm, by

$$\|x(t)\| = \left\{ \sum_{i=1}^{n} x_i^2(t) \right\}^{1/2},$$

and for each t this is a mapping of $x(t)$ into the nonnegative real numbers. It has the properties

(i) $\|x(t)\| = 0$ if and only if $x(t) = 0$, that is, each $x_i(t)$ is zero;

(ii) $\|kx(t)\| = |k| \, \|x(t)\|$ for any real or complex scalar k; and

(iii) $\|x(t) + y(t)\| \leq \|x(t)\| + \|y(t)\|$.

However, any result given will not depend on the norm chosen (Sanchez, 1968). If we let $\|\cdot\|$ be a given norm on $\mathbb{R}^n$. Then for each matrix $A \in \mathbb{R}^{n\times n}$, the quantity $\|A\|$ defined by

$$\|A\| = \sup_{x\neq 0, x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|\leq 1} \|Ax\|,$$

is called the induced (matrix) norm of A corresponding to the vector norm $\|\cdot\|$.

Frequently we will be considering a given function f mapping a subset of $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$. If we denote a point in $\mathbb{R}^{n+1}$ by (t, x), where t is a real and $x = (x_1, x_2, ..., x_n)^T$, then its image, wherever defined, may be denoted by

$$y = (y_1, y_2, ..., y_n)^T = f(t, x) = (f_1(t, x), ..., f_n(t, x))^T.$$

In particular, if $x = x(t) = (x_1(t), ..., x_n(t))^T$, then $y = y(t) = f(t, x(t))$ is an element in $\mathbb{R}^n$ dependent on the real variable t. If $f(t, x(t))$ is continuous for (say) $t_1 \leq t \leq t_2$, then we can define the integral

$$\int_{t_1}^{t_2} f(s, x(s))ds = \left(\int_{t_1}^{t_2} f_1(s, x(s))ds, \int_{t_1}^{t_2} f_2(s, x(s))ds, \ldots, \int_{t_1}^{t_2} f_n(s, x(s))ds\right)^T,$$

and the usual rules of integration will hold.

2.2 Ordinary Differential Equations

Differential equations occur in connection with numerous problems that are encountered in the various branches of science and engineering, such as the problem of determining the motion of a projectile, rocket, satellite, or planet, the problem of determining of charge or current in an electric circuit, and so on. In those situations, the objects involved obey certain scientific laws. These laws involve various rate of change of one or more quantities with respect to other quantities. Recall that such change of rates are expressed mathematically by derivatives. In the mathematical formulation of each situation, the various rates of change are thus expressed by various derivatives and the scientific laws themselves become mathematical equations involving derivatives, that is, differential equations.

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a *differential equation*. It is called an *ordinary differential equation* if the equation is involving ordinary derivatives of one or more dependent variables, (an) unknown function(s), with respect to a single independent variable. In addition, initial conditions, which the unknown function is required to satisfy, may be given. With such an equation, the object is two-fold : (i) to find the unknown function or a class of functions satisfying the equation, and (ii) whether (i) is possible or not, to gain some information about the behavior of any function satisfying the equation. Since the order of an ordinary differential equation is the order of the highest derivative of the unknown function appearing in it, therefore the general form of an ordinary differential equation of kth order is

$$F(t, x, \dot{x}, ..., x^{(k)}) = 0 \tag{2.1}$$

where $x = x(t) = (x_1(t), ..., x_n(t))^T$ is an unknown function, and F is a function defined on some subset of $\mathbb{R}^{n(k+1)+1}$.

A function $x = \varphi(t) = (\varphi_1(t), ..., \varphi_n(t))^T,\ r_1 < t < r_2$, which when substituted in (2.1) reduces it to an identity, is called a solution of (2.1), and (r_1, r_2) is its interval of definition. When we say that we shall solve a differential equation we mean that we shall find one or more of its solutions. Furthermore, suppose we have solved a differential equation, this does not necessarily mean that we have found a *formula* for the solution. Comparatively few differential equations have solutions so expressible; in fact, a closed-form solution is really a luxury in differential equations. Ross, 1984, showed certain types of differential equations that do have such closed-form solutions. For those equations of which exact methods are unavailable are solved approximately by various methods (Ross, 1984, and Sanchez, 1968). Since very little can be said about the equation in the form given in (2.1), so let us assume that we can solve (locally) for $x^{(k)}$. We obtain

$$x^{(k)} = G(t, x, \dot{x}, ..., x^{(k-1)}), \tag{2.2}$$

the kth order equation in *normal* form. In this case, since $x^{(k)}$ is an n-dimensional vector, the function G is a mapping from some subset of $\mathbb{R}^{nk+1}$ into $\mathbb{R}^n$.

Given an equation in the form (2.2) with $k > 1$, the following substitution reduces it to an equation with $k = 1$. In fact, let

$$y_1 = x$$

and

$$y_2 = \dot{x}, ..., y_k = x^{(k-1)}$$

then

$$\begin{aligned} \dot{y}_1 &= \dot{x} = y_2, \\ \dot{y}_2 &= \ddot{x} = y_3, \\ &\ldots \\ \dot{y}_{k-1} &= x^{(k-1)} = y_k, \end{aligned}$$

and

$$\dot{y}_k = x^{(k)} = G(t, x, \dot{x}, ..., x^{(k-1)}) = G(t, y_1, y_2, ..., y_k) = G(t, y).$$

This is the first-order system $\dot{y} = f(t, y)$, when $y = (y_1, y_2, ..., y_k)^T$ and $f(t, y) = (f_1(t, y), ..., f_k(t, y))^T = (y_2, ..., y_k, G(t, y))^T$. Note that if x is an n-dimensional vector, then y is a $(k \times n)$-dimensional vector. It follows that we need only consider the first-order equation

$$\dot{x} = f(t, x) \tag{2.3}$$

where $x = x(t)$ is an unknown n-dimensional vector function, and $f(t, x)$ is a mapping from a subset of $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$. We refer to (2.3) as a *vector field*. If, in addition, f is differentiable, the derivative of f is given by

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial x_1} & \ldots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

A linear ordinary differential equation of order n, in the dependent variable x and the independent variable t, is an equation that is in, or can be expressed, in the form

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n(t)x = b(t) \tag{2.4}$$

where a_0 is not identically zero. Since (2.2) and (2.3) are equivalent, (2.4) can be expressed in the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t) \tag{2.5}$$

where $a_{ij}(t), i, j = 1, ..., n$, and $b_i(t), i = 1, ..., n$, are continuous real-valued functions on $r_1 < t < r_2$ and $x(t) = (x_1(t), ..., x_n(t))^T$ is an unknown n-dimensional vector. If we denote by $A(t)$ the $n \times n$ matrix $(a_{ij}(t))$, and by $B(t)$ the vector $(b_1(t), ..., b_n(t))^T$, the system (2.5) can be conveniently expressed as

$$\dot{x} = A(t)x + B(t). \tag{2.6}$$

Hence $f(t,x) = A(t)x + B(t)$ and is defined in the infinite slab

$$B = \{(t,x)\,|r_1 < t < r_2, -\infty < x_i < \infty\,, i = 1, ..., n\}.$$

If $B(t) = 0$ for all t, then the linear system (2.6) is called *homogeneous*; otherwise, the system is said to be *nonhomogeneous.* If f_i do not depend explicitly on t, the system

$$\dot{y} = f(y) \tag{2.7}$$

is called *autonomous* or *time-invariance.* Otherwise, (2.3), it is called *nonautonomous* or *time-varying.* However, any nonautonomous sytem (2.3) with $x \in \mathbb{R}^n$ can be written as an autonomous system (2.7) with $y \in \mathbb{R}^n$ simply by letting $y_1 = t, \dot{y}_1 = 1, y_2 = x_1, \dot{y}_2 = \dot{x}_1$, and so on.

A *nonlinear ordinary differential equation* is an ordinary differential equation that is not linear. A *dynamical system* is a system characterized by a set of related variables, which can change with time in a manner which is, at least in principle, predictable provided that the external influences acting on the system are known. A dynamical system described by a nonlinear differential equation is so called a *nonlinear dynamical system.* When a differential equation is used to model the evolution of a state variable for a physical process, a fundamental problem is to determine the future values of the state variable from its initial value. The mathematical model is then given by a pair of equations

$$\frac{dx}{dt} = f(t,x) \ , \tag{2.8}$$

$$x(t_0) = x_0 \tag{2.9}$$

where the second equation is called an *initial condition.* While the pair of equations is called an initial value problem. By a solution of (2.8) we mean a map, x, from some interval $I \subset \mathbb{R}$ into $\mathbb{R}^n$, which will be represented as follows

$$x : I \to \mathbb{R}^n$$

and

$$t \mapsto x(t)$$

such that $x(t)$ satisfies (2.8), i.e.,

$$\dot{x}(t) = f(t, x(t)).$$

In this case $x(\cdot, x_0) : I \to \mathbb{R}^n$ defines a *solution curve, trajectory,* or *orbit* of the differential equation (2.8) based at x_0.The map x has the geometrical interpretation of a curve in $\mathbb{R}^n$, and (2.8) gives the tangent vector at each point of the curve. We will refer to the space of dependent variables of (2.8) (i.e., $\mathbb{R}^n$) as the *phase space* or the *state space* of (2.8), and our goal will be to understand the geometry of solution curve in phase space. It may be useful to distinguish a solution curve by a particular point in phase space that it passes through at a specific time, i.e., for a solution $x(t)$ we have, (2.9), $x(t_0) = x_0$. We refer to this specifying an initial condition. This is often included in the expression for a solution by writing $x(t, t_0, x_0)$. In some situations explicitly displaying the initial condition may be unimportant, in which case we will denote the solution simply as $x(t)$. Still, in other situations the initial time may be always understood to be a specific value, in this case we would denote the solution as $x(t, x_0)$. Unfortunately, there are no known method of solving equation (2.8). However, it is not necessary, in most application, to find the solutions of (2.8) explicitly. For example, in 2-dimensional space, let $x_1(t)$ and $x_2(t)$ denote the populations, at time t, of two species competing amongst themselves for the limited food and living space in their microcosm. Suppose, moreover, that the rates of growth of $x_1(t)$ and $x_2(t)$ are governed by the differential equation (2.8). In this case, we are not really interested in the values of $x_1(t)$ and $x_2(t)$ at every time t. Rather, we are interested in the qualitative properties of $x_1(t)$ and $x_2(t)$. Specially, we wish to answer the following questions.

- Do there exist ξ_1 and ξ_2 at which the two species coexist together in a steady state? This is to say, are there numbers ξ_1, ξ_2 such that $x_1(t) \equiv \xi_1$ and $x_2(t) \equiv \xi_2$ is a solution of (2.8)? Such values ξ_1, ξ_2, if they exist, are called equilibrium points of (2.8).

- Suppose that the two species are coexisting in equilibrium. Suddenly, we add a few members of species 1 to the microcosm (i.e., the system is perturbed). Will $x_1(t)$ and $x_2(t)$ remain close to their equilibrium value for all future time? Or perhaps the extra few members give species 1 a large advantage and it will proceed to annihilate species 2.

- Suppose that $x_1(t)$ and $x_2(t)$ have arbitrary values at $t = 0$. What happens as t approaches infinity? Will one species ultimately emerge victorious, or will the struggle for existence end in a draw?

More generally, we are interested in determining the following properties of solutions of (2.8).

- Do there exist equilibrium values $x_0 = (x_{10}, ..., x_{n0})^T$ for which $x(t) \equiv x_0$ is a solution of (2.8)?

