การมีอยู่ของผลเฉลยสำหรับชั้นของสมการอินทิโกร ดิฟเฟอเรนเชียลกึ่งเชิงเส้นแบบพาราโบลาที่มี การประวิงและการควบคุมเหมาะที่สุด

นาย เกียรติ แสงอรุณ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

สาขาวิชาคณิตศาสตร์ประยุกต์

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EXISTENCE OF SOLUTIONS FOR A CLASS OF SEMILINEAR INTEGRODIFFERENTIAL EQUATIONS OF PARABOLIC TYPE WITH DELAY AND OPTIMAL CONTROL

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วิทยานิพนธ์ฉบับนี้ ศึกษาปัญหาสำหรับชั้นของระบบซึ่งถูกครอบงำด้วยสมการ อินทิโกรดิฟเฟอเรนเชียลที่มีการประวิงบนปริภูมิบานาค และได้มีการศึกษาปัญหาการควบคุมเหมาะที่สุดที่สมนัยกันด้วย

ในส่วนแรกได้พิจารณาปัญหาสำหรับชั้นของระบบ ซึ่งถูกครอบงำด้วยสมการอินทิโกรดิฟเฟอเรนเชียลที่มีการประวิงบนปริภูมิบานาค ได้พิสูจน์การมีอยู่หนึ่งเดียว และความไม่อิสระอย่างต่อเนื่องของผลเฉลย

ในส่วนที่สอง ได้ศึกษาปัญหาการควบคุมเหมาะที่สุดแบบลากรองจ์ ที่สมนัยกับปัญหาในส่วนแรก ได้หาการมีอยู่ของการควบคุมเหมาะที่สุดของระบบที่ถูกควบคุม

ในตอนท้าย ได้แสดงตัวอย่างสมการเชิงอนุพันธ์ย่อยกึ่งเชิงเส้นแบบพาราโบลาที่มีการประวิง ซึ่งเป็นผลลัพธ์จากการศึกษา

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INTEGRODIFFERENTIAL EQUATIONS / EVOLUTION EQUATIONS / MILD SOLUTIONS / EVOLUTION OPERATORS / PARABOLIC EQUATIONS / DELAY / OPTIMAL CONTROL

This thesis studies the problems for a class of systems governed by semilinear integrodifferential equations with delay on Banach spaces and corresponding optimal control problems.

The first part concerns about the problems for a class of systems governed by integrodifferential equations with delay on Banach spaces. The existence, uniqueness and continuous dependence of solutions are proved.

The second part deals with a corresponding Lagrange optimal control problems. The existence of optimal controls for controlled systems is solved.

Finally, the results are illustrated by examples from semilinear partial differential equations of parabolic type with delay.

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Kiat Sangaroon

Contents

Chapter I

Introduction

In many disciplines (e.g. physics, chemistry, biology, engineering and economics), it is frequently necessary to study a system which is evolving with time. In principle, if we know the initial state of such a system and the laws which describe how the state of the system changes with time, then we should be able to determine the state at any given time. The mathematical model for such a system is an *evolution equation*, which often takes the form of a partial differential equation (PDE).

Infinite dimensional systems can be used to describe many phenomena in the real world. As is well known, heat conduction, properties of elasticplastic material, fluid dynamics, diffusion-reaction processes, etc., all lie within this area. The object that we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the *state*. We are interested in the case where the state satisfies proper differential equations that are derived from certain physical laws, such as Newton's law, Fourier's law etc. The space in which the state exists is called the *state space*, and the equation that the state satisfies is called the *state equation*. By an infinite dimensional system we mean one whose corresponding state space is infinite dimensional. In particular, we are interested in the case where the state equation is one of the following types : partial differential equation, functional differential equation, integrodifferential equation or abstract evolution equation. In general, under broad assumptions, these equations can be reformulated as ordinary differential equations on abstract spaces, for example, Banach spaces. This is where semigroup theory plays a central role and provides a unified and powerful tool for the study of existence, uniqueness and continuous dependence of solutions on parameter (well posedness).

Many classical methods have been developed for solving particular types of PDE, but complications soon arise when the equations are nonlinear, as opposed to linear. Many textbooks and research papers introduce solutions of problems involving evolution equations via the theory of semigroups of operators both linear and nonlinear. For an example fo the linear case, the questions of existence and uniqueness of the solution of the initial value problem for the abstract evolution equation

$$\begin{cases} \frac{dx(t)}{dt} + Ax(t) = f(t), \quad 0 < t \leq T \\ x(0) = x_0 \end{cases} \tag{1.1}$$

has been investigated by many authors, e.g., Ahmed (1991), Belleni-Morante and McBride (1998), Goldstein (1985), and Pazy (1983). Moreover, in the case $A = A(t)$, the equation (1.1) has been investigated by Amann (1978), Friedman (1969), Pazy (1983) and Tanabe (1997). For the semilinear case, the questions of existence and uniqueness of the solution of the semilinear initial value problem

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t, x(t)), \quad 0 < t \leq T \\ x(0) = x_0 \end{cases}$$

has been investigated by many authors, e.g., Amann (1978) and Fattorini (1999).

In the previous paragraphs we considered system whose future behavior depended only on the present state and did not depend on how that state was achieved. There are many examples in economics, biology, control etc. where the past influences the future significantly. One type of such systems is described by differential delay equations.

More recently, delay parabolic problems have been studied by many authors using semigroup methods. Some authors investigated more general delay problem, some investigated concrete delay partial differential equation (e.g., Travis and Webb (1974,1978) and Wu (1996)). Xiang (1988, 1991 and 1994), Xiang and Ahmed (1992) and Xiang and Yang (1990) discussed semilinear parabolic delay problems and obtained some results on the existence of solutions.

In this thesis, we establish an existence result of mild solutions for a class of semilinear integrodifferential equations with delay in Banach spaces. Our approach will be based on techniques and results of the theory of semigroup of operators, the theory of evolution operators in Banach spaces and the contraction mapping theorem (Banach's fixed point theorem).

The problem of existence of solutions for a system governed by an integro-differential equations has been studied by several authors. However, most of the works concentrated on time-invariant systems, that is, system with generating operator being time independent. We refer to the works of Da Prato and Iannelli (1985) and Webb (1978, 1979). Moreover, some results on time-invariant systems with delay were obtained by Ahmed (1991) and Kunisch and Mastinsek (1990).

For time variant systems without delay, the existence problem was investigated by Heard and Rankin (1988) and Grimmer (1982). Grimmer studied the existence of a resolvent operator for an integrodifferential equation :

$$\begin{cases} \frac{dx(t)}{dt} = A(t)x(t) + \int_0^b B(t,s)x(s)ds + f(x), \\ x(0) = x_0 \in X, t \geq 0 \end{cases}$$

in a Banach space X.

Ahmed (1991) discussed about the existence of solutions for the system given by

$$\begin{cases} \frac{dx(t)}{dt} + Ax(t) = f(x(t)) + \int_{-a}^t h(t-s)g(x(s))ds, \quad 0 \leq t \leq T \\ x(t) = \varphi(t), \quad -a \leq t \leq 0, \end{cases} \tag{1.2}$$

where $-A$ is the infinitesimal generator of analytic semigroup $\{T(t), t \geq 0\}$ in a Banach space X, f and g are nonlinear functions from X_α into X satisfying Lipschitz and growth conditions, $h \in L^1([0, a+T], \mathbb{R})$ and $\varphi \in C([-a, 0], X_\alpha)$. X_α denotes the Banach space $D(A^\alpha)$ endowed with the graph norm $\|\cdot\|_\alpha$ defined by

$$\|x\|_\alpha = \|A^\alpha x\|_X + \|x\|_X, x \in X_\alpha.$$

Ahmed only gives an existence result of a mild solution by using a generalized contraction mapping theorem.

Here, we consider the existence of mild solution for a system governed by an integrodifferential equation with delay which is different from theirs.

This thesis is motivated by Ahmed (1991), that is, we study the system governed by (1.2), in the case where the generating operator is time dependent ($A = A(t)$). In fact, we study the existence of mild solution for the system governed by

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(x(t)) + \int_{-a}^t h(t-s)g(x(s))ds, \quad 0 \leq t \leq T \\ x(t) = \varphi(t), \quad -a \leq t \leq 0, \end{cases} \tag{1.3}$$

where $A(t) : D(A(t)) \subset X \to X$ is a linear, not necessarily unbounded operator on a Banach X. The main assumption is that the family operators $\{A(t)\}_{t\in[0,T]}$ generates the evolution system $\{U(t,s), 0 \leq s \leq t \leq T\}$, which is a two parameter family of bounded linear operators. We obtain some results under the following assumptions :

(1) f and g are Lipschitz continuous on X_α (domain of A^α) and map into a Banach space X.

(2) $\varphi \in C([-a, 0], X_\alpha)$.

(3) $h \in L^p([0, a+T], \mathbb{R})$.

We also consider (1.3) in the case when $f = f(t, x(t))$ and $g = g(t, x(t))$. We only use the original contraction mapping theorem and develop a step by step approach to prove the existence and continuous dependence of solution. We need not proof and make use of a priori estimates, such as Gronwall's Lemma. Instead, we develop a step by step approach, which yields a mild solution after a finite number of iterations. Such a method is particularly useful for applications and computations. Furthermore, it easily and clearly gives good estimates of solutions showing continuous dependence of solutions on initial conditions.

Moreover, semigroup theory has also found extension applications in the study of control theory. In control theory, optimal control problems are minimum problems which describe the behavior of systems that can be modified by the action of an operator. Two kinds of variables (or sets of variables) are involved: one of them describes the state of the system and cannot be modified directly by the operator, it is called the *state variable*; the second one, on the contrary, is under the direct control of the operator and is used to modify the state of the system, it is called the *control variable.*

The operator tries to modify the state of the system indirectly, acting on control variables; only these may act on the system, through a link control-state, usually called *controlled equation.* Finally, the operator, acting directly on controls and indirectly on states through the state equation, must achieve a goal usually written as a minimization of a functional which depends on the control that has been chosen and on the corresponding state : the so-called *cost functional.* Just to give a simple example, consider a car which is driven only by acting directly on controls that are the accelerator, the brakes and the steering-gear; the state of the car could be for instance its position and velocity which, state equations are then the equations of mechanics which, to a given choice of acceleration and steering angle, associate the position and velocity of the car, also taking into account the specifications of the engine (technological constraints, nonlinear behaviors,...). Assume that, for instance, the driver wants to minimize the fuel consumption to run along a given path; he has then to choose the best driving strategy to minimize the cost functional, which is in this case the total fuel consumption.

A recent, most productive development is the theory of optimal control of distributed parameter systems. It is well known that evolution systems are an important class of distributed parameter system and the optimal control of infinite dimensional system is a remarkable subject in control theory. Some authors including us investigated the existence of solution (e.g., Ahmed (1991), Ahmed and Teo (1981), Fattorini (1999), Lions (1971) and Li and Yong (1995).

Recent results on delay differential equations and control theory to many phenomena in the real world began to be described. Some authors investigated the existence of solutions, for example, Ahmed and Xiang (1996), Bensoussan, Da Prato, Delfour and Mitter (1992), Wu (1996), Da Prato and Ichikawa (1993), Xiang (2000) and Xiang and Kuang (2000). To the knowledge of the author, optimal controls for integrodifferential equation with delay have not been intensively studied. Here we consider the optimal control problem for a class of delay systems which is different from the delay equations studied by these authors.

In this thesis we shall study the optimal control problem

$$\text{minimize} \int_0^T l(t, x^u(t), u(t))dt$$

subject to $u \in U_{ad}$ which is called the set of admissible controls and $x \in X_\alpha$ satisfying the state system which is obtained by (1.3), that is,

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t,x(t)) + \int_{-a}^t h(t-s)g(s,x(s))ds + B(t)u(t), 0 \leq t \leq T \\ x(t) = \varphi(t), \quad -a \leq t \leq 0, \varphi \in C([-a,0], X_\alpha) \end{cases}$$

where $f, g : [0,T] \times X_\alpha \to X$ is Hölder continuous with respect to t and Lipschitz continuous with respect to x, $h \in L^1([0, a+T], \mathbb{R})$, $B(t) \in \mathcal{L}(E, X)$, $0 \leq t \leq T$ and E is separable reflexive Banach space (control space). By Balder's results, again using step by step approach we obtain existence result of optimal controls.

The thesis is organized as follows. Chapter II collects some basic concepts and results from functional analysis, semigroup theory and evolution equations that are necessary for the presentation of the theory in later chapters. In Chapter III we study the problems of existence, uniqueness and continuous dependence of mild solutions for the system governed by an integrodifferential equation with delay. Chapter IV concerns the problem of existence of optimal control of abstract semilinear integrodifferential equation on Banach spaces, and we considere a Lagrange problem for a class of semilinear integrodifferential equation with delay.

Chapter V concerns some examples from semilinear partial differential equations of parabolic type with delay as an application which extends some results of chapter III and IV.

Chapter II

Mathematical Preliminaries

In this chapter we recall some basic concepts and results that are necessary for the presentation of the theories in later chapters. Most proofs for the standard results will be omitted.

2.1 Operators in Banach Spaces

- **Banach Spaces**

Definition 2.1.1. Let X be a (real) vector space. A *norm* $\|\cdot\|$ on X is a mapping from X into $\mathbb{R}$ satisfying the four conditions:

(i) $\|x\| \geq 0$ for all $x \in X$.

(ii) $\|x\| = 0$ if and only if $x = 0$, the zero element of X.

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in X, \alpha \in \mathbb{R}$.

(iv) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *normed vector space*. Often when the norm being used is obvious, we simply say that X is a normed vector space (or X is a normed space).

Definition 2.1.2. (i) A normed vector space X is *complete* if every Cauchy sequence $\{x_n\} \subset X$ is convergent (to limit which belongs to the space).

(ii) A normed vector space X is called a *Banach space* if it is complete.

For example, given $-\infty < a < b < \infty$, let $C([a, b], \mathbb{R})$ denote the set of all real-valued continuous functions defined on $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. Thus $f \in C([a, b], \mathbb{R})$ is continuous on the right at a, continuous on the left at b and (two-sided) continuous at c for any $c : a < c < b$. The vector space operations on

$C([a,b],\mathbb{R})$ are, as usual,

$$(f+g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x) \text{ for all } x \in [a,b];$$

where $f, g \in C([a,b],\mathbb{R})$ and $\alpha \in \mathbb{R}$. With respect to

$$\|f\| = \max\{|f(x)|, x \in [a,b]\},$$

$C([a,b],\mathbb{R})$ becomes a Banach space (Belleni-Morante, 1979, pp.15-16).

- **Linear Operators**

Definition 2.1.3. Let X and Y be two given Banach spaces over the same field $\mathbb{R}$ of real numbers (or $\mathbb{C}$ of complex numbers), and let D be a subset of X. A mapping A that sends each $x \in D$ into a unique $y \in Y$ is called an *operator* with *domain* $D(A) = D$ and *range*

$$R(A) = \{y : y \in Y; y = A(x), x \in D(A)\}$$

contained in Y. We may write this as $A : D(A) \subset X \to Y$. The element $y = A(x)$ is the *image* of $x \in D(A)$ under the mapping A. In the following, we shall generally use the word 'operator' for mappings between X and Y with $X \neq \mathbb{R}$ and reserve the word 'function' for mappings from a subset of $\mathbb{R}$ into Y.

We remark that, for simplicity, we write Ax instead of $A(x)$. (The symbol Ax is justified for linear operators by definition 2.1.6.)

If the operator A is one-to-one (1-1), i.e., if

$$x_1, x_2 \in D(A), Ax_1 = Ax_2 \Longrightarrow x_1 = x_2,$$

(where the symbol $\Longrightarrow$ means "implies"), we can define the *inverse operator*

$$A^{-1} : Y \supset D(A^{-1}) = R(A) \to D(A) = R(A^{-1}) \subset X$$

by

$$A^{-1}y = x \Leftrightarrow Ax = y, x \in D(A), y \in R(A).$$

Definition 2.1.4. A is said to be *densely defined* if $D(A)$ is dense in X (that is, if every element x in X is the limit of a sequence of elements in $D(A)$, i.e., the closure of $D(A)$, $\overline{D(A)} = X$).

Definition 2.1.5. Let X and Y be two Banach spaces and let the operator $A : D(A) \subset X \to Y$. The operator A is *bounded* on $D(A)$ if there exists a positive constant M such that

$$\|Ax\|_Y \leq M \|x\|_X \text{ for all } x \in D(A).$$

Definition 2.1.6. Let operator $A : D(A) \subset X \to Y$ where $D(A)$ is a subspace of the (real) vector space X. Then A is said to be a *linear operator* if

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

for all $x, y \in D(A)$ and $\alpha, \beta \in \mathbb{R}$.

When we say that A is a linear operator, it will always be understood that $D(A)$ is a subspace of the appropriate vector space X (i.e. $\alpha x + \beta y \in D(A)$ for all $x, y \in D(A)$ and $\alpha, \beta \in \mathbb{R}$).

The following result gives several characterizations of bounded linear operators between Banach space.

Theorem 2.1.7. Let X and Y be two Banach spaces and $A : X \to Y$ be a linear operator. Then the following statements are equivalent:

(i) A is continuous at 0, the zero vector in X.

(ii) A is continuous on X.

(iii) A is bounded on X.

(Belleni-Morante, 1998, pp. 30-31)

Because of the above result, linear bounded operators are also called *continuous linear operator*. Now, for any Banach spaces X and Y, let $\mathcal{L}(X, Y)$ be the set of all bounded linear operator from X to Y, and $\mathcal{L}(X)$ if $X = Y$. For any $\alpha, \beta \in \mathbb{R}$ and $A, B \in \mathcal{L}(X, Y)$, we define $\alpha A + \beta B$ as follows:

$$(\alpha A + \beta B)(x) = \alpha Ax + \beta Bx \text{ for all } x \in X.$$

Then, $\mathcal{L}(X, Y)$ is also a vector space. By Theorem 2.1.7, the norm of the bounded linear operator A, denoted by $\|A\|$, is defined by

$$\|A\| = \sup \left\{ \frac{\|Ax\|_Y}{\|x\|_X}, x \in X, \|x\| \neq 0 \right\}. \tag{2.1}$$

An equivalent $\|A\|$ is

$$\|A\| = \sup \left\{ \|Ax\|_Y : x \in X, \|x\| \leq 1 \right\}.$$

It is not hard to show that $\|\cdot\|$ defined by (2.1) is norm on $\mathcal{L}(X,Y)$. Moreover, we can show that $\mathcal{L}(X,Y)$ is a Banach space under this norm (Belleni-Morante, 1979, pp.41-43).

We can introduce different notions of convergence in the space of bounded linear operators. The natural one based on the norm in $\mathcal{L}(X,Y)$ is called uniform convergence.

Definition 2.1.8. Let $\{A_n\}$ be a sequence of bounded linear operators in $\mathcal{L}(X,Y)$ so that

$$\|A_n - A\|_{\mathcal{L}(X,Y)} \to 0 \text{ as } n \to \infty$$

then we say that A_n *converges uniformly* to A as $n \to \infty$.

Frequently in applications we will use a different kind of convergence.

Definition 2.1.9. Let $\{A_n\}$ be a sequence of bounded linear operators in $\mathcal{L}(X,Y)$ so that

$$\|A_n x - Ax\|_Y \to 0 \text{ as } n \to \infty \text{ for all } x \in X$$

then we say that A_n *converges strongly* to A as $n \to \infty$.

If a bounded linear operator depends on a parameter t from some interval of $\mathbb{R}$, we can define strong continuity, and uniform continuity with respect to t in an analogous manner.

Definition 2.1.10. Let $T(t) \in \mathcal{L}(X,Y)$ for every $t \in [a,b]$. $T(t)$ is said to be *uniformly continuous* at $t_0 \in [a,b]$, if

$$\|T(t) - T(t_0)\|_{\mathcal{L}(X,Y)} \to 0 \text{ as } t \to t_0.$$

Definition 2.1.11. Let $T(t) \in \mathcal{L}(X,Y)$ for every $t \in [a,b]$. $T(t)$ is said to be *strongly continuous* at t_0, if

$$\|T(t)x - T(t_0)x\|_Y \to 0 \text{ for all } x \in X \text{ as } t \to t_0.$$

Example 2.1.12. Consider the operator e^{At}, where $A \in \mathcal{L}(X)$, and e^{At} is defined by

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^nt^n}{n!} + \cdots,$$

then

$$e^{At} - e^{At_0} = e^{At_0}(e^{A(t-t_0)} - I).$$

Hence

$$\left\|e^{At} - e^{At_0}\right\|_{\mathcal{L}(X)} \leq \left\|e^{At_0}\right\|_{\mathcal{L}(X)} \left\|e^{A(t-t_0)} - I\right\|_{\mathcal{L}(X)}.$$

But

$$\begin{aligned} \left\|e^{A(t-t_0)} - I\right\|_{\mathcal{L}(X)} &\leq \left\|A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \cdots\right\|_{\mathcal{L}(X)} \\ &\leq k\,|t-t_0| + \frac{k^2\,|t-t_0|^2}{2!} + \cdots \\ &= e^{k|t-t_0|} - 1, \end{aligned}$$

where $\|A\|_{\mathcal{L}(X)} = k$. Moreover $\left\|e^{At_0}\right\|_{\mathcal{L}(X)} \leq e^{k|t_0|}$ and so

$$\left\|e^{At} - e^{At_0}\right\|_{\mathcal{L}(X)} \leq e^{k|t_0|}(e^{k|t-t_0|} - 1),$$

i.e.,

$$\left\|e^{At} - e^{At_0}\right\|_{\mathcal{L}(X)} \to 0 \text{ as } t \to t_0.$$

Hence for $t \in [a, b], e^{At}$ is uniformly continuous. □

Theorem 2.1.13. (Uniform Boundedness Theorem) Let X and Y be Banach spaces and let $\mathcal{A}$ be a subset of $\mathcal{L}(X, Y)$ such that $\{Ax, A \in \mathcal{A}\}$ is bounded in Y for each $x \in X$. Then $\|A\| \leq C$ $(A \in \mathcal{A})$ for some constant C; that is, $\mathcal{A}$ is bounded in $\mathcal{L}(X, Y)$.

- **Closed Operators**

Let X and Y be two Banach spaces. The cartesian product $X \times Y$ of X and Y is the set (of all ordered pairs)

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

Addition and scalar multiplication are defined in $X \times Y$ by

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ \alpha(x_1, y_1) &= (\alpha x_1, \alpha y_1) \end{aligned}$$

for all $x_1, x_2 \in X, y_1, y_2 \in Y$ and $\alpha \in \mathbb{R}$. Then $X \times Y$ is a Banach space with norm

$$\|(x, y)\| = \|x\|_X + \|y\|_Y, x \in X, y \in Y \tag{2.2}$$

(Belleni-Morante, 1998, p.27)

Definition 2.1.14. Let X and Y be two Banach spaces. For any operator $A : D(A) \subset X \to Y$, the *graph of* A is the subset $G(A)$ of $X \times Y$ defined by

$$G(A) = \{(x, y) : y = Ax, x \in D(A)\}.$$

Definition 2.1.15. Let X and Y be Banach spaces. A linear operator $A : D(A) \subset X \to Y$ is said to be closed if its graph is a closed subspace of $X \times Y$. Equivalently, A is closed if $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$ implies $x \in D(A)$ and $Ax = y$.

We remark that the norm (2.2) is often called the graph norm on $X \times Y$, for reasons which should now be fairly clear.

Theorem 2.1.16. (Closed Graph Theorem) Let X and Y be Banach spaces and suppose $A : X \to Y$ is a linear operator and A closed in $X \times Y$. Then A is bounded, that is, A is continuous.
(Yosida, 1980, p.79).

- **Resolvent Operators**

Definition 2.1.17. Let X be a Banach space and let $A : D(A) \subset X \to X$ be a linear, not necessarily bounded, operator. The resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X, that is, the resolvent set $\rho(A)$ of A is given by

$$\rho(A) = \left\{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(X)\right\},$$

I is the identity linear operator in X. When $\lambda \in \rho(A), R(\lambda, A) = (\lambda I - A)^{-1}$ is called the *resolvent operator* of A at λ.

If A is closed linear operator then for all $\lambda, \mu \in \rho(A), R(\lambda, A) - R(\mu, A) = -(\lambda - \mu)\, R(\lambda, A)R(\mu, A)$ (see Belleni-Morante, 1998, p.50).

- **Fixed Point Theorems**

In this section we discuss some fixed point theorems and their applications. Fixed points have long been used in analysis to solve various kinds of differential and integral equations.

Definition 2.1.18. Let X be a Banach space and let $A : X \to X$ be an operator (not necessarily linear). A *fixed point* of A is a point $x \in X$ such that

$$Ax = x.$$

In other words, a fixed point of A is a solution of the equation

$$Ax = x, x \in X.$$

Definition 2.1.19. Let X be a Banach space and let $A : X \to X$ be an operator. The operator A is called *Lipschitz continuous* (or, briefly, A is Lipschitz) if

$$\|Ax - Ay\| \leq L \|x - y\| \tag{2.3}$$

for some constant L and all $x, y \in X$. If $0 \leq L < 1$ then A is called a *contraction*.

