CONTACT AND LIE-BÄCKLUND TRANSFORMATIONS OF THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS

Mr. Ekkarath Thailert

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I

เอกรัฐ ไทยเลิศ : การแปลงแบบคอนแทคท์และการแปลงแบบลีย์-แบคก์ลันด์ของสมการนาเวียร์-สโตกส์ชนิดสองมิติ (CONTACT AND LIE-BÄCKLUND TRANSFORMATIONS OF THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS)

อ. ที่ปรึกษา : PROF. DR. SERGEY V. MELESHKO, 78 หน้า

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วิทยานิพนธ์ศึกษาสมการนาเวียร์-สโตกส์แบบสองมิติ ซึ่งพบว่าสมการนี้มีความสัมพันธ์กับสมการที่เขียนอยู่ในรูปของฟังก์ชันเส้นกระแส ในส่วนแรกเริ่มด้วยการศึกษาการหาการแปลงแบบคอนแทคท์ที่ยอมรับโดยสมการที่เขียนให้อยู่ในรูปฟังก์ชันเส้นกระแส ผลลัพธ์ของส่วนนี้แสดงว่าการแปลงแบบคอนแทคท์คือการต่อออกไปของการแปลงแบบจุดอันดับหนึ่ง ในส่วนที่สองได้ศึกษาการแปลงแบบลีย์-แบคก์ลันด์อันดับสอง ซึ่งผลลัพธ์ที่ได้มีลักษณะเดียวกันกับส่วนแรกคือ การแปลงแบบลีย์-แบคกลันด์อันดับสองแท้ที่จริงแล้วคือการต่อออกไปของการแปลงแบบจุดอันดับหนึ่ง

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TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS / GROUP ANALYSIS / CONTACT AND LIE-BÄCKLUND TRANSFORMATIONS /

In this thesis the two-dimensional Navier-Stokes equations are studied. These equations are considered in equivalent form written as one equation for the streamline function. The first part of the research is devoted to finding contact transformations admitted by the equation for the streamline function. As a result of the calculations it is obtained that admitted contact transformations are the prolongations of point transformations. The second part of the thesis is focused on seeking second order Lie-Bäcklund transformations admitted by the equation for the streamline function. The calculations show that the second order Lie-Bäcklund transformations are also the first prolongations of point transformations.

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Ekkarath Thailert

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Chapter I

Introduction

Almost all important models in physics take the form of nonlinear partial differential equations. After modelling one needs to find solutions of these equations. There are two ways for finding solutions: applying numerical methods and methods for constructing exact solutions. One of the methods for constructing exact solutions is group analysis of differential equations. The idea of group analysis was originally formulated by the outstanding mathematician of the 19th century, Sophus Lie. However, systematic applications of Lie groups to differential equations started less than 50 years ago (cf. Ovsiannikov (1978)).

The theory of Lie groups for constructing exact solutions of partial differential equations has been applied to many differential equations (most of its applications can be found in the Handbook of Group Analysis, (1994),(1995),(1996)). A historical review of the development of group analysis can be found in Ibragimov (1999). Roughly, these applications are separated into applications of point transformations, contact transformations and Lie-Bäcklund transformations. Many applications of group analysis are done by point transformations. Contact transformations and Lie-Bäcklund transformations are less studied. The goal of these transformations is to reduce an initial system of partial differential equations to a system for which it is easier to find solution.

Note that contact transformations were studied in Pucci and Saccomandi (1994), Momoniat (2001), Ibragimov and Khabirov (2000), Wafo and Mahomed (2002), Rudra (1999), Popovych (1995), Ludlow, Clarkson and Bassom (1999).

In almost all of these studies the contact transformations are trivial: they are prolongations of the admitted point transformations.

The targets of contact and Lie-Bäcklund transformations are similar, one has to find the coefficients of their generators. The step of the method for finding these coefficients are similar, i.e., the first step consists of constructing the determining equation. After that one has to split the determining equation with respect to parametric derivatives. Each step has a large amount of analytical calculations. For this purpose it is necessary to use a computer system. A brief review of computer systems can be found, for example, in Davenport (1993). In our calculations the REDUCE system (cf. Hearn (1999)) was used. REDUCE is a system for carrying out algebraic operations accurately, no matter how complicated the expressions become. It can manipulate polynomials in a variety of forms, expanding, substituting and factoring them as required. REDUCE can also do differentiation and integration.

In this thesis the two-dimensional Navier-Stokes equations are studied. The Navier-Stokes equations are fluid dynamics equations and they play a central role in a variety of research within applied mathematics, physics and engineering.

The Navier-Stokes equations in the two-dimensional case are

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right),$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

where ρ is the density, μ is the coefficient of viscosity, p is the pressure, t is time, u, v are component of velocity in x and y axis, respectively. In the two-dimensional case these equations are equivalent to the following equation

$$\psi_{yyt}+\psi_{xxt}-\psi_y\left(\psi_{xyy}+\psi_{xxx}\right)+\psi_x\left(\psi_{yyy}+\psi_{yxx}\right)=\left(\psi_{yyyy}+\psi_{xxxx}\right)+2\psi_{xxyy}, \quad (1.1)$$

where ψ is a streamline function and $u = \psi_y,\ v = -\psi_x$.

The first group classification of the Navier-Stokes equations in the three-dimensional case was done in Bytev (1972). Several papers Puchnachov (1974), Lloyd (1981), Boidvert and Ames (1983), Grauel and Steeb (1985), Fushchich and Popovych (1994), Popovych (1995), Meleshko and Puchnachov (1999) are devoted to invariant and partially invariant solutions of the Navier-Stokes equations. The solutions in these articles are based on admitted Lie-groups of point transformations.

This thesis is devoted to the application of contact and Lie-Bäcklund transformations to the Navier-Stokes equations in the two-dimensional case.

The result of the thesis research is: the Lie-group of contact transformations and Lie-Bäcklund transformations of second order admitted by equation (1.1) are prolongations of the Lie-group of point transformations.

This thesis is organized as follows. Chapter II mainly introduces background knowledge and notations of Lie-groups in each applications, namely point transformations, contact transformations and Lie-Bäcklund transformations. Chapter III is devoted to the Navier-Stokes equations in two dimensions, dimensional analysis of the two-dimensional Navier-Stokes equations and writing them through the streamline function. The result of the research can be found in chapters IV and V. These chapters are devoted to finding contact and Lie-Bäcklund transformations. The computer programs for seeking contact and Lie-Bäcklund transformations are shown in appendix A and appendix B, respectively.

Chapter II

Group Analysis Method

In this chapter, Group analysis method for finding point transformations, contact transformations and Lie-Bäcklund transformations is discussed. The introduction to this method can be found in textbooks (cf. Ovsiannikov (1978), handbook of Lie group analysis (1994), (1995), (1996)).

2.1 Lie-point Transformations

Consider transformations

$$\bar{z}^i = g^i(z; a) \tag{2.1}$$

where $i = 1, 2, ..., N$, $z \in V \subset Z = R^N$, $a \in \Delta$ is a parameter and Δ is a symmetric interval of R^1. The set V is an open set in Z.

Definition 1. *A set of transformations (2.1) is called a local one-parameter Lie group if it has the following properties*

1. $g(z; 0) = z$ *for all* $z \in V$.

2. $g(g(z; a), b) = g(z; a + b)$ *for all* $a, b, a + b \in \Delta, z \in V$.

3. If for $a \in \Delta$ *one have* $g(z; a) = z$ *for all* $z \in V$*, then* $a = 0$.

4. $g \in C^\infty(V, \Delta)$.

If $z = (x, u)$, then one uses the notation $g = (f, \varphi)$. Here $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the vector of the independent variables, and $u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m$ is the vector of the dependent variables. The transformation of the independent variables x, and dependent variables u has the form

$$\bar{x}^i = f^i(x, u; a), \;\; \bar{u}^k = \varphi^k(x, u; a) \tag{2.2}$$

where $i = 1, 2, ..., n,\; k = 1, 2, ..., m,\; (x, u) \in V \subset Z = R^n \times R^m$, the set V is an open set in Z, $a \in \Delta$ is a parameter and Δ is a symmetric interval of R^1. Transformation (2.2) is called a one-parameter group of point transformation.

For Lie groups of point transformations, the functions f^i and φ^k can written by Taylor series expansion with respect to the parameter a in the neighborhood of $a = 0$,

$$\bar{x}^i = f^i(x, u; a) \approx x + a\xi^i(x, u), \;\; \bar{u}^k = \varphi^k(x, u; a) \approx u + a\eta^k(x, u), \tag{2.3}$$

where

$$\xi^i(x, u) = \left.\frac{\partial f^i(x, u; a)}{\partial a}\right|_{a=0}, \;\; \eta^k(x, u) = \left.\frac{\partial \varphi^k(x, u; a)}{\partial a}\right|_{a=0}.$$

In order to apply the Lie group of transformations (2.2) to the study of a differential equation one needs to know how this group acts on the functions $u(x)$ and its derivatives. Assume that $u_0(x)$ is given. The transformed function $\bar{u}_a(\bar{x})$ is obtained in the following way. After substituting $u_0(x)$ into (2.2) one has

$$\bar{x}^i = f^i(x, u_0(x); a), \tag{2.4}$$

$$\bar{u}^k = \varphi^k(x, u_0(x); a).$$

Using the inverse function theorem for (2.4) one can find $x = \theta(\bar{x}; a)$. The transformed function is

$$\bar{u}_a(\bar{x}) = \varphi(\theta(\bar{x}; a), u_0(\theta^i(\bar{x}; a)); a). \tag{2.5}$$

2.1.1 Admitted Lie Group

A relation between differential equations and Lie groups is the following.

Definition 2. *A group of transformations, which transforms a solution $u_0(x)$ to the new solution $u_a(x)$ of the same equation is called an admitted Lie group of transformations.*

Let us consider the equation

$$F(x, u) = 0. \tag{2.6}$$

After applying on admitted Lie group of transformations, one has

$$F(\bar{x}, \bar{u}_a(\bar{x})) = 0. \tag{2.7}$$

Differentiating with respect to the group parameter a and substituting $a = 0$, one has

$$\left. \left(\frac{\partial F}{\partial x^i} \frac{\partial \bar{x}^i}{\partial a} + \frac{\partial F}{\partial u^k} \frac{\partial \bar{u}_a^k}{\partial a} \right) \right|_{a=0} = 0 \tag{2.8}$$

or

$$\left. \frac{\partial F}{\partial x^i} \right|_{a=0} \left(\left. \frac{\partial f^i}{\partial a} \right|_{a=0} \right) + \left. \frac{\partial F}{\partial u^k} \right|_{a=0} \left(\left. \frac{\partial \varphi_a^k}{\partial a} \right|_{a=0} \right) = 0. \tag{2.9}$$

Applying (2.3), one obtains

$$\xi^i(x, u) \frac{\partial F}{\partial x^i}(x, u) + \eta^k \frac{\partial F}{\partial u^k}(x, u) = 0. \tag{2.10}$$

The last equation can be expressed as an action of the infinitesimal generator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^k(x, u) \frac{\partial}{\partial u^k}. \tag{2.11}$$

2.1.2 Prolongation of a Lie Group

Because one apply group analysis to differential equations, one also need to know the transformation of the derivatives. For the sake of simplicity one consider here the case where the number of dependent and independent variables is equal to one. The transformation of the first derivative can be found as follows. Let us differentiate (2.5) with respect to $\bar{x}$

$$\bar{u}_{\bar{x}} = \frac{\partial \bar{u}_a(\bar{x})}{\partial \bar{x}} = \frac{\partial \varphi}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \theta}{\partial \bar{x}} = \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x}\right)\frac{\partial \theta}{\partial \bar{x}}. \tag{2.12}$$

Substituting $x = \theta(\bar{x}, a)$ into (2.4) implies the identity

$$\bar{x} = f(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{2.13}$$

Differentiating this identity with respect to x one obtains

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right)\frac{\partial \theta}{\partial \bar{x}} = 1. \tag{2.14}$$

Since

$$\frac{\partial f}{\partial x}(\theta(\bar{x}, 0), u_0(\theta(\bar{x}, 0)); 0) = 1, \ \frac{\partial f}{\partial u}(\theta(\bar{x}, 0), u_0(\theta(\bar{x}, 0)); 0) = 0, \tag{2.15}$$

then $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x) \neq 0$ in the neighborhood of $a = 0$. Hence,

$$\bar{u}_{\bar{x}} = \frac{\frac{\partial \varphi}{\partial x}(x, u_0; a) + \frac{\partial \varphi}{\partial u}(x, u_0; a)u_0'(x)}{\frac{\partial f}{\partial x}(x, u_0; a) + \frac{\partial f}{\partial u}(x, u_0; a)u_0'(x)} \equiv \omega(x, u_0(x), u_0'(x); a). \tag{2.16}$$

This is the first prolongation of the transformations (2.2). As before, the function ω can be written by Taylor series expansion with respect to the parameter a in the neighborhood of the point $a = 0$:

$$\omega(x, u, u_x; a) \approx u_x + a\zeta \tag{2.17}$$

where

$$\zeta = \left.\frac{\partial \omega}{\partial a}\right|_{a=0} = D_x\eta - u_x D_x\xi, \ \xi = \left.\frac{\partial f}{\partial a}\right|_{a=0}, \ \eta = \left.\frac{\partial \varphi}{\partial a}\right|_{a=0},$$
$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \dots .$$

The first prolongation of the generator (2.11) is given by

$$\underset{1}{X} = X + \zeta\frac{\partial}{\partial u_x} .$$

The second and higher prolongations can be found in the same way.

For the case where the number of the dependent and independent variables is greater than one, the first prolongation of the generator (2.11) is given by

$$\underset{1}{X} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \ (\alpha = 1, ..., m). \tag{2.18}$$

Here

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \ (i = 1, 2, ..., n),$$

$$D_i = \frac{\partial}{\partial x_i} + \sum_\alpha u_i^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\alpha,\beta} u_{i\beta}^\alpha \frac{\partial}{\partial u_\beta^\alpha} + \ldots \, .$$

The second prolongation of the generator X is

$$\underset{2}{X} = \underset{1}{X} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha}, \ (\alpha = 1, ..., m) \tag{2.19}$$

where

$$\zeta_{i_1 i_2}^\alpha = D_{i_2}(\zeta_{i_1}^\alpha) - u_{j i_1}^\alpha D_{i_2}(\xi^j), \ (i_1, i_2 = 1, 2, ..., n).$$

In the general case, a k-th prolongation of the generator X is

$$\underset{k}{X} = \underset{k-1}{X} + \zeta_{i_1 \ldots i_k}^\alpha \frac{\partial}{\partial u_{i_1 \ldots i_k}^\alpha} \tag{2.20}$$

where

$$\zeta_{i_1 \ldots i_k}^\alpha = D_{i_k}(\zeta_{i_1 \ldots i_k}^\alpha) - u_{j i_1 \ldots i_{k-1}}^\alpha D_{i_k}(\xi^j), \ (i_1, i_2, ..., i_k = 1, 2, ..., n).$$

2.1.3 Lie Equations

There is a theorem, which explains a correspondence between a one-parameter group G and its infinitesimal generator.

Theorem 1 (Lie). *Let a function $g(z; a)$ satisfy the group properties and have the expansion*

$$\bar{z} = g(z; a) = z + \xi(z)a + O(a) \approx z + \xi(z)a \tag{2.21}$$

where

$$\xi(z) = \left.\frac{\partial g(z;a)}{\partial a}\right|_{a=0}.$$

Then it solves the Cauchy problem

$$\frac{d\bar{z}}{da} = \xi(\bar{z}), \;\; \bar{z}\mid_{a=0} = z. \tag{2.22}$$

Conversely, given $\xi(z)$, the solution of the Cauchy problem (2.22) satisfies the group properties. Equations (2.22) are called Lie equations.

2.1.4 Determining Equations

Let F be a differential function of order p. The equation

$$F\left(x, u, u_{(1)}, u_{(2)}, ..., u_{(p)}\right) = 0 \tag{2.23}$$

composes a manifold in the space of the variables $x, u, u_{(1)}, u_{(2)}, ..., u_{(p)}$. The manifold defined by equation (2.23) and its total derivations

$$D_iF = 0, D_iD_jF = 0, ...$$

is called a differential manifold $[F]$. For equation (2.23) one has to prolong the group of transformations (2.2) to derivatives up to order p. The prolonged transformations form again a one-parameter group, which is denoted by $G_{(p)}$. This group acts on the variables $x, u, u_{(1)}, u_{(2)}, ..., u_{(p)}$. The infinitesimal generator of the group $G_{(p)}$ is

$$\underset{p}{X} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + ... + \zeta_{i_1...i_p}^\alpha \frac{\partial}{\partial u_{i_1...i_p}^\alpha}.$$

In order to find the infinitesimal generator of the group admitted by a differential equation (2.23) one can use the following theorem.

Theorem 2. *The differential equation (2.23) admits the group G with the generator X if and only if the following equations hold:*

$$\underset{p}{X} F\Big|_{[F]} = 0. \tag{2.24}$$

Equations (2.24) are called determining equations.

2.1.5 Multi-Parameter Lie-Group of Transformations.

Let O be a ball with center at the origin of the space R^r. Assume that (O, ψ) is a pair with the mapping $\psi : O \times O \longrightarrow R^r$. This pair is called a local multi-parameter Lie group with the multiplication law ψ if it has the following properties:

1. $\psi(a, 0) = \psi(0, a) = a$ for all $a \in O$.
2. $\psi(\psi(a, b), c) = \psi(a, \psi(b, c))$ for all $a, b, c \in O$ for which $\psi(a, b), \psi(b, c) \in O$.
3. $\psi \in C^\infty(O, O)$.

Let V be an open set in Z. Consider transformations

$$\bar{z}^i = g^i(z; a) \tag{2.25}$$

where $i = 1, 2, ..., N$, $z \in V \subset Z = R^N$, $a \in \Delta$ and a vector-parameter $a \in O$.

Definition 3. *A set of transformations (2.25) is called a local r-parameter group G^r if it has the following properties:*

1. $g(z, 0) = z$ for all $z \in V$.

2. $g(g(z, a), b) = g(z, \psi(a, b))$ for all $a, b, \psi(a, b) \in O$, $z \in V$.

3. If for $a \in O$ one has $g(z, a) = z$ for all $z \in V$, then $a = 0$.

Note that if one fixes all parameters except one, for example a_k, then the multi-parameter Lie group of transformations (2.25) composes a one-parameter Lie group. Conversely, in group analysis it is proven that any r-parameter group is a union of one-parameter subgroups belonging to it.

2.1.6 Lie Algebra

Let

$$X_1 = \xi_1^i(z)\frac{\partial}{\partial z^i},\ X_2 = \xi_2^i(z)\frac{\partial}{\partial z^i}$$

be two infinitesimal generators. One define the operator

$$[X_1, X_2] = X_1X_2 - X_2X_1$$

or

$$[X_1, X_2] = \left(X_1\left(\xi_2^i\right) - X_2\left(\xi_1^i\right)\right)\frac{\partial}{\partial z^i}.$$

This operator $[X_1,\ X_2]$ is called a commutator of X_1 and X_2. A commutator satisfies the axioms :

1. bilinearity:

$[cX_1, X_2] = [X_1, cX_2] = c\,[X_1, X_2]\,,$

$[X, X_1 + X_2] = [X, X_1] + [X, X_2]\,, [X_1 + X_2, X] = [X_1, X] + [X_2, X],$

where c is an arbitrary constant;

2. skew-symmetry: $[X_1, X_2] = -\,[X_2, X_1],$

3. Jacobi's identity:

$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$

Definition 4. *A Lie algebra is a vector space L of infinitesimal operators such that L contains the commutator $[X_1, X_2]$ of any operators $X_1, X_2 \in L$.*

Let $X_1, X_2, ..., X_r$ be a basis of an r-dimensional Lie algebra L_r. then each X in L_r can be decomposed

$$X = \sum_{\mu=1}^{r} c^{\mu}X_{\mu},$$

where $c^{\mu} = constant$. In particular,

$$[X_i, X_j] = \sum_{\mu=1}^{r} c_{ij}^{\mu}X_{\mu},\ \ i, j = 1, 2..., r.$$

In the theory of group analysis it is proven that if a set of operators $X_k,\ (k = 1, 2, ...r)$ composes a Lie algebra L_r, then there exists a local r-parameter Lie group G^r of transformations with the generators $X_k,\ (k = 1, 2, ..., r)$.

2.2 Tangent Transformations

Let $p = p_\alpha^k$ be a vector of derivatives of the functions u^j with respect to x_i:

$$p_\alpha^k \equiv \frac{\partial^{|\alpha|} u^k}{\partial x^\alpha} = \frac{\partial^{|\alpha|} u^k}{\partial x_1^{\alpha_1} ... x_n^{\alpha_n}}.$$

Here $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is multi-index, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha, j = (\alpha_1, \alpha_2, ..., \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, ..., \alpha_n)$.