- Let $\phi(t)$ be a solution of (2.8)-(2.9). Suppose that $\psi(t)$ is a second solution with $\psi(0)$ very close to $\phi(0)$; that is, $\psi_j(0)$ is very close to $\phi_j(0), j = 1, ..., n$. Will $\psi(t)$ remain close to $\phi(t)$ for all future time, or will $\psi(t)$ diverge from $\phi(t)$ as t approaches infinity? This question is often referred to as the problem of *stability* (We discuss this problem, in detail, later in chapter 3).

Note that if $x(\cdot)$ is a solution of (2.8)-(2.9) over $[0, T]$ and f is continuous, then $x(\cdot)$ also satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \ t \in [t_0, T], t_0 \geq 0. \tag{2.10}$$

On the other hand, if $x(\cdot) \in \mathbb{C}^n[0, T]$ satisfies (2.10), then clearly $x(\cdot)$ is actually differentiable everywhere and satisfies (2.8)-(2.9). Then (2.8)-(2.9) and (2.10) are equivalent in the sense that every solution of (2.8)-(2.9) is also a solution of (2.10) and vice versa.

2.3 Existence and Uniqueness Theorems

The two fundamental questions which much now be considered are (i) what conditions will ensure the existence of a solution of an initial value problem for a general first order differential equation? and (ii) what conditons will ensure the uniqueness of a solution of an initial value problem for a general first order differential equation? These questions are of prime improtance practically, because in physical situations modeled by initial value problems is normally expected that a solution can be found (*it exists*) and, furthermore, there is only one solution (*it is unique*). If no solution exists where one is expected this will indicate a failure of a mathematical model used to derive the differential equation. Should more than one solution exist this will either indicate an important feature of a physical problem, or a failure of the model. In either event such a problem will require further investigation, and this could lead to some modification of the mathematical model.

Since the continuity of $f(t, x)$ in its arguments only ensures that there is at least one solution. It is not sufficient to ensure uniqueness of the solution. Extra conditions must be imposed on the function f. The following theorem utilize a condition on $f(t, x)$ called a Lipschitz condition.

Theorem 2.3.1. (Local Existence and Uniqueness) Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\,\|x - y\| \tag{2.11}$$

for all $x, y \in B = \{x \in \mathbb{R}^n \,|\|x - x_0\| \leq r\}$ for all $t \in [t_0, t_1]$. Then, there exists some $\delta > 0$ such that the state equation

$$\dot{x} = f(t, x), \text{ with } x(t_0) = x_0$$

has a unique solution over $[t_0, t_0 + \delta]$.

Proof. (Khalil, 1996).

The key assumption in Theorem 2.3.1 is the Lipschitz condition (2.11). A function satisfying (2.11) is said to be *Lipschitz* in x, and the positive constant L is called a *Lipschitz constant*. We also use the words *locally Lipschitz* and *globally Lipschitz* to indicate the domain over which the Lipschitz condition holds. Let us introduce the terminology first for the case when f depends only on x. A function $f(x)$ is said to be *locally Lipschitz* on a domain (open and connected set) $D \subset \mathbb{R}^n$ if each point of D has a neighborhood D_0 with some Lipschitz constant L_0. We say that f is Lipschitz on a set W if it satisfies (2.11) for all points in W, with the same Lipschitz constant L. A locally Lipschitz function on a domain D is not necessarily Lipschitz on D, since the Lipschitz condition may not hold uniformly (with the same constant L) for all points in D. However, a locally Lipschitz function on a domain D is Lipschitz on every compact (closed and bounded) subset of D (Khalil, 1996). A function $f(x)$ is said to be *globally Lipschitz* if it is Lipschitz on $\mathbb{R}^n$. The same terminology is extended to a function $f(t, x)$, provided the Lipschitz condition holds uniformly in t for all t in a given interval of time.

Theorem 2.3.1 is a local theorem since it guarantees existence and uniqueness only over an interval $[t_0, t_0 + \delta]$, where δ may be very small. In other words, we have no control on δ; hence, we cannot ensures existence and uniqueness over a given time interval $[t_0, t_1]$. However, one may try to extend the interval of existence by repeating applications of the local theorem. Starting with a time t_0 with an initial state $x(t_0) = x_0$, Theorem 2.3.1 shows that there is a positive constant δ (depending on x_0) such that the state equation (2.8)-(2.9) has a unique solution over the time interval $[t_0, t_0+\delta]$. Now, taking $t_0+\delta$ as a new initial state, one may try to apply Theorem 2.3.1 again to extablish existence of the solution beyond $t_0+\delta$ and so on. More convenient, the theorem below establishes the existence of a unique solution over $[t_0, t_1]$ where t_1 may be arbitrarily large.

Theorem 2.3.2. (Global Existence and Uniqueness) Suppose $f(t,x)$ is piecewise continuous in t and satisfies

$$\|f(t,x) - f(t,y)\| \leq L\,\|x-y\|$$

and

$$\|f(t,x_0)\| \leq h$$

for all $x, y \in \mathbb{R}^n$, for all $t \in [t_0, t_1], h \geq 0$. Then, the state equation

$$\dot{x} = f(t,x), \text{ with } x(t_0) = x_0$$

has a unique solution over $[t_0, t_1]$.

Proof. (Khalil, 1996)

By uniqueness, it is meant that if two solutions $x = \varphi(t)$ and $x = \psi(t)$ of the equation (2.8) both satisfy $\varphi(t_0) = \psi(t_0) = x_0$, then these solutions are identical in their common intervals of definition. In fact, the theorem states that through every point of $I \times D$ there passes one and only one integral curve. A solution of (2.8) through a point (initial condition) $(t_0, x_0) \in I \times D$ will be denoted by $x(t, x_0)$. We shall also sometimes write $\dot{x}$ for $\frac{dx}{dt}$. Suppose further that we are given two solutions $\varphi_1(t)$, $r_1 < t < r_2$, and $\varphi_2(t)$, $s_1 < t < s_2$, of the differential equation (2.8) and both solutions satisfy the initial condition

(2.9). Therefore $\varphi_1(t_0) = \varphi_2(t_0) = x_0$. If the equation satisfies the hypotheses of Theorem 2.3.1, then in a neighborhood of (t_0, x_0) we have uniqueness, and the two solution overlap. For instance, if $r_1 < s_1 < t_0 < r_2 < s_2$, then $\varphi_1(t) = \varphi_2(t)$ for $s_1 < t_0 < r_2$. However, we can define a new solution $\varphi(t)$ on $r_1 < t_0 < s_2$, that containing both $\varphi_1(t)$ and $\varphi_2(t)$ as follows:

$$\varphi(t) = \begin{cases} \varphi_1(t), & \text{if } r_1 < t_0 < r_2 \\ \varphi_2(t), & \text{if } s_1 < t_0 < s_2 \end{cases} .$$

This $\varphi(t)$ is a solution, since $\varphi_1(t)$ and $\varphi_2(t)$ are solutions. $\varphi(t)$ agrees with common values on $s_1 < t_0 < r_2$ and it is defined on the larger interval. This same procedure of tacking together solutions would apply if we were given a finite number of solutions $\varphi_1(t), \cdots, \varphi_m(t)$ such that $\varphi_1(t_0) = \cdots = \varphi_m(t_0) = x_0$. We could then define a new solution $\varphi(t)$ satisfying $\varphi(t_0) = x_0$ and whose interval of definition contains those $\varphi_1(t), \cdots, \varphi_m(t)$.

Theorem 2.3.3. Suppose the hypotheses of Theorem 2.3.1 are satisfied for the differential equation (2.8). Then given the initial value (t_0, x_0), there exists a solution $\varphi(t)$ of (2.8) defined on $m_1 < t < m_2$, satisfying $\varphi(t_0) = x_0$. Furthermore, if $\psi(t)$ is any other solution and $\psi(t_0) = x_0$, then its interval of definition is contained in (m_1, m_2).

Proof. (Khalil, 1996).

The interval (m_1, m_2) is then called the *maximum interval* of existence corresponding to the initial value (t_0, x_0), while the solution $\varphi(t)$ is called the *maximal solution*.

From now on, we shall assume that the f_i's are continuous and satisfy standard conditions so that the solution of (2.1) exists and is unique. Throughout this thesis, we consider the behavior of the zero solution of the system (2.8). Since any solution of (2.8) can be shifted to the origin. In fact, if C is a class of solutions of (2.8) which remains in D and $x_0(t)$ is an element of C. Setting

$x = y + x_0(t)$, the system (2.8) is transferred into

$$\begin{aligned} \frac{dy}{dt} &= \frac{dx}{dt} - \frac{dx_0(t)}{dt} = f(t,x) - f(t,x_0(t)) \\ &= f\left(t, y + x_0(t)\right) - f\left(t, x_0(t)\right) \triangleq g(t,y) \end{aligned} \tag{2.12}$$

If we denote by $g(t,y)$ the right-hand side of (2.12), clearly $g(t,0) \equiv 0$ and the zero solution $y(t) \equiv 0$ of (2.12) corresponds to $x_0(t)$. In which case, the point $y(t) \equiv 0$ is called a *critical* point. In fact, any point x_0 in $I \times D$ at which f vanishes is called a critical point of (2.8). In other words, if a system starts at a critical point, it remains in that state thereafter. Moreover, a critical point x_0 of (2.8) is called an *isolated* point if there exists a neighborhood of x_0 containing no other critical points. Other terms often substituted for the term critical point are *equilibrium point*, *fixed point*, *stationary point*, *rest point*, *sigularity*, or *steady state*. In this thesis, we always use the term *equilibrium point*. We shall assume that this has been done for any given system so that we then have $f(t,0) = 0, t \geq t_0$ and D is a domain such that $||x|| < H, H > 0$.We shall also assume that there is no other constant solution in the neighborhood of the origin, so this is an *isolated* equilibrium point.

Chapter III

Lyapunov Stability and Control Theory

3.1 Definitions

Consider the nonlinear dynamical system

$$\dot{x}(t) = f(t, x(t)), \forall t \geq 0 \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$, and $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the Lipschitz condition (2.11) so that the solution exists and is unique corresponding to each initial condition. Let $x(t, t_0, x_0)$ be the solution of (3.1) corresponding to the initial condition $x(t_0) = x_0$, evaluated at time t. In other words, x satisfies the equation

$$\frac{d}{dt}x(t, t_0, x_0) = f(t, x(t, t_0, x_0)), \forall t \geq t_0 \tag{3.2}$$

and satisfies

$$x(t_0, t_0, x_0) = x_0, \forall x_0 \in \mathbb{R}^n.$$

The intuitive idea of stability in a dynamical system is that for *small* perturbations from the equilibrium state at some time t_0, subsequent motion $x(t), t \geq t_0$, should not be too *large*. Suppose that Fig. 3.1 (Barnett and Cameron, 1993). represents a ball resting in equilibrium on a sheet of metal bent into various shapes with cross-sections as shown. If frictional forces can be neglected then small perturbations lead to:

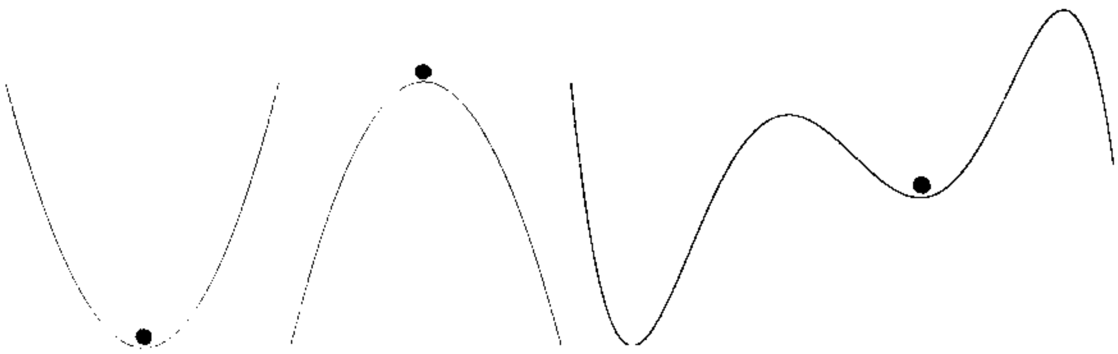

Figure 3.1: A ball resting in equilibrium on a sheet of metal bent into various shapes with cross-section.