Now (2.3) of course implies A is continuous, and more importantly provides a uniform modulus of continuity. It turns out to be useful to consider also operators A satisfying a variant of (2.3), namely

$$\|Ax - Ay\| \leq L \|x - y\|^{\gamma}, 0 < \gamma \leq 1.$$

Such an operator is said to be *Hölder continuous with exponent* γ. We denote the family of all Hölder continuous with exponent γ on X by $C^{\gamma}(X, X)$

The following theorem implies that a contraction mapping on Banach space always has a unique fixed point.

Theorem 2.1.20 (The Contraction Mapping Theorem)
Let X be a Banach space and let $A : X \to X$ be a contraction. Then the equation

$$Ax = x$$

has a unique solution in X, i.e. A has a unique fixed point x. Further, this fixed point may be obtained by the method of successive approximations as follow :

$$x_0 \in X \text{ arbitrary}, x_n = Ax_{n-1} \ (n \geq 1); \ x = \lim_{n \to \infty} x_n.$$

(Belleni-Morante, 1998, pp.54-55)

Corollary 2.1.21 Let X_0 be a closed subset of the Banach space X and assume that A maps X_0 into itself and is a contraction on X_0. The equation $Ax = x$ has a unique solution $x \in X_0$.
(Belleni-Morante, 1998, p.55)

2.2 Semigroups of Linear Operators

- **Definitions and Basic Properties of Semigroups**

Let X be a Banach space and $\{T(t), t \geq 0\}$ a family of bounded linear operators on X, that is, for each $t \geq 0$, $T(t) \in \mathcal{L}(X)$ where $\mathcal{L}(X)$ denotes the space of bounded linear operators in X.

Definition 2.2.1. The family of operators $\{T(t), t \geq 0\}$ is said to be a *semigroup of bounded linear operators on* X if

(*i*) $T(0) = I$, (I is the identity operator on X).

(*ii*) $T(t+s) = T(t)T(s) = T(s)T(t)$ for all $t, s \geq 0$.

The semigroup $\{T(t), t \geq 0\}$ is said to be *uniformly continuous* if $t \longmapsto T(t)$ is continuous on $[0, \infty)$ in the uniform operator topology, that is,

$$\lim_{t \to 0} \|T(t) - I\|_{\mathcal{L}(X)} = 0.$$

Equivalently, from the definition it is clear that if $\{T(t), t \geq 0\}$ is a uniformly continuous semigroup of bounded linear operators then

$$\lim_{t \to t_0} \|T(t) - T(t_0)\|_{\mathcal{L}(X)} = 0,$$

for all $t_0 \in [0, \infty)$.

Definition 2.2.2. The operator $A : X \supset D(A) \to R(A) \subset X$ defined by

$$D(A) = \left\{ x \in X : \lim_{t \to 0^+} A_t x \text{ exists in } X \right\}$$

$$Ax = \lim_{t \to 0^+} A_t x \text{ for } x \in D(A),$$

where, for $t > 0$, $A_t x = \frac{T(t)x - x}{t}$, $x \in X$, is called the *infinitesimal generator* of the semigroup $\{T(t), t \geq 0\}$ on X.

Theorem 2.2.3. A linear operator $A : X \supset D(A) \to R(A) \subset X$ is the infinitesimal generator of uniformly continuous semigroup of operator $\{T(t), t \geq 0\}$ in X if, and only if A is a bounded linear operator.
(Ahmed, 1991, p.4)

Definition 2.2.4. (C_0*-semigroup*) The semigroup $\{T(t), t \geq 0\}$ is said to be *strongly continuous* at the origin if for each $x \in X$,

$$\lim_{t \to 0^+} \|T(t)x - x\|_X = 0$$

That is, $t \mapsto T(t)x$ is continuous from the right at $t = 0$ for each $x \in X$.

A strongly continuous semigroup of bounded linear operators on X will be called a *semigroup of class* C_0 or simply a C_0*-semigroup.*

It readily follows from the semigroup property that strong right continuity at origin implies strong right continuity for every $t > 0$, we have only to note that $T(t+h)x - T(t)x = T(t)(T(h)x - x)$ for $h > 0$. To obtain left continuity, we have to invoke the uniform boundedness principle.

Theorem 2.2.5. *(Properties of C_0-Semigroups)* Let X be a Banach space and $\{T(t), t \geq 0\}$ a C_0-semigroup on X with A as its infinitesimal generator. Then,

(1) There exist constant $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \text{ for all } t \geq 0.$$

(2) For each $x \in X, t \mapsto T(t)x$ is a continuous X-valued function on $[0, \infty)$.

(3) For $x \in X, t \in [0, \infty)$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(\tau)x d\tau = T(t)x.$$

(4) For $x \in X$ and $t > 0, \int_0^t T(\tau)x d\tau \in D(A)$ and

$$A\left(\int_0^t T(\tau)x d\tau\right) = T(t)x - x.$$

(5) For $x \in D(A)$, $T(t)x \in D(A)$ and $\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$.

(6) For $x \in D(A)$, $0 \leq s \leq t$,

$$\int_s^t AT(\tau)x d\tau = \int_s^t T(\tau)Ax d\tau = T(t)x - T(s)x.$$

(7) (i) $\overline{D(A)} = X$ and

(ii) A is closed operator or equivalently its graph $\Gamma(A) = \{(x, y) \in X \times X : y = Ax\}$is a closed subset of $X \times X$.

(8) Let B be the infinitesimal generator of C_0-semigroup $\{S(t), t \geq 0\}$. If $A = B$, then $T(t) = S(t)$ for all $t \geq 0$, that is, each C_0-semigroup generator generates a unique semigroup.

(9) The set $\cap_{n=1}^{\infty} D(A^n)$ is dense in X.

(Ahmed, 1991, pp.5-11)

Let $\{T(t), t \geq 0\}$ be a C_0-semigroup. From Theorem 2.2.5 (1) it follows that there are constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \text{ for all } t \geq 0. \tag{2.4}$$

Let $\mathcal{G}(M, \omega)$ denote the class of C_0-semigroup $\{T(t), t \geq 0\}$.

If $\omega = 0$, $T(t)$ is called a *uniformly bounded semigroup.* i.e., $\|T(t)\| \leq M$.

If $\omega = 0$, $M = 1$, $T(t)$ is called a *contraction semigroup*, i.e., $\|T(t)\| \leq 1$.

If $M = 1$, $T(t)$ is called a *quasi-contraction semigroup*, i.e., $\|T(t)\| \leq e^{\omega t}$.

(In fact, the set $\mathcal{G}(M, \omega)$ can also be defined for $\omega < 0$; in this case we have $\|T(t)\| \leq Me^{-|\omega|t}$.)

We can now give necessary and sufficient conditions for an operator A to generate a semigroup which belongs to the class $\mathcal{G}(M, \omega)$.

Theorem 2.2.6. *(Hille - Yosida Theorem)* A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ satisfying $\|T(t)\| \leq Me^{\omega t}$ for some $\omega \geq 0$ and all $t \geq 0$ if and only if

(i) A is closed linear operator whose domain $D(A)$ is dense in X.

(ii) For all real numbers $\lambda > \omega, \lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \text{ for } n = 1, 2, \ldots \tag{2.5}$$

(Pazy, 1983, p.20)

Remark 2.2.7 The condition that every real $\lambda, \lambda > \omega$, is in the resolvent set of A together with the estimate (2.5) imply that every complex λ satisfying $Re\lambda > \omega$ is in the resolvent set of A and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re\lambda - \omega)^n} \text{ for } \mathrm{Re}\lambda > \omega, n = 1, 2, \ldots .$$

(Pazy, 1983, pp.20-21)

- **Differentiable and Analytic Semigroups**

Definition 2.2.8. A C_0-semigroup $\{T(t), t \geq 0\}$ in a Banach space X is said to be *differentiable* if , for each $x \in X$, $T(t)x$ is differentiable for all $t > 0$.

We have seen in Theorem 2.2.5(5) that if $T(t)$ is a C_0-semigroup with infinitesimal generator A and $x \in D(A)$ then $t \mapsto T(t)x$ is differentiable for $t > 0$.

Theorem 2.2.9. If $\{T(t), t \geq 0\}$ is differentiable semigroup with A being its infinitesimal generator then it is differentiable infinitely many times and, for each $n = 1, 2, \ldots$

(i) $\frac{d^n}{dt^n}T(t) = T^{(n)}(t) = A^n T(t) \in \mathcal{L}(X)$ for $t > 0$.

(ii) $T^{(n)}(t) = \left(AT\left(\frac{t}{n}\right)\right)^n$ for $t > 0$.

(iii) $T^{(n)}(t)$ is uniformly continuous for $t > 0$.

(Ahmed, 1991, pp. 73-74).

Definition 2.2.10. Let $\triangle = \{z \in \mathbb{C} : \theta_1 < \arg z < \theta_2;\ \theta_1 < 0 < \theta_2\}$ and suppose $T(z) \in \mathcal{L}(X)$ for all $z \in \triangle$. The family $\{T(z), z \in \triangle\}$ is called an *analytic semigroup* in $\triangle$ if it satisfies the following properties :

(i) $z \longmapsto T(z)$ is analytic in $\triangle$, that is for each $x^* \in X^*$ and $x \in X$, the scalar valued function $z \longmapsto x^*(T(z)x)$ is analytic in the usual sense uniformly with respect to $x^* \in B_1(X^*) = \{x^* : \|x^*\|_{X^*} \leq 1\}$ and $x \in B_1(X) = \{x : \|x\|_X \leq 1\}$.

(ii) $T(0) = I$ and $lim_{z \to 0_{z \in \triangle}} T(z)x = x$ for each $x \in X$.

(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \triangle$.

A complete characterization of analytic semigroups is given in the following theorem.

Theorem 2.2.11. Let A be the infinitesimal generator of a uniformly bounded C_0-semigroup $\{T(t), t \geq 0\}$ with $0 \in \rho(A)$. The following statements are equivalent:

(a) $T(t)$ can be extended to an analytic semigroup from $[0, \infty)$ to a sector around it, given by,

$$\triangle_\delta = \{z : |\arg z| < \delta\} \text{ for some } \delta > 0,$$

and $\|T(z)\|$ is uniformly bounded in every closed subsector $\triangle_{\delta'} \subset \triangle_\delta, \delta' < \delta$.

(b) There exists a constant $C > 0$ such that, for every $\sigma > 0$ and $\tau \neq 0$

$$\|R(\sigma + i\tau, A)\|_{\mathcal{L}(X)} \leq \frac{C}{|\tau|}.$$

(c) There exist $0 < \delta < \frac{\pi}{2}$ and $M \geq 1$ such that

$$\rho(A) \supset \Sigma = \left\{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\}$$

and

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \text{ for all } \lambda \in \Sigma, \lambda \neq 0.$$

(*d*) $T(t)$ is differentiable for $t > 0$ and there exists a constant $C > 0$, such that

$$\|AT(t)\| \leq \frac{C}{t} \text{ for } t > 0.$$

(Ahmed, 1991, p.82)

- **Fractional Powers of Closed Operators**

We concentrate mainly on fractional powers of operators A for which $-A$ is the infinitesimal generator of an analytic semigroup. The result of this section will be used in the study of solutions of semilinear initial value problems. Throughout this section, we will use the following general assumption.

Assumption 2.2.12. Let A be a densely defined closed linear operator with $D(A)$ and $R(A)$ in X for which the resolvent set

$$\rho(A) \supset \Sigma^{+} = \{\lambda \in \mathbb{C} : 0 < \omega < |\arg \lambda| \leq \pi\} \cup V$$

where V is a neighborhood of zero in $\mathbb{C}$ and

$$\|R(\lambda, A)\| \leq \frac{M}{1 + |\lambda|} \text{ for } \lambda \in \Sigma^{+}.$$

For an operator A satisfying Assumption 2.2.12 and $\alpha > 0$ we define

$$A^{-\alpha} = \frac{1}{2\pi i} \int_{C} z^{-\alpha} (A - zI)^{-1} dz \tag{2.6}$$

where the path C runs in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, $\omega < \theta < \pi$, avoiding the negative real axis and the origin and $z^{-\alpha}$ is taken to be positive for real positive values of z. The integral (2.6) converges in the uniform operator topology in $\mathcal{L}(X)$ for every $\alpha > 0$ and thus defines a bounded linear operator $A^{-\alpha}$. If $\alpha = n$, an integer, the integrand is an analytic function in $\sum^{+}$ and the path of integration C can be transformed to a small circle around the origin. Then using the residue theorem it follows that

$$A^{-n} = \frac{1}{2\pi i} \int_{C} z^{-n} (A - zI)^{-1} dz$$

and thus for positive integral values of α the definition (2.6) coincides with the classical definition of $(A^{-1})^{n}$.

If $\omega < \frac{\pi}{2}$, i.e., if $-A$ is infinitesimal generator of an analytic semigroup $\{T(t),\ t \geq 0\}$ we obtain still another representation of $A^{-\alpha}$. In fact, suppose A satisfy the assumption 2.1.12 with $0 < \omega < \frac{\pi}{2}$ and let $\{T(t),\ t \geq 0\}$ be the semigroup corresponding to the operator $-A$. Then for every $0 \leq \alpha$ we have

$$A^{-\alpha} = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)dt & \text{for } \alpha > 0 \\ I & \text{for } \alpha = 0 \end{cases}.$$

(see Pazy, 1983, p. 70)

Theorem 2.2.13 There exists a constant $0 < C < \infty$ such that

$$\left\| A^{-\alpha} \right\|_{\mathcal{L}(X)} \leq C \text{ for all } 0 \leq \alpha \leq 1.$$

(Pazy, 1983, p.71)

Let A satisfy Assumption 2.2.12 with $0 < \omega < \frac{\pi}{2}$, that is, $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t),\ t \geq 0\}$. It is known that $A^{-\alpha}$ is a one-to-one, bounded linear operator on X (Pazy, 1983, p.72). For every $\alpha \geq 0$, we define

$$A^{\alpha} = \begin{cases} (A^{-\alpha})^{-1} & \text{for } \alpha > 0 \\ I & \text{for } \alpha = 0 \end{cases}. \tag{2.7}$$

Theorem 2.2.14. The operator A^{α}, $0 \leq \alpha \leq 1$, as defined by (2.7) , satisfies the following properties :

(a) A^{α} is a closed operator with domain $D(A^{\alpha}) = R(A^{-\alpha})$ the range of $A^{-\alpha}$.

(b) $0 < \beta \leq \alpha$ implies $D(A^{\alpha}) \subset D(A^{\beta})$.

(c) $\overline{D(A^{\alpha})} = X$ for every $\alpha \geq 0$.

(d) If α, β are real then

$$A^{\alpha+\beta}x = A^{\alpha}A^{\beta}x$$

for every $x \in D(A^{\gamma})$ where $\gamma = \max\{\alpha, \beta, \alpha+\beta\}$.
(Pazy, 1983, p.72).

Theorem 2.2.15 Let $-A$ be the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ in X. If $0 \in \rho(A)$ then,

(a) $T(t) : X \to D(A^{\alpha})$ for every $t > 0$ and $\alpha \geq 0$.

(*b*) For every $x \in D(A^\alpha)$ we have

$$T(t)A^\alpha x = A^\alpha T(t)x \text{ for all } \alpha \geq 0.$$

(*c*) For every $t > 0$, $A^\alpha T(t) \in \mathcal{L}(X)$ and

$$\|A^\alpha T(t)\|_{\mathcal{L}(X)} \leq M_\alpha t^{-\alpha} e^{-\delta t}, \delta \geq 0$$

for some constant $M_\alpha > 0$.

(*d*) For $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$ then

$$\|T(t)x - x\| \leq C_\alpha t^\alpha \|A^\alpha x\|$$

for some constant $C_\alpha > 0$.
(Pazy, 1983, pp.74-75)

2.3 Differential Equations on Banach Spaces

- **The Homogeneous Initial Value Problem**

At the beginning of this section, we motivate the study of semigroups of operators via differential equations on Banach spaces which are abstracts formulation of initial value problem for partial differential equations.

Let X be a Banach space and let $A : X \supset D(A) \to R(X) \subset X$ be a given operator and consider the homogeneous initial value problem in X given by

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) \ , \ t > 0 \\ x(0) = x_0. \end{cases} \tag{2.8}$$

where $x_0 \in X$.

Definition 2.3.1. A *(classical) solution* of (2.8) is a function $x : [0, \infty) \to X$ such that

(*i*) $x \in C([0,\infty), X) \cap C^1((0,\infty), X)$.
(*ii*) $x(t) \in D(A)$ for all $t > 0$.
(*iii*) (2.8) is satisfied, i.e., $\begin{cases} \frac{dx(t)}{dt} = Ax(t) \ , \ t > 0 \\ x(0) = x_0. \end{cases}$

Theorem 2.3.2. Let $\overline{D(A)} = X, \rho(A) \neq \emptyset$. Then (2.8) has a unique classical solution $x(t)$ which is continuously differentiable on $[0, \infty)$, for every initial value

$x_0 \in D(A)$ if and only if, A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X.
(Pazy, 1983, pp.102-104)

Theorem 2.3.3. (i) If A is the infinitesimal generator of a differentiable semigroup $\{T(t), t \geq 0\}$ in X then for every $x_0 \in X$, (2.8) has a unique (classical) solution $x(t) = T(t)x_0, t > 0$.

(ii) If A is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ then for every $x_0 \in X$, (2.8) has a unique (classical) solution $x(t) = T(t)x_0, t > 0$.

Proof. (i) Since $\{T(t), t \geq 0\}$ is a differentiable semigroup for $t > 0$, the X-valued function $t \to T(t)x_0$ is differentiable for every $x_0 \in X$ and

$$\frac{d}{dt}T(t)x_0 = AT(t)x_0 \text{ for } t > 0.$$

Further, by Theorem 2.2.9(iii), $AT(t)x_0$ is Lipschitz continuous for $t > 0$ and hence we conclude that $x(t) = T(t)x_0, t > 0$, is the unique (classical) solution of (2.8)

(ii) This follows from the simple fact that, for analytic semigroups, $T(t)x \in D(A)$ for every $x \in X$ and $t > 0$; and consequently every analytic semigroup is also a differentiable semigroup. □

If A is the infinitesimal generator of a C_0-semigroup which is not differentiable then, in general, if $x \notin D(A)$, the initial value problem (2.8) does not have a solution. The function $t \mapsto T(t)x_0$ is then a *generalized solution* of (2.8) which we will call a *mild solution*.

- **The Nonhomogeneous Initial Value Problems**

Consider the nonhomogeneous initial value problem:

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + f(t)\ ,\ t > 0 \\ x(0) = x_0. \end{cases} \tag{2.9}$$

where $f : [0, T) \to X, x_0 \in X$.

Definition 2.3.4. A function $x : [0, T) \to X$ is a *(classical) solution* of (2.9) if

(i) $x \in C([0, T), X) \cap C^1((0, T), X)$.

(ii) $x(t) \in D(A)$ for all $t \in (0, T)$.

(iii) (2.9) is satisfied on $[0, T)$.

Theorem 2.3.5. *(Existence and Uniqueness)* Let A be the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$. If $f \in L^1([0,T], X)$ then for every $x_0 \in X$, (2.9) has at most one solution. If it has solution, this solution is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, 0 < t \leq T. \tag{2.10}$$

Proof. Define $z(s) = T(t-s)x(s)$ for $0 < s \leq t < \infty$. Since $x(s) \in D(A)$ for $s > 0$ and $x \in C^1((0,T), X)$, z is differentiable and it is given by

$$\begin{aligned}
\frac{dz(s)}{ds} &= -AT(t-s)x(s) + T(t-s)\,\dot{x}\,(s) \\
&= -AT(t-s)x(s) + T(t-s)\,(Ax(s) + f(s)) \\
&= -AT(t-s)x(s) + AT(t-s)x(s) + T(t-s)f(s) \\
&= T(t-s)f(s) \text{ for } s > 0.
\end{aligned}$$

The second equation follows from the fact that x is solution of (2.9) in the sense of definition 2.3.4, and the third follows from the fact that $x(s) \in D(A)$ for $s > 0$ and $T(t)$ commutes with A on $D(A)$. Since $f \in L^1([0,T], X)$ and $\|T(t)\| \leq Me^{\omega t}, t > 0$, the last expression is integrable. Integrating this over interval $[0,t]$ we obtain

$$z(t) - z(0) = \int_0^t T(t-s)f(s)ds$$

or

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds.$$

The uniqueness follows from the uniqueness of the semigroup associated to the operator A. This complete the proof. $\square$

Definition 2.3.6. Let A be the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$. Let $x_0 \in X$ and $f \in L^1([0,T], X)$. A function $x \in C([0,T], X)$ given by (2.10) is called a *mild solution* of (2.9) on $[0,T]$.

The definition of the mild solution of (2.9) coincides when $f \equiv 0$ with the definition of $T(t)x_0$ as the mild solution of the corresponding homogeneous equation. It is therefore clear that not every mild solution of (2.9) is indeed a (classical) solution even in the case $f \equiv 0$.

Theorem 2.3.7. Let A be the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$. Then the following results hold (*i*) If $x_0 \in D(A)$ and $f \in C^1([0,T],X)$ then (2.9) has a unique (classical) solution.

(*ii*) If $x_0 \in D(A)$ and $f \in L^1([0,T],X)$ satisfying (*a*) $f(t) \in D(A)$ for $t \in (0,T)$ and (*b*) $Af \in L^1([0,T],X)$, then (2.9) has a unique (classical) solution. (Ahmed, 1991, pp.154-155)

2.4 Evolution Equations

Let X be a Banach space. For every t, $0 \leq t \leq T$ let $A(t) : D(A(t)) \subset X \to X$ be a linear operator in X and let $f(t)$ be an X-valued function. Consider the linear initial value problem :

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t)\ ,\ 0 \leq s < t \leq T \\ x(s) = x_0. \end{cases} \tag{2.11}$$

The initial value problem (2.11) is called an *evolution problem.*

Definition 2.4.1. An X-valued function $x : [s,T] \to X$ is a *classical solution* of (2.11) if $x \in C([s,T],X) \cap C^1((s,T],X)$, $x(t) \in D(A(t))$ *for* $s < t \leq T$, and satisfies (2.11).

Definition 2.4.2. A two parameter family of bounded linear operators $U(t,s), 0 \leq s \leq t \leq T$, on X is called an *evolution system (or evolution operator)* if the following three conditions are satisfied :

(*i*) $U(s,s) = I$.

(*ii*) $U(t,s) = U(t,r)U(r,s)$ for $0 \leq s \leq r \leq t \leq T$.

(*iii*) $(t,s) \longmapsto U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$ (that is, $U(\cdot,\cdot)x \in C(\overline{\triangle},X)$ for every $x \in X, \triangle = \{(t,s) \in [0,T]^2 : 0 \leq s < t \leq T\}$).

In the following we shall present sufficient conditions for the existence of an evolution operator U, in the parabolic case. We will need the following assumptions :

Assumption 2.4.3.

(A1) : $\{A(t), 0 \leq t \leq T\}$ is a family of closed densely defined linear operator in X such that the domain $D(A(t))$ of $A(t)$ is independent of t, i.e. $D(A(t)) = D(A)$.

(A2) : For each $t \in [0,T]$, the resolvent $R(\lambda, A(t))$ of $A(t)$ exists for all λ with $\operatorname{Re}\lambda \leq 0$ and there is a constant M such that

$$\|R(\lambda, A(t))\|_{\mathcal{L}(X)} \leq \frac{M}{1+|\lambda|}$$

for $\operatorname{Re}\lambda \leq 0, t \in [0,T]$.

(A3) : There exist constants L and $0 < \alpha \leq 1$ such that

$$\left\|(A(t) - A(s))A(\tau)^{-1}\right\|_{\mathcal{L}(X)} \leq L\,|t-s|^{\alpha}$$

for $s, t, \tau \in [0,T]$.

Here $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. We note that (**A1**) and (**A2**) imply that for every $t \in [0,T], -A(t)$ is the infinitesimal generator of an analytic semigroup $\left\{e^{-\tau A(t)}, 0 \leq \tau < \infty\right\}$ in $\mathcal{L}(X)$. Moreover, there exist positive constants C and δ such that

$$\left\|e^{-\tau A(t)}\right\|_{\mathcal{L}(X)} \leq Ce^{-\delta\tau}, \tau \geq 0 \tag{2.12}$$

and

$$\left\|A(t)e^{-\tau A(t)}\right\|_{\mathcal{L}(X)} \leq \frac{Ce^{-\delta\tau}}{\tau}, \tau > 0$$

for all $0 \leq t \leq T$ (see Amann, 1978, pp.7-8).

For each $\alpha > 0$, the inequality (2.12) implies the fractional power $A^{-\alpha}(t)$ exists and is given by

$$A^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \tau^{\alpha-1} e^{-\tau A(t)} d\tau, \; 0 \leq t \leq T.$$

It is known that $A^{-\alpha}(t)$ is a one-to-one, bounded linear operator on X. We define positive fractional powers of $A(t)$ by

$$A^{\alpha}(t) = \left[A^{-\alpha}(t)\right]^{-1}.$$

For $\alpha = 0$, we set $A^0(t) = I$.

Theorem 2.4.4. *(i)* $A^{\alpha}(t)$ is a closed bijective linear operator in X.