Here one study the case $z = (x, u, p)$ and $g = (f, \varphi, \omega)$. The Lie group of transformations of the independent variables x, dependent variables u and derivatives p :

$$\bar{x}^i = f^i(x, u, p; a), \ \bar{u}^k = \varphi^k(x, u, p; a), \ \bar{p}_\alpha^k = \omega_\alpha^k(x, u, p; a) \tag{2.26}$$

is called a one-parameter group of tangent transformations if it preserves the tangent conditions

$$d\bar{u}^k - \bar{p}_j^k d\bar{x}_j = 0, \ d\bar{p}_\alpha^k - \bar{p}_{\alpha,j}^k d\bar{x}_j = 0. \tag{2.27}$$

Similar to point transformations, the functions f^i, φ^k and ω_α^k can be written by using Taylor series expansion with respect to the parameter a in the neighborhood of $a = 0$:

$$\begin{aligned} \overline{x}^i &= f^i\left(x, u, p : a\right) \approx x + a\xi^i\left(x, u, p\right), \\ \overline{u}^k &= \varphi^k\left(x, u, p : a\right) \approx u + a\eta^k\left(x, u, p\right), \\ \overline{p}_\alpha^k &= \omega_\alpha^k\left(x, u, p : a\right) \approx p + a\zeta_\alpha^k\left(x, u, p\right), \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} \xi^i\left(x, u, p\right) &= \left.\frac{\partial f^i\left(x, u, p; a\right)}{\partial a}\right|_{a=0}, \\ \eta^k\left(x, u, p\right) &= \left.\frac{\partial \varphi^k\left(x, u, p; a\right)}{\partial a}\right|_{a=0}, \\ \zeta_\alpha^k\left(x, u, p\right) &= \left.\frac{\partial \omega_\alpha^k\left(x, u, p; a\right)}{\partial a}\right|_{a=0}. \end{aligned} \tag{2.29}$$

2.2.1 Contact Transformations

Contact transformations are a special case of tangent transformations, for which the transformations of the independent and dependent variables and the first derivatives only depend on the dependent and independent variables and the first derivatives.

The prolongation formulae of tangent transformations, has the same form as the prolongation formulae of point transformations. The difference is that the coefficients of an infinitesimal generator of point transformations depend on the dependent and independent variables, but for tangent transformations they depend on the dependent and independent variables and the derivatives.

For contact transformations the following is known.

Theorem 3. *If the number of dependent variables is $m > 1$, then a group of contact transformations (2.26) is the first prolongation of a Lie group of point transformations.*

Proof. Taking differentials of (2.28), one has

$$\begin{aligned} d\overline{x}^i &= \frac{\partial f^i}{\partial x^j}dx^j + \frac{\partial f^i}{\partial u^\beta}du^\beta + \frac{\partial f^i}{\partial p_j^\beta}dp_j^\beta, \\ d\overline{u}^k &= \frac{\partial \varphi^k}{\partial x^j}dx^j + \frac{\partial \varphi^k}{\partial u^\beta}du^\beta + \frac{\partial \varphi^k}{\partial p_j^\beta}dp_j^\beta, \\ d\overline{p}_i^k &= \frac{\partial \omega_i^k}{\partial x^j}dx^j + \frac{\partial \omega_i^k}{\partial u^\beta}du^\beta + \frac{\partial \omega_i^k}{\partial p_j^\beta}dp_j^\beta, \end{aligned} \tag{2.30}$$

where $i, j = 1, 2, ..., n,\ k, \beta = 1, 2, ..., m$. After substituting (2.30) into the first equation of (2.27), one obtains

$$\left(\frac{\partial \varphi^k}{\partial x^i}dx^i + \frac{\partial \varphi^k}{\partial u^\beta}du^\beta + \frac{\partial \varphi^k}{\partial p_j^\beta}dp_j^\beta\right) - \omega_\gamma^k\left(\frac{\partial f^\gamma}{\partial x^i}dx^i + \frac{\partial f^\gamma}{\partial u^\beta}du^\beta + \frac{\partial f^\gamma}{\partial p_j^\beta}dp_j^\beta\right) = 0, \tag{2.31}$$

$$i, j, \gamma = 1, 2, ..., n;\ k, \beta = 1, 2, ..., m.$$

Since $du^\beta = p_i^\beta dx^i$, equation (2.31) become

$$\left[\frac{\partial \varphi^k}{\partial x^i} + p_i^\beta \frac{\partial \varphi^k}{\partial u^\beta} - \omega_\gamma^k \left(\frac{\partial f^\gamma}{\partial x^i} + p_j^\beta \frac{\partial f^\gamma}{\partial u^\beta}\right)\right] dx^i + \left[\frac{\partial \varphi^k}{\partial p_j^\beta} - \omega_\gamma^k \frac{\partial f^\gamma}{\partial p_j^\beta}\right] dp_j^\beta = 0, \quad (2.32)$$

$$i, j, \gamma = 1, 2, ..., n; \;\; k, \beta = 1, 2, ..., m.$$

The left side of these equations is a linear form with respect to the differentials dx^i, dp_j^β. Since dx^i, dp_j^β are arbitrary, this implies that the coefficients of these forms have to be equal to zero,

$$\frac{\partial \varphi^k}{\partial x^i} + p_i^\beta \frac{\partial \varphi^k}{\partial u^\beta} - \omega_\gamma^k \left(\frac{\partial f^\gamma}{\partial x^i} + p_j^\beta \frac{\partial f^\gamma}{\partial u^\beta}\right) = 0, \;\; i, j, \gamma = 1, 2, ..., n; \;\; k, \beta = 1, 2, ..., m, \quad (2.33)$$

$$\frac{\partial \varphi^k}{\partial p_j^\beta} - \omega_\gamma^k \frac{\partial f^\gamma}{\partial p_j^\beta} = 0, \;\; i, j, \gamma = 1, 2, ..., n; \;\; k, \beta = 1, 2, ..., m. \quad (2.34)$$

Differentiating (2.34) with respect to the parameter a and substituting $a = 0$ into them, one obtains

$$\frac{\partial}{\partial p_j^\beta}\left(\left.\frac{\partial \varphi^k}{\partial a}\right|_{a=0}\right) - \omega_\gamma^k\,|_{a=0} \left(\frac{\partial}{\partial p_j^\beta}\left(\left.\frac{\partial f^\gamma}{\partial a}\right|_{a=0}\right)\right) - \left(\left.\frac{\partial f^\gamma}{\partial p_j^\beta}\right|_{a=0}\right)\left(\left.\frac{\partial \omega_\gamma^k}{\partial a}\right|_{a=0}\right) = 0, \quad (2.35)$$

$$i, j, \gamma = 1, 2, ..., n; \;\; k, \beta = 1, 2, ..., m.$$

Here commutativity of the second order derivatives and the operations $\frac{\partial}{\partial a}()\,|_{a=0}$ and $\frac{\partial}{\partial p_j^\beta}$ were used. Using (2.29) this implies

$$\frac{\partial \eta^k}{\partial p_j^\beta} = p_i^k \frac{\partial \xi^i}{\partial p_j^\beta}, \;\; i, j = 1, 2, ..., n; \;\; k, \beta = 1, 2, ..., m. \quad (2.36)$$

Replacing the symbol β, j, by l,s, one can write

$$\frac{\partial \eta^k}{\partial p_s^l} = p_i^k \frac{\partial \xi^i}{\partial p_s^l}, \;\; i, j = 1, 2, ..., n; \;\; k, \beta = 1, 2, ..., m. \quad (2.37)$$

Differentiating (2.36) and (2.37) with respect to p_s^l and p_j^β, respectively, one has

$$\frac{\partial^2 \eta^k}{\partial p_j^\beta \partial p_s^l} = \frac{\partial p_i^k}{\partial p_s^l}\frac{\partial \xi^i}{\partial p_j^\beta} + p_i^k \frac{\partial^2 \xi^i}{\partial p_j^\beta \partial p_s^l}, \;\; i, j, s = 1, 2, ..., n; \;\; k, \beta, l = 1, 2, ..., m, \quad (2.38)$$

$$\frac{\partial^2\eta^k}{\partial p_s^l \partial p_j^\beta} = \frac{\partial p_i^k}{\partial p_j^\beta}\frac{\partial \xi^i}{\partial p_s^l} + p_i^k \frac{\partial^2 \xi^i}{\partial p_s^l \partial p_j^\beta},\ i,j,s = 1,2,...,n;\ k,\beta,l = 1,2,...,m. \quad (2.39)$$

Subtracting (2.39) from (2.38), one gets

$$\frac{\partial p_i^k}{\partial p_s^l}\frac{\partial \xi^i}{\partial p_j^\beta} = \frac{\partial p_i^k}{\partial p_j^\beta}\frac{\partial \xi^i}{\partial p_s^l},\ i,j,s = 1,2,...,n;\ k,\beta,l = 1,2,...,m. \quad (2.40)$$

Equations (2.40) are valid for every k and l. If $k \neq l$ or $i \neq s$, then $\frac{\partial p_i^k}{\partial p_s^l} = 0$, which implies

$$0 = \frac{\partial p_i^k}{\partial p_j^\beta}\frac{\partial \xi^i}{\partial p_s^l} = \frac{\partial p_1^k}{\partial p_j^\beta}\frac{\partial \xi^1}{\partial p_s^l} + \frac{\partial p_2^k}{\partial p_j^\beta}\frac{\partial \xi^2}{\partial p_s^l} + ... + \frac{\partial p_n^k}{\partial p_j^\beta}\frac{\partial \xi^n}{\partial p_s^l}, \quad (2.41)$$
$$i,j,s = 1,2,...,n,\ k,\beta,l = 1,2,...,m.$$

For $\beta = k$, equation (2.41) can be rewritten as

$$\frac{\partial \xi^j}{\partial p_s^l} = 0,\ \forall\ j,s = 1,2,...,n;\ l = 1,2,...,m. \quad (2.42)$$

This means that ξ^j only depends on the variables x and u. Substituting (2.42) into (2.38), one obtains

$$\frac{\partial \eta^k}{\partial p_j^\beta} = 0,\ \forall\ \beta = 1,2,...,n;\ k,j = 1,2,...,m.$$

This also means that η^k only depends on the variables x and u. Therefore, a Lie group of contact transformations where the number of dependent variables is greater than one is a prolongation of a Lie group of point transformations.

In the case where $m = 1$ the following theorem is valid.

Theorem 4. *The infinitesimal generator*

$$X = \xi^i(x,u,p)\frac{\partial}{\partial x^i} + \eta(x,u,p)\frac{\partial}{\partial u} + \zeta_i(x,u,p)\frac{\partial}{\partial p_i},\ i = 1,2,...,n \quad (2.43)$$

is a generator of a Lie group of contact transformations (2.26) if and only if there exists a function $W = W(x,u,p)$ *such that*

$$\xi^i = \frac{\partial W}{\partial p_i},\ \eta = W - p_i\frac{\partial W}{\partial p_i},\ \zeta_i = \frac{\partial W}{\partial x_i} + p_i\frac{\partial W}{\partial u},\ i = 1,2,...,n. \quad (2.44)$$

The function W *occurring in this theorem is called the characteristic function of the group of contact transformations (2.26).*

Proof. Note that up to formula (2.36) the proof of the previous theorem is valid for $m \geq 1$. Equations (2.40) for $m = 1$ are reduced to

$$\frac{\partial \xi^i}{\partial p_k} = \frac{\partial \xi^k}{\partial p_i}, \; i, k = 1, 2, ..., n. \tag{2.45}$$

The general solution of (2.45) can be written in the form

$$\xi^i = \frac{\partial U}{\partial p_i}, \; i = 1, 2, ..., n, \tag{2.46}$$

with some function $U(x, u, p)$. Substituting (2.46) into (2.36), one obtains

$$\frac{\partial \eta}{\partial p_k} - p_j \left(\frac{\partial^2 U}{\partial p_j \partial p_k} \right) = 0, \; j, k = 1, 2, ..., n. \tag{2.47}$$

Since $p_j \left(\frac{\partial^2 U}{\partial p_j \partial p_k} \right) = \frac{\partial}{\partial p_k} \left(p_j \frac{\partial U}{\partial p_j} \right) - \frac{\partial U}{\partial p_k}$, equations (2.47) become

$$\frac{\partial}{\partial p_k} \left(\eta - p_j \frac{\partial U}{\partial p_j} + U \right) = 0, \; j, k = 1, 2, ..., n. \tag{2.48}$$

The general solution of (2.48) can be written in the form

$$\eta - p_j \frac{\partial U}{\partial p_j} + U = H\left(x, u\right), \; j = 1, 2, ..., n. \tag{2.49}$$

Let

$$W\left(x, u, p\right) = -U + H(x, u). \tag{2.50}$$

Then

$$\frac{\partial U}{\partial p_i} = \frac{\partial W}{\partial p_i}, \; i = 1, 2, ..., n. \tag{2.51}$$

From (2.46) and (2.51) one gets

$$\xi^i = -\frac{\partial W}{\partial p_i}, \; i = 1, 2, ..., n. \tag{2.52}$$

After substituting (2.50) and (2.51) into (2.49), one obtains

$$\eta = -p_i \frac{\partial W}{\partial p_i} + W, \; i = 1, 2, ..., n. \tag{2.53}$$

Considering (2.33) where the number of the dependent variables is $m = 1$, one can write

$$\frac{\partial \varphi}{\partial x^j} + p_j \frac{\partial \varphi}{\partial u} - \omega_i \left(\frac{\partial f^i}{\partial x^j} + p_j \frac{\partial f^i}{\partial u} \right) = 0, \ i, j = 1, 2, ..., n. \tag{2.54}$$

Differentiating (2.54) with respect to the parameter a and substituting $a = 0$ into them, one has

$$\begin{aligned} &\left(\frac{\partial}{\partial a} \left(\frac{\partial \varphi}{\partial x^j} \right) \right) \Big|_{a=0} + p_j \left(\frac{\partial}{\partial a} \left(\frac{\partial \varphi}{\partial u} \right) \right) \Big|_{a=0} - \\ &- \left(\frac{\partial}{\partial a} \left(\omega_i \frac{\partial f^i}{\partial x^j} \right) \right) \Big|_{a=0} - p_j \left(\frac{\partial}{\partial a} \left(\omega_i \frac{\partial f^i}{\partial u} \right) \right) \Big|_{a=0} = 0. \end{aligned} \tag{2.55}$$

Considering each term in (2.55), one has

$$\begin{aligned} \left(\frac{\partial}{\partial a} \left(\frac{\partial \varphi}{\partial x^j} \right) \right) \Big|_{a=0} &= \left(\frac{\partial}{\partial x^j} \left(\frac{\partial \varphi}{\partial a} \right) \right) \Big|_{a=0} = \frac{\partial}{\partial x^j} \left(\frac{\partial \varphi}{\partial a} \right) \Big|_{a=0} \\ &= \frac{\partial \eta}{\partial x^j}, \ j = 1, 2, ..., n, \end{aligned} \tag{2.56}$$

$$\begin{aligned} p_j \left(\frac{\partial}{\partial a} \left(\frac{\partial \varphi}{\partial u} \right) \right) \Big|_{a=0} &= p_j \left(\frac{\partial}{\partial u} \left(\frac{\partial \varphi}{\partial a} \right) \right) \Big|_{a=0} = p_j \left(\frac{\partial}{\partial u} \left(\frac{\partial \varphi}{\partial a} \Big|_{a=0} \right) \right) \\ &= p_j \frac{\partial \eta}{\partial u}, \ j = 1, 2, ..., n, \end{aligned} \tag{2.57}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial a} \left(\omega_i \frac{\partial f^i}{\partial x^j} \right) \right) \Big|_{a=0} = \left(\omega_i \frac{\partial}{\partial a} \left(\frac{\partial f^i}{\partial x^j} \right) + \frac{\partial f^i}{\partial x^j} \frac{\partial \omega_i}{\partial a} \right) \Big|_{a=0} \\ &= (\omega_i \, |_{a=0}) \left(\frac{\partial}{\partial x^j} \left(\frac{\partial f^i}{\partial a} \right) \right) \Big|_{a=0} + \left(\frac{\partial f^i}{\partial x^j} \right) \Big|_{a=0} \left(\frac{\partial \omega_i}{\partial a} \right) \Big|_{a=0} \\ &= (\omega_i \, |_{a=0}) \left(\frac{\partial}{\partial x^j} \left(\frac{\partial f^i}{\partial a} \Big|_{a=0} \right) \right) + \left(\frac{\partial \left(f^i \, |_{a=0} \right)}{\partial x^j} \right) \left(\frac{\partial \omega_i}{\partial a} \Big|_{a=0} \right) \\ &= p_i \frac{\partial \xi^i}{\partial x^j} + \frac{\partial x^i}{\partial x^j} \zeta_i = p_i \frac{\partial \xi^i}{\partial x^j} + \zeta_j, \ i, j = 1, 2, ..., n, \end{aligned} \tag{2.58}$$

$$\begin{aligned} &p_j \left(\frac{\partial}{\partial a} \left(\omega_i \frac{\partial f^i}{\partial u} \right) \right) \Big|_{a=0} = p_j \left(\omega_i \frac{\partial}{\partial a} \left(\frac{\partial f^i}{\partial u} \right) + \frac{\partial f^i}{\partial u} \frac{\partial \omega_i}{\partial a} \right) \Big|_{a=0} \\ &= p_j \left(\omega_i \, |_{a=0} \left(\frac{\partial}{\partial u} \left(\frac{\partial f^i}{\partial a} \Big|_{a=0} \right) \right) + \left(\frac{\partial \left(f^i \, |_{a=0} \right)}{\partial u} \right) \left(\frac{\partial \omega_i}{\partial a} \Big|_{a=0} \right) \right) \\ &= p_j \left(p_i \frac{\partial \xi^i}{\partial u} + \frac{\partial x^i}{\partial u} \zeta_i \right) = p_j p_i \frac{\partial \xi^i}{\partial u}, \ i, j = 1, 2, ..., n. \end{aligned} \tag{2.59}$$

Substituting (2.56)-(2.59) into (2.55), one obtains

$$\frac{\partial \eta}{\partial x^j} + p_j \frac{\partial \eta}{\partial u} - p_i \frac{\partial \xi^i}{\partial x^j} - \zeta_j - p_j p_i \frac{\partial \xi^i}{\partial u} = 0, \ i, j = 1, 2, ..., n. \tag{2.60}$$

Using (2.52) and (2.53), equation (2.60) becomes

$$\zeta_i = \frac{\partial W}{\partial x^i} + p_i \frac{\partial W}{\partial u}, \ i = 1, 2, ..., n. \tag{2.61}$$

Therefore there exists a function $W = W(x, u, p)$ which satisfies (2.44).

2.3 Lie-Bäcklund Transformations

Let us consider the transformations (2.26). Assume that for some N the functions $f^i, \varphi^k, \omega^k_\alpha$ with $|\alpha| \leq N$ depend only the independent variables, dependent variables and derivatives up to N-th order.

Theorem 5 (Bäcklund). *There are no tangent transformations of finite order N other than tangent transformations which are prolongations of contact $(m > 1)$ or point transformations $(m = 1)$.*

This means that for the case $m > 1$, a tangent transformation is the N-th prolongation of a group of point transformations, and for $m = 1$ the tangent transformation is a contact transformation.

The Bäcklund theorem requires for tangent transformations which are not prolongation of contact or point transformations that for any N, one of the functions ω^k_α(with $|\alpha| \leq N$) must include derivatives of order greater than N. This means that in the transformation process one must include an infinite number of derivatives and leads us to the notion of Lie-Bäcklund transformations.

Definition 5. *A transformation*

$$\begin{aligned}
\bar{x} &= f(x, u, \underset{1}{u}, \underset{2}{u}, ...; a),\\
\bar{u} &= \varphi(x, u, \underset{1}{u}, \underset{2}{u}, ...; a),\\
\underset{1}{\bar{u}} &= \underset{1}{\omega}(x, u, \underset{1}{u}, \underset{2}{u}, ...; a),\\
\underset{2}{\bar{u}} &= \underset{2}{\omega}(x, u, \underset{1}{u}, \underset{2}{u}, ...; a),\\
&.............................
\end{aligned} \tag{2.62}$$

where any function in (2.62) is a function of a finite number of derivatives and satisfies the tangent conditions is called a Lie-Bäcklund transformation.

For Lie groups of finite order tangent transformations, the number of variables involved in the transformation is finite. This allows using Taylor series expansion. In Lie-Bäcklund transformations the number of variables involved in transformation is infinite. Thus for Lie-Bäcklund transformations one has to use the new object formal power series.

2.3.1 Formal One-Parameter Group

Let Z be the space of the sequences $z = (z^1, z^2, ...)$. Consider sequences of formal power series in one symbol a:

$$f^i\,(z; a) = \sum_{k=0}^{\infty} A_k^i\,(z) a^k, i = 1, 2, ... \,. \tag{2.63}$$

where each coefficient is an analytic function. Assume that

$$A_0^i\,(z) = z^i, i = 1, 2, ... \,. \tag{2.64}$$

and denote the coefficients

$$\xi^i\,(z) = A_1^i\,(z)\,. \tag{2.65}$$

Let λ and μ be constant. Linear combination and multiplication of formal series is defined by

$$\begin{aligned} \lambda\left(\sum_{k=0}^{\infty} A_k a^k\right) + \mu\left(\sum_{k=0}^{\infty} B_k a^k\right) &= \sum_{k=0}^{\infty} (\lambda A_k + \mu B_k) a^k, \\ \left(\sum_{k_1=0}^{\infty} A_{k_1} a^{k_1}\right)\left(\sum_{k_2=0}^{\infty} B_{k_2} a^{k_2}\right) &= \sum_{k=0}^{\infty}\left(\sum_{k_1+k_2=k} A_{k_1} B_{k_2}\right) a^k, \end{aligned} \tag{2.66}$$

and the results are again formal power series.