(3.1a) oscillatory motion about equilibrium;

(3.1b) the ball moving away without returning to equilibrium;

(3.1c) oscillatory motion about equilibrium, unless the initial perturbation is so large that the ball is forced to oscillate about the new equilibrium position on the left, or to fall off at the right.

If friction is taken into account then the oscillatory motions steadily decrease until the equilibrium state is returned to. Clearly there is no single concept of stability, and very many different definitions are possible, see for example, Bellman,1953, p. 30). We shall consider for the present only the following fundamental statements. The solution $x(t) = x(t, t_0, x_0)$ is said to be *Lyapunov-stable*, or simply *stable* (see Fig. 3.2 (Wiggins, 1986)) if for given $\varepsilon > 0$ and each $t_0 \in [0, \infty)$ there exists a $\delta = \delta(\varepsilon, t_0)$ such that whenever $\|x_0 - x_1\| \leq \delta(\varepsilon, t_0)$ implies

$$\|x(t, t_0, x_0) - x(t, t_0, x_1)\| < \varepsilon, \forall t \geq t_0.$$

It is *unstable* if it is not stable. The solution $x(t) = x(t, t_0, x_0)$ of (3.2) is *asymptotically stable* (see Fig. 3.3 (Wiggins, 1986)) if it is stable and in addition there exists $b > 0$ such that $\|x_0 - x_1\| \leq b$ implies

$$\lim_{t \to \infty} \|x(t, t_0, x_0) - x(t, t_0, x_1)\| = 0.$$

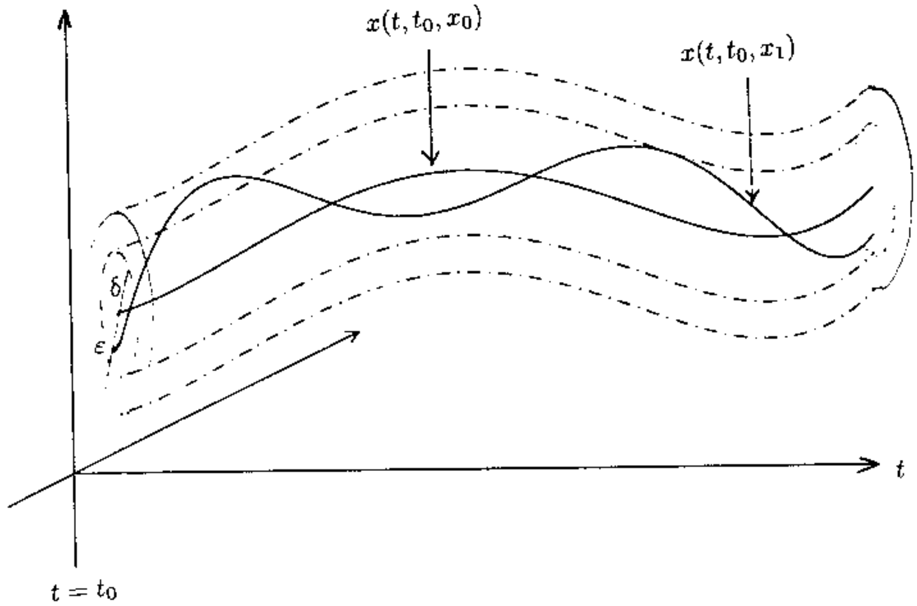

Figure 3.2: Lyapunov Stability of a solution.

Geometrically, the definitions say that $x(t) = x(t, t_0, x_0)$ is stable if any other solution whose initial data is sufficiently close to that of $x(t)$ remain in a *tube* enclosing $x(t)$. If the diameter of the tube approaches zero as t approaches infinity, then $x(t)$ is asymptotically stable. Recall that a vector $x_0 \in \mathbb{R}^n$ is an equilibrium of the system (3.2) if

$$f(t, x_0) = 0, \forall t \geq 0.$$

In other words, if the system starts at an equilibrium, it stays there. Throughout this chapter it is assumed that the origin is an equilibrium of the system (3.2). If the equilibrium under study is not the origin, one can always redefine the coordinates on $\mathbb{R}^n$ in such a way that the equilibrium of interest becomes the new origin (see Chapter 2). Thus, without loss of generality, it is assumed that

$$f(t, 0) = 0, \forall t \geq 0.$$

This is equivalent to the statement

$$x(t, t_0, 0) = 0, \forall t \geq t_0.$$

Lyapunov stability is concerned with the behavior of the trajectories of a system when its initial state is near equilibrium. In other words, it is concerned with the

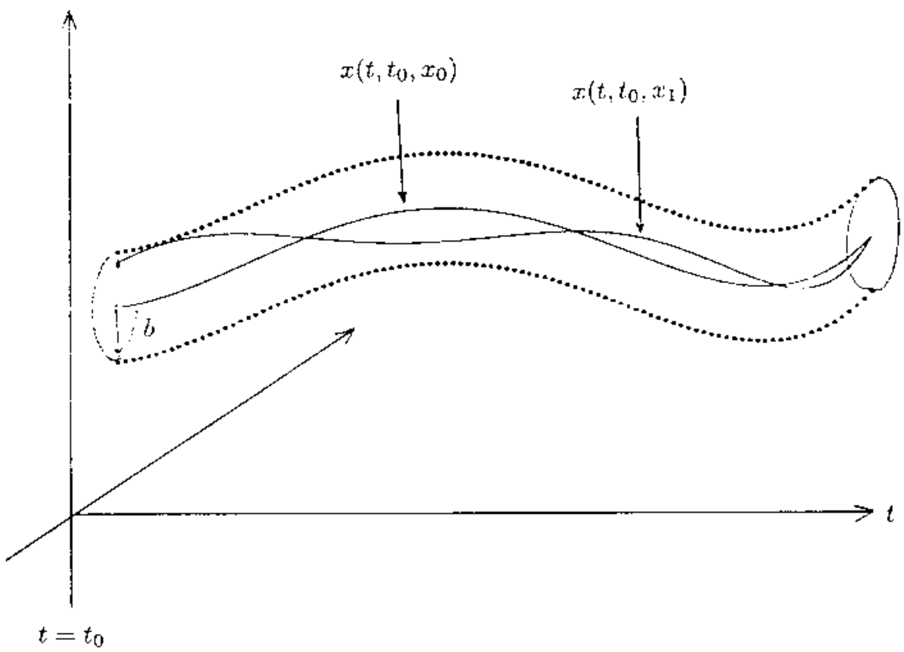

Figure 3.3: Asymptotic stability of a solution.

behavior of the function $x(t, t_0, x_0)$ when $x_0 \neq 0$ but is *close* to it. Now, according to the definitions given above, we state the difinitions for the equilibrium zero as follows:

Definition 3.1.1. The equilibrium zero is

- *stable* (see Fig. 3.4 (Afanas'ev, Kolmanovskii, and Nosov, 1995)) if, for each $\varepsilon > 0$ and each $t_0 \in [0, \infty)$ there exists a $\delta = \delta(\varepsilon, t_0)$ such that whenever $\|x_0\| \leq \delta(\varepsilon, t_0)$ implies

$$\|x(t, t_0, x_0)\| < \varepsilon, \forall t \geq t_0.$$

- *asymptotically stable* (see Fig. 3.5 (Afanas'ev, Kolmanovskii, and Nosov, 1995))) if it is stable and in addition there exists $\rho > 0$ such that $\|x_0\| \leq \rho$ implies

$$\lim_{t \to \infty} \|x(t, t_0, x_0)\| = 0.$$

- *exponentially stable* (see Fig. 3.6 (Afanas'ev and others, 1995) if there exist positive constants M, δ and γ such that

$$\|x(t, t_0, x_0)\| \leq M \|x_0\| e^{-\gamma(t-t_0)}, \forall t \geq t_0, \forall x_0 \in B_\delta.$$

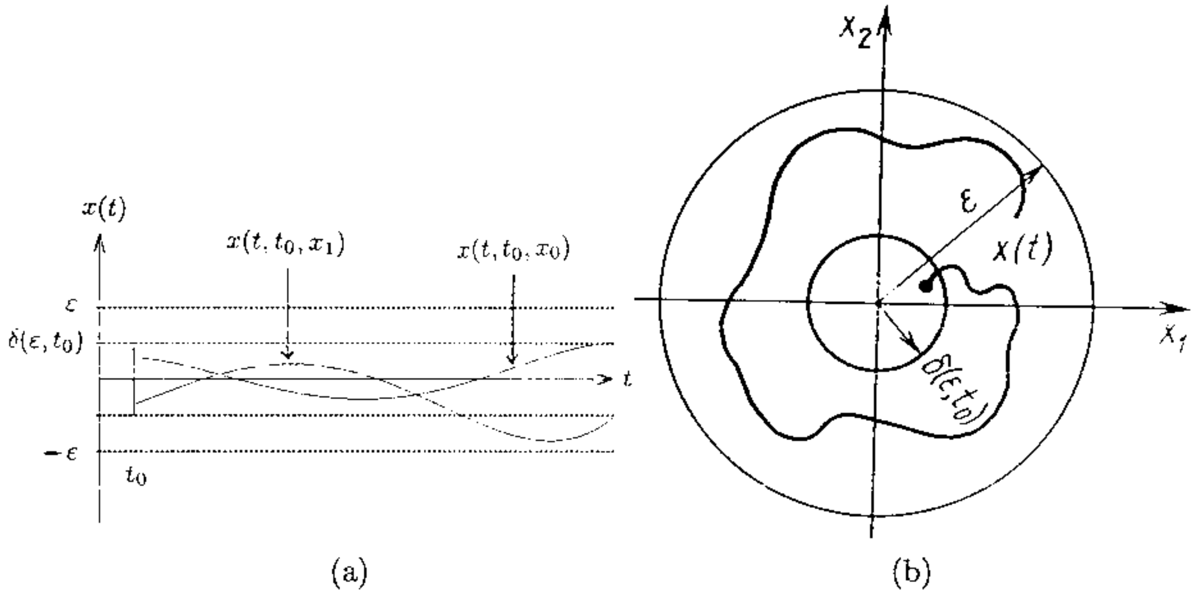

Figure 3.4: (a) Lypunov stability of the trivial solution. (b) Lyapunov-stable phase trajectory.

Occasionally, however, the case when x_0 is *far* from zero is also of interest.

Definition 3.1.2. The equilibrium zero is *globally exponentially stable* if there exist positive constants M and γ such that

$$\|x(t, t_0, x_0)\| \leq M \|x_0\| e^{-\gamma(t-t_0)}, \forall t \geq t_0, \forall x_0 \in \mathbb{R}^n.$$

In other words, the definition says that for any chosen $x_0 \in \mathbb{R}^n$ the solution is bounded by an exponential function depending on its intial state.

The study of nonlinear dynamical system is carried out by one of two Lyapunov's method. One is the Lyapunov's linearization method, and the other is the Lyapunov's direct method. The stability criteria for the linear system are discussed in many literatures e.g. Bellman, 1953; Chicone, 1999; Curtain and Pritchard, 1977; Khalil, 1996; Ross, 1984; and Sanchez, 1968.