(ii) $\overline{D(A^{\alpha}(t))} = X$.

(iii) $0 \leq \alpha \leq \beta$ implies $D(A^{\beta}(t)) \subset D(A^{\alpha}(t))$.

(iv) If $\alpha, \beta \in \mathbb{R}$ then

$$A^{\alpha+\beta}(t)x = A^{\alpha}(t)A^{\beta}(t)x = A^{\beta}(t)A^{\alpha}(t)x$$

for every $x \in D(A^{\gamma}(t))$ where $\gamma = \max\{\alpha, \beta, \alpha+\beta\}$.

(v) For $0 \leq \alpha < \beta \leq 1$ and $s, t \in [0,T]$,

$$D(A^{\beta}(s)) \subset D(A^{\alpha}(t))$$

and

$$\left\|A^{\alpha}(s)A^{-\beta}(t)\right\| \leq C(\alpha, \beta).$$

(Amann, 1978,p.8)

In the following we let $\|x\|_{\alpha} = \|A^{\alpha}x\|$ for $x \in D(A^{\alpha})$ and $0 \leq \alpha \leq 1$, and we denote by X_{α} the Banach space $(D(A^{\alpha}), \|\cdot\|_{\alpha})$. Then $X_{\beta} \hookrightarrow X_{\alpha}$ for $0 \leq \alpha \leq \beta \leq 1$ (with $X_0 = X$). (Amann, 1978, p.8)

Theorem 2.4.5. Under the Assumptions (**A1**)-(**A3**) there is a unique evolution operator $U(t,s)$ on $0 \leq s \leq t \leq T$, satisfying:

(i) $\|U(t,s)\| \leq C$ for $0 \leq s \leq t \leq T$.

(ii) For $0 \leq s < t \leq T, U(t,s) : X \to D(A)$ i.e., $U(t,s)X \subset D(A)$ and $t \mapsto U(t,s)$ is strongly differentiable in X. The derivative $\frac{\partial}{\partial t}U(t,s) \in \mathcal{L}(X)$ and it is strongly continuous on $0 \leq s < t \leq T$. Moreover,

$$\frac{\partial}{\partial t}U(t,s) + A(t)U(t,s) = 0 \text{ for } 0 \leq s < t \leq T$$

$$\left\|\frac{\partial}{\partial t}U(t,s)\right\| = \|A(t)U(t,s)\| \leq \frac{C}{t-s}$$

and

$$\left\|A(t)U(t,s)A(s)^{-1}\right\| \leq C \text{ for } 0 \leq s \leq t \leq T.$$

(iii) For every $v \in D(A)$ and $t \in (0,T)$, $U(t,s)v$ is differentiable with respect to s on $0 \leq s \leq t \leq T$ and

$$\frac{\partial}{\partial s}U(t,s)v = U(t,s)A(s)v.$$

(Pazy, 1983, pp. 150-151)

Theorem 2.4.6. Let $\{A(t)\}_{t\in[0,T]}$ satisfy the Assumptions (**A1**)-(**A3**). For every $0 \leq s < T$ and $x_0 \in X$ the initial value problem

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = 0\ , \ \ s < t \leq T \\ x(s) = x_0. \end{cases} \tag{2.13}$$

has a unique solution x given by $x(t) = U(t,s)x_0$, where $U(t,s)$ is the evolution operator constructed above.

(Pazy, 1983, p. 164)

Definition 2.4.7. Let $\{A(t)\}_{t\in[0,T]}$ satisfy the Assumptions (**A1**)-(**A3**) and $U(t,s)$, $0 \leq s \leq t \leq T$ be the evolution operator provided by Theorem 2.4.5. A function $x \in C([0,T],X)$ is said to be *mild solution* of (2.11) if x satisfies the integral equation

$$x(t) = U(t,s)x_0 + \int_s^t U(t,\tau)f(\tau)d\tau.$$

Theorem 2.4.8. Let $\{A(t), 0 \leq t \leq T\}$ satisfy the Assumptions (**A1**)-(**A3**) and let $U(t,s)$ be the evolution operator provided by Theorem 2.4.5. If f is Hölder continuous on $[0,T]$ then the IVP

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t)\ ,\ 0 < t \leq T \\ x(0) = x_0. \end{cases}$$

has, for every $x_0 \in X$, a unique classical solution x given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(\tau)d\tau, 0 \leq t \leq T.$$

Moreover, $x \in C^1([0,T],X)$, provided $x_0 \in D(A)$.
(Amann, 1978, p.8)

In the following theorem we collect the most important regularity properties of the evolution operator. For abbreviation we denote the norm in $\mathcal{L}(X_\alpha, X_\beta)$ by $\|\cdot\|_{\alpha,\beta}$.

Theorem 2.4.9. *(Properties of the evolution operator)*

(i) Suppose that $0 \leq \alpha \leq \beta < 1$. Then

$$\|U(t,s)\|_{\alpha,\beta} \leq C(\alpha,\beta,\gamma)(t-s)^{-\gamma}$$

for $\beta - \alpha < \gamma < 1$ and $0 \leq s < t \leq T$. Moreover, if $0 \leq \alpha < \beta \leq 1$, then

$$\|U(t,s)\|_{\beta,\alpha} \leq C(\alpha,\beta).$$

(ii) Suppose that $0 \leq \alpha < \beta \leq 1$. Then

$$\|U(t,\tau) - U(s,\tau)\|_{\beta,\alpha} \leq C(\alpha,\beta,\gamma)\,|t-s|^\gamma$$

for $0 \leq \gamma < \beta - \alpha$ and $(t,\tau),(s,\tau) \in \overline{\Delta}, \Delta = \{(t,s) \in [0,T]^2 : 0 \leq s < t \leq T\}$.

(iii) Let $0 \leq \alpha < 1, 0 \leq \sigma < T$ and $f \in C([\sigma,T],X)$. Then

$$\left\| \int_\sigma^t U(t,\tau)f(\tau)d\tau - \int_\sigma^s U(s,\tau)f(\tau)d\tau \right\|_\alpha \leq C(\alpha,\gamma)\,|s-t|^\gamma \max_{\sigma\leq\tau\leq T} \|f(\tau)\|_X$$

for $0 \leq \gamma < 1-\alpha$ and $\sigma \leq s, t \leq T$.

(iv) Let $0 \leq \alpha < \beta \leq 1$ and let

$$K(x,f)(t) = U(t,0)x + \int_0^t U(t,\tau)f(\tau)d\tau, 0 \leq t \leq T.$$

Then K is a continuous linear operator from $X_\beta \times C([0,T],X)$ into $C^\theta([0,T],X_\alpha)$ for every $\theta \in [0, \beta-\alpha]$.
(Amann, 1978, pp. 9-10)

Chapter III

Semilinear Integrodifferential Equations

At the beginning of this chapter, we motivated the study of evolution operator via initial value problems for PDEs. We shall now show that the theory developed in the previous chapter answers all the basic questions of existence, uniqueness and continuous dependence of solution for the system governed by an integrodifferential equation with delay :

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(x(t)) + \int_{-a}^{t} h(t-s)g(x(s))ds \ , \ t \in [0,T] \\ x(t) = \varphi(t) \ , \ t \in [-a,0]. \end{cases}$$

We use the following notation : Let X be a Banach space with norm $\|\cdot\|$, and let T be fixed positive number. Let $\{A(t), 0 \leq t \leq T\}$ be a family of closed linear operators in X satisfying the Assumption 2.4.3 :

(A1) : The domain $D(A(t))$ of $A(t)$ is dense in X and does not depend on t, i.e. $D(A(t)) = D(A)$.

(A2) : For each $t \in [0,T]$, the resolvent $R(\lambda : A(t))$ of $A(t)$ exists for all λ with $\operatorname{Re}\lambda \leq 0$ and there is a constant C that is independent of λ and t such that

$$\|R(\lambda : A(t))\|_{\mathcal{L}(X)} \leq \frac{C}{1+|\lambda|}$$

for $\operatorname{Re}\lambda \leq 0, t \in [0,T]$.

(A3) : There exist constants L and $0 < \alpha \leq 1$ such that

$$\left\|(A(t) - A(s))A(\tau)^{-1}\right\|_{\mathcal{L}(X)} \leq L\,|t-s|^{\alpha}$$

for $s, t, \tau \in [0,T]$.

Throughout this chapter we presuppose hypothesis (**A1**)-(**A3**). In addition we make the following assumptions. Suppose

(F1) $f : X_\alpha \to X$ is Lipschitz continuous, that is, there exists a constant $L > 0$ such that

$$\|f(x) - f(y)\|_X \le L\,\|x - y\|_\alpha$$

for all $x, y \in X_\alpha$.

(F2) $f : [0,T] \times X_\alpha \to X$ is Hölder continuous with respect to t and Lipschitz continuous with respect to x, that is, there exist constants C and $0 < \theta \le 1$ such that

$$\|f(t_1, x_1) - f(t_2, x_2)\|_X \le C\left\{|t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha\right\}$$

for $t_1, t_2 \in [0,T]$ and $x_1, x_2 \in X_\alpha$.

3.1 A Class of Semilinear Evolution Equations

Before studying integrodifferential equations, we discuss the semilinear initial value problems (IVPs) of the form:

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(x(t)) \ , \ 0 < t \le T \\ x(0) = x_0 \end{cases} \tag{3.1}$$

with $x_0 \in X_\alpha$ for some $\alpha \in [0,1)$.

We will develop a step by step approach to discuss the regularity, existence and continuous dependence of mild solution for (3.1). The main idea is also the basis of studying integrodifferential equations.

Definition 3.1.1. A function $x \in C([0,T], X_\alpha)$ is said to be *classical solution* of (3.1) if $x \in C([0,T], X_\alpha) \cap C^1((0,T], X)$ with $x(0) = x_0$ such that $x(t) \in D(A)$ and $\frac{dx(t)}{dt} + A(t)x(t) = f(x(t))$ for $0 < t \le T$.

Definition 3.1.2. A function $x \in C([0,T], X_\alpha)$ is said to be *mild solution* of (3.1) if x satisfies the integral equation

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(x(\tau))d\tau \ , \ t \in [0,T]. \tag{3.2}$$

• Regularity of Mild Solutions

We now turn to consider the conditions of f that will ensure that the mild solution of (3.1) is a classical solution.

Theorem 3.1.3. Let $\{A(t)\}_{t\in[0,T]}$ satisfy the assumptions **(A1)-(A3)**, $U(t,s)$, $0 \leq s \leq t \leq T$ be an evolution system, f satisfy the assumption (**F1**). Then for every $x_0 \in X_\beta$, $0 \leq \alpha < \beta < 1$, the initial value problem (3.1) is equivalent to the integral equation (3.2).

Proof Let x be a solution of (3.1). Then the function

$$V(t,\tau) = U(t,\tau)x(\tau)$$

is differentiable with respect to τ for $0 \leq \tau \leq t \leq T$ and

$$\begin{aligned}
\frac{\partial V(t,\tau)}{\partial \tau} &= \frac{\partial U(t,\tau)}{\partial \tau} x(\tau) + U(t,\tau)\,\dot{x}\,(\tau) \\
&= U(t,\tau)A(\tau)x(\tau) + U(t,\tau)\left(-A(\tau)x(\tau) + f(x(\tau))\right) \\
&= U(t,\tau)f(x(\tau)).
\end{aligned}$$

Integrating the above equation with respect to τ from 0 to t, we get

$$V(t,t) - V(t,0) = \int_0^t U(t,\tau)f(x(\tau))d\tau$$

or

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(x(\tau))d\tau\ ,\ t \in [0,T].$$

Conversely, suppose that $x \in C([0,T], X_\alpha)$ is a mild solution of (3.1). Then we have

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(x(\tau))d\tau$$

for $0 \leq t \leq T$. Thus by Theorem 2.4.9 (*ii*) and (*iii*), we get $x \in C^\theta([0,T], X_\alpha)$, $\theta \in [0, \beta-\alpha)$, *i.e.*,

$$\|x(t_1) - x(t_2)\|_\alpha \leq C\,|t_1 - t_2|^\theta$$

for $t_1, t_2 \in [0,T]$ and $\theta \in [0, \beta-\alpha)$. Since f satisfies (**F1**), then

$$\begin{aligned}
\|f(x(t_1)) - f(x(t_2))\|_X &\leq L\,\|x(t_1) - x(t_2)\|_\alpha \\
&\leq LC\,|t_1 - t_2|^\theta .
\end{aligned}$$

Hence $f(x(\cdot))$ is Hölder continuous with exponent θ, $0 \leq \theta < \beta - \alpha$, *i.e.*, $f(x(\cdot)) \in C^{\theta}([0,T], X_{\alpha})$. By Theorem 2.4.8, we get, for every $x_0 \in X_{\beta} = D(A^{\beta})$, $x \in C^1((0,T], X)$. Therefore, $x \in C([0,T], X_{\alpha}) \cap C^1((0,T], X)$, i.e., x is classical solution. □

Remark 3.1.4. As usual, Theorem 3.1.3 still holds if f satisfies assumption (**F2**). It is easy to show that $f(\cdot, x(\cdot))$ is Hölder continuous, in fact,

$$\|f(t_1, x(t_1)) - f(t_2, x(t_2))\|_X \leq C_1 \left\{ |t_1 - t_2|^{\theta_1} + \|x(t_1) - x(t_2)\|_{\alpha} \right\}$$

$$\begin{aligned} &\leq C_1 \left\{ |t_1 - t_2|^{\theta_1} + C_2 |t_1 - t_2|^{\theta_2} \right\} \\ &\leq C_3 \left\{ |t_1 - t_2|^{\theta_1} + |t_1 - t_2|^{\theta_2} \right\}, C_3 = \max\{C_1, C_1 C_2\} \\ &\leq C |t_1 - t_2|^{\theta}, \ \theta = \min\{\theta_1, \theta_2\}. \end{aligned}$$

- **Existence, Uniqueness and Continuous Dependence of Mild Solutions**

We prove theorem that employs the Lipschitz condition to show existence, uniqueness and continuous dependence of solutions.

Theorem 3.1.5. Let $\{A(t)\}_{t \in [0,T]}$ satisfy the assumptions **(A1)-(A3)**, $U(t,s)$, $0 \leq s \leq t \leq T$ be the evolution system, f satisfy the assumption (**F1**). Then for every $x_0 \in X_{\beta}$, $0 \leq \alpha < \beta < 1$, the initial value problem (3.1) has a unique mild solution $x \in C([0,T], X_{\alpha})$.

Proof We define an operator G_1 on $C([0,T], X_{\alpha})$ by:

$$(G_1 x^1)(t) = U(t,0)x_0 + \int_0^t U(t,\tau) f(x^1(\tau)) d\tau \ , \ 0 \leq t \leq T.$$

Then, it follows from Theorem 2.4.9(ii) and (iii) that G_1 maps $C([0,T], X_{\alpha})$ into itself, in fact, $\|(G_1 x^1)(t_1) - (G_1 x^1)(t_2)\|_{\alpha}$

$\leq \|U(t_1,0)x_0 - U(t_2,0)x_0\|_{\alpha} + \left\| \int_0^{t_1} U(t_1,\tau) f(x^1(\tau)) d\tau - \int_0^{t_2} U(t_2,\tau) f(x^1(\tau)) d\tau \right\|_{\alpha}$

$= \|(U(t_1,0) - U(t_2,0))\, x_0\|_{\alpha} + \left\| \int_0^{t_1} U(t_1,\tau) f(x^1(\tau)) d\tau - \int_0^{t_2} U(t_2,\tau) f(x^1(\tau)) d\tau \right\|_{\alpha}$

$\leq C_1(\alpha,\beta,\gamma_1) |t_1 - t_2|^{\gamma_1} + C_2(\alpha,\gamma_2) |t_1 - t_2|^{\gamma_2} \max_{0 \leq \tau \leq T} \|f(x^1(\tau))\|_X$

for $0 \leq \gamma_1 < \beta - \alpha$ and $0 \leq \gamma_2 < 1 - \alpha$ and $(t_1, 0), (t_2, 0) \in \overline{\triangle}$,

$\triangle = \{(t,s) \in [0,T] \times [0,T] : 0 \leq s < t \leq T\}$.

$\leq C(\alpha,\beta,\gamma) |t_1 - t_2|^{\gamma}, \gamma = \min\{\gamma_1, \gamma_2\}$.

Now, we claim that if $T > 0$ is small enough, then G_1 is a contraction. To prove this, suppose that $x_1^1, x_2^1 \in C([0,T], X_\alpha)$ and $t \in [0,T]$. Then, by Theorem 2.4.9 (i), we have

$$\|(G_1x_1^1)(t) - (G_1x_2^1)(t)\|_\alpha \leq \int_0^t \|U(t,\tau)\,(f(x_1^1(\tau)) - f(x_2^1(\tau)))\|_\alpha \, d\tau$$

$$\leq C(0,\alpha,\gamma) \int_0^t (t-\tau)^{-\gamma} \|f(x_1^1(\tau)) - f(x_2^1(\tau))\|_X \, d\tau \ , \alpha < \gamma < 1$$

$$\leq \ C(0,\alpha,\gamma)\, L \int_0^t (t-\tau)^{-\gamma} \|x_1^1(\tau) - x_2^1(\tau)\|_\alpha \, d\tau$$

$$\leq C(0,\alpha,\gamma)\, L \sup_{0\leq\tau\leq T} \|x_1^1(\tau) - x_2^1(\tau)\|_\alpha \int_0^t (t-\tau)^{-\gamma} d\tau$$

$$\leq C(0,\alpha,\gamma)\, L \tfrac{T^{1-\gamma}}{1-\gamma} \|x_1^1 - x_2^1\|_{C([0,T],X_\alpha)} .$$

Maximizing the left hand side with respect to t, we have

$$\left\|G_1x_1^1 - G_1x_2^1\right\|_{C([0,T],X_\alpha)} \leq C(0,\alpha,\gamma)\, L \frac{T^{1-\gamma}}{1-\gamma} \left\|x_1^1 - x_2^1\right\|_{C([0,T],X_\alpha)},$$

where L and C are independent of T and thus G_1 is a contraction, provided $T > 0$ is so small that

$$C(0,\alpha,\gamma) L \frac{T^{1-\gamma}}{1-\gamma} < 1.$$

Given any $T > 0$ we select $T_0 > 0$ so small that

$$C(0,\alpha,\gamma) L \frac{{T_0}^{1-\gamma}}{1-\gamma} < 1. \tag{3.3}$$

By the contraction mapping theorem, we can conclude that G_1 has a unique fixed point $x^1 \in C([0,T_0], X_\alpha)$ such that $G_1 x^1 = x^1$, that is,

$$x^1(t) = U(t,0)x_0 + \int_0^t U(t,\tau) f(x^1(\tau)) d\tau \ , \ t \in [0,T_0].$$

Thus the IVP (3.1) has a unique mild solution $x^1 \in C([0,T_0], X_\alpha)$.

Now, we define an operator G_2 on $C([T_0, 2T_0], X_\alpha)$ by

$$(G_2x^2)(t) = U(t,T_0)x^1(T_0) + \int_{T_0}^t U(t,\tau) f(x^2(\tau)) d\tau$$

for $T_0 \leq t \leq 2T_0$. By the same argument as before, one can show that G_2 maps $C([T_0, 2T_0], X_\alpha)$ into itself. Let us show that G_2 is a contraction mapping in $C([T_0, 2T_0], X_\alpha)$. Suppose that $x_1^2, x_2^2 \in C([T_0, 2T_0], X_\alpha)$ and $t \in [T_0, 2T_0]$. Then

$$\|(G_2x_1^2)(t) - (G_2x_2^2)(t)\|_\alpha \leq \int_{T_0}^t \|U(t,\tau)\,(f(x_1^2(\tau)) - f(x_2^2(\tau)))\|_\alpha \, d\tau$$

$$\leq C(0,\alpha,\gamma) \int_{T_0}^t (t-\tau)^{-\gamma} \|f(x_1^2(\tau)) - f(x_2^2(\tau))\|_X \, d\tau \ , \alpha < \gamma < 1$$

$$\leq \ C(0,\alpha,\gamma)\, L \int_{T_0}^t (t-\tau)^{-\gamma} \|x_1^2(\tau) - x_2^2(\tau)\|_\alpha \, d\tau$$

$$\leq C(0,\alpha,\gamma)\, L \sup_{T_0\leq\tau\leq 2T_0} \|x_1^2(\tau)-x_2^2(\tau)\|_\alpha \int_{T_0}^t (t-\tau)^{-\gamma} d\tau$$
$$\leq C(0,\alpha,\gamma)\, L \frac{T_0{}^{1-\gamma}}{1-\gamma} \|x_1^2 - x_2^2\|_{C([T_0,2T_0],X_\alpha)}\,.$$

Therefore,

$$\|G_2 x_1^2 - G_2 x_2^2\|_{C([T_0,2T_0],X_\alpha)} \leq C(0,\alpha,\gamma)\, L \frac{T_0{}^{1-\gamma}}{1-\gamma} \|x_1^2 - x_2^2\|_{C([T_0,2T_0],X_\alpha)}\,,$$

where L and C are independent of T_0. From (3.3), we get G_2 is a contraction mapping in $C([T_0,2T_0],X_\alpha)$. Hence G_2 has a unique fixed point $x^2 \in C([T_0,2T_0],X_\alpha)$ such that $G_2 x^2 = x^2$, that is,

$$x^2(t) = U(t,T_0)x^1(T_0) + \int_{T_0}^t U(t,\tau) f(x^2(\tau))d\tau\ ; T_0 \leq t \leq 2T_0.$$

Thus, the IVP (3.1) has a unique mild solution $x^2 \in C([T_0,2T_0],X_\alpha)$. Now, let us show that the IVP (3.1) has a unique mild solution $x \in C([0,2T_0],X_\alpha)$. Define

$$x(t) = \begin{cases} x^1(t)\ , & 0 \leq t \leq T_0 \\ x^2(t)\ , & T_0 \leq t \leq 2T_0 \end{cases}$$

where

$$x^1(t) = U(t,0)x_0 + \int_0^t U(t,\tau) f(x^1(\tau))d\tau$$

and

$$x^2(t) = U(t,T_0)x^1(T_0) + \int_{T_0}^t U(t,\tau) f(x^2(\tau))d\tau.$$

Then

$$\begin{aligned} x^2(t) &= U(t,T_0)\left(U(T_0,0)x_0 + \int_0^{T_0} U(T_0,\tau) f(x^1(\tau))d\tau\right) + \int_{T_0}^t U(t,\tau) f(x^2(\tau))d\tau \\ &= U(t,0)x_0 + \int_0^{T_0} U(t,\tau) f(x^1(\tau))d\tau + \int_{T_0}^t U(t,\tau) f(x^2(\tau))d\tau \end{aligned}$$

and we get

$$x(t) = \begin{cases} U(t,0)x_0 + \int_0^t U(t,\tau) f(x^1(\tau))d\tau\ , 0 \leq t \leq T_0 \\ U(t,0)x_0 + \int_0^{T_0} U(t,\tau) f(x^1(\tau))d\tau + \int_{T_0}^t U(t,\tau) f(x^2(\tau))d\tau, T_0 \leq t \leq 2T_0 \end{cases},$$

that is,

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau) f(x(\tau))d\tau\ , 0 \leq t \leq 2T_0.$$

Therefore, the IVP (3.1) has a unique mild solution $x \in C([0,2T_0],X_\alpha)$.

Repeating the above procedure in intervals $[0,3T_0], [0,4T_0], \ldots$, after finite n steps such that $nT_0 \geq T$. We can construct a unique mild solution on the full interval $[0,T]$, that is, $x \in C([0,T],X_\alpha)$,

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau) f(x(\tau))d\tau\ , 0 \leq t \leq T. \qquad \square$$

Corollary 3.1.6. Under the same assumptions of Theorem 3.1.5, for every $x_0 \in X_\beta, 0 \leq \alpha < \beta < 1$, the initial value problem (3.1) has a unique classical solution.

Proof The assertion follows immediately from Theorem 3.1.5 and Theorem 3.1.3. □

Corollary 3.1.7. Let $\{A(t)\}_{t\in[0,T]}$ satisfy the assumptions **(A1)-(A3)**, $U(t,s)$, $0 \leq s \leq t \leq T$ be the evolution system and f satisfy the assumption (**F2**). Then for every $x_0 \in X_\beta,\ 0 \leq \alpha < \beta < 1$, the initial value problem

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t, x(t))\ ,\ 0 \leq t \leq T \\ x(0) = x_0 \end{cases} \tag{3.4}$$

has a unique classical solution.

Proof The proof is similar to the proof of Theorem 3.1.5, for (3.4) has a unique mild solution. Theorem 3.1.3 and Remark 3.1.4 imply that (3.4) has a unique classical solution. □

In the following we denote by $D = \cup X_\beta, \beta \in (\alpha, 1]$. For $x_0 \in D$ the IVP (3.1) is solvable. Hence, by Corollary 3.1.6 we can define a map

$$F : D \to C([0,T], X_\alpha) \cap C^1((0,T], X),$$

which assigns to every $x_0 \in D$ the unique solution $F(x_0)$ of (3.1). This map is called the *solution operator* for the IVP (3.1).

In the remainder of this section we study continuous dependence for the solution operator F.

