ANALYSIS OF STATISTICAL DISTRIBUTIONS APPEARING IN MEDICAL ULTRASOUND IMAGES

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การวิเคราะห์การแจกแจงทางสถิติที่ปรากฏใน
ภาพคลื่นเสียงความถี่สูงทางการแพทย์

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ANALYSIS OF STATISTICAL DISTRIBUTIONS APPEARING IN MEDICAL ULTRASOUND IMAGES

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for a Master's Degree.

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มังกร ดำเนตร : การวิเคราะห์การแจกแจงทางสถิติที่ปรากฏในภาพคลื่นเสียงความถี่สูงทางการแพทย์ (ANALYSIS OF STATISTICAL DISTRIBUTIONS APPEARING IN MEDICAL ULTRASOUND IMAGES) อาจารย์ที่ปรึกษา : ผู้ช่วยศาสตราจารย์ ดร.เจษฎา ตัณฑนุช, 124 หน้า.

ภาพคลื่นเสียงความถี่สูงทางการแพทย์/การแจกแจงเรย์ลี/การแจกแจงริเชียน/การแจกแจงเค/การแจกแจงนาคากามิ

วิทยานิพนธ์นี้มีจุดมุ่งหมายต้องการศึกษาการแจกแจงทางสถิติที่เกี่ยวข้องกับสัญญาณที่ปรากฏในภาพถ่ายคลื่นเสียงความถี่สูงทางการแพทย์ การศึกษาขยายแนวคิดในการใช้การแจกแจงทางสถิติพื้นฐานได้แก่ การแจกแจงเรย์ลี และการแจกแจงริเชียน ในการบอกคุณลักษณะของสัญญาณที่ปรากฏในภาพถ่ายคลื่นเสียงความถี่สูงทางการแพทย์ จากการศึกษาพบว่าการแจกแจงเค การแจกแจงโฮโมไดน์เค และการแจกแจงริเชียนซึ่งพารามิเตอร์ถูกมอดูเลทด้วยการแจกแจงเกาส์เซียนแบบผกผันสามารถอธิบายคุณลักษณะได้ดีกว่า และการแจกแจงนาคากามิซึ่งพารามิเตอร์ถูกมอดูเลทด้วยการแจกแจงเกาส์เซียนแบบผกผันวางนัยทั่วไปทำได้ดีที่สุดเพราะมีความทั่วไปมากกว่าและพารามิเตอร์ถูกมอดูเลทด้วยการแจกแจงที่มีความทั่วไปมากกว่า

ในการศึกษาครั้งนี้ได้พัฒนาโปรแกรมเพื่อแสดงกราฟของการแจกแจงต่าง ๆ ที่เกี่ยวข้อง รวมทั้งโปรแกรมหาค่าพารามิเตอร์ที่สำคัญของการแจกแจงทางสถิติเหล่านั้น ได้แก่ ค่าเฉลี่ย ความแปรปรวน ความเบ้ และเคอร์โทซิส

สาขาวิชาคณิตศาสตร์	ลายมือชื่อนักศึกษา มังกร ดำเนตร
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MEDICAL ULTRASOUND IMAGE/ RAYLEIGH DISTRIBUTION/ RICIAN DISTRIBUTION/ K-DISTRIBUTION/ NAKAGAMI DISTRIBUTION

The purpose of this thesis is to study the statistical distributions of signals appearing in medical ultrasound images. The study extended concepts of using basic statistical distributions to characterize signals in medical ultrasound images, i.e. Rayleigh and Rician distributions. K-distribution and homodyned K-distribution and Rician distribution with modulated inverse Gaussian distribution parameter were claimed to be better in giving physical explanation of the signals. Nakagami distribution with modulated generalized inverse Gaussian parameters is the most generalized distribution than other distributions mentioned before. The distribution is able to explain physical phenomenon with the specification of the density of random scatterers and coherent component. Also its parameters are modulated by more generalized distribution, generalized inverse Gaussian distribution.

In this study, the Python code was developed for displaying the graphs of the proposed distributions. Also the software code for finding the important parameters of the statistical distribution, i.e. mean, variance, skewness and kurtosis, was implemented.

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CHAPTER I

INTRODUCTION

Ultrasound denotes sound waves of frequencies higher than the upper audible limit of human hearing. It is considered as a mechanical disturbance that moves as a pressure wave through a medium. Medical ultrasound imaging is an ultrasound-based diagnostic imaging technique using a mechanical disturbance that moves as a high frequency pressure wave through muscles and internal organs and producing an image from the echoes. An image of soft tissue in a body can be produced by transforming the ultrasound echo signals received to electric voltage and then processing the detected voltage to the brightness at each point of the image.

The ultrasonic wave encounters multiple interfaces that implies a masking effect for those reflectors and scattering, which produce coherent and incoherent (diffuse scattering) effects. The incoherent effect causes the degradation of a medical image quality, which is called *noise*. To enhance the quality of the appearance image one needs to understand the process of imaging. However, there are many types of noise which contribute to the reduction of the quality of a medical ultrasound image. Generally, noise patterns in ultrasound images are considered as noises generated from two sources. One arises from noise generated by electronic devices (electrical noise). The other one arises from noise-like variation called speckle (noise). In practice, speckle noise is the dominating disturbance which is clearly noticeable on a medical ultrasound image.(Mamou and Oelze, 2013)

Speckle noise has specific characteristics which can be described by some probability distribution functions. The mathematical model of the speckle noise probability distribution is derived from the envelope of the voltage signal received from an ultrasound transducer. An understanding of the characteristics of ultrasound waves and their behavior in various media were studied (Destrempes and Cloutier, 2010 and Cai, 2016, Jensakda and Tanthanuch, 2018, Seekot and Tanthanuch, 2008, Tanthanuch, Kaptsov and Meleshko, 2019, Tanthanuch, Seekot, and Schulz, 2012). The probability distributions found in many researches explaining the ultrasound noise characteristic are Gaussian distribution, Poisson distribution, Rayleigh distribution, Rician distribution, K-distribution, homodyned K-distribution, generalized K-distribution, ascending order K distribution, Nakagami distribution, Nakagami-gamma distribution, Rician inverse Gaussian distribution and Nakagami-generalized inverse Gaussian distribution.

In this research, many probability distributions related to ultrasound imaging will be studied. Then the probability distribution properly being best describing image noise will be proposed.

1.1 Research objectives

1. To study about probability distributions related ultrasound noise imaging.
2. To proposed another probability distribution which gives a better explanation for ultrasound image noise.

1.2 Scope and limitations

The research works on the ultrasound image of human tissue of class 1 scattering, which is composed of a large number of scattering cells, the scatterers per resolution cell is 25 or higher. The backscattered ultrasound echoes from the cells, both the magnitudes and the phases, are statistically independent and uniform.

The calculation in this thesis performed by

- Laptop HP ProBook computer with Intel CPU I5 6200U 12GB RAM;
- Operation System: Microsoft Windows 10 Pro;
- Programming Language: Python 3.7 with Jupyter Notebook Editor.

1.3 Research procedure

The research work will proceed as follows:

1. Study the theory and application of statistical distribution to ultrasound image
2. Analyze and construct the probability distribution to explain ultrasound image noise
3. Validate and verify the model obtained

1.4 Expected results

This research would be able to have a concept to find an appropriate and effective distributions to explain ultrasound image noise.

CHAPTER II

LITERATURE REVIEW

In this chapter, the knowledge of basic mathematics and statistics related with applications of probability distributions to ultrasound imaging is presented. Most contents of probability and random variables comes from Grimmett and Stirzaker (2001), Grimmett and Welsh (1986) and Destrempes and Cloutier (2010).

2.1 Probability

The mathematical theory of probability starts with the idea of an experiment (or trial), being a course of action whose consequence is not predetermined; this experiment is reformulated as a mathematical object called a *random variable* in a *probability space*.

Definition 2.1. If A is some event, the occurrence or non-occurrence of A depends upon the *chain* of circumstances involved. This chain is called an *experiment* or *trial*; the result of an experiment is called its *outcome*. The set of all possible outcomes of an experiment is called the *sample space* and is denoted by Ω.

Definition 2.2. A *Bernoulli trial* or *binomial trial* is a random experiment/trial with exactly two possible outcomes, "success" and "failure", i.e.

$$\Omega = \{\text{success}, \text{failure}\}.$$

Definition 2.3. A non-empty collection $\mathcal{F}$ of subsets of the sample space Ω is called an *event space* of $\mathcal{F}$.

Definition 2.4. A collection $\mathcal{F}$ of subsets of Ω is called a *σ-field* if it satisfies the following conditions:

1. $\emptyset \in \mathcal{F}$;
2. if $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{F}$;
3. if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, where A^c is the complement of A.

Definition 2.5. A *probability measure* $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying

1. $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$;
2. if $A_1, A_2, \ldots$ is a collection of disjoint members of $\mathcal{F}$, in that $A_i \cap A_j = \emptyset$ for all pairs i, j satisfying $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$, comprising a set Ω, a σ-field $\mathcal{F}$ of subsets of Ω, and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, is called a *probability space*.

The above definitions give more efficient tools to measure the likelihoods of the occurrences of events. Compared to the classical concept, the experience of most scientific experimentation is that the proportion of times that A occurs settles down to some value as N becomes larger; that is, writing $N(A)$ for the number of occurrences of A in the N trials, the ratio $N(A)/N$ appears to converge to a constant limit as N increases. The ultimate value of this ratio as being the probability $\mathbb{P}(A)$ that A occurs on any particular trial. In practice, N may be taken to be large but finite, and the ratio $N(A)/N$ may be taken as an approximation to $\mathbb{P}(A)$. Some individuals refer informally to $\mathbb{P}$ as a *probability distribution*.

Definition 2.6. *Conditional Probability* is a measure of the probability of one event occurring with some relationship to one or more other events. The conditional probability of A given B, or the probability of A under the condition B, is usually written as $\mathbb{P}(A|B)$, or sometimes $\mathbb{P}_B(A)$ or $\mathbb{P}(A/B)$, defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)},$$

where $\mathbb{P}(A \cap B)$ is the probability that both events A and B occur.

2.2 Random Variables and their distribution

Quantities governed by randomness correspond to functions on the probability space called *random variables*. The value taken by a random variable is subject to chance, and the associated likelihoods are described by a function called the *distribution function*.

Definition 2.7. *A random variable* is a function $X : \Omega \to \mathbb{R}$ with the property that

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$$

for each $x \in \mathbb{R}$. Such a function is said to be *$\mathcal{F}$-measurable*.

Distribution is a statistical concept used in data research. It is a listing or function showing all the possible values of the statistical data and how often they occur. Every random variable has a *distribution function*. Distribution functions are very important and useful.

Definition 2.8. The *distribution function* of a random variable X is the function

$$F : \mathbb{R} \to [0, 1]$$

given by $F(x) = \mathbb{P}(X \leq x)$.

The distribution function satisfies the following conditions.

Theorem 2.1. *A distribution function F has the following properties:*

1. $\lim_{x \to -\infty} F(x) = 0$,
2. $\lim_{x \to \infty} F(x) = 1$,
3. *F is nondecreasing, i.e. if $x < y$ then $F(x) \leq F(y)$,*
4. *F is right-continuous, i.e. $F(x + h) \to F(x)$ as $h \to 0^+$.*

Moreover, some conditions for the relation of the distribution function of a random variable X and the probability measure $\mathbb{P}$ are as follows:

Theorem 2.2. *Let F be the distribution function of X. Then*

1. $\mathbb{P}(X > x) = 1 - F(x)$,
2. $\mathbb{P}(x < X \leq y) = F(y) - F(x)$,
3. $\mathbb{P}(X = x) = F(x) - \lim_{y \to x^-} F(y)$.

2.3 Discrete and continuous random variables

Random variables can be classified into two basic categories, *discrete* and *continuous*.

Definition 2.9. The random variable X is called *discrete* if it takes values in some countable subset $\{x_1, x_2, \ldots\}$, only, of $\mathbb{R}$. The discrete random variable X has *(probability) mass function* $f : \mathbb{R} \to [0, 1]$ given by $f(x) = \mathbb{P}(X = x)$.

Definition 2.10. The random variable X is called *continuous* if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)du \qquad x \in \mathbb{R},$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the *(probability) density function* of X.

2.4 Expectation

Let $x_1, x_2, \ldots, x_N$ be the numerical outcomes of N repetitions of some experiment. The average of this outcomes is

$$m = \frac{1}{N}\sum_i x_i.$$

Consider the N discrete random variables with a common mass function f. For each possible value x, about $Nf(x)$ of the outcome X_i, $i = 1, \ldots, N$, will take that value x. So the average value is

$$m = \frac{1}{N}\sum_x xNf(x) = \sum_x xf(x),$$

where the summation is over all possible values of the X_i. This average is called the *expectation* or *mean value* of the underlying distribution with mass function f.

Definition 2.11. The *mean value*, or *expectation*, or *expected value* of a discrete random variable X with mass function f is defined to be

$$\mathbb{E}(X) = \sum_{x:f(x)>0} xf(x), \tag{2.1}$$

whenever this summation is absolutely convergent.

The expectation (2.1) of a discrete variable X or an average of the possible values of X may be written in form

$$\mathbb{E}(X) = \sum_x x\mathbb{P}(X = x),$$

which means each value being weighted by its probability.

Lemma 2.3. *If X has a mass function f and g:$\mathbb{R} \to \mathbb{R}$, then*

$$\mathbb{E}(g(X)) = \sum_x g(x)f(x),$$

whenever this sum is absolutely convergent.

Definition 2.12. If k is a positive integer, the kth *moment* m_k of X is defined to be

$$m_k = \mathbb{E}(X^k) = \sum_x x^k \mathbb{P}(X = x).$$

The kth *central moment* σ_k is

$$\sigma_k = \mathbb{E}\left[(X - m_1)^k\right].$$

Definition 2.13. The two moments of most use are

- $m_1 = \mathbb{E}(X)$, called the *mean* (or *expectation*) of X, and
- $\sigma_2 = \mathbb{E}\left[(X - \mathbb{E}X)^2\right] = \mathbb{E}(X^2) - (m_1)^2$, called *variance* of X.

These two quantities are measures of the mean and dispersion of X; that is, m_1 is the average value of X, and σ_2 measures the amount by which X tends to deviate from the average. The term *variance* was first introduced in 1918 by Ronald Fisher (Lovric, 2011). The mean m_1 is often denoted μ, and the variance of X is often denoted $\mathrm{var}(X)$. The positive square root $\sigma = \sqrt{\mathrm{var}(X)}$ is called the standard deviation.

For continuous variables, expectations are defined as integrals.

Definition 2.14. The *expectation* of a continuous random variable X with density function f is given by

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx,$$

whenever the integral converges absolutely.

By the definition (2.12) of moment for a discrete random variable, we define the kth moment of a continuous variable X as the following.

$$m_k = \mathbb{E}(X^k) = \int_{-\infty}^{\infty} x^k f(x) dx, \tag{2.2}$$

whenever the integral exists.

Therefore, we can define *mean* and *variance* for a continuous random variable similar to definition 2.13 as $\mu = \mathbb{E}(X)$ and $\sigma_2 = \mathbb{E}\left[(X - \mathbb{E}X)^2\right]$, respectively, where the kth moment is defined by (2.2).

2.5 Skewness

Skewness describes which side of a distribution has a longer tail. It is a measure of distributional asymmetry. If the long tail is on the right, then the skewness is rightward or positive; if the long tail is on the left, then the skewness is leftward or negative.

Skewness is measured by the third standardized moment

$$\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right],$$

where X is a random variable with mean μ and standard deviation σ. However, there are some alternative definitions. One may see the other definitions in Lovric (2011).

2.6 Kurtosis

In 1905, Pearson mentioned the measure which can explain whether the frequency towards the mean is emphasized more or less than that required by the Gaussian law. He defined $\kappa = \dfrac{m_4}{(m_2)^2}$ to compare other distributions to the normal distribution and called $\kappa - 3$ the *degree of kurtosis*, (or sometimes called *excess* *kurtosis*). Kurtosis is a statistical measure that is used to describe the size of the center and the tails on a distribution. High kurtosis is linked to high concentration of mass in the center and/or the tails of the distribution (Lovric, 2011).

2.7 Harmonic Waves

2.7.1 One-Dimensional Waves and Harmonic Waves

The mathematical expression for wave motion and the harmonic wave can be described as the follows.

Definition 2.15. Let ψ be an n-dimensional vector, say $\psi = \psi(x_1, x_2, ..., x_n)$. The **Laplacian** $\nabla^2\psi$ of ψ is defined by

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + ... + \frac{\partial^2\psi}{\partial x_n^2},$$

Note that

1. We may sometimes replace x_n with variable t.
2. ∇^2 is $\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \ldots + \dfrac{\partial^2}{\partial x_n^2}$.

Definition 2.16. The **differential wave equation** which describes propagation of waves with speed v respect to time t is given by

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}.$$

In particular, if $n = 1$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}.$$

is called the one-dimensionl differential wave equation.

Definition 2.17. A function $y = f(x \pm vt)$ is said to be a function of **one-dimensional wave** which moves along the x-axis at speed v relative to coordinate (x, y) if it satisfies the one-dimensional differential wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \tag{2.3}$$

Definition 2.18. Harmonic wave is a particular wave which involves the sine function

$$y = A\sin[c(x \pm vt)]$$

or cosine function

$$y = A\cos[c(x \pm vt)],$$

where A and c are constants, A is called the amplitude of the wave

The harmonic wave invdring sine function $y = A\sin[c(x \pm vt)]$ satisfies the one-dimensional differential wave equation since

$$\frac{\partial^2 y}{\partial x^2} = -Ac^2\sin[c(x \pm vt)]$$

and

$$\frac{\partial^2 y}{\partial t^2} = -Ac^2v^2\sin[c(x \pm vt)].$$

here, we can consider the proposed equation as a one-dimensional wave.

It is also similar for the case of $y = A\cos[c(x \pm vt)]$.

Denote $\omega = cv$, which is called the *angular frequency*. With these relationships, one can show that the equivalence of the following general forms for harmonic waves:

$$y = A\sin[cx \pm \omega t].$$

In the cosine function is also similar.

2.7.2 Superposition of harmonic Waves of the Same Frequency

For the case that two or more such waves arrive at the same point in space or exist together along the same direction, several important cases of the combined effects of two or more harmonic waves are called *the superposition of waves*. The superposition principle says that the resultant displacement is the sum of the separate displacements of the constituent waves. Here we consider the superposition of harmonic waves of different amplitudes and phases but with the same frequency. This case leads to an important difference between the irradiance attainable from *randomly phased* and *coherent harmonic waves*.

The superposition of harmonic waves may be expressed in terms of equation

$$y = y_1 + y_2,$$

where y_1 and y_2 are the independent waves which exist together in the space.

The time variations of the harmonic waves at the given point (two harmonic waves with different phases) can be expressed by

$$\begin{aligned} y_1 &= A_1 \sin(\omega t + \alpha_1), \\ y_2 &= A_2 \sin(\omega t + \alpha_2), \end{aligned}$$

where A_1, A_2 are amplitudes of y_1 and y_2, α_1, α_2 are phases of y_1 and y_2 respectively.

By the superposition principle, the resultant y_R at the point is

$$y_R = y_1 + y_2 = A_1 \sin(\omega t + \alpha_1) + A_2 \sin(\omega t + \alpha_2).$$

The proposed y_R can be derived to

$$y_R = (A_1 \cos\alpha_1 + A_2 \cos\alpha_2) \sin\omega t + (A_1 \sin\alpha_1 + A_2 \sin\alpha_2) \cos\omega t.$$

For the simplicity, we may consider the coefficients of $\sin \omega t$ and $\cos \omega t$ as terms of cosine and sine with magnitude A and phase α, respectively.

The coefficients are

$$A \cos \alpha = A_1 \cos \alpha_1 + A_2 \cos \alpha_2$$

and

$$A \sin \alpha = A_1 \sin \alpha_1 + A_2 \sin \alpha_2.$$

The quantities A and α are defined by this technique, hence

$$y_R = A \cos \alpha \sin \omega t + A \sin \alpha \cos \omega t$$

or

$$y_R = A \sin(\omega t + \alpha).$$

We claim that the wave y_R is also harmonic wave of the same frequency ωt, with amplitude A and phase α. Moreover, by mathematical induction, the sum of N harmonic waves of identical frequency is again a harmonic wave of the same frequency, with amplitude given by A and phase given by α.

2.7.3 Coherence and Incoherence

Before giving descriptions of *coherence* and *incoherence*, we first say about *monochromatic waves*. Monochromatic wave is represented by a wave function with harmonic time dependence, its amplitude and phase are generally position dependent. The term coherence and incoherence are used to describe the correlation between phases of monochromatic waves. In the mathematics point of view, the wave function is a harmonic function of time with the same frequency at all positions.

Definition 2.19 (relative phase)**.** Let

$$\begin{aligned} y_1(x) &= A_1 \sin(\omega x + \alpha_1) \qquad \text{and} \\ y_2(x) &= A_2 \sin(\omega x + \alpha_2) \end{aligned}$$

be harmonic waves with the same frequency ω. The **relative phase** of y_1 and y_2 is given by $\alpha_1 - \alpha_2$.

Definition 2.20 (coherent)**.** Waves are said to be **coherent** if they have a **constant relative phase**, which also implies that they have the same frequency. In the superposition of coherent waves, individual amplitudes add together. Waves add constructively or subtract destructively, depending on their relative phase, as shown in figure 2.1.

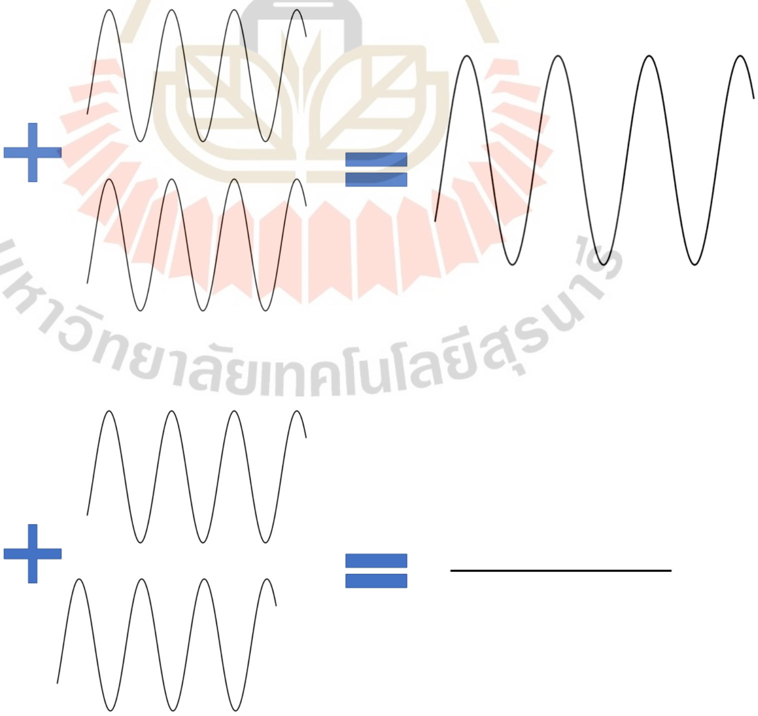

Figure 2.1 Waves add constructively or subtract destructively.

Definition 2.21 (incoherent)**.** Waves are said to be **incoherent** if they have **random relative phase**. Which means that they are combined with a lots of different phases, as an example in figure 2.2

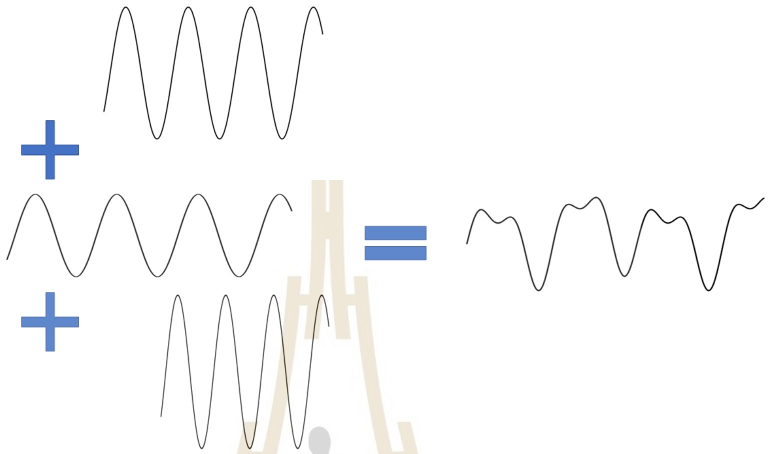

Figure 2.2 The figure of waves that combine with lots of different phases nearly cancel out and yield very low irradiance (incoherent).

Definition 2.22 (speckle)**. Speckle** is a random pattern which has a negative impact on **coherent imaging**, including ultrasound imaging. It is a result of the superposition of many waves, which have different phases (incoherent).

Speckle occurs in an ultrasound image because the ultrasonic wave encounters rough surfaces that result in the scattering of waves. Each scattered wave from a rough surface has a different phase which leads to the forming of speckles.

2.8 Distributions related Ultrasound Image

Well known distributions and distributions used in ultrasound image processing are the following.

2.8.1 Bernoulli distribution

The *Bernoulli distribution* admits only two possible outcomes, for example $\Omega = \{\text{Yes}, \text{No}\}$ and $\Omega = \{\text{True}, \text{False}\}$. It is usually considered as $\Omega = \{0, 1\}$, where the event "1" is often called a *success*, the other "0", a *failure*. Further $f(1) = p$ and $f(0) = 1 - p$, where p is the probability of the event occurring and $0 \leq p \leq 1$. The mean and variance of a Bernoulli random variable are p and $p(1-p)$, respectively. The variance is maximum if $p = 0.5$. The probability mass function is

$$f(k) = p^k(1-p)^{1-k}, \quad k = 0, 1.$$

2.8.2 Binomial distribution

Let $0 \leq k \leq n$, and consider $f(k)$. Exactly $\binom{n}{k}$ points in Ω give a total of k wanted events; each of these points occurs with probability $p^k(1-p)^{n-k}$, and so

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{if} \quad 0 \leq k \leq n. \tag{2.4}$$

The random variable X is said to have the *binomial distribution* with parameters n and p. This distribution is the discrete probability distribution of the number of successes in a sequence of n independent experiments. If there are n independent Bernoulli random variables, each with success probability p, then the total number of successes has the binomial distribution.

A binomial random variable is often transformed into a proportion by dividing by n. The resulting random variable k/n is called the binomial proportion and the probability function (2.4) shifted on to $0, 1/n, 2/n, \ldots, 1$.

The binomial distribution is the basis for the popular binomial test of statistical significance, where the binomial test is an exact test of the statistical significance of deviations from a theoretically expected distribution of observations into two categories.

2.8.3 Poisson distribution

If a random variable X takes values in the set $\{0, 1, 2, \ldots\}$ with mass function

$$f(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots,$$

where $\lambda > 0$, then X is said to have the *Poisson distribution* with parameter λ.