3.2 The Lyapunov's linearization method

One of the most useful results of Lyapunov stability theory is the Lyapunov's linearization method. Its great value lies in the fact that, under certain condi-

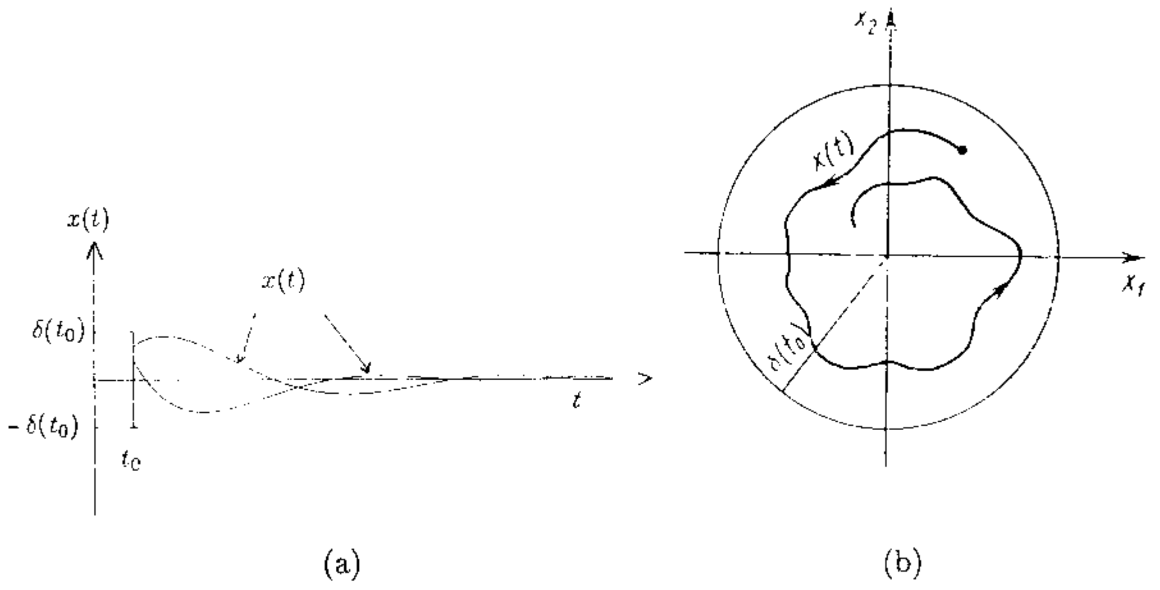

Figure 3.5: Asymptotic stability of the zero solution.

tions, it enables one to draw conclusions about nonlinear system by studying the behavior of a linear system. We begin by defining precisely the concept of linearizing a nonlinear system around an equilibrium. Consider first the autonomous system

$$\frac{dx_i}{dt} = f_i(x_1, x_2, ..., x_n), \;\; i = 1, 2, ..., n \tag{3.3}$$

where $f : D \to \mathbb{R}^n$ is a continuously differentiable map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Let $x_i \equiv 0, \;\; i = 1, 2, ..., n$ be an equilibrium point of system (3.3), i.e. $f_i(0, 0, ..., 0) = 0, \; i = 1, 2, ..., n$. We shall assume that the function $f_i(x_1, x_2, ..., x_n)$ can be differentiated a sufficiently large number of times at the origin of coordinates. By Mean Value Theorem,

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)x$$

where z_i is a point on the line segment connecting x to the origin. The above equality is valid for any point $x \in D$ such that the line segment connection x to the origin lies entirely in D. Since $f(0) = 0$, we can write $f_i(x)$ as

$$f_i(x) = \frac{\partial f_i}{\partial x}(0)x + \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

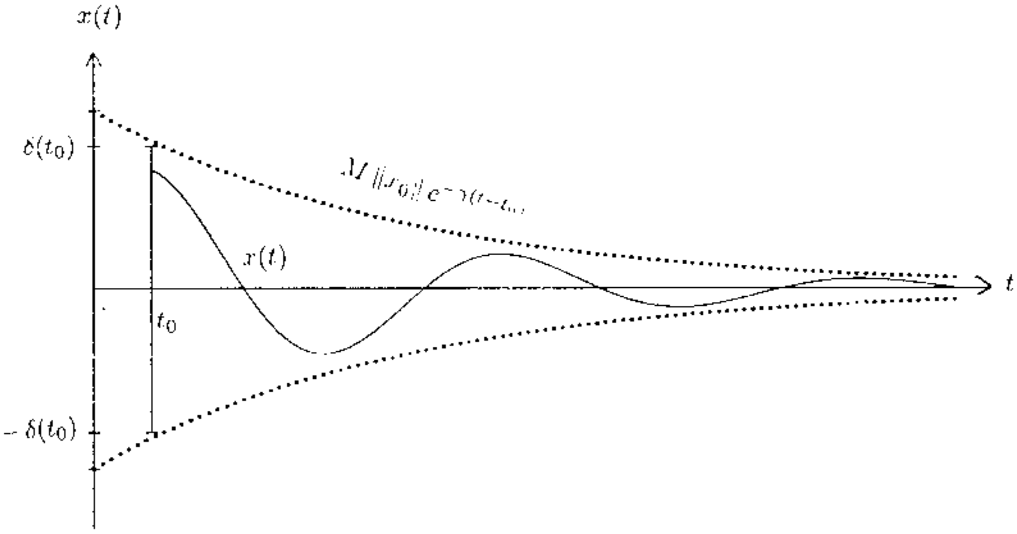

Figure 3.6: Exponential stability of the zero solution.

Hence,

$$f(x) = Ax + g(x)$$

where

$$A = \frac{\partial f}{\partial x}(0), \text{ and } g(x) = \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

The function $g_i(x)$ satisfies

$$|g_i(x)| \leq \left\|\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right\| \|x\|$$

By continuity of $\left(\frac{\partial f}{\partial x}\right)$, we see that

$$\frac{\|g_i(x)\|}{\|x\|} \to 0 \text{ as } \|x\| \to 0$$

Alternatively, one can think of

$$f(x) = Ax + g(x)$$

as the Taylor's series expansion of $f(\cdot)$ around the point $x = 0$. With this notation, the system

$$\dot{z}(t) = Az(t), \text{ where } A = \frac{\partial f}{\partial x}(0). \tag{3.4}$$

is referred to as the *linearization* or the *linearized system* of (3.3) around the equilibrium zero. Precisely, the Jacobian matrix A is defined by

$$A = \left[\frac{\partial f_i}{\partial x_j}(x_0)\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=x_0}.$$

The development for nonautonomous system is similar. Consider the nonlinear nonautonomous system

$$\dot{x}(t) = f(t, x(t)) \tag{3.5}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is continuously differentiable and $D = \{x \in \mathbb{R}^n \mid \|x\| < H\}$. Suppose the origin $x = 0$ is an equilibrium point for the system at $t = 0$; that is,

$$f(t, 0) = 0, \forall t \geq 0. \tag{3.6}$$

Furthermore, suppose the Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is bounded and Lipschitz on D, uniformly in t; thus,

$$\left\| \frac{\partial f_i}{\partial x}(t, x_1) - \frac{\partial f_i}{\partial x}(t, x_2) \right\| \leq L_1 \|x_1 - x_2\|, \forall x_1, x_2 \in D, \forall t \geq 0$$

for all $1 \leq i \leq n$. By the Mean Value Theorem

$$f_i(t, x) = f_i(t, 0) + \frac{\partial f_i}{\partial x}(t, z_i)x$$

where z_i is a point on the line segment connection x to the origin. Since $f(t, 0) = 0$ we can write $f_i(t, x)$ as

$$f_i(t, x) = \frac{\partial f_i}{\partial x}(t, z_i)x = \frac{\partial f_i}{\partial x}(t, 0)x + \left[\frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0)\right] x.$$

Hence,

$$f(t, x) = A(t)x + g(t, x)$$

where

$$A(t) = \frac{\partial f_i}{\partial x}(t, 0), \tag{3.7}$$

a constant matrix and

$$g_i(t,x) = \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right] x. \tag{3.8}$$

The function $g(t,x)$ satisfies

$$\|g(t,x)\| \leq \left(\sum_{i=1}^{n} \left\|\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right\|^2\right)^{1/2} \cdot \|x\| \leq L \|x\|^2, L = \sqrt{n}L_1$$

which implies

$$\lim_{\|x\|\to 0} \frac{\|g(t,x)\|}{\|x\|} = 0. \tag{3.9}$$

Therefore, in a small neighborhood of the origin, we may approximate the nonlinear nonautonomous system (3.5) by its linearization about the origin. The system

$$\dot{z}(t) = A(t)z(t), \text{ where } A(t) = \frac{\partial f}{\partial x}(t,0).$$

is referred to as the *linearization* or the *linearized system* of (3.5) around the equilibrium zero. The following theorem spells out conditions under which we can draw conclusions about stability of the origin as an equilibrium point for the nonlinear system by investigating its stability as an equilibrium point for the linear system.

Theorem 3.2.1. Consider the nonlinear system (3.5). Suppose that (3.6) holds and that $f(\cdot,\cdot)$ is continuously differentiable. Define $A(t)$, $g(t,x)$ as in (3.7), (3.8), respectively, and assume that (i) (3.9) holds, and (ii) $A(\cdot)$ is bounded. Under these conditions, if zero is an exponentially stable equilibrium of the linear system

$$\dot{z}(t) = A(t)z(t)$$

then it is also an exponentially stable equilibrium of the system (3.5).

Proof. (Khalil, 1996).

3.3 The Lyapunov's Direct Method

The Lyapunov's direct method also called the Lyapunov's second method developed by the Russian mathematician, A. M. Lyapunov. The idea behind the various Lyapunov theorems on stability is as follows: Consider a system which is *isolated* in the sense that there are no external forces acting on the system. Suppose that one can identify the various equilibrium states of the system, and that 0 is one of the equilibria (possibly the only equilibrium). Now suppose that it is possible to define, in some sense, the *total energy* of the system, which is a function having the property that it is zero at the origin and positive everywhere else. (In other words, the energy function has either a global or a local minimum at 0). If the system, which was originally in the equilibrium state 0, is perturbed to a new nonzero initial state (where the energy level is positive, by definition), the following possibilities occur; (*i*) If the *system dynamics* are such that the energy of the system is nonincreasing with time, then the energy level of the system never increase beyond the initial positive value. Depending on the nature of energy function, this may be sufficient to conclude that the equilibrium 0 is stable. (*ii*) If the dynamics are such that the energy of the system is monotonically decreasing with time and the energy eventually reduces to zero, this may be sufficient to conclude that the equilibrium 0 is asymptotically stable. (*iii*) If the energy function continues to increase beyond its initial value, then one may be able to conclude that the equilibrium 0 is unstable.

Such an approach to analyzing the qualitative behavior of mechanical system was pioneered by Lagrange, who showed that an equilibrium of a conservative mechanical system is stable if it corresponds to a local minimum of the potential energy function, and that it is unstable if it corresponds to a local maximum of the potential energy function. The genius of Lyapunov lay in his ability to extract from this type of reasoning a general theory that is applicable to any differential equation. This theory requires one to search for a function which satisfies some prespecified properties. This function is now commonly known as a Lyapunov function, and is a generalization of the energy of a dynamical system. Consider the following nonautonomous system described by nonlinear equations

$$\dot{x} = f(t,x)\ ,\ f(t,0) = 0 \tag{3.10}$$

subject to $x(t_0) = x_0$ (modifications for nonautonomous case are straightforward). The aim is to determine the stability nature of the equilibrium point at the origin of (3.10) without obtaining the solution $x(t)$. For the Lyapunov's direct method, we need to construct the Lyapunov function in order to investigate the stability of the equilibrium point of (3.10).