Formula (2.63) may be thought of as the transformation which transforms the sequence $z = (z^1, z^2, ...)$ into the sequence $\overline{z} = (\overline{z}^1, \overline{z}^2, ...)$ with

$$\overline{z}^i = f^i\,(z; a)\,.$$

Let us do one more transformation with the parameter b:

$$\overline{\overline{z}}^i = f^i(\overline{z}; b) = \sum_{k=0}^{\infty} A_k^i(\overline{z}) b^k.$$

Definition 6. *A sequence $f(z;a) = (f^1(z;a), f^2(z;a), ...)$ of formal power series (2.63) is called a formal one-parameter group, if the coefficients of these series satisfy the conditions*

$$A_k^i(f(z;a)) = \sum_{l=0}^{\infty} \frac{(k+l)!}{k!l!} A_{k+l}^i(z) a^l, i = 1, 2, ...; k = 0, 1, 2, ... \tag{2.67}$$

Consider

$$f^i(f(z;a); b) = \sum_{k=0}^{\infty} A_k^i(f(z;a)) b^k \tag{2.68}$$

and

$$\begin{aligned} f^i(z; a+b) &= \sum_{k=0}^{\infty} A_k^i(z)(a+b)^k \\ &= \sum_{k=0}^{\infty} A_k^i(z) \sum_{k_1+k_2=k} \frac{k!}{k_1!k_2!} a^{k_1} b^{k_2} \\ &= \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{(k_1+k_2)}{k_1!k_2!} A_{k_1+k_2}^i(z) a^{k_1} b^{k_2} \\ &= \sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{(k+l)!}{k!l!} A_{k+l}^i(z) a^l \right) b^k. \end{aligned} \tag{2.69}$$

Comparing (2.68) and (2.69), one has that (2.67) is equivalent to the following:

$$f^i(f(z;a); b) = f^i(z; a+b), i = 1, 2, ... \tag{2.70}$$

For the formal power series (2.63), one introduce the notation

$$f^i(z;a)|_{a=0} \equiv A_0^i(z). \tag{2.71}$$

From (2.64), one can write (2.71) as

$$f^i(z;a)\Big|_{a=0} = z^i. \tag{2.72}$$

Since

$$\left.\frac{df^i(z;a)}{da}\right|_{a=0} = \left.\left(\sum_{k=1}^{\infty} k A_k^i(z) a^{k-1} \right)\right|_{a=0} = A_1^i(z),$$

and by (2.65), this implies

$$\xi^i(z) = \left.\frac{df^i(z;a)}{da}\right|_{a=0}. \tag{2.73}$$

For a formal series one can apply Lie's theorem.

Theorem 6. *The sequence of formal series (2.63), constituting a formal one-parameter group with (2.73), satisfies the differential equations*

$$\frac{df^i}{da} = \xi^i(f), i = 1, 2, \dots . \tag{2.74}$$

Conversely, for every sequence $\xi(z) = (\xi^1(z), \xi^2(z), \dots)$ *of analytic functions* $\xi^i(z)$ *of finite number of variables* z^i*, there is one, and only one solution of (2.74) given by formal power series (2.63) with initial condition (2.72).*

Like for contact transformations one can construct an infinitesimal generator by differentiating (2.62) with respect to the parameter a, and setting it to zero. The infinitesimal generator

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \dots \tag{2.75}$$

is called a Lie-Bäcklund operator.

In practice for relating the coefficients of the Lie-Bäcklund operator one can use the following theorem.

Theorem 7. *The formal one-parameter group* G *is a group of Lie-Bäcklund transformations if and only if the coefficients of the corresponding generator* $(\xi^i, \eta^\alpha, \zeta^\alpha_{i_1}, \zeta^\alpha_{i_1i_2}, \dots)$ *satisfy the conditions*

$$\zeta^\alpha_{i_1i_2\dots i_s} = D_{i_1}\zeta^\alpha_{i_2\dots i_s} - u^\alpha_{ji_2\dots i_s}D_{i_1}\xi^j,\ \ s = 1, 2, \dots . \tag{2.76}$$

Let us consider the operator

$$X_* = \xi^i_*D_i \equiv \xi^i_*\frac{\partial}{\partial x^i} + \xi^i_*u^\alpha_j\frac{\partial}{\partial u^\alpha} + \xi^i_*u^\alpha_{ij}\frac{\partial}{\partial u^\alpha_j} + \dots . \tag{2.77}$$

with some coefficients $\xi^i_* = \xi^i_*(z)$

Theorem 8. *Every operator X_* is a Lie-Bäcklund operator.*

Let us consider a Lie-Bäcklund operator (2.75). Subtracting (2.75) from the operator (2.77) with $\xi_*^i = \xi^i$, one obtains the Lie Bäcklund operator

$$X - \xi^i D_i = \left(\eta^\alpha - \xi^i u_i^\alpha\right) \frac{\partial}{\partial u^\alpha} + ... \,. \tag{2.78}$$

This operator can be rewritten in the form

$$\tilde{X} = \tilde{\eta}^\alpha \frac{\partial}{\partial u^\alpha} + ... \,. \tag{2.79}$$

where $\tilde{\eta}^\alpha = \eta^\alpha - \xi^i u_i^\alpha$. The Lie-Bäcklund operators (2.79) are called canonical operators. The prolongation formulas (2.76) for canonical Lie-Bäcklund operators

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + ... \tag{2.80}$$

acquire a simple form

$$\zeta^\alpha_{i_1 i_2 ... i_s} = D_{i_1} ... D_{i_s}(\eta^\alpha). \tag{2.81}$$

Two Lie-Bäcklund operators, X and Y, are equivalent whenever $X - Y \in L_*$. Here

$$L_* = \{X_* \mid X_* = \xi_*^i D_i\}.$$

Chapter III

Navier-Stokes Equations

3.1 Continuity Equation

Consider a two-dimensional fluid flow. Let u and v be velocity in the x and y directions, respectively, and ρ be density of the fluid. The mass of fluid passing through a section of area A per unit time, $\rho u A$ is called the mass flow rate. Consider an elementary rectangle of fluid of side dx, side dy and thickness b. For the x direction, the fluid mass stored in the fluid element per unit time can be obtained by subtracting the inlet mass flow rate from the outlet mass flow rate, i.e.,

$$\rho ubdy - \left[\rho u + \frac{\partial(\rho u)}{\partial x}dx\right] bdy = -\frac{\partial(\rho u)}{\partial x}dxdy. \tag{3.1}$$

Similarly, in the y direction, the fluid mass stored in it per unit time is

$$-\frac{\partial(\rho v)}{\partial y}bdxdy. \tag{3.2}$$

The mass of fluid element in unit time, $\partial(\rho bdxdy)/\partial t$ is obtained by combining the fluid mass stored in the fluid element per unit time in the x and y directions:

$$-\frac{\partial(\rho u)}{\partial x}bdxdy - \frac{\partial(\rho v)}{\partial y}bdxdy = \frac{\partial(\rho bdxdy)}{\partial t}$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0. \tag{3.3}$$

Equation (3.3) is called the continuity equation. In the case of an incompressible fluid, where ρ is constant, equation (3.3) is reduced to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.4}$$

This equation is applicable to both steady and unsteady flows.

3.2 Motion Equations

Applying Newton's second law to the elementary rectangle of fluid of side dx, side dy and thickness b and force $F = (F_x, F_y)$ acting on this element one has

$$\begin{aligned} \rho b dx dy \frac{\partial u}{\partial t} &= F_x, \\ \rho b dx dy \frac{\partial v}{\partial t} &= F_y. \end{aligned} \tag{3.5}$$

The left-hand side of equations (3.5) expresses the inertial force. Let velocity $u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$. Since $u = u(x, y, t)$, the differential du can be expressed by the following equation:

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

Thus,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}. \tag{3.6}$$

Substituting (3.6) into (3.5), one has

$$\begin{aligned} \rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) b dx dy &= F_x, \\ \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) b dx dy &= F_y. \end{aligned} \tag{3.7}$$

Because the force F acting on the elements consist of the body force $F_B = (B_x, B_y)$, pressure force $F_p = (P_x, P_y)$ and viscous force $F_s = (S_x, S_y)$,

$$\begin{aligned} F_x &= B_x + P_x + S_x, \\ F_y &= B_y + P_y + S_y. \end{aligned} \tag{3.8}$$

The body force F_B acts directly throughout the mass

$$\begin{aligned} B_x &= X \rho b dx dy, \\ B_y &= Y \rho b dx dy, \end{aligned} \tag{3.9}$$

here X and Y are body forces acting on the mass of fluid in x and y axis components. Examples of such forces are the gravitational force, the centrifugal force, the electromagnetic force, etc.

The pressure force F_p is obtained by subtracting the inlet pressure force from the outlet pressure force, i.e.,

$$\begin{aligned} P_x &= pbdy - \left(p + \frac{\partial p}{\partial x}dx\right) bdy = -\frac{\partial p}{\partial x} bdxdy, \\ P_y &= -\frac{\partial p}{\partial y} bdxdy. \end{aligned} \tag{3.10}$$

All fluids are viscous. The viscous force F_s is obtained as follows

$$\begin{aligned} S_x &= \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) bdxdy, \\ S_y &= \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) bdxdy, \end{aligned} \tag{3.11}$$

where the coefficient of viscosity μ is constant.

Substituting (3.9),(3.10) and (3.11) into (3.7), one obtains

$$\begin{aligned} \rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= \rho X - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) &= \rho Y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \end{aligned} \tag{3.12}$$

These equations together with the continuity equation (3.4) are called the two-dimensional Navier-Stokes equations. In this research, it is assumed that the body force $F_B = 0$. Thus, in this thesis the following equations are studied

$$\begin{aligned} \rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) &= -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{3.13}$$

3.3 Dimensional Analysis of the Two-Dimensional Navier-Stokes Equations

Let us consider the non dimensional variables $v^*, u^*, p^*, x^*, y^*, t^*$ by setting

$$v^* = \frac{v}{V},\ u^* = \frac{u}{V},\ p^* = \frac{p}{P},\ x^* = \frac{x}{L},\ y^* = \frac{y}{L},\ t^* = \frac{t}{T}, \tag{3.14}$$

where V, P, L, T are velocity, pressure, length and time units, respectively.

Equation (3.14), can be written as

$$v = Vv^*\ , u = Vu^*\ , p = Pp^*,\ x = Lx^*,\ y = Ly^*,\ t = Tt^*.$$

Differentiating theses functions with respect to the independent variables, one has

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{V}{L}\frac{\partial v^*}{\partial x^*}, & \frac{\partial v}{\partial t} &= \frac{V}{T}\frac{\partial v^*}{\partial t^*}, & \frac{\partial p}{\partial x} &= \frac{P}{L}\frac{\partial p^*}{\partial x^*}, \\ \frac{\partial u}{\partial y} &= \frac{V}{L}\frac{\partial u^*}{\partial y^*}, & \frac{\partial u}{\partial t} &= \frac{V}{T}\frac{\partial u^*}{\partial t^*}, & \frac{\partial p}{\partial y} &= \frac{P}{L}\frac{\partial p^*}{\partial y^*}. \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{V}{L^2}\frac{\partial^2 v^*}{\partial (x^*)^2}\ , \\ \frac{\partial^2 u}{\partial y^2} &= \frac{V}{L^2}\frac{\partial^2 u^*}{\partial (y^*)^2}\ . \end{aligned} \tag{3.16}$$

Substituting (3.15),(3.16) into (3.13), one obtains

$$\begin{aligned} \frac{\rho V}{T}\frac{\partial u^*}{\partial t^*} + \frac{\rho V^2}{L}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) &= -\frac{P}{L}\frac{\partial p^*}{\partial x^*} + \frac{\mu V}{L^2}\left(\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2}\right), \\ \frac{\rho V}{T}\frac{\partial v^*}{\partial t^*} + \frac{\rho V^2}{L}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) &= -\frac{P}{L}\frac{\partial p^*}{\partial y^*} + \frac{\mu V}{L^2}\left(\frac{\partial^2 v^*}{\partial (x^*)^2} + \frac{\partial^2 v^*}{\partial (y^*)^2}\right), \end{aligned} \tag{3.17}$$

$$\frac{V}{L}\frac{\partial u^*}{\partial x^*} + \frac{V}{L}\frac{\partial v^*}{\partial y^*} = 0. \tag{3.18}$$

Multiplying equation (3.17) by $L/\rho V^2$ and equation (3.18) by L/V, one has

$$\begin{aligned} \frac{L}{VT}\frac{\partial u^*}{\partial t^*} + \left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) &= -\frac{P}{\rho V^2}\frac{\partial p^*}{\partial x^*} + \frac{\mu}{\rho V L}\left(\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2}\right), \\ \frac{L}{VT}\frac{\partial v^*}{\partial t^*} + \left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) &= -\frac{P}{\rho V^2}\frac{\partial p^*}{\partial y^*} + \frac{\mu}{\rho V L}\left(\frac{\partial^2 v^*}{\partial (x^*)^2} + \frac{\partial^2 v^*}{\partial (y^*)^2}\right), \\ \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} &= 0. \end{aligned} \tag{3.19}$$

Choosing the units L, V, T, P such that

$$\frac{\mu}{\rho VL} = 1, \ \frac{P}{\rho V^2} = 1, \ \frac{L}{VT} = 1$$

one gets

$$\begin{gathered}
\frac{\partial u^*}{\partial t^*} + \left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \left(\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2}\right), \\
\frac{\partial v^*}{\partial t^*} + \left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*} + \left(\frac{\partial^2 v^*}{\partial (x^*)^2} + \frac{\partial^2 v^*}{\partial (y^*)^2}\right), \\
\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0.
\end{gathered} \tag{3.20}$$

After omitting (*), one obtains

$$\frac{\partial u}{\partial t} + \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial^2 u}{\partial (x)^2} + \frac{\partial^2 u}{\partial (y)^2}\right), \tag{3.21}$$

$$\frac{\partial v}{\partial t} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \left(\frac{\partial^2 v}{\partial (x)^2} + \frac{\partial^2 v}{\partial (y)^2}\right), \tag{3.22}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.23}$$

Equations (3.21)-(3.23) are the dimensionless form of the two-dimensional Navier-Stokes equations.

3.4 Two-Dimensional Navier-Stokes Equations Written Through Streamline Function

Let us consider the two-dimensional Navier-Stokes equations (3.21)-(3.23). In the two-dimensional case there are three dependent variables: u, v, p, and three independent variables: t, x, y. Since by the Bäcklund theorem, a group of tangent transformations of equations (3.21)-(3.23) is the first prolongation of a Lie group of point transformations. This research is devoted to contact transformations. In order to study contact transformations of the two-dimensional Navier-Stokes equations one has to reformulate them in equivalent form with only one dependent variable. This can be done as the follows.

Excluding the pressure from equations (3.21),(3.22), by differentiating with respect to x and y respectively, one has

$$u_{ty} + (uu_{xy} + u_x u_y) + (vu_{yy} + u_y v_y) + p_{xy} = u_{xxy} + u_{yyy}, \tag{3.24}$$

$$v_{tx} + (uv_{xx} + v_x u_x) + (vu_{yx} + v_y v_x) + p_{yx} = v_{xxx} + v_{yyx}, \tag{3.25}$$

$$\begin{aligned} &u_{ty} - v_{tx} + u\,(u_{xy} - v_{xx}) + (u_x + v_y)\,(u_y - v_x) + v\,(u_{yy} - v_{yx}) \\ &= (u_{yyy} - v_{xxx}) + (u_{xxy} - v_{yyx})\,. \end{aligned} \tag{3.26}$$

Note that the general solution of equation (3.23) is

$$u = -\psi_y,\ \ v = \psi_x, \tag{3.27}$$

where ψ is an arbitrary function of t, x, y. This function is called a streamline function. After substituting (3.27) into (3.26) one obtains

$$\psi_{yyt}+\psi_{xxt}-\psi_y\,(\psi_{xyy} + \psi_{xxx})+\psi_x\,(\psi_{yyy} + \psi_{yxx}) = (\psi_{yyyy} + \psi_{xxxx})+2\psi_{xxyy}. \tag{3.28}$$

In this thesis the last equation (3.28) is studied.

Chapter IV

Seeking Contact Transformation

The main goal of the research is to find contact transformations of equation (3.28). According to theorem2, first one has to construct the determining equation. The generator X for equation (3.28) is chosen

$$X = \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^t \frac{\partial}{\partial t} + \zeta^{\psi} \frac{\partial}{\partial \psi} + \zeta^{\psi_x} \frac{\partial}{\partial \psi_x} + \zeta^{\psi_y} \frac{\partial}{\partial \psi_y} + \zeta^{\psi_t} \frac{\partial}{\partial \psi_t}, \tag{4.1}$$

where the coefficients depend on the independent variables x, y, t, the dependent variables ψ and the first derivatives ψ_x, ψ_y, ψ_t. Because equation (3.28) is a fourth-order equation, the infinitesimal generator has to be prolonged up to fourth order

$$\underset{4}{X} = X + \zeta^{\psi_{xx}} \frac{\partial}{\partial \psi_{xx}} + \zeta^{\psi_{xy}} \frac{\partial}{\partial \psi_{xy}} + \zeta^{\psi_{xt}} \frac{\partial}{\partial \psi_{xt}} + ... + \zeta^{\psi_{tttt}} \frac{\partial}{\partial \psi_{tttt}}, \tag{4.2}$$

where the coefficients of (4.2) are defined by (2.20). For example,

$$\zeta^{\psi_x} = D_x \zeta^{\psi} - \psi_x D_x \xi^x - \psi_y D_x \xi^y - \psi_t D_x \xi^t,$$

$$\zeta^{\psi_y} = D_y \zeta^{\psi} - \psi_x D_y \xi^x - \psi_y D_y \xi^y - \psi_t D_y \xi^t,$$

$$\zeta^{\psi_t} = D_t \zeta^{\psi} - \psi_x D_t \xi^x - \psi_y D_t \xi^y - \psi_t D_t \xi^t,$$

with

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xt} \frac{\partial}{\partial \psi_t} + ..., \\ D_y &= \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{yt} \frac{\partial}{\partial \psi_t} + ..., \\ D_t &= \frac{\partial}{\partial t} + \psi_t \frac{\partial}{\partial \psi} + \psi_{tt} \frac{\partial}{\partial \psi_t} + ... \end{aligned} \tag{4.3}$$

The full expression for ζ^{ψ_x}, ζ^{ψ_y} and ζ^{ψ_t} are

$$
\begin{aligned}
\zeta^{\psi_x} = & \frac{\partial \zeta^\psi}{\partial x} + \psi_x \frac{\partial \zeta^\psi}{\partial \psi} + \psi_{xx} \frac{\partial \zeta^\psi}{\partial \psi_x} + \psi_{xy} \frac{\partial \zeta^\psi}{\partial \psi_y} + \psi_{xt} \frac{\partial \zeta^\psi}{\partial \psi_t} \\
& - \psi_x \left(\frac{\partial \xi^x}{\partial x} + \psi_x \frac{\partial \xi^x}{\partial \psi} + \psi_{xx} \frac{\partial \xi^x}{\partial \psi_x} + \psi_{xy} \frac{\partial \xi^x}{\partial \psi_y} + \psi_{xt} \frac{\partial \xi^x}{\partial \psi_t} \right) \\
& - \psi_y \left(\frac{\partial \xi^y}{\partial x} + \psi_x \frac{\partial \xi^y}{\partial \psi} + \psi_{xx} \frac{\partial \xi^y}{\partial \psi_x} + \psi_{xy} \frac{\partial \xi^y}{\partial \psi_y} + \psi_{xt} \frac{\partial \xi^y}{\partial \psi_t} \right) \\
& - \psi_t \left(\frac{\partial \xi^t}{\partial x} + \psi_x \frac{\partial \xi^t}{\partial \psi} + \psi_{xx} \frac{\partial \xi^t}{\partial \psi_x} + \psi_{xy} \frac{\partial \xi^t}{\partial \psi_y} + \psi_{xt} \frac{\partial \xi^t}{\partial \psi_t} \right),
\end{aligned}
$$

$$
\begin{aligned}
\zeta^{\psi_y} = & \frac{\partial \zeta^\psi}{\partial y} + \psi_y \frac{\partial \zeta^\psi}{\partial \psi} + \psi_{yx} \frac{\partial \zeta^\psi}{\partial \psi_x} + \psi_{yy} \frac{\partial \zeta^\psi}{\partial \psi_y} + \psi_{yt} \frac{\partial \zeta^\psi}{\partial \psi_t} \\
& - \psi_x \left(\frac{\partial \xi^x}{\partial y} + \psi_y \frac{\partial \xi^x}{\partial \psi} + \psi_{yx} \frac{\partial \xi^x}{\partial \psi_x} + \psi_{yy} \frac{\partial \xi^x}{\partial \psi_y} + \psi_{yt} \frac{\partial \xi^x}{\partial \psi_t} \right) \\
& - \psi_y \left(\frac{\partial \xi^y}{\partial y} + \psi_y \frac{\partial \xi^y}{\partial \psi} + \psi_{yx} \frac{\partial \xi^y}{\partial \psi_x} + \psi_{yy} \frac{\partial \xi^y}{\partial \psi_y} + \psi_{yt} \frac{\partial \xi^y}{\partial \psi_t} \right) \\
& - \psi_t \left(\frac{\partial \xi^t}{\partial y} + \psi_y \frac{\partial \xi^t}{\partial \psi} + \psi_{yx} \frac{\partial \xi^t}{\partial \psi_x} + \psi_{yy} \frac{\partial \xi^t}{\partial \psi_y} + \psi_{yt} \frac{\partial \xi^t}{\partial \psi_t} \right),
\end{aligned}
$$

$$
\begin{aligned}
\zeta^{\psi_t} = & \frac{\partial \zeta^\psi}{\partial t} + \psi_t \frac{\partial \zeta^\psi}{\partial \psi} + \psi_{tx} \frac{\partial \zeta^\psi}{\partial \psi_x} + \psi_{ty} \frac{\partial \zeta^\psi}{\partial \psi_y} + \psi_{tt} \frac{\partial \zeta^\psi}{\partial \psi_t} \\
& - \psi_x \left(\frac{\partial \xi^x}{\partial t} + \psi_t \frac{\partial \xi^x}{\partial \psi} + \psi_{tx} \frac{\partial \xi^x}{\partial \psi_x} + \psi_{ty} \frac{\partial \xi^x}{\partial \psi_y} + \psi_{tt} \frac{\partial \xi^x}{\partial \psi_t} \right) \\
& - \psi_y \left(\frac{\partial \xi^y}{\partial t} + \psi_t \frac{\partial \xi^y}{\partial \psi} + \psi_{tx} \frac{\partial \xi^y}{\partial \psi_x} + \psi_{ty} \frac{\partial \xi^y}{\partial \psi_y} + \psi_{tt} \frac{\partial \xi^y}{\partial \psi_t} \right) \\
& - \psi_t \left(\frac{\partial \xi^t}{\partial t} + \psi_t \frac{\partial \xi^t}{\partial \psi} + \psi_{tx} \frac{\partial \xi^t}{\partial \psi_x} + \psi_{ty} \frac{\partial \xi^t}{\partial \psi_y} + \psi_{tt} \frac{\partial \xi^t}{\partial \psi_t} \right).
\end{aligned}
$$

The other coefficients are obtained in a similar way by using prolongation formulas (2.20).