The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or specified intervals such as distance, area, volume or space, if these events occur with a known constant rate and independently of the time since the last event.

Suppose that in the binomial distribution, n becomes large while p becomes small, by (2.4), let $n \to \infty$ and $p \to 0$ in such a way that np approaches a non-zero constant λ. Then,

$$\binom{n}{k} p^k (1-p)^{n-k} \sim \frac{1}{k!} \left(\frac{np}{1-p} \right)^k (1-p)^n \to \frac{\lambda^k}{k!} e^{-\lambda}, \quad \text{for} \quad k = 0, 1, 2, \ldots$$

2.8.4 Negative binomial distribution

Let X_r be the waiting time for the rth success of Bernoulli trials of a random variable X_r, which takes values 1 and 0 with probabilities p and $q(= 1-p)$, respectively. Check that X_r has mass function

$$\mathbb{P}(X_r = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, ...;$$

it is said to have the *negative binomial distribution* with parameters r and p. The random variable X_r is the sum of r independent geometric variables. Note that if $r = 1$, it becomes the geometric distribution.

The negative binomial distribution is a discrete probability distribution of the number of successes in a sequence of independent and identically distributed *Bernoulli trials* before a specified (non-random) number of failures r occurs, in which the probability of success is the same every time the experiment is conducted.

2.8.5 Continuous uniform distribution

A random variable X is uniform on $[a, b]$ if it has distribution function

$$F(x) = \begin{cases} 0 & \text{if } x \leq a, \\ \dfrac{x-a}{b-a} & \text{if } a < x \leq b, \\ 1 & \text{if } x > b. \end{cases}$$

Roughly speaking, X takes any value between a and b with equal probability. The *continuous uniform distribution* or *rectangular distribution* is a family of symmetric probability distributions such that all members of the family are equally probable.

2.8.6 Exponential distribution

A random variable X is *exponential* with parameter $\lambda > 0$ if it has distribution function

$$F(x) = 1 - e^{-\lambda x}, \qquad x \geq 0.$$

The exponential distribution is the probability distribution that describes the time between events in a *Poisson point process*. The Poisson point process is a type of random mathematical object that consists of points randomly located on a mathematical space. The Poisson point process is often defnned on the real line, where it can be considered as a stochastic process. This distribution proves to be the cornerstone of the theory of *Markov processes* in continuous time.

2.8.7 Normal distribution

The most important continuous distribution is the *normal* (or *Gaussian*) distribution, which has two parameters μ and σ^2 and density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty.$$

The normal distribution arises in many ways. In particular it can be obtained as a continuous limit of the binomial distribution as $n \to \infty$.

Insurance companies use normal distributions to model certain average cases.

2.8.8 Log-normal distribution

A log-normal distribution is a statistical distribution of logarithmic values from a related normal distribution, i.e. for a variable x, $y = \ln(x)$ is normal distributed,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), \quad x > 0.$$

2.8.9 Gamma distribution

The random variable X has the *gamma* distribution with parameters $\lambda, t > 0$, if it has density

$$f(x) = \frac{1}{\Gamma(t)}\lambda^t x^{t-1} e^{-\lambda x}, \quad x \geq 0.$$

Here $\Gamma(t)$ is the *gamma function*

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

The gamma distribution is a two-parameter family of continuous probability distributions. The exponential distribution can be considered as a special case of the gamma distribution.

2.8.10 Inverse Gaussian distribution

The inverse Gaussian distribution (also known as *Wald distribution*) is a two-parameter family of continuous probability distributions with density similar to that of gamma distribution, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2 x^3}} \exp\left\{ -\frac{1}{2x} \left(\frac{x-\mu}{\sigma\mu} \right)^2 \right\}, \quad x > 0.$$

The inverse Gaussian is used in situations of extreme skewness.

2.8.11 Generalized Inverse Gaussian distribution

The generalized inverse Gaussian distribution (GIG) (also known as *Sichel distribution*) is a three-parameter family of continuous probability distributions with probability density function

$$f(x) = \frac{(\psi/\chi)^{(\lambda/2)}}{2K_\lambda(\sqrt{\chi\psi})} x^{(\lambda-1)} e^{-(\psi\chi+\chi/x)/2}, \quad x > 0.$$

where $K_\lambda(\cdot)$ stands for the modified Bessel function of the third kind, $\psi > 0$, $\chi > 0$ and λ is a real parameter (Embrechts, 1983). It is used in ruin problems, geostatistics, statistical linguistics, finance, etc. The inverse Gaussian distribution is a special case of the generalized inverse Gaussian distribution for $\lambda = -\frac{1}{2}$.

2.8.12 Rayleigh distribution

The Rayleigh distribution is considered to describe laser speckle patterns and speckle patterns in SAR images. In these cases, an echo is considered to be a sum of a large number of small amplitudes of coherent and incoherent sine waves which can be considered as independent, identically distributed (i.i.d.) random variables. By the superposition principle, the result is also a sine wave whose amplitude is the length of a two dimensional vector of two identical, zero mean

Gaussian random variables. If these two components are uncorrelated (or independent), then the distribution of the received envelope A of a signal is a Rayleigh distribution.

Histogram of intensity values of tissues with high cell concentration per resolution cell such as liver and blood, can be fitted by Rayleigh distribution. This situation can be understood by assuming that the returned diffuse signal is a linear summation of many small amplitude incoherent sine waves, or in the other word, the returned diffuse signal is a random walk process of many small amplitude, single frequency sine waves whose phases are uniform random variables, i.e.,

$$\mathcal{E}_N(t) = \sum k = 1N\mathcal{E}_k \cos(\omega t + \theta_k). \tag{2.5}$$

In the ultrasound imaging process, it is assumed that the superposition of received echo ultrasound is a linear summation. The envelope detection process in B-scanning is a nonlinear step which yields essentially the magnitude of the complex detected voltage. Wagner et. al. (1983) showed that the Rayleigh distribution with PDF

$$P_{Ra}(x) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}, \tag{2.6}$$

where $x > 0$ and parameter σ^2 is a variance, governs the fist-order behavior of the magnitude.

2.8.13 Rician distribution

Specular reflection consists of sound interactions with smooth tissue interfaces where the surface features are much larger than the wavelength of ultrasound. The ultrasound beam must strike the interface at a perpendicular angle to generate these reflections. These specular echoes originate from high acoustic impedance difference at boundaries such as fluid-tissue or bone-tissue interfaces, and produce a bright interface in the image. Thus shape geometry of organ in ultrasound image is generated by specular echoes.

Assume that specular echo has a large amplitude (when compared to the diffuse scattering echo) sine wave. Consider the sum of a specular echo and diffuse echoes, the resulting amplitude is still considered as the length of a two dimensional vector of two Gaussian random variables. However, only one of them has mean zero. If the two components are independent, then distribution of amplitude x is a Rician distribution,

$$P(x;x_0,\sigma) = \frac{x}{\sigma^2} e^{-(x_0^2+x^2)/2\sigma^2} I_0\left(\frac{x_0 A}{\sigma^2}\right),$$

where $x \geq 0$, $x_0 > 0$ represent the amplitude of the specular echo, σ is a parameter, and I_0 is the modified Bessel functions of the first kind of zero order defined by

$$I_0(s) = \frac{1}{\pi}\int_0^{\pi} e^{s\cos\theta}\, d\theta. \tag{2.7}$$

2.8.14 K-distribution

Jakeman (1980) considered noise by the concept of a random walk. Then the resultant of the received ultrasound signal at the transducer face is possible to be non-Gaussian. He proposed that the modified Bessel function or K-distribution with PDF

$$P(x;\alpha,b) = 2\left(\frac{x}{2}\right)^{\alpha} \frac{b^{\alpha+1}}{\Gamma(\alpha)} K_{\alpha-1}(bx), \tag{2.8}$$

where $x > 0$ and $\Gamma(\cdot)$ is the gamma function, α and $b = \sqrt{\dfrac{4\alpha}{E\{x^2\}}}$ are parameters, is more appropriate to model the amplitude statistics of the scattered radiation in medical ultrasound images.

2.8.15 Homodyned-K distribution

The random variable $x \geq 0$ has the Homodyned-K distribution with parameters $\alpha, b > 0$, if it has p.d.f.

$$P(x;\alpha,\sigma,s) = x\int_0^\infty tJ_0(xt)J_0(st)\left(1+\frac{t^2\sigma^2}{2\alpha}\right)^{-\alpha} dt, \tag{2.9}$$

where $J_0(\cdot)$ is the zero-th order Bessel function of the first kind and $\alpha > 0$.

2.8.16 Nakagami distribution

Koundal, Gupta and Singh (2015) mentioned that the Nakagami model is more general than the Rayleigh distribution for statistical modeling of speckle in ultrasound images. They presented the Nakagami-based noise removal method, which is more efficient to enhance thyroid ultrasound images and to improve clinical diagnosis. The statistics of log-compressed medical ultrasound images in that research are derived from the Nakagami distribution. The PDF of the Nakagami distribution is

$$P(x;m,\sigma^2) = \frac{2m^m}{\Gamma(m)(2\sigma^2)^m}x^{2m-1}e^{-mx^2/(2\sigma^2)}, \tag{2.10}$$

where $x > 0$, $\Gamma(\cdot)$ is the gamma function, $m \geq 0.5$ is the Nakagami shape parameter defined by $\dfrac{E\{x^2\}}{\mathrm{Var}\{x^2\}}$ and $2\sigma^2$ is the scaling parameter defined by $E\{x^2\}$.

One can see that Rayleigh distribution is the limiting distribution of K-distribution and it is a specific case of the Nakagami distribution when $m = 1$.

2.8.17 Compound probability distribution

The distribution H, which assumes that a random variable X is distributed according to some parametrized distribution F with an unknown parameter θ that is again distributed according to some other distribution G, is known as a *mixed, mixture* or *compound* distribution.(Lovric, 2011)

The resulting distribution H is said to be the distribution that results from compounding F with G, which means the distribution F is compounded by the distribution G and the parameter's distribution G is known as the compounding (mixing or latent) distribution. Technically, the unconditional distribution H results from marginalizing over G, i.e., from integrating out the unknown parameter(s) θ. Its probability density function is given by:

$$P_H(x) = \int P_F(x|\theta)P_G(\theta)d\theta$$

2.9 Application of probability distribution to ultrasound imaging

This section proposes the relation between the ultrasound image and statistical distributions.

2.9.1 B-Mode imaging

An ultrasound transducer converts electrical energy into ultrasound energy and vice versa by arrays of piezoelectric crystals. An image of soft tissue in a body can be produced by transforming the ultrasound echo signals received to electricity voltage and processing the detected voltage to the brightness at each point of the image defined by their x- and y-coordinates. To be precise, the gray-scale intensity of each pixel of a B-mode or brightness mode image is obtained from the envelope of the received voltage. The envelope of the signal can be derived by applying the Hilbert transform to the signal.

In practice, not only the pure echo signals from the surface pass through the piezoelectric receiver but there is signal contamination as well. The diffusion scattering of ultrasound signals can be modeled by some probability distribution.

The mathematics and statistical models related with the diffusion scattering of ultrasound signals are presented in the following content.

2.9.2 Ultrasound scattering classes of biologic system

Note that contents in this section follows Greenleaf and Sehgal (1992).

The concentration or size of scattering centers relative to the resolution ce1l of the imaging system is used to develop a hierarchy of scattering classes that correlates with a hierarchy of biologic classes.

Class 0 scattering is associated with macromolecules (size: 10^4 -10^5 Å) such as proteins and lipids, the most common molecule in the body. Class 0 scattering causes absorption of sound in various degrees and variations in propagation speed. Absorption of sound is related closely to the concentration of proteins in water. This type of scattering is due to macromolecular effects, which produce absorption and sound speed dispersion. (Szabo, 2014)

Class 1 scattering, or diffusion scattering, is associated with cells, depending on their concentration. Class 1 scattering occurs when the concentration of scatterers per resolution cell is high (25 or higher). This occurs in tissues such as blood or liver and results in speckle, the fine-grained noise familiar from laser light.

Class 2 scattering, or coherent/specular scattering, is associated with tissues in which the structural architecture (connective tissue) or other components such as lipids are scattered throughout the tissue in concentrations lower than 1 per resolution cell. These elements scatter independently and cause scattering distinguishable from speckle produced by the cell components themselves.

Class 3 scattering is caused by the borders of the organs and vessels. Class 3 scattering cause specular or mirror-like reflection.

Class 4 scattering is caused by motion that produces a Doppler shift which is associated with ultrasonic signals scattering from interfaces within blood, heart, lung, and gut which move to accomplish their function. This class is not considered in B-mode imaging.

In this thesis, we will scope on class 1 scattering only.

2.9.3 Some probability distribution functions of ultrasound echo envelope

Note that contents in this section follow Destrempes and Cloutier (2010).

Random walks are stochastic processes formed by successive summation of independent, identically distributed random variables (Lawler and Limic, 2010). An n-dimensional random walk is an object moving in the Euclidean space of dimension n by discrete independent random steps according to a specific probability distribution. The accumulation of the random scattering can be modeled by a random walk of component phasors.

The square of the amplitude of a random walk, which is the distance of the moving object to the origin, corresponds to the the intensity of the received signal from scatterers.

For two dimensional ultrasound imaging, the dimension of the corresponding random walk is $n = 2$. In the case of no compression of filters, the amplitude corresponds to the gray level of the B-mode image. The mean intensity according to the intensity distribution is considered as the signal intensity averaged over space.

Two examples of relation between probability distribution and random walk are given as follows.

If an n-dimensional random walk is defined as

$$\mathbf{A} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathbf{a}_j, \tag{2.11}$$

where N is the number of steps and $\mathbf{a}_j$ is an independent random vector representing independent phase and amplitude, for $j = 1, \ldots, N$.

Theorem 2.4 (Central Limit)**.** *Let N tend to infinity in the random walk of equation (2.11). The distribution of the amplitude of the resulting random walk is a Rayleigh distribution with parameter*

$$\sigma^2 = \frac{\overline{a^2}}{n},$$

where $\overline{a^2}$ is the mean intensity of the random step $\mathbf{a}_j$ before scaling by factor $\dfrac{1}{\sqrt{N}}$.

In the case that an n-dimensional random walk is defined as

$$\mathbf{A} = \overrightarrow{\varepsilon} + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathbf{a}_j, \tag{2.12}$$

where $\overrightarrow{\varepsilon}$ is a constant vector.

Theorem 2.5 (Central Limit)**.** *Let N tend to infinity in the random walk of equation (2.12). The distribution of the amplitude of the resulting random walk is a Rician distribution with parameter*

$$\varepsilon = \|\overrightarrow{\varepsilon}\|, \quad \sigma^2 = \frac{\overline{a^2}}{n},$$

where $\overline{a^2}$ is the mean intensity of the random step $\mathbf{a}_j$ before scaling by factor $\dfrac{1}{\sqrt{N}}$ and ε represents coherent signal component.

2.10 Software related with statistical distributions

There are many softwares for helping in manipulation with statistical distributions, e.g. RStudio, SPSS, Microsoft Excel. However, for the software code development, the programming language Python is very well-known in the present. World-Class software companies employ Python for working in many projects (Reynolds, ND). Python was first developed in the late 1980s. After decades of development, Python is now version 3.8 which is available in mathematics, statistics, science and data science works. It has many libraries which can be used in the Android platform, for websites, hardware interface, image processing, sound processing, artificial intelligence (AI), machine learning, network and database (Chityala and Pudipeddi, 2014). In this thesis, the Python software code was developed for implementing the analysis of statistical distributions appearing in medical ultrasound images.

2.11 Related researches

Seekot and Tanthanuch (2008) and Tanthanuch, Seekot, and Schulz (2012) considered speckle noise in ultrasound image to be explained by Rayleigh distribution. They considered the noise model as an additive one. Calculus of variational approach was used to reduce noise in ultrasound image.

Tanthanuch, Kaptsov and Meleshko (2019) used group classification and conservation laws in cylindrical coordinates to classify ultrasound Rayleigh noise reduction model.

Jensakda and Tanthanuch (2018) considered ultrasound noise reduction model via Rayleigh distribution, K-distribution and Nakagami distribution. The comparison of noise reduction performances were presented

Destrempes and Cloutier (2010) presented an overview of the statistical distributions based on their compound representation. The development of applying the probability distributions to describe the ultrasound scatterers distributed were discussed as

- Wagner et al. (1983, quoted in Destrempes and Cloutier, 2010) claimed that Rayleigh distribution corresponds to the distribution of the gray level in an unfiltered B-mode image.

- Insana et al. (1986, quoted in Destrempes and Cloutier, 2010) showed that the Rician distribution corresponds to a high density of random scatterers but combined with the presence of a coherent signal component of power ε^2. The Rayleigh distribution is the special case of the Rician distribution, where $\varepsilon = 0$.

- Shankar et al. (1993, quoted in Destrempes and Cloutier, 2010) introduced K-distribution which corresponds to a variable density α of random scatterers, with no coherent signal component.

- Jakeman (1980, quoted in Destrempes and Cloutier, 2010) and Jakeman and Tough (1987, quoted in Destrempes and Cloutier, 2010) studied the homodyned K-distribution. The homodyned K-distribution corresponds to the general case of a variable effective density of random scatterers with or without a coherent signal component The K-distribution is a special case of the homodyned K-distribution, and the Rayleigh and the Rice distributions are limiting cases of the two mentioned distributions

- Barakat (1986, quoted in Destrempes and Cloutier, 2010) developed works of Jakeman and Tough (1987, quoted in Destrempes and Cloutier, 2010). The work was equivalent to modulate both the coherent signal component ε and the diffuse signal power $2\sigma^2$ of the Rician distribution by a gamma distribution. This gave rise to the generalized K-distribution.

- Eltoft (2005, quoted in Destrempes and Cloutier, 2010) introduced the Rician inverse Gaussian distribution. It corresponded to a model in which both the coherent signal component ε and the diffuse signal power $2\sigma^2$ of a Rician distribution are modulated by an inverse Gaussian distribution, instead of a gamma distribution.

- Shankar (2000, quoted in Destrempes and Cloutier, 2010) referred the Nakagami distribution in modeling the gray level of the speckle pattern in a B-mode image. The Nakagami distribution was first introduced by Nakagami in 1943, which is a two-parameter distribution.

- Shankar (2003, quoted in Destrempes and Cloutier, 2010) also proposed the Nakagami-gamma distribution, which can be viewed as the marginal distribution of a model in which the Rician distribution is approximated by a Nakagami distribution, and in which its total signal power $\Omega = \varepsilon^2 + 2\sigma^2$ is modulated by a gamma distribution.

- Agrawal and Karmeshu (2006, quoted in Destrempes and Cloutier, 2010) introduced the Nakagami-generalized inverse Gaussian distribution, which corresponds to a model in which the approximating Nakagami distribution has its total signal power Ω modulated by a generalized inverse Gaussian distribution instead of a gamma distribution.

Cai (2016) characterized the Medical Ultrasound Echo Signals with the ascending order K distribution which manifests the feature that the greatest amount of signal appears at the original point in the distribution function.

CHAPTER III

RESEARCH METHODOLOGY

This chapter presents the process used in this research. The process comprises of 3 parts as follows:

1. analysis of statistical distributions;
2. the development of the software code in statistical compound distribution functions;
3. the interpretation of the obtained results.

3.1 Analysis of statistical distributions

Here, by the literature review, it was found that most used parameters of the statistical distributions link with the statistical moments, i.e. raw moments (sometimes called *crude moments*), central moments (or moments about the mean) and standardized moments.

Definition 3.1 (moments)**.** Here the definitions of three types of moments are as follows:

1. The *raw moments* of a statistical distribution are defined as the expectation values of a random variable X.
2. For a continuous random variable, the *central moments* are defined by

$$\mu_n = \mathbb{E}\left[(X - \mathbb{E}(X))^n\right] = \int_{-\infty}^{\infty} (x-\mu)^n f(x)dx,$$

where $f(x)$ is the density function of X.

3. The *standardized moment* of degree k is $\frac{\sigma_k}{\sigma^k}$ where σ_k is the k-moment and σ is the standard deviation.

Table 3.1 Significance of moments.

Moment ordinal	**Moment**		
	Raw	**Central**	**Standardized**
1	Mean	0	0
2	-	Variance	1
3	-	-	Skewness
4	-	-	Kurtosis

By the context mentioned above, the software code for finding moments was considered to be developed.

3.2 The development of the software code in statistical compound distribution functions

For the Python software, there are some statistical libraries, e.g. scipy.stats. However, when applied scipy.stats.skewness and scipy.stats.kurtosis to our considered statistical distributions, it presented the wrong values. There were also some examples of error in using that commands in the internet (https://stackoverflow.com/questions/45483890/how-to-correctly-use-scipys-skew-and-kurtosis-functions), thus we developed the software code for finding the skewness and the kurtosis. The Python software code for finding the skewness and the kurtosis is presented in section A.2, Appendix A. Since the calculation of statistical distributions deals with the mathematics functions and integrations, library numpy and library scipy.integrate were used. For more details of the coding software code in this thesis, some libraries and commands are presented as follows.

3.2.1 Statistical distributions functions

In the scipy.stats (the Python statistical library), there are statistical distribution functions used in this thesis, i.e. Normal distribution (scipy.stats.norm), Rayleigh distribution (scipy.stats.rayleigh), Rician distribution (scipy.stats.rice), Nakagami distribution (scipy.stats.nakagami),Gamma distribution (scipy.stats.gamma), inverse Gaussian distribution (scipy.stats.invgauss) and generalized inverse Gaussian distribution (scipy.stats.geninvgauss). For some other statistical distribution and compound distribution functions used in this thesis, we thus needed to develop the software cood. The code was done for the following distributions:

- K-distribution;
- homodyned-K distribution;
- Generalized K-distribution;
- Rician inverse Gaussian distribution;
- Nakagami gamma distribution;
- Nakagami generalized inverse Gaussian distribution.

The Python software code for the statistical compound distribution functions is presented in A.1, Appendix A.

3.2.2 The development of the software code in statistical parameter analysis

For the statistical parameters *statistical moments*, *central moments*, *standardized moments*, *expected value*, *variance*, *skewness* and *kurtosis* of statistical distributions functions in section 3.2.1, the software code of the mentioned parameter was developed. The Python software code is shown in section A.1, Appendix A.

3.2.3 Graph display

Python has many libraries for plotting a graph. Here we employed `matplotlib.pyplot`. Some examples of using `matplotlib.pyplot` to plot the statistical distribution curves are presented in section A.2, Appendix A.

3.3 The interpretation of the obtained results

In this section, the comparison of the statistical distributions dealing with the medical ultrasound image is presented.

CHAPTER IV

RESULTS AND DISCUSSION

This chapter presents the results from the process proposed in Chapter III Research Methodology. The main goal in this section is to discuss the output from Python software developed in this thesis. For The code, is presented in the Appendix.

4.1 Analysis and graphs of the statistical distributions related with medical ultrasound image

Analysis and graphs of the statistical distributions related with medical ultrasound image obtained by Python are presented as follows. For the sake of simplicity, some notations of statistical distribution presented in Chapter II are changed according to the physical interpretation of ultrasound signal.

4.1.1 Rayleigh distribution

the Rayleigh distribution corresponds to the distribution of the gray level of each pixel in the image (or an amplitude of the ultrasound signal before transformed to the image) in the case of a high density of random scatterers with no coherent signal component (Wagner et al., 1983). The probability density function of the n-dimensional Rayleigh distribution is defined by

$$P_{Ra}\left(x|\sigma^2\right) = \frac{2}{\Gamma(n/2)}\left(\frac{1}{\sigma^2}\right)^{n/2} x^{n-1}\exp\left(-\frac{x^2}{2\sigma^2}\right),$$

where x is the amplitude of the signal, $\sigma > 0$ is a shape parameter and n is the dimension.

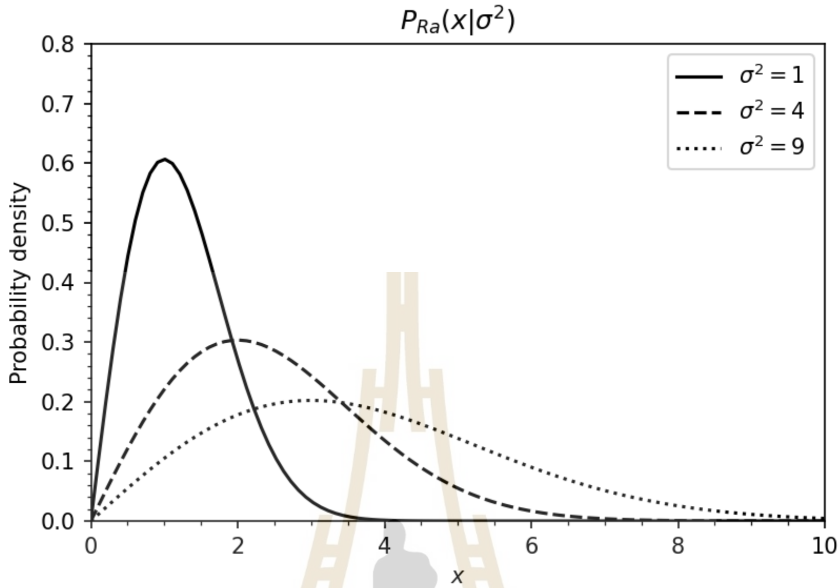

Figure 4.1 Graph of Rayleigh distribution with $n = 2$, variances $\sigma^2 = 1, 4$ and 9.

4.1.2 Rician distribution

Similar to the Rayleigh distribution, the Rician distribution also corresponds to the distribution of the amplitude. However, the Rician distribution has a parameter which can be used for describing an appearing coherent signal component. The probability density function of the n-dimensional Rician distribution is defined by

$$P_{Ri}\left(x|\varepsilon,\sigma^2\right) = \left(\frac{\varepsilon}{\sigma^2}\right)\left(\frac{x}{\varepsilon}\right)^{n/2} \mathsf{B}_{n/2-1}\left(\frac{\varepsilon}{\sigma^2}x\right)\exp\left(-\frac{\varepsilon^2+x^2}{2\sigma^2}\right),$$

where x is the amplitude of the signal, $\sigma > 0$ is the shape parameter, $\varepsilon \geq 0$ is a real parameter, n is the dimension and B_p is the modified Bessel function of the first kind of order p.

The particular case of Rician distribution, where $\varepsilon = 0$, becomes the Rayleigh distribution.