Theorem 3.1.8. Suppose the assumptions of Theorem 3.1.5 hold. Then, there exists $K_\beta > 0$ such that for $0 \leq t \leq T, u, v \in X_\beta\, (\beta \in (\alpha, 1])$

$$\|Fu - Fv\|_{C([0,T],X_\alpha)} \leq K_\beta \|u - v\|_\beta\,.$$

Proof Let $x_1^1 = Fu, x_2^1 = Fv, u, v \in X_\beta$. Suppose that $0 \leq \alpha < \alpha_1 < \beta \leq 1$,

$$\|U(t,\tau)\|_{0,\alpha_1} \leq C(0, \alpha_1, \beta)(t-\tau)^{-\beta}, 0 \leq \tau < t \leq T.$$

We claim that if $T_0, 0 \leq T_0 \leq T$, is so small enough, then

$$\left\|x_1^1 - x_2^1\right\|_{C([0,T_0],X_{\alpha_1})} \leq M_1 \|u - v\|_\beta\,.$$

To prove this, suppose that $x_1^1, x_2^1 \in C([0,T_0], X_\alpha)$ and let $N_0 > 0$ be as in Theorem 2.2.13 it follows that

$$\begin{aligned}
&\|x_1^1(t) - x_2^1(t)\|_{\alpha_1} \\
&\le \|U(t,0)u - U(t,0)v\|_{\alpha_1} + \textstyle\int_0^t \|U(t,\tau)f(x_1^1(\tau)) - U(t,\tau)f(x_2^1(\tau))\|_{\alpha_1} d\tau \\
&\le \|U(t,0)\|_{\beta,\alpha_1} \|u-v\|_\beta + \textstyle\int_0^t \|U(t,\tau)\|_{0,\alpha_1} \|f(x_1^1(\tau)) - f(x_2^1(\tau))\|_X d\tau \\
&\le C_1(\beta,\alpha_1)\|u-v\|_\beta + C(0,\alpha_1,\beta)\textstyle\int_0^t (t-\tau)^{-\beta} \|f(x_1^1(\tau)) - f(x_2^1(\tau))\|_X d\tau \\
&\le C_1(\beta,\alpha_1)\|u-v\|_\beta + L\,C(0,\alpha_1,\beta)\textstyle\int_0^t (t-\tau)^{-\beta} \|x_1^1(\tau) - x_2^1(\tau)\|_\alpha d\tau \\
&= C_1(\beta,\alpha_1)\|u-v\|_\beta + L\,C(0,\alpha_1,\beta)\textstyle\int_0^t (t-\tau)^{-\beta} \|A^\alpha(x_1^1(\tau) - x_2^1(\tau))\|_X d\tau \\
&\le C_1(\beta,\alpha_1)\|u-v\|_\beta + \\
&\qquad L\,C(0,\alpha_1,\beta)\textstyle\int_0^t (t-\tau)^{-\beta} \|A^{\alpha-\alpha_1}A^{\alpha_1}(x_1^1(\tau) - x_2^1(\tau))\|_X d\tau \\
&\le C_1(\beta,\alpha_1)\|u-v\|_\beta + \\
&\qquad L\,C(0,\alpha_1,\beta)\,N_0\textstyle\int_0^t (t-\tau)^{-\beta} \|A^{\alpha_1}(x_1^1(\tau) - x_2^1(\tau))\|_X d\tau, \\
&= C_1(\beta,\alpha_1)\|u-v\|_\beta + L\,C(0,\alpha_1,\beta)\,N_0\textstyle\int_0^t (t-\tau)^{-\beta} \|x_1^1(\tau) - x_2^1(\tau)\|_{\alpha_1} d\tau \\
&\le C_1(\beta,\alpha_1)\|u-v\|_\beta + L\,C(0,\alpha_1,\beta)\,N_0\tfrac{T_0{}^{1-\beta}}{1-\beta} \|x_1^1(\tau) - x_2^1(\tau)\|_{\alpha_1}.
\end{aligned}$$

Therefore,

$$\|Fu - Fv\|_{C([0,T_0],X_{\alpha_1})} \le 2C_1(\beta,\alpha_1)\|u-v\|_\beta = M_1\|u-v\|_\beta\,,$$

provided $T_0 > 0$ is so small that

$$C(0,\alpha_1,\beta)L\,N^2\frac{T_0{}^{1-\beta}}{1-\beta} \le \frac{1}{2} \tag{3.5}$$

where $M_1 = 2C_1(\beta,\alpha_1)$ and $N = \max\{1, N_0\}$.

Now, let $0 \le \alpha < \alpha_2 < \alpha_1 < \beta \le 1$. The solution x_i^2 $(i = 1,2)$ can be written as

$$x_i^2(t) = U(t,T_0)x_i^1(T_0) + \textstyle\int_{T_0}^t U(t,\tau)f(x_i^2(\tau))d\tau,$$

for $t \in [T_0, 2T_0]$. Let us show that

$$\|Fu - Fv\|_{C([T_0,2T_0],X_{\alpha_2})} \le M_2\|u-v\|_\beta\,.$$

Suppose that $u, v \in X_\beta$, $x_1^2 = Fu$ and $x_2^2 = Fv$. Then

$$\begin{aligned}
&\|x_1^2(t) - x_2^2(t)\|_{\alpha_2} \\
&\le \|U(t,T_0)x_1^1(T_0) - U(t,T_0)x_2^1(T_0)\|_{\alpha_2} + \textstyle\int_{T_0}^t \|U(t,\tau)\left(f(x_1^2(\tau)) - f(x_2^2(\tau))\right)\|_{\alpha_2} d\tau
\end{aligned}$$

$\leq \|U(t,T_0)\|_{\alpha_1,\alpha_2} \|x_1^1(T_0) - x_2^1(T_0)\|_{\alpha_1} +$

$N_0 \int_{T_0}^t \|U(t,\tau)\left(f(x_1^2(\tau)) - f(x_2^2(\tau))\right)\|_{\alpha_1} d\tau, N_0 > 0$

$\leq C_2(\alpha_1,\alpha_2)M_1 \|u-v\|_\beta + N_0 \int_{T_0}^t \|U(t,\tau)\|_{0,\alpha_1} \|f(x_1^2(\tau)) - f(x_2^2(\tau))\|_X d\tau$

$\leq C_2(\alpha_1,\alpha_2)M_1 \|u-v\|_\beta +$

$N_0\, C(0,\alpha_1,\beta) \int_{T_0}^t (t-\tau)^{-\beta} \|f(x_1^2(\tau)) - f(x_2^2(\tau))\|_X d\tau$

$\leq C_2(\alpha_1,\alpha_2)M_1 \|u-v\|_\beta + LN_0\, C(0,\alpha_1,\beta) \int_{T_0}^t (t-\tau)^{-\beta} \|x_1^2(\tau) - x_2^2(\tau)\|_\alpha d\tau$

$\leq C_2(\alpha_1,\alpha_2)M_1 \|u-v\|_\beta + LN_0^2\, C(0,\alpha_1,\beta) \int_{T_0}^t (t-\tau)^{-\beta} \|x_1^2(\tau) - x_2^2(\tau)\|_{\alpha_2} d\tau$

$\leq \frac{M_2}{2} \|u-v\|_\beta + LN^2\, C(0,\alpha_1,\beta) \frac{T_0{}^{1-\beta}}{1-\beta} \max_{T_0 \leq t \leq 2T_0} \|x_1^2(t) - x_2^2(t)\|_{\alpha_2},$

where $M_2 = 2C_2M_1$ and $N = \max\{1, N_0\}$. Therefore,

$\max_{T_0 \leq t \leq 2T_0} \|x_1^2(t) - x_2^2(t)\|_{\alpha_2}$

$\leq \frac{M_2}{2} \|u-v\|_\beta + LN^2\, C(0,\alpha_1,\beta) \frac{T_0{}^{1-\beta}}{1-\beta} \max_{T_0 \leq t \leq 2T_0} \|x_1^2(t) - x_2^2(t)\|_{\alpha_2}.$

From (3.5), we get,

$$\max_{T_0 \leq t \leq 2T_0} \left\|x_1^2(t) - x_2^2(t)\right\|_{\alpha_2} \leq \frac{M_2}{2} \|u-v\|_\beta + \frac{1}{2} \max_{T_0 \leq t \leq 2T_0} \left\|x_1^2(t) - x_2^2(t)\right\|_{\alpha_2}$$

and

$$\|Fu - Fv\|_{C([T_0,2T_0],X_{\alpha_2})} \leq M_2 \|u-v\|_\beta .$$

For step by step, the ith step, choosing $0 \leq \alpha < \ldots < \alpha_{i+1} < \alpha_i < \ldots < \beta < 1$, we have $\max_{(i-1)T_0 \leq t \leq iT_0} \|x_1^i(t) - x_2^i(t)\|_{\alpha_i} \leq M_i \|u-v\|_\beta$ .After finite n steps, we have

$$\max_{(n-1)T_0 \leq t \leq T} \|x_1^n(t) - x_2^n(t)\|_\alpha \leq M_n \|u-v\|_\beta .$$

Hence,

$$\begin{aligned}
\|Fu - Fv\|_{C([0,T],X_\alpha)} &\leq \sum_{i=1}^{n-1} \|Fu - Fv\|_{C([(i-1)T_0,iT_0],X_\alpha)} + \\
&\quad \|Fu - Fv\|_{C([(n-1)T_0,T],X_\alpha)} \\
&\leq N_0 \sum_{i=1}^{n-1} \|Fu - Fv\|_{C([(i-1)T_0,iT_0],X_{\alpha_i})} + \\
&\quad \|Fu - Fv\|_{C([(n-1)T_0,T],X_\alpha)} \\
&\leq N_0 \sum_{i=1}^{n-1} M_i \|u-v\|_\beta + M_n \|u-v\|_\beta \\
&\leq K_\beta \|u-v\|_\beta
\end{aligned}$$

where $K_\beta = N_0 \sum_{i=1}^{n-1} M_i + M_n$. □

3.2 A Class of Integrodifferential Equations

In this section we wish to consider the questions of existence, uniqueness and regularity of mild solution for a system governed by an integrodifferential equation with delay :

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(x(t)) + \int_{-a}^{t} h(t-s)g(x(s))ds \ , \ t \in [0,T] \\ x(t) = \varphi(t) \ , \ t \in [-a,0]. \end{cases} \tag{3.6}$$

Evolution operator $U(\cdot,\cdot)$ and step by step approach are successfully used to this class of equation, under one of the following assumption:

(G1) $g : X_\alpha \to X$ is Lipschitz continuous, that is, there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\|_X \leq L\|x - y\|_\alpha$$

for all $x, y \in X_\alpha$.

(G2) $g : [0,T] \times X_\alpha \to X$ is Hölder continuous with respect to t and Lipschitz continuous with respect to x, that is, there exist constants C and $0 < \theta \leq 1$ such that

$$\|g(t_1,x_1) - g(t_2,x_2)\|_X \leq C\left\{|t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha\right\}$$

for $t_1, t_2 \in [0,T]$ and $x_1, x_2 \in X_\alpha$.

Definition 3.2.1. A function $x : [-a,T] \to X_\alpha$ is a *(classical) solution* of (3.6) if $x \in C([-a,T], X_\alpha) \cap C^1((0,T], X)$ with $x(t) \in D(A), t \in [0,T]$ and (3.6) is satisfied.

Definition 3.2.2. A function $x \in C([-a,T], X_\alpha)$ is said to be a *mild solution* of (3.6) corresponding to $\varphi \in C([-a,0], X_\alpha)$ if x satisfies the following integral equation

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)f(x(\tau))d\tau + \\ \qquad \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right)d\tau \ , \ t \in [0,T] \\ \varphi(t) \ , \ t \in [-a,0]. \end{cases} \tag{3.7}$$

- **Regularity of mild solutions**

Theorem 3.2.3. Let $\{A(t)\}_{t\in[0,T]}$ satisfy the assumptions **(A1)-(A3)**, $U(t,s)$, $0 \leq s \leq t \leq T$ be the evolution system, f and g satisfy the assumptions **(F1)** and **(G1)** respectively. Then for every $\varphi \in C([-a,0],X_\alpha), \varphi(0) \in X_\beta, 0 \leq \alpha < \beta < 1$ and $h \in L^p([0,a+T],\mathbb{R})$, $p > 1$, the evolution equation (3.6) is equivalent to (3.7).

Proof Let x be a solution of (3.6). Define $V(t,\tau) = U(t,\tau)x(\tau)$ for $0 < \tau \leq t < \infty$. Since $x(\tau) \in D(A)$ and $x \in C^1((0,T],X)$, V is differentiable and it is given by

$$\begin{aligned}
\frac{\partial V(t,\tau)}{\partial \tau} &= \frac{\partial U(t,\tau)}{\partial \tau}x(\tau) + U(t,\tau)\dot{x}(\tau) \\
&= U(t,\tau)A(\tau)x(\tau) + \\
&\quad U(t,\tau)\left(-A(\tau)x(\tau) + f(x(\tau)) + \int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right) \\
&= U(t,\tau)f(x(\tau)) + U(t,\tau)\int_{-a}^{\tau} h(\tau-s)g(x(s))ds.
\end{aligned}$$

Integrating this from 0 to t yields,

$$V(t,t) - V(t,0) = \int_0^t U(t,\tau)f(x(\tau))d\tau + \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right)d\tau$$

or

$$\begin{aligned}
x(t) = U(t,0)\varphi(0) + \textstyle\int_0^t U(t,\tau)f(x(\tau))d\tau + \\
\textstyle\int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right)d\tau \ , \ t \in [0,T].
\end{aligned}$$

Conversely, suppose that $x \in C([-a,T],X_\alpha)$ is a mild solution of (3.6). Then we have

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)f(x(\tau))d\tau + \\ \qquad \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right)d\tau \ , \ t \in [0,T] \\ \varphi(t) \ , \ t \in [-a,0]. \end{cases}$$

Let us show that the right hand side of (3.6) is Hölder continuous. First, to show $x \in C^\gamma([0,T],X_\alpha), \gamma \in [0,\beta-\alpha)$. Let

$$\|f(x(s))\|_X \leq N_1 \text{ and } \|g(x(s))\|_X \leq N_2.$$

Then, for $t_1, t_2 \in [0,T]$, $0 \leq \gamma_1 < \beta - \alpha, 0 < \gamma_2 < 1-\alpha, 0 < \gamma_3 < 1-\alpha$, it follows from Theorem 2.4.9 (i) and (iii) that

$$\begin{aligned}
&\|x(t_1)-x(t_2)\|_\alpha \\
&\leq \|U(t_1,0)\varphi(0)-U(t_2,0)\varphi(0)\|_\alpha + \left\|\int_0^{t_1} U(t_1,\tau)f(x(\tau))d\tau - \int_0^{t_2} U(t_2,\tau)f(x(\tau))d\tau\right\|_\alpha \\
&\quad + \left\|\int_0^{t_1} U(t_1,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right)d\tau - \right. \\
&\qquad\qquad \left.\int_0^{t_2} U(t_2,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right)d\tau\right\|_\alpha \\
&\leq C_1(\alpha,\beta,\gamma_1)\,|t_1-t_2|^{\gamma_1}\,\|\varphi(0)\|_\beta + C_2(\alpha,\gamma_2)\,|t_1-t_2|^{\gamma_2}\max_{0\leq\tau\leq T}\|f(x(\tau))\|_X + \\
&\quad C_3(\alpha,\gamma_3)\,|t_1-t_2|^{\gamma_3}\max_{0\leq\tau\leq T}\left\|\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right\|_X \\
&\leq C_1(\alpha,\beta,\gamma_1)\,|t_1-t_2|^{\gamma_1}\,\|\varphi(0)\|_\beta + N_1C_2(\alpha,\gamma_2)\,|t_1-t_2|^{\gamma_2} + \\
&\quad C_3(\alpha,\gamma_3)\,|t_1-t_2|^{\gamma_3}\max_{0\leq\tau\leq T}\int_{-a}^{\tau}|h(\tau-s)|\,\|g(x(s))\|_X\,ds \\
&\leq C_1(\alpha,\beta,\gamma_1)\,|t_1-t_2|^{\gamma_1}\,\|\varphi(0)\|_\beta + N_1C_2(\alpha,\gamma_2)\,|t_1-t_2|^{\gamma_2} + \\
&\quad N_2C_3(\alpha,\gamma_3)\,|t_1-t_2|^{\gamma_3}\max_{0\leq\tau\leq T}\int_{-a}^{\tau}|h(\tau-s)|\,ds \\
&\leq C_1(\alpha,\beta,\gamma_1)\,|t_1-t_2|^{\gamma_1}\,\|\varphi(0)\|_\beta + N_1C_2(\alpha,\gamma_2)\,|t_1-t_2|^{\gamma_2} + \\
&\quad N_2C_3(\alpha,\gamma_3)\,|t_1-t_2|^{\gamma_3}\max_{0\leq\tau\leq T}\int_0^{a+\tau}|h(\theta)|\,ds \\
&\leq C_1(\alpha,\beta,\gamma_1)\,|t_1-t_2|^{\gamma_1}\,\|\varphi(0)\|_\beta + N_1C_2(\alpha,\gamma_2)\,|t_1-t_2|^{\gamma_2} + N_2C_3(\alpha,\gamma_3)\,\widetilde{h}\,|t_1-t_2|^{\gamma_3} \\
&= M_1(\alpha,\beta,\gamma_1)\,|t_1-t_2|^{\gamma_1} + M_2\,|t_1-t_2|^{\gamma_2} + M_3\,|t_1-t_2|^{\gamma_3} \\
&\leq M\left(|t_1-t_2|^{\gamma_1} + |t_1-t_2|^{\gamma_2} + |t_1-t_2|^{\gamma_3}\right),\ M=\max\{M_1,M_2,M_3\} \\
&\leq M\,|t_1-t_2|^{\gamma},\ \gamma=\min\{\gamma_1,\gamma_2,\gamma_3\}.
\end{aligned}$$

Therefore $x \in C^\gamma([0,T],X_\alpha), \gamma \in [0,\beta-\alpha)$. Since f and g satisfy **(F1)** and **(G1)** respectively, then

$$\|f(x(t_1)) - f(x(t_2))\|_X \leq L_1\,\|x(t_1)-x(t_2)\|_\alpha \leq ML_1\,|t_1-t_2|^\gamma$$

and

$$\|g(x(t_1)) - g(x(t_2))\|_X \leq L_2\,\|x(t_1)-x(t_2)\|_\alpha \leq ML_2\,|t_1-t_2|^\gamma.$$

Hence, $f(x(\cdot))$ and $g(x(\cdot))$ are Hölder continuous. Now, let

$$H(x(t)) = \int_{-a}^{t} h(t-s)g(x(s))ds, t \in [0,T].$$

We now show that $H(x(\cdot))$ is Hölder continuous. It follows from Hölder inequality that

$$\begin{aligned}
\|H(x(t_1)) - H(x(t_2))\|_X &= \left\|\int_{-a}^{t_1} h(t_1-s)g(x(s))ds - \int_{-a}^{t_2} h(t_2-s)g(x(s))ds\right\|_X \\
&= \left\|\int_{a+t_1}^{0} h(\theta_1)g(x(t_1-\theta_1))(-d\theta_1) - \int_{a+t_2}^{0} h(\theta_2)g(x(t_2-\theta_2))(-d\theta_2)\right\|_X
\end{aligned}$$

$$= \left\| \int_0^{a+t_1} h(\theta_1) g(x(t_1-\theta_1)) d\theta_1 - \int_0^{a+t_2} h(\theta_2) g(x(t_2-\theta_2)) d\theta_2 \right\|_X$$
$$= \left\| \int_0^{a+t_1} h(\theta) g(x(t_1-\theta)) d\theta - \int_0^{a+t_2} h(\theta) g(x(t_2-\theta)) d\theta \right\|_X$$
$$= \left\| \int_0^{a+t_1} h(\theta) g(x(t_1-\theta)) d\theta - \int_0^{a+t_1} h(\theta) g(x(t_2-\theta)) d\theta - \int_{a+t_1}^{a+t_2} h(\theta) g(x(t_2-\theta)) d\theta \right\|_X, (\text{ for } t_1 < t_2)$$
$$\leq \int_0^{a+t_1} |h(\theta)| \, \|g(x(t_1-\theta)) - g(x(t_2-\theta))\|_X \, d\theta + \int_{a+t_1}^{a+t_2} |h(\theta)| \, \|g(x(t_2-\theta))\|_X \, d\theta$$
$$\leq L \int_0^{a+t_1} |h(\theta)| \, \|x(t_1-\theta) - x(t_2-\theta)\|_\alpha \, d\theta + N_2 \int_{a+t_1}^{a+t_2} |h(\theta)| \, d\theta$$
$$\leq L \int_0^{a+t_1} |h(\theta)| \; M \, |t_1-t_2|^\gamma \, d\theta + N_2 \left(\int_{a+t_1}^{a+t_2} |h(\theta)|^p \, d\theta \right)^{1/p} \left(\int_{a+t_1}^{a+t_2} 1^q d\theta \right)^{1/q}$$
$$\leq L \, M \, |t_1-t_2|^\gamma \int_0^{a+T} |h(\theta)| \, d\theta + N_2 \widetilde{h}_p \, |t_1-t_2|^{1/q}$$
$$\leq L \, M \widetilde{h} \, |t_1-t_2|^\gamma + N_2 \widetilde{h}_p \, |t_1-t_2|^{1/q}$$
$$\leq M_3 \left(|t_1-t_2|^\gamma + |t_1-t_2|^{1/q} \right), M_3 = \max\left\{ LM\widetilde{h}, N_2\widetilde{h}_p \right\}$$
$$\leq M_4 \, |t_1-t_2|^\eta \, , \eta = \min\left\{ \gamma, \tfrac{1}{q} \right\}.$$

Thus, the right hand side of (3.6) is Hölder continuous. By Theorem 2.4.8, we get, for $\varphi(0) \in X_\beta, x \in C^1((0,T], X)$. □

Remark 3.2.4. As usual, Theorem 3.2.3 still holds if f and g satisfy the assumptions (**F2**) and (**G2**) respectively.

- **Existence, Uniqueness and Continuous Dependence of Mild Solutions**

Theorem 3.2.5. Let $\{A(t)\}_{t\in[0,T]}$ satisfy the assumptions **(A1)-(A3)**, $U(t,s)$, $0 \leq s \leq t \leq T$ be the evolution system, f and g satisfy the assumptions (**F1**) and (**G1**) respectively and $h \in L^p([0,a+T],\mathbb{R})$, $p > 1$. Then for every $\varphi \in C([-a,0],X_\alpha)$ and $\varphi(0) \in X_\beta, 0 \leq \alpha < \beta < 1$, the evolution equation (3.6) has a unique mild solution $x \in C([-a,T],X_\alpha)$.

Proof Given T_0 with $0 \leq T_0 \leq T$, let $\Omega_1 = \{x \in C([-a,T_0],X_\alpha) : x(t) = \varphi(t), -a \leq t \leq 0\}$. Then Ω_1 is a closed subset of $C([-a,T_0],X_\alpha)$. We define an operator G_1 on Ω_1 by

$$(G_1x^1)(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau) f(x^1(\tau)) d\tau + \\ \qquad \int_0^t U(t,\tau) \left(\int_{-a}^{\tau} h(\tau-s) g(x^1(s)) ds \right) d\tau \; , \; t \in [0,T_0] \\ \varphi(t) \; , \; t \in [-a,0]. \end{cases}$$

To show G_1 maps Ω_1 into itself. Suppose that $x^1 \in \Omega_1$ and $t_1, t_2 \in [0, T_0]$. Then

$$\begin{aligned}
&\|(G_1x^1)(t_1) - (G_1x^1)(t_2)\|_\alpha \\
&\le \|U(t_1,0)\varphi(0) - U(t_2,0)\varphi(0)\|_\alpha + \left\| \int_0^{t_1} U(t_1,\tau) f(x^1(\tau)) d\tau - \right. \\
&\quad \left. \int_0^{t_2} U(t_2,\tau) f(x^1(\tau)) d\tau \right\|_\alpha + \left\| \int_0^{t_1} U(t_1,\tau) \left(\int_{-a}^{\tau} h(\tau - s) g(x^1(s)) ds \right) d\tau - \right. \\
&\quad \left. \int_0^{t_2} U(t_2,\tau) \left(\int_{-a}^{\tau} h(\tau - s) g(x^1(s)) ds \right) d\tau \right\|_\alpha \\
&\le C_1(\alpha,\beta,\gamma_1) |t_1 - t_2|^{\gamma_1} \|\varphi(0)\|_\beta + C_2(\alpha,\gamma_2) |t_1 - t_2|^{\gamma_2} \max_{0\le\tau\le T_0} \|f(x^1(\tau))\|_X + \\
&\quad C_3(\alpha,\gamma_3) |t_1 - t_2|^{\gamma_3} \max_{0\le\tau\le T_0} \left\| \int_{-a}^{\tau} h(\tau - s) g(x^1(s)) ds \right\|_X , \\
&\quad 0 \le \gamma_1 < \beta - \alpha, 0 < \gamma_2 < 1 - \alpha, 0 < \gamma_3 < 1 - \alpha, 0 \le t_1, t_2 \le T_0. \\
&\le M_1 |t_1 - t_2|^{\gamma_1} + M_2 |t_1 - t_2|^{\gamma_2} + M_3 |t_1 - t_2|^{\gamma_3} \\
&\le M \left(|t_1 - t_2|^{\gamma_1} + |t_1 - t_2|^{\gamma_2} + |t_1 - t_2|^{\gamma_3} \right), M = \max\{M_1, M_2, M_3\} \\
&\le K |t_1 - t_2|^{\gamma}, \gamma = \min\{\gamma_1, \gamma_2, \gamma_3\}.
\end{aligned}$$

Since $\varphi \in C([-a, 0], X_\alpha)$, then we get

$$\left\|(G_1x^1)(t_1) - (G_1x^1)(t_2)\right\|_\alpha = \|\varphi(t_1) - \varphi(t_2)\|_\alpha ,$$

for $t_1, t_2 \in [-a, 0]$. Therefore G_1 maps Ω_1 into itself.