Defintion 3.3.1. The *Lyapunov function* is a real-value function $V(t,x)$ satisfying the following properties : (i). $V(t,x)$ and all its partial derivatives $\frac{\partial V}{\partial t}$ and $\frac{\partial V}{\partial x_i}$ are continuous. (ii). $V(t,x)$ is positive definite, i.e. $V(t,0) = 0$ and $V(t,x) > 0$ for all $x \neq 0$ in some neighborhood $\|x\| \leq k$ of the origin. (iii). The derivative of V along the solution of (3.10), namely

$$\begin{aligned} \dot{V} = \frac{dV}{dt} &= \nabla_t V(t,x) + \nabla_x^T V(t,x) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt} \\ &= \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) \end{aligned} \tag{3.11}$$

is negative semidefinite (i.e. $\dot{V}(0) = 0$, and for all in $\|x\| \leq k$, $\dot{V}(x) \leq 0$).

A function $V(t,x)$ satisfying $V(t,0) = 0$ and $V(t,x) > 0$ for $x \neq 0$, is said to be positive definite. If it satisfies a weaker condition $V(t,x) \geq 0$ for $x \neq 0$, it is said to be positive semidefinite. A function $V(t,x)$ is said to be negative definite or negative semidefinite if $-V(t,x)$ is positive definite or positive semidefinite, respectively. If $V(t,x)$ does not have a definite sign as per one of these four cases, it is said to be indefinite. In definition 3.3.1, the property (i) ensures that V is a smooth function and generally has the shape of bowl near the equilibrium (see Fig. 3.7 (Afanas'ev, Kolmanovskii, and Nosov, 1995). The property (ii) means that, like energy, $V > 0$ if any state is different from zero, but $V = 0$ when the state is zero. The property (iii) guarantees that any trajectory moves so as never to climb higher on the bowl than where it started out. If property (iii) is made stronger so that the derivative of V along the solution of (3.10) is negative definite for

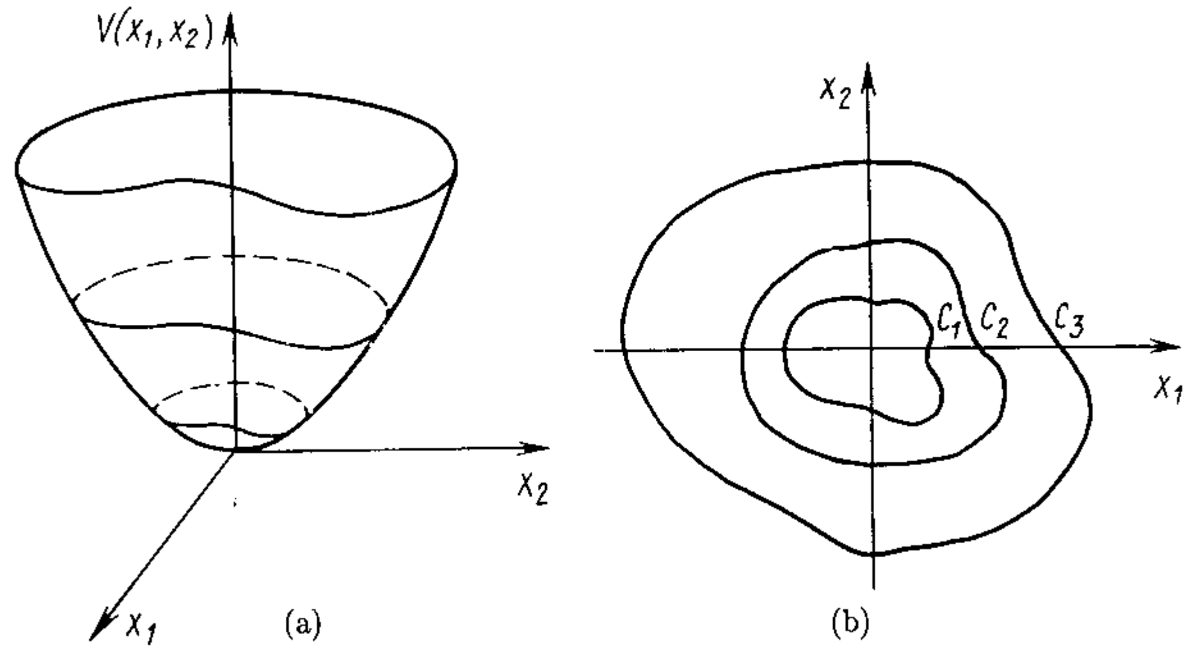

Figure 3.7: (a) Example of Lyapunov function. (b) Level curves for the Lyapunov function $V(t,x) = C_i$, $(i = 1,2,3)$,$C_1 < C_2 < C_3$.

$\|x\| \neq 0$, then the trajectory must be drawn to the origin. the Lyapunov stability states that, given the system of equation (3.10) with $f(t,0) = 0$, if there exists a Lyapunov function for this equation, then the origin is a stable equilibrium point; in addition, if $\dot{V} < 0$, then the stability is asymptotic. Notice that in (3.11) the f_i are the components of f in (3.10), so $\dot{V}$ can be determined directly from the system equations. Note in particular that a function $\phi : [0,\infty) \to [0,\infty)$ is of class $\mathbb{K}$ if it is continuous, strictly increasing, $\phi(0) = 0$, and $\phi(t) > 0$, for all $t > 0$. furthermore, if there exists $a(\cdot) \in \mathbb{K}$ such that

$$a(\|x\|) \leq V(t,x) \tag{3.12}$$

then (3.12) is positive definite (i.e., it is equivalent to the property (ii)). The followings are some theorems on Lyapunov's direct method spelling out conditions under which we can draw conclusions about stability of the origin as an equilibrium point of the nonlinear system.

Theorem 3.3.2. (Lyapunov's first theorem) Suppose there exists a Lyapunov function $V(t,x)$ such that

$$w_1(\|x\|) \leq V(t,x) \qquad (3.13)$$

where $w_1(\cdot) \in \mathbb{K}$ and the derivative of V along the solutions of (3.10) satisfies

$$\dot{V}(t,x) \leq 0. \qquad (3.14)$$

Then the trivial solution of the system (3.10) is stable.

Proof. (Yoshizawa, 1966).

Theorem 3.3.3. (K.P.Peridskii) If in addition to the conditons (3.13) and (3.14) the following inequality holds:

$$V(t,x) \leq w_2(\|x\|), \qquad (3.15)$$

where $w_2(\cdot) \in \mathbb{K}$, then the trivial solution of the system (3.10) is uniformly stable.

Proof. (Khalil, 1996).

Theorem 3.3.4. (Lyapunov's second theorem) suppose there exists a Lyapunov function $V(t,x)$ such that

$$w_1(\|x\|) \leq V(t,x) \leq w_2(\|x\|) \qquad (3.16)$$

and the derivative of V along the solutions of (3.10) satisfies

$$\dot{V}(t,x) \leq -w_3(\|x\|). \qquad (3.17)$$

where where $w_1(\cdot), w_2(\cdot), w_3(\cdot) \in \mathbb{K}$. Then the trivial solution of the system (3.10) is uniformly asymptotically stable.

Proof. (Khalil, 1996).

Theorem 3.3.5. Assume there exitst a sufficiently smooth function $V(t,x)$ and positive constants $\lambda_1, \lambda_2, \lambda_3, p, \varepsilon$ and β, with $\beta > \frac{\lambda_3}{\lambda_2}$, such that for all $x \in \mathbb{R}^n, t \leq t_0$

$$\lambda_1 \|x\|^p \leq V(t,x) \leq \lambda_2 \|x\|^p$$

and the derivative of V along the solutions of (3.10) satisfies

$$\dot{V}(t,x) \leq -\lambda_3 \|x\|^p + \varepsilon e^{-\beta t}.$$

Then the system (3.10) is asymptotically stable.

Proof. (Sun and others, 1998).

3.4 Introduction to Control Theory

In most branches of applied mathematics, the aim is to *analyze* a given situation. To take a very simple example, if a mass is suspended by a string from a fixed point then the assumptions might be that air resistance, the mass of the string and the dimensions of the body could all be neglected, and that gravitational attraction is constant. A familiar mathematical problem would then be to determine the nature of small motions about the equilibrium position. We should need a bridge—a mathematical description, between the real world and the mathematical theory to put control theory into practice. Thus in order to be able to obtain a mathematical description, or *model*, of the real-life situation it is necessary to make certain simplifying assumptions so that established laws from science, engineering, economic theory, etc., can be used. Mathematical methods can then be applied to investigate the properties of the model, and the conclusions reached will reflect reality only insofar as the accuracy of the model permits. Of course the more realistic the model, the more difficult in general will it be to solve the resultant mathematical equations. Many problems of great importance in the contemporary world require a quite different approach, the aim being or *control* a system to behave in some desired manners. Here *system* is used to mean a collection of objects which are related by interactions and produce various

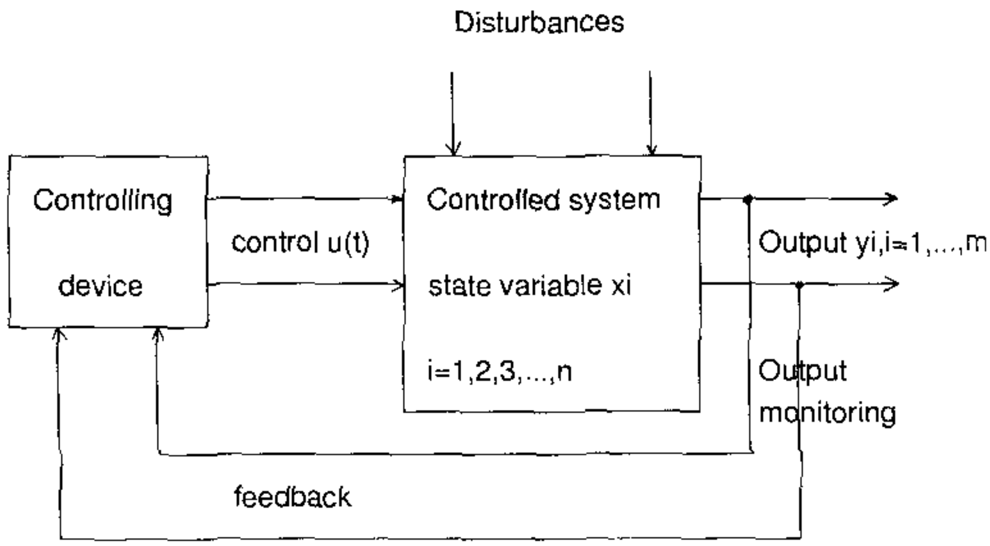

Figure 3.8: The main features of a control systems

outputs in response to different inputs. These situations can include for example industrial complexes such as chemical plants or steelworks; electro-mechanical machines such as motors; biological systems such as the bacteria colony; and economic structures of countries or regions. The complexity of many systems in the present-day world is such that it is desirable for control to be carried out automatically without direct human interactions. The main feature of a control system can be represented as in Fig. 3.8 (Barnett and Cameron, 1993). The state variables x_i describe the internal energy or states of the system, and provide the information which (together with the knowledge of the model describing the system) enables us to predict the future behavior from the knowledge of the inputs. The n-dimensional space containing x_is is state space. In practice, it is often not possible to determine the values of the state variables directly, perhaps for the reasons of expense or inaccessibility. Instead only a set of output variables y_i, which depend in some way on the x_i, is measured and almost invariably $m \leq n$. For example, the state of the economy of a country is described by a great many variables, but it is only possible to measure a few of these, such as the volume of production, the number in employment, the value of gold reserves, and so on. In general the objective is to govern a system to perform in some required manners by suitably manipulating the control variables u_i. To achieve this requires a controlling device, or *controller*. Systems are often subject to external disturbances of unpredictable nature, for example wind gusts during aircraft landing, or variations in the cost of raw materials for a manufacturing process. We shall assume that all our system models have the property that, given an initial state and any input then the resulting state and output at some specified later time are uniquely determined. A control system is a mathematical model which describes the relationship between the states/outputs, $x(t)$, and the controls/inputs, $u(t)$, by the differential equation