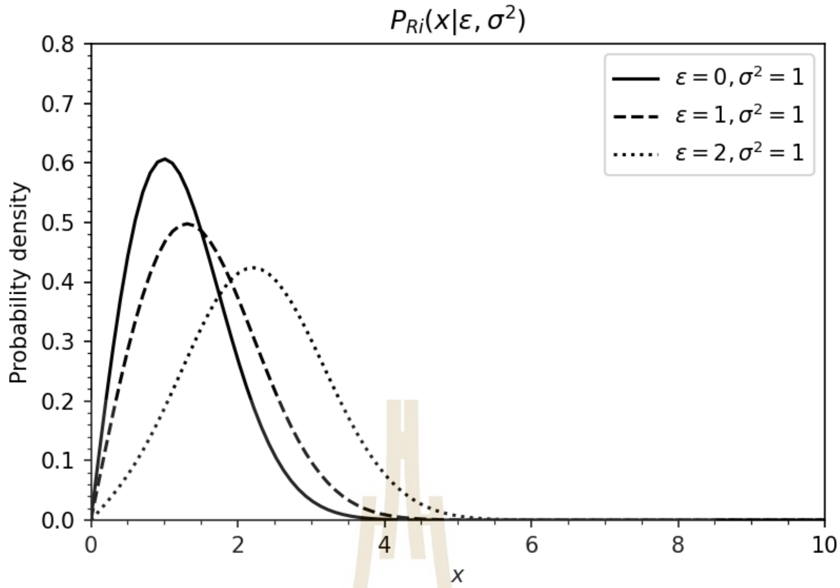

Figure 4.2 Graph of Rician distribution with $n = 2$, variances $\sigma^2 = 1$ and parameter $\varepsilon = 0, 1$ and 2.

4.1.3 K-distribution

Jakeman (1980) claimed that K-distribution corresponds to a variable density α of random scatterers, without coherent signal component. The K-distribution is defined by

$$P_K\left(x|\sigma^2,\alpha\right) = \frac{4x^{\alpha-1+n/2}}{\left(2\sigma^2\right)^{(\alpha+n/2)/2}\Gamma(\alpha)\Gamma(n/2)},$$

where x is the amplitude of the signal, $\alpha > 0, \sigma > 0$ are parameter, K_p is the modified Bessel function of the second kind of order p and Γ is a gamma function.

One may consider that the received echo ultrasound wave is the superposition of k arrival signals, which each signal reflections from the surface according to a Poisson process. The probability of the occurrence signal that we observe from now until time t can be approximated as the cumulative distribution function of the Poisson distribution, which implies the gamma distribution (Christou, ND).

Then the K-distribution can be represented in the compound form of Rayleigh distribution with modulated gamma distribution parameter:

$$P_k\left(x|\sigma^2,\alpha\right) = \int_0^\infty P_{Ra}\left(x|\sigma^2 w\right)\mathcal{G}\left(w|\alpha,1\right)dw,$$

where $\mathcal{G}$ is the gamma distribution.

The parameter α can be viewed as the scatterers per resolution cell multiplied by a coefficient depending on the scanning geometry and parameters, and the backscatter coefficient statistics (Destrempes and Cloutier, 2010).

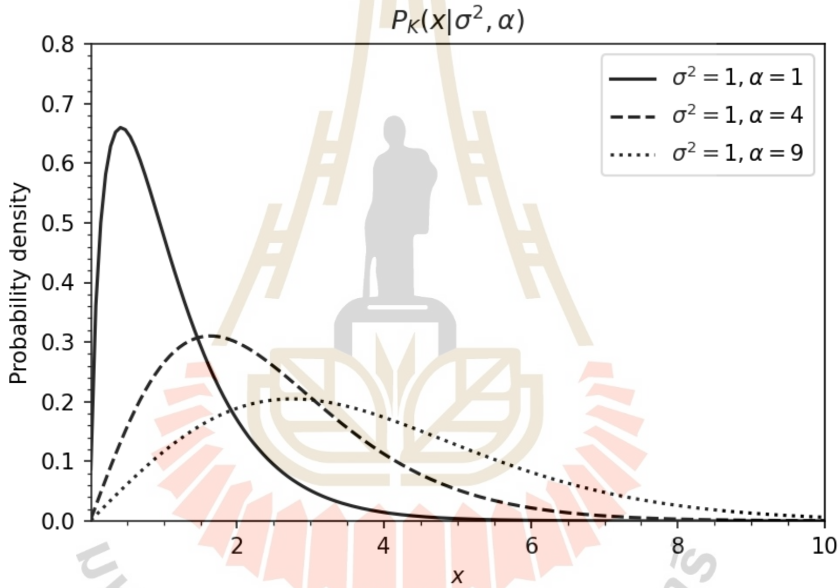

Figure 4.3 Graph of K-distribution with $n = 2$, variances $\sigma^2 = 1$ and parameter $\alpha = 0.1, 3.0$ and 9.0.

the Homodyned K-distribution corresponds to a variable density α of random scatterers similar to K-distribution but it has a parameter which is able to describe an appearing coherent signal component ε. the Homodyned K-distribution is defined by a Rician distribution with modulated gamma distribution parameter:

$$P_{HK}\left(x|\varepsilon,\sigma^2,\alpha\right) = \int_0^\infty P_{Ri}\left(x|\varepsilon,\sigma^2 w\right)\mathcal{G}\left(w|\alpha,1\right)dw.$$

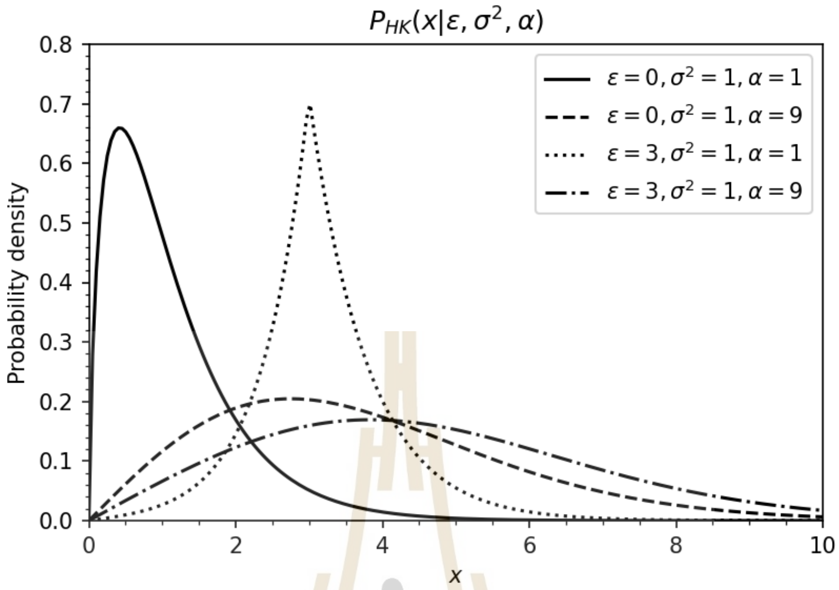

Figure 4.4 Graph of homodyned K-distribution with $n = 2$, variances $\sigma^2 = 1$, parameter $\varepsilon = 0$ and 1, and parameter $\alpha = 1$ and 9.

4.1.4 Generalized K-distribution

By modulating both parameters of the Rician distribution, the coherent signal component ε and the variance σ^2, one obtains the Generalized K-distribution,

$$P_{GK}\left(x|\varepsilon, \sigma^2, \alpha\right) = \int_0^\infty P_{Ri}\left(x|\varepsilon w, \sigma^2 w\right) \mathcal{G}\left(w|\alpha, 1\right) dw.$$

The generalized K-distribution can be represented as a Rician distribution with both the mean-square noise component and the coherent amplitude varying according to a gamma distribution.

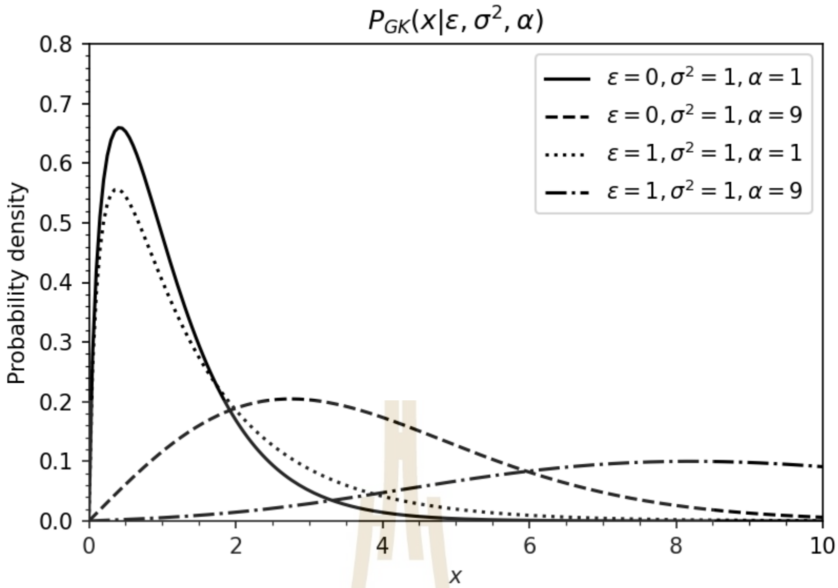

Figure 4.5 Graph of Generalized K-distribution with modulated gamma distribution parameters, with $n = 2$, coherent parameter $\varepsilon = 0$ and 1 and variance $\sigma^2 = 1$, and shape parameter $\alpha = 1$ and 9.

4.1.5 Rician distribution with modulated inverse Gaussian distribution parameters

The Gaussian or normal distribution describes a Brownian motion's level at a considering time, the inverse Gaussian describes the distribution of the time a Brownian motion with a fixed positive level (Folks and Chhikara, 1978). Therefore, the inverse Gaussian distribution is more appropriate to describe the random reflexive ultrasound signal phenomena than the Gaussian distribution. the Inverse Gaussian distribution is a distribution with 2 parameters, mean μ and shape parameter λ. In the case that $\mu = \sqrt{\lambda}$, the inverse Gaussian distribution has variance $\sqrt{\lambda}$ also. on the other hand, the gamma distribution which has shape parameter α and scale parameter equal to 1, its mean and variance are exact the same α. That is an intuitive idea to assume that gamma distribution can be generalized to inverse Gaussian distribution. The Rician distribution with modulated inverse Gaussian distribution parameters can be defined by

$$P_{RiIG}\left(x|\varepsilon,\sigma^2,\lambda\right) = \int_0^\infty P_{Ri}\left(x|\varepsilon w,\sigma^2 w\right)\mathcal{IG}\left(w|\sqrt{\lambda},\lambda\right)dw,$$

where $\mathcal{IG}$ is an inverse Gaussian distribution function. The distribution proposed is more general to describe the physical meaning of an ultrasound signal than homodyned K-distribution. Homodyned K-distribution has an only one modulated parameter, whereas this distribution has more modulated parameters, which are the coherent signal component ε and variance σ^2.

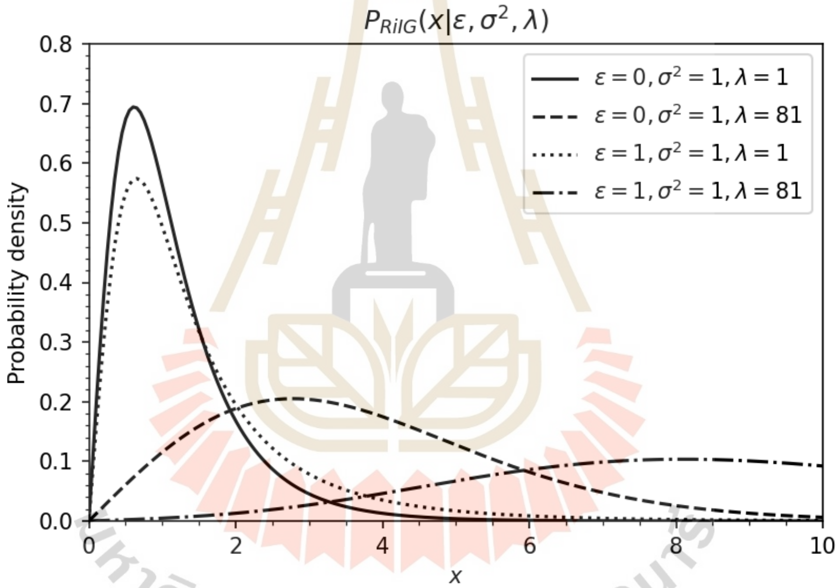

Figure 4.6 Graph of Rician distribution with modulated inverse Gaussian distribution parameters, with $n = 2$, variance $\sigma^2 = 1$ and 4, shape parameter $\alpha = \sqrt{\lambda}$, parameter $\varepsilon = 0$ and 1, and parameter $\lambda = 1$ and 9.

4.1.6 Nakagami distribution

the Nakagami distribution is defined by equation (2.10) (page 24). Destrempes and Cloutier (2010) show that the Nakagami distribution can be approximated to the Rayleigh distribution, Rician distribution and K-distribution, which dependiy on the parameters Ω and m. Thus the Nakagami distribution is claimed to be better in describing the ultrasound signal with coherent signal component ε, which has a high density of random scatterers. It was found that the total signal power Ω of the Nakagami distribution is equal to $\varepsilon^2 + 2\sigma^2$, where ϵ is the coherent signal component and σ^2 is the variance of the distribution.

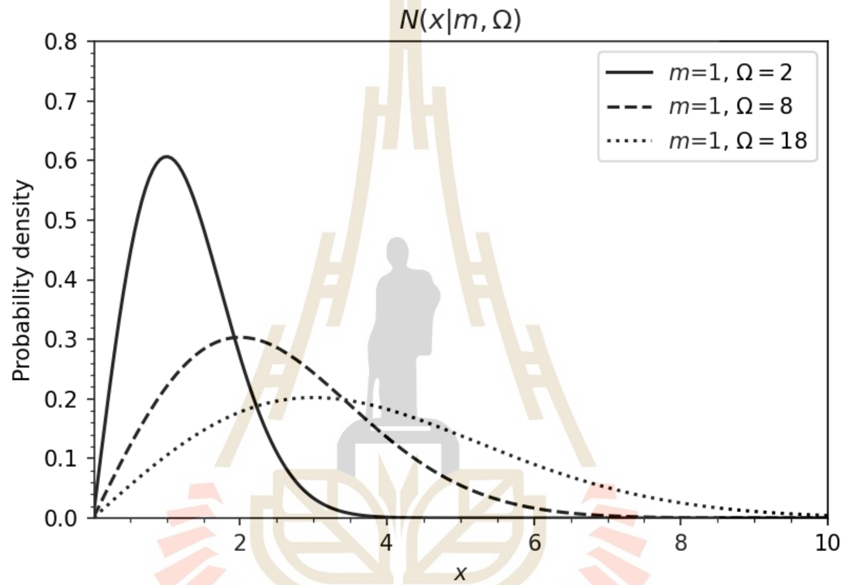

Figure 4.7 Graph of Nakagami distribution with parameter $\Omega = 0.5$ and parameter $m = 0.5, 1.0$ and 1.5.

4.1.7 Nakagami distribution with modulated gamma distribution parameter

In order to extend the concept using Nakagami distribution to be able to specify a density α of random scatterers, Nakagami distribution with modulated gamma distribution parameter was introduced (Shankar, 2003). The compound representation of the distribution is

$$P_{NG}\left(A|m,\Omega,\alpha\right)=\int_0^\infty N\left(A|m,\Omega w\right)\mathcal{G}\left(w|\alpha,1\right)dw,$$

where N is the Nakagami distribution and $\mathcal{G}\left(w|\alpha,1\right)$ is the gamma distribution with mean and variance α.

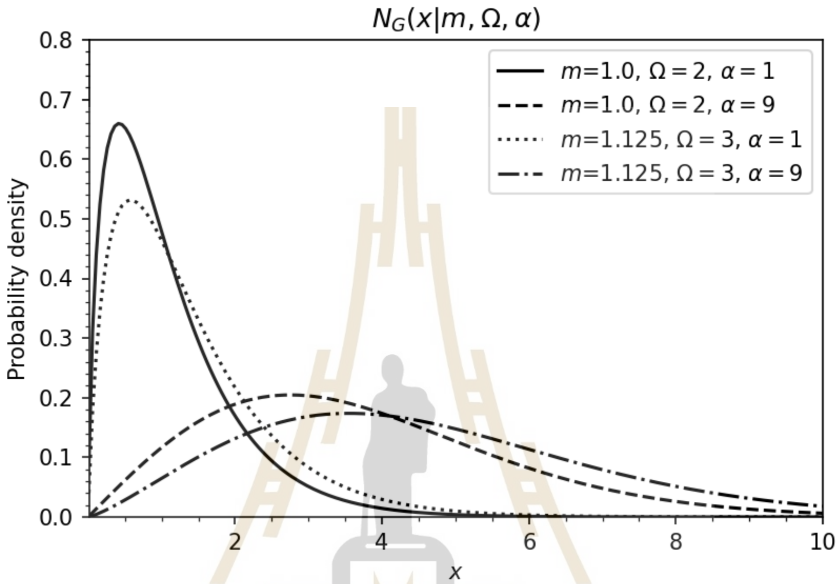

Figure 4.8 Graph of Nakagami distribution with modulated gamma distribution parameter, $m=1$, $\Omega=2$ and a density parameter $\alpha=1,2,3$.

4.1.8 Nakagami distribution with modulated generalized inverse Gaussian parameters

For the sake of the simplicity, the generalized inverse Gaussian distribution (GIG) is redefined in terms of 3 parameters θ, μ and λ as

$$GIG(w|\theta,\mu,\lambda)=\frac{1}{2\mu^\theta K_\theta(\lambda/\mu)}w^{\theta-1}\exp\left[-\frac{1}{2}\left(\frac{\lambda}{w}+\frac{\lambda}{\mu^2}w\right)\right],$$

where w is a random variable, K_θ is the modified Bessel function of second kind.

The Nakagami distribution with modulated generalized inverse Gaussian parameters is the distribution with 4 parameters, i.e. m, Ω, θ and λ, defined by

$$P_{NGIG}\left(x|m,\Omega,\theta,\lambda\right)=\int_0^\infty N\left(x|m,\Omega w\right)GIG\left(w|\theta,\sqrt{\lambda},\lambda\right)dw,$$

where N is Nakagami distribution and $GIG\left(w|\theta,\sqrt{\lambda},\lambda\right)$ is the gamma distribution with mean $\mu=\sqrt{\lambda}$.

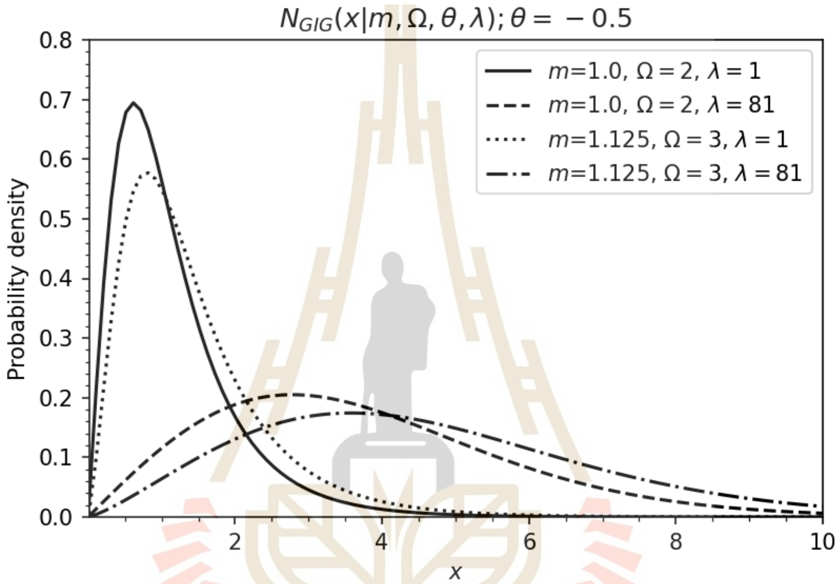

Figure 4.9 Graph of Nakagami distribution with modulated generalized inverse Gaussian parameters, $m=1$, $\Omega=2$ and a density parameter $\alpha=1,2,3$.

4.2 Parameters of the statistical distributions related with medical ultrasound image

Normally, mean, variance, skewness and kurtosis are main distribution parameters that we consider for each distribution. In this section, graphs of the relation between a number of parameters, σ, ε, α, λ, m or Ω and the distribution parameters mean, variance, skewness and kurtosis are presented.

4.2.1 Gaussian distribution

Gaussian or normal distribution is the very well-known and standard distribution. The graphs of parameters of the distribution are presented as follows:

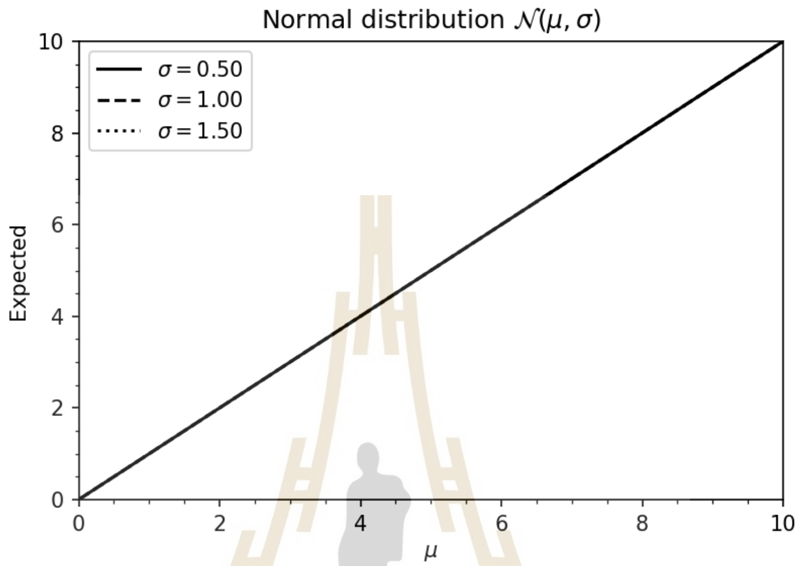

Figure 4.10 Graph of expectation value with standard deviation $\sigma = 0.5, 1, 1.5$ vary mean μ of the Gaussian distribution.

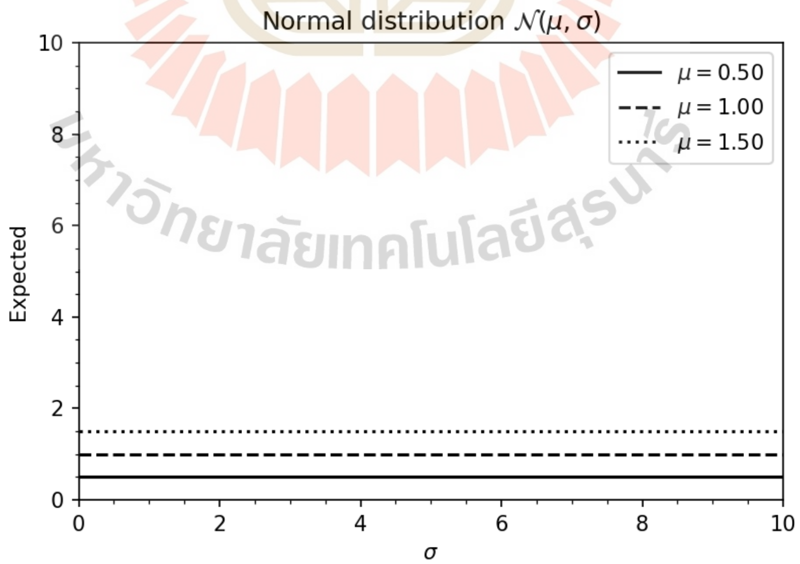

Figure 4.11 Graph of expectation value with mean $\mu = 0.5, 1, 1.5$ and vary standard deviation σ of the Gaussian distribution.

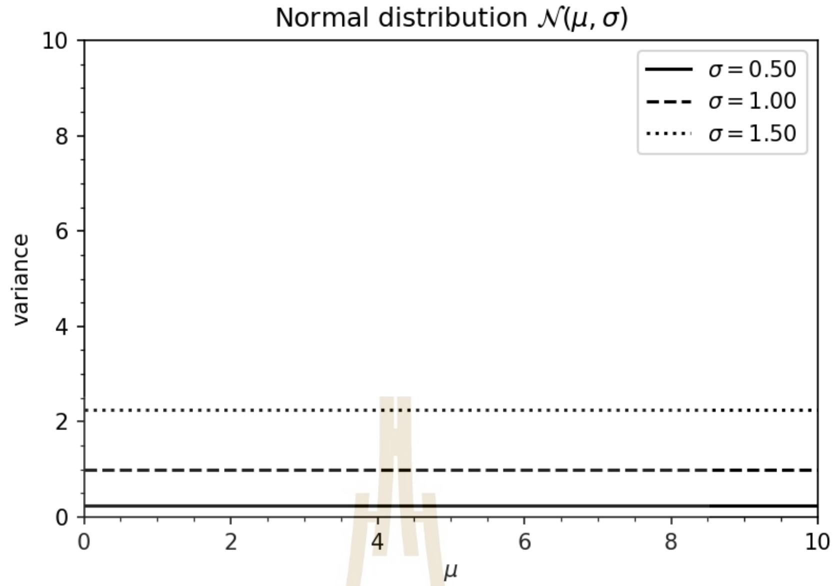

Figure 4.12 Graph of variance value with standard deviation $\sigma = 0.5, 1, 1.5$ and vary mean σ of the Gaussian distribution.

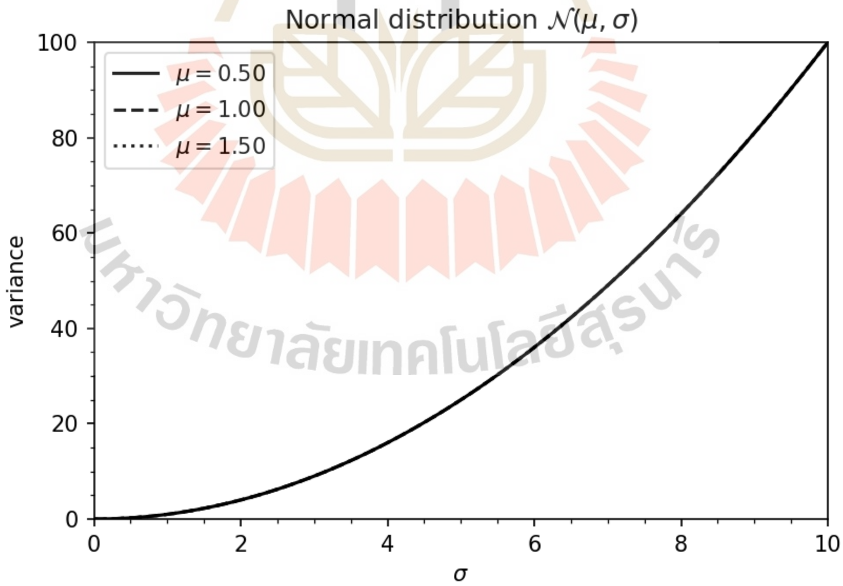

Figure 4.13 Graph of variance value with mean $\mu = 0.5, 1, 1.5$ and vary standard deviation σ of the Gaussian distribution.