Note that by virtue in the strength of theorem4 for contact transformations, in the case where $m = 1$ one has

$$
\begin{aligned}
& \xi^x = \frac{\partial W}{\partial \psi_x}, \ \xi^y = \frac{\partial W}{\partial \psi_y}, \ \xi^t = \frac{\partial W}{\partial \psi_t}, \\
& \zeta^\psi = W - \psi_x \frac{\partial W}{\partial \psi_x} - \psi_y \frac{\partial W}{\partial \psi_y} - \psi_t \frac{\partial W}{\partial \psi_t},
\end{aligned}
\tag{4.4}
$$

with a characteristic function $W(x, y, t, \psi, \psi_x, \psi_y, \psi_t)$. Thus, all coefficients in the prolonged generator (4.2) are rewritten through the characteristic function W.

Let us consider the manifold $[F]$ defined by equation (3.28) with the function

$$F\left(x, y, t, \psi, \psi_x, ..., \psi_{tttt}\right) = \psi_{yyt} + \psi_{xxt} - \psi_y\left(\psi_{xyy} + \psi_{xxx}\right) + \psi_x\left(\psi_{yyy} + \psi_{yxx}\right) - \left(\psi_{yyyy} - \psi_{xxxx}\right) + 2\psi_{xxyy} = 0. \tag{4.5}$$

The manifold $[F]$ is considered in the space of variables $x, y, t, \psi, \psi_x, ..., \psi_{tttt}$.

On the manifold described by equation (3.28) one can choose the independent coordinates $x, y, t, \psi, \psi_x, \psi_y, ..., \psi_{xxxx}, \psi_{xxyy}$, and the dependent coordinate ψ_{yyyy} :

$$\psi_{yyyy} = \psi_{yyt} + \psi_{xxt} - \psi_y\left(\psi_{xyy} + \psi_{xxx}\right) + \psi_x\left(\psi_{yyy} + \psi_{yxx}\right) + \psi_{xxxx} + 2\psi_{xxyy}. \tag{4.6}$$

After substituting the dependent variable ψ_{yyyy} into the equation $\underset{4}{X} F = 0$ one obtains the equation which contains the unknown function W, its derivatives $\xi^x, \xi^y, \xi^t, \zeta^\psi$ and the variables $x, y, t, \psi, \psi_x, \psi_y, ..., \psi_{xxxx}, \psi_{xxyy}$. The next step is to analyze the determining equation

$$\underset{4}{X} F \left.\right|_{[F]} = 0. \tag{S}$$

Because this equation is cumbersome, one only explain here the main steps of the calculations.

After differentiating the determining equation with respect to ψ_{xx} and ψ_{xxxx}, respectively, one has

$$\frac{\partial^2 W}{\partial \psi_x^2} = 0. \tag{4.7}$$

The general solution of (4.7) is

$$W = a_1\psi_x + a_2, \tag{4.8}$$

where a_1 and a_2 are functions of $x, y, t, \psi, \psi_x, \psi_y$ and ψ_t. Substituting (4.8) into the determining equation (S) and differentiating it with respect to ψ_{xy} and ψ_{xxyt}, respectively, one obtains

$$\frac{\partial a_1}{\partial \psi_t} = 0. \tag{4.9}$$

Differentiating (S) with respect to ψ_{xyy} and ψ_{yyy}, respectively, one gets

$$\frac{\partial a_1}{\partial \psi_y} = 0. \tag{4.10}$$

From (4.9) and (4.10), one obtains that the function a_1 does not depend on ψ_y and ψ_t. That is, the function a_1 is only a function of x, y, t and ψ.

Differentiating (S) with respect to ψ_{yyyt} and ψ_{yt}, respectively, one obtains

$$\frac{\partial^2 a_2}{\partial \psi_t^2} = 0. \tag{4.11}$$

Hence,

$$a_2 = a_3\psi_t + a_4, \tag{4.12}$$

where a_3 and a_4 are functions of x, y, t, ψ and ψ_y. Substituting (4.12) into (S) and differentiating (S) with respect to ψ_{xx} and ψ_{xxx}, respectively, one has

$$\frac{\partial a_1}{\partial \psi} = 0. \tag{4.13}$$

Hence, the function a_1 does not depend on ψ. Differentiating (S) with respect to ψ_{xxy} and ψ_{yyt}, respectively, one gets

$$\frac{\partial a_3}{\partial \psi_y} = 0. \tag{4.14}$$

Thus, $a_3 = a_3(x, y, t, \psi)$. Differentiating (S) with respect to ψ_{yyy} twice, one obtains

$$\frac{\partial^2 a_4}{\partial \psi_y^2} = 0.$$

Therefore

$$a_4 = a_5\psi_y + a_6, \tag{4.15}$$

where a_5 and a_6 are functions of x, y, t and ψ. Differentiating (S) with respect to ψ_y, ψ_t and ψ_{xx}, respectively, one has

$$\frac{\partial a_3}{\partial \psi} = 0. \tag{4.16}$$

Differentiating (S) with respect to ψ_x, ψ_y and ψ_{yyy}, respectively, one gets

$$\frac{\partial a_5}{\partial \psi} = 0. \tag{4.17}$$

Hence, the functions a_3 and a_5 do not depend on ψ. Differentiating (S) with respect to ψ_t and ψ_{yyy}, respectively, one obtains

$$\frac{\partial a_3}{\partial x} = 0, \tag{4.18}$$

and differentiating (S) with respect to ψ_t and ψ_{xxx}, respectively, one has

$$\frac{\partial a_3}{\partial y} = 0. \tag{4.19}$$

This means that the function a_3 is only a function of t. Differentiating (S) with respect to ψ_{yy} twice, one gets

$$\frac{\partial^2 a_6}{\partial \psi^2} = 0. \tag{4.20}$$

The general solution of (4.20) is

$$a_6 = a_7\psi + a_8 \tag{4.21}$$

where a_7 and a_8 are functions of x, y and t. Substituting (4.21) into (S), and differentiating (S) with respect to ψ and ψ_{yyyy}, respectively, one obtains

$$\frac{\partial a_7}{\partial x} = 0, \tag{4.22}$$

and differentiating (S) with respect to ψ and ψ_{xxxx}, respectively, one has

$$\frac{\partial a_7}{\partial y} = 0. \tag{4.23}$$

Thus, $a_7 = a_7(t)$. Differentiating (S) with respect to ψ_y and ψ_{yy}, respectively, one gets

$$\frac{\partial^2 a_5}{\partial x \partial y} = 0. \tag{4.24}$$

The general solution of (4.24) is

$$a_5 = a_9 + a_{10}, \tag{4.25}$$

where a_9 is a function of x and t, a_{10} is a function of y and t. Substituting (4.25) into (S) and differentiating (S) with respect to y, ψ_y and ψ_{xxy}, respectively, one gets

$$\frac{\partial^2 a_1}{\partial y^2} = 0. \tag{4.26}$$

The general solution of 4.26 is

$$a_1 = a_{11} y + a_{12}, \tag{4.27}$$

where a_{11} and a_{12} are a functions of x and t. Differentiating (S) with respect to ψ_x and ψ_{xx}, respectively, one obtains

$$\frac{\partial a_{11}}{\partial x} = 0. \tag{4.28}$$

Thus, the function a_{11} does not depend on x. Differentiating (S) with respect to x and ψ_{xxt}, one has

$$\frac{\partial^2 a_{12}}{\partial x^2} = 0.$$

This implies

$$a_{12} = a_{13} x + a_{14}, \tag{4.29}$$

where a_{13} and a_{14} are only functions of t. Differentiating (S) with respect to y, ψ_y and ψ_{xyy}, one gets

$$\frac{\partial^2 a_{10}}{\partial y^2} = 0,$$

the general solution of which is

$$a_{10} = a_{15}y + a_{16}, \tag{4.30}$$

where $a_{15} = a_{15}(t)$ and $a_{16} = a_{16}(t)$. Differentiating (S) with respect to y twice and with respect to ψ_{xxx}, one obtains

$$\frac{\partial^3 a_8}{\partial y^3} = 0.$$

Thus,

$$a_8 = a_{17}y^2 + a_{18}y + a_{19}, \tag{4.31}$$

where a_{17}, a_{18} and a_{19} are functions of x and t. Differentiating (S) with respect to x and y, one has

$$\frac{\partial a_{17}}{\partial x} = 0. \tag{4.32}$$

The last equation means that the function a_{17} only depends on t. Differentiating (S) with respect to x, y and ψ_{xxy}, one gets

$$\frac{\partial^2 a_{18}}{\partial x^2} = 0.$$

Hence,

$$a_{18} = a_{20}x + a_{21}, \tag{4.33}$$

where a_{20} and a_{21} only depend on t. Differentiating (S) with respect to x, ψ_x and ψ_{xyy}, one obtains

$$\frac{\partial^2 a_9}{\partial x^2} = 0. \tag{4.34}$$

The general solution of (4.34) is

$$a_9 = a_{22}x + a_{23}, \tag{4.35}$$

where $a_{22} = a_{22}(t)$ and $a_{23} = a_{23}(t)$. Differentiating (S) with respect to ψ_x and ψ_{xyy}, one has

$$a_{11} + a_{22} = 0.$$

Substituting in (S), this means that

$$a_{11} = -a_{22}. \tag{4.36}$$

Differentiating (S) with respect to ψ_x and ψ_{yyy}, respectively, one obtains

$$-a_{13} + a_{15} - a_7 = 0,$$

or

$$a_{13} = a_{15} - a_7. \tag{4.37}$$

Differentiating (S) with respect to ψ_y and ψ_{xxx}, one gets

$$a_7 = 0. \tag{4.38}$$

Differentiating (S) with respect to ψ_{xxx}, one obtains

$$\frac{\partial a_{15}}{\partial t} = 0. \tag{4.39}$$

Thus, the function a_{15} does not depend on t. Differentiating (S) with respect to y and ψ_{yyy}, one has

$$a_{20} = 0. \tag{4.40}$$

Differentiating (S) with respect to x and ψ_y, one gets

$$\frac{\partial^4 a_{19}}{\partial x^4} = 0.$$

The general solution of the last equation is

$$a_{19} = a_{24}x^3 + a_{25}x^2 + a_{26}x + a_{27}, \tag{4.41}$$

where a_{24}, a_{25}, a_{26} and a_{27} are functions of t. Differentiating (S) with respect to x twice and with respect to ψ_{yyy}, one obtains

$$a_{24} = 0. \tag{4.42}$$

Differentiating (S) with respect to x and ψ_{yyy}, one has

$$\frac{\partial a_{22}}{\partial t} + 2a_{25} = 0,$$

or

$$a_{25} = -\frac{1}{2}\frac{\partial a_{22}}{\partial t}. \tag{4.43}$$

Differentiating (S) with respect to ψ_{xxt}, one gets

$$\frac{\partial a_3}{\partial t} - 2a_{15} = 0.$$

Solving the last equation for a_{15} thus means that

$$a_{15} = -\frac{1}{2}\frac{\partial a_3}{\partial t}, \tag{4.44}$$

and differentiating (S) with respect to ψ_{xx}, one obtains

$$\frac{\partial^2 a_3}{\partial t^2} = 0. \tag{4.45}$$

The general solution of (4.45) is

$$a_3 = a_{28}t + a_{29}, \tag{4.46}$$

where a_{28} and a_{29} are constants. After differentiating (S) with respect to ψ_{yyy}, one has

$$\frac{\partial a_{16}}{\partial t} + \frac{\partial a_{23}}{\partial t} + a_{26} = 0.$$

That is

$$a_{26} = -\left(\frac{\partial a_{16}}{\partial t} + \frac{\partial a_{23}}{\partial t}\right). \tag{4.47}$$

Differentiating (S) with respect to y and ψ_{xxx}, one gets

$$\frac{\partial a_{22}}{\partial t} + 2a_{17} = 0.$$

Hence,

$$a_{17} = -\frac{1}{2}\frac{\partial a_{22}}{\partial t}. \tag{4.48}$$

Differentiating (S) with respect to ψ_{xxx}, one obtains

$$-\frac{\partial a_{14}}{\partial t}+a_{21}=0,$$

or

$$a_{21}=\frac{\partial a_{14}}{\partial t}. \tag{4.49}$$

After all substitutions, the determining equation (S) becomes

$$S=\frac{\partial^2 a_{22}}{\partial t^2}=0.$$

That is

$$a_{22}=a_{30}t+a_{31}. \tag{4.50}$$

Changing

$$a_{30}=c_1,\ a_{31}=c_2,\ a_{28}=c_3,\ a_{29}=c_4,\ a_{14}=f_1,\ a_{16}+a_{23}=f_2,\ a_{27}=f_3.$$

One finds from equation (4.8) that

$$\begin{aligned} W=&\left(\frac{c_3}{2}x-(c_1t+c_2)\,y+f_1\right)\psi_x+\left((c_1t+c_2)\,x+\frac{c_3}{2}y+f_2\right)\psi_y\\ &+(c_3t+c_4)\,\psi_t+y\frac{df_1}{dt}-x\frac{df_2}{dt}-\frac{c_1}{2}\left(x^2+y^2\right)+f_3, \end{aligned} \tag{4.51}$$

where f_1, f_2 and f_3 are functions of t, c_1, c_2 and c_3 are constant. Therefore, after substituting the function W in (4.51) into (4.4), the unknown functions are

$$\begin{aligned} \xi^x &= (c_1t+c_2)\,y-\frac{c_3}{2}x-f_1,\\ \xi^y &= -\left((c_1t+c_2)\,x+\frac{c_3}{2}y+f_2\right),\\ \xi^t &= -\left(c_3t+c_4\right),\\ \zeta^\psi &= y\frac{df_1}{dt}-x\frac{df_2}{dt}-\frac{c_1}{2}\left(x^2+y^2\right)+f_3. \end{aligned} \tag{4.52}$$

Observe that in equation (4.52), the function ξ^x, ξ^y, ξ^t and ζ^ψ depend only on the independent variables x, y and t but do not depend on first derivatives. Thus the transformations obtained are not proper contact transformation: they are the first prolongation of point transformations. This is the result of the research done.

Chapter V

Lie-Bäcklund Transformations

This part of the research is devoted to finding Lie-Bäcklund transformations admitted by equation (3.28). The process of finding Lie-Bäcklund transformations is similar to the study of contact transformations. First one has to construct the determining equation. The generator Y for this case is taken in a canonical form.

$$Y = \zeta^{\psi}\frac{\partial}{\partial\psi} + \zeta^{\psi_x}\frac{\partial}{\partial\psi_x} + \zeta^{\psi_y}\frac{\partial}{\partial\psi_y} + ... + \zeta^{\psi_{tttt}}\frac{\partial}{\partial\psi_{tttt}}. \tag{5.1}$$

In this thesis one suppose that the coefficient ζ^{ψ} depends on the independent variables x, y, t, the dependent variable ψ and the derivatives up to second order: $\psi_x, \psi_y, \psi_t, \psi_{xx}, \psi_{xy}, \psi_{xt}, \psi_{yy}, \psi_{tt}$. The coefficients $\zeta^{\psi_x}, \zeta^{\psi_y}, ..., \zeta^{\psi_{tttt}}$ are defined by the prolongation formulas (2.76), for example,

$$\zeta^{\psi_x} = D_x\zeta^{\psi}, \quad \zeta^{\psi_y} = D_y\zeta^{\psi}, \quad \zeta^{\psi_t} = D_t\zeta^{\psi}, \tag{5.2}$$

where the differential operators D_x, D_y, D_t are defined as in the case of contact transformations (4.3). For example,

$$\begin{aligned}
\zeta^{\psi_x} =& \frac{\partial\zeta^{\psi}}{\partial x} + \psi_x\frac{\partial\zeta^{\psi}}{\partial\psi} + \psi_{xx}\frac{\partial\zeta^{\psi}}{\partial\psi_x} + \psi_{xy}\frac{\partial\zeta^{\psi}}{\partial\psi_y} + \psi_{xt}\frac{\partial\zeta^{\psi}}{\partial\psi_t} + \psi_{xxx}\frac{\partial\zeta^{\psi}}{\partial\psi_{xx}} \\
&+ \psi_{xxy}\frac{\partial\zeta^{\psi}}{\partial\psi_{xy}} + \psi_{xxt}\frac{\partial\zeta^{\psi}}{\partial\psi_{xt}} + \psi_{xyy}\frac{\partial\zeta^{\psi}}{\partial\psi_{yy}} + \psi_{xyt}\frac{\partial\zeta^{\psi}}{\partial\psi_{yt}} + \psi_{xtt}\frac{\partial\zeta^{\psi}}{\partial\psi_{tt}}, \\
\zeta^{\psi_y} =& \frac{\partial\zeta^{\psi}}{\partial y} + \psi_y\frac{\partial\zeta^{\psi}}{\partial\psi} + \psi_{yx}\frac{\partial\zeta^{\psi}}{\partial\psi_x} + \psi_{yy}\frac{\partial\zeta^{\psi}}{\partial\psi_y} + \psi_{yt}\frac{\partial\zeta^{\psi}}{\partial\psi_t} + \psi_{yxx}\frac{\partial\zeta^{\psi}}{\partial\psi_{xx}} \\
&+ \psi_{yxy}\frac{\partial\zeta^{\psi}}{\partial\psi_{xy}} + \psi_{yxt}\frac{\partial\zeta^{\psi}}{\partial\psi_{xt}} + \psi_{yyy}\frac{\partial\zeta^{\psi}}{\partial\psi_{yy}} + \psi_{yyt}\frac{\partial\zeta^{\psi}}{\partial\psi_{yt}} + \psi_{ytt}\frac{\partial\zeta^{\psi}}{\partial\psi_{tt}}, \\
\zeta^{\psi_t} =& \frac{\partial\zeta^{\psi}}{\partial t} + \psi_t\frac{\partial\zeta^{\psi}}{\partial\psi} + \psi_{tx}\frac{\partial\zeta^{\psi}}{\partial\psi_x} + \psi_{ty}\frac{\partial\zeta^{\psi}}{\partial\psi_y} + \psi_{tt}\frac{\partial\zeta^{\psi}}{\partial\psi_t} + \psi_{txx}\frac{\partial\zeta^{\psi}}{\partial\psi_{xx}} \\
&+ \psi_{txy}\frac{\partial\zeta^{\psi}}{\partial\psi_{xy}} + \psi_{txt}\frac{\partial\zeta^{\psi}}{\partial\psi_{xt}} + \psi_{tyy}\frac{\partial\zeta^{\psi}}{\partial\psi_{yy}} + \psi_{tyt}\frac{\partial\zeta^{\psi}}{\partial\psi_{yt}} + \psi_{ttt}\frac{\partial\zeta^{\psi}}{\partial\psi_{tt}}.
\end{aligned}$$

Let us analyze the manifold $[F]$ defined by the fourth-order equation (3.28) for Lie-Bäcklund transformations of second order. Since the coefficients of the $\underset{4}{X}F$ contain derivatives up to sixth-order, then for $[F]$ one needs to use prolongations of the manifold defined by equation (3.28) up to sixth-order:

$$\begin{aligned} D_xF = 2\,&(\psi_{yytx} + \psi_{xxtx} - \psi_y\,(\psi_{xyyx} + \psi_{xxxx}) - \psi_{yx}\,(\psi_{xyy} + \psi_{xxx}) \\ &+ \psi_x\,(\psi_{yyyx} + \psi_{yxxx}) + \psi_{xx}\,(\psi_{yyy} + \psi_{yxx}) - (\psi_{yyyyx} - \psi_{xxxxx}) \\ &+ 2\psi_{xxyyx}) = 0, \end{aligned}$$

$$\begin{aligned} D_yF = 2\,&(\psi_{yyty} + \psi_{xxty} - \psi_y\,(\psi_{xyyy} + \psi_{xxxy}) - \psi_{yy}\,(\psi_{xyy} + \psi_{xxx}) \\ &+ \psi_x\,(\psi_{yyyy} + \psi_{yxxy}) + \psi_{xy}\,(\psi_{yyy} + \psi_{yxx}) - (\psi_{yyyyy} - \psi_{xxxxy}) \\ &+ 2\psi_{xxyyy}) = 0, \end{aligned}$$

$$\begin{aligned} D_tF = 2\,&(\psi_{yytt} + \psi_{xxtt} - \psi_y\,(\psi_{xyyt} + \psi_{xxxt}) - \psi_{yt}\,(\psi_{xyy} + \psi_{xxx}) \\ &+ \psi_x\,(\psi_{yyyt} + \psi_{yxxt}) + \psi_{xt}\,(\psi_{yyy} + \psi_{yxx}) - (\psi_{yyyyt} - \psi_{xxxxt}) \\ &+ 2\psi_{xxyyt}) = 0, \end{aligned}$$

$$\begin{aligned} D_xD_xF = 4\,&(\psi_{yytxx} + \psi_{xxtxx} - \psi_y\,(\psi_{xyyxx} + \psi_{xxxxx}) - 2\psi_{yx}\,(\psi_{xyyx} + \psi_{xxxx}) \\ &- \psi_{yxx}\,(\psi_{xyy} + \psi_{xxx}) + \psi_x\,(\psi_{yyyxx} + \psi_{yxxxx}) + 2\psi_{xx}\,(\psi_{yyyx} + \psi_{yxxx}) \\ &+ \psi_{xxx}\,(\psi_{yyy} + \psi_{yxx}) + \psi_{xxyyxx}) - 2\,(\psi_{yyyyxx} - \psi_{xxxxxx}) = 0, \\ &\vdots \end{aligned}$$

$$\begin{aligned} D_tD_tF = 4\,&(\psi_{yyttt} + \psi_{xxttt} - \psi_y\,(\psi_{xyytt} + \psi_{xxxtt}) - 2\psi_{yt}\,(\psi_{xyyt} + \psi_{xxxt}) \\ &- \psi_{ytt}\,(\psi_{xyy} + \psi_{xxx}) + \psi_x\,(\psi_{yyytt} + \psi_{yxxtt}) + 2\psi_{xt}\,(\psi_{yyyt} + \psi_{yxxt}) \\ &+ \psi_{xtt}\,(\psi_{yyy} + \psi_{yxx}) + \psi_{xxyytt}) - 2\,(\psi_{yyyytt} - \psi_{xxxxtt}) = 0. \end{aligned}$$