$$\dot{x} = f(t, x(t), u(t)), t \in I \subset \mathbb{R}$$

where $x(t) \in \mathbb{X}, u(t) \in \mathbb{U}$. The set $\mathbb{X}$ is called the set of states/outputs and $\mathbb{U}$ the set of controls/inputs. A control $u(t)$ that is restricted to take values in a preassigned subset Ω of the control space $\mathbb{U}$ is called *admissible*. If the controller operates according to some pre-set pattern without taking account of the output or state, the system is called *open loop*, because the *loop* in Fig. 3.8 is not completed. In an open–loop control system, the input or control, $u(t)$ is selected based on the goals for the system and all available a priori knowledge about the system. If however there is *feedback* of information concerning the outputs to the controller, which the control $u(t)$ is modified in accordance with the information of the output, the system is closed loop. Simple illustrations of open and closed loop system are provided by traffic lights which change at fixed intervals of time, and those which are controlled by some device which measures traffic flow and reacts accordingly. Consider the following nonlinear dynamical system

$$\dot{x} = f(t, x(t), u(t)), \forall t \geq 0, \tag{3.18}$$

where t denotes time; $x(t)$ denotes the value of the state function $x(\cdot)$ at time t and is an n-dimensional vector; $u(t)$ denotes the value of the control function $u(\cdot)$ and is m-dimensional vector; and the function f associate with each value of t, $x(t)$ and $u(t)$, a corresponding n-dimensional vector. Following the common convention, this is denoted as: $t \in [0, \infty), x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$, and $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The system (3.18) is said to be *forced*, or to have an input. While a system described by an equation of the form

$$\dot{x} = f(t, x(t)), \forall t \geq 0, \tag{3.19}$$

is said to be *unforced.* Note that the distinction is not too precise. In the system (4.18), if $u(\cdot)$ is specified, then it is possible to define a function $f_u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_u(t, x) = f(t, x, u(t)). \tag{3.20}$$

In this case (3.18) becomes

$$\dot{x} = f_u(t, x), \forall t \geq 0. \tag{3.21}$$

Moreover, if $u(\cdot)$ is clear from the context, the subscript u of f_u is often omitted. In this case there is no distinction between (3.21) and (3.20). Thus it is safer to think of (3.19) as (i) there is no external input to the system, or (ii) there is an external input, which is kept fixed throughout the study. In control theory one is interested in the question of how the input of the system should be chosen to assure that the corresponding output has the desired properties. Depending on the properties involved one can define some of the specific problems as follow.

Controllability The controllability problem is concerned with the question of existence of controls which steer an arbitrary state of the system into a given one. The system is called controllable if it is possible to force the system from an arbitrary state to a desired state in finite time by some admissible control. In practice, one requires various types of steering the states:

- From an arbitrary state to the origin (*Null-Controllability*). The system is null-controllable if every state $x \in \mathbb{X}$ can be controllable to zero by some control $u(t) \in \mathbb{U}$ in some finite time $T > 0$, i.e., one can find a finite time $T > 0$ and an admissible control $u(t)$ such that the solution $x(t)$ corresponding to this control satisfies $x(t_0) = x_0$, and $x(T) = 0$.

- From the origin to an arbitrary state (*Reachability*). The system is reachable if one can find a finite time $T > 0$, and admissible control $u(t)$ such that the solution $x(t)$ of the system satisfies $x(t_0) = 0$, and $x(T) = x_0$.
- From any state to an arbitrary state (*Global Controllability*). The system is globally controllable if one can find a finite time $T > 0$ and an admissible control $u(t)$ such that the solution $x(t)$ of control system satisfies $x(t_0) = x_0$, and $x(T) = x_1$.
- From one state to a neighborhood of another state (*Approximate Controllability*). The system is approximately controllable if one can find a finite time $T > 0$, and an control $u(t)$ such that $x(t_0) = x_0$, and $x(T) = B_{x_1}$.
- etc.

Stabilizability The stabilization problem is how the feedback control $u(t) = h(t, x(t))$ can be determined in order to stabilize the closed-loop system

$$\dot{x} = f(t, x(t), h(t, x(t)))$$

in the sense of Lyapunov.

Optimality In optimal control problem one is looking for an admissible control $u(\cdot)$ which not only steers a state x_0 to a state x_1 but also does it in the minimal time, or with minimal resources. Reality often puts some practical constraints onto the problem of control optimization. This enforces us occasionally to relax the requirement of optimality, under which it is called near optimal or suboptimal.

Chapter IV

Problem Formulation and Main Results

In recent decades, the stability problems of nonlinear systems have been extensively studied (Bellman, 1963; Curtain and Pritchard, 1977; Lakshminkantham, 1989; Zabczyk, 1992). It is well known that the study of stability theory of nonlinear dynamical systems is carried out by one of the two Lyapunov methods. The first one is the Lyapunov's linearization method. The other is the Lyapunov's direct method based on the construction of the Lyapunov function. The stability problem has motivated the study of Lyapunov function in both finite (Bellman, 1953; Sun and others, 1998; Yoshizawa, 1966) and infinite dimensional spaces (Curtain and Pritchard, 1977; Sattayatham and Huawu, 1999). The Lyapunov's direct method is used with this work. It is the purpose of this work to investigate the exponential stabilization for nonlinear dynamical systems with control constraint. This chapter is organized as follows. In section 4.2, we propose a theorem, which is a criterion for the exponential stability. Based on this theorem, a bounded and continuous state feedback control is proposed to guarantee the exponential stability. This is described in section 4.3. In section 4.4, a numerical example is given to illustrate the use of our main result.

4.1 Problem Formulation

Consider a class of uncertain nonlinear dynamical systems described by the following state equations :

$$\begin{cases} \dot{x}(t) = f(t,x) + F(t,x) \cdot \Phi(t,x,u), \;\; t \geq t_0 \geq 0 \\ x(t_0) = x_0 \end{cases} \tag{4.1}$$

where $t \in \mathbb{R}$ is time, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, and $\Phi(t,x,u)$ represents the system uncertainties. The assumption (B3), below, is a general structure condition on $\Phi(t,x,u)$. Moreover, the function $\Phi(\cdot,\cdot,\cdot)$: $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $F(\cdot,\cdot) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $f(\cdot,\cdot) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be continuous. The corresponding system of (4.1) without uncertainties, called the nominal system, is described by

$$\begin{cases} \dot{x}(t) = f(t,x), \;\; t \geq t_0 \geq 0 \\ x(t_0) = x_0. \end{cases} \tag{4.2}$$

We assume further that the equation (4.2) has a unique solution corresponding to each initial condition and the origin is the unique equilibrium point. The state feedback controller can be represented by a nonlinear function in the form

$$u(t) = -\gamma(t,x)K^T(t,x).$$

Now, the question is how to synthesize a state feedback controller $u(t)$ that can guarantee the exponential stability of nonlinear dynamical system (4.1) in the presence of uncertainties $\Phi(t,x,u)$. Before giving our synthesis approach, we prove a sufficient conditions for the global exponential stability of system (4.2).

4.2 Sufficient Conditions for Exponential Stability

Theorem 4.1. Assume there exist a sufficiently smooth function $V(t,x)$, positive constants $\lambda_1, \lambda_2, \lambda_3, p$, and q such that for all $x(t) \in D$, D is an open connected subset of $\mathbb{R}^n$ containing the $B_1(0)$, for all $t \geq t_0 \geq 0$

$$\lambda_1 \|x(t)\|^p \leq V(t, x(t)) \leq \lambda_2 \|x(t)\|^q \tag{4.3}$$

and the derivative of V along the solution of (4.2) satisfies

$$\frac{dV(t,x(t))}{dt} = \nabla_t V(t,x(t)) + \nabla_x^T V(t,x(t)) \cdot f(t,x(t)) \leq -\lambda_3 \|x(t)\|^q. \tag{4.4}$$

Then the equilibrium point of the system (4.2) is exponentially stable if $p \leq q$ and asymptotically stable if $p > q$ for all $x(t_0) \in B_1(0)$.

Proof. Let

$$Q(t, x(t)) = V(t, x(t)) e^{\frac{\lambda_3}{\lambda_2} t}. \tag{4.5}$$

Then, from (4.5), (4.4) and (4.3), we have

$$\begin{aligned} \dot{Q}(t, x(t)) &= \dot{V}(t, x(t)) e^{\frac{\lambda_3}{\lambda_2} t} + \frac{\lambda_3}{\lambda_2} V(t, x(t)) e^{\frac{\lambda_3}{\lambda_2} t} \\ &\leq -\lambda_3 \|x(t)\|^q e^{\frac{\lambda_3}{\lambda_2} t} + \frac{\lambda_3}{\lambda_2} \lambda_2 \|x(t)\|^q e^{\frac{\lambda_3}{\lambda_2} t} \\ &\leq 0. \end{aligned} \tag{4.6}$$

Integrate both sides of (4.6), we have, for all $t \geq t_0 \geq 0$

$$Q(t, x(t)) \leq Q(t_0, x(t_0)) = V(t_0, x(t_0)) e^{\frac{\lambda_3}{\lambda_2} t_0} \leq \lambda_2 \|x(t_0)\|^q e^{\frac{\lambda_3}{\lambda_2} t_0} \tag{4.7}$$

Hence, it follows from (4.3), (4.5), and (4.7), we get

$$\begin{aligned} \|x(t)\| &\leq \left(\frac{V(t, x(t))}{\lambda_1}\right)^{1/p} = \left(\frac{Q(t, x(t)) e^{-\lambda_2 t}}{\lambda_1}\right)^{1/p} \\ &\leq \left(\frac{\lambda_2 \|x(t_0)\|^q e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}}{\lambda_1}\right)^{1/p} \\ &= \left(\frac{\lambda_2}{\lambda_1}\right)^{1/p} \|x(t_0)\|^{q/p} e^{-\frac{\lambda_3}{\lambda_2 p}(t-t_0)}. \end{aligned} \tag{4.8}$$

If $p \leq q$ and $x(t_0) \in B_1(0)$, we have $\|x(t_0)\|^{q/p} \leq \|x(t_0)\|$, hence

$$\|x(t)\| \leq k \|x(t_0)\| e^{-\gamma(t-t_0)}, \forall x(t_0) \in B_1(0)$$

where $k = \left(\frac{\lambda_2}{\lambda_1}\right)^{1/p}, \gamma = \frac{\lambda_3}{\lambda_2 p}$. Then the equilibrium zero of (4.2) is exponentially stable. If $p > q$, from (4.8), we have, $\forall x(t_0) \in B_1(0)$,

$$\lim_{t\to\infty} \|x(t)\| = 0. \tag{4.9}$$

Now let $\varepsilon > 0$ be given. Choose $\delta(\varepsilon, t_0) < \left(\frac{\varepsilon}{k\exp(\gamma t_0)}\right)^{p/q}$. From (4.8) again, if for all $\|x(t_0)\| \leq \delta(\varepsilon, t_0)$, we have

$$\|x(t)\| < k\frac{\varepsilon}{k\exp(\gamma t_0)} e^{-\gamma(t-t_0)} = \varepsilon e^{-\gamma t} \leq \varepsilon, \forall t \geq t_0 \geq 0.$$

That is, in the case of $p > q$, the equilibrium zero of (4.2) is stable and by (4.9), it is asymptotically stable. This completes our proof. □

4.3 Stabilization Problem

We shall use Theorem 4.1 to find the condition on $u(t)$ that can guarantee the exponential stability of nonlinear dynamical system (4.1). Let us introduce for system (4.1) the following assumptions :

(B1) *The components of the control vector are physically limited by*

$$|u_i| < c_i\ ,\ \ \forall i = 1, 2, ..., m \tag{4.10}$$

with $c_i > 0$, $\forall i = 1, 2, ..., m$.