Now, we claim that if $T_0 > 0$ is small enough, then G_1 is a contraction in Ω_1. To prove this, suppose $x_1^1, x_2^1 \in \Omega_1$. Then

$$\begin{aligned}
&\|(G_1x_1^1)(t) - (G_1x_2^1)(t)\|_\alpha \\
&\le \int_0^t \|U(t,\tau)\left(f(x_1^1(\tau)) - f(x_2^1(\tau))\right)\|_\alpha d\tau + \\
&\quad \int_0^t \left\| U(t,\tau) \left(\int_{-a}^{\tau} h(\tau - s) g(x_1^1(s)) ds - \int_{-a}^{\tau} h(\tau - s) g(x_2^1(s)) ds \right) \right\|_\alpha d\tau \\
&\le \int_0^t \|U(t,\tau)\|_{0,\alpha} \|f(x_1^1(\tau)) - f(x_2^1(\tau))\|_X d\tau + \\
&\quad \int_0^t \|U(t,\tau)\|_{0,\alpha} \left(\int_{-a}^{\tau} \|h(\tau - s)\left(g(x_1^1(s)) - g(x_2^1(s))\right)\|_X ds \right) d\tau \\
&\le C(0,\alpha,\gamma) \int_0^t (t-\tau)^{-\gamma} \|f(x_1^1(\tau)) - f(x_2^1(\tau))\|_X d\tau + \\
&\quad C(0,\alpha,\gamma) \int_0^t (t-\tau)^{-\gamma} \left(\int_{-a}^{\tau} |h(\tau - s)| \|g(x_1^1(s)) - g(x_2^1(s))\|_X ds \right) d\tau \\
&\quad \alpha \le \gamma < 1, 0 \le \tau < t < T_0. \\
&\le C(0,\alpha,\gamma) L_1 \int_0^t (t-\tau)^{-\gamma} \|x_1^1(\tau) - x_2^1(\tau)\|_\alpha d\tau + \\
&\quad C(0,\alpha,\gamma) L_2 \int_0^t (t-\tau)^{-\gamma} \left(\int_{-a}^{\tau} |h(\tau - s)| \|x_1^1(s) - x_2^1(s)\|_\alpha ds \right) d\tau
\end{aligned}$$

$$\leq C(0,\alpha,\gamma)L_1 \sup_{0\leq\tau\leq T_0} \left\|x_1^1(\tau) - x_2^1(\tau)\right\|_\alpha \int_0^t (t-\tau)^{-\gamma} d\tau + C(0,\alpha,\gamma)L_2 \cdot$$
$$\int_0^t (t-\tau)^{-\gamma} \left(\int_{-a}^{\tau} |h(\tau-s)| \sup_{-a\leq s\leq\tau} \left\|x_1^1(s) - x_2^1(s)\right\|_\alpha ds\right) d\tau$$
$$\leq C(0,\alpha,\gamma)L_1 \left\|x_1^1 - x_2^1\right\|_{C([0,T_0],X_\alpha)} \frac{T_0^{1-\gamma}}{1-\gamma} +$$
$$C(0,\alpha,\gamma)L_2\widetilde{h} \left\|x_1^1 - x_2^1\right\|_{C([-a,T_0],X_\alpha)} \frac{T_0^{1-\gamma}}{1-\gamma}$$
$$\leq M(0,\alpha,\gamma) \left\|x_1^1 - x_2^1\right\|_{C([-a,T_0],X_\alpha)} \frac{T_0^{1-\gamma}}{1-\gamma},$$

where $M(0,\alpha,\gamma) = CL_1 + CL_2\widetilde{h}$. Hence,

$$\left\|G_1x_1^1 - G_1x_2^1\right\|_{C([-a,T_0],X_\alpha)} \leq M(0,\alpha,\gamma)\frac{T_0^{1-\gamma}}{1-\gamma} \left\|x_1^1 - x_2^1\right\|_{C([-a,T_0],X_\alpha)},$$

where M is independent of T_0 and thus G_1 is a contraction mapping in Ω_1, provided $T_0 > 0$ is so small that

$$M(0,\alpha,\gamma)\frac{T_0^{1-\gamma}}{1-\gamma} < 1. \tag{3.8}$$

By the contraction mapping theorem, we can conclude that G_1 has a unique fixed point $x \in \Omega_1$ such that $G_1x = x$, that is

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)f(x(\tau))d\tau + \\ \qquad \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right)d\tau, \ t \in [0,T_0] \\ \\ \varphi(t), \ t \in [-a,0]. \end{cases}$$

Therefore the system (3.6) has a unique mild solution $x \in \Omega_1$.

Now, define an operator G_2 on $\Omega_2 = \{x \in C([-a,2T_0],X_\alpha) : x(t) = x^1(t), -a \leq t \leq T_0\}$ by

$$(G_2x^2)(t) = \begin{cases} U(t,T_0)x^1(T_0) + \int_{T_0}^t U(t,\tau)f(x^2(\tau))d\tau + \\ \qquad \int_{T_0}^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x^2(s))ds\right)d\tau, \ t \in [T_0,2T_0] \\ \\ x^1(t), \ t \in [-a,T_0], \end{cases}$$

where

$$x^1(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)f(x^1(\tau))d\tau + \\ \qquad \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x^1(s))ds\right)d\tau, \ t \in [0,T_0] \\ \\ \varphi(t), \ t \in [-a,0]. \end{cases}$$

By the same argument as before, one can show that G_2 maps Ω_2 into itself.

Let us show that G_2 is a contraction mapping in Ω_2. Suppose that $x_1^2, x_2^2 \in \Omega_2$. Then

$$
\begin{aligned}
&\left\|(G_2x_1^2)(t) - (G_2x_2^2)(t)\right\|_\alpha \\
&\quad \le \int_{T_0}^t \left\|U(t,\tau)\left(f(x_1^2(\tau)) - f(x_2^2(\tau))\right)\right\|_\alpha d\tau + \\
&\quad\quad \int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^\tau h(\tau-s)g(x_1^2(s))ds - \int_{-a}^\tau h(\tau-s)g(x_2^2(s))ds\right)\right\|_\alpha d\tau \\
&\quad \le \int_{T_0}^t \|U(t,\tau)\|_{0,\alpha} \left\|f(x_1^2(\tau)) - f(x_2^2(\tau))\right\|_X d\tau + \\
&\quad\quad \int_{T_0}^t \|U(t,\tau)\|_{0,\alpha} \left(\int_{-a}^\tau \left\|h(\tau-s)\left(g(x_1^2(s)) - g(x_2^2(s))\right)\right\|_X ds\right) d\tau \\
&\quad \le C(0,\alpha,\gamma)\int_{T_0}^t (t-\tau)^{-\gamma}\left\|f(x_1^2(\tau)) - f(x_2^2(\tau))\right\|_X d\tau + \\
&\quad\quad C(0,\alpha,\gamma)\int_{T_0}^t (t-\tau)^{-\gamma}\left(\int_{-a}^\tau |h(\tau-s)|\left\|g(x_1^2(s)) - g(x_2^2(s))\right\|_X ds\right) d\tau, \\
&\quad\quad \alpha \le \gamma < 1, 0 \le \tau < t < T_0. \\
&\quad \le C(0,\alpha,\gamma)L_1\int_{T_0}^t (t-\tau)^{-\gamma}\left\|x_1^2(\tau) - x_2^2(\tau)\right\|_\alpha d\tau + \\
&\quad\quad C(0,\alpha,\gamma)L_2\int_{T_0}^t (t-\tau)^{-\gamma}\left(\int_{-a}^\tau |h(\tau-s)|\left\|x_1^2(s) - x_2^2(s)\right\|_\alpha ds\right) d\tau \\
&\quad \le C(0,\alpha,\gamma)L_1\sup_{T_0\le\tau\le 2T_0}\left\|x_1^2(\tau) - x_2^2(\tau)\right\|_\alpha \int_{T_0}^t (t-\tau)^{-\gamma}d\tau + C(0,\alpha,\gamma)L_2\cdot \\
&\quad\quad \int_{T_0}^t (t-\tau)^{-\gamma}\left(\int_{-a}^\tau |h(\tau-s)|\sup_{-a\le s\le\tau}\left\|x_1^2(s) - x_2^2(s)\right\|_\alpha ds\right) d\tau \\
&\quad \le C(0,\alpha,\gamma)L_1\frac{T_0{}^{1-\gamma}}{1-\gamma}\left\|x_1^2 - x_2^2\right\|_{C([T_0,2T_0],X_\alpha)} + \\
&\quad\quad C(0,\alpha,\gamma)L_2\widetilde{h}\frac{T_0{}^{1-\gamma}}{1-\gamma}\left\|x_1^2 - x_2^2\right\|_{C([T_0,2T_0],X_\alpha)} \\
&\quad = M(0,\alpha,\gamma)\frac{T_0{}^{1-\gamma}}{1-\gamma}\left\|x_1^2 - x_2^2\right\|_{C([T_0,2T_0],X_\alpha)},
\end{aligned}
$$

where $M(0,\alpha,\gamma) = CL_1 + CL_2\widetilde{h}$. Therefore,

$$
\left\|G_2x_1^2 - G_2x_2^2\right\|_{C([-a,2T_0],X_\alpha)} \le M\frac{T_0{}^{1-\gamma}}{1-\gamma}\left\|x_1^2 - x_2^2\right\|_{C([-a,2T_0],X_\alpha)},
$$

where M is independent of T_0. From (3.8), we get G_2 is a contraction mapping in Ω_2. By the contraction mapping theorem, we can conclude that G_2 has a unique fixed point $x^2 \in \Omega_2$ such that $G_2x^2 = x^2$, that is

$$
x^2(t) = \begin{cases}
U(t,T_0)x^1(T_0) + \int_{T_0}^t U(t,\tau)f(x^2(\tau))d\tau + \\
\qquad \int_{T_0}^t U(t,\tau)\left(\int_{-a}^\tau h(\tau-s)g(x^2(s))ds\right)d\tau\ ,\ T_0 \le t \le 2T_0 \\
x^1(t)\ ,\ -a \le t \le T_0.
\end{cases}
$$

Define,

$$
x(t) = \begin{cases} x^1(t), -a \le t \le T_0 \\ x^2(t), -a \le t \le 2T_0 \end{cases}
$$

Then, we get

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)f(x(\tau))d\tau + \\ \qquad \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(x(s))ds\right)d\tau \ , \ t \in [0, 2T_0] \\ \\ \varphi(t) \ , \ t \in [-a, 0]. \end{cases}$$

Therefore, the system (3.6) has a unique mild solution $x \in \Omega_2$.

Repeating the above procedure in intervals $[-a, 3T_0], [-a, 4T_0], \ldots$, after finite n steps such that $nT_0 \geq T$, we can construct a unique mild solution $x \in C([-a,T], X_\alpha)$. □

Corollary 3.2.6. Under the same assumptions of Theorem 3.2.3, for every $\varphi \in C([-a,0], X_\alpha), \varphi(0) \in X_\beta, 0 \leq \alpha < \beta < 1$ and $h \in L^p([0, a+T], \mathbb{R})$, $p > 1$, the evolution equation (3.6) has a unique classical solution.

Proof The assertion follows immediately from Theorem 3.2.5 and Theorem 3.2.3. □

Corollary 3.2.7. Let $\{A(t)\}_{t\in[0,T]}$ satisfy the assumptions **(A1)-(A3)**, $U(t,s)$, $0 \leq s \leq t \leq T$ be the evolution system, f and g satisfy the assumptions **(F2)** and **(G2)**, respectively and $h \in L^p([0,a+T], \mathbb{R})$, $p > 1$. Then for every $\varphi \in C([-a,0], X_\alpha), \varphi(0) \in X_\beta, 0 \leq \alpha < \beta < 1$ the evolution equation:

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t,x(t)) + \int_{-a}^{t} h(t-s)g(s,x(s))ds \ , \ t \in [0,T] \\ x(t) = \varphi(t) \ , \ t \in [-a, 0] \end{cases} \tag{3.9}$$

has a unique classical solution.

Proof By a proof is similar to the proof of Theorem 3.2.5 one show that (3.9) has a unique mild solution. By Theorem 3.2.3 and Remark 3.2.4, this solution is a classical solution. □

In the following we denote by $D = \cup D_\beta, D_\beta = \{\varphi \in C([-a,0], X_\alpha), \varphi(0) \in X_\beta, \beta \in (\alpha, 1]\}$. For $\varphi \in D$ the evolution equation (3.6) is solvable. Hence, by Corollary 3.2.6 we can define a map

$$F : D \to C([-a,T], X_\alpha),$$

which assigns to every $\varphi \in C([-a,0], X_\alpha)$ the unique solution $F(\varphi)$ of (3.6). This map is called the *solution operator* for the evolution equation (3.6).

In the remainder of this section we study continuous dependence for the solution operator F.

Theorem 3.2.8. Under the assumption of Theorem 3.2.5, there exists $K_\beta < \infty$ such that

$$\|F\varphi_1 - F\varphi_2\|_{C([-a,T],X_\alpha)} \leq K_\beta \left\{ \|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} + \|\varphi_1(0) - \varphi_2(0)\|_\beta \right\}$$

for $\varphi_1, \varphi_2 \in D_\beta$.

Proof Given any $T > 0$. Let $0 \leq \alpha < \alpha_1 < \beta < 1$. We claim that if we select $T_0, 0 < T_0 \leq T$ so small enough, then

$$\|F\varphi_1 - F\varphi_2\|_{C([-a,T_0],X_\alpha)} \leq K_1 \left\{ \|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} + \|\varphi_1(0) - \varphi_2(0)\|_\beta \right\}$$

for $\varphi_1, \varphi_2 \in D_\beta$. To prove this, suppose that $\varphi_1, \varphi_2 \in D_\beta$ and $x_1^1 = F\varphi_1, x_2^1 = F\varphi_2$. The solution $x_i^1 = F\varphi_i$ $(i = 1, 2)$, can be written as

$$x_i^1(t) = \begin{cases} U(t,0)\varphi_i(0) + \int_0^t U(t,\tau) f(x_i^1(\tau)) d\tau + \\ \qquad \int_0^t U(t,\tau) \left(\int_{-a}^{\tau} h(\tau - s) g(x_i^1(s)) ds \right) d\tau \ , \ t \in [0, T_0] \\ \\ \varphi_i(t) \ , \ t \in [-a, 0]. \end{cases}$$

By Theorem 2.4.9, there exists $C(0, \alpha_1, \beta)$ such that

$$\|U(t,\tau)\|_{0,\alpha_1} \leq C(0, \alpha_1, \beta)(t - \tau)^{-\beta}$$

for $-a \leq \tau < t \leq T_0$. Consider,

$$\begin{aligned}
&\|x_1^1(t) - x_2^1(t)\|_{\alpha_1} = \|(F\varphi_1)(t) - (F\varphi_2)(t)\|_{\alpha_1} \\
&\leq \|U(t,0)\left(\varphi_1(0) - \varphi_2(0)\right)\|_{\alpha_1} + \int_0^t \|U(t,\tau)\left(f(x_1^1(\tau)) - f(x_2^1(\tau))\right)\|_{\alpha_1} d\tau + \\
&\quad \int_0^t \left\| U(t,\tau) \left(\int_{-a}^{\tau} h(\tau - s)\left(g(x_1^1(s)) - g(x_2^1(s))\right) ds \right\|_{\alpha_1} d\tau \\
&\leq \|U(t,0)\|_{\beta,\alpha_1} \|\varphi_1(0) - \varphi_2(0)\|_\beta + \int_0^t \|U(t,\tau)\|_{0,\alpha_1} \|f(x_1^1(\tau)) - f(x_2^1(\tau))\|_X d\tau \\
&\quad + \int_0^t \|U(t,\tau)\|_{0,\alpha_1} \left(\int_{-a}^{\tau} |h(\tau - s)| \, \|g(x_1^1(s)) - g(x_2^1(s))\|_X ds \right) d\tau \\
&= \|U(t,0)\|_{\beta,\alpha_1} \|\varphi_1(0) - \varphi_2(0)\|_\beta + \int_0^t \|U(t,\tau)\|_{0,\alpha_1} \|f(x_1^1(\tau)) - f(x_2^1(\tau))\|_X d\tau \\
&\quad + \int_0^t \|U(t,\tau)\|_{0,\alpha_1} \left(\int_{-a}^{0} |h(\tau - s)| \, \|g(x_1^1(s)) - g(x_2^1(s))\|_X ds \right) d\tau \\
&\quad + \int_0^t \|U(t,\tau)\|_{0,\alpha_1} \left(\int_0^{\tau} |h(\tau - s)| \, \|g(x_1^1(s)) - g(x_2^1(s))\|_X ds \right) d\tau \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Thus,

$$\left\|x_1^1(t) - x_2^1(t)\right\|_{\alpha_1} \le I_1 + I_2 + I_3 + I_4 \tag{3.10}$$

for $0 \le t \le T$. We estimate each of the terms of (3.10) separately.

$I_1 \le C_1(\beta, \alpha_1)\, \|\varphi_1(0) - \varphi_2(0)\|_\beta$.

$I_2 \le C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \|f(x_1^1(\tau)) - f(x_2^1(\tau))\|_X \, d\tau$

$\le C(0, \alpha_1, \beta) L_1 \int_0^t (t-\tau)^{-\beta} \|x_1^1(\tau) - x_2^1(\tau)\|_\alpha \, d\tau$

$= C(0, \alpha_1, \beta) L_1 \int_0^t (t-\tau)^{-\beta} \|A^{\alpha-\alpha_1} A^{\alpha_1} (x_1^1(\tau) - x_2^1(\tau))\|_X \, d\tau$

$\le N_0 L_1 C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \|A^{\alpha_1} (x_1^1(\tau) - x_2^1(\tau))\|_X \, d\tau, N_0 > 0$

$= N_0 L_1 C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \|x_1^1(\tau) - x_2^1(\tau)\|_{\alpha_1} \, d\tau$

$\le N L_1 C(0, \alpha_1, \beta) \frac{T_0^{1-\beta}}{1-\beta} \max_{0 \le t \le T_0} \|x_1^1(t) - x_2^1(t)\|_{\alpha_1}$

where $N = \max\{1, N_0\}$.

$I_3 \le C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \left(\int_{-a}^0 |h(\tau - s)|\, \|g(x_1^1(s)) - g(x_2^1(s))\|_X \, ds \right) d\tau$

$\le C(0, \alpha_1, \beta) L_2 \int_0^t (t-\tau)^{-\beta} \left(\int_{-a}^0 |h(\tau - s)|\, \|x_1^1(s) - x_2^1(s)\|_\alpha \, ds \right) d\tau$

$= C(0, \alpha_1, \beta) L_2 \int_0^t (t-\tau)^{-\beta} \left(\int_{-a}^0 |h(\tau - s)|\, \|\varphi_1(s) - \varphi_2(s)\|_\alpha \, ds \right) d\tau$

$\le C(0, \alpha_1, \beta) L_2 \int_0^t (t-\tau)^{-\beta} \left(\|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} \, d\tau \int_0^{a+T} |h(\theta)| \, d\theta \right) d\tau$

$\le \widetilde{h} L_2 C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} \, d\tau$

$\le \widetilde{h} L_2 C(0, \alpha_1, \beta) \frac{{T_0}^{1-\beta}}{1-\beta} \|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)}$.

$I_4 \le L_2 C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \left(\int_0^\tau |h(\tau - s)|\, \|x_1^1(s) - x_2^1(s)\|_\alpha \, ds \right) d\tau$

$= L_2 C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta}$.

$\left(\int_0^\tau |h(\tau - s)|\, \|A^{\alpha-\alpha_1} A^{\alpha_1} (x_1^1(s) - x_2^1(s))\|_X \, ds \right) d\tau$

$\le N_0 L_2 C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \left(\int_0^\tau |h(\tau - s)|\, \|x_1^1(s) - x_2^1(s)\|_{\alpha_1} \, ds \right) d\tau,$

$N_0 > 0$

$\le N_0 L_2 C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \max_{0 \le t \le T_0} \|x_1^1(t) - x_2^1(t)\|_{\alpha_1} \int_0^{a+T} |h(\theta)| \, d\theta d\tau$

$\le \widetilde{h} N_0 L_2 C(0, \alpha_1, \beta) \int_0^t (t-\tau)^{-\beta} \max_{0 \le t \le T_0} \|x_1^1(t) - x_2^1(t)\|_{\alpha_1} \, d\tau$

$\le \widetilde{h} N L_2 C(0, \alpha_1, \beta) \frac{T_0^{1-\beta}}{1-\beta} \max_{0 \le t \le T_0} \|x_1^1(t) - x_2^1(t)\|_{\alpha_1}$

where $N = \max\{1, N_0\}$. Combining (3.10) with these estimates it follows that

$\|x_1^1(t) - x_2^1(t)\|_{\alpha_1} \le C_1(\beta, \alpha_1)\, \|\varphi_1(0) - \varphi_2(0)\|_\beta +$

$N L_1 C(0, \alpha_1, \beta) \frac{T_0^{1-\beta}}{1-\beta} \max_{0 \le t \le T_0} \|x_1^1(t) - x_2^1(t)\|_{\alpha_1} +$

$$\widetilde{h}L_2C(0,\alpha_1,\beta)\frac{T_0{}^{1-\beta}}{1-\beta}\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+$$
$$\widetilde{h}NL_2C(0,\alpha_1,\beta)\frac{T_0^{1-\beta}}{1-\beta}\max_{0\le t\le T_0}\|x_1^1(t)-x_2^1(t)\|_{\alpha_1}$$
$$\le C_1(\beta,\alpha_1)\|\varphi_1(0)-\varphi_2(0)\|_\beta+M_1(0,\alpha_1,\beta)\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+$$
$$M(0,\alpha_1,\beta)N^2\frac{T_0^{1-\beta}}{1-\beta}\max_{0\le t\le T_0}\|x_1^1(t)-x_2^1(t)\|_{\alpha_1}$$

where $M=(L_1+\widetilde{h}L_2)C(0,\alpha_1,\beta)$. Therefore,

$$\|F\varphi_1-F\varphi_2\|_{C([0,T_0],X_{\alpha_1})}\le H_1\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}, \tag{3.11}$$

provided $T_0>0$ is so small that

$$MN^2\frac{T_0^{1-\beta}}{1-\beta}\le\frac{1}{2} \tag{3.12}$$

where $H_1=\max\{2M_1,2C_1\}$. Hence,

$$\begin{aligned}\|F\varphi_1-F\varphi_2\|_{C([-a,T_0],X_\alpha)} &\le \|F\varphi_1-F\varphi_2\|_{C([-a,0],X_\alpha)}+\|F\varphi_1-F\varphi_2\|_{C([0,T_0],X_\alpha)}\\ &\le \|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+N_0\|F\varphi_1-F\varphi_2\|_{C([0,T_0],X_{\alpha_1})}\\ &\le \|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+N_0H_1\Big\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}\\ &\quad+\|\varphi_1(0)-\varphi_2(0)\|_\beta\Big\}\\ &\le (1+N_0H_1)\Big\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}\\ &\quad+\|\varphi_1(0)-\varphi_2(0)\|_\beta\Big\}\\ &\le K_1\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}\end{aligned}$$

where $N_0>0, K_1=1+NH_1$ and $N=\max\{1,N_0\}$.