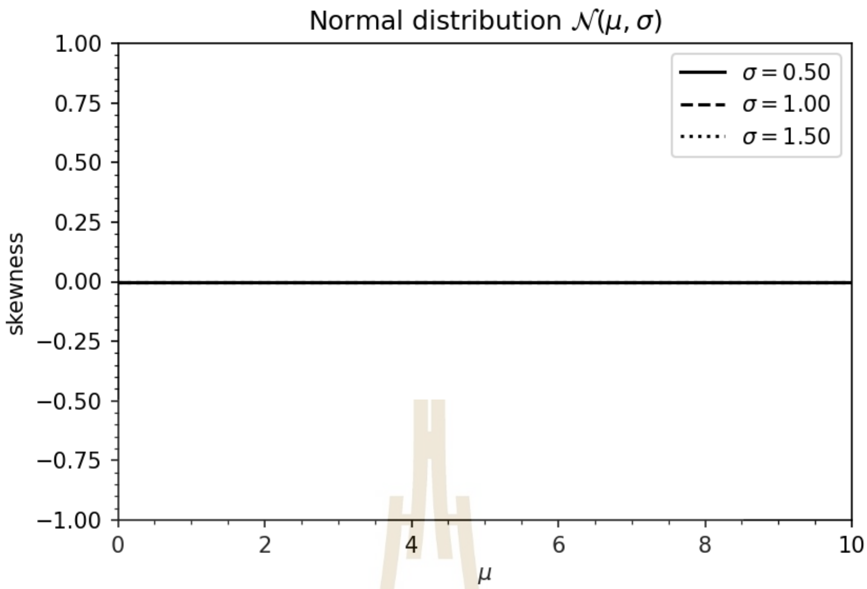

Figure 4.14 Graph of skewness value with standard deviation $\sigma = 0.5, 1, 1.5$ and vary mean σ of Gaussian distribution.

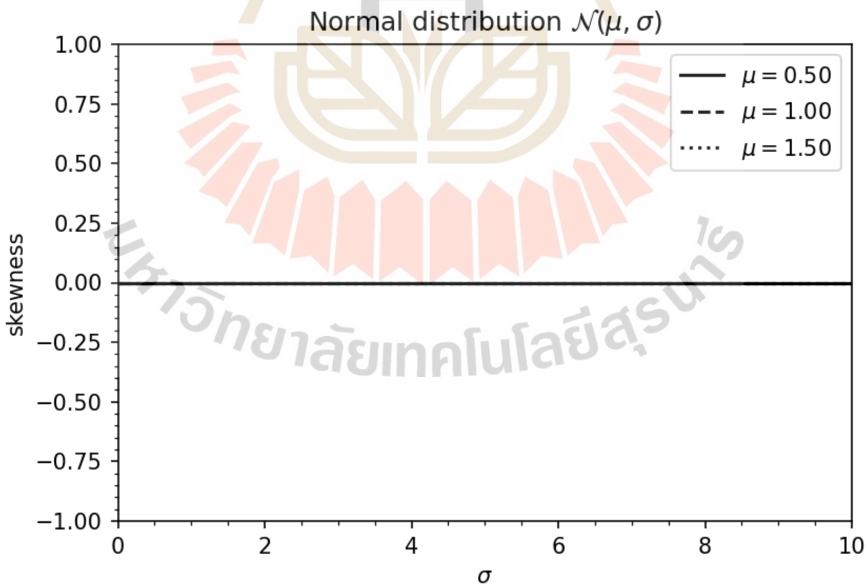

Figure 4.15 Graph of skewness value with mean $\mu = 0.5, 1, 1.5$ and vary standard deviation σ of Gaussian distribution.

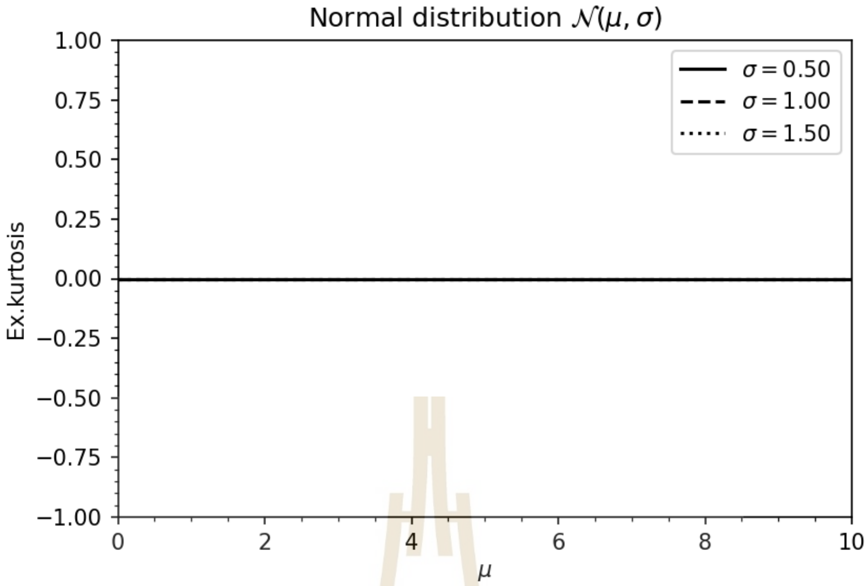

Figure 4.16 Graph of excess kurtosis value with standard deviation $\sigma = 0.5, 1, 1.5$ and vary mean σ of Gaussian distribution.

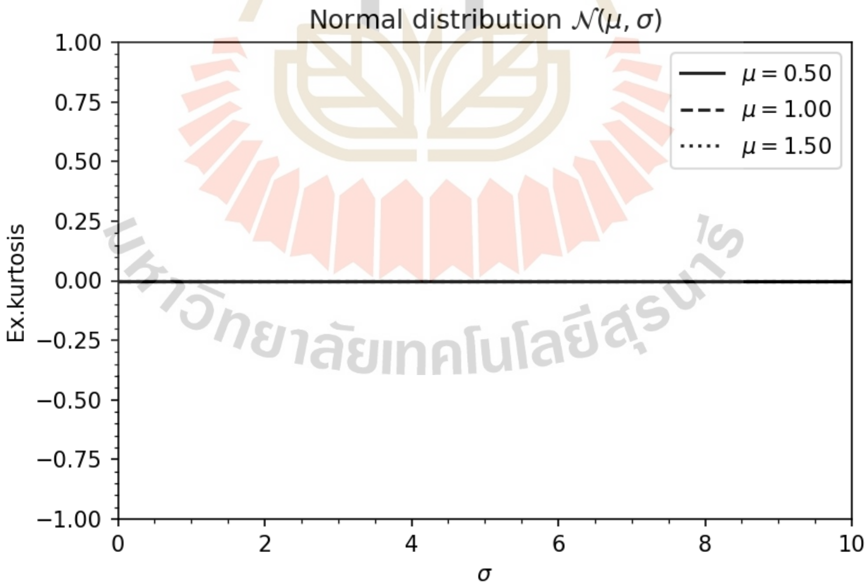

Figure 4.17 Graph of excess kurtosis value with mean $\mu = 0.5, 1, 1.5$ and vary standard deviation σ of Gaussian distribution.

4.2.2 Rayleigh distribution

The graphs of parameters of the Rayleigh distribution are presented as follows:

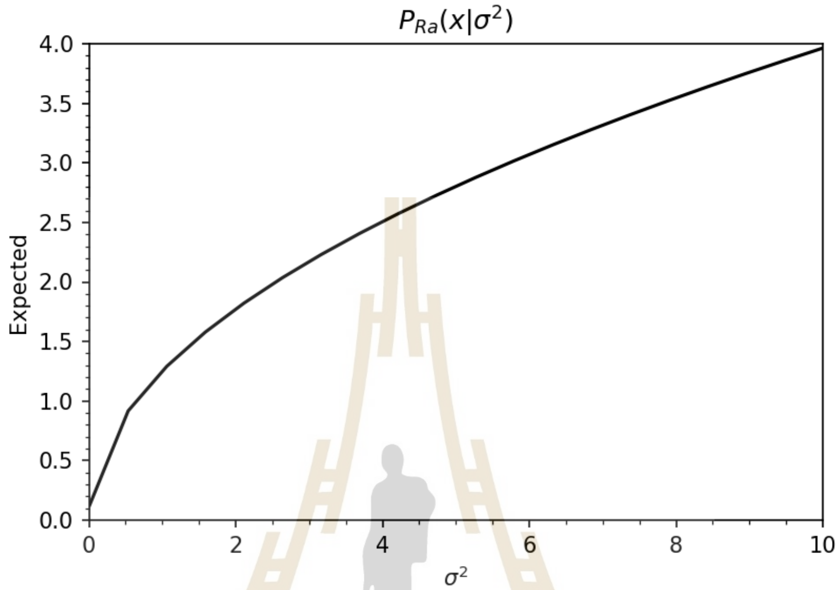

Figure 4.18 Graph of expectation value with varying variance σ^2 of Rayleigh distribution.

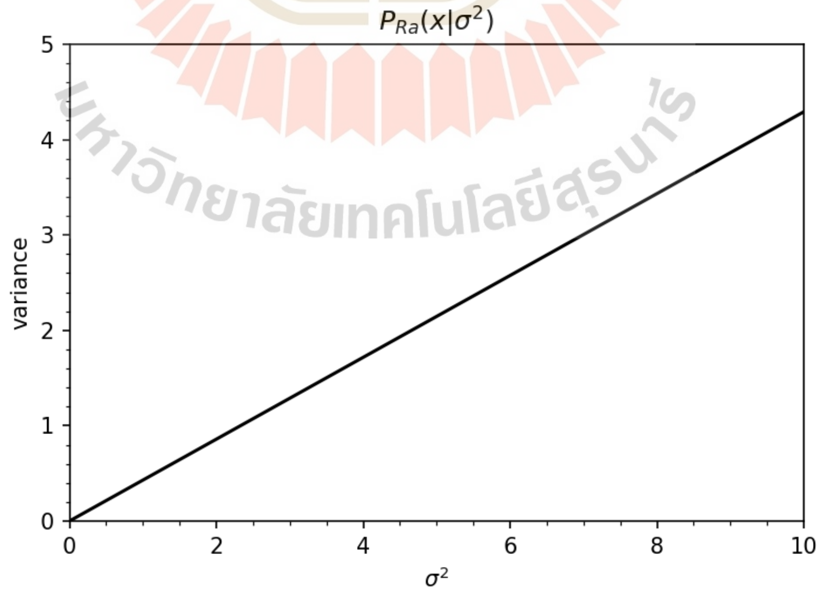

Figure 4.19 Graph of variance with varying variance σ^2 of Rayleigh distribution.

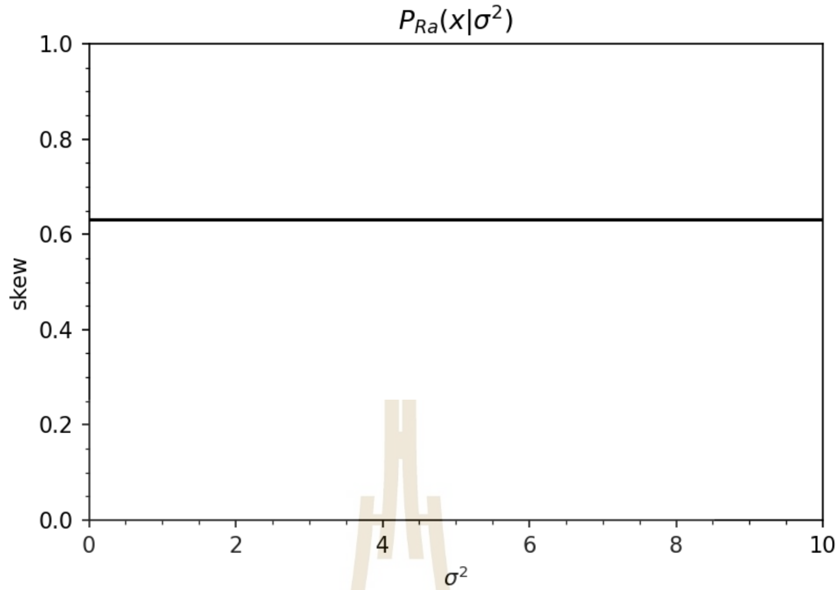

Figure 4.20 Graph of skewness with varying variance σ^2 of Rayleigh distribution.

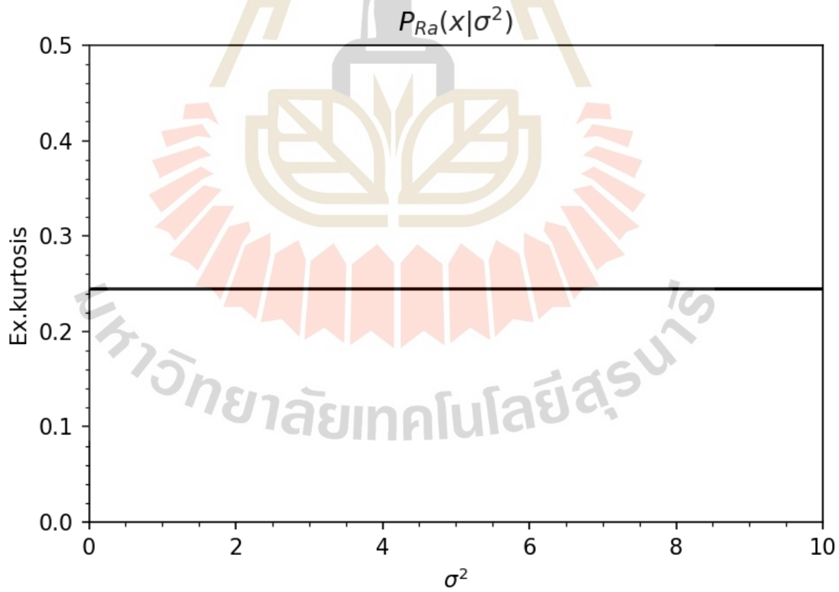

Figure 4.21 Graph of excess kurtosis with varying variance σ^2 of Rayleigh distribution.

4.2.3 Rician distribution

The graphs of parameters of Rician distribution are presented as follows:

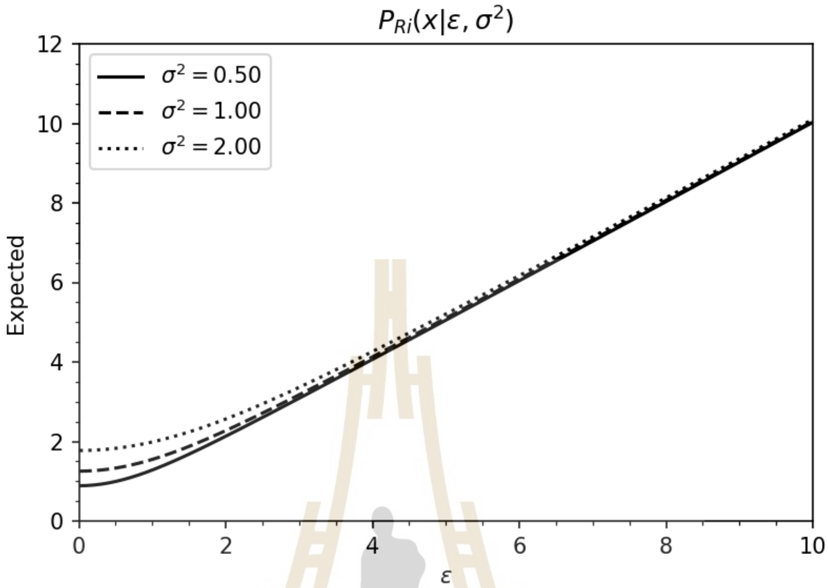

Figure 4.22 Graph of expectation value with $\sigma^2 = 0.5, 1, 2$ and varying ϵ of Rician distribution.

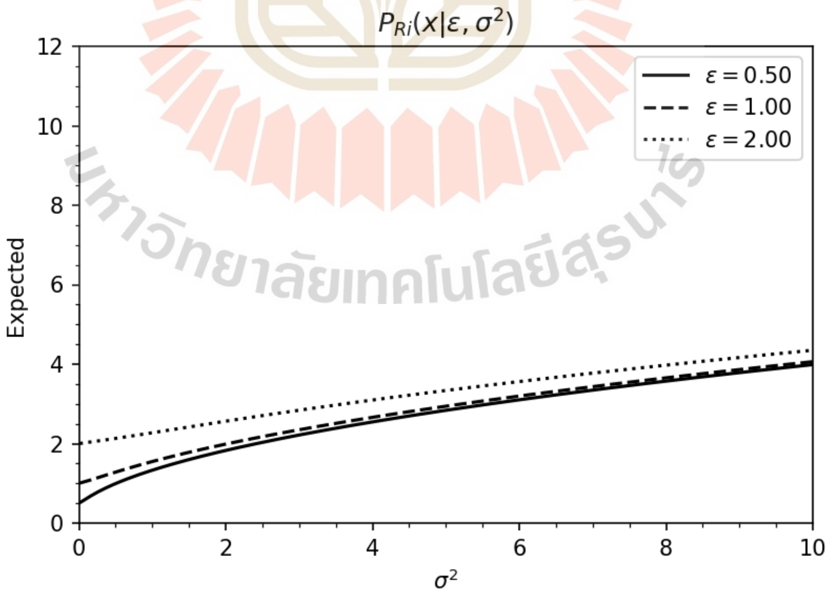

Figure 4.23 Graph of expectation value with $\epsilon = 0.5, 1, 2$ and varying σ^2 of Rician distribution.

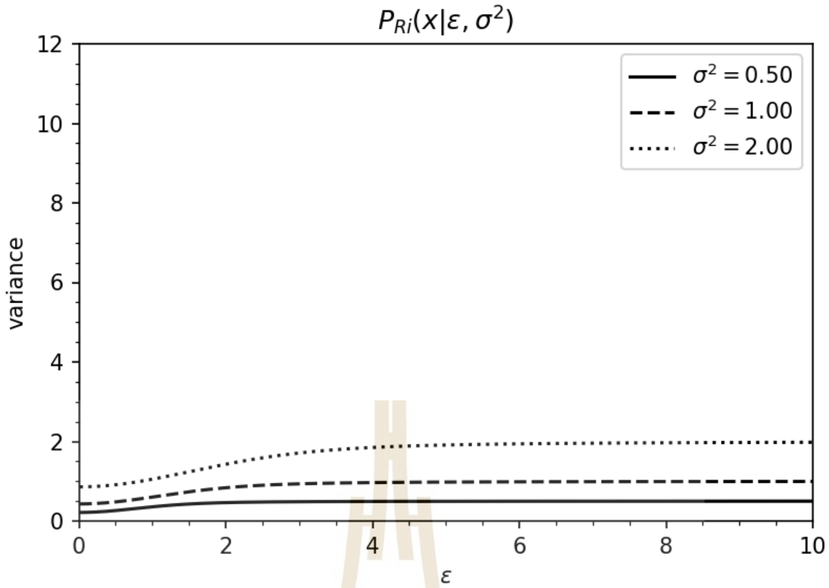

Figure 4.24 Graph of variance value with $\sigma^2 = 0.5, 1, 2$ and varying ϵ of Rician distribution.

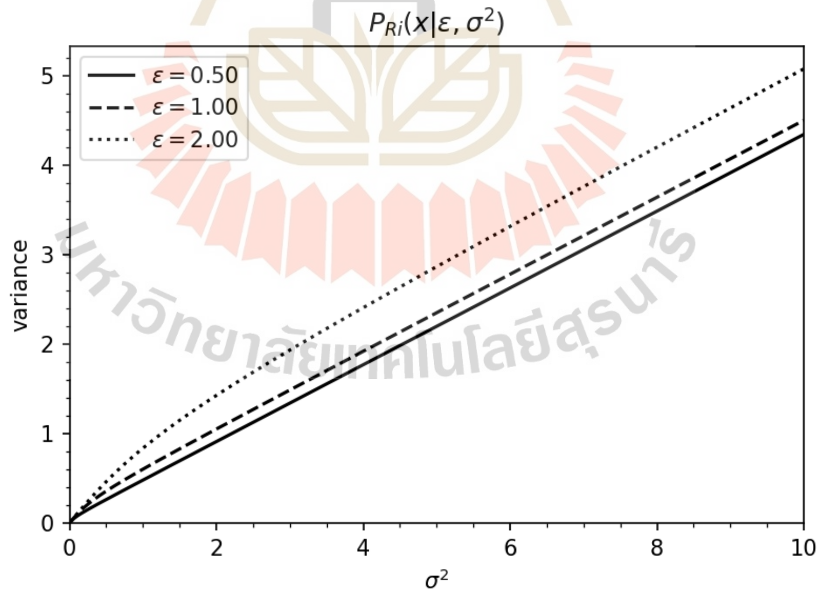

Figure 4.25 Graph of variance value with $\epsilon = 0.5, 1, 2$ and varying σ^2 of Rician distribution.

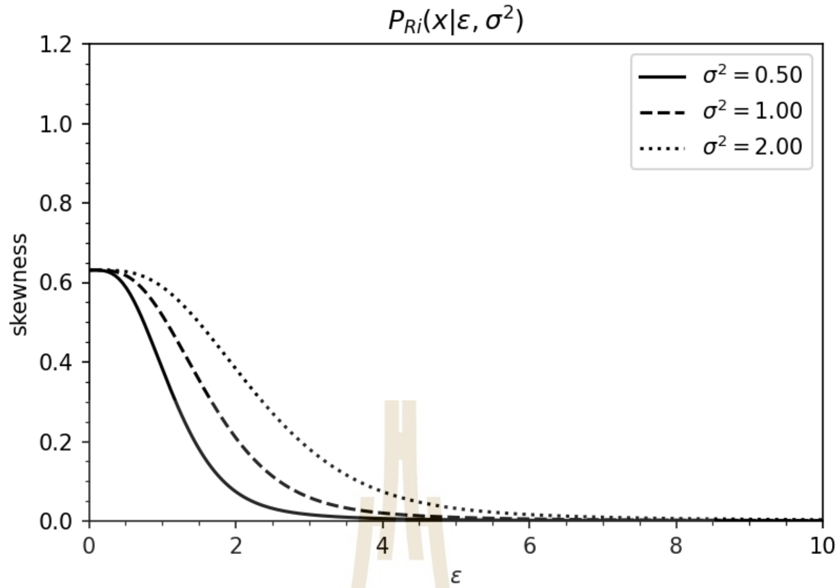

Figure 4.26 Graph of skewness value with $\sigma^2 = 0.5, 1, 2$ and varying ϵ of Rician distribution.

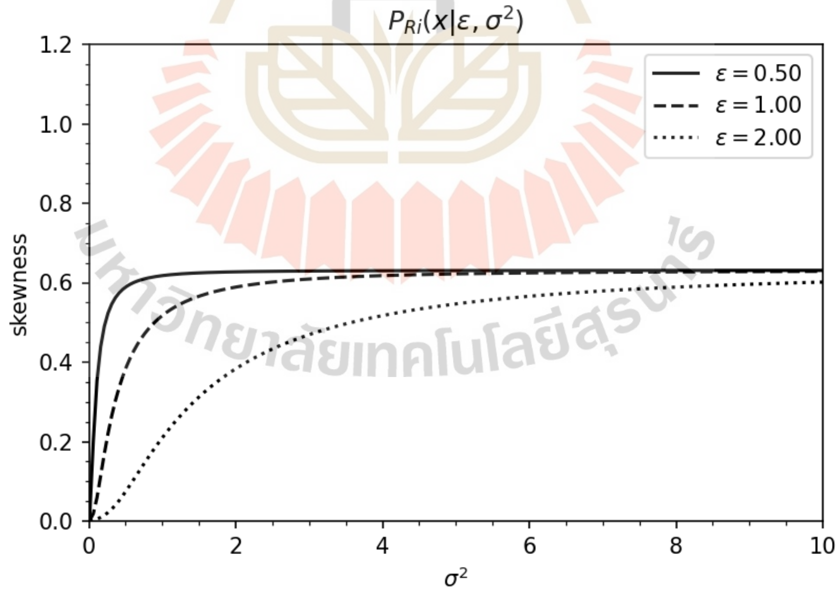

Figure 4.27 Graph of skewness value with $\epsilon = 0.5, 1, 2$ and varying σ^2 of Rician distribution.

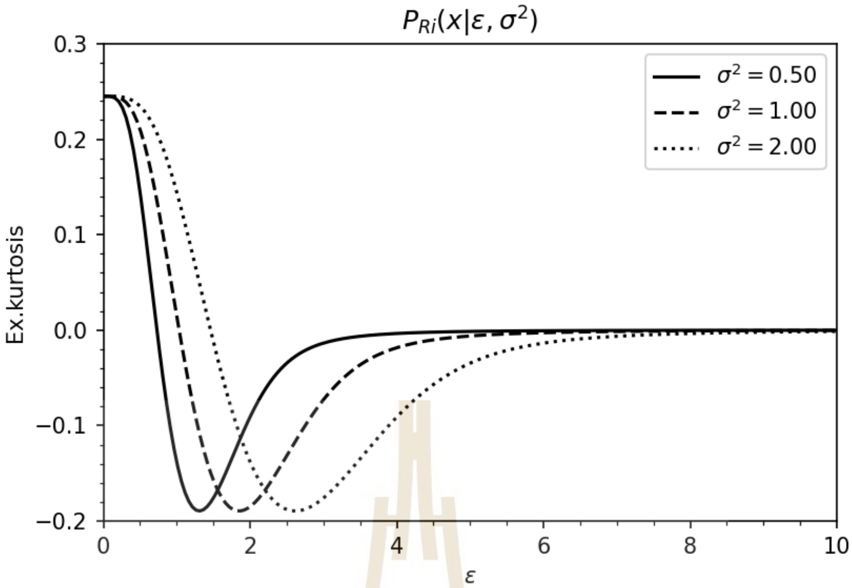

Figure 4.28 Graph of excess kurtosis value with $\sigma^2 = 0.5, 1, 2$ and varying ϵ of Rician distribution.

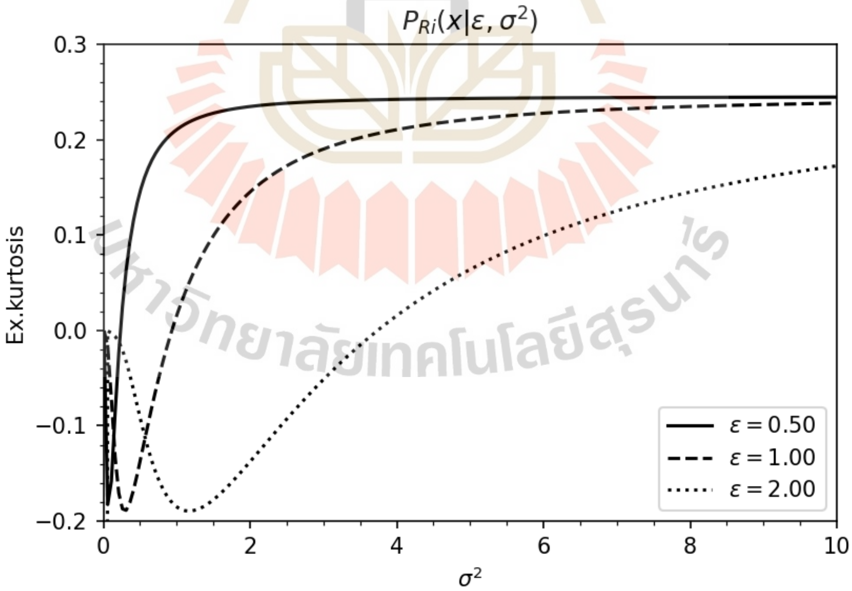

Figure 4.29 Graph of excess kurtosis value with $\epsilon = 0.5, 1, 2$ and varying σ^2 of Rician distribution.