(B2) There exist a sufficiently smooth function $W(t, x)$, positive constants λ_1, λ_2, p, and q such that for all $x \in D \subset \mathbb{R}^n$, for all $t \geq t_0 \geq 0$, we have

$$\lambda_1\|x\|^p \leq W(t, x) \leq \lambda_2\|x\|^q \tag{4.11}$$

and the derivative of W along the solution of $\dot{x}(t) = f(t, x)$ satisfies

$$\frac{dW(t, x(t))}{dt} = \nabla_t W(t, x(t)) + \nabla_x^T W(t, x(t)) \cdot f(t, x(t)) \leq 0. \tag{4.12}$$

Remark :The nominal system $\dot{x}(t) = f(t, x)$ is stable in the sense of Lyapunov with (B2). (See Yoshizawa, 1966, p. 32)

(B3) There exist positive continuous functions $f_1(t,x)$, $f_2(t,x)$, $f_3(t,x)$ and a positive constants β, λ_3 and α with $\lambda_3 > \alpha \geq 1$ such that

$$y^T\Phi_1(t,x,y) \geq -f_1(t,x)\|y\| + f_2(t,x)\|y\|^2 - f_3(t,x)\|y\|^3 + \frac{\lambda_3 f_2(t,x)\|x(t)\|^q}{2f_3(t,x)\left[\|\nabla_x^T W(t,x)F(t,x)\| + [f_1(t,x)]^{-1}\|x(t)\|^q + e^{-\beta t}\right]} \quad \forall y \in \mathbb{R}^m, \forall x \in D \subset \mathbb{R}^n, \forall t \geq t_0 \geq 0, \tag{4.13}$$

where

$$f_2^2(t,x) \geq 4\,f_1(t,x)f_3(t,x), \quad \forall x \in D \subset \mathbb{R}^n, \forall t \geq t_0 \geq 0, \tag{4.14}$$

$$f_1(t,x)\left\|\nabla_x^T W(t,x)F(t,x)\right\| \leq \alpha\|x(t)\|^q \tag{4.15}$$

$$\Phi_1(t,x,y) := \Phi(t,x,\frac{2c_1}{\pi}\tan^{-1}y_1, \frac{2c_2}{\pi}\tan^{-1}y_2, ..., \frac{2c_m}{\pi}\tan^{-1}y_m), \tag{4.16}$$

and

$$y := [y_1, y_2, \cdots, y_m]^T \in \mathbb{R}^m. \tag{4.17}$$

Lemma 4.2. Under the assumptions (B2) and (B3),

$$f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 - \alpha\|x(t)\|^q \leq 0$$

where $\gamma(t,x) := \frac{f_2(t,x)}{2f_3(t,x)[\|\nabla_x^T W(t,x)F(t,x)\| + \varepsilon^*(t,x)]}$, $\varepsilon^*(t,x) := [f_1(t,x)]^{-1}\|x(t)\|^q + e^{-\beta t}$, $K(t,x) := F^T(t,x)\nabla_x W(t,x)$ and $\beta > 0, \alpha \geq 1$.

Proof. $f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 - \alpha\|x(t)\|^q =$

$$\begin{aligned} &= f_1\|K\| - \frac{f_2^2\|K\|^2}{2f_3(\|K\| + \varepsilon^*)} + \frac{f_2^2\|K\|^3}{4f_3(\|K\| + \varepsilon^*)^2} - \alpha\|x(t)\|^q \\ &= \frac{4f_1f_3\|K\|^3 + 4f_1f_3 2\|K\|^2\varepsilon^* + 4f_1f_3\|K\|\varepsilon^{*2} - 2f_2^2\|K\|^3}{4f_3(\|K\| + \varepsilon^*)^2} \\ &\quad + \frac{-2f_2^2\|K\|^2\varepsilon^* + f_2^2\|K\|^3}{4f_3(\|K\| + \varepsilon^*)^2} - \alpha\|x(t)\|^q \end{aligned}$$

$$
\begin{aligned}
&= \frac{-\|K\|^3 [f_2^2 - 4f_1f_3] - 2\varepsilon^* \|K\|^2 [f_2^2 - 4f_1f_3]}{4f_3 (\|K\| + \varepsilon^*)^2} + \frac{4f_3f_1 \|K\| \varepsilon^{*2}}{4f_3 (\|K\| + \varepsilon^*)^2} \\
&\quad -\alpha\|x(t)\|^q \leq \frac{f_1 \|K\| \varepsilon^{*2} - \alpha\|x(t)\|^q (\|K\| + \varepsilon^*)^2}{(\|K\| + \varepsilon^*)^2} \\
&= \frac{f_1 \|K\| \varepsilon^{*2} - \alpha\|x(t)\|^q \|K\|^2 - 2\alpha\|x(t)\|^q \|K\| \varepsilon^* - \alpha\|x(t)\|^q\varepsilon^{*2}}{(\|K\| + \varepsilon^*)^2} \\
&= \frac{\|K\| (f_1)^{-1}\|x(t)\|^{2q} + 2 \|K\| \|x(t)\|^q e^{-\beta t} + \|K\| f_1 (e^{-\beta t})^2}{(\|K\| + \varepsilon^*)^2} \\
&\quad \frac{-\alpha\|x(t)\|^q \|K\|^2 - 2\alpha \|K\| (f_1)^{-1}\|x(t)\|^{2q} - 2\alpha\|x(t)\|^q \|K\| e^{-\beta t}}{(\|K\| + \varepsilon^*)^2} \\
&\quad \frac{-\alpha\|x(t)\|^{3q}(f_1)^{-2} - 2\alpha\|x(t)\|^{2q}(f_1)^{-1}e^{-\beta t} - \alpha\|x(t)\|^q(e^{-\beta t})^2}{(\|K\| + \varepsilon^*)^2} \\
&\leq 0. \ \square
\end{aligned}
$$

Theorem 4.3. The System (4.1) satisfying the assumptions (B1) - (B3) is exponentially stable if $p \leq q$ and asymptotically stable if $p > q$ under the control

$$u_i(t) = \frac{2c_i}{\pi} \tan^{-1}[y_i(t)], \quad \forall i = 1, 2, ..., m. \tag{4.18}$$

Here

$$[y_1(t), y_2(t), ..., y_m(t)] = -\gamma(t, x) K^T(t, x), \tag{4.19}$$

$$\gamma(t, x) := \frac{f_2(t, x)}{2f_3(t, x) [\|\nabla_x^T W(t, x) F(t, x)\| + \varepsilon^*(t, x)]}, \tag{4.20}$$

$$\varepsilon^*(t, x) := [f_1(t, x)]^{-1} \|x(t)\|^q + e^{-\beta t}, \tag{4.21}$$

and

$$K(t, x) := F^T(t, x) \nabla_x W(t, x) \tag{4.22}$$

with $1 \leq \alpha < \lambda_3$.

Proof. By (4.1) and (4.16)-(4.18), one has

$$
\begin{aligned}
\dot{x}(t) &= f(t, x) + F(t, x) \cdot \Phi(t, x, u_1, u_2, ..., u_m) \\
&= f(t, x) + F(t, x) \cdot \Phi \left(t, x, \frac{2c_1}{\pi} \tan^{-1} y_1, \frac{2c_2}{\pi} \tan^{-1} y_2, ..., \frac{2c_m}{\pi} \tan^{-1} y_m \right)
\end{aligned}
$$

$$= f(t,x) + F(t,x) \cdot \Phi_1(t,x,y), \quad \forall x \in D \subset \mathbb{R}^n, \; t \geq t_0 \geq 0.$$

Let $W(t,x)$ be a Lyapunov function candidate of (4.1) with (4.18)-(4.22). The time derivative of $W(t,x)$ along the trajectories of the closed-loop system, using (B2), is given by

$$\begin{aligned} \dot{W} &= \nabla_t W + \nabla_x^T W \left[f + F \cdot \Phi_1 \right] \\ &\leq \nabla_x^T W F \cdot \Phi_1. \end{aligned} \tag{4.23}$$

From (4.13), (4.19), (4.20), and (4.21), we have

$$y^T \cdot \Phi_1 \geq -f_1\gamma \|K\| + f_2\gamma^2 \|K\|^2 - f_3\gamma^3 \|K\|^3 + \lambda_3\gamma\|x(t)\|^q.$$

Multiply both sides by $-\frac{1}{\gamma}$ and from (4.19), and (4.22), we have

$$K^T \cdot \Phi_1 = \nabla_x^T W F \cdot \Phi_1 \leq f_1 \|K\| - f_2\gamma \|K\|^2 + f_3\gamma^2 \|K\|^3 - \lambda_3\|x(t)\|^q. \tag{4.24}$$

Substitute (4.24) into (4.23), we get

$$\begin{aligned} \dot{W} &\leq f_1 \|K\| - f_2\gamma \|K\|^2 + f_3\gamma^2 \|K\|^3 - \lambda_3\|x(t)\|^q + \alpha\|x(t)\|^q - \alpha\|x(t)\|^q \\ &= -(\lambda_3 - \alpha)\|x(t)\|^q + f_1 \|K\| - f_2\gamma \|K\|^2 + f_3\gamma^2 \|K\|^3 - \alpha\|x(t)\|^q \end{aligned} \tag{4.25}$$

Simplifying (4.25) by using (4.20), (4.21), (4.22), we get, by Lemma 4.2,

$$\dot{W} \leq -(\lambda_3 - \alpha)\|x(t)\|^q. \tag{4.26}$$

By virtue of theorem 4.1, the proof is completed. □

4.4 Example

Consider the following uncertain nonlinear system:

$$\begin{aligned} \dot{x}\,(t) &= \begin{pmatrix} x_2 - x_1^3 \\ -2x_1 - \frac{x_2^3}{2} \end{pmatrix} + \\ &\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \left\{ a(t)u + b(t)u^2 + c(t)\tan u - 17 \right\} \end{aligned} \tag{4.27}$$

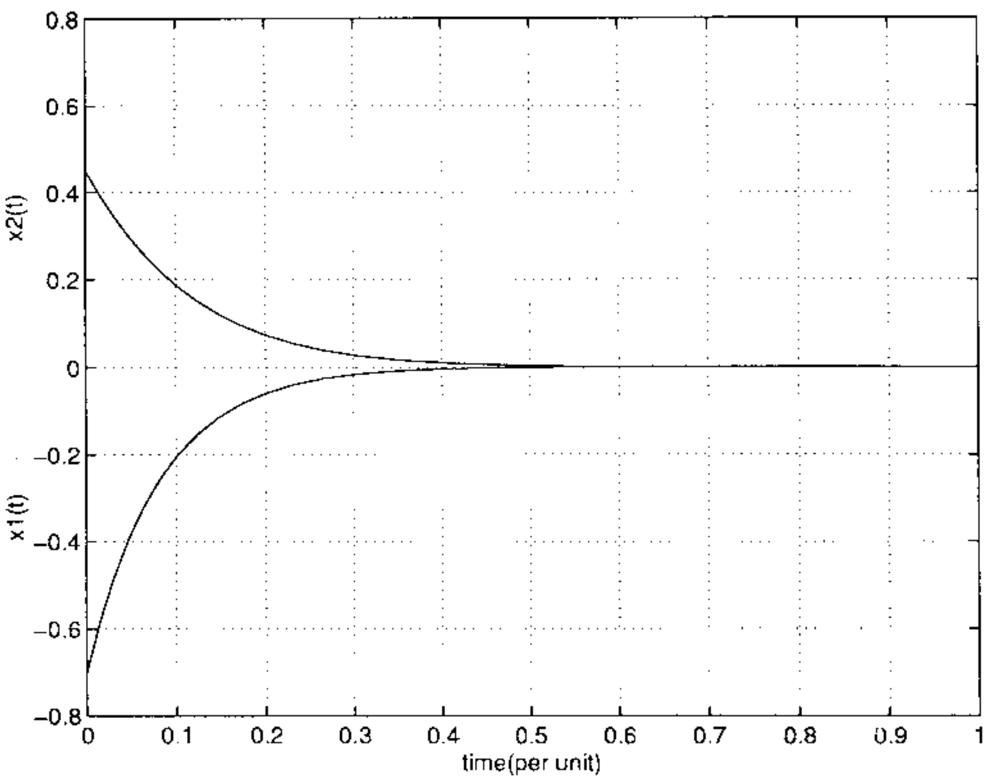

Figure 4.1: The state trajectories of the feedback-controlled system for (4.27).