From these equations one can define $\psi_{yyyyx}, \psi_{yyyyy}, \psi_{yyyyt}, \psi_{yyyyxx}, \psi_{yyyyxy}, ..., \psi_{yyyytt}$:

$$\begin{aligned} \psi_{yyyyx} = \psi_{yytx} &+ \psi_{xxtx} - \psi_y\,(\psi_{xyyx} + \psi_{xxxx}) - \psi_{yx}\,(\psi_{xyy} + \psi_{xxx}) \\ &+ \psi_x\,(\psi_{yyyx} + \psi_{yxxx}) + \psi_{xx}\,(\psi_{yyy} + \psi_{yxx}) + \psi_{xxxxx} + 2\psi_{xxyyx}, \end{aligned}$$

$$\psi_{yyyyy} = \psi_{yyty} + \psi_{xxty} - \psi_y\left(\psi_{xyyy} + \psi_{xxxy}\right) - \psi_{yy}\left(\psi_{xyy} + \psi_{xxx}\right)$$
$$+ \psi_x\left(\psi_{yyyy} + \psi_{yxxy}\right) + \psi_{xy}\left(\psi_{yyy} + \psi_{yxx}\right) + \psi_{xxxxy} + 2\psi_{xxyyy},$$
$$\psi_{yyyyt} = \psi_{yytt} + \psi_{xxtt} - \psi_y\left(\psi_{xyyt} + \psi_{xxxt}\right) - \psi_{yt}\left(\psi_{xyy} + \psi_{xxx}\right)$$
$$+ \psi_x\left(\psi_{yyyt} + \psi_{yxxt}\right) + \psi_{xt}\left(\psi_{yyy} + \psi_{yxx}\right) + \psi_{xxxxt} + 2\psi_{xxyyt},$$
$$\psi_{yyyyxx} = 2\left(\psi_{yytxx} + \psi_{xxtxx} - \psi_y\left(\psi_{xyyxx} + \psi_{xxxxx}\right) - 2\psi_{yx}\left(\psi_{xyyx} + \psi_{xxxx}\right)\right.$$
$$- \psi_{yxx}\left(\psi_{xyy} + \psi_{xxx}\right) + \psi_x\left(\psi_{yyyxx} + \psi_{yxxxx}\right) + 2\psi_{xx}\left(\psi_{yyyx} + \psi_{yxxx}\right),$$
$$\left. + \psi_{xxx}\left(\psi_{yyy} + \psi_{yxx}\right) + \psi_{xxyyxx}\right) + \psi_{xxxxxx},$$
$$\vdots$$
$$\psi_{yyyytt} = 2\left(\psi_{yyttt} + \psi_{xxttt} - \psi_y\left(\psi_{xyytt} + \psi_{xxxtt}\right) - 2\psi_{yt}\left(\psi_{xyyt} + \psi_{xxxt}\right)\right.$$
$$- \psi_{ytt}\left(\psi_{xyy} + \psi_{xxx}\right) + \psi_x\left(\psi_{yyytt} + \psi_{yxxtt}\right) + 2\psi_{xt}\left(\psi_{yyyt} + \psi_{yxxt}\right)$$
$$\left. + \psi_{xtt}\left(\psi_{yyy} + \psi_{yxx}\right) + \psi_{xxyytt}\right) + \psi_{xxxxtt}. \tag{5.3}$$

This means that in the determining equation

$$\underset{4}{Y} F \Big|_{[F]} = 0, \tag{S}$$

the determining equation contains only one unknown function ζ^{ψ}. The left side of the determining equation is a polynomial function with respect to parametric derivatives up to six-order. The main derivatives are $\psi_{yyyyx}, \psi_{yyyyy}, \psi_{yyyyt}, \psi_{yyyyxx}, \psi_{yyyyxy}, ..., \psi_{yyyytt}$.

The determining equation for Lie-Bäcklund transformations is more complicated than that for contact transformations. The determining equation (S) is a fourth-order polynomial equation with respect to parametric derivatives up to six-order. It can be split in a similar way as is explained in the example below (see program in appendix B).

The main idea of splitting the determining equation is the following. Assume that the determining equation S is a second-degree polynomial function of the variables x and y :

$$S = Ax^2 + By^2 + Cxy + Dx + Ey + F = 0, \tag{5.4}$$

where A, B, C, D, E and F are some expressions which do not depend on x and y. Differentiating (5.4) with respect to x twice and multiplying by $1/2$, one obtains

$$\frac{1}{2}\frac{\partial^2(S)}{\partial x^2} = A = 0. \tag{5.5}$$

Substituting it into (5.4), one gets

$$S = By^2 + Cxy + Dx + Ey + F = 0. \tag{5.6}$$

In the Reduce-program substitution is implemented by subtraction of Ax^2 from S. Differentiating (5.6) with respect to x and y, respectively, one has

$$\frac{\partial^2(S)}{\partial x \partial y} = C = 0. \tag{5.7}$$

Substituting it into (5.6), one obtains

$$S = By^2 + Dx + Ey + F = 0. \tag{5.8}$$

Differentiating (5.8) with respect to y twice and multiplying by $1/2$, one gets

$$\frac{1}{2}\frac{\partial^2(S)}{\partial y^2} = B = 0. \tag{5.9}$$

Hence,

$$S = Dx + Ey + F = 0. \tag{5.10}$$

Differentiating (5.10) with respect to x, one has

$$\frac{\partial S}{\partial x} = D = 0. \tag{5.11}$$

In a similar way $E = 0$, thus

$$S = F = 0. \tag{5.12}$$

Therefore all coefficients of the polynomial (5.4) are equal to zero. Note once more that these coefficients are expressions that do not depend on x and y. This is a simple example for splitting the determining equation. Moreover, if the determining equation is not a second order polynomial, then it can be split in the same way. In the determining equation, the parametric derivatives play the role of the variables x and y.

The result of calculations using the Reduce program in Appendix B show that

$$\begin{aligned} \zeta^{\psi} = & -\left((c_1 t + c_2)y - \frac{c_3}{2}x - f_1\right)\psi_x + \left((c_1 t + c_2)x + \frac{c_3}{2}y + f_2\right)\psi_y \\ & + (c_3 t + c_4)\,\psi_t + \left(y\frac{df_1}{dt} - x\frac{df_2}{dt} - \frac{c_1}{2}\left(x^2 + y^2\right) + f_3\right) \end{aligned} \tag{5.13}$$

where f_1, f_2, f_3 and g_4 are functions of t, c_1, c_2, c_3 and c_4 are constant.

The obtained canonical operator is $Y = \zeta^{\psi}\partial_{\psi}$. Note that the first prolongation of the point transformations is

$$\begin{aligned} X = & \left((c_1 t + c_2)y - \frac{c_3}{2}x - f_1\right)\frac{\partial}{\partial x} - \left((c_1 t + c_2)x + \frac{c_3}{2}y + f_2\right)\frac{\partial}{\partial y} \\ & - (c_3 t + c_4)\frac{\partial}{\partial t} + \left(y\frac{df_1}{dt} - x\frac{df_2}{dt} - \frac{c_1}{2}\left(x^2 + y^2\right) + f_3\right)\frac{\partial}{\partial \psi}. \end{aligned} \tag{5.14}$$

Hence,

$$\begin{aligned} X - Y = & \left((c_1 t + c_2)y - \frac{c_3}{2}x - f_1\right)\frac{\partial}{\partial x} - \left((c_1 t + c_2)x + \frac{c_3}{2}y + f_2\right)\frac{\partial}{\partial y} \\ & - (c_3 t + c_4)\frac{\partial}{\partial t} + \left[\left((c_1 t + c_2)y - \frac{c_3}{2}x - f_1\right)\psi_x \right. \\ & \left. - \left((c_1 t + c_2)x + \frac{c_3}{2}y + f_2\right)\psi_y - (c_3 t + c_4)\,\psi_t\right]\frac{\partial}{\partial \psi} + \ldots \\ = & \ \xi^x\frac{\partial}{\partial x} + \xi^y\frac{\partial}{\partial y} + \xi^t\frac{\partial}{\partial t} + \left(\xi^x\psi_x + \xi^y\psi_y + \xi^t\psi_t\right)\frac{\partial}{\partial \psi} + \ldots \\ = & \ \xi^x D_x + \xi^y D_y + \xi^t D_t. \end{aligned}$$

Thus the canonical operator Y is equivalent to the generator of the first prolongation of point transformations. This means that the Lie-Bäcklund transformations which are restricted to the independent variables x, y, t, the dependent variable ψ and the derivatives up to second order: $\psi_x, \psi_y, \psi_t, \psi_{xx}, \psi_{xy}, \psi_{xt}, \psi_{yy}, \psi_{tt}$ are prolongations of point transformations. This is the result of the research.

Chapter VI

Conclusion

6.1 Thesis Summary

This thesis is devoted to the study of contact and second order Lie-Bäcklund transformations of the two-dimensional Navier-Stokes equations.

6.1.1 Problem

The two-dimensional Navier-Stokes equations considered in the thesis are written in the form

$$\psi_{yyt} + \psi_{xxt} - \psi_y \left(\psi_{xyy} + \psi_{xxx}\right) + \psi_x \left(\psi_{yyy} + \psi_{yxx}\right) = \left(\psi_{yyyy} + \psi_{xxxx}\right) + 2\psi_{xxyy}$$

where ψ is a streamline function, which depends on t, x, y. The coefficients of the generator

$$X = \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^t \frac{\partial}{\partial t} + \zeta^\psi \frac{\partial}{\partial \psi} + \zeta^{\psi_x} \frac{\partial}{\partial \psi_x} + \zeta^{\psi_y} \frac{\partial}{\partial \psi_y} + \zeta^{\psi_t} \frac{\partial}{\partial \psi_t}$$

of contact transformations are defined through a characteristic function $W(x, y, t, \psi, \psi_x, \psi_y, \psi_t)$ by formulas (4.4). Prolongations of the generator X are defined by (2.20). Similarly, for finding admitted Lie-Bäcklund transformations, the problem is to find the coefficient ζ^ψ of the canonical operator:

$$Y = \zeta^\psi \frac{\partial}{\partial \psi}. \tag{6.1}$$

In this research it is assumed that the coefficient ζ^ψ depends on the independent variables x, y, t, the dependent variable ψ and the derivatives up to second order:

$\psi_x, \psi_y, \psi_t, \psi_{xx}, \psi_{xy}, \psi_{xt}, \psi_{yy}, \psi_{tt}$. The coefficients $\zeta^{\psi_x}, \zeta^{\psi_y}, ..., \zeta^{\psi_{tttt}}$ are defined by the prolongation formulas (2.76).

6.1.2 Result

1. For contact transformations, it is found that the characteristic function is

$$W = \left(\frac{c_3}{2}x - (c_1t + c_2)\, y + f_1\right)\psi_x + \left((c_1t + c_2)\, x + \frac{c_3}{2}y + f_2\right)\psi_y$$
$$+ (c_3t + c_4)\,\psi_t + y\frac{df_1}{dt} - x\frac{df_2}{dt} - \frac{c_1}{2}\left(x^2 + y^2\right) + f_3.$$

The coefficients of the generator X are

$$\xi^x = (c_1t + c_2)\, y - \frac{c_3}{2}x - f_1,$$
$$\xi^y = -\left((c_1t + c_2)\, x + \frac{c_3}{2}y + f_2\right),$$
$$\xi^t = -(c_3t + c_4)\,,$$
$$\zeta^\psi = y\frac{df_1}{dt} - x\frac{df_2}{dt} - \frac{c_1}{2}\left(x^2 + y^2\right) + f_3,$$

where f_1, f_2, f_3 are arbitrary functions of t and c_1, c_2, c_3 are arbitrary constants. This means that contact transformations are the first prolongation of point transformations.

2. As the result of calculations it is obtained that the coefficient ζ^ψ of the Lie-Bäcklund operator (6.1) is

$$\zeta^\psi = -\left((c_1t + c_2)y - \frac{c_3}{2}x - f_1\right)\psi_x + \left((c_1t + c_2)x + \frac{c_3}{2}y + f_2\right)\psi_y$$
$$+ (c_3t + c_4)\,\psi_t + \left(y\frac{df_1}{dt} - x\frac{df_2}{dt} - \frac{c_1}{2}\left(x^2 + y^2\right) + f_3\right).$$

This also means that the second order Lie-Bäcklund transformations are the first prolongation of point transformations.

6.1.3 Limitations

In this thesis, group analysis was applied to the Navier-Stokes equations in the two-dimensional case written through the streamline function. If was assumed that the coefficient ζ^ψ of a canonical Lie-Bäcklund operator depends on derivatives not higher than second order.

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Appendix

Appendix A

Program of Seeking Contact Transformations

A.1 The REDUCE program for finding the determining equations of equation (3.28)

In the REDUCE program the following symbols were used to represent the various variables used in the research

`ux` $= \psi$, `uy` $= \psi_y$, `ut` $= \psi_t$, `uxx` $= \psi_{xx}$,..., `utttt` $= \psi_{tttt}$,
`zeta_u` $= \zeta^{\psi}$, `zeta_ux` $= \zeta^{\psi_x}$,
`zeta_uy` $= \zeta^{\psi_y}$, `zeta_ut` $= \zeta^{\psi_t}$,
`zeta_uxx` $= \zeta^{\psi_{xx}}$,..., `zeta_uyyyy` $= \zeta^{\psi_{yyyy}}$,
`psix` $= \xi^x$, `psiy` $= \xi^y$, `psit` $= \xi^t$,
`equation` $= F$, `generator` $= X$, `determin` $= S$

and for `x,y,t,w` the same notation is used. Next one explain identifier and main commands.

The commands

```
depend psix,x,y,t,u,ut,ux,uy$
depend psiy,x,y,t,u,ut,ux,uy$
depend psit,x,y,t,u,ut,ux,uy;$
depend zeta_u,x,y,t,u,ut,ux,uy$
depend w,x,y,t,u,ut,ux,uy$

depend ff,x,y,t,u,
       ux,uy,ut,
       uxx,uxy,uxt,uyy,uyt,utt,
       uxxx,uxxy,uxxt,uxyy,uxyt,uxtt,
       uyyt,uytt,uttt,uyyy,
       uxxxx,uxxxy,uxxxt,uxxyy,uxxyt,uxxtt,
       uxyyy,uxyyt,uxytt,uxttt,uyyyy,uyyyt,uyytt,uyttt,utttt$
```

mean that `psix, psiy, psit` and `w` are functions of the dependent variables, depending on `x, y, t, u, ut, ux, uy`. The function `ff` depend on `x, y, t, u, ut, ux, uy ,..., utttt`.

The command

```
equation :=uyyyy+uxxxx+2*uxxyy-uyyt-uxxt+uy*(uxyy+uxxx)-ux*(uyyy+uxxy)$
```

corresponds to (4.5).

The command

```
generator:=zeta_ux*df(ff,ux)+zeta_uy*df(ff,uy)+zeta_uxxt*df(ff,uxxt)+
           zeta_uyyt*df(ff,uyyt)+zeta_uxyy*df(ff,uxyy)+zeta_uxxy*df(ff,uxxy)+
           zeta_uxxx*df(ff,uxxx)+zeta_uyyy*df(ff,uyyy)+zeta_uyyyy*df(ff,uyyyy)+
           zeta_uxxxx*df(ff,uxxxx)+zeta_uxxyy*df(ff,uxxyy)$
```

defines the infinitesimal generator of the function ff.
The command

```
determin := sub(ff=equation,generator)$
```

substitutes the equation into the generator.
The commands

```
dx :=  df(ff,x)+ux*df(ff,u)+
       uxx*df(ff,ux)+uxy*df(ff,uy)+uxt*df(ff,ut)+
       uxxx*df(ff,uxx)+uxxy*df(ff,uxy)+uxyy*df(ff,uyy)+
       uxxt*df(ff,uxt)+uxyt*df(ff,uyt)+uxtt*df(ff,utt)+
       uxxxx*df(ff,uxxx)+uxxxy*df(ff,uxxy)+uxxyy*df(ff,uxyy)+
       uxxxt*df(ff,uxxt)+uxxyt*df(ff,uxyt)+uxxtt*df(ff,uxtt)+
       uxyyy*df(ff,uyyy)+uxyyt*df(ff,uyyt)+uxytt*df(ff,uytt)+
       uxttt*df(ff,uttt)+
       uxxxxx*df(ff,uxxxx)+uxxxxy*df(ff,uxxxy)+uxxxxt*df(ff,uxxxt)+
       uxxxyy*df(ff,uxxyy)+uxxxyt*df(ff,uxxyt)+uxxxtt*df(ff,uxxtt)+
       uxxyyy*df(ff,uxyyy)+uxxyyt*df(ff,uxyyt)+uxxytt*df(ff,uxytt)+
       uxxttt*df(ff,uxttt)+uxyyyy*df(ff,uyyyy)+uxyyyt*df(ff,uyyyt)+
       uxyytt*df(ff,uyytt)+uxyttt*df(ff,uyttt)+uxtttt*df(ff,utttt)$

dy :=  df(ff,y)+uy*df(ff,u)+
       uxy*df(ff,ux)+uyy*df(ff,uy)+uyt*df(ff,ut)+
       uxxy*df(ff,uxx)+uxyy*df(ff,uxy)+uyyy*df(ff,uyy)+
       uxyt*df(ff,uxt)+uyyt*df(ff,uyt)+uytt*df(ff,utt)+
       uxxxy*df(ff,uxxx)+uxxyy*df(ff,uxxy)+uxyyy*df(ff,uxyy)+
       uxxyt*df(ff,uxxt)+uxyyt*df(ff,uxyt)+uxytt*df(ff,uxtt)+
       uyyyy*df(ff,uyyy)+uyyyt*df(ff,uyyt)+uyytt*df(ff,uytt)+
       uyttt*df(ff,uttt)+
       uxxxxy*df(ff,uxxxx)+uxxxyy*df(ff,uxxxy)+uxxxyt*df(ff,uxxxt)+
       uxxyyy*df(ff,uxxyy)+uxxyyt*df(ff,uxxyt)+uxxytt*df(ff,uxxtt)+
       uxyyyy*df(ff,uxyyy)+uxyyyt*df(ff,uxyyt)+uxyytt*df(ff,uxytt)+
       uxyttt*df(ff,uxttt)+uyyyyy*df(ff,uyyyy)+uyyyyt*df(ff,uyyyt)+
       uyyytt*df(ff,uyytt)+uyyttt*df(ff,uyttt)+uytttt*df(ff,utttt)$

dt :=  df(ff,t)+ut*df(ff,u)+
       uxt*df(ff,ux)+uyt*df(ff,uy)+utt*df(ff,ut)+
       uxxt*df(ff,uxx)+uxyt*df(ff,uxy)+uyyt*df(ff,uyy)+
       uxtt*df(ff,uxt)+uytt*df(ff,uyt)+uttt*df(ff,utt)+
       uxxxt*df(ff,uxxx)+uxxyt*df(ff,uxxy)+uxyyt*df(ff,uxyy)+
       uxxtt*df(ff,uxxt)+uxytt*df(ff,uxyt)+uxttt*df(ff,uxtt)+
       uyyyt*df(ff,uyyy)+uyytt*df(ff,uyyt)+uyttt*df(ff,uytt)+
       utttt*df(ff,uttt)+
       uxxxxt*df(ff,uxxxx)+uxxxyt*df(ff,uxxxy)+uxxxtt*df(ff,uxxxt)+
       uxxyyt*df(ff,uxxyy)+uxxytt*df(ff,uxxyt)+uxxttt*df(ff,uxxtt)+
       uxyyyt*df(ff,uxyyy)+uxyytt*df(ff,uxyyt)+uxyttt*df(ff,uxytt)+
       uxtttt*df(ff,uxttt)+uyyyyt*df(ff,uyyyy)+uyyytt*df(ff,uyyyt)+
       uyyttt*df(ff,uyytt)+uytttt*df(ff,uyttt)+uttttt*df(ff,utttt)$
```