4.2.4 K-distribution

The graphs of parameters of K-distribution are presented as follows:

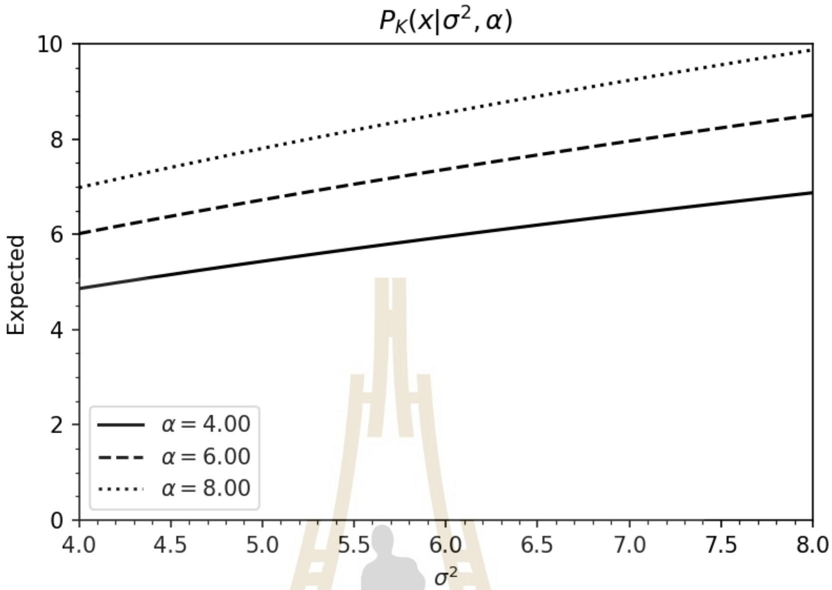

Figure 4.30 Graph of expectation value with $\alpha = 4, 6, 8$ and varying σ^2 of K-distribution.

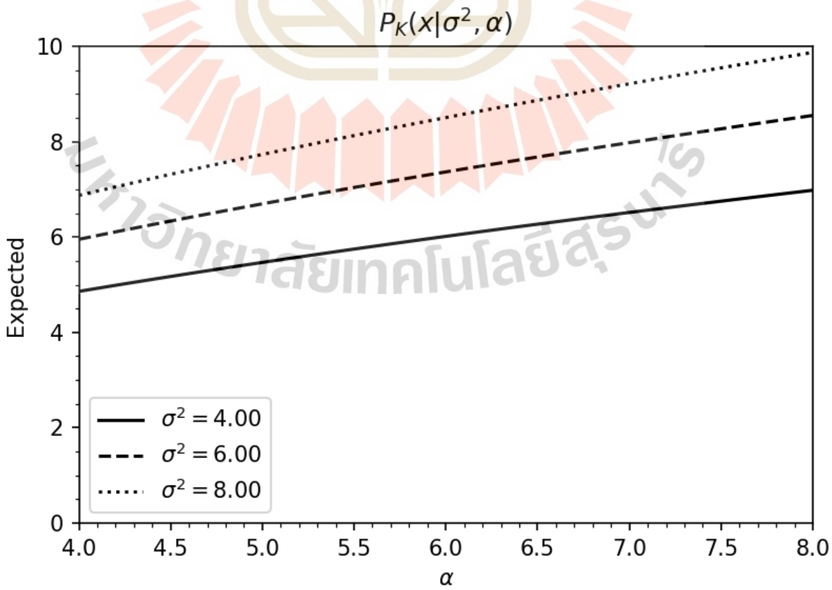

Figure 4.31 Graph of expectation value with $\sigma^2 = 4, 6, 8$ and varying α of K-distribution.

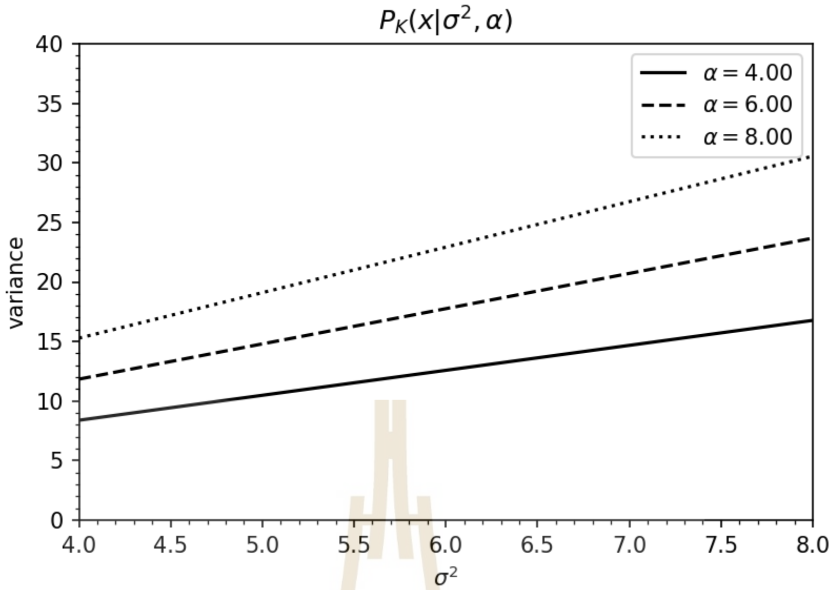

Figure 4.32 Graph of variance value with $\alpha = 4, 6, 8$ and varying σ^2 of K-distribution.

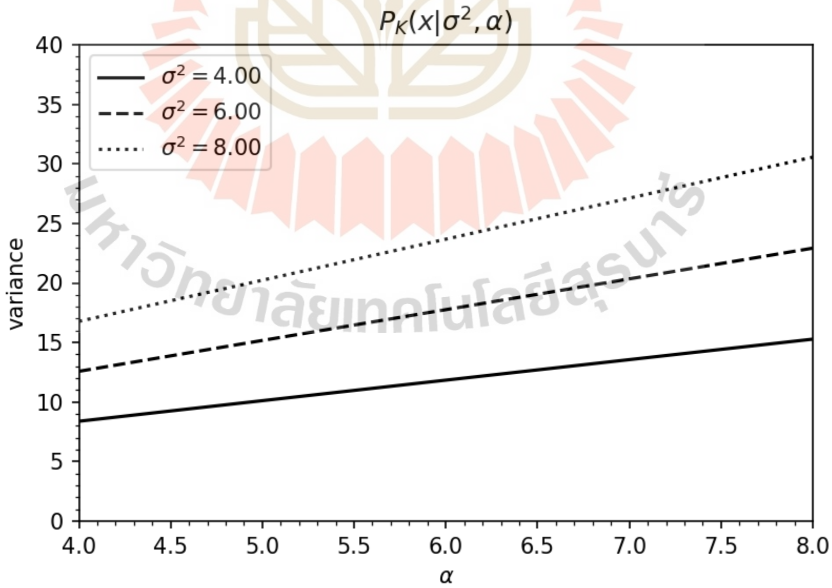

Figure 4.33 Graph of variance value with $\sigma^2 = 4, 6, 8$ and varying α of K-distribution.

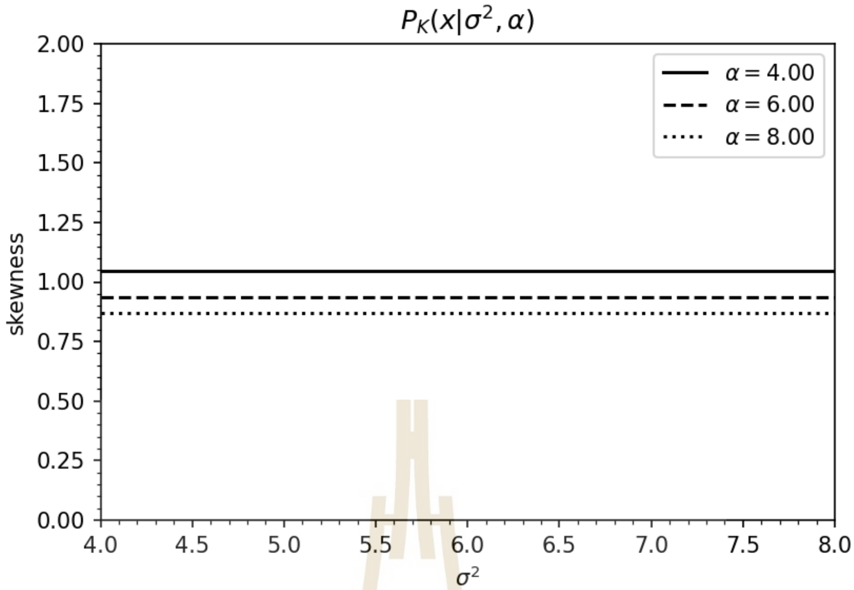

Figure 4.34 Graph of skewness value with $\alpha = 4, 6, 8$ and varying σ^2 of K-distribution.

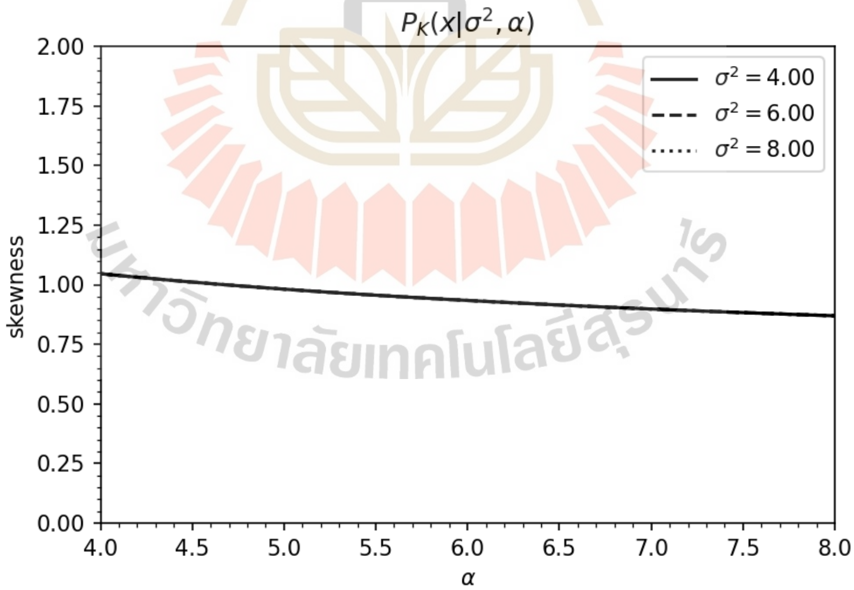

Figure 4.35 Graph of skewness value with $\sigma^2 = 4, 6, 8$ and varying α of K-distribution.

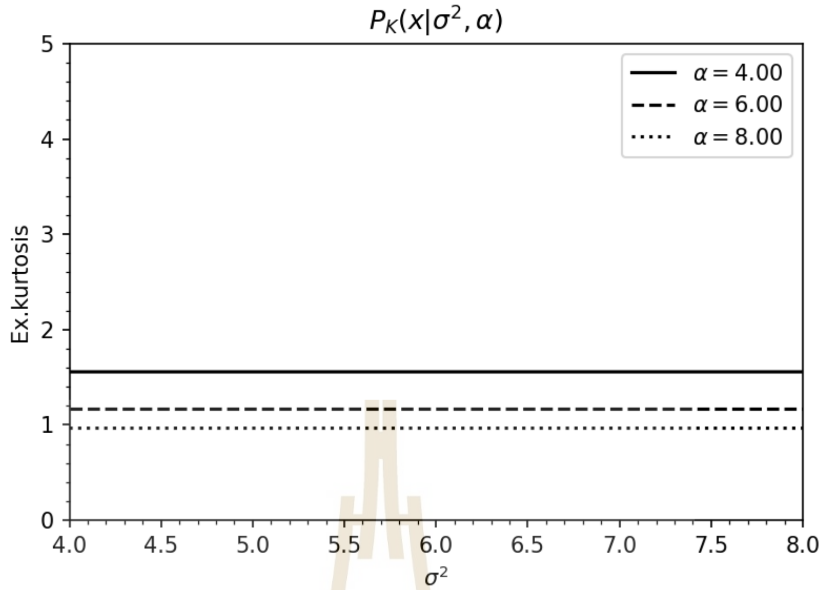

Figure 4.36 Graph of excess kurtosis value with standard deviation $\alpha = 4, 6, 8$ and varying σ^2 of K-distribution.

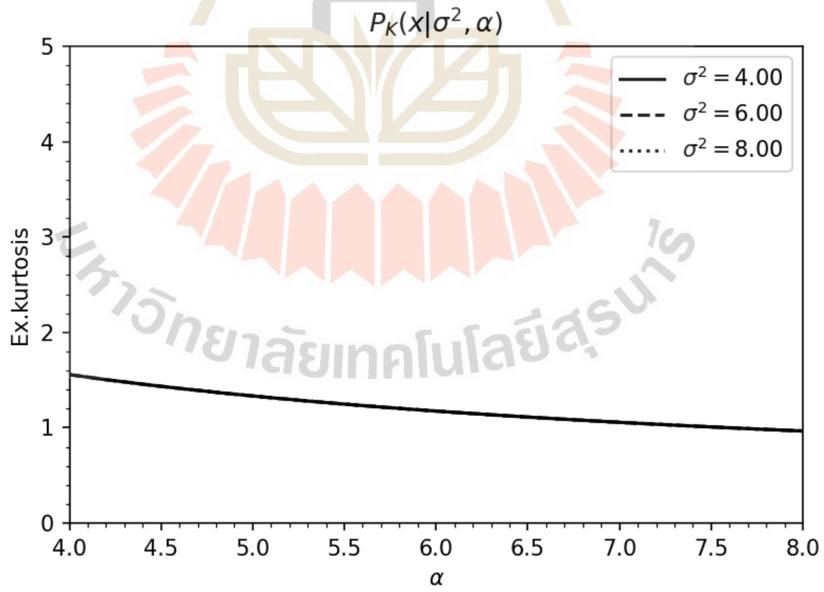

Figure 4.37 Graph of excess kurtosis value with $\sigma^2 = 4, 6, 8$ and varying α of K-distribution.

4.2.5 Homodyned K-distribution

The graphs of parameters of Homodyned K-distribution are presented as follows:

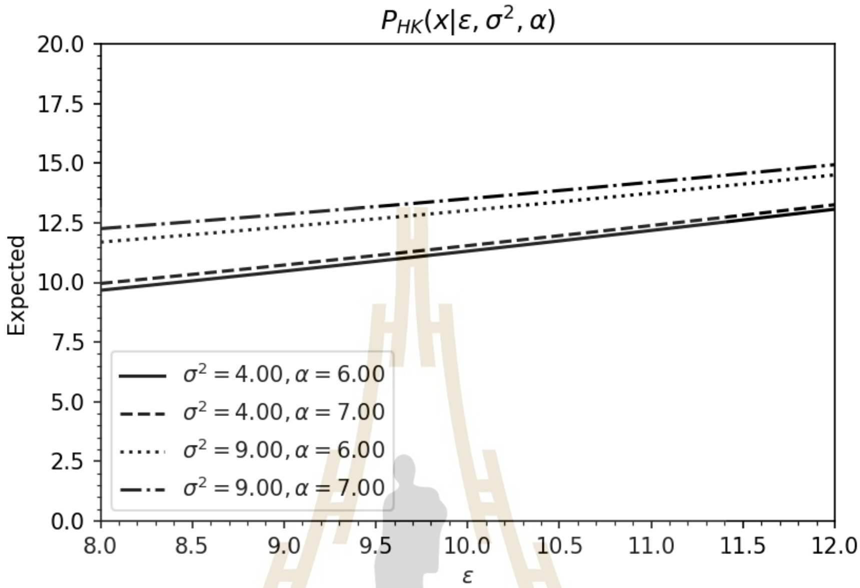

Figure 4.38 Graph of expectation value with $\sigma^2 = 4, 9, \alpha = 6, 7$ and varying ϵ of the Homodyned K-distribution.

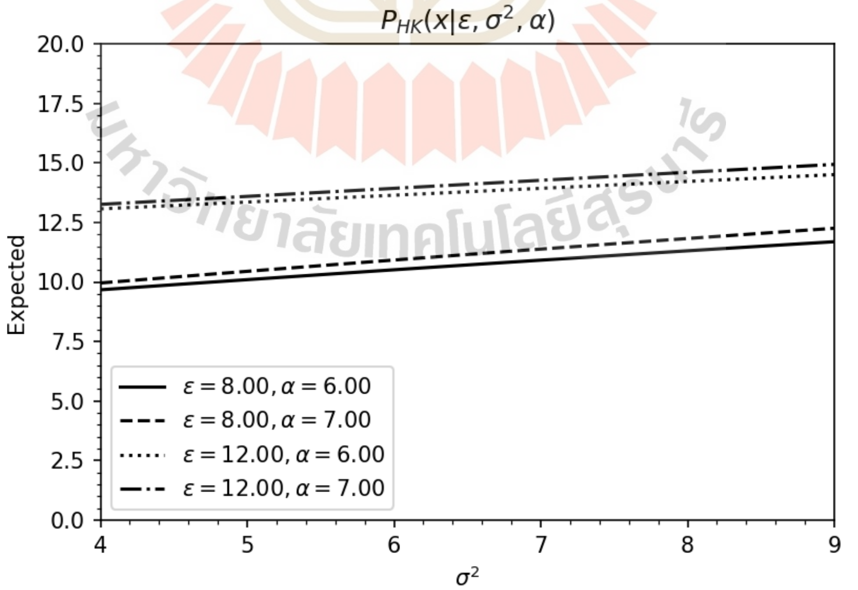

Figure 4.39 Graph of expectation value with $\epsilon = 8, 12, \alpha = 6, 7$ and varying σ^2 of the Homodyned K-distribution.

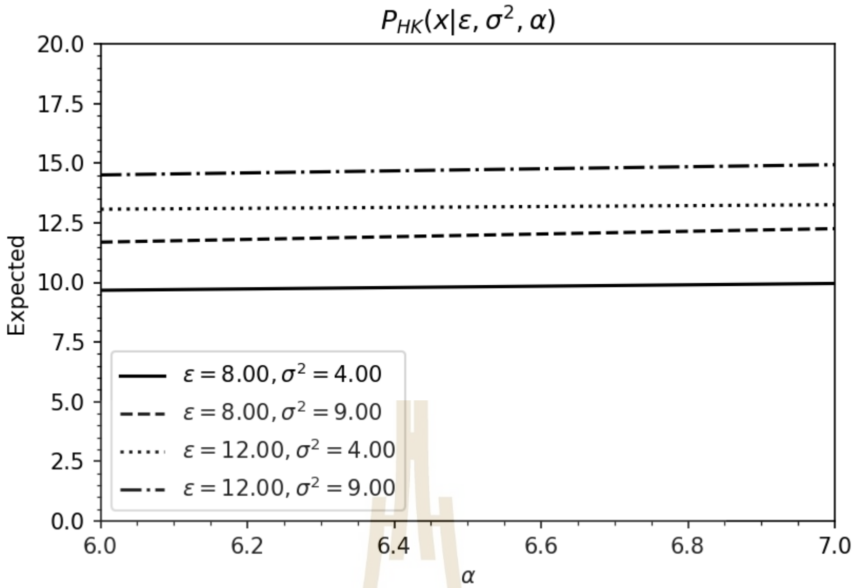

Figure 4.40 Graph of expectation value with $\epsilon = 8, 12, \sigma^2 = 4, 9$ and varying α of the Homodyned K-distribution.

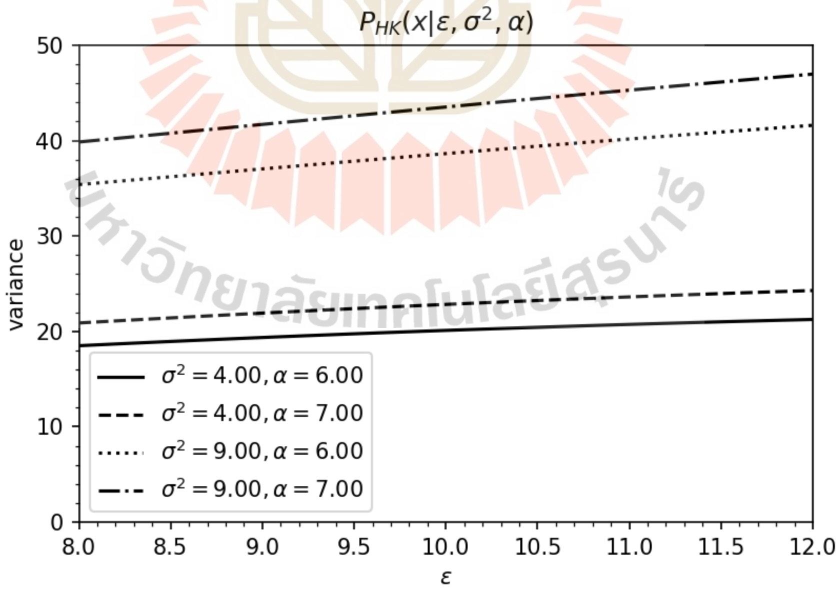

Figure 4.41 Graph of variance value with $\sigma^2 = 4, 9, \alpha = 6, 7$ and varying ϵ of the Homodyned K-distribution.

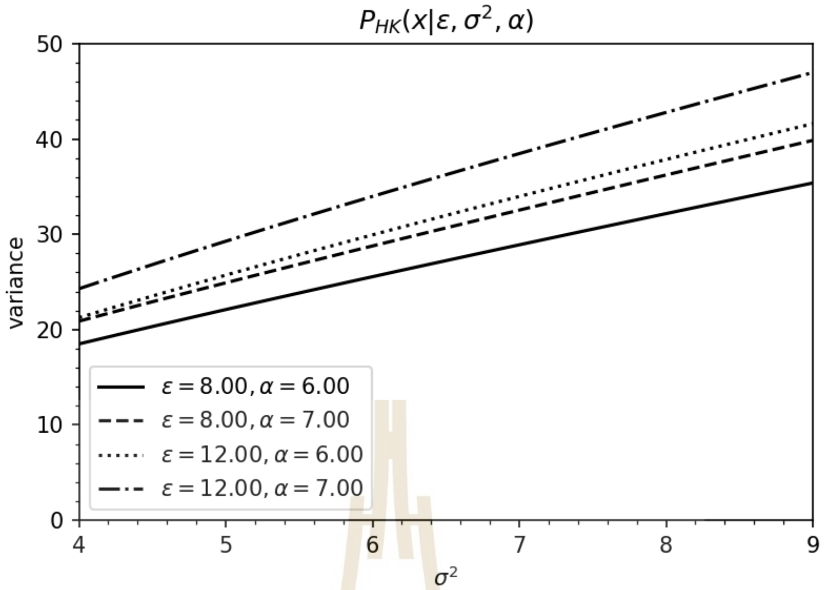

Figure 4.42 Graph of variance value with $\epsilon = 8, 12, \alpha = 6, 7$ and varying σ^2 of the Homodyned K-distribution.

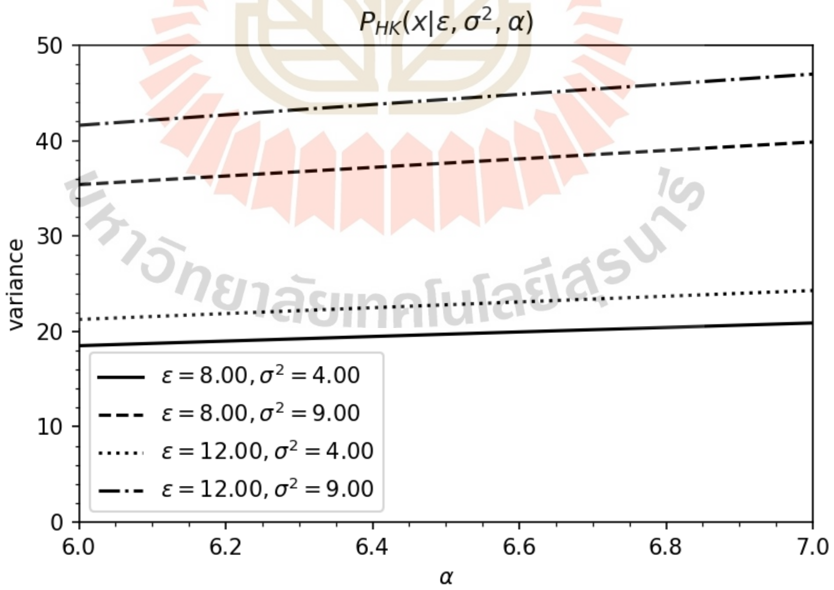

Figure 4.43 Graph of variance value with $\epsilon = 8, 12, \sigma^2 = 4, 9$ and varying α of the Homodyned K-distribution.

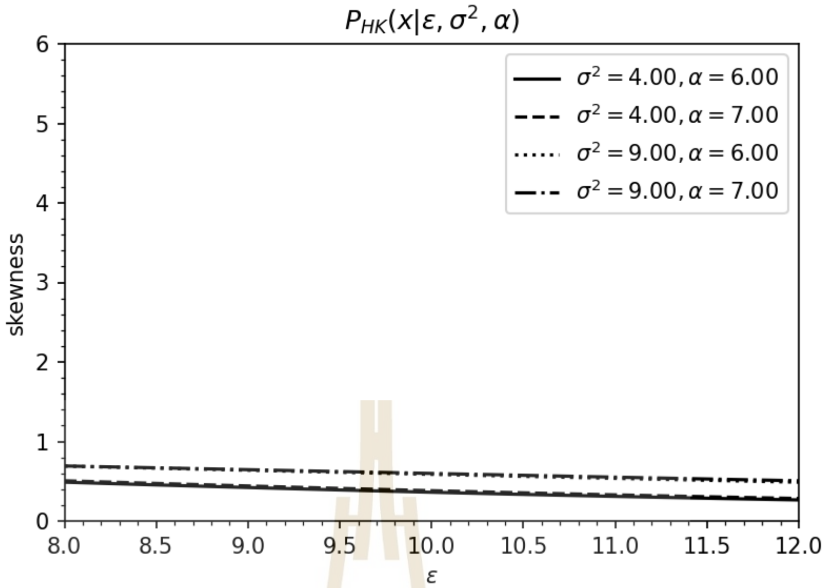

Figure 4.44 Graph of skewness value with $\sigma^2 = 4, 9, \alpha = 6, 7$ and varying ϵ of the Homodyned K-distribution.