Now, let $0\le\alpha<\alpha_2<\alpha_1<\beta<1$ and

$$\|U(t,T_0)\|_{\alpha_1,\alpha_2}\le C_2(\alpha_1,\alpha_2).$$

To show that

$$\|F\varphi_1-F\varphi_2\|_{C([-a,2T_0],X_\alpha)}\le K_2\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\},$$

for $\varphi_1,\varphi_2\in D_\beta$. To prove this, suppose that $\varphi_1,\varphi_2\in D_\beta$ and $x_1^2\in F\varphi_1, x_2^2\in F\varphi_2$. The solution $x_i^2=F\varphi_i,(i=1,2)$, can be written as

$$x_i^2(t)=\begin{cases}U(t,T_0)x_i^1(T_0)+\int_{T_0}^t U(t,\tau)f(x_i^2(\tau))d\tau+\\ \qquad\int_{T_0}^t U(t,\tau)\left(\int_{-a}^\tau h(\tau-s)g(x_i^2(s))ds\right)d\tau, \\ \qquad\qquad T_0\le t\le 2T_0\\ x_i^1(t)\ ,\ -a\le t\le T_0.\end{cases}$$

Consider,

$$
\begin{aligned}
&\|x_1^2(t) - x_2^2(t)\|_{\alpha_2} \\
&\le \|U(t,T_0)(x_1^1(T_0) - x_2^1(T_0))\|_{\alpha_2} + \int_{T_0}^t \|U(t,\tau)(f(x_1^2(\tau)) - f(x_2^2(\tau)))\|_{\alpha_2} d\tau \\
&\quad + \int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^{\tau} h(\tau - s)(g(x_1^2(s)) - g(x_2^2(s)))\right) ds\right\|_{\alpha_2} d\tau \\
&\le \|U(t,T_0)\|_{\alpha_1,\alpha_2} \|x_1^1(T_0) - x_2^1(T_0)\|_{\alpha_1} + \\
&\quad \int_{T_0}^t \|U(t,\tau)(f(x_1^2(\tau)) - f(x_2^2(\tau)))\|_{\alpha_2} d\tau + \\
&\quad \int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^{T_0} h(\tau - s)(g(x_1^2(s)) - g(x_2^2(s)))\right) ds\right\|_{\alpha_2} d\tau + \\
&\quad \int_{T_0}^t \left\|U(t,\tau)\left(\int_{T_0}^{\tau} h(\tau - s)(g(x_1^2(s)) - g(x_2^2(s)))\right) ds\right\|_{\alpha_2} d\tau \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}
$$

for $T_0 \le t \le 2T_0$. Thus,

$$
\left\|x_1^2(t) - x_2^2(t)\right\|_{\alpha_2} \le I_1 + I_2 + I_3 + I_4. \tag{3.13}
$$

We estimate each of terms of (3.13) separately.

$$
\begin{aligned}
I_1 &\le C_2(\alpha_1,\alpha_2)\|x_1^1(T_0) - x_2^1(T_0)\|_{\alpha_1} \\
&\le C_2(\alpha_1,\alpha_2)K_1\left\{\|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} + \|\varphi_1(0) - \varphi_2(0)\|_\beta\right\}. \\
I_2 &= \int_{T_0}^t \|U(t,\tau)(f(x_1^2(\tau)) - f(x_2^2(\tau)))\|_{\alpha_2} d\tau \\
&= \int_{T_0}^t \|A^{\alpha_2-\alpha_1}A^{\alpha_1}U(t,\tau)(f(x_1^2(\tau)) - f(x_2^2(\tau)))\|_X d\tau \\
&\le N_0 \int_{T_0}^t \|U(t,\tau)(f(x_1^2(\tau)) - f(x_2^2(\tau)))\|_{\alpha_1} d\tau, N_0 > 0 \\
&\le N_0 \int_{T_0}^t \|U(t,\tau)\|_{0,\alpha_1} \|f(x_1^2(\tau)) - f(x_2^2(\tau))\|_X d\tau \\
&\le N_0 L_1 C(0,\alpha_1,\beta) \int_{T_0}^t (t-\tau)^{-\beta} \|x_1^2(\tau) - x_2^2(\tau)\|_\alpha d\tau \\
&\le N_0^2 L_1 C(0,\alpha_1,\beta) \int_{T_0}^t (t-\tau)^{-\beta} \|x_1^2(\tau) - x_2^2(\tau)\|_{\alpha_2} d\tau \\
&\le N^2 L_1 C(0,\alpha_1,\beta)\tfrac{T_0^{1-\beta}}{1-\beta} \max_{T_0 \le t \le 2T_0} \|x_1^2(t) - x_2^2(t)\|_{\alpha_2}
\end{aligned}
$$

where $N = \max\{1, N_0\}$.

$$
\begin{aligned}
I_3 &= \int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^{T_0} h(\tau - s)(g(x_1^2(s)) - g(x_2^2(s)))\right) ds\right\|_{\alpha_2} d\tau \\
&= \int_{T_0}^t \left\|A^{\alpha_2-\alpha_1}A^{\alpha_1}U(t,\tau)\left(\int_{-a}^{T_0} h(\tau - s)(g(x_1^2(s)) - g(x_2^2(s)))\right) ds\right\|_X d\tau \\
&\le N_0 \int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^{T_0} h(\tau - s)(g(x_1^2(s)) - g(x_2^2(s)))\right) ds\right\|_{\alpha_1} d\tau, N_0 > 0 \\
&\le N_0 \int_{T_0}^t \|U(t,\tau)\|_{0,\alpha_1} \left(\int_{-a}^{T_0} |h(\tau - s)| \|g(x_1^2(s)) - g(x_2^2(s))\|_X ds\right) d\tau
\end{aligned}
$$

Consider,

$$\begin{aligned}
&\|x_1^2(t)-x_2^2(t)\|_{\alpha_2}\\
&\le \|U(t,T_0)\,(x_1^1(T_0)-x_2^1(T_0))\|_{\alpha_2}+\int_{T_0}^t \|U(t,\tau)\,(f(x_1^2(\tau))-f(x_2^2(\tau)))\|_{\alpha_2}\,d\tau\\
&\quad+\int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)\,(g(x_1^2(s))-g(x_2^2(s)))\right)ds\right\|_{\alpha_2} d\tau\\
&\le \|U(t,T_0)\|_{\alpha_1,\alpha_2}\,\|x_1^1(T_0)-x_2^1(T_0)\|_{\alpha_1}+\\
&\quad\int_{T_0}^t \|U(t,\tau)\,(f(x_1^2(\tau))-f(x_2^2(\tau)))\|_{\alpha_2}\,d\tau+\\
&\quad\int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^{T_0} h(\tau-s)\,(g(x_1^2(s))-g(x_2^2(s)))\right)ds\right\|_{\alpha_2} d\tau+\\
&\quad\int_{T_0}^t \left\|U(t,\tau)\left(\int_{T_0}^{\tau} h(\tau-s)\,(g(x_1^2(s))-g(x_2^2(s)))\right)ds\right\|_{\alpha_2} d\tau\\
&= I_1+I_2+I_3+I_4
\end{aligned}$$

for $T_0 \le t \le 2T_0$. Thus,

$$\left\|x_1^2(t)-x_2^2(t)\right\|_{\alpha_2} \le I_1+I_2+I_3+I_4. \tag{3.13}$$

We estimate each of terms of (3.13) separately.

$$\begin{aligned}
I_1 &\le C_2(\alpha_1,\alpha_2)\,\|x_1^1(T_0)-x_2^1(T_0)\|_{\alpha_1}\\
&\le C_2(\alpha_1,\alpha_2)K_1\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}.\\
I_2 &= \int_{T_0}^t \|U(t,\tau)\,(f(x_1^2(\tau))-f(x_2^2(\tau)))\|_{\alpha_2}\,d\tau\\
&= \int_{T_0}^t \|A^{\alpha_2-\alpha_1}A^{\alpha_1}U(t,\tau)\,(f(x_1^2(\tau))-f(x_2^2(\tau)))\|_X\,d\tau\\
&\le N_0\int_{T_0}^t \|U(t,\tau)\,(f(x_1^2(\tau))-f(x_2^2(\tau)))\|_{\alpha_1}\,d\tau,\ N_0>0\\
&\le N_0\int_{T_0}^t \|U(t,\tau)\|_{0,\alpha_1}\,\|f(x_1^2(\tau))-f(x_2^2(\tau))\|_X\,d\tau\\
&\le N_0L_1C(0,\alpha_1,\beta)\int_{T_0}^t (t-\tau)^{-\beta}\,\|x_1^2(\tau)-x_2^2(\tau)\|_\alpha\,d\tau\\
&\le N_0^2L_1C(0,\alpha_1,\beta)\int_{T_0}^t (t-\tau)^{-\beta}\,\|x_1^2(\tau)-x_2^2(\tau)\|_{\alpha_2}\,d\tau\\
&\le N^2L_1C(0,\alpha_1,\beta)\tfrac{T_0^{1-\beta}}{1-\beta}\max_{T_0\le t\le 2T_0}\|x_1^2(t)-x_2^2(t)\|_{\alpha_2}
\end{aligned}$$

where $N=\max\{1,N_0\}$.

$$\begin{aligned}
I_3 &= \int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^{T_0} h(\tau-s)\,(g(x_1^2(s))-g(x_2^2(s)))\right)ds\right\|_{\alpha_2} d\tau\\
&= \int_{T_0}^t \left\|A^{\alpha_2-\alpha_1}A^{\alpha_1}U(t,\tau)\left(\int_{-a}^{T_0} h(\tau-s)\,(g(x_1^2(s))-g(x_2^2(s)))\right)ds\right\|_X d\tau\\
&\le N_0\int_{T_0}^t \left\|U(t,\tau)\left(\int_{-a}^{T_0} h(\tau-s)\,(g(x_1^2(s))-g(x_2^2(s)))\right)ds\right\|_{\alpha_1} d\tau,\ N_0>0\\
&\le N_0\int_{T_0}^t \|U(t,\tau)\|_{0,\alpha_1}\left(\int_{-a}^{T_0} |h(\tau-s)|\,\|g(x_1^2(s))-g(x_2^2(s))\|_X\,ds\right)d\tau
\end{aligned}$$

$$\leq N_0L_2C(0,\alpha_1,\beta)\int_{T_0}^{t}(t-\tau)^{-\beta}\left(\int_{-a}^{T_0}|h(\tau-s)|\,\|x_1^2(s)-x_2^2(s)\|_{\alpha}\,ds\right)d\tau$$

$$= N_0L_2C(0,\alpha_1,\beta)\int_{T_0}^{t}(t-\tau)^{-\beta}\left(\int_{-a}^{T_0}|h(\tau-s)|\,\|x_1^1(s)-x_2^1(s)\|_{\alpha}\,ds\right)d\tau$$

$$\leq N_0L_2C(0,\alpha_1,\beta)\int_{T_0}^{t}(t-\tau)^{-\beta}\left(\int_{-a}^{0}|h(\tau-s)|\,\|x_1^1(s)-x_2^1(s)\|_{\alpha}\,ds+\right.$$

$$\left.\int_{0}^{T_0}|h(\tau-s)|\,\|x_1^1(s)-x_2^1(s)\|_{\alpha}\,ds\right)d\tau$$

$$\leq N_0L_2C(0,\alpha_1,\beta)\int_{T_0}^{t}(t-\tau)^{-\beta}\left(\int_{-a}^{0}|h(\tau-s)|\,\|\varphi_1(s)-\varphi_2(s)\|_{\alpha}\,ds+\right.$$

$$\left.\int_{0}^{T_0}|h(\tau-s)|\,N_0\,\|x_1^1(s)-x_2^1(s)\|_{\alpha_1}\,ds\right)d\tau$$

$$\leq N_0L_2C(0,\alpha_1,\beta)\int_{T_0}^{t}(t-\tau)^{-\beta}\left(\widetilde{h}\,\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\right.$$

$$\left.\widetilde{h}N_0K_1\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_{\beta}\right\}\right)d\tau$$

$$\leq N_0L_2C(0,\alpha_1,\beta)\tfrac{T_0^{1-\beta}}{1-\beta}\left((1+N_0K_1)\,\widetilde{h}\,\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\right.$$

$$\left.\widetilde{h}N_0K_1\,\|\varphi_1(0)-\varphi_2(0)\|_{\beta}\right)$$

$$= \widetilde{h}N_0L_2C(0,\alpha_1,\beta)\tfrac{T_0^{1-\beta}}{1-\beta}\left((1+N_0K_1)\,\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\right.$$

$$\left.N_0K_1\,\|\varphi_1(0)-\varphi_2(0)\|_{\beta}\right)$$

$$\leq \widetilde{h}NL_2C(0,\alpha_1,\beta)\,(1+NK_1)\,\tfrac{T_0^{1-\beta}}{1-\beta}\left(\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\right.$$

$$\left.\|\varphi_1(0)-\varphi_2(0)\|_{\beta}\right)$$

where $N=\max\{1,N_0\}$.

$$I_4=\int_{T_0}^{t}\left\|U(t,\tau)\left(\int_{T_0}^{\tau}h(\tau-s)\left(g(x_1^2(s))-g(x_2^2(s))\right)ds\right\|_{\alpha_2}d\tau$$

$$=\int_{T_0}^{t}\left\|A^{\alpha_2-\alpha_1}A^{\alpha_1}U(t,\tau)\left(\int_{T_0}^{\tau}h(\tau-s)\left(g(x_1^2(s))-g(x_2^2(s))\right)\right)ds\right\|_{X}d\tau$$

$$\leq N_0\int_{T_0}^{t}\left\|U(t,\tau)\left(\int_{T_0}^{\tau}h(\tau-s)\left(g(x_1^2(s))-g(x_2^2(s))\right)\right)ds\right\|_{\alpha_1}d\tau,\ N_0>0$$

$$\leq N_0\int_{T_0}^{t}\|U(t,\tau)\|_{0,\alpha_1}\left(\int_{T_0}^{\tau}|h(\tau-s)|\,\|g(x_1^2(s))-g(x_2^2(s))\|_{X}\,ds\right)d\tau$$

$$\leq N_0L_2C(0,\alpha_1,\beta)\int_{T_0}^{t}(t-\tau)^{-\beta}\left(\int_{T_0}^{\tau}|h(\tau-s)|\,\|x_1^2(s)-x_2^2(s)\|_{\alpha}\,ds\right)d\tau$$

$$\leq N_0^2L_2C(0,\alpha_1,\beta)\int_{T_0}^{t}(t-\tau)^{-\beta}\left(\int_{T_0}^{\tau}|h(\tau-s)|\,\|x_1^2(s)-x_2^2(s)\|_{\alpha_2}\,ds\right)d\tau$$

$$\leq \widetilde{h}N^2L_2C(0,\alpha_1,\beta)\tfrac{T_0^{1-\beta}}{1-\beta}\max_{T_0\leq t\leq 2T_0}\|x_1^2(t)-x_2^2(t)\|_{\alpha_2}$$

where $N=\max\{1,N_0\}$. Combining (3.13) with these estimates it follows that

$$\|x_1^2(t)-x_2^2(t)\|_{\alpha_2}$$

$$\leq C_2(\alpha_1,\alpha_2)K_1\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_{\beta}\right\}+$$

$$N^2 L_1 C(0,\alpha_1,\beta)\tfrac{T_0^{1-\beta}}{1-\beta}\max_{T_0\le t\le 2T_0}\|x_1^2(t)-x_2^2(t)\|_{\alpha_2}+$$
$$\widetilde{h}NL_2C(0,\alpha_1,\beta)\tfrac{T_0^{1-\beta}}{1-\beta}\left(1+NK_1\right)\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}$$
$$\widetilde{h}N^2L_2C(0,\alpha_1,\beta)\tfrac{T_0^{1-\beta}}{1-\beta}\max_{T_0\le t\le 2T_0}\|x_1^2(t)-x_2^2(t)\|_{\alpha_2}$$
$$= M_2\left(\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right)+$$
$$MN^2\tfrac{T_0^{1-\beta}}{1-\beta}\max_{T_0\le t\le 2T_0}\|x_1^2(t)-x_2^2(t)\|_{\alpha_2},$$

where $M=(L_1+\widetilde{h}L_2)C(0,\alpha_1,\beta)$ and $M_2=C_2K_1+\widetilde{h}NL_2C\frac{T_0^{1-\beta}}{1-\beta}\left(1+NK_1\right)$. From (3.12), we get

$$\begin{aligned}\max_{T_0\le t\le 2T_0}\left\|x_1^2(t)-x_2^2(t)\right\|_{\alpha_2} \le\ & M_2\left(\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right)\\ &+\frac{1}{2}\max_{T_0\le t\le 2T_0}\left\|x_1^2(t)-x_2^2(t)\right\|_{\alpha_2}.\end{aligned}$$

Hence,

$$\|F\varphi_1-F\varphi_2\|_{C([T_0,2T_0],X_{\alpha_2})}\le H_2\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\},$$

where $H_2=2M_2$. Therefore,

$$\begin{aligned}\|F\varphi_1-F\varphi_2\|_{C([-a,2T_0],X_\alpha)} \le\ & \|F\varphi_1-F\varphi_2\|_{C([-a,T_0],X_\alpha)}+\\ & \|F\varphi_1-F\varphi_2\|_{C([T_0,2T_0],X_\alpha)}\\ \le\ & K_1\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}+\\ & N_0H_2\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}\\ =\ & K_2\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}\end{aligned}$$

where $K_2=K_1+NH_2$ and $N=\max\{1,N_0\}$.

For step by step, the ith step, choosing $0\le\alpha<\ldots<\alpha_{i+1}<\alpha_i<\ldots<\beta<1$, we have

$$\|x_1^i-x_2^i\|_{C([(i-1)T_0,iT_0],X_{\alpha_i})}\le H_i\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}$$

and then

$$\|F\varphi_1-F\varphi_2\|_{C([-a,iT_0],X_\alpha)}\le K_i\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}.$$

After finite n steps, we have

$$\|x_1^n-x_2^n\|_{C([(n-1)T_0,T],X_\alpha)}\le H\left\{\|\varphi_1-\varphi_2\|_{C([-a,0],X_\alpha)}+\|\varphi_1(0)-\varphi_2(0)\|_\beta\right\}.$$

Therefore,

$$
\begin{aligned}
\|F\varphi_1 - F\varphi_2\|_{C([-a,T],X_\alpha)} &\le \|F\varphi_1 - F\varphi_2\|_{C([-a,(n-1)T_0],X_\alpha)} + \\
&\quad \|F\varphi_1 - F\varphi_2\|_{C([(n-1)T_0,T],X_\alpha)} \\
&\le K_{n-1}\left\{\|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} + \|\varphi_1(0) - \varphi_2(0)\|_\beta\right\} + \\
&\quad H\left\{\|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} + \|\varphi_1(0) - \varphi_2(0)\|_\beta\right\} \\
&= K_\beta\left\{\|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} + \|\varphi_1(0) - \varphi_2(0)\|_\beta\right\},
\end{aligned}
$$

where $K_\beta = K_{n-1} + H$. □

Chapter IV

Existence of Optimal Controls

4.1 Introduction

We recall some basic concepts and results that are necessary for the presentation of the theories in this chapter.

Definition 4.1.1. Let X be a Banach space and X^* be its dual. A sequence $\{x_n\} \subset X$ is said to be *weakly convergent* to $x \in X$ if

$$\lim_{n\to\infty} f(x_n) = f(x), \forall f \in X^*.$$

In this case x is called *weak limit* of the sequence and the notation $x_n \xrightarrow{w} x$ or $x_n \rightharpoonup x$ are used.

Theorem 4.1.2. If $\{x_n\}$ is bounded sequence in the reflexive Banach space X, then $\{x_n\}$ has a weakly convergent subsequence.

If, in addition, each weakly convergent subsequence of $\{x_n\}$ has the same limit x, then $x_n \xrightarrow{w} x$ as $n \to \infty$.
(Zeidler II/A (1990), p. 258)

Definition 4.1.3. Let X, Y be Banach spaces. We define the following properties for an operator $B : X \to Y$:

(i) B is *continuous* iff $x_n \to x$ implies

$$Bx_n \to Bx \text{ as } n \to \infty.$$

(ii) B is *strongly continuous* iff $x_n \xrightarrow{w} x$ as $n \to \infty$ implies

$$Bx_n \to Bx \text{ as } n \to \infty.$$

(iii) B is *compact* iff B is continuous and maps bounded sets into relatively compact sets.

Definition 4.1.4. Let X be Banach space.

(i) A subset C of X is *convex* if, whenever $x, y \in C, tx + (1-t)y \in C$ for all $t \in [0,1]$, i.e. if the points x and y belong to C, then the segment jointing them also belongs to C.

(ii) A functional $f : C \to \mathbb{R}$ on a convex set C is called *convex* iff

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $t \in [0,1]$ and $x, y \in C$.

Definition 4.1.5. Let X be Banach space. The functional $F : M \subset X \to \mathbb{R}$ is called *sequentially lower semicontinuous* at the point $x \in M$ iff

$$F(x) \leq \underline{\lim}_{n \to \infty} F(x_n)$$

for each sequence $\{x_n\}$ in M with $x_n \to x$ as $n \to \infty$. Furthermore, $F : M \subset X \to \mathbb{R}$ is called sequentially lower semicontinuous iff it is sequentially lower semicontinuous at each point of M.

Definition 4.1.6. Let X be Banach space. The functional $F : M \subset X \to \mathbb{R}$ is called *weakly sequentially lower semicontinuous* on M iff for each $x \in M$ and each sequence $\{x_n\}$ in M

$$x_n \xrightarrow{w} x \text{ as } n \to \infty \text{ implies } F(x) \leq \underline{\lim}_{n \to \infty} F(x_n).$$

Theorem 4.1.7. Suppose that the functional $F : M \subset X \to \mathbb{R}$ has the following two properties :

(i) M is a nonempty bounded closed convex set in the reflexive Banach space X.

(ii) F is weakly sequentially lower semicontinuous on M.

Then F has a minimum.
(Zeidler II/B (1990), p. 512)

Theorem 4.1.8. Let $F : M \subset X \to \mathbb{R}$ be a functional on the convex closed subset M of the Banach space X. Then F is weakly sequentially lower semicontinuous if one of the following two conditons is satisfied :

(i) F is continuous and convex.

(ii) F is lower semicontinuous and convex.

(Zeidler II/B (1990), pp. 514-515)

Let $(\Omega, \mathcal{T}, \mu)$ be a finite nonatomic measure space, X a separable Banach space and E a separable reflexive Banach space whose dual we denote by E'. Let $l : \Omega \times X \times E \to (-\infty, \infty]$ be a given measurable function. The associated integral functional $I_l : L^1(\Omega, X) \times L^1(\Omega, E) \to [-\infty, \infty]$ is defined by

$$I_l(x, v) = \int_\Omega l(t, x(t), v(t))\mu(dt).$$

We equip $L^1(\Omega, X)$ with the L^1-norm and $L^1(\Omega, E)$ with the weak topology $\sigma\left(L^1(\Omega, E), L^\infty(\Omega, E')\right)$.

Theorem 4.1.9. The following three conditions:

$$l(t, \cdot, \cdot) \text{ is sequentially l.s.c. on } X \times E \text{ a.e.,}$$
$$l(t, x, \cdot) \text{ is convex on } E \text{ for every } x \in X \text{ a.e.,}$$

there exist $M > 0$ and $\psi \in L^1(\Omega, \mathbb{R})$ such that

$$l(t, x, v) \geq \psi(t) - M\left(\|x\| + \|v\|\right) \text{ for all } x \in X, v \in E \text{ a.e.}$$

are sufficient for sequential strong-weak lower semicontinuity of I_l on $L^1(\Omega, X) \times L^1(\Omega, E)$. Moreover they are also necessary, provided that $I_l(\overline{x}, \overline{v}) < \infty$ for some $\overline{x} \in L^1(\Omega, X), \overline{v} \in L^1(\Omega, E)$.

(Balder, 1987, pp. 1399-1400)

4.2 Controlled System

The dynamics of many physical system, such as visco elastic fluid or thermodynamics are governed by integrodifferential equations on Banach space. The abstract mathematical model for all systems can be described as follows :

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t, x(t)) + \int_{-a}^{t} h(t-s)g(s, x(s))ds \ , \ t \in [0, T] \\ x(t) = \varphi(t) \ , \ t \in [-a, 0], \end{cases}$$

where $A(t)$ is typically a linear unbounded operator in a suitable Banach space, f, g and h are nonlinear operators. A corresponding control system may be described as:

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t, x(t)) + \int_{-a}^{t} h(t-s)g(s, x(s))ds + B(t)u(t)\ ,\ t \in [0, T] \\ x(t) = \varphi(t)\ ,\ t \in [-a, 0], \end{cases} \tag{4.1}$$

where B is an operator and u is the control.

To work with the general model we shall consider a general Banach space X to be the state space and E a separable reflexive Banach space where the controls take their values from. For any Banach space Y and any interval $[0, T]$, $T < \infty, L^p([0, T], Y), 1 \le p < \infty$, will denote the Banach space of strongly measurable Y-valued functions having p-th power summable norms. For any two Banach spaces X and Y, $\mathcal{L}(X, Y)$ will denote the space of bounded linear operators from X to Y. We shall introduce further notations in the sequel as required.

In this section, we consider the system (4.1) and discuss the questions of existence and uniqueness of mild solutions. For the existence of solutions for the controlled system (4.1), we shall introduce the following assumptions.

(A) X is a separable reflexive Banach space, $A(t), 0 \le t \le T$ satisfy the assumptions **(A1)**-**(A3)** and $U(t, s), 0 \le s \le t \le T$ is the evolution operator corresponding to $A(t)$.

(B1) E is a reflexive Banach space from which the controls u take the values, $B \in L^\infty([0, T], \mathcal{L}(E, X))$.

(H) $h \in L^1([0, a+T], \mathbb{R})$.

Definition 4.2.1. For any $u \in L^p([0, T], E), 1 < p < \infty$, if there exists a $T = T(u) > 0$ and $x \in C([-a, T], X_\alpha)$ such that

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)B(\tau)u(\tau)d\tau + \int_0^t U(t,\tau)f(\tau, x(\tau))d\tau \\ \qquad + \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(s, x(s))ds\right)d\tau, t \in [0, T] \\ \\ \varphi(t), \quad -a \le t \le 0, \end{cases} \tag{4.2}$$

then system (4.1) is called mildly solvable with respect to u on $[-a, T]$ and $x \in C([-a, T], X_\alpha)$ is said to be a *mild solution* with respect to u on $[-a, T]$.