define the operators `dx,dy` and `dt` respectively. They correspond to (4.3).
The commands

```
zeta_ut:=sub(ff=zeta_u,dt)-ut*sub(ff=psit,dt)-ux*sub(ff=psix,dt)-uy*sub(ff=psiy,dt)$
zeta_ux:=sub(ff=zeta_u,dx)-ut*sub(ff=psit,dx)-ux*sub(ff=psix,dx)-uy*sub(ff=psiy,dx)$
zeta_uy:=sub(ff=zeta_u,dy)-ut*sub(ff=psit,dy)-ux*sub(ff=psix,dy)-uy*sub(ff=psiy,dy)$

zeta_uxx:=sub(ff=zeta_ux,dx)-uxt*sub(ff=psit,dx)-uxx*sub(ff=psix,dx)-uxy*sub(ff=psiy,dx)$
zeta_uxy:=sub(ff=zeta_ux,dy)-uxt*sub(ff=psit,dy)-uxx*sub(ff=psix,dy)-uxy*sub(ff=psiy,dy)$
zeta_uxt:=sub(ff=zeta_ux,dt)-uxt*sub(ff=psit,dt)-uxx*sub(ff=psix,dt)-uxy*sub(ff=psiy,dt)$
zeta_uyy:=sub(ff=zeta_uy,dy)-uyt*sub(ff=psit,dy)-uxy*sub(ff=psix,dy)-uyy*sub(ff=psiy,dy)$
zeta_uyt:=sub(ff=zeta_uy,dt)-uyt*sub(ff=psit,dt)-uxy*sub(ff=psix,dt)-uyy*sub(ff=psiy,dt)$
zeta_utt:=sub(ff=zeta_ut,dt)-utt*sub(ff=psit,dt)-uxt*sub(ff=psix,dt)-uyt*sub(ff=psiy,dt)$
zeta_uxxx:=sub(ff=zeta_uxx,dx)-uxxt*sub(ff=psit,dx)-uxxx*sub(ff=psix,dx)-uxxy*sub(ff=psiy,dx)$
zeta_uxxy:=sub(ff=zeta_uxx,dy)-uxxt*sub(ff=psit,dy)-uxxx*sub(ff=psix,dy)-uxxy*sub(ff=psiy,dy)$
```

```
zeta_uxxt:=sub(ff=zeta_uxx,dt)-uxxt*sub(ff=psit,dt)-uxxx*sub(ff=psix,dt)-uxxy*sub(ff=psiy,dt)$
zeta_uxyy:=sub(ff=zeta_uxy,dy)-uxyt*sub(ff=psit,dy)-uxxy*sub(ff=psix,dy)-uxyy*sub(ff=psiy,dy)$
zeta_uxyt:=sub(ff=zeta_uxy,dt)-uxyt*sub(ff=psit,dt)-uxxy*sub(ff=psix,dt)-uxyy*sub(ff=psiy,dt)$
zeta_uxtt:=sub(ff=zeta_uxt,dt)-uxtt*sub(ff=psit,dt)-uxxt*sub(ff=psix,dt)-uxyt*sub(ff=psiy,dt)$
zeta_uyyy:=sub(ff=zeta_uyy,dy)-uyyt*sub(ff=psit,dy)-uxyy*sub(ff=psix,dy)-uyyy*sub(ff=psiy,dy)$
zeta_uyyt:=sub(ff=zeta_uyy,dt)-uyyt*sub(ff=psit,dt)-uxyy*sub(ff=psix,dt)-uyyy*sub(ff=psiy,dt)$
zeta_uytt:=sub(ff=zeta_uyt,dt)-uytt*sub(ff=psit,dt)-uxyt*sub(ff=psix,dt)-uyyt*sub(ff=psiy,dt)$
zeta_uttt:=sub(ff=zeta_utt,dt)-uttt*sub(ff=psit,dt)-uxtt*sub(ff=psix,dt)-uytt*sub(ff=psiy,dt)$
zeta_uyyyy:=sub(ff=zeta_uyyy,dy)-uyyyt*sub(ff=psit,dy)-uxyyy*sub(ff=psix,dy)
           -uyyyy*sub(ff=psiy,dy)$
zeta_uxxxx:=sub(ff=zeta_uxxx,dx)-uxxxt*sub(ff=psit,dx)-uxxxx*sub(ff=psix,dx)
           -uxxxy*sub(ff=psiy,dx)$
zeta_uxxyy:=sub(ff=zeta_uxxy,dy)-uxxyt*sub(ff=psit,dy)-uxxxy*sub(ff=psix,dy)
           -uxxyy*sub(ff=psiy,dy)$
```

define the coefficients of the generator and they correspond to(2.20).
The commands

```
psix:=-df(w,ux)$
psiy:=-df(w,uy)$
psit:=-df(w,ut)$
zeta_u:=w-ux*df(w,ux)-uy*df(w,uy)-ut*df(w,ut)$
```

define `psix, psiy, psit,zeta_u` which correspond to equation (4.4).
The command

```
uyyyy:=-uxxxx-2*uxxyy+uyyt+uxxt-uy*(uxyy+uxxx)+ux*(uyyy+uxxy)$
```

corresponds to equation (4.6).
The commands

```
factor  u,ux,uy,ut,uxx,uxy,uxt,uyy,uyt,utt,
        uxxx,uxxy,uxxt,uxyy,uxyt,uxtt,uyyt,uytt,uttt,uyyy,
        uxxxx,uxxxy,uxxxt,uxxyy,uxxyt,uxxtt,uxyyy,uxyyt,uxytt,uxttt,
        uyyyt,uyytt,uyttt,utttt$
sss:=determin$
```

factor function `u,ux,...,utttt` from the determining equation.

The next step is to analyze the determining equation for finding the function `w` in (4.4).
The commands

```
df(sss,uxx,uxxxx);
%4*df(w,ux,2)
depend a1,x,y,t,u,ut,uy;
depend a2,x,y,t,u,ut,uy;
w:=a1*ux + a2;
df(w,ux,2);
```

correspond to equation (4.8).
The commands

```
df(sss,uxxyt,uxy);
%4*df(a1,ut)
nodepend a1,ut;
df(a1,ut);
```

correspond to equation (4.9).
The commands

```
df(sss,uxyy,uyyy);
%6*df(a1,uy)
nodepend a1,uy; df(a1,uy);
```

correspond to equation (4.10).

The commands

```
df(sss,uyyyt,uyt);
%4*ux*df(a1,ut,2) + 4*df(a2,ut,2)
%df(a2,ut,2)
depend a3,x,y,t,u,uy; depend a4,x,y,t,u,uy; a2:=a3*ut+a4;
4*ux*df(a1,ut,2) + 4*df(a2,ut,2);
```

correspond to equation (4.12).

The commands

```
df(sss,uxxx,uxx);
%10*df(a1,u)
nodepend a1,u;
df(a1,u);
```

correspond to equation (4.13).

The commands

```
df(sss,uxxy,uyyt);
%2*df(a3,uy)
nodepend a3,uy;
df(a3,uy);
```

correspond to equation (4.14).

The commands

```
df(sss,uyyy,2);
%6*ut*df(a3,uy,2) + 6*df(a4,uy,2)
depend a5,x,y,t,u;
depend a6,x,y,t,u;
a4:=a5*uy+a6;
6*ut*df(a3,uy,2) + 6*df(a4,uy,2);
```

correspond to equation (4.15).

The commands

```
df(sss,uxxx,uy,ut);
%df(a3,u)
nodepend a3,u;
df(sss,uxxx,uy,ut);
df(a3,u);
```

correspond to equation (4.16).

The commands

```
df(sss,uyyy,ux,uy);
%df(a5,u);
nodepend a5,u; df(a5,u);
```

correspond to equation (4.17).

The commands

```
df(sss,uxxx,ut);
%df(a3,y)
nodepend a3,y; df(a3,y);
```

correspond to equation (4.19).

The commands

```
df(sss,uyyy,ut);
%-df(a3,x)
nodepend a3,x; df(a3,x);
```

correspond to equation (4.18).

The commands

```
df(sss,uyy,2);
%6*df(a6,u,2);
depend a7,x,y,t; depend a8,x,y,t; a6:=a7*u+a8; df(a6,u,2);
```

correspond to equation (4.21).

The commands

```
df(sss,uyyy,u);
%df(a7,x)
nodepend a7,x; df(a7,x);
```

correspond to equation (4.22).

The commands

```
df(sss,uxxx,u);
%df(a7,y)
nodepend a7,y; df(a7,y);
```

correspond to equation (4.23).

The commands

```
df(sss,uyy,uy);
%2*df(a5,x,y)
depend a9,x,t; depend a10,y,t;
a5:=a9+a10;
df(a5,x,y);
```

correspond to equation (4.25).

The commands

```
df(sss,uxxy,uy,y);
%2*df(a1,y,2)
depend a11,x,t;
depend a12,x,t;
a1:=a11*y+a12;
df(a1,y,2);
```

correspond to equation (4.27).

The commands

```
df(sss,uxx,ux);
% - 2*df(a11,x)
nodepend a11,x; df(a11,x);
```

correspond to equation (4.28).

The commands

```
df(sss,uxxt,x);
% - 2*df(a12,x,2)
depend a13,t;
depend a14,t;
a12:=a13*x+a14;
df(a12,x,2);
```

correspond to equation (4.29).

The commands

```
df(sss,uxyy,y,uy);
% - df(a10,y,2)
depend a15,t;
depend a16,t;
a10:=a15*y+a16;
df(a10,y,2);
```

correspond to equation (4.30).

The commands

```
df(sss,uxxx,y,2);
%df(a8,y,3)
depend a17,x,t;
depend a18,x,t;
depend a19,x,t;
a8:=a17*y*y+a18*y+a19;
df(a8,y,3);
```

correspond to equation (4.31).

The commands

```
df(sss,uxxx,x,y);
%2*df(a17,x)
nodepend a17,x;
df(a17,x);
```

correspond to equation (4.32).

The commands

```
df(sss,x,y,uxxy);
% - df(a18,x,2)
depend a20,t;
depend a21,t;
a18:=a20*x+a21;
df(a18,x,2);
```

correspond to equation (4.33).

The commands

```
df(sss,x,ux,uxyy);
% - 2*df(a9,x,2)
depend a22,t;
depend a23,t;
a9:=a22*x+a23;
df(a9,x,2);
```

correspond to equation (4.35).

The commands

```
df(sss,ux,uxyy);
% - 2*(a11 + a22)
a11:=-a22;
```

correspond to equation (4.36).

The commands

```
df(sss,ux,uyyy);
% - a13 + a15 - a7
a13:=a15-a7;
```

correspond to equation (4.37).

The commands

```
df(sss,uxxx,uy);
% - 2*a7
a7:=0;
```

correspond to equation (4.38).

The commands

```
df(sss,uxx);
% - 2*df(a15,t)
nodepend a15,t; df(a15,t);
```

correspond to equation (4.39).

The commands

```
df(sss,y,uyyy);
% - a20
a20:=0;
```

correspond to equation (4.40).

The commands

```
df(sss,x,uy);
%df(a19,x,4)
depend a24,t;
depend a25,t;
depend a26,t;
depend a27,t;
a19:=a24*x**3+a25*x**2+a26*x+a27;
df(a19,x,4);
```

correspond to equation (4.41).

The commands

```
df(sss,x,2,uyyy);
% - 6*a24
a24:=0;
```

correspond to equation (4.42).

The commands

```
df(sss,x,uyyy);
% - df(a22,t) - 2*a25
a25:=-(1/2)*df(a22,t); - df(a22,t) - 2*a25;
```

correspond to equation (4.43).

The commands

```
df(sss,uxxt);
% - df(a3,t) + 2*a15
a15:=(1/2)*df(a3,t); - df(a3,t) + 2*a15;
```

correspond to equation (4.44).
The commands

```
df(sss,uxx);
% - df(a3,t,2)
a3:=a28*t+a29;
 - df(a3,t,2);
```

correspond to equation (4.46).
The commands

```
df(sss,uyyy);
% - (df(a16,t) + df(a23,t) + a26)
a26:=-(df(a16,t) + df(a23,t));
- (df(a16,t) + df(a23,t) + a26);
```

correspond to equation (4.47).
The commands

```
df(sss,uxxx,y);
%df(a22,t) + 2*a17
a17:=-(1/2)*df(a22,t); df(a22,t) + 2*a17;
```

correspond to equation (4.48).
The commands

```
df(sss,uxxx);
 % - df(a14,t) + a21
a21:=df(a14,t);
- df(a14,t) + a21;
```

correspond to equation (4.49).
The commands

```
%2*df(a22,t,2)
a22:=a30*t+a31;
df(a22,t,2);
```

correspond to equation (4.50).

A.2 The Program of Finding Contact Transformations

```
depend psix,x,y,t,u,ut,ux,uy$
depend psiy,x,y,t,u,ut,ux,uy$
depend psit,x,y,t,u,ut,ux,uy;$
depend zeta_u,x,y,t,u,ut,ux,uy$
depend w,x,y,t,u,ut,ux,uy$

depend ff,x,y,t,u,
       ux,uy,ut,
       uxx,uxy,uxt,uyy,uyt,utt,
```

```
        uxxx,uxxy,uxxt,uxyy,uxyt,uxtt,
        uyyt,uytt,uttt,uyyy,
        uxxxx,uxxxy,uxxxt,uxxyy,uxxyt,uxxtt,
        uxyyy,uxyyt,uxytt,uxttt,uyyyy,uyyyt,uyytt,uyttt,utttt$

equation :=uyyyy+uxxxx+2*uxxyy-uyyt-uxxt+uy*(uxyy+uxxx)-ux*(uyyy+uxxy)$

generator:=zeta_ux*df(ff,ux)+zeta_uy*df(ff,uy)+zeta_uxxt*df(ff,uxxt)+
               zeta_uyyt*df(ff,uyyt)+zeta_uxyy*df(ff,uxyy)+zeta_uxxy*df(ff,uxxy)+
               zeta_uxxx*df(ff,uxxx)+zeta_uyyy*df(ff,uyyy)+zeta_uyyyy*df(ff,uyyyy)+
               zeta_uxxxx*df(ff,uxxxx)+zeta_uxxyy*df(ff,uxxyy)$

determin := sub(ff=equation,generator)$

dx :=   df(ff,x)+ux*df(ff,u)+
        uxx*df(ff,ux)+uxy*df(ff,uy)+uxt*df(ff,ut)+
        uxxx*df(ff,uxx)+uxxy*df(ff,uxy)+uxyy*df(ff,uyy)+
        uxxt*df(ff,uxt)+uxyt*df(ff,uyt)+uxtt*df(ff,utt)+
        uxxxx*df(ff,uxxx)+uxxxy*df(ff,uxxy)+uxxyy*df(ff,uxyy)+
        uxxxt*df(ff,uxxt)+uxxyt*df(ff,uxyt)+uxxtt*df(ff,uxtt)+
        uxyyy*df(ff,uyyy)+uxyyt*df(ff,uyyt)+uxytt*df(ff,uytt)+
        uxttt*df(ff,uttt)+
        uxxxxx*df(ff,uxxxx)+uxxxxy*df(ff,uxxxy)+uxxxxt*df(ff,uxxxt)+
        uxxxyy*df(ff,uxxyy)+uxxxyt*df(ff,uxxyt)+uxxxtt*df(ff,uxxtt)+
        uxxyyy*df(ff,uxyyy)+uxxyyt*df(ff,uxyyt)+uxxytt*df(ff,uxytt)+
        uxxttt*df(ff,uxttt)+uxyyyy*df(ff,uyyyy)+uxyyyt*df(ff,uyyyt)+
        uxyytt*df(ff,uyytt)+uxyttt*df(ff,uyttt)+uxtttt*df(ff,utttt)$

dy :=   df(ff,y)+uy*df(ff,u)+
        uxy*df(ff,ux)+uyy*df(ff,uy)+uyt*df(ff,ut)+
        uxxy*df(ff,uxx)+uxyy*df(ff,uxy)+uyyy*df(ff,uyy)+
        uxyt*df(ff,uxt)+uyyt*df(ff,uyt)+uytt*df(ff,utt)+
        uxxxy*df(ff,uxxx)+uxxyy*df(ff,uxxy)+uxyyy*df(ff,uxyy)+
        uxxyt*df(ff,uxxt)+uxyyt*df(ff,uxyt)+uxytt*df(ff,uxtt)+
        uyyyy*df(ff,uyyy)+uyyyt*df(ff,uyyt)+uyytt*df(ff,uytt)+
        uyttt*df(ff,uttt)+
        uxxxxy*df(ff,uxxxx)+uxxxyy*df(ff,uxxxy)+uxxxyt*df(ff,uxxxt)+
        uxxyyy*df(ff,uxxyy)+uxxyyt*df(ff,uxxyt)+uxxytt*df(ff,uxxtt)+
        uxyyyy*df(ff,uxyyy)+uxyyyt*df(ff,uxyyt)+uxyytt*df(ff,uxytt)+
        uxyttt*df(ff,uxttt)+uyyyyy*df(ff,uyyyy)+uyyyyt*df(ff,uyyyt)+
        uyyytt*df(ff,uyytt)+uyyttt*df(ff,uyttt)+uytttt*df(ff,utttt)$

dt :=   df(ff,t)+ut*df(ff,u)+
        uxt*df(ff,ux)+uyt*df(ff,uy)+utt*df(ff,ut)+
        uxxt*df(ff,uxx)+uxyt*df(ff,uxy)+uyyt*df(ff,uyy)+
        uxtt*df(ff,uxt)+uytt*df(ff,uyt)+uttt*df(ff,utt)+
        uxxxt*df(ff,uxxx)+uxxyt*df(ff,uxxy)+uxyyt*df(ff,uxyy)+
        uxxtt*df(ff,uxxt)+uxytt*df(ff,uxyt)+uxttt*df(ff,uxtt)+
        uyyyt*df(ff,uyyy)+uyytt*df(ff,uyyt)+uyttt*df(ff,uytt)+
        utttt*df(ff,uttt)+
        uxxxxt*df(ff,uxxxx)+uxxxyt*df(ff,uxxxy)+uxxxtt*df(ff,uxxxt)+
        uxxyyt*df(ff,uxxyy)+uxxytt*df(ff,uxxyt)+uxxttt*df(ff,uxxtt)+
        uxyyyt*df(ff,uxyyy)+uxyytt*df(ff,uxyyt)+uxyttt*df(ff,uxytt)+
        uxtttt*df(ff,uxttt)+uyyyyt*df(ff,uyyyy)+uyyytt*df(ff,uyyyt)+
        uyyttt*df(ff,uyytt)+uytttt*df(ff,uyttt)+uttttt*df(ff,utttt)$

zeta_ut:=sub(ff=zeta_u,dt)-ut*sub(ff=psit,dt)-ux*sub(ff=psix,dt)-uy*sub(ff=psiy,dt)$
zeta_ux:=sub(ff=zeta_u,dx)-ut*sub(ff=psit,dx)-ux*sub(ff=psix,dx)-uy*sub(ff=psiy,dx)$
zeta_uy:=sub(ff=zeta_u,dy)-ut*sub(ff=psit,dy)-ux*sub(ff=psix,dy)-uy*sub(ff=psiy,dy)$

zeta_uxx:=sub(ff=zeta_ux,dx)-uxt*sub(ff=psit,dx)-uxx*sub(ff=psix,dx)-uxy*sub(ff=psiy,dx)$
zeta_uxy:=sub(ff=zeta_ux,dy)-uxt*sub(ff=psit,dy)-uxx*sub(ff=psix,dy)-uxy*sub(ff=psiy,dy)$
zeta_uxt:=sub(ff=zeta_ux,dt)-uxt*sub(ff=psit,dt)-uxx*sub(ff=psix,dt)-uxy*sub(ff=psiy,dt)$
zeta_uyy:=sub(ff=zeta_uy,dy)-uyt*sub(ff=psit,dy)-uxy*sub(ff=psix,dy)-uyy*sub(ff=psiy,dy)$
zeta_uyt:=sub(ff=zeta_uy,dt)-uyt*sub(ff=psit,dt)-uxy*sub(ff=psix,dt)-uyy*sub(ff=psiy,dt)$
zeta_utt:=sub(ff=zeta_ut,dt)-utt*sub(ff=psit,dt)-uxt*sub(ff=psix,dt)-uyt*sub(ff=psiy,dt)$
zeta_uxxx:=sub(ff=zeta_uxx,dx)-uxxt*sub(ff=psit,dx)-uxxx*sub(ff=psix,dx)-uxxy*sub(ff=psiy,dx)$
zeta_uxxy:=sub(ff=zeta_uxx,dy)-uxxt*sub(ff=psit,dy)-uxxx*sub(ff=psix,dy)-uxxy*sub(ff=psiy,dy)$
zeta_uxxt:=sub(ff=zeta_uxx,dt)-uxxt*sub(ff=psit,dt)-uxxx*sub(ff=psix,dt)-uxxy*sub(ff=psiy,dt)$
zeta_uxyy:=sub(ff=zeta_uxy,dy)-uxyt*sub(ff=psit,dy)-uxxy*sub(ff=psix,dy)-uxyy*sub(ff=psiy,dy)$
zeta_uxyt:=sub(ff=zeta_uxy,dt)-uxyt*sub(ff=psit,dt)-uxxy*sub(ff=psix,dt)-uxyy*sub(ff=psiy,dt)$
zeta_uxtt:=sub(ff=zeta_uxt,dt)-uxtt*sub(ff=psit,dt)-uxxt*sub(ff=psix,dt)-uxyt*sub(ff=psiy,dt)$
zeta_uyyy:=sub(ff=zeta_uyy,dy)-uyyt*sub(ff=psit,dy)-uxyy*sub(ff=psix,dy)-uyyy*sub(ff=psiy,dy)$
zeta_uyyt:=sub(ff=zeta_uyy,dt)-uyyt*sub(ff=psit,dt)-uxyy*sub(ff=psix,dt)-uyyy*sub(ff=psiy,dt)$
zeta_uytt:=sub(ff=zeta_uyt,dt)-uytt*sub(ff=psit,dt)-uxyt*sub(ff=psix,dt)-uyyt*sub(ff=psiy,dt)$
zeta_uttt:=sub(ff=zeta_utt,dt)-uttt*sub(ff=psit,dt)-uxtt*sub(ff=psix,dt)-uytt*sub(ff=psiy,dt)$
zeta_uyyyy:=sub(ff=zeta_uyyy,dy)-uyyyt*sub(ff=psit,dy)-uxyyy*sub(ff=psix,dy)
```

```
            -uyyyy*sub(ff=psiy,dy)$
zeta_uxxxx:=sub(ff=zeta_uxxx,dx)-uxxxt*sub(ff=psit,dx)-uxxxx*sub(ff=psix,dx)
            -uxxxy*sub(ff=psiy,dx)$
zeta_uxxyy:=sub(ff=zeta_uxxy,dy)-uxxyt*sub(ff=psit,dy)-uxxxy*sub(ff=psix,dy)
            -uxxyy*sub(ff=psiy,dy)$

psix:=-df(w,ux)$
psiy:=-df(w,uy)$
psit:=-df(w,ut)$
zeta_u:=w-ux*df(w,ux)-uy*df(w,uy)-ut*df(w,ut)$

%manifold
uyyyy:=-uxxxx-2*uxxyy+uyyt+uxxt-uy*(uxyy+uxxx)+ux*(uyyy+uxxy)$

factor  u,ux,uy,ut,uxx,uxy,uxt,uyy,uyt,utt,
        uxxx,uxxy,uxxt,uxyy,uxyt,uxtt,uyyt,uytt,uttt,uyyy,
        uxxxx,uxxxy,uxxxt,uxxyy,uxxyt,uxxtt,uxyyy,uxyyt,uxytt,uxttt,
        uyyyt,uyytt,uyttt,utttt$
sss:=determin$

%Start do splitting
df(sss,uxx,uxxxx);
%4*df(w,ux,2)
depend a1,x,y,t,u,ut,uy;
depend a2,x,y,t,u,ut,uy;
w:=a1*ux + a2;
df(w,ux,2);

df(sss,uxxyt,uxy);
%4*df(a1,ut)
nodepend a1,ut;
df(a1,ut);

df(sss,uxyy,uyyy);
%6*df(a1,uy)
nodepend a1,uy;
df(a1,uy);

df(sss,uyyyt,uyt);
%4*ux*df(a1,ut,2) + 4*df(a2,ut,2)
%df(a2,ut,2)
depend a3,x,y,t,u,uy;
depend a4,x,y,t,u,uy;
a2:=a3*ut+a4;
4*ux*df(a1,ut,2) + 4*df(a2,ut,2);

df(sss,uxxx,uxx);
%10*df(a1,u)
nodepend a1,u;
df(a1,u);

df(sss,uxxy,uyyt);
%2*df(a3,uy)
nodepend a3,uy;
df(a3,uy);

df(sss,uyyy,2);
%6*ut*df(a3,uy,2) + 6*df(a4,uy,2)
depend a5,x,y,t,u;
depend a6,x,y,t,u; a4:=a5*uy+a6;
6*ut*df(a3,uy,2) + 6*df(a4,uy,2);

df(sss,uxxx,uy,ut);
%df(a3,u)
nodepend a3,u;
df(sss,uxxx,uy,ut);
df(a3,u);

df(sss,uyyy,ux,uy);
%df(a5,u);
nodepend a5,u; df(a5,u);

df(sss,uxxx,ut);
%df(a3,y)
nodepend a3,y; df(a3,y);

df(sss,uyyy,ut);
```

```
%-df(a3,x)
nodepend a3,x; df(a3,x);

df(sss,uyy,2);
%6*df(a6,u,2);
depend a7,x,y,t;
depend a8,x,y,t; a6:=a7*u+a8;
df(a6,u,2);

df(sss,uyyy,u);
%df(a7,x)
nodepend a7,x; df(a7,x);

df(sss,uxxx,u);
%df(a7,y)
nodepend a7,y; df(a7,y);

df(sss,uyy,uy);
%2*df(a5,x,y)
depend a9,x,t;
depend a10,y,t;
a5:=a9+a10;
df(a5,x,y);

df(sss,uxxy,uy,y);
%2*df(a1,y,2)
depend a11,x,t;
depend a12,x,t;
a1:=a11*y+a12;
df(a1,y,2);

df(sss,uxx,ux);
% - 2*df(a11,x)
nodepend a11,x;
df(a11,x);

df(sss,uxxt,x);
% - 2*df(a12,x,2)
depend a13,t;
depend a14,t;
a12:=a13*x+a14;
df(a12,x,2);

df(sss,uxyy,y,uy);
% - df(a10,y,2)
depend a15,t;
depend a16,t;
a10:=a15*y+a16;
df(a10,y,2);

df(sss,uxxx,y,2);
%df(a8,y,3)
depend a17,x,t;
depend a18,x,t;
depend a19,x,t;
a8:=a17*y*y+a18*y+a19;
df(a8,y,3);

df(sss,uxxx,x,y);
%2*df(a17,x)
nodepend a17,x;
df(a17,x);

df(sss,x,y,uxxy);
% - df(a18,x,2)
depend a20,t;
depend a21,t;
a18:=a20*x+a21;
df(a18,x,2);

df(sss,x,ux,uxyy);
% - 2*df(a9,x,2)
depend a22,t;
depend a23,t;
a9:=a22*x+a23;
df(a9,x,2);
```

```
df(sss,ux,uxyy);
% - 2*(a11 + a22)
a11:=-a22;

df(sss,ux,uyyy);
% - a13 + a15 - a7
a13:=a15-a7;

df(sss,uxxx,uy);
% - 2*a7
a7:=0;

df(sss,uxx);
% - 2*df(a15,t)
nodepend a15,t;
df(a15,t);

df(sss,y,uyyy);
% - a20
a20:=0;

df(sss,x,uy);
%df(a19,x,4)
depend a24,t;
depend a25,t;
depend a26,t;
depend a27,t;
a19:=a24*x**3+a25*x**2+a26*x+a27;
df(a19,x,4);

df(sss,x,2,uyyy);
% - 6*a24
a24:=0;

df(sss,x,uyyy);
% - df(a22,t) - 2*a25
a25:=-(1/2)*df(a22,t);
- df(a22,t) - 2*a25;

df(sss,uxxt);
% - df(a3,t) + 2*a15
a15:=(1/2)*df(a3,t);
- df(a3,t) + 2*a15;

df(sss,uxx);
% - df(a3,t,2)
a3:=a28*t+a29;
 - df(a3,t,2);

df(sss,uyyy);
% - (df(a16,t) + df(a23,t) + a26)
a26:=-(df(a16,t) + df(a23,t));
- (df(a16,t) + df(a23,t) + a26);

df(sss,uxxx,y);
%df(a22,t) + 2*a17
a17:=-(1/2)*df(a22,t);
df(a22,t) + 2*a17;

df(sss,uxxx);
% - df(a14,t) + a21
a21:=df(a14,t);
- df(a14,t) + a21;

%2*df(a22,t,2)
a22:=a30*t+a31;
df(a22,t,2);

w:=w; psix:=psix; psiy:=psiy; psit:=psit; zeta_u:=zeta_u;

end;
```