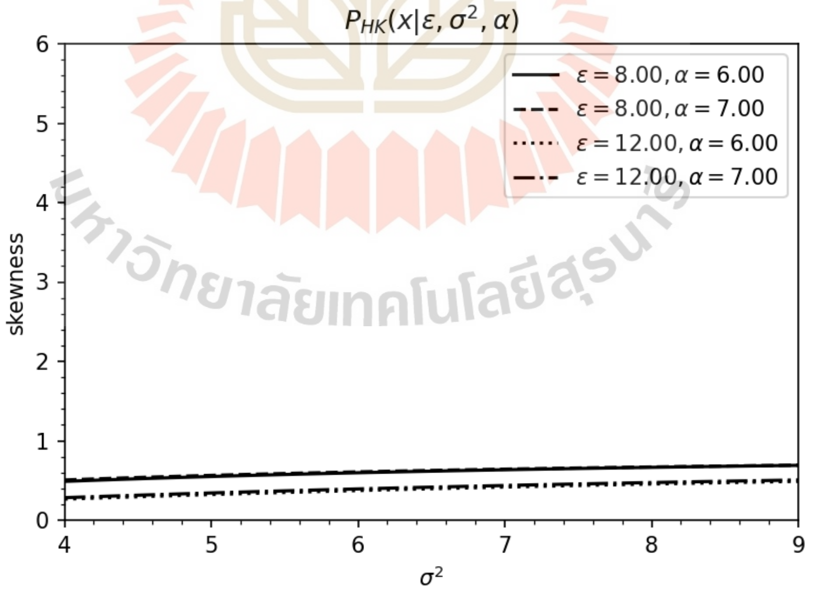

Figure 4.45 Graph of skewness value with $\epsilon = 8, 12, \alpha = 6, 7$ and varying σ^2 of the Homodyned K-distribution.

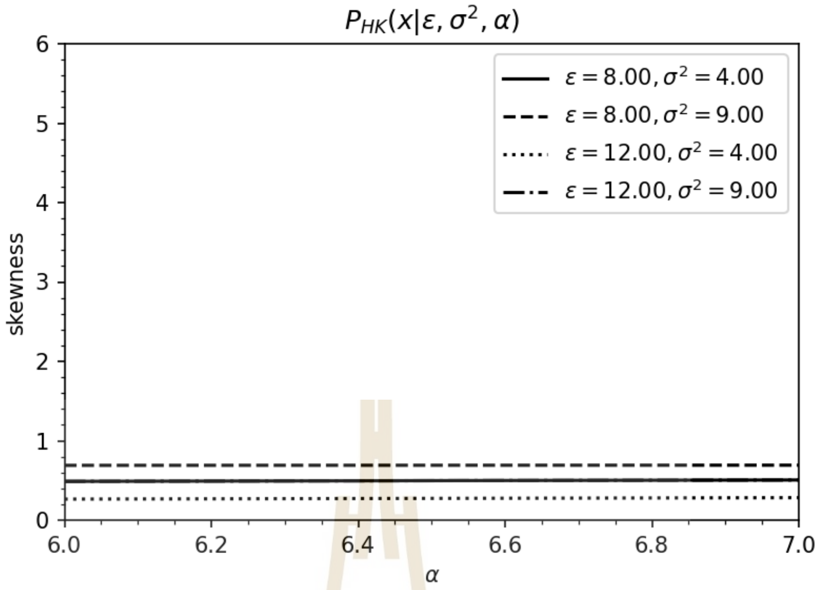

Figure 4.46 Graph of skewness value with $\epsilon = 8, 12, \sigma^2 = 4, 9$ and varying α of the Homodyned K-distribution.

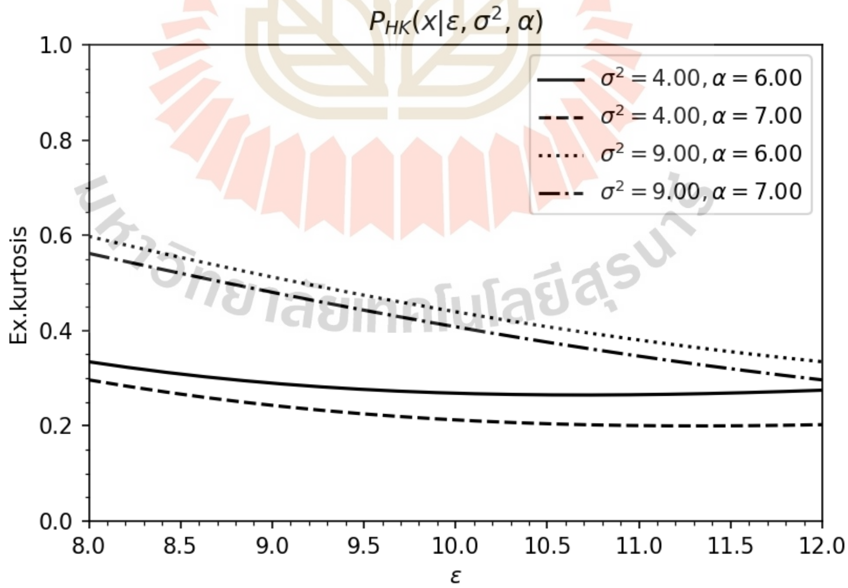

Figure 4.47 Graph of excess kurtosis value with $\sigma^2 = 4, 9, \alpha = 6, 7$ and varying ϵ of the Homodyned K-distribution.

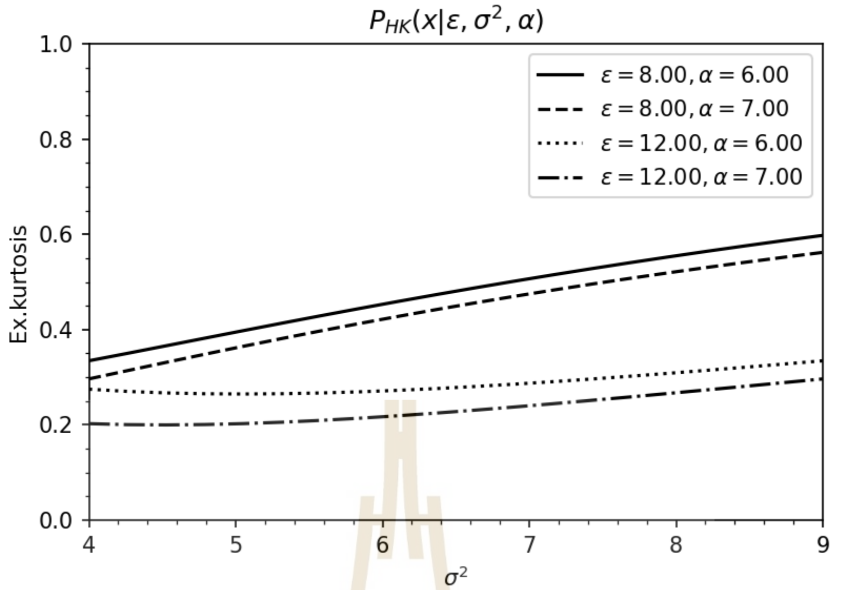

Figure 4.48 Graph of excess kurtosis value with $\epsilon = 8, 12, \alpha = 6, 7$ and varying σ^2 of the Homodyned K-distribution.

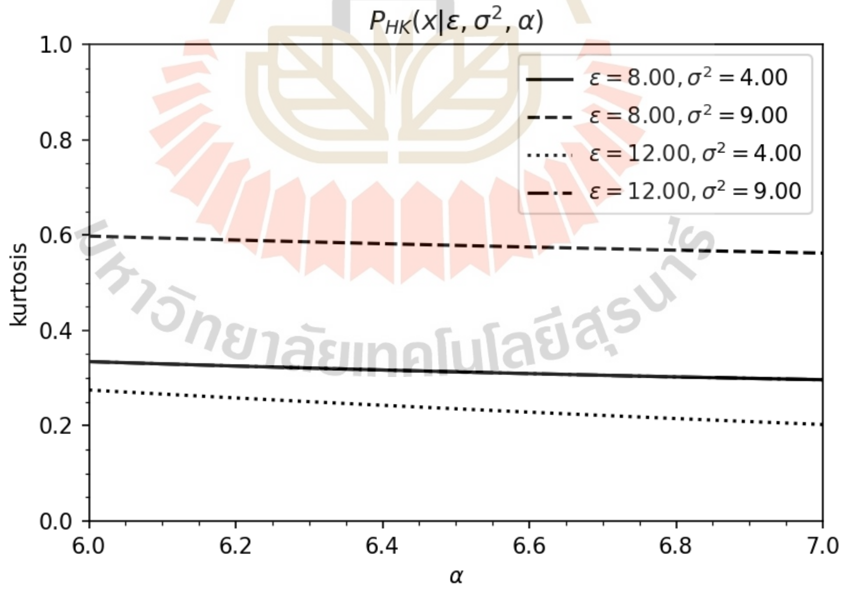

Figure 4.49 Graph of excess kurtosis value with $\epsilon = 8, 12, \sigma^2 = 4, 9$ and varying α of the Homodyned K-distribution.

4.2.6 Generalized K-distribution

The graphs of parameters of Generalized K-distribution are presented as follows:

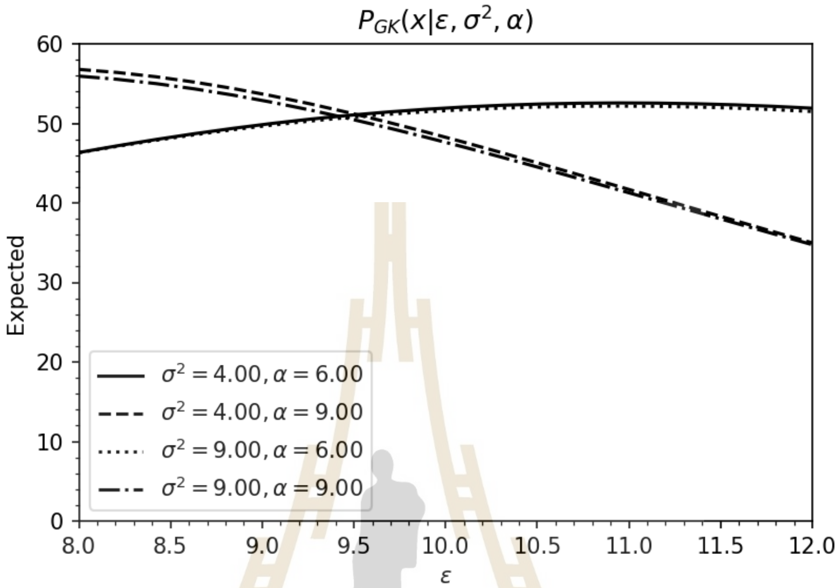

Figure 4.50 Graph of expectation value with $\sigma^2 = 4, 9, \alpha = 6, 9$ and varying ϵ of the Generalized K-distribution.

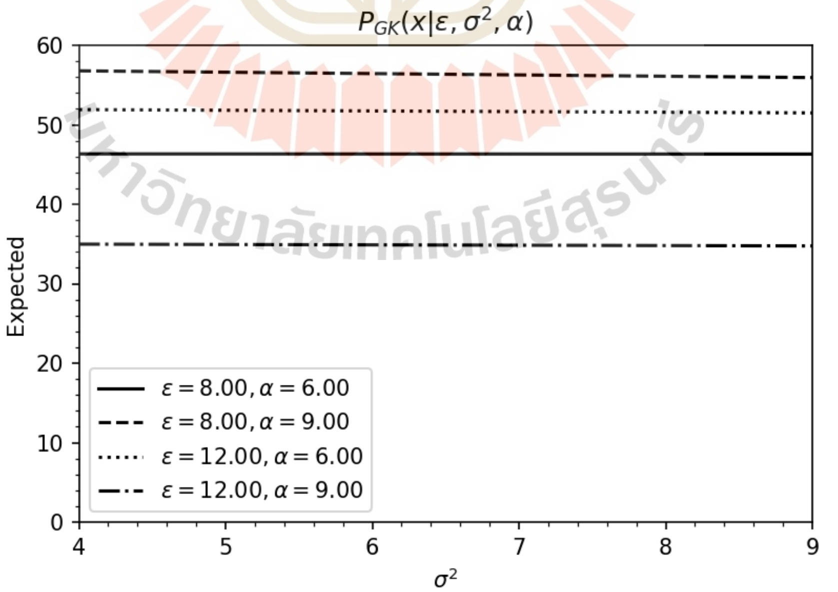

Figure 4.51 Graph of expectation value with $\epsilon = 8, 12, \alpha = 6, 9$ and varying σ^2 of the Generalized K-distribution.

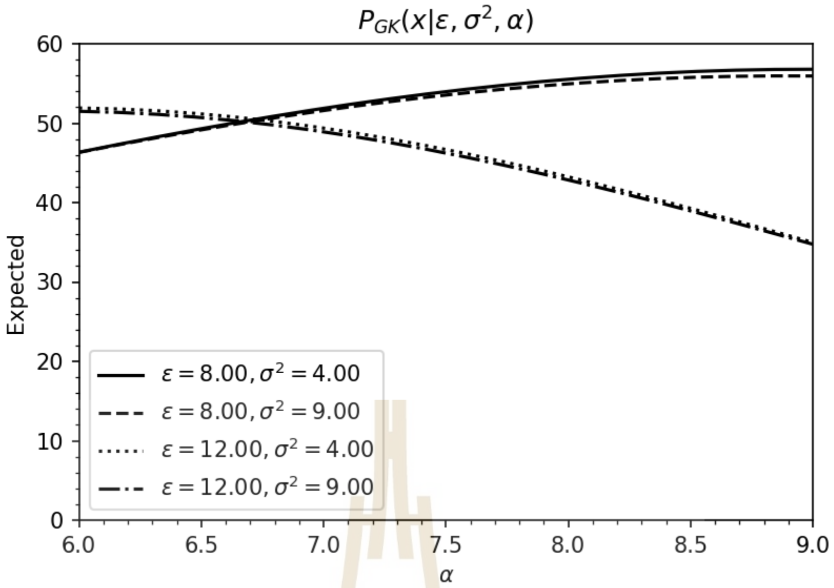

Figure 4.52 Graph of expectation value with $\epsilon = 8, 12, \sigma^2 = 4, 9$ and varying α of the Generalized K-distribution.

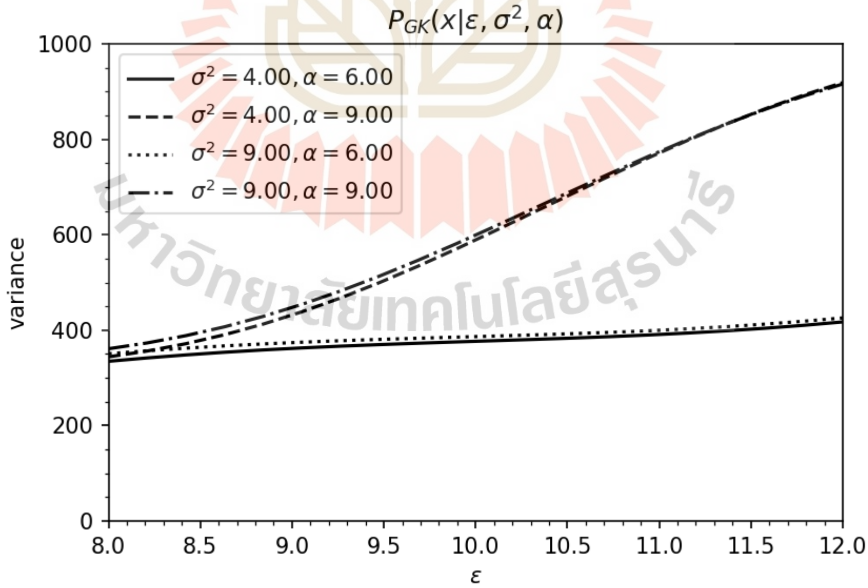

Figure 4.53 Graph of variance value with $\sigma^2 = 4, 9, \alpha = 6, 9$ and varying ϵ of the Generalized K-distribution.

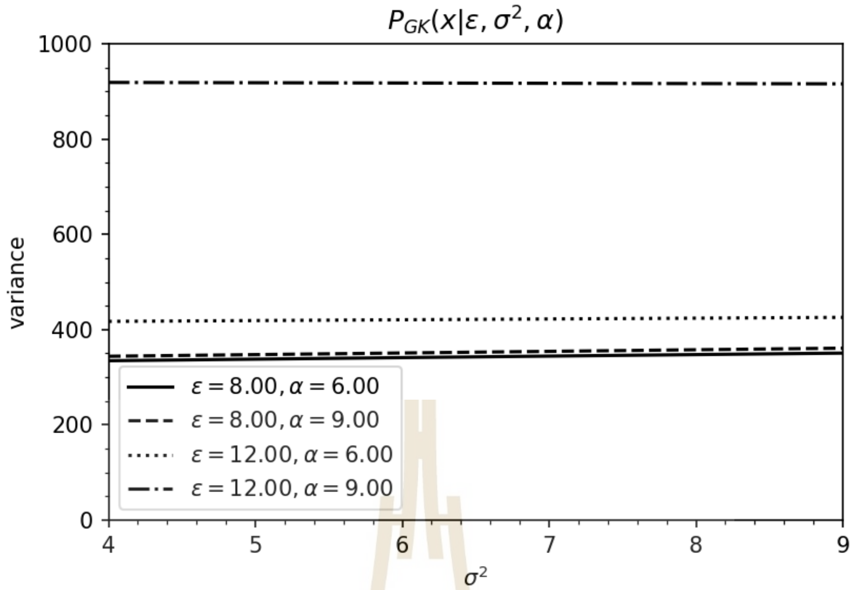

Figure 4.54 Graph of variance value with $\epsilon = 8, 12, \alpha = 6, 9$ and varying σ^2 of the Generalized K-distribution.

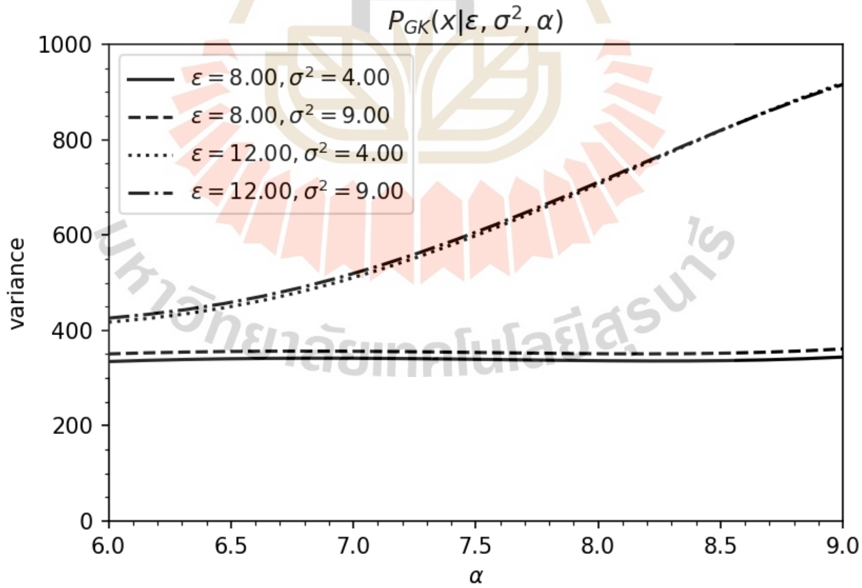

Figure 4.55 Graph of variance value with $\epsilon = 8, 12, \sigma^2 = 4, 9$ and varying α of the Generalized K-distribution.

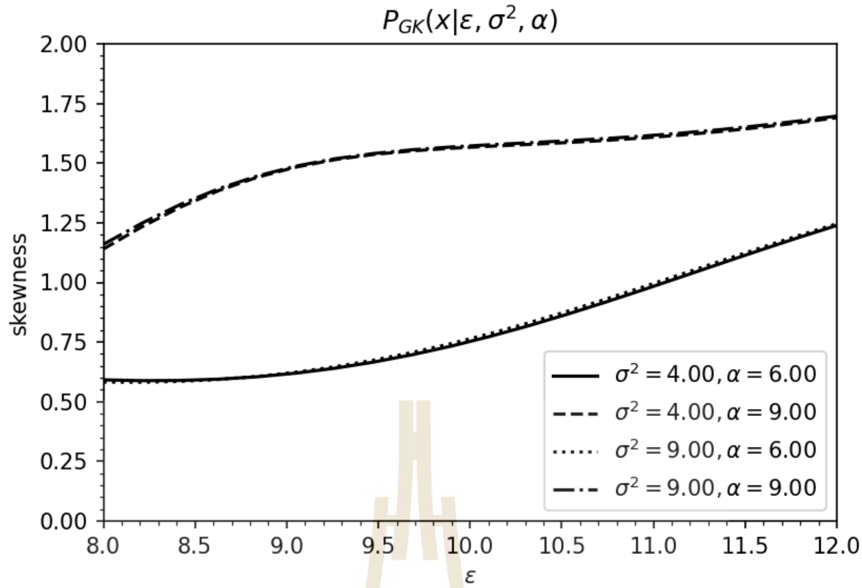

Figure 4.56 Graph of skewness value with $\sigma^2 = 4, 9, \alpha = 6, 9$ and varying ϵ of the Generalized K-distribution.

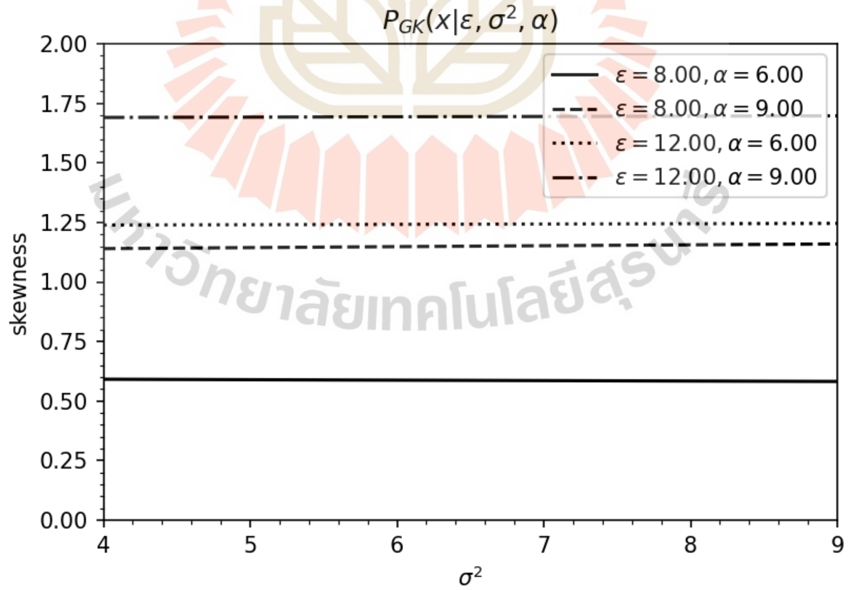

Figure 4.57 Graph of skewness value with $\epsilon = 8, 12, \alpha = 6, 9$ and varying σ^2 of the Generalized K-distribution.

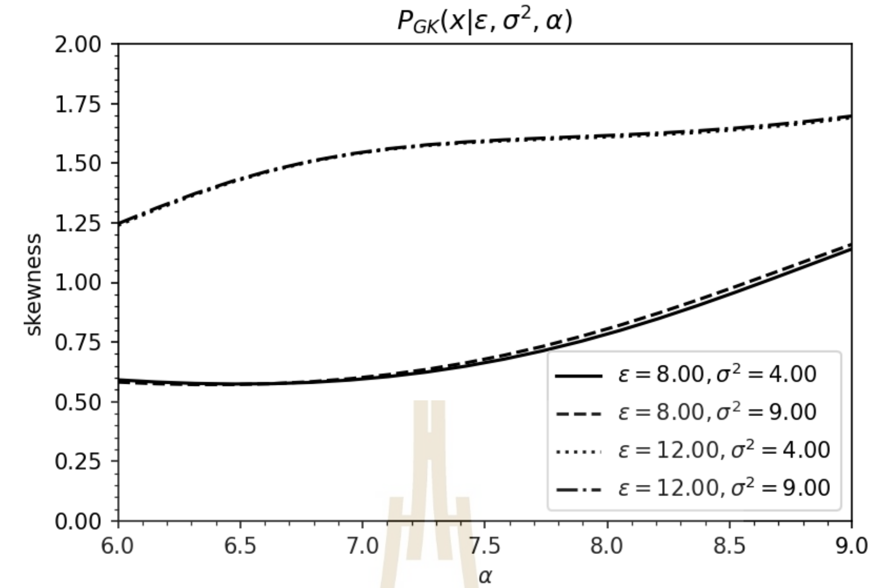

Figure 4.58 Graph of skewness value with $\epsilon = 8, 12, \sigma^2 = 4, 9$ and varying α of the Generalized K-distribution.

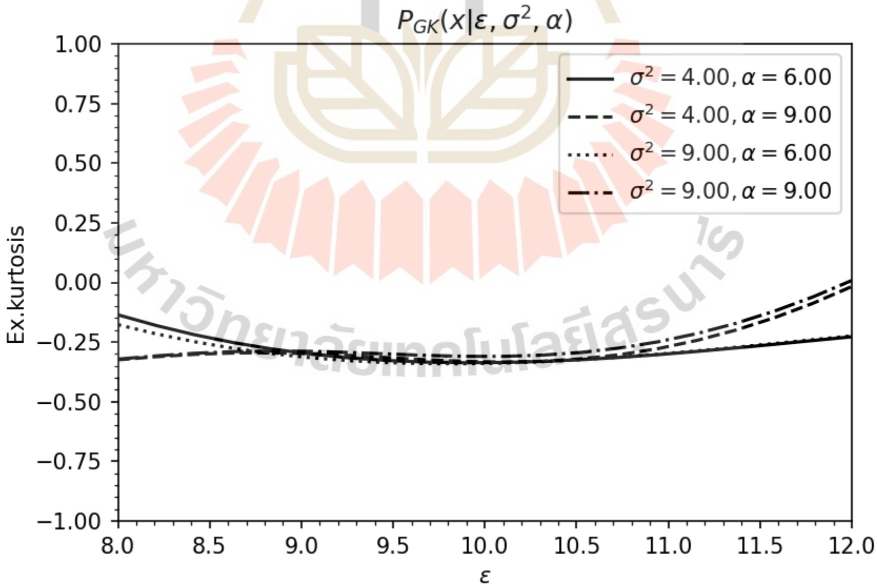

Figure 4.59 Graph of excess kurtosis value with $\sigma^2 = 4, 9, \alpha = 6, 9$ and varying ϵ of the Generalized K-distribution.