where $u \in \mathbb{R}, x := (x_1, x_2)^T \in \mathbb{R}^2, -1 \leq a(t) \leq 1, -1 \leq b(t) \leq 1$, and $5 \leq c(t) \leq 6$ for all $t \geq t_0 \geq 0$. The coefficients $a(t), b(t)$, and $c(t)$ are arbitrarily chosen to satisfy (4.13), (4.14) and (4.15). The control u is limited by $-\frac{\pi}{2} < u(t) < \frac{\pi}{2}$, and

$$f(t,x) = \begin{pmatrix} x_2 - x_1^3 \\ -2x_1 - \frac{x_2^3}{2} \end{pmatrix}, \quad F(t,x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\Phi(t,x,u) = \left\{a(t)u + b(t)u^2 + c(t)\tan u - 17\right\}.$$

Choose a positive functional

$$W(t,x) = 2x_1^2 + x_2^2.$$

Then (4.11) and (4.12) are satisfied with $\lambda_1 = 1, \lambda_2 = 2, p = 2$, and $q = 2$. In fact,

$$\lambda_1 \|x\|^p = x_1^2 + x_2^2 \leq W(t,x) = 2x_1^2 + x_2^2 \leq 2(x_1^2 + x_2^2) = \lambda_2 \|x\|^q.$$

and

$$\begin{aligned}
\nabla_x^T W(t,x) f(t,x) &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) W(t,x) \cdot f(t,x) \\
&= (4x_1\ ,\ 2x_2) \begin{pmatrix} x_2 - x_1^3 \\ -2x_1 - \frac{x_2^3}{2} \end{pmatrix} \\
&= 4x_1\left(x_2 - x_1^3\right) + \ 2x_2 \left(-2x_1 - \frac{x_2^3}{2}\right) \\
&= -4x_1^4 - x_2^4 \leq 0.
\end{aligned}$$

From (4.16), we have

$$\begin{aligned}
\Phi_1(t,x,y) \quad &: \quad = \Phi(t,x,\tan^{-1} y) \\
&= a(t)\tan^{-1} y + b(t)(\tan^{-1} y)^2 + c(t)y - 17
\end{aligned}$$

Hence, in (4.13), we have

$$\begin{aligned}
y^T \cdot \Phi_1(t,x,y) &= \left[a(t)\tan^{-1} y + b(t)(\tan^{-1} y)^2\right] y + c(t)y^2 - 17y \\
&\geq -\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right)|y| + 5|y|^2 - 17y \\
&\geq -\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right)|y| + 5|y|^2 - |y|^3 - 17y.
\end{aligned}$$

This suggests that in (4.13) we choose

$$f_1(t,x) = \frac{\pi}{2} + \frac{\pi^2}{4}, \quad f_2(t,x) = 5\,, \text{ and }, \ f_3(t,x) \ = 1.$$

It follows that (4.14) is satisfied. In fact,

$$\begin{aligned}
f_2^2(t,x) &= 25 \geq 4\, f_1(t,x) f_3(t,x) = 4\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right) \cdot 1 \\
&\approx 16.15.
\end{aligned}$$

According to (4.21) with $\beta = 1$, we have

$$\varepsilon^*(t,x) := \frac{x_1^2 + x_2^2}{2 + \frac{\pi}{2} + \frac{\pi^2}{4}} + e^{-t}.$$

By (4.22) and (4.20), we obtain

$$K(t,x) = 4x_1^2 + 2x_2^2,$$

and

$$\gamma(t,x) = \frac{5}{2\left(|K(t,x)| + \varepsilon^*(t,x)\right)}.$$

With (4.19), (4.13) become

$$\begin{aligned} y^T \cdot \Phi_1(t,x,y) &\geq -\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right)|y| + 5|y|^2 - |y|^3 + 17\gamma K \\ &= -\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right)|y| + 5|y|^2 - |y|^3 + \frac{17 \cdot 5\ (4x_1^2 + 2x_2^2)}{2\left(|K(t,x)| + \varepsilon^*(t,x)\right)} \\ &\geq -\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right)|y| + 5|y|^2 - |y|^3 + \frac{17 \cdot 5 \cdot 2\ (x_1^2 + x_2^2)}{2\left(|K(t,x)| + \varepsilon^*(t,x)\right)} \end{aligned}$$

which imply that (4.13) holds with $\lambda_3 = 34$. Choosing $1 \leq \alpha = 17 \leq \lambda_3$, such that (4.15) holds, i.e.,

$$f_1 \left\|\nabla_x^T W(t,x) F(t,x)\right\| \simeq 4.03\left(4x_1^2 + 2x_2^2\right) \leq 17\left(x_1^2 + x_2^2\right) = \alpha\|x\|^q.$$

Finally, owing to (4.18) and (4.19), it can be obtained that

$$\begin{aligned} u(t) &= \tan^{-1}[y(t)] \\ &= \tan^{-1}\left[-\gamma(t,x)K^T(t,x)\right]. \end{aligned} \tag{4.28}$$

By Theorem 4.3, we conclude that (4.27) with the bounded control (4.28) is exponentially stable. With $a(t) = b(t) = 1$, $c(t) = 5, x_1(0) = -0.70$, and $x_2(0) = 0.45$, the state trajectories of the feedback-controlled system is depicted in Fig. 4.1 (see Appendix A for Matlab scripts to solve (4.27) and sketch the figure 4.1). It can be seen from equation (4.28) that $u(t)$ is bounded by $-\frac{\pi}{2} < u(t) < \frac{\pi}{2}$.□

Chapter V

Conclusion

In this thesis, the exponential stabilization of nonlinear time-varying differential equations with control constraint has been investigated. We have proposed a bounded and continuous state feedback control for the exponential stability for the closed-loop system. A numerical example has also been given to demonstrate the use of our main result.

The result of this thesis will be useful for those who are dealing with dynamical systems governed by nonlinear differential equations with input nonlinearly uncertainties. However, it has some limitation. For example, we have to construct a Lyapunov function satisfying the conditions (4.3) and (4.4) and three positive continuous functions must be found with some positive constants to meet the conditions (4.13)-(4.15).

Besides, recently, the stability problem of retarded systems has been widely investigated. In further research, one may consider the issue of exponential stability of a class of uncertain systems with time-delays. The systems which are described by functional differential equations with uncertainties in both the current and delayed state. For example, one may consider the delay system

$$\dot{x}(t) = A(t)x(t) + f(t, x(t-h(t))), t \geq 0. \tag{5.1}$$

where $A(t) \in L(X,X), f(\cdot) : [0,\infty) \times X \to X$, X is a Banach space, and $h(\cdot) : [0,\infty) \to \mathbb{R}^+, 0 \leq h(t) \leq a, f(t,0) = 0$. One may first assume the autonomous case that $A(t) = A$ (later $A = A(t)$ in Banach (reflexive) space) is a stable operator, X is a Hilbert space and

$$\|f(t,x)\| \le a(t)\, \|x\|^m .$$

Then find conditions on $a(t), m$ and a such that the system (5.1) is asymptotically stable. After that apply the stability result above to stabilize the control system

$$\dot{x}(t) = Ax(t) + Bu(t) + f(t, x(t - h(t), u(t - h(t)))),$$

by using feedback control $u(t) = g(x(t))$. Moreover, the stability and stabilization of the results of this thesis and the results above to the case of discrete-time systems

$$x(k+1) = f(k, x(k), u(k)), k \in Z^+.$$

would be of interest.

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Appendix

Appendix

Matlab Scripts

The function ODE23 solve differential equations, low order method. ODE23 integrates a system of ordinary differential equations using 2nd and 3rd order Runge-Kutta formulas. The matlab command

$$[T, X] = ODE23('xprime', T0, Tfinal, X0)$$

integrates the system of ordinary differential equations described by the M-file $xprime.m$, over the interval $T0$ to $Tfinal$, with initial conditions $X0$. Also the command

$$[T, X] = ODE23(F, T0, Tfinal, X0, TOL, 1)$$

uses tolerance TOL and displays status while the integration proceeds.

INPUT:

F - String containing name of user-supplied problem description.

Call: xprime = fun(t,x) where F = 'fun'.

t - Time (scalar).

x - Solution column-vector.

$xprime$ - Returned derivative column-vector; $xprime(i) = dx(i)/dt$.

$t0$ - Initial value of t.

$tfinal$- Final value of t.

$x0$ - Initial value column-vector.

tol - The desired accuracy. (Default: tol = 1.e-3).

$trace$ - If nonzero, each step is printed. (Default: trace = 0).

OUTPUT:

T - Returned integration time points (column-vector).

X - Returned solution, one solution column-vector per tout-value.

PLOT:

Plot vectors or matrices. PLOT(X,Y) plots vector X versus vector Y. If X or Y is a matrix, then the vector is plotted versus the rows or columns of the matrix, whichever line up.

$$PLOT(X1, Y1, S1, X2, Y2, S2, X3, Y3, S3, ...)$$

combines the plots defined by the (X,Y,S) triples, where the X's and Y's are vectors or matrices and the S's are strings. For example,

$$PLOT(X, Y, 'y-', X, Y, 'go')$$

plots the data twice, with a solid yellow line interpolating green circles at the data points.

XLABEL:

X-axis labels for 2-D and 3-D plots. XLABEL('text') adds text beside the X-axis on the current axis.

YLABEL:

Y-axis labels for 2-D and 3-D plots. YLABEL('text') adds text beside the Y-axis on the current axis.

The following are matlab programs used to solve the system *(4.27)*. The first program, *equations.m*, describes the state equations of (4.27). The second program, *solve_it.m*, calls for ODE23.M to solve and plots the solutions.

equations.m

function xprime $= equations(t, x)$

$k = 4 * x(1)\hat{}2 + 2 * x(2)\hat{}2;$

$esp = (x(1)\hat{}2 + x(2)\hat{}2)/(2 + pi/2 + pi\hat{}2/4) + exp(-t);$

$gam = 5/(2 * (abs(k) + eps));$

$u = atan(-gam * k);$

```
delt = u + u^2 + 5 * tan(u) - 17;
g1 = x(2) - x(1)^3;
g2 = -2 * x(1) - 0.5 * x(2)^3;
xprime = [g1; g2] + [x(1); x(2)] * delt;
```

solve_it.m

```
[t, w]=ode23( 'equations',0,3,[-0.7 0.45]');
x = w(:, 1);
y = w(:, 2);
k = 4 * x.^2 + 2 * y.^2;
esp = (x.^2 + y.^2)./(2 + pi/2 + pi^2/4) + exp(-t);
gam = 5./(2 * (abs(k) + eps));
plot(t, x, 'w', t, y, 'w'); axis([0 1-0.8 0.8]);grid
xlabel('time ( perunit )');
ylabel('x1(t)        x2(t)');
```

Curriculum Vitae

Name : Rattikan Saelim

Date of Birth : May 21,1974 Age : 27

Place of Birth : Satun, Thailand

Religion : Buddhism

Marital status : Single

Education : Bachelor of Science (Applied Mathematics), Prince of Songkla University, Pattani Campus (1993-1996).

Experience : Lecturer at Prince of Songkla University, Pattani Campus (1997-1998).

Area of interest : Stability theory of linear and nonlinear differential equations.