Appendix B

Program of Seeking Bäcklund Transformations

B.1 The REDUCE program for finding the determining equations of equation (3.28)

In the REDUCE program the following symbols were used to represent the various variables used in the research

`x(1)` $= x$, `x(2)` $= y$, `x(3)` $= t$,
`u(0,0,0)` $= \psi$, `u(1,0,0)` $= \psi_x$, `u(0,1,0)` $= \psi_y$, `u(0,0,1)` $= \psi_t$,
`u(2,0,0)` $= \psi_{xx}$, `u(1,1,0)` $= \psi_{xy}$,..., `u(0,0,2)` $= \psi_{tt}$,
`zeta_u(0,0,0)` $= \zeta^{\psi}$, `zeta_u(0,0,0)` $= \zeta^{\psi_x}$, `zeta_u(0,1,0)` $= \zeta^{\psi_y}$,
`zeta_u(0,0,1)` $= \zeta^{\psi_t}$, `zeta_u(2,0,0)` $= \zeta^{\psi_{xx}}$,..., `zeta_u(0,0,4)` $= \zeta^{\psi_{tttt}}$,
`equation` $= F\_eq$, `generator` $= Y$, `determin` $= S$.

Next the identificators and main commands are explained.

The command

```
equation:=u(0,4,0)+u(4,0,0)+2*u(2,2,0)-u(0,2,1)-u(2,0,1)+u(0,1,0)
          *(u(1,2,0)+u(3,0,0))-u(1,0,0)*(u(0,3,0)+u(2,1,0))$
```

corresponds to equation (4.5).

The command

```
operator x, u, psix, dx, zeta_u, Uyyyy, eq ,uu;
```

declares `x, u, psix, dx, zeta_u, Uyyyy, eq, uu` are operators

The procedure

```
% Declare dependent variables.
PROCEDURE Dep_v(f,ndmax); begin
 integer ndmax;
 write f," depend:";
  for i:=1:3 do <<depend f,x(i) ; write x(i)>>;
      depend f,u(0,0,0); write u(0,0,0);
    for nd:=1:ndmax do begin
        b:=0; c:=0;
        for a:=nd step -1 until 0 do begin
            b:=nd-a;
            if b=0 then << depend f,u(a,b,c); write u(a,b,c) >>;
            if b>0 then << depend f,u(a,b,c); write u(a,b,c);
              while b>0 do << b:=b-1; c:=c+1; depend f,u(a,b,c);
              write u(a,b,c)>>;c:=0>>;
        end;
    end;
  end;
```

and the commands

```
% M:=Maximum order of independent variables of zeta_u
% R:=Requirment of maximum order of dx(i)
% O:=Order of equation
O:=4; M:=2; R:=M+3; Dep_v(zeta_u,M); Dep_v(F_eq,O); Dep_v(ff,R);
```

are used to define the coefficient zeta_u depend on the independent variables x(1),x(2),x(3), dependent variable u(0,0,0) and the derivatives up to the second order u(1,0,0), u(0,1,0), u(0,0,1), u(2,0,0), u(1,1,0) ,..., u(0,0,2). The function F_eq depends on the independent variables x(1),x(2),x(3), dependent variable u(0,0,0) and the derivatives up to the fourth order u(1,0,0), u(0,1,0), u(0,0,1), u(2,0,0), u(1,1,0) ,..., u(0,0,4). The function ff depends on the independent variables x(1),x(2),x(3), dependent variable u(0,0,0) and the derivatives up to the fifth order u(1,0,0), u(0,1,0), u(0,0,1), u(2,0,0), u(1,1,0) ,..., u(0,0,5).

The subroutine

```
generator:=zeta_u(0,0,0)*df(F_eq,u(0,0,0))$
for nd:=1:O do begin
  b:=0; c:=0;
  for a:=nd step -1 until 0 do begin
      b:=nd-a;
   if b=0 then <<generator:=generator+zeta_u(a,b,c)*df(F_eq,u(a,b,c))>>;
   if b>0 then <<generator:=generator+zeta_u(a,b,c)*df(F_eq,u(a,b,c)) ;
               while b>0 do << b:=b-1; c:=c+1;
               generator:=generator+zeta_u(a,b,c)*df(F_eq,u(a,b,c))>>;
               c:=0>>;
  end;
end;
```

defines the canonical operator (5.1).

The subroutine

```
% Define dx(i)
dx(1):=df(ff,x(1))+u(1,0,0)*df(ff,u(0,0,0))$
dx(2):=df(ff,x(2))+u(0,1,0)*df(ff,u(0,0,0))$
dx(3):=df(ff,x(3))+u(0,0,1)*df(ff,u(0,0,0))$
 for i:=1:3 do begin
  for nd:=1:R do begin
     b:=0; c:=0;
   for a:=nd step -1 until 0 do begin
         b:=nd-a;
         if i=1 then  begin
          if b=0 then <<dx(i):=dx(i)+u(a+1,b,c)*df(ff,u(a,b,c)) >>;
          if b>0 then <<dx(i):=dx(i)+u(a+1,b,c)*df(ff,u(a,b,c));
                      while b>0 do << b:=b-1; c:=c+1;
                      dx(i):=dx(i)+u(a+1,b,c)*df(ff,u(a,b,c)) >>; c:=0>> ;
         end;
         if i=2 then  begin
          if b=0 then <<dx(i):=dx(i)+u(a,b+1,c)*df(ff,u(a,b,c)) >>;
          if b>0 then <<dx(i):=dx(i)+u(a,b+1,c)*df(ff,u(a,b,c));
                      while b>0 do << b:=b-1; c:=c+1;
                      dx(i):=dx(i)+u(a,b+1,c)*df(ff,u(a,b,c)) >>; c:=0>> ;
          end;
          if i=3 then  begin
           if b=0 then <<dx(i):=dx(i)+u(a,b,c+1)*df(ff,u(a,b,c)) >>;
           if b>0 then <<dx(i):=dx(i)+u(a,b,c+1)*df(ff,u(a,b,c));
                      while b>0 do << b:=b-1; c:=c+1;
                      dx(i):=dx(i)+u(a,b,c+1)*df(ff,u(a,b,c)) >>; c:=0>> ;
          end;
   end;
 end;
 write "dx",i, ":=", dx(i);
end;
```

defines the differential operator (4.3).
The procedure

```
%  Define differential manifold ,Uyyyy(a,b,c)
eq(0,0,0):=equation;
PROCEDURE  DF_Uyyyy(a,b,c); begin
     eq(x,y,z):=eq(0,0,0);
         if a>0 then for i:=1:a do
         <<eq(a,b,c) := sub(ff=eq(x,y,z),dx(1)); eq(x,y,z) := eq(a,b,c)>>;
         if b>0 then for j:=1:b do
         <<eq(a,b,c) := sub(ff=eq(x,y,z),dx(2)); eq(x,y,z) := eq(a,b,c)>>;
         if c>0 then for k:=1:c do
         <<eq(a,b,c) := sub(ff=eq(x,y,z),dx(3)); eq(x,y,z) := eq(a,b,c)>>;
     write "eq(" ,a ,"," ,b, "," ,c, "):=" ,eq(a,b,c);
     uu(a,b+4,c):= u(a,b+4,c) - (eq(a,b,c) / df(eq(a,b,c),u(a,b+4,c)));
     write "uu(",a ,"," ,b+4 ,"," ,c,"):=",uu(a,b+4,c);
end;
```

and subroutines

```
 for nd:=1:M do begin
   b:=0; c:=0;
   for a:=nd step -1 until 0 do begin
       b:=nd-a;
       if b=0 then << DF_Uyyyy(a,b,c) >>;
       if b>0 then << DF_Uyyyy(a,b,c);
          while b>0 do << b:=b-1; c:=c+1;
          DF_Uyyyy(a,b,c)>>;c:=0>>;
   end;
 end;

u(0,4,0):=-u(4,0,0)-2*u(2,2,0)+u(0,2,1)+u(2,0,1)-u(0,1,0)*(u(1,2,0)
        +u(3,0,0))+u(1,0,0)*(u(0,3,0)+u(2,1,0));
% Instead uu by
 for nd:=1:M do begin
  b:=0; c:=0;
  for a:=nd step -1 until 0 do begin
  b:=nd-a;
   if b=0 then <<  u(a,b+4,c):=uu(a,b+4,c);
      write "u(",a ,"," ,b+4 ,"," ,c,"):=",u(a,b+4,c)  >>;
   if b>0 then <<  u(a,b+4,c):=uu(a,b+4,c);
      write "u(",a ,"," ,b+4 ,"," ,c,"):=",u(a,b+4,c);
       while b>0 do
       << b:=b-1; c:=c+1; u(a,b+4,c):=uu(a,b+4,c);
       write "u(",a ,"," ,b+4 ,"," ,c,"):=",u(a,b+4,c)>>;
   c:=0>>;
  end;
 end;
```

correspond to equation (5.3). The determining equation $YF|_{[F]=0} = 0$ can be found by the following command.

```
determin := determin$
```

One can write the determining equation into "data_determining" file. The procedure program, for splitting using the idea in chapter V can be written by the commands:

```
perator fu,sk; sk(1):=u(0,3,3); sk(2):=u(1,3,2); sk(3):=u(2,3,1);
sk(4):=u(3,3,0); sk(5):=u(0,2,4); sk(6):=u(1,2,3);
sk(7):=u(2,2,2); sk(8):=u(3,2,1); sk(9):=u(4,2,0);
sk(10):=u(0,1,5); sk(11):=u(1,1,4); sk(12):=u(2,1,3);
sk(13):=u(3,1,2); sk(14):=u(4,1,1); sk(15):=u(5,1,0);
sk(16):=u(0,0,6); sk(17):=u(1,0,5); sk(18):=u(2,0,4);
sk(19):=u(3,0,3); sk(20):=u(4,0,2); sk(21):=u(5,0,1);
sk(22):=u(6,0,0); sk(23):=u(0,3,2); sk(24):=u(1,3,1);
sk(25):=u(2,3,0); sk(26):=u(0,2,3); sk(27):=u(1,2,2);
sk(28):=u(2,2,1); sk(29):=u(3,2,0); sk(30):=u(0,1,4);
```

```
sk(31):=u(1,1,3); sk(32):=u(2,1,2); sk(33):=u(3,1,1);
sk(34):=u(4,1,0); sk(35):=u(0,0,5); sk(36):=u(1,0,4);
sk(37):=u(2,0,3); sk(38):=u(3,0,2); sk(39):=u(4,0,1);
sk(40):=u(5,0,0); sk(41):=u(0,3,1); sk(42):=u(1,3,0);
sk(43):=u(0,2,2); sk(44):=u(1,2,1); sk(45):=u(2,2,0);
sk(46):=u(0,1,3); sk(47):=u(1,1,2); sk(48):=u(2,1,1);
sk(49):=u(3,1,0); sk(50):=u(0,0,4); sk(51):=u(1,0,3);
sk(52):=u(2,0,2); sk(53):=u(3,0,1); sk(54):=u(4,0,0);
sk(55):=u(0,3,0); sk(56):=u(0,2,1); sk(57):=u(1,2,0);
sk(58):=u(0,1,2); sk(59):=u(1,1,1); sk(60):=u(2,1,0);
sk(61):=u(0,0,3); sk(62):=u(1,0,2); sk(63):=u(2,0,1);
sk(64):=u(3,0,0); sk(65):=u(0,0,2); sk(66):=u(0,1,1);
sk(67):=u(1,0,1);

ms:=67; off nat; in data_determining;

%SQUARE TERMS
    for m:=1:ms do for l:=m:ms do begin     %2
       if m=l then h:=2 else h:=1;
       ss:=df(df(oper,sk(m)),sk(l))/h;
       oper:=oper-ss*sk(m)*sk(l);
       if not (ss=0) then
       <<
        fu(m,l) := num (ss);
       write "fu(",m,",",l,") := ",fu(m,l);
       >>;
    end;    %2

%LINEAR TERMS
    for m:=1:ms do begin                %3
        ss:=df(oper,sk(m));
        oper:=oper-ss*sk(m);
        if not (ss=0) then begin            %4
            fu(m) := num (ss);
            if not (fu(m)=0) then
             write ("fu(",m,") := ",fu(m));
        end;                                                %4
    end;                                                    %3
    oper := num (oper);

end;
```

These commands are saved in the "splitting" file.