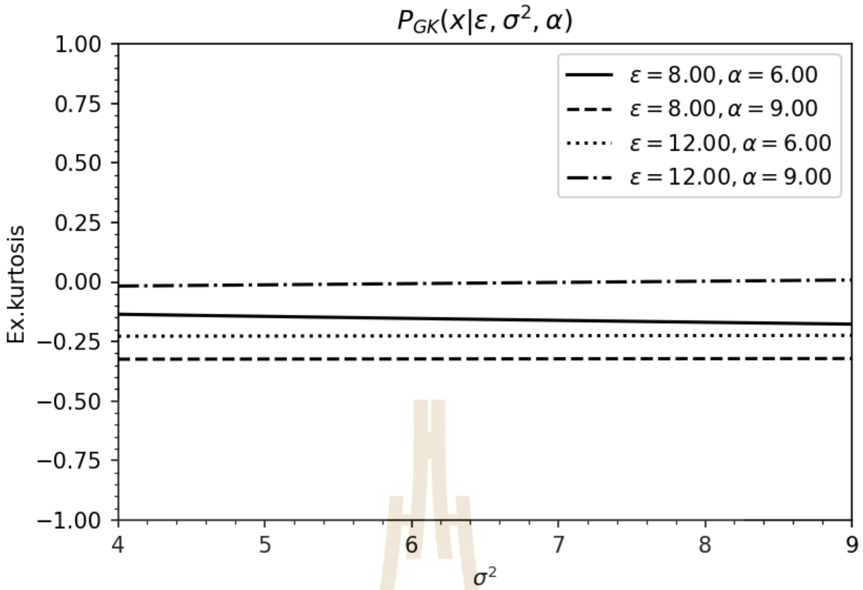

Figure 4.60 Graph of excess kurtosis value with $\epsilon = 8, 12, \alpha = 6, 9$ and varying σ^2 of the Generalized K-distribution.

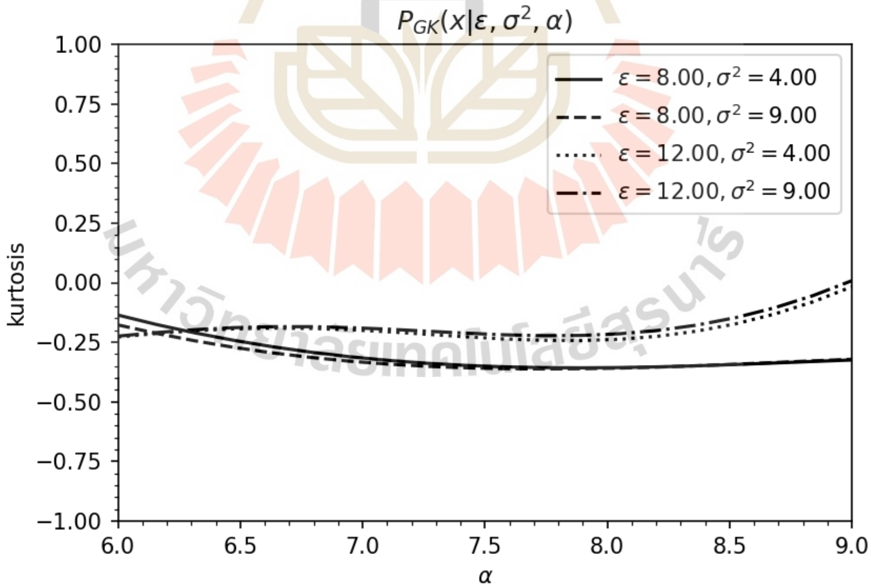

Figure 4.61 Graph of excess kurtosis value with $\epsilon = 8, 12, \sigma^2 = 4, 9$ and varying α of the Generalized K-distribution.

4.2.7 Rician distribution with modulated inverse Gaussian distribution parameters

The graphs of parameters of the Rician distribution with modulated inverse Gaussian distribution parameters are presented as follows:

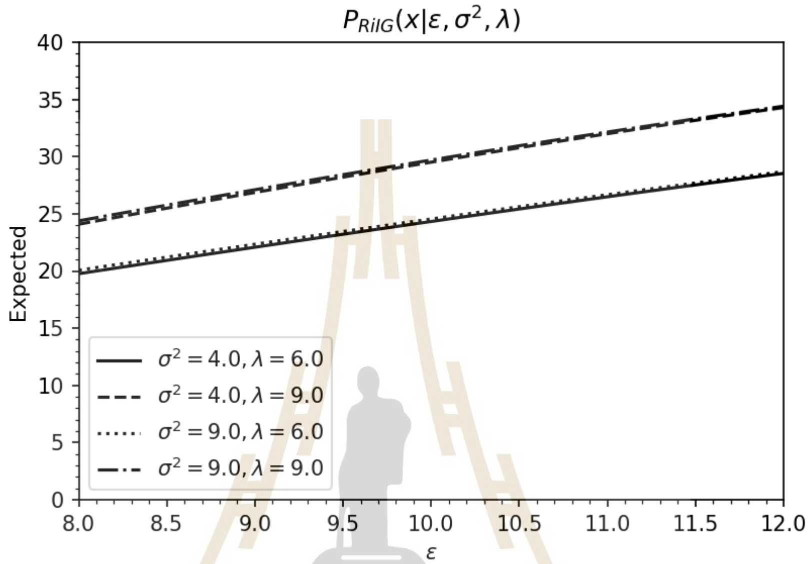

Figure 4.62 Graph of expectation value with $\sigma^2 = 4, 9, \lambda = 6, 9$ and varying ϵ of the Rician distribution with modulated inverse Gaussian distribution parameters.

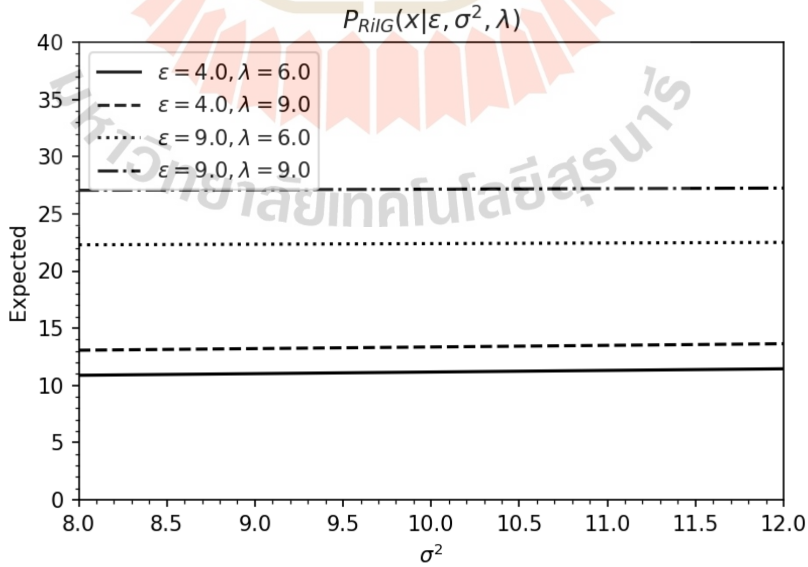

Figure 4.63 Graph of expectation value with $\epsilon = 8, 12, \lambda = 6, 9$ and varying σ^2 of the Rician distribution with modulated inverse Gaussian distribution parameters.

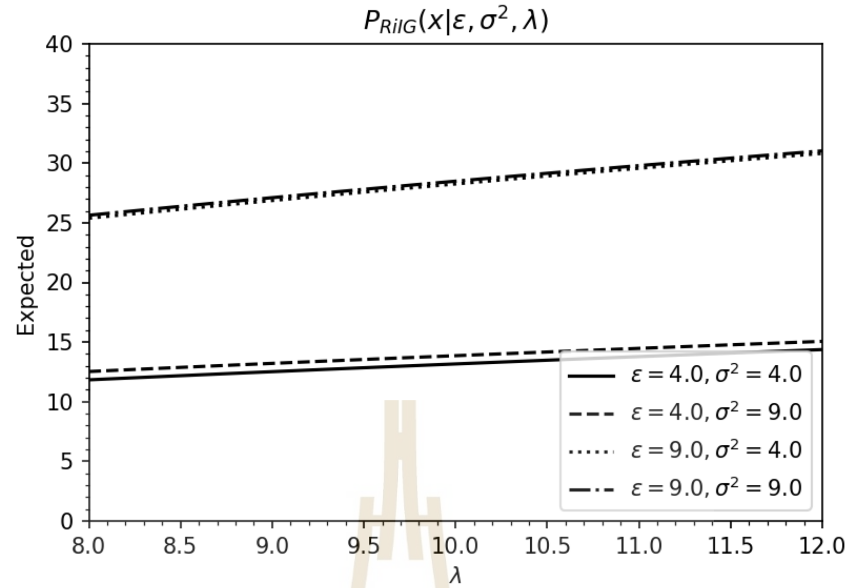

Figure 4.64 Graph of expectation value with $\epsilon = 4, 9, \sigma^2 = 4, 9$ and varying λ of the Rician distribution with modulated inverse Gaussian distribution parameters.

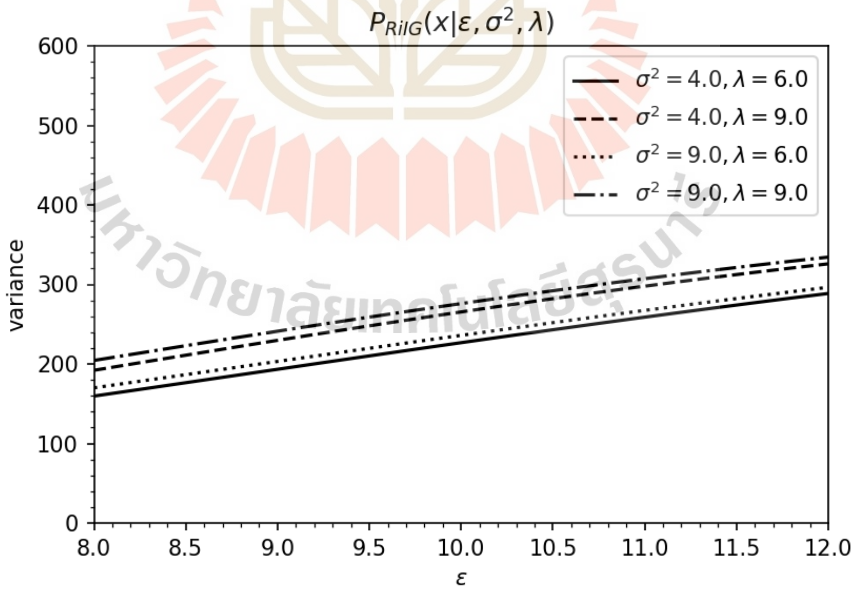

Figure 4.65 Graph of variance value with $\sigma^2 = 4, 9, \lambda = 6, 9$ and varying ϵ of the Rician distribution with modulated inverse Gaussian distribution parameters.

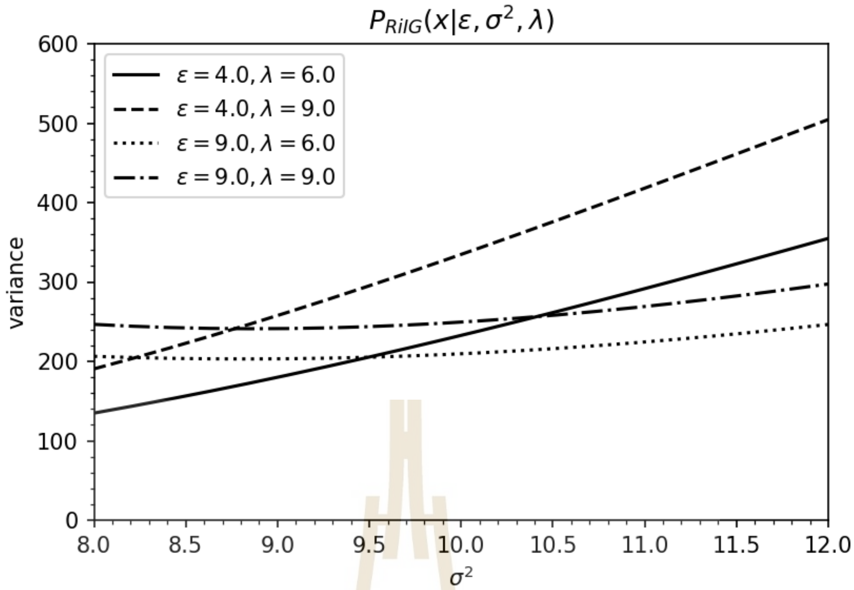

Figure 4.66 Graph of variance value with $\epsilon = 4, 9, \lambda = 6, 9$ and varying σ^2 of the Rician distribution with modulated inverse Gaussian distribution parameters.

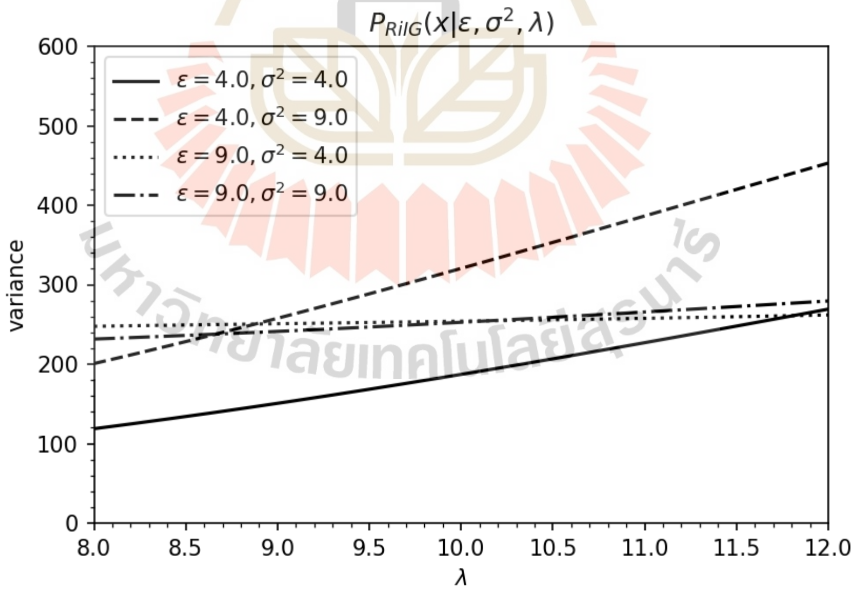

Figure 4.67 Graph of variance value with $\epsilon = 4, 9, \sigma^2 = 4, 9$ and varying λ of the Rician distribution with modulated inverse Gaussian distribution parameters.

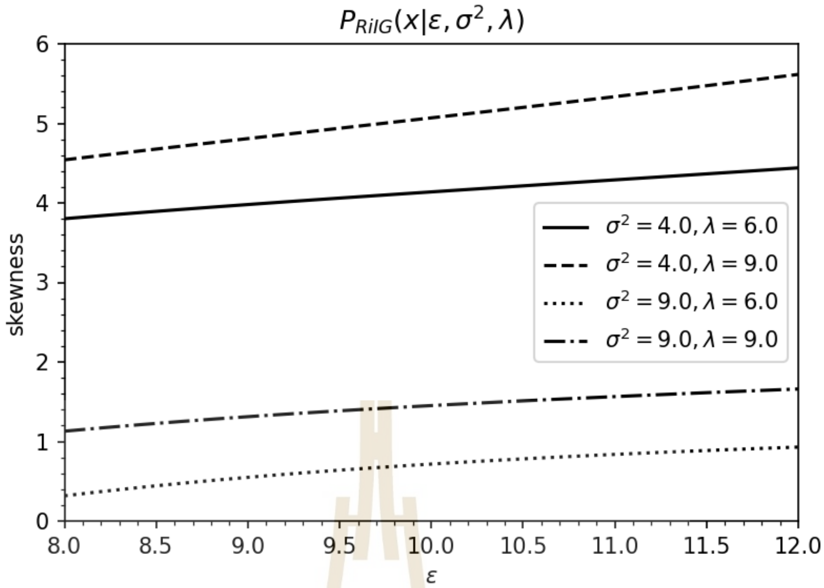

Figure 4.68 Graph of skewness value with $\sigma^2 = 4, 9, \lambda = 6, 9$ and varying ϵ of the Rician distribution with modulated inverse Gaussian distribution parameters.

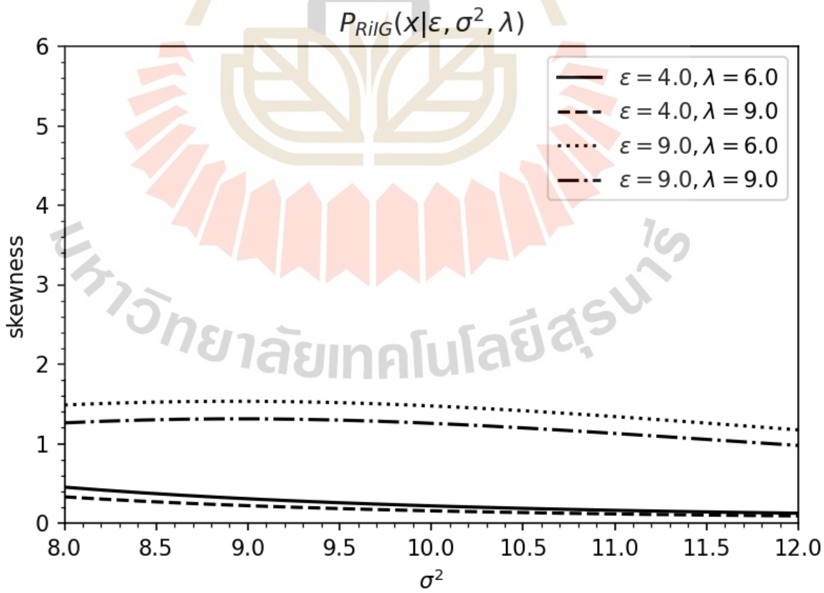

Figure 4.69 Graph of skewness value with $\epsilon = 4, 9, \lambda = 6, 9$ and varying σ^2 of the Rician distribution with modulated inverse Gaussian distribution parameters.

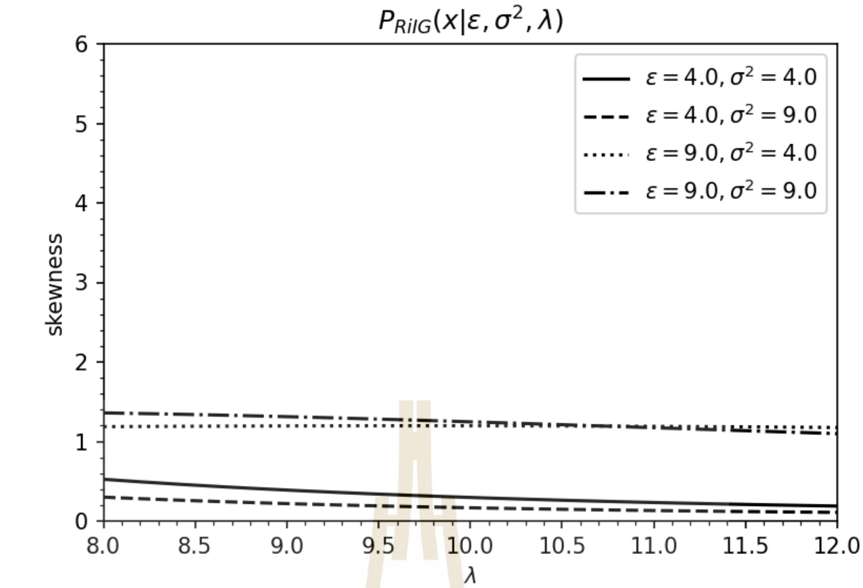

Figure 4.70 Graph of skewness value with $\epsilon = 4, 9, \sigma^2 = 4, 9$ and varying λ of the Rician distribution with modulated inverse Gaussian distribution parameters.

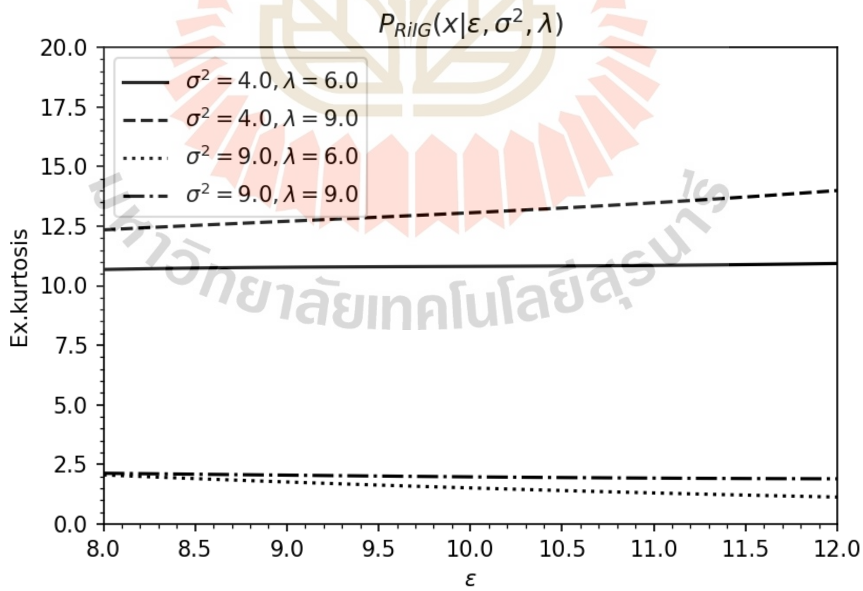

Figure 4.71 Graph of excess kurtosis value with $\sigma^2 = 4, 9, \lambda = 6, 9$ and varying ϵ of the Rician distribution with modulated inverse Gaussian distribution parameters.

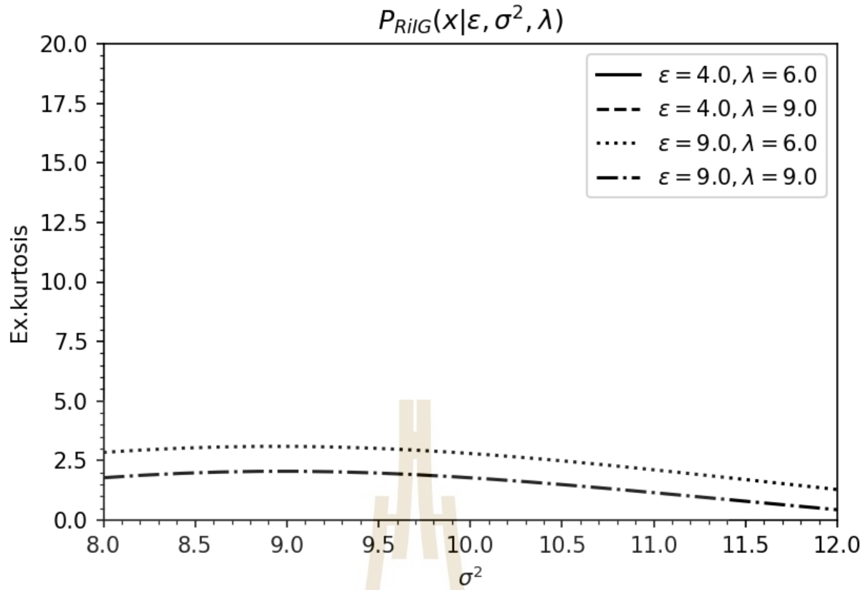

Figure 4.72 Graph of excess kurtosis value with $\epsilon = 4, 9, \lambda = 6, 9$ and varying σ^2 of the Rician distribution with modulated inverse Gaussian distribution parameters.

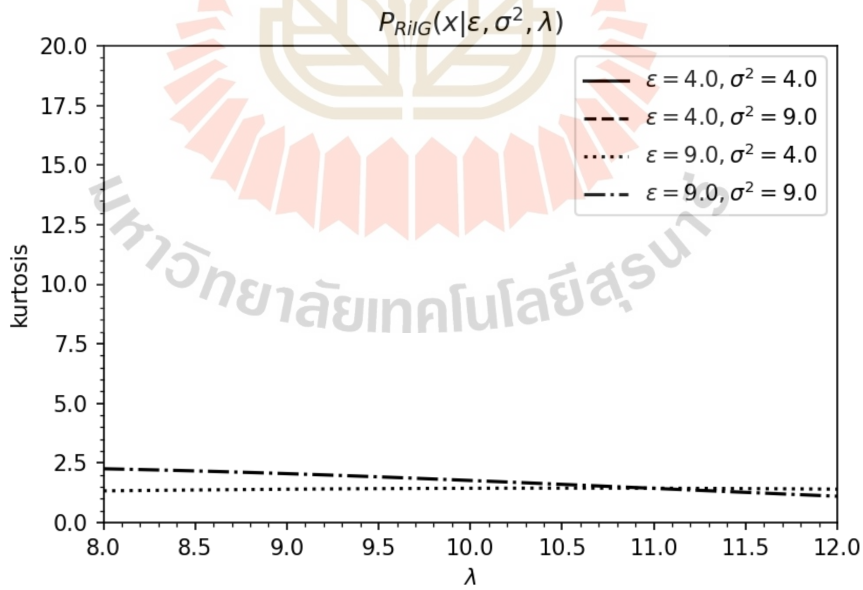

Figure 4.73 Graph of excess kurtosis value with $\epsilon = 4, 9, \sigma^2 = 4, 9$ and varying λ of the Rician distribution with modulated inverse Gaussian distribution parameters.

4.2.8 Nakagami distribution

The graphs of parameters of Nakagami distribution are presented as follows:

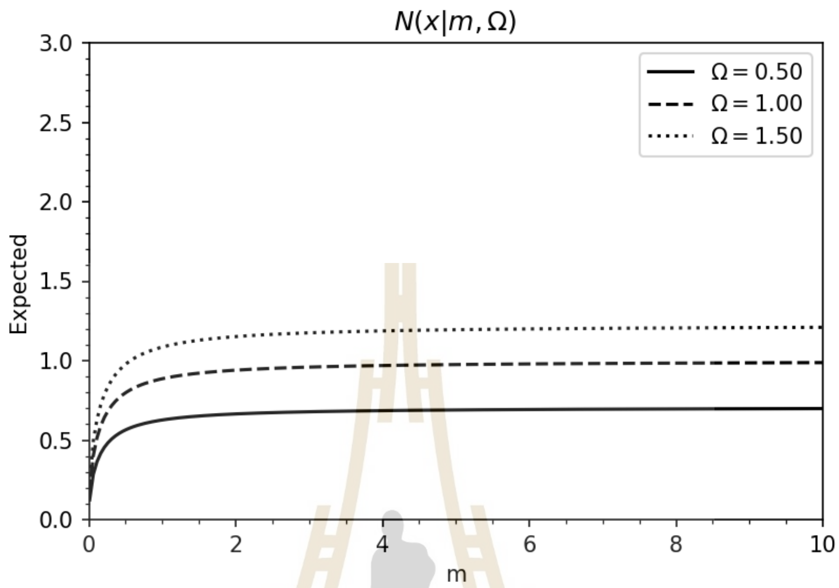

Figure 4.74 Graph of expectation value with $\Omega = 0.5, 1, 1.5$ and varying m of the Nakagami distribution.