Theorem 4.2.2. Suppose **(F2)**, **(G2)** (in chapter III), **(A)**, **(B1)** and **(H)** hold, $\varphi \in C([-a, 0], X_\alpha), \varphi(0) \in X_\beta, (\alpha < \beta)$ and $p > \frac{1}{1-\alpha}$. Then for each $u \in$

$L^p([0,T],E), 1 < p < \infty$, the controlled system (4.1) has a unique mild solution $x \in C([-a,T],X_\alpha)$.

Proof. We define an operator G on $C([-a,T],X_\alpha)$ by:

$$(Gx)(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)f(\tau,x(\tau))d\tau + \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)\cdot \right. \\ \qquad \left. g(s,x(s))ds\right)d\tau + \int_0^t U(t,\tau)B(\tau)u(\tau)d\tau, t \in [0,T], \\ \\ \varphi(t), \quad t \in [-a,0]. \end{cases}$$

It is enough to show that $\int_0^t U(t,\tau)B(\tau)u(\tau)d\tau$ is continuous. By assumption (**B1**) it can be seen that $B(\cdot)u(\cdot) \in L^p([0,T],X)$. Consider, for $0 < t_1 < t_2 < T$

$$\left\|\int_0^{t_2} U(t_2,\tau)B(\tau)u(\tau)d\tau - \int_0^{t_1} U(t_1,\tau)B(\tau)u(\tau)d\tau\right\|_\alpha$$

$$= \left\|\int_0^{t_1} U(t_2,\tau)B(\tau)u(\tau)d\tau + \int_{t_1}^{t_2} U(t_2,\tau)B(\tau)u(\tau)d\tau - \int_0^{t_1} U(t_1,\tau)B(\tau)u(\tau)d\tau\right\|_\alpha$$

$$\leq \int_0^{t_1} \|(U(t_2,\tau) - U(t_1,\tau))\,B(\tau)u(\tau)\|_\alpha\, d\tau + \int_{t_1}^{t_2} \|U(t_2,\tau)B(\tau)u(\tau)\|_\alpha\, d\tau$$

$$= I_1 + I_2.$$

Thus,

$$\left\|\int_0^{t_2} U(t_2,\tau)B(\tau)u(\tau)d\tau - \int_0^{t_1} U(t_1,\tau)B(\tau)u(\tau)d\tau\right\|_\alpha \leq I_1 + I_2. \tag{4.3}$$

We estimate each of the terms of (4.3) separately. Let $0 \leq \gamma_1 < \beta - \alpha, \beta < \gamma_2 < 1$ and $\alpha < \gamma_3 < 1$.

$$\begin{aligned} I_1 &= \int_0^{t_1} \|(U(t_2,\tau) - U(t_1,\tau))\,B(\tau)u(\tau)\|_\alpha\, d\tau \\ &= \int_0^{t_1} \|(U(t_2,t_1) - I)\,U(t_1,\tau)B(\tau)u(\tau)\|_\alpha\, d\tau \\ &\leq \int_0^{t_1} \|U(t_2,t_1) - I\|_{\beta,\alpha} \|U(t_1,\tau)B(\tau)u(\tau)\|_\beta\, d\tau \\ &\leq C_1(\alpha,\beta,\gamma_1)\,|t_2 - t_1|^{\gamma_1} \int_0^{t_1} \|U(t_1,\tau)\|_{0,\beta} \|B(\tau)u(\tau)\|_X\, d\tau \\ &\leq C_1(\alpha,\beta,\gamma_1)\,|t_2 - t_1|^{\gamma_1}\, C_2(0,\beta,\gamma_2) \int_0^{t_1} (t_1 - \tau)^{-\gamma_2} \|B(\tau)u(\tau)\|_X\, d\tau \\ &\leq M_1\,|t_2 - t_1|^{\gamma_1} \left(\int_0^{t_1} (t_1 - \tau)^{-\frac{p\gamma_2}{p-1}} d\tau\right)^{\frac{p-1}{p}} \left(\int_0^{t_1} \|B(\tau)u(\tau)\|_X^p\, d\tau\right)^{\frac{1}{p}} \\ &\leq M_2\,|t_2 - t_1|^{\gamma_1} \left(\frac{(p-1)T^{\frac{p(1-\gamma_2)-1}{p-1}}}{p(1-\gamma_2)-1}\right)^{\frac{p-1}{p}} = M_3\,|t_2 - t_1|^{\gamma_1}. \end{aligned}$$

$$
\begin{aligned}
I_2 &= \int_{t_1}^{t_2} \|U(t_2,\tau)B(\tau)u(\tau)\|_\alpha \, d\tau \\
&\le \int_{t_1}^{t_2} \|U(t_2,\tau)\|_{0,\alpha} \|B(\tau)u(\tau)\|_X \, d\tau \\
&\le C_3(0,\alpha,\gamma_3) \int_{t_1}^{t_2} (t_2-\tau)^{-\gamma_3} \|B(\tau)u(\tau)\|_X \, d\tau \\
&\le C_3(0,\alpha,\gamma_3) \left(\int_{t_1}^{t_2} (t_2-\tau)^{-\frac{p\gamma_3}{p-1}} d\tau \right)^{\frac{p-1}{p}} \left(\int_{t_1}^{t_2} \|B(\tau)u(\tau)\|_X^p \, d\tau \right)^{\frac{1}{p}} \\
&\le M_4 \left(\frac{(p-1)(t_2-t_1)^{\frac{p(1-\gamma_3)-1}{p-1}}}{p(1-\gamma_3)-1} \right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Hence $I_2 \le M_5(t_2-t_1)^{\frac{p(1-\alpha)-1}{p}}$ provided $|t_2-t_1|<1$. Combining (4.3) with these estimates it follows that

$$
\begin{aligned}
& \left\| \int_0^{t_2} U(t_2,\tau)B(\tau)u(\tau)d\tau - \int_0^{t_1} U(t_1,\tau)B(\tau)u(\tau)d\tau \right\|_\alpha \\
\le\ & M_3 |t_2-t_1|^{\gamma_1} + M_5(t_2-t_1)^{\frac{p(1-\alpha)-1}{p}} \\
\le\ & M |t_2-t_1|^{\gamma}
\end{aligned}
$$

where $M = \max\{M_3, M_5\}$ and $\gamma = \min\{\gamma_1, \frac{p(1-\alpha)-1}{p}\}$.

An argument analogous to the proof of Theorem 3.2.5 implies that (4.1) has a unique mild solution $x \in C([-a,T], X_\alpha)$. □

Remark 4.2.3. It can be seen from the proof of the Theorem 4.2.2 that $B : L^p([0,T],E) \to L^p([0,T],X)$ is linear and bounded, the theorem is valid.

4.3 Existence of Optimal Controls

In this section, we wish to prove the existence of optimal controls of the controlled system with time delay (4.1) for the Lagrange problem. By Theorem 4.2.2, the controlled system (4.1) is mildly solvable on $[-a,T]$ for every $u \in \mathcal{U}_{ad}$.

We consider the Lagrange problem:

(P) Find $u^0 \in \mathcal{U}_{ad}$ such that

$$J(u^0) \le J(u), \forall u \in \mathcal{U}_{ad},$$

where

$$J(u) = \int_{[0,T]} l(t, x^u(t), u(t))dt.$$

Here x^u denotes the mild solution of controlled system (4.1) corresponding to the control $u \in \mathcal{U}_{ad}$. $\{u, x^u\}$ is called an *admissible state-control pair*, or simply admissible pair.

For the existence of solutions for problem (**P**), we shall introduce the following assumptions.

(U) E is a separable reflexive Banach space, $\mathcal{U}_{ad} = L^p([0,T], E), 1 < p < \infty$.

(B2) $B(t) \in \mathcal{L}(E, X)), t \in [0,T]$ and $\widetilde{B} : L^p([0,T], E) \to L^p([0,T], X)$ linear and bounded is given by

$$(\widetilde{B}u)(t) = B(t)u(t), \quad t \in [0,T]$$

and $\widetilde{B}$ is strongly continuous.

(L) $l : [0,T] \times X_\alpha \times E \to \mathbb{R} \cup \{\infty\}$ is Borel measurable satisfying the following conditions:

(1) $l(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X_\alpha \times E$ for almost all $t \in [0,T]$

(2) $l(t, x, \cdot)$ is convex on E for each $x \in X_\alpha$ and almost all $t \in [0,T]$.

(3) There exist $b \geq 0, c > 0$ and $\phi \in L^1([0,T], \mathbb{R})$ such that

$$l(t, x, v) \geq \phi(t) + b \|x\|_\alpha + c \|v\|_E^p .$$

Theorem 4.3.1. Under the assumption (**F2**), (**G2**) (in chapter III), (**A**), (**H**), (**U**), (**B2**), (**L**),and $p > \frac{1}{1-\alpha}$, the optimal control problem (**P**) has a solution, that is, there exists an admissible state-control pair $\left\{u^0, x^{u^0}\right\}$ such that

$$J(u^0) = \int_{[0,T]} l(t, x^{u^0}(t), u^0(t))dt \leq J(u), \forall u \in \mathcal{U}_{ad}.$$

Proof. Note that our assumption (**L**) implies the assumptions of Balder (Theorem 4.1.9). Hence by Balder's result we conclude that

$$(u, x) \longmapsto \int_{[0,T]} l(t, x^u(t), u(t))dt$$

is sequentially lower semicontinuous in the weak topology $L^p([0,T], E) \subset L^1([0,T], E)$ and the strong topology of $L^1([0,T], X)$.

If $\inf\{J(u), u \in \mathcal{U}_{ad}\} = \infty$, there is nothing to prove. So we assume

$$\inf\{J(u), u \in \mathcal{U}_{ad}\} = m < \infty.$$

It follows from (**L**)-(3) that

$$\begin{aligned} J(u) &= \int_{[0,T]} l(t, x^u(t), u(t))dt, \forall u \in \mathcal{U}_{ad} \\ &\geq \int_{[0,T]} \phi(t)dt + \int_{[0,T]} b\,\|x(t)\|_\alpha\, dt + \int_{[0,T]} c\,\|u(t)\|_E^p\, dt \\ &\geq -\gamma \\ &> -\infty. \end{aligned}$$

Hence $m \geq -\gamma > -\infty$. Let $\{u^n\}$ be a minimizing sequence of J, i.e., $J(u^n) \to m$ as $n \to \infty$. By virtue of (**L**)-(3),

$$J(u^n) \geq \int_{[0,T]} \phi(t)dt + b\int_{[0,T]} \|x(t)\|_\alpha\, dt + c\int_{[0,T]} \|u^n(t)\|_E^p\, dt$$

or

$$m \geq -\gamma + c\,\|u^n\|_{L^p([0,T],E)}\,.$$

Therefore, $\{u^n\}$ is a bounded sequence in $L^p([0,T],E)$. That is, $\|u^n\|_E \leq M$ for all n. Hence $\{u^n\}$ is contained in a bounded subset of $L^p([0,T],E)$. Since $L^p([0,T],E)$ is a separable reflexive Banach space, it has a subsequence relabelled as $\{u^n\}$ and there is an element $u^0 \in \mathcal{U}_{ad}$ such that

$$u^n \xrightarrow{w} u^0$$

in $L^p([0,T],E)$. By strong continuity of $\widetilde{B}$, we have

$$\widetilde{B}u^n \xrightarrow{s} \widetilde{B}u^0 \text{ in } L^p([0,T],X).$$

Let $\{x^n\} \subset C([0,T],X_\alpha)$ denote the corresponding sequence of solutions of the integral equation

$$x^n(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)B(\tau)u^n(\tau)d\tau + \int_0^t U(t,\tau)f(\tau,x^n(\tau))d\tau + \\ \qquad \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(s,x^n(s))ds\right)d\tau, t \in [0,T] \\ \\ \varphi(t), t \in [-a,0]. \end{cases}$$

Let x^0 denote the solution corresponding to u^0, that is,

$$x^0(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)B(\tau)u^0(\tau)d\tau + \int_0^t U(t,\tau)f(\tau,x^0(\tau))d\tau + \\ \qquad \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau-s)g(s,x^0(s))ds\right)d\tau, t\in[0,T] \\ \\ \varphi(t), t\in[-a,0]. \end{cases}$$

By assumptions (**F2**) and (**G2**) on f and g respectively, we have

$$\left\|f(\tau,x^n(\tau)) - f(\tau,x^0(\tau))\right\|_X \le L_1\left\|x^n(\tau)-x^0(\tau)\right\|_\alpha; 0\le\tau\le T,$$

and

$$\left\|g(s,x^n(s)) - g(s,x^0(s))\right\|_X \le L_2\left\|x^n(s)-x^0(s)\right\|_\alpha; 0\le s\le T.$$

Next, we shall show that

$$\left\|x^n - x^0\right\|_{C([-a,T],X_\alpha)} \to 0.$$

By using Hölder's inequality, we have

$$\begin{aligned} &\left\|\int_0^t U(t,\tau)B(\tau)\left(u^n(\tau)-u^0(\tau)\right)d\tau\right\|_\alpha \\ \le\ & \int_0^t \|U(t,\tau)\|_{0,\alpha}\left\|B(\tau)\left(u^n(\tau)-u^0(\tau)\right)\right\|_X d\tau \\ \le\ & C(0,\alpha,\gamma)\int_0^t (t-\tau)^{-\gamma}\left\|B(\tau)\left(u^n(\tau)-u^0(\tau)\right)\right\|_X d\tau \\ \le\ & C(0,\alpha,\gamma)\left(\int_0^t (t-\tau)^{-\gamma\frac{p}{p-1}}d\tau\right)^{\frac{p-1}{p}}\left(\int_0^t\left\|B(\tau)\left(u^n(\tau)-u^0(\tau)\right)\right\|_X^p d\tau\right)^{\frac{1}{p}} \\ \le\ & C(0,\alpha,\gamma)\left(\frac{(p-1)T^{\frac{p(1-\gamma)-1}{p-1}}}{p(1-\gamma)-1}\right)^{\frac{p-1}{p}}\left\|\widetilde{B}u^n - \widetilde{B}u^0\right\|_{L^p([0,T],X)} \\ =\ & M(0,\alpha,\gamma)\left\|\widetilde{B}u^n - \widetilde{B}u^0\right\|_{L^p([0,T],X)} \end{aligned}$$

where $\alpha<\gamma<1-\frac{1}{p}$. Hence,

$$\left\|\int_0^t U(t,\tau)B(\tau)\left(u^n(\tau)-u^0(\tau)\right)d\tau\right\|_\alpha \le M(0,\alpha,\gamma)\left\|\widetilde{B}u^n - \widetilde{B}u^0\right\|_{L^p([0,T],X)}$$

Since

$$\widetilde{B}u^n \xrightarrow{s} \widetilde{B}u^0 \text{ in } L^p([0,T],E)$$

and an argument analogous to the proof of Theorem 3.2.8 that there exists a constant $M^* > 0$, independent of n such that

$$\left\|x^n(t) - x^0(t)\right\|_\alpha \le M^* \left\|\widetilde{B}u^n - \widetilde{B}u^0\right\|_{L^p([0,T],X)}$$

for $t \in [0,T]$. Hence

$$\begin{aligned}\left\|x^n - x^0\right\|_{C([-a,T],X_\alpha)} &\le \max_{-a\le t\le 0} \left\|x^n(t) - x^0(t)\right\|_\alpha + \max_{0\le t\le T} \left\|x^n(t) - x^0(t)\right\|_\alpha \\ &\le M^* \left\|\widetilde{B}u^n - \widetilde{B}u^0\right\|_{L^p([0,T],X)}.\end{aligned}$$

This implies that as $n \to \infty$,

$$\left\|x^n - x^0\right\|_{C([-a,T],X_\alpha)} \to 0$$

i.e. $x^n \xrightarrow{s} x^0$ in $C([-a,T],X_\alpha)$.

By sequentially lower semicontinuous of J,

$$m \le J(x^{u^0}, u^0) \le \varliminf_{n\to\infty} J(x^{u^n}, u^n) = m.$$

This means that the optimal control problem (**P**) has a solution. □

Chapter V

Applications

5.1 Introduction

In the previous chapter we have applied the theory of evolution systems to obtain existence and uniqueness results of solutions for semilinear integrodifferential equations with delay and discussed the existence of optimal controls for a Lagrange problem. In this chapter we will apply these abstract results to partial differential equations.

We turn now to the description of the main concrete Banach spaces that will be used in the sequel. In doing so we will use the following notations; $x = (x_1, x_2, \ldots, x_n)$ is a variable point in the n-dimensional Eucledean space $\mathbb{R}^n$. An n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called a multi-index and we define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \text{ for } x = (x_1, x_2, \ldots, x_n).$$

Denoting $D_k = \frac{\partial}{\partial x_k}$ and $D = (D_1, D_2, \ldots, D_n)$ we have

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Let Ω be a fixed domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. We will usually assume that $\partial\Omega$ is smooth. This will mean that $\partial\Omega$ is of the class C^k for some suitable $k \geq 1$. Recall that $\partial\Omega$ is of the class C^k if for each point $x \in \partial\Omega$ there is a ball B with center at x such that $\partial\Omega \cap B$ can be represented in the form $x_i = \varphi(x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_n)$ for some i with φ, k-times continuously differentiable.

By $C^m(\Omega)$ $\left(C^m\left(\overline{\Omega}\right)\right)$ we denote the set of all $m-$times continuously differentiable real-valued functions in Ω $\left(\overline{\Omega}\right)$. $C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of those functions which have compact support in Ω.

For $u \in C^m(\Omega)$ and $1 \le p < \infty$ we define

$$\|u\|_{m,p} = \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha u|^p \, dx\right)^{1/p}. \tag{5.1}$$

Denoting by $\widetilde{C}_p^m(\Omega)$ the subset of $C^m(\Omega)$ consisting of those functions u for which $\|u\|_{m,p} < \infty$ we define $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ to be the completion in the norm $\|\cdot\|_{m,p}$ of $\widetilde{C}_p^m(\Omega)$ and $C_0^m(\Omega)$ respectively.

It is well known that $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are Banach spaces and obviously $W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega)$. For $p = 2$ we denote $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$.

The spaces $W^{m,p}(\Omega)$ defined above, consist of functions $u \in L^p(\Omega)$ whose derivatives $D^\alpha u$, in the sense of distributions, of order $k \le m$ are in $L^p(\Omega)$.

If Ω is a bounded domain then the Hölder inequality implies

$$W^{m,p}(\Omega) \subset W^{m,r}(\Omega) \text{ for } 1 \le r \le p.$$

5.2 Semilinear Parabolic Equations

In the present section we will give some examples to demonstrate some results of chapter III and IV.

We denote by Ω a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. We let

$$A(t, y, D)x = \sum_{|\alpha| \le 2m} a_\alpha(t, y) D^\alpha x$$

where (t, y) denotes a point of $[0, T] \times \Omega$. We will make the following assumptions:

(H1) The operators $A(t, y, D), t \ge 0$, are uniformly strongly elliptic in Ω, i.e., there is a constant $c > 0$ such that

$$(-1)^m \operatorname{Re}\left(\sum_{|\alpha| = 2m} a_\alpha(t, y)\xi^\alpha\right) \ge c\,|\xi|^{2m}$$

for every $y \in \overline{\Omega}, 0 \le t \le T$ and $\xi \in \mathbb{R}^n$.

(H2) The coefficients $a_\alpha(t, y)$ are smooth functions of the variables $y \in \overline{\Omega}$ for every $0 \leq t \leq T$ and satisfy for some constants $c_1 > 0$ and $0 < \beta \leq 1$

$$|a_\alpha(t, y) - a_\alpha(s, y)| \leq c_1 |t - s|^\beta$$

for $y \in \overline{\Omega}, 0 \leq s, t \leq T$ and $|\alpha| \leq 2m$.

Definition 5.2.1. Let $A(t) = A(t, y, D)$ be a strongly elliptic operator of order $2m$ on a bounded domain Ω in $\mathbb{R}^n$, we associate a family of linear operators $A_p(t)$, $0 \leq t \leq T$, in $L^p(\Omega)$, $1 < p < \infty$ by setting

$$D(A_p(t)) = D = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$$

and

$$A_p(t)u = A(t, y, D)u \text{ for } u \in D.$$

Lemma 5.2.2. Under the assumptions (**H1**) and (**H2**), there is a constant $k \geq 0$ such that the family of operators $\{A_p(t) + kI\}_{t\in[0,T]}$ satisfies the conditions (**A1**)-(**A3**) of chapter III.

Proof. (see Pazy, 1983. pp. 227-228.)

Example 5.2.3. Consider the initial value problem

$$\begin{cases} \frac{\partial x(t,y)}{\partial t} + \sum_{|\alpha|\leq 2m} a_\alpha(t, y) D^\alpha x(t, y) + kx(t, y) = f_1(t, y, x(t, y)) + \\ \qquad kx(t, y) + \int_{-a}^{t} h(t - s) g_1(s, y, x(s, y)) ds, \; y \in \Omega, 0 < t \leq T, \\ x(t, y) = \varphi(t, y), y \in \Omega, -a \leq t \leq 0, \\ D^\alpha x(t, y) = 0, y \in \partial\Omega, 0 \leq t \leq T, 0 \leq |\alpha| \leq m - 1, \end{cases} \tag{5.2}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $\varphi \in C_0^2([-a, 0] \times \Omega)$, $h \in L^1([0, T + a], \mathbb{R})$ and (**H1**) and (**H2**) are satisfied.

Let $X = L^p(\Omega)$, define $D = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ and

$$A_p(t)x = \sum_{|\alpha|\leq 2m} a_\alpha(t, y) D^\alpha x$$

for $x \in D$. Then it follows from Lemma 5.2.2 that the family $\{A_p(t) + kI\}_{t\in[0,T]}$ satisfies assumptions (**A1**)-(**A3**). For $t \in [-a, 0]$, we define a function $t \mapsto \varphi(t)$ by

$$\varphi(t)(y) = \varphi(t, y), y \in \Omega.$$

It is well known that $C_0^2(\Omega) \hookrightarrow X_1$. Since $X_1 \hookrightarrow X_\alpha$ for all $0 < \alpha < 1$, it follows that $\varphi(t)(\cdot)$ lies in all the spaces D_β discussed in Chapter III.

Suppose $f_1 : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants N_1 and $0 < \gamma \leq 1$ such that

$$\left|f_1(t,y,\xi) - f_1(s,y,\overline{\xi})\right| \leq N_1 \left\{|t-s|^\gamma + \left|\xi - \overline{\xi}\right|\right\}.$$

For $x \in L^p(\Omega)$, define

$$f(t,x)(y) = f_1(t,y,x(y)) + kx(y)$$

To show that f satisfies (**F2**). By the assumption of f_1, we have

$$\begin{aligned}
&\|f(t_1,x_1) - f(t_2,x_2)\|_X \\
&= \left(\int_\Omega |f(t_1,x_1)(y) - f(t_2,x_2)(y)|^p \, dy\right)^{\frac{1}{p}} \\
&= \left(\int_\Omega |f_1(t_1,y,x_1(y)) + kx_1(y) - f_1(t_2,y,x_2(y)) - kx_2(y)|^p \, dy\right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega (N_1 (|t_1 - t_2|^\gamma + |x_1(y) - x_2(y)|) + k\,|x_1(y) - x_2(y)|)^p \, dy\right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega (N_1 + k)^p \left(|t_1 - t_2|^\gamma + |x_1(y) - x_2(y)|\right)^p \, dy\right)^{\frac{1}{p}} \\
&\leq (N_1 + k) \left(\int_\Omega (|t_1 - t_2|^\gamma + |x_1(y) - x_2(y)|)^p \, dy\right)^{\frac{1}{p}} \\
&\leq (N_1 + k) \left\{\left(\int_\Omega |t_1 - t_2|^{\gamma p} \, dy\right)^{\frac{1}{p}} + \left(\int_\Omega |x_1(y) - x_2(y)|^p \, dy\right)^{\frac{1}{p}}\right\} \\
&= (N_1 + k) \left\{N\,|t_1 - t_2|^\gamma + \left(\int_\Omega |x_1(y) - x_2(y)|^p \, dy\right)^{\frac{1}{p}}\right\} \\
&\leq C_0 \left\{|t_1 - t_2|^\gamma + \|x_1 - x_2\|_X\right\},
\end{aligned}$$

where $C_0 = \max\left\{(N_1 + k)\,N, N_1 + k\right\}$. Introducing fractional power spaces, we have $X_\alpha \hookrightarrow X$ for $\alpha \in (0,1]$ (such as $\alpha = \frac{1}{2}$). Hence there exists a constant C_1 such that

$$\|f(t_1,x_1) - f(t_2,x_2)\|_X \leq C_1 \left(|t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha\right)$$

provided $x_1, x_2 \in X_\alpha$.