B.2 The Program of Finding Lie-Bäcklund Transformations

```
operator x, u, psix, dx, zeta_u, Uyyyy, eq ,uu;

PROCEDURE Dep_v(f,ndmax); begin
 integer ndmax;
 write f," depend:";
   for i:=1:3 do <<depend f,x(i) ; write x(i)>>;
      depend f,u(0,0,0); write u(0,0,0);
      for nd:=1:ndmax do begin
           b:=0; c:=0;
           for a:=nd step -1 until 0 do begin
              b:=nd-a;
                  if b=0 then << depend f,u(a,b,c);  write u(a,b,c);
                          factor df(f,u(a,b,c));>>;
                  if b>0 then << depend f,u(a,b,c);  write u(a,b,c);
                          factor df(f,u(a,b,c));
                            while b>0 do << b:=b-1; c:=c+1;
                            depend f,u(a,b,c);  write u(a,b,c)>>;
                          factor df(f,u(a,b,c));
                            c:=0>>;
           end;
      end;
```

```
end;

% M:=Maximum order of independent variables of zeta_u
% R:=Requirment of maximum order of dx(i)
% O:=Order of equation
O:=4; M:=2; R:=M+3; Dep_v(zeta_u(0,0,0),M); Dep_v(F_eq,O);
Dep_v(ff,R);

%Start do splitting
%df(zeta_u(0,0,0),u(0,0,2),2):=0$
depend
b1,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(1,0,1),u(0,2,0),u(0,1,1);
depend
b2,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(1,0,1),u(0,2,0),u(0,1,1);
zeta_u(0,0,0):=b1*u(0,0,2)+b2; df(zeta_u(0,0,0),u(0,0,2),2);

%df(zeta_u(0,0,0),u(0,1,1),2):=0$
depend
b3,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(1,0,1),u(0,2,0);
depend
b4,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(1,0,1),u(0,2,0);
depend
b5,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(1,0,1),u(0,2,0);
depend
b6,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(1,0,1),u(0,2,0);
zeta_u(0,0,0):=u(0,0,2)*(b3*u(0,1,1)+b4) + (b5*u(0,1,1)+b6);
df(zeta_u(0,0,0),u(0,1,1),2);

%df(zeta_u(0,0,0),u(1,0,1),2):=0$
depend
b7,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(0,2,0);
depend
b8,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(0,2,0);
depend
b9,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(0,2,0);
depend
b10,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(0,2,0);
depend
b11,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(0,2,0);
depend
b12,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(0,2,0);
depend
b13,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(0,2,0);
depend
b14,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(1,1,0),u(0,2,0);
zeta_u(0,0,0):=u(0,0,2)*u(0,1,1)*(b7*u(1,0,1)+b8)+u(0,0,2)*(b9*u(1,0,1)+b10)
               +u(0,1,1)*(b11*u(1,0,1)+b12) +b13*u(1,0,1)+b14;
df(zeta_u(0,0,0),u(1,0,1),2);

%fu(23,56) := 4*(u(1,0,1)*b7 + b8)$
%(u(1,0,1)*b7 + b8):=0;
b7:=0; b8:=0;

%fu(24,56) := 4*(u(0,0,2)*b7 + b11)$
b11:=0;

%fu(24,58) := 4*(u(0,1,1)*b7 + b9)$
b9:=0;

%fu(23,55) := 4*(df(b7,u(0,2,0))*u(1,0,1)*u(0,1,1) + df(b8,u(0,2,0))*u(0,1,1)
% + df(b9,u(0,2,0))*u(1,0,1) + df(b10,u(0,2,0)))$
%df(b10,u(0,2,0)):=0;
nodepend b10,u(0,2,0);

%fu(23,57) := 4*(df(b7,u(1,1,0))*u(1,0,1)*u(0,1,1) + df(b8,u(1,1,0))*u(0,1,1)
% + df(b9,u(1,1,0))*u(1,0,1) + df(b10,u(1,1,0)))$
%df(b10,u(1,1,0)):=0;
nodepend b10,u(1,1,0);

%fu(23,60) := 4*(df(b7,u(2,0,0))*u(1,0,1)*u(0,1,1) + df(b8,u(2,0,0))*u(0,1,1)
% + df(b9,u(2,0,0))*u(1,0,1) + df(b10,u(2,0,0)))$
%df(b10,u(2,0,0)):=0;
nodepend b10,u(2,0,0);

%fu(23,66) := 4*(df(b7,u(1,0,0))*u(1,1,0)*u(1,0,1) + df(b7,u(0,1,0))*u(1,0,1)
%*u(0,2,0) + 2*df(b7,u(0,0,1))*u(1,0,1)*u(0,1,1) + df(b7,u(0,
%0,0))*u(1,0,1)*u(0,1,0) + df(b7,x(2))*u(1,0,1) + df(
```

```
b8%,u(1,0,0))*u(1,1,0) + df(b8,u(0,1,0))*u(0,2,0) + 2*df(b8,
%u(0,0,1))*u(0,1,1) + df(b8,u(0,0,0))*u(0,1,0) + df(b8,x(2
%)) + df(b9,u(0,0,1))*u(1,0,1) + df(b10,u(0,0,1)))$
%df(b10,u(0,0,1)):=0;
nodepend b10,u(0,0,1);

%fu(24,55) := 4*(df(b7,u(0,2,0))*u(0,1,1)*u(0,0,2) + df(b9,u(0,2,0))*u(0,0,2)
% + df(b11,u(0,2,0))*u(0,1,1) + df(b13,u(0,2,0)))$
%df(b13,u(0,2,0)):=0;
nodepend b13,u(0,2,0);

%fu(24,64) := 4*(df(b7,u(2,0,0))*u(1,0,1)*u(0,0,2) + df(b8,u(2,0,0))*u(0,0,2)
% + df(b11,u(2,0,0))*u(1,0,1) + df(b12,u(2,0,0)))$
%df(b12,u(2,0,0)):=0;
nodepend b12,u(2,0,0);

%fu(24,66) := 4*(df(b11,u(1,0,0))*u(1,1,0) + df(b11,u(0,1,0))*u(0,2,0) + 2*df(
%b11,u(0,0,1))*u(0,1,1) + df(b11,u(0,0,0))*u(0,1,0) + df(
%b11,x(2)) + df(b13,u(0,0,1)))$
%df(b13,u(0,0,1)):=0;
nodepend b13,u(0,0,1);

%fu(24,67) := 4*(df(b11,u(1,0,0))*u(2,0,0) + df(b11,u(0,1,0))*u(1,1,0) + 2*df(
%b11,u(0,0,1))*u(1,0,1) + df(b11,u(0,0,0))*u(1,0,0) + df(
%b11,x(1)) + df(b12,u(0,0,1)))$
%df(b12,u(0,0,1)):=0;
nodepend b12,u(0,0,1);

%fu(25,63) := 4*(df(b7,u(1,1,0))*u(0,1,1)*u(0,0,2) + df(b9,u(1,1,0))*u(0,0,2)
% + df(b11,u(1,1,0))*u(0,1,1) + df(b13,u(1,1,0)))$
%df(b13,u(1,1,0)):=0;
nodepend b13,u(1,1,0);

%fu(24,57) := 4*(df(b7,u(1,1,0))*u(0,1,1)*u(0,0,2) + df(b7,u(0,2,0))*u(1,0,1)
%*u(0,0,2) + df(b8,u(0,2,0))*u(0,0,2) + df(b9,u(1,1,0))*u(0,0
%,2) + df(b11,u(1,1,0))*u(0,1,1) + df(b11,u(0,2,0))*u(1,0,1)
% + df(b12,u(0,2,0)) + df(b13,u(1,1,0)))$
%df(b12,u(0,2,0)):=0;
nodepend b12,u(0,2,0);

%fu(28,57) := 4*( - df(b7,u(1,1,0))*u(1,0,1)*u(0,0,2) + df(b7,u(0,2,0))*u(0,1
%,1)*u(0,0,2) - df(b8,u(1,1,0))*u(0,0,2) + df(b9,u(0,2,0))
%*u(0,0,2) - df(b11,u(1,1,0))*u(1,0,1) + df(b11,u(0,2,0))*u(0,1
%,1) - df(b12,u(1,1,0)) + df(b13,u(0,2,0)))$
%df(b12,u(1,1,0)):=0;
nodepend b12,u(1,1,0);

%fu(24,60) := 4*(df(b7,u(2,0,0))*u(0,1,1)*u(0,0,2) + df(b7,u(1,1,0))*u(1,0,1)
%*u(0,0,2) + df(b8,u(1,1,0))*u(0,0,2) + df(b9,u(2,0,0))*u(0,0
%,2) + df(b11,u(2,0,0))*u(0,1,1) + df(b11,u(1,1,0))*u(1,0,1)
% + df(b12,u(1,1,0)) + df(b13,u(2,0,0)))$
%df(b13,u(2,0,0)):=0;
nodepend b13,u(2,0,0);

%fu(41,64) := 2*df(b12,u(1,0,0))$
%df(b12,u(1,0,0)):=0;
nodepend b12,u(1,0,0);

%fu(42,62) := 4*df(b10,u(0,1,0))$
%df(b10,u(0,1,0)):=0;
nodepend b10,u(0,1,0);

%fu(43,57) := 6*df(b10,u(1,0,0))$
%df(b10,u(1,0,0)):=0;
nodepend b10,u(1,0,0);

%fu(25,57) := 4*df(b14,u(2,0,0),u(1,1,0))$
%df(b14,u(2,0,0),u(1,1,0)):=0;
depend
b15,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(1,1,0),u(0,2,0);
depend
b16,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(2,0,0),u(0,2,0);
b14:=b15+b16; df(b14,u(2,0,0),u(1,1,0));

%fu(41,67) := 2*b10$
b10:=0;
```

```
%fu(55,62) :=  - df(b13,u(0,1,0))$
%df(b13,u(0,1,0)):=0;
nodepend b13,u(0,1,0);

%fu(29,55) :=  - 4*df(b15,u(1,1,0),u(0,2,0))$
%df(b15,u(1,1,0),u(0,2,0)):=0;
depend
b17,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(0,2,0);
depend
b18,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(1,1,0);
b15:=b17+b18; df(b15,u(1,1,0),u(0,2,0));

%fu(25,67) := 4*df(b18,u(1,1,0),u(0,0,1))$
%df(b18,u(1,1,0),u(0,0,1)):=0;
depend b19,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1);
depend b20,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(1,1,0);
b18:=b19+b20; df(b18,u(1,1,0),u(0,0,1));

%fu(58,60) :=  - df(b12,u(0,1,0)) - 2*df(b16,u(2,0,0),u(0,0,1))$
df(b12,u(0,1,0),u(2,0,0));
%df(b16,u(2,0,0),2,u(0,0,1)):=0;
depend
b21,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(0,2,0);
depend
b22,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(0,2,0);
depend
b23,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(2,0,0),u(0,2,0);
b16:=b21+b22*u(2,0,0)+b23; df(b16,u(2,0,0),2,u(0,0,1));

%fu(28) := 4*( - df(b12,u(0,1,0))*u(0,2,0) - df(b12,u(0,0,0))*u(0,1,0) - df(
%b12,x(2)) + df(b13,u(1,0,0))*u(2,0,0) + df(b13,u(0,0,0))*u(1,
%0,0) + df(b13,x(1)))$
%df(fu(28),u(2,0,0));
%4*df(b13,u(1,0,0))$
nodepend b13,u(1,0,0);

%df(fu(28),u(0,2,0));
%- 4*df(b12,u(0,1,0))$
nodepend b12,u(0,1,0);

%df(fu(58,60),u(0,0,1));
% - 2*df(b22,u(0,0,1),2)$
depend b24,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,2,0);
depend b25,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,2,0);
b22:=b24*u(0,0,1)+b25;

% df(fu(25,66),u(2,0,0));
% - 4*df(b24,u(0,2,0))$
nodepend b24,u(o,2,0);

% df(fu(42,59),u(1,1,0));
% - df(b20,u(1,1,0),3)$
depend b26,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
depend b27,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
depend b28,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b20:=(0.5)*b26*(u(1,1,0)**2)+b27*u(1,1,0)+b28;

%fu(58,60) :=  - 2*b24$
b24:=0;

% df(fu(24),u(1,0,0));
%4*df(b12,u(0,0,0))$
nodepend b12,u(0,0,0);

%df(fu(28),u(1,0,0));
%4*df(b13,u(0,0,0))$
nodepend b13,u(0,0,0);

%fu(43) := 2*(df(b12,u(0,0,0))*u(0,1,0) + df(b12,x(2)))$
%df(b12,x(2)):=0;
nodepend b12,x(2);

%fu(52) := 2*(2*df(b12,u(0,0,0))*u(0,1,0) + 2*df(b12,x(2)) - df(b13,u(0,0,0))*
%u(1,0,0) - df(b13,x(1)))$
%df(b13,x(1)):=0;
nodepend b13,x(1);
```

```
%2*(df(b25,u(0,2,0)) + df(b23,u(2,0,0),u(0,2,0)))
depend b29,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(2,0,0);
b25:= -(1/2)*df(b23,u(2,0,0))+b29; 2*(df(b25,u(0,2,0)) +
df(b23,u(2,0,0),u(0,2,0)));

%fu(49,59) :=  - b26$
b26:=0;

%fu(49,49) := 4*df(b23,u(2,0,0),2) + 3*b26$
depend b30,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,2,0);
depend b31,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,2,0);
b23:=b30*u(2,0,0)+b31; 4*df(b23,u(2,0,0),2) + 3*b26;

%df(fu(24),x(2));
%4*df(b13,x(2),2)$
depend b32,x(3);
depend b33,x(3);
b13:=b32*x(2)+b33;

%fu(41,60) := df(b30,u(0,2,0))$
nodepend b30,u(0,2,0);

%fu(42,59) := 8*(df(b17,u(0,2,0),u(0,0,1)) + df(b21,u(0,2,0),u(0,0,1)))$
%b34:=b17+b21;
depend b34,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1),u(0,2,0);
b21:=b34-b17;

%fu(24) := 4*(df(b12,x(1)) + b32)$
%df(fu(24),x(1));
%4*df(b12,x(1),2)
depend b35,x(3);
depend b36,x(3);
b12:=b35*x(1)+b36;

%fu(25,66) :=  - 4*df(b34,u(0,2,0),u(0,0,1))$
depend b37,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1);
depend b38,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,2,0);
b34:=b37+b38;

%fu(48,60) := 2*( - df(b29,u(2,0,0),2)*u(2,0,0) - 2*df(b29,u(2,0,0)))$
depend b39,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
depend b40,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b29:=b39+(b40/u(2,0,0));

%fu(24) := 4*(b32 + b35)$
b32:=-b35;

%fu(25,55) :=  - 4*(df(b31,u(0,2,0),2) + df(b38,u(0,2,0),2))$
%b41:=b31+b38;
depend b41,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,2,0);
b31:=b41-b38;

%fu(41,66) := 4*(df(b19,u(0,0,1),2) + df(b37,u(0,0,1),2))$
%b42:=b19+b37;
depend b42,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0),u(0,0,1);
b19:=b42-b37;

%fu(42,64) := 2*(df(b27,u(1,0,0)) + df(b30,u(0,1,0)) + 2*df(b39,u(0,1,0)))$
%b43:=b27+b30;
depend b43,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b27:=b43-b30;

%fu(25,55) :=  - 4*df(b41,u(0,2,0),2)$
depend b44,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
depend b45,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b41:=b44*u(0,2,0)+b45;

%fu(44,67) := 4*df(b42,u(0,0,1),2)$
depend b46,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
depend b47,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b42:=b46*u(0,0,1)+b47;

%From output_bl13
%fu(42,55) := 2*( - 3*df(b30,u(0,1,0)) + 2*df(b44,u(1,0,0)) + 3*df(b43,u(0,1,0)))$
%b48:=b43-b30;
depend b48,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b30:=b43-b48;
```

```
%fu(55,65) :=  - df(b46,u(0,1,0))$
nodepend b46,u(0,1,0);

%fu(55,56) := 7*df(b44,u(0,1,0)) + 6*df(b46,u(0,1,0))$
nodepend b44,u(0,1,0);

%fu(59,66) :=  - 2*df(b46,u(1,0,0))$
nodepend b46,u(1,0,0);

%fu(56,64) := df(b44,u(1,0,0)) + 2*df(b46,u(1,0,0))$
nodepend b44,u(1,0,0);

%fu(63,67) :=  - df(b44,u(1,0,0)) - 3*df(b46,u(1,0,0)) + x(1)*b35 + b36$
%df(fu(63,67),x(1));
% - df(b44,u(1,0,0),x(1)) - 3*df(b46,u(1,0,0),x(1)) + b35$
b35:=0; b36:=0;

%fu(42,55) := 6*df(b48,u(0,1,0))$
nodepend b48,u(0,1,0);

%fu(42,57) := 10*df(b48,u(1,0,0))$
nodepend b48,u(1,0,0);

%fu(42,67) :=  - df(b48,u(1,0,0)) + b33$
b33:=0;

%fu(66,66) :=  - 2*df(b46,u(0,0,0))$
nodepend b46,u(0,0,0);

%fu(58) :=  - 2*(df(b46,u(0,0,0))*u(0,1,0) + df(b46,x(2)))$
nodepend b46,x(2);

%fu(62) :=  - 2*(df(b46,u(0,0,0))*u(1,0,0) + df(b46,x(1)))$
nodepend b46,x(1);

%fu(42,60) := 2*( - df(b48,u(1,0,0)) + 3*df(b48,u(0,1,0)) + 2*df(b39,u(1,0,0)) +
%df(b43,u(1,0,0)))$
%2*df(b39,u(1,0,0)) + df(b43,u(1,0,0)):=0
%b49:=2*b39+b43;
depend b49,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b43:=b49-(2*b39);

%fu(42,60) := 2*df(b49,u(1,0,0))$
nodepend b49,u(1,0,0);

%fu(45,60) :=  - df(b49,u(0,1,0))$
nodepend b49,u(0,1,0);

% df(fu(41),u(0,1,0));
%2*df(b44,u(0,0,0))$
nodepend b44,u(0,0,0);

% df(fu(44),u(0,1,0));
%2*df(b48,u(0,0,0))$
nodepend b48,u(0,0,0);

%fu(55,55) := 3*(df(b28,u(0,1,0),2) + df(b45,u(0,1,0),2) + df(b47,u(0,1,0),2) + df
%(b40,u(0,1,0),2))$
%b50:=b28+b45;
depend b50,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b28:=b50-b45;
%b51:=b47+b40;
depend b51,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b47:=b51-b40;

%df(fu(25),u(0,1,0));
%2*df(b49,u(0,0,0))$
nodepend b49,u(0,0,0);

%fu(41) := 2*df(b44,x(2))$
nodepend b44,x(2);

%fu(55,55) := 3*(df(b50,u(0,1,0),2) + df(b51,u(0,1,0),2))$
%b52:=b50+b51;
depend b52,x(1),x(2),x(3),u(0,0,0),u(1,0,0),u(0,1,0);
b50:=b52-b51;
```

```
% df(fu(44),x(2));
%2*df(b48,x(2),2)$
depend b53,x(1),x(3);
depend b54,x(1),x(3);
b48:=b53*x(2)+b54;

%fu(55,55) := 3*df(b52,u(0,1,0),2)$
depend b55,x(1),x(2),x(3),u(0,0,0),u(1,0,0);
depend b56,x(1),x(2),x(3),u(0,0,0),u(1,0,0);
b52:=b55*u(0,1,0)+b56;

%fu(44) := 2*(df(b44,x(1)) + b53)$
b53:=-df(b44,x(1));

%df(fu(25),x(2),2);
%2*df(b49,x(2),3)$
depend b57,x(1),x(3);
depend b58,x(1),x(3);
depend b59,x(1),x(3);
b49:=(1/2)*b57*(x(2)**2)+b58*x(2)+b59;

%fu(55,57) := 6*df(b55,u(1,0,0))$
nodepend b55,u(1,0,0);

%fu(57,57) := 3*(df(b55,u(1,0,0),2)*u(0,1,0) + df(b56,u(1,0,0),2))$
%df(b56,u(1,0,0),2)):=0;
depend b60,x(1),x(2),x(3),u(0,0,0);
depend b61,x(1),x(2),x(3),u(0,0,0);
b56:=b60*u(1,0,0)+b61;

% df(fu(29),x(2),2);
%2*df(b57,x(1))$
nodepend b57,x(1);

% df(fu(42),u(2,0,0),x(2),2);
%2*b57$
b57:=0;

%df(fu(56),u(1,1,0));
2*b44$ b44:=0;

%df(fu(56),u(2,0,0));
% - df(b44,x(1))*x(2) + b54$
b54:=0;

% df(fu(59),u(1,0,0));
% - 2*df(b55,u(0,0,0))$
nodepend b55,u(0,0,0);

%df(fu(59),u(0,1,0));
% - 2*df(b60,u(0,0,0))$
nodepend b60,u(0,0,0);

%From output_bl21
%fu(25) := 2*b58$
b58:=0;

%df(fu(42),u(2,0,0));
%x(2)*b58 + b59$
b59:=0;

%df(fu(56),x(1));
%2*df(b55,x(2),x(1))$
depend b62,x(1),x(3);
depend b63,x(2),x(3);
b55:=b62+b63;

% df(fu(66),u(0,1,0));
% - 2*df(b61,u(0,0,0),2)$
depend b64,x(1),x(2),x(3);
depend b65,x(1),x(2),x(3);
b61:=b64*u(0,0,0)+b65;

% df(fu(42),x(2));
%4*df(b60,x(2),2)$
depend b66,x(1),x(3);
```

```
depend b67,x(1),x(3);
b60:=b66*x(2)+b67;

% df(fu(55),u(0,0,0));
% - df(b64,x(1))$
nodepend b64,x(1);

% df(fu(56),x(2));
%2*df(b63,x(2),2)$
depend b68,x(3);
depend b69,x(3);
b63:=b68*x(2)+b69;

%df(fu(54),x(2));
%4*df(b66,x(1))$
nodepend b66,x(1);

%fu(56) :=  - df(b46,x(3)) + 2*b68$
b68:=(1/2)*df(b46,x(3));

%df(fu(66),x(2));
% - 2*df(b64,x(2),2)$
depend b70,x(3);
depend b71,x(3);
b64:=b70*x(2)+b71;

%df(fu(42),x(1));
%4*df(b62,x(1),2)$
depend b72,x(3);
depend b73,x(3);
b62:=b72*x(1)+b73;

%df(fu(45),x(1));
%4*df(b67,x(1),2)$
depend b74,x(3);
depend b75,x(3);
b67:=b74*x(1)+b75;

%df(fu(57),u(0,1,0),x(2));
%2*b70$
b70:=0;

%fu(42) := 4*(b66 + b72)$
b66:=-b72;

%fu(45) := 2*( - df(b46,x(3)) + 2*b74)$
b74:=(1/2)*df(b46,x(3)) ;

%df(fu(64),x(1),2);
%2*df(b65,x(2),x(1),2)$
depend b76,x(1),x(3);
depend b77,x(2),x(3);
depend b78,x(2),x(3);
b65:=b76+b77+b78*x(1);

%From output_bl26
%df(fu(55),u(1,0,0));
% - 2*b71$
b71:=0;

% df(fu(55),x(2),2);
% - 2*df(b78,x(2),2)$
depend b79,x(3);
depend b80,x(3);
b78:=b79*x(2)+b80;

% df(fu(60),x(1),2);
% - 2*df(b76,x(1),3)$
depend b81,x(3);
depend b82,x(3);
depend b83,x(3);
b76:=(1/2)*b81*(x(1)**2)+b82*x(1)+b83;

% df(fu(55),x(2));
% - df(b46,x(3),2) - 2*b79$
b79:=-(1/2)*df(b46,x(3),2);
```

```
%df(fu(55),x(1));
% - (df(b72,x(3)) + b81)$
b81:=-df(b72,x(3));

% df(fu(57),x(1));
% - df(b46,x(3),2)$
b46:=b84*x(3)+b85;

% df(fu(57),x(2),2);
%df(b77,x(2),3)$
depend b86,x(3);
depend b87,x(3);
depend b88,x(3);
b77:=(1/2)*b86*(x(2)**2)+b87*x(2)+b88;

%fu(55) :=  - (df(b69,x(3)) + df(b73,x(3)) + b80 + b82)$
%b89:=b69+b73;
depend b89,x(3);
b73:=b89-b69;
%b90:=b80+b82;
b82:=b90-b80;

%fu(55) :=  - (df(b89,x(3)) + b90)$
b90:=-df(b89,x(3));

% df(fu(57),x(2));
%df(b72,x(3)) + b86$
b86:=-df(b72,x(3));

%fu(57) := df(b72,x(3))*x(2) - df(b75,x(3)) + x(2)*b86 + b87$
b87:=df(b75,x(3));

%oper := 2*df(b72,x(3),2)$
b72:=b91*x(3)+b92;

in splitting;
write "zeta_u(0,0,0):=",zeta_u(0,0,0);
end;
```

Curriculum Vitae

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