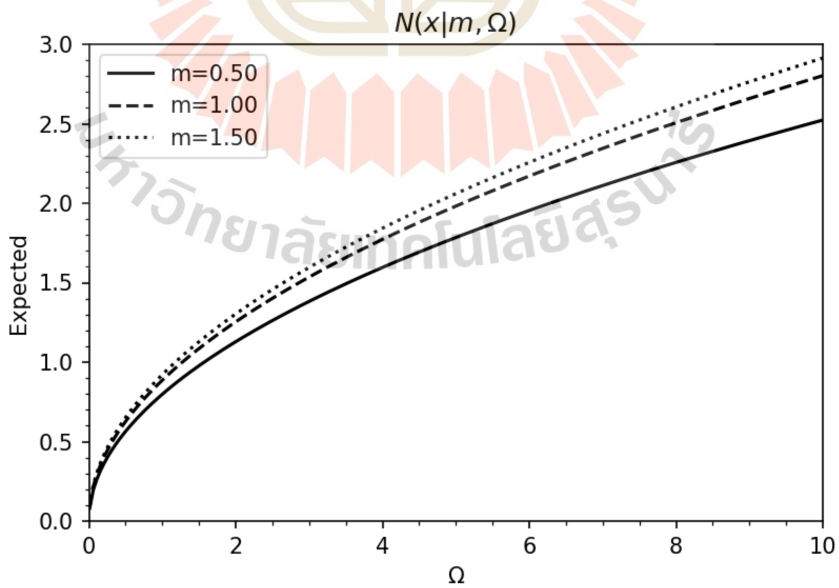

Figure 4.75 Graph of expectation value with $\Omega = 0.5, 1, 1.5$ and varying m of the Nakagami distribution.

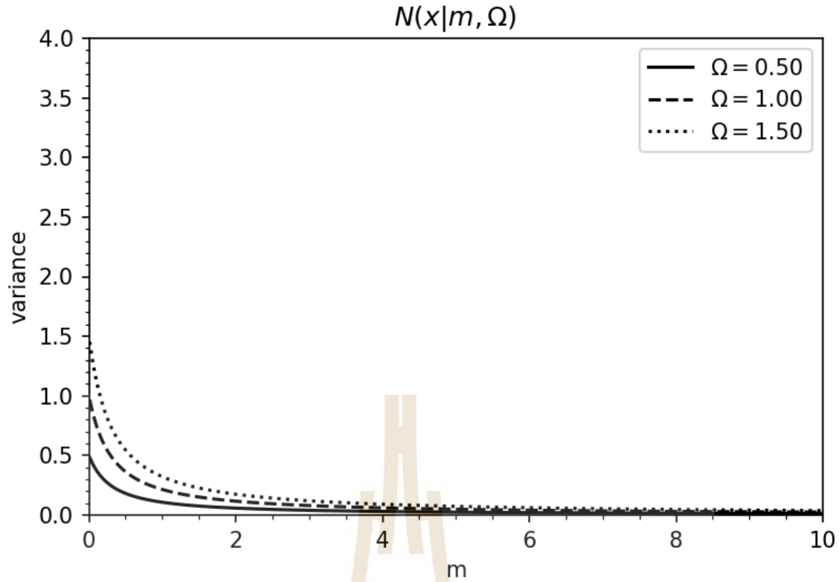

Figure 4.76 Graph of variance value with $\Omega = 0.5, 1, 1.5$ and varying m of the Nakagami distribution.

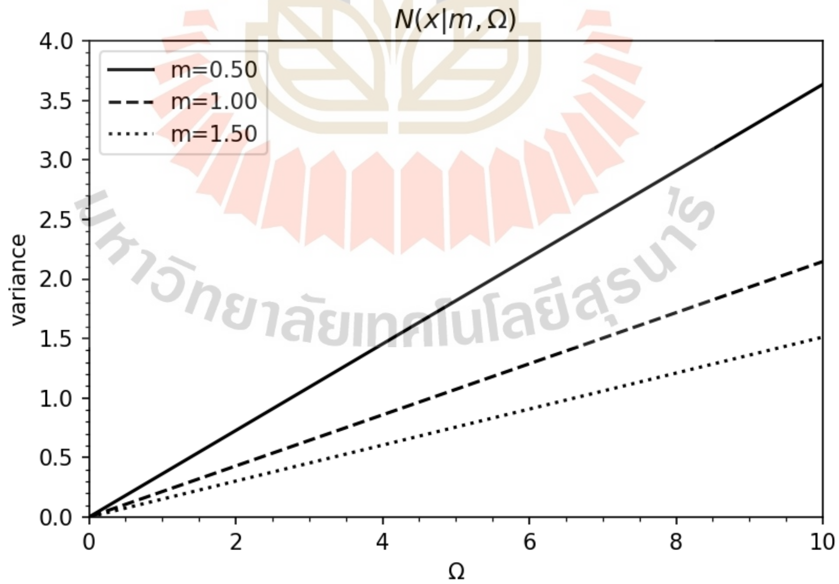

Figure 4.77 Graph of variance value with $m = 0.5, 1, 1.5$ and varying Ω of the Nakagami distribution.

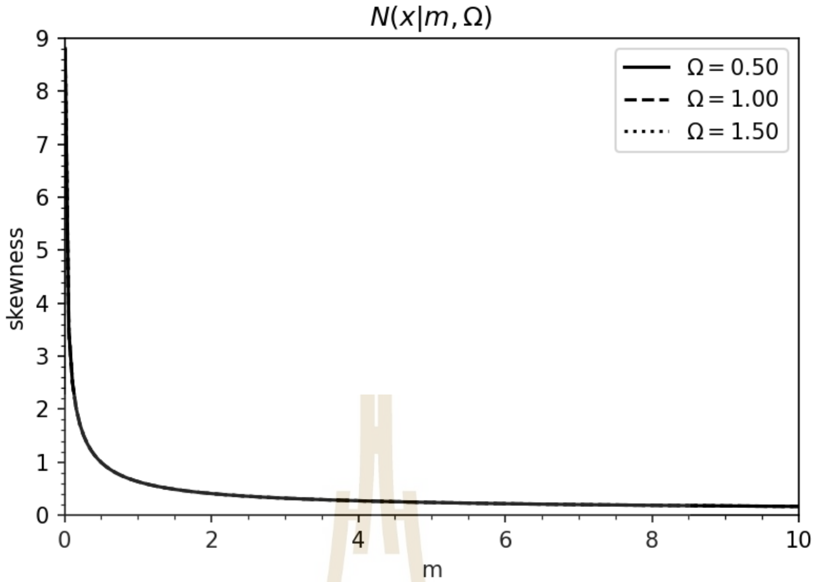

Figure 4.78 Graph of skewness value with $\Omega = 0.5, 1, 1.5$ and varying m of the Nakagami distribution.

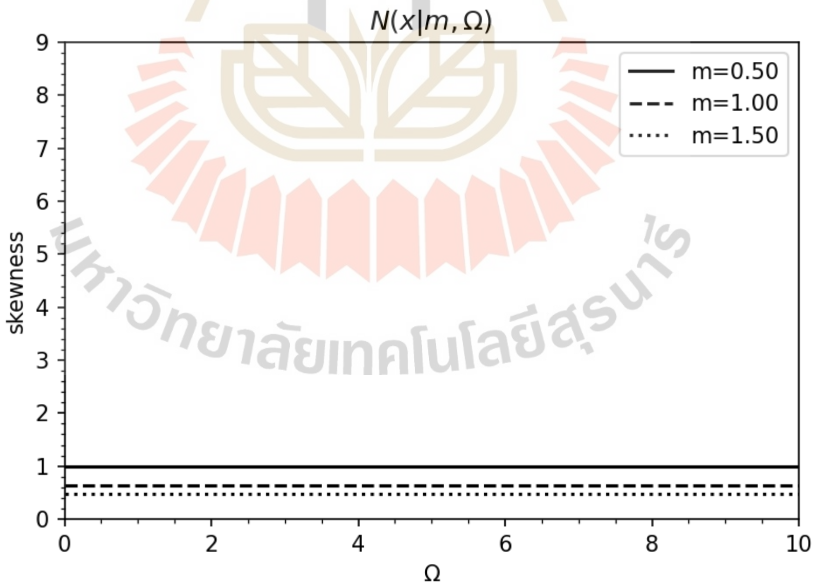

Figure 4.79 Graph of skewness value with $m = 0.5, 1, 1.5$ and varying Ω of the Nakagami distribution.

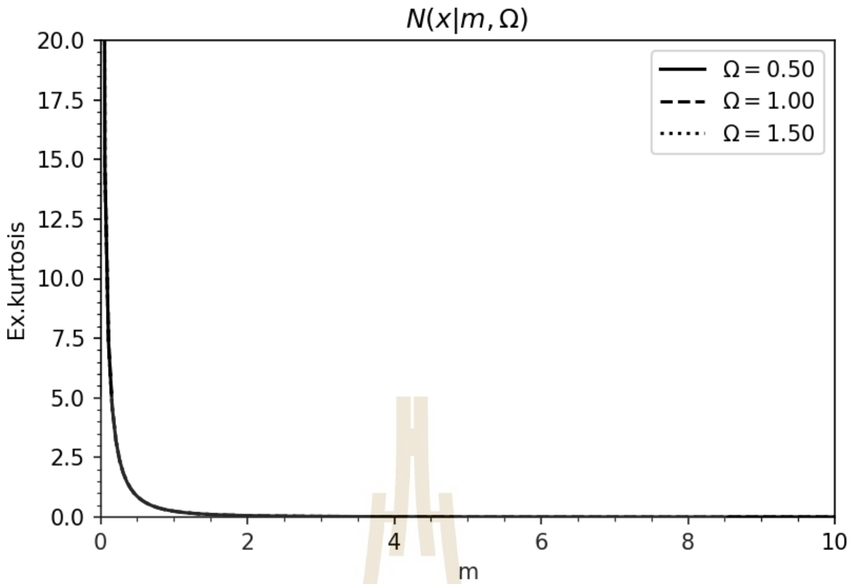

Figure 4.80 Graph of excess kurtosis value with $\Omega = 0.5, 1, 1.5$ and varying m of the Nakagami distribution.

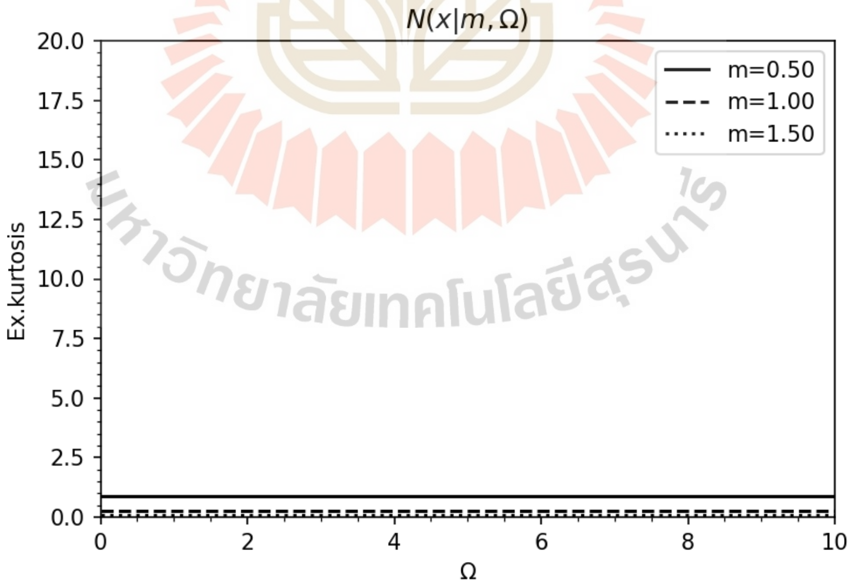

Figure 4.81 Graph of excess kurtosis value with $m = 0.5, 1, 1.5$ and varying Ω of the Nakagami distribution.

4.2.9 Nakagami distribution with modulated gamma distribution parameter

The graphs of parameters of Nakagami distribution with modulated gamma distribution parameter are presented as follows:

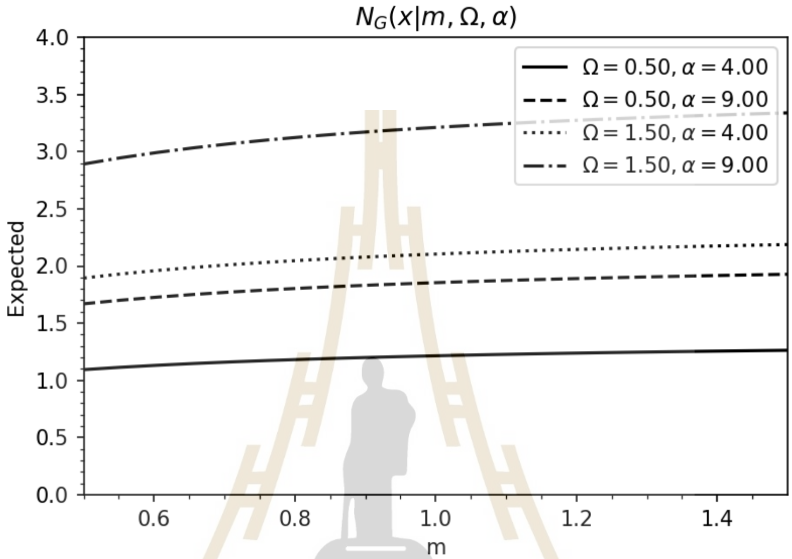

Figure 4.82 Graph of expectation value with $\Omega = 0.5, 1.5, \alpha = 4, 9$ and varying m of the Nakagami distribution with modulated gamma distribution parameter.

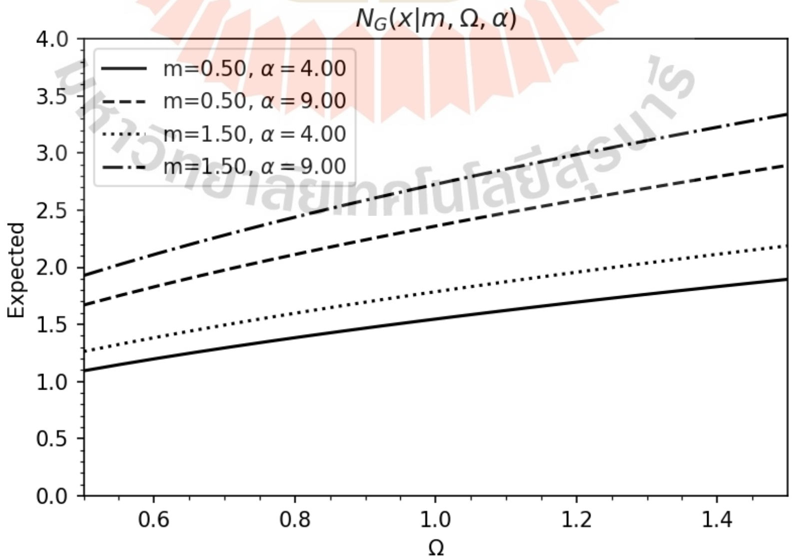

Figure 4.83 Graph of expectation value with $m = 0.5, 1.5, \alpha = 4, 9$ and varying Ω of the Nakagami distribution with modulated gamma distribution parameter.

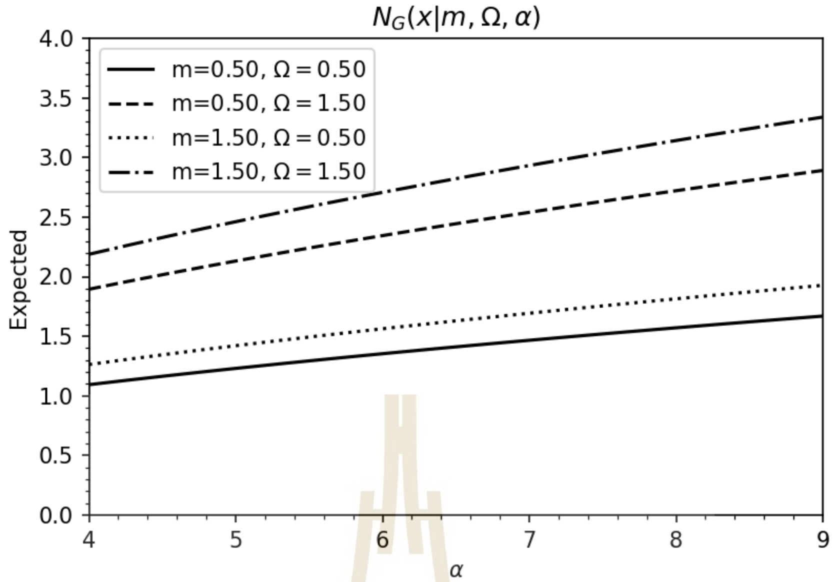

Figure 4.84 Graph of expectation value with $m = 0.5, 1.5, \Omega = 0.5, 1.5$ and varying α of the Nakagami distribution with modulated gamma distribution parameter.

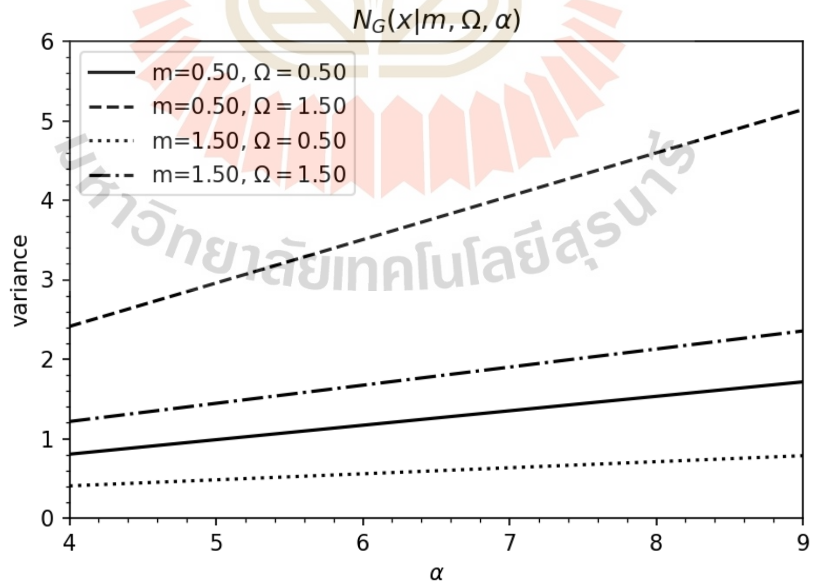

Figure 4.87 Graph of variance value with $m = 0.5, 1.5, \Omega = 0.5, 1.5$ and varying α of the Nakagami distribution with modulated gamma distribution parameter.

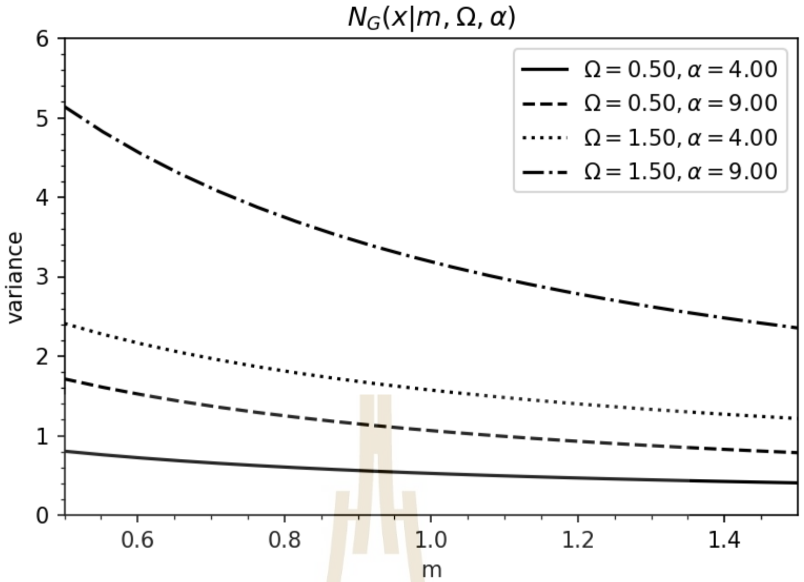

Figure 4.85 Graph of variance value with $\Omega = 0.5, 1.5, \alpha = 4, 9$ and varying m of the Nakagami distribution with modulated gamma distribution parameter.

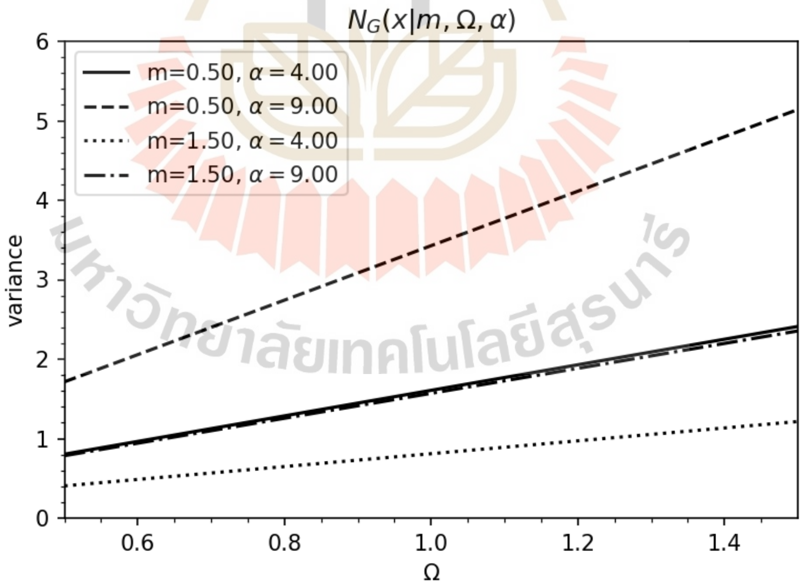

Figure 4.86 Graph of variance value with $m = 0.5, 1, 5, \alpha = 4, 9$ and varying Ω of the Nakagami distribution with modulated gamma distribution parameter.

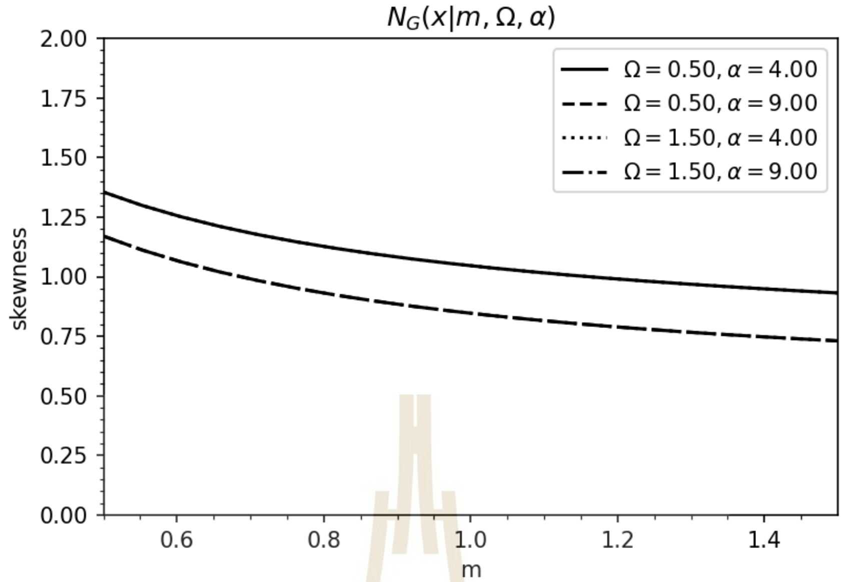

Figure 4.88 Graph of skewness value with $\Omega = 0.5, 1.5, \alpha = 4, 9$ and varying m of the Nakagami distribution with modulated gamma distribution parameter.

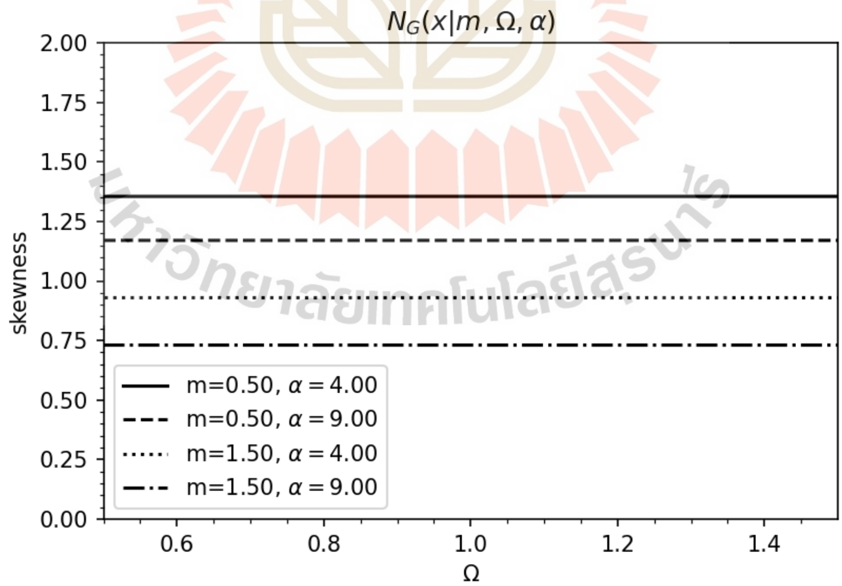

Figure 4.89 Graph of skewness value with $m = 0.5, 1.5, \alpha = 4, 9$ and varying Ω of the Nakagami distribution with modulated gamma distribution parameter.

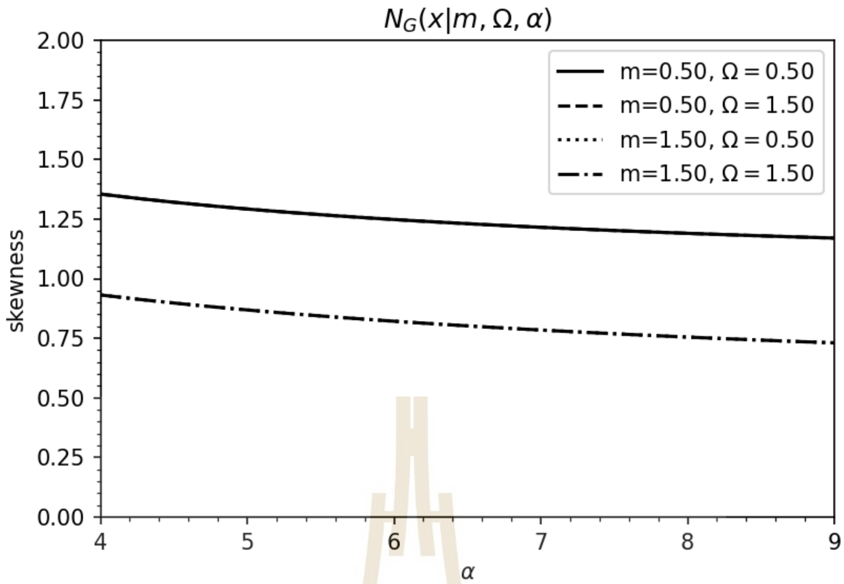

Figure 4.90 Graph of skewness value with $m = 0.5, 1.5, \Omega = 0.5, 1.5$ and varying α of the Nakagami distribution with modulated gamma distribution parameter.