Similarly to f_1, if we let $g_1 : [-a,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function and satisfy :

$$\left|g_1(t,y,\xi) - g_1(s,y,\overline{\xi})\right| \leq N_2 \left\{|t-s|^\gamma + \left|\xi - \overline{\xi}\right|\right\},$$

for some N_2, then we can prove g that

$$\|g(t_1,x_1)-g(t_2,x_2)\|_X \le C_2\left\{|t_1-t_2|^\gamma + \|x_1-x_2\|_\alpha\right\}.$$

Hence, the initial value problem (5.2) can be written as

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t,x(t)) + \int_{-a}^{t} h(t-s)g(s,x(s))ds \ , \ t\in[0,T] \\ x(t)=\varphi(t) \ , \ t\in[-a,0], \end{cases} \tag{5.3}$$

where $A(t) = A_p(t) + kI$. By Corollary 3.2.7, we have the system (5.3) has a unique mild solution. Hence the problem (5.2) has a generalized solution $x \in C([-a,T], L^p(\Omega))$. □

Example 5.2.4. Consider the following problem

$$\begin{cases} \frac{\partial x(t,y)}{\partial t} + \sum_{|\alpha|\le 2m} a_\alpha(t,y)D^\alpha x(t,y) + kx(t,y) = f_1(t,y,x(t,y)) + \\ \qquad kx(t,y) + \int_{-a}^{t} h(t-s)g_1(s,y,x(s,y))ds + \\ \qquad \int_\Omega K(y,\xi)u(t,\xi)d\xi, \ y\in\Omega, 0<t\le T, \\ \\ x(t,y)=\varphi(t,y), y\in\Omega, -a\le t\le 0, \\ \\ D^\alpha x(t,y) = 0, y\in\partial\Omega, 0\le t\le T, 0\le|\alpha|\le m-1, \end{cases} \tag{5.4}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $\varphi \in C_0^2([-a,0]\times\Omega)$, $u\in L^2([0,T]\times\Omega)$, $h\in L^1([0,T+a],\mathbb{R})$ and $K:\overline{\Omega}\times\overline{\Omega}\to\mathbb{R}$ is continuous.

Let $X = L^2(\Omega)$, define $D = W^{2m,2}(\Omega)\cap W_0^{m,2}(\Omega)$ and

$$A_p(t)x = \sum_{|\alpha|\le 2m} a_\alpha(t,y)D^\alpha x$$

for $x\in D$. Then it follows from Lemma 5.2.2 that the family $\{A_p(t)+kI\}_{t\in[0,T]}$ satisfy assumptions (**A1**)-(**A3**). Suppose f_1, f, g_1 and g are defined as in example 5.2.3. Hence f and g satisfy

$$\|f(t_1,x_1)-f(t_2,x_2)\|_X \le N\left\{|t_1-t_2|^\gamma + \|x_1-x_2\|_\alpha\right\},$$

and

$$\|g(t_1,x_1)-g(t_2,x_2)\|_X \le M\left\{|t_1-t_2|^\gamma + \|x_1-x_2\|_\alpha\right\}$$

respectively. Let $K(y,\xi)$ be a real continuous function from $\Omega\times\Omega\to\mathbb{R}$ such that

$$\int_\Omega\int_\Omega |K(y,\xi)|^2\, dy d\xi < \infty.$$

Then the integral operator B defined by

$$B(t)u(t)(y) = \int_{\overline{\Omega}} K(y,\xi)u(t,\xi)d\xi, t \in [0,T]$$

is compact as an operator $\in \mathcal{L}\left(L^2(\overline{\Omega}), L^2(\overline{\Omega})\right)$ and strongly continuous (see Renardy and Rogers, 1993 pp. 262-263). Now problem (5.4) can be written as

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t,x(t)) + \int_{-a}^{t} h(t-s)g(s,x(s))ds + B(t)u(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T] \\ \\ x(t) = \varphi(t)\ ,\ t \in [-a,0]. \qquad \square \end{cases} \tag{5.5}$$

As a direct consequence of Theorem 4.2.2, we have:

Theorem 5.2.5. Under the assumptions stated above, system (5.5) has a unique mild solution. Hence the problem (5.4) has a generalized solution $x \in C([-a,T], L^2(\Omega))$.

Now, let the function $l : [0,T] \times L^2(\Omega) \times L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$, defined by

$$l(t,x,u) = a\int_{\Omega} |x(y)|^2\, dy + b\int_{\Omega} |u(y)|^2\, dy$$

where a, b are positive. The cost functional is given by

$$\begin{aligned} J(u) &= \int_0^T l(t, x^u(t), u(t))dt \\ &= \int_0^T \int_{\Omega} \left(a\,|x(t,y)|^2 + b\,|u(t,y)|^2\right) dydt \end{aligned}$$

where x^u denotes the mild solution of (5.5) corresponding to the control u. We consider the Lagrange problem $(\mathbf{P})$:

$(\mathbf{P})$: Find $u^0 \in L^2([0,T], L^2(\Omega))$ such that

$$J(u^0) \leq J(u), \forall u \in L^2([0,T], L^2(\Omega)).$$

Similarly to the discussion in Theorem 5.2.5, applying Theorem 4.2.1 we have the following existence result for optimal control:

Theorem 5.2.6. Under assumptions as in Theorem 5.2.5, there exists a $u^0 \in L^2([0,T] \times \Omega, \mathbb{R})$ such that

$$J(u^0) \leq J(u), \forall u \in L^2([0,T] \times \Omega, \mathbb{R}).$$

Proof. It is sufficient to show that $l(\cdot)$ satisfies assumption (**L**) (in chapter IV, section 4.3). For convexity of l, first, we have to show that $|x|^2$ is convex. Let $l_1(x) = |x|^2$. Then, for $t \in [0,1]$, we have

$$\begin{aligned} l_1(tx_1 + (1-t)x_2) &= |tx_1 + (1-t)x_2|^2 \\ &\leq t^2 |x_1|^2 + 2t(1-t)|x_1|^2 |x_2|^2 + (1-t)^2 |x_2|^2 . \end{aligned}$$

Since,

$$\begin{aligned} t|x_1|^2 + (1-t)|x_2|^2 - t^2|x_1|^2 - 2t(1-t)|x_1|^2|x_2|^2 - (1-t)^2|x_2|^2 \\ = t(1-t)\left(|x_1|^2 + 2|x_1||x_2| + |x_2|^2\right) \\ = t(1-t)\left(|x_1| - |x_2|\right)^2 \\ \geq 0. \end{aligned}$$

Hence, $l_1(tx_1 + (1-t)x_2) \leq t|x_1|^2 + (1-t)|x_2|^2, |x|^2$ is convex. It follows that $x \longmapsto \int_\Omega |x(y)|^2\, dy$ is convex and continuous in $L^2(\Omega)$. By Theorem 4.1.8, $x \longmapsto \int_\Omega |x(y)|^2\, dy$ is weakly sequentially lower semicontinuous. Similarly, $u \longmapsto \int_\Omega |u(y)|^2\, dy$ is weakly sequentially lower semicontinuous. Hence, l satisfies assumption (**L**). By Theorem 4.3.1 the optimal control problem (**P**) has a solution. □

Chapter VI

Conclusion

6.1 Thesis Summary

In this thesis, we have studied the existence of solutions for a class of semilinear integrodifferential equations with delay in Banach spaces and sufficient conditions for the existence of optimal controls for a Lagrange problem, in the case of the generating operator being time dependent.

- **Problems**

We used the contraction mapping theorem, the theory of evolution operators and developed a step by step approach to prove existence, uniqueness and continuous dependence of mild solution for a class of semilinear integrodifferential equations with delay and optimal control.

First, we considered

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(x(t)) + \int_{-a}^{t} h(t-s)g(x(s))ds \ , \ t \in [0,T] \\ x(t) = \varphi(t) \ , \ t \in [-a,0]. \end{cases} \tag{6.1}$$

Under the following assumptions :

(F1) $f : X_\alpha \to X$ is Lipschitz continuous, that is, there exists a constant $L > 0$ such that

$$\|f(x) - f(y)\|_X \leq L\,\|x-y\|_\alpha$$

for all $x, y \in X_\alpha$.

(F2) $f : [0,T] \times X_\alpha \to X$ is Hölder continuous with respect to t and Lipschitz continuous with respect to x, that is, there exist constants C and $0 < \theta \leq 1$ such that

$$\|f(t_1,x_1) - f(t_2,x_2)\|_X \leq C\left\{|t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha\right\}$$

for $t_1, t_2 \in [0, T]$ and $x_1, x_2 \in X_\alpha$.

(G1) $g : X_\alpha \to X$ is Lipschitz continuous, that is, there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\|_X \leq L \|x - y\|_\alpha$$

for all $x, y \in X_\alpha$.

(G2) $g : [0, T] \times X_\alpha \to X$ is Hölder continuous with respect to t and Lipschitz continuous with respect to x, that is, there exist constants C and $0 < \theta \leq 1$ such that

$$\|g(t_1, x_1) - g(t_2, x_2)\|_X \leq C \left\{ |t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha \right\}$$

for $t_1, t_2 \in [0, T]$ and $x_1, x_2 \in X_\alpha$.

This thesis has considered the following problems :

1. Regularity of mild solutions for system (6.1).
2. Regularity of mild solutions for system (6.1), in the case $f, g : [0, T] \times X_\alpha \to X$.
3. Existence, uniqueness and continuous dependence of mild solution for the system (6.1). We also considered (6.1) in the case $f, g : [0, T] \times X_\alpha \to X$.

The second set of questions investigated deals with the optimal control problem

$$\text{minimize} \int_0^T l(t, x^u(t), u(t))dt$$

subject to $u \in U_{ad}$ (admissible controls) and $x \in X_\alpha$ satisfying the controlled system which is obtained from (6.1), that is,

$$\begin{cases} \frac{dx(t)}{dt} + A(t)x(t) = f(t, x(t)) + \int_{-a}^{t} h(t-s)g(s, x(s))ds + B(t)u(t), 0 \leq t \leq T \\ x(t) = \varphi(t), \quad -a \leq t \leq 0, \varphi \in C([-a, 0], X_\alpha). \end{cases} \tag{6.2}$$

That is, to find $u^0 \in U_{ad}$ such that

$$J(u^0) \leq J(u), \forall u \in \mathcal{U}_{ad},$$

where

$$J(u) = \int_{[0,T]} l(t, x^u(t), u(t))dt.$$

Here x^u denotes the mild solution of controlled system (6.2) corresponding to the control $u \in \mathcal{U}_{ad}$.

We imposed the following hypotheses

(A) X is a separable reflexive Banach space, $A(t), 0 \leq t \leq T$ satisfy the assumptions (**A1**)-(**A3**) and $U(t,s), 0 \leq s \leq t \leq T$ is the corresponding evolution operator.

(B1) E is another reflexive Banach space from which the controls u take the values, $B \in L^\infty([0,T], \mathcal{L}(E,X))$.

(H) $h \in L^1([0, a+T], \mathbb{R})$.

(U) E is a separable reflexive Banach space, $\mathcal{U}_{ad} = L^p([0,T], E), 1 < p < \infty$.

(B2) $B(t) \in \mathcal{L}(E,X), t \in [0,T], \widetilde{B} : L^p([0,T], E) \to L^p([0,T], X)$ given by

$$(\widetilde{B}u)(t) = B(t)u(t), t \in [0,T]$$

and $\widetilde{B}$ is strongly continuous.

(L) $l : [0,T] \times X_\alpha \times E \to \mathbb{R} \cup \{\infty\}$ is Borel measurable satisfying the following conditions:

1. $l(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $X_\alpha \times E$ for almost all $t \in [0,T]$.
2. $l(t,x,\cdot)$ is convex on E for each $x \in X_\alpha$ and almost all $t \in [0,T]$.
3. There exist $b \geq 0, c > 0$ and $\phi \in L^1([0,T], \mathbb{R})$ such that

$$l(t,x,v) \geq \phi(t) + b\|x\|_\alpha + c\|v\|_E^p.$$

This thesis has considered the following problems :

1. Existence and uniqueness of mild solutions for the control system (6.2)
2. Existence of optimal controls of the control system (6.2) for the Lagrange problem (**P**) :

(P) Find $u^0 \in \mathcal{U}_{ad}$ such that

$$J(u^0) \leq J(u), \forall u \in \mathcal{U}_{ad},$$

where

$$J(u) = \int_{[0,T]} l(t, x^u(t), u(t))dt.$$

- **Results**

The main results of this thesis is summarized as follows :

(1) Under hypotheses (**F1**), (**G1**) and $h \in L^p([0, a+T], \mathbb{R}), p > 1$, for every $\varphi \in C([-a, 0], X_\alpha), \varphi(0) \in X_\beta, 0 \leq \alpha < \beta < 1$ the evolution equation (6.1) is equivalent to the integral equation

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \int_0^t U(t,\tau)f(x(\tau))d\tau + \\ \qquad\qquad \int_0^t U(t,\tau)\left(\int_{-a}^{\tau} h(\tau - s)g(x(s))ds\right)d\tau \ , \ t \in [0,T] \\ \varphi(t) \ , \ t \in [-a, 0]. \end{cases}$$

(see Theorem 3.2.3)

(2) If f and g satisfy (**F2**) and (**G2**) respectively then (1) still holds (see Remark 3.2.4).

(3) Under the same assumptions of (1), for every $\varphi \in C([-a, 0], X_\alpha), \varphi(0) \in X_\beta, 0 \leq \alpha < \beta < 1$,

- the evolution equation (6.1) has a unique mild solution (see Theorem 3.2.5), and
- the evolution equation (6.1) has a unique classical solution (see Theorem 3.2.6).

(4) If f and g satisfy (**F2**) and (**G2**) respectively, then (3) still holds (see Corollary 3.2.7).

(5) Under the same assumptions of (1), there exists $K < \infty$ such that

$$\|F\varphi_1 - F\varphi_2\|_{C([-a,T],X_\alpha)} \leq K_\beta \left\{ \|\varphi_1 - \varphi_2\|_{C([-a,0],X_\alpha)} + \|\varphi_1(0) - \varphi_2(0)\|_\beta \right\}$$

for $\varphi_1, \varphi_2 \in D_\beta$ where

$$F : D = \cup D_\beta \to C([-a, T], X_\alpha),$$

which assigns to every $\varphi \in C([-a, 0], X_\alpha)$ the unique solution $G(\varphi)$ of (6.1) and $D_\beta = \{\varphi \in C([-a, 0], X_\alpha), \varphi(0) \in X_\beta, \beta \in (\alpha, 1]\}$ (see Theorem 3.2.8).

(6) Suppose (**F2**), (**G2**) (in chapter III), (**A**), (**B1**) and (**H**) hold, $\varphi \in C([-a,0], X_\alpha), \varphi(0) \in X_\beta, (\alpha < \beta)$ and $p > \frac{1}{1-\alpha}$. Then for each $u \in \mathcal{U}_{ad}$ the controlled system (6.2) has a unique mild solution $x \in C([-a,T], X_\alpha)$ (see Theorem 4.2.2).

(7) Under the assumptions (**F2**), (**G2**) (in chapter III), (**A**), (**H**), (**U**), (**B2**), (**L**),and $p > \frac{1}{1-\alpha}$, the optimal control problem (**P**) has a solution, that is, there exists an admissible state-control pair $\left\{u^0, x^{u^0}\right\}$ such that

$$J(u^0) = \int_{[0,T]} l(t, x^{u^0}(t), u^0(t))dt \leq J(u), \forall u \in \mathcal{U}_{ad}.$$

(see Theorem 4.3.1)

6.2 Applications

All results of the abstract framework in this thesis can be applied to semilinear integrodifferential equation with delay. Two examples are presented for illustration. The first example is concerned about the existence of solutions for system governed by $2m$-order semilinear integrodifferential equations of parabolic type with delay. The second, deals with the corresponding controlled system for the Lagrange problem.

6.3 Discussion and Recommendations

We studied a class of semilinear integrodifferential equations with the generating operator being time dependent and obtained the existence and uniqueness of mild solutions. At first, we only used the contraction mapping theory and developed the step by step approach to prove the existence , uniqueness and continuous dependence. We did not need any other estimate method, such as Gronwall Lemma. Furthermore, we discussed the corresponding control system for the Lagrange problem. By Balder's results, again using step by step approach we obtained existence result of optimal controls.

Based on the results and the approach of this thesis, we can continue to discuss related problems, such as :

1. the existence of mild solutions for integrodifferential equations with operator valued h,

2. relaxed optimal control problems for integrodifferential equations

3. integrodifferential inclusion, and so on.

Furthermore, we can consider other application problems and computation algorithm.

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Appendices

Appendix A

Parabolic Equations

A *partial differential equation* (PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Fix $k \geq 1$ and let U denote an open subset in $\mathbb{R}^n$.

Definition A-1. An expression of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0, x \in U \qquad \text{(A.1)}$$

is called a k^{th}*−order partial differential equation*, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{U} \to \mathbb{R}$$

is given, and

$$u : U \to \mathbb{R}$$

is the unknown.

Definition A-2.

(i) The partial differential equation (A.1) is called *linear* if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

for given functions a_α $(|\alpha| \leq k)$ and f. This linear PDE is *homogeneous* if $f = 0$.

(ii) The PDE (A.1) is *semilinear* if it has the form

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0 \left(D^{k-1}u, \ldots, Du, u, x \right) = 0.$$

The linear partial differential operator

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha \qquad \text{(A.2)}$$

is said to be *elliptic* at a point $x^0 \in U$ if $\sum_{|\alpha|=k} a_\alpha(x^0)\xi^\alpha \neq 0$ for any real $\xi \neq 0$ Suppose now that $k = 2m$, m an integer. The operator (A.2) is said to be *strongly elliptic* at x^0 if

$$(-1)^m \operatorname{Re}\left(\sum_{|\alpha|=2m} a_\alpha(x^0)\xi^\alpha\right) > 0 \quad \text{for any real } \xi \neq 0.$$

If the operator is elliptic (strongly elliptic) at each point of U, then it is said to be elliptic (strongly elliptic) in U.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the differential operators

$$Lu = \frac{\partial u}{\partial x} + A(t,x,D)u = \frac{\partial u}{\partial t} + \sum_{|\alpha|\leq 2m} a_\alpha(t,x)D^\alpha u$$

where the coefficients $a_\alpha(t,x)$ are sufficiently smooth function of the variables x in $\overline{\Omega}$ for every $0 \leq t \leq T$.

Definition A-3. L is said to be *parabolic* at a point (x^0, t^0) if $A(t^0, x, D)$ is strongly elliptic at x^0. L is *parabolic* on a set A if L is parabolic at each point of A.

Appendix B

Banach Space Valued Functions

A Banach space setting of evolution equations requires taking the derivative in the Banach space. Hence, integration of Banach space valued function is an important tool in this setting. Throughout this section, we let X be a Banach space.

Definition B-1. Let I be an interval in $\mathbb{R}$, let $(X, \|.\|)$ be a (real) Banach space and let $u : I \to X$ (so that $u(t) \in X$ for all $t \in I$). u is *strongly continuous at* $t_0 \in I$ if, given $\varepsilon > 0$ there exists a $\delta > 0, \delta = \delta(\varepsilon, t_0)$, such that

$$\|u(t) - u(t_0)\|_X < \varepsilon$$

whenever $t \in I$ and $|t - t_0| < \delta$. u is *strongly continuous on* I if it is strongly continuous at t_0 for every $t_0 \in I$. The set of all functions $u : I \to X$ which are strongly continuous on I will be denoted by $C(I, X)$.

Remark B-2. (i) When t_0 is an endpoint of I, the definition deals with one-sided continuity. For example, when $I = [a, b]$, we have continuity on the right at a and on the left at b. When t_0 is not an endpoint of I, the definition deals with two-sided continuity at t_0.

(ii) The word "strongly" is used to emphasize the difference between this type of continuity and *weak continuity*; see [Li ,X. and Yong, J.] p. 47. However, we shall omit "strongly" when there is no possibility of confusion.

Definition B-3. Let $u : I \to X$. u is (strongly) *uniformly continuous on* I if, given $\varepsilon > 0$ there exists a $\delta > 0, \delta = \delta(\varepsilon)$, such that for any t_0 and $t \in X$ with $|t - t_0| < \delta$,

$$\|u(t) - u(t_0)\|_X < \varepsilon.$$

Definition B-4. Let I be an interval in $\mathbb{R}$, let $(X, \|.\|)$ be a (real) Banach space and let $u : I \to X$. We say that u is *(strongly) differentiable at* $c \in I$ if there exists an element $v \in X$ such that

$$\frac{u(c+h)-u(c)}{h} \to v \text{ as } h \to 0,$$

i.e., given $\varepsilon > 0$, a positive $\delta = \delta(\varepsilon, c)$ can be found such that

$$\left\| \frac{u(c+h)-u(c)}{h} - v \right\|_X < \varepsilon$$

whenever $c + h \in I$ and $0 < |h| < \delta$. The (necessarily unique) element v is called the *(strong) derivative* of u at c and is denoted by $u'(c)$ or by $\left[\frac{du}{dt}\right]_{t=c}$. u is *(strongly) differentiable on* I if u is (strongly) differentiable at c for all $c \in I$.

The set of all functions $u : I \to X$ which are (strongly) continuously differentiable on I will be denoted by $C^1(I, X)$. Hence each $u \in C^1(I, X)$ has a strong derivative u', which is strongly continuous on I.

We are also interested in measurable vector-valued functions. Let X be a Banach space and I a finite Lebesgue measurable subset of $\mathbb{R}$. Let f be a vector-valued function defined on I with value $f(t) \in X$.

Definition B-5. (i) The function $f : I \to X$ is called a *simple function* if there exist finite many measurable set $E_i \subset I$, mutually disjoint, and $x_i \in X$ such that

$$f(t) = \sum_{i=1}^{n} x_i \chi_{E_i}(t); t \in I,$$

where ${}_i\chi_{E_i}$ is the characteristic function on E_i :

$$\chi_{E_i}(t) = \begin{cases} 1, t \in E_i \\ 0, t \notin E_i. \end{cases}$$

(ii) The function $f : I \to X$ is said to be *strongly measurable* if there exists a sequence of simple functions $f_n : I \to X$ such that

$$\lim_{n\to\infty} \|f_n(t) - f(t)\|_X = 0, \quad a.e.\ t \in I \tag{B.1}$$

Strongly measurable vector-valued functions have properties analogous to those of measurable scalar-valued functions. If f is the strong limit a.e. of a sequence $\{f_n\}$ of strongly measurable functions, then f is strongly measurable.

Now, for any simple function $f(\cdot) = \sum_{i=1}^{n} x_i \chi_{E_i}(\cdot)$, we define its *Bochner integral* by

$$\int_E f(t)dt = \sum_{i=1}^{n} x_i m(E \cap E_i)$$

for any measurable set E, $m(E \cap E_i)$ is the Lebesgue measure of the set $E \cap E_i$.

Definition B-6. Let $f : I \to X$ be strongly measurable. We say that f is *Bochner integral* if there exists a sequence of simple functions $f_n : I \to X$ such that (B.1) holds and the sequence $\int_I f_n(t)dt$ is strongly convergent in X. In this case, we define the Bochner integrable of f by

$$\int_I f(t)dt = \lim_{n\to\infty} \int_I f_n(t)dt.$$

Thus, by definition, the Bochner integral of f over any measurable set $E \subset I$ is

$$\int_E f(t)dt = \lim_{n\to\infty} \int_E f_n(t)dt.$$

It can be shown that the integral is well defined in that it is independent of the choice of the sequence $\{f_n\}$. A necessary and sufficient conditions that f be Bochner integrable is that f is strongly measurable and

$$\int_I \|f(t)\|_X \, dt < \infty.$$

We denote the set of all Bochner integrable functions on I to X by $B(I, X)$. If X is the field of scalars, the Bochner integral reduces to the usual Lebesgue integral. $B(I, X)$ becomes a linear vector space under the natural definition of addition and scalar multiplication.

The integral $B \int_E f(t)dt$ for E any measurable set defines a linear transformation from $B(I, X)$ into X. Moreover, the following property holds for the Bochner integral :

Theorem B-7. If $f \in B(I, X)$, then

$$\left\| \int_E f(t)dt \right\| \leq \int_E \|f(t)\|_X \, dt, \quad E \text{ measurable.}$$

The Bochner integral possesses almost the same properties as the Lebesgue integral. We omit the exact statement here.

Definition B-8. Let Ω be a nonempty measurable set in $\mathbb{R}^N, N \geq 1$. For X a Banach space, we denote by $L^p(\Omega, X)$ the space of (equivalence classes of) strongly measurable function $f : \Omega \to X$ such that

$$\int_\Omega \|f(t)\|_X^p \, dt < \infty \quad \text{for } 1 \leq p < \infty.$$

This space is a Banach space when endowed with the norm

$$\|f\|_{L^p(\Omega,X)} = \left(\int_\Omega \|f(t)\|_X^p \, dt\right)^{1/p}$$

Moreover, $L^p(\Omega, X)$ is separable and $C_0^\infty(\Omega, X)$, the space of infinitely differentiable function with compact support is dense in $L^p(\Omega, X)$ for $1 \leq p < \infty$.

Theorem B-9. (Hölder Inequality) Let $1 < p, q < \infty$ be given with $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\Omega, X)$ and $g \in L^q(\Omega, X)$. Then

$$\int_\Omega \|f(t)g(t)\|_X \, dt \leq \left(\int_\Omega \|f(t)\|_X^p \, dt\right)^{1/p} \left(\int_\Omega \|g(t)\|_X^q \, dt\right)^{1/q}$$

where all the integrals exist.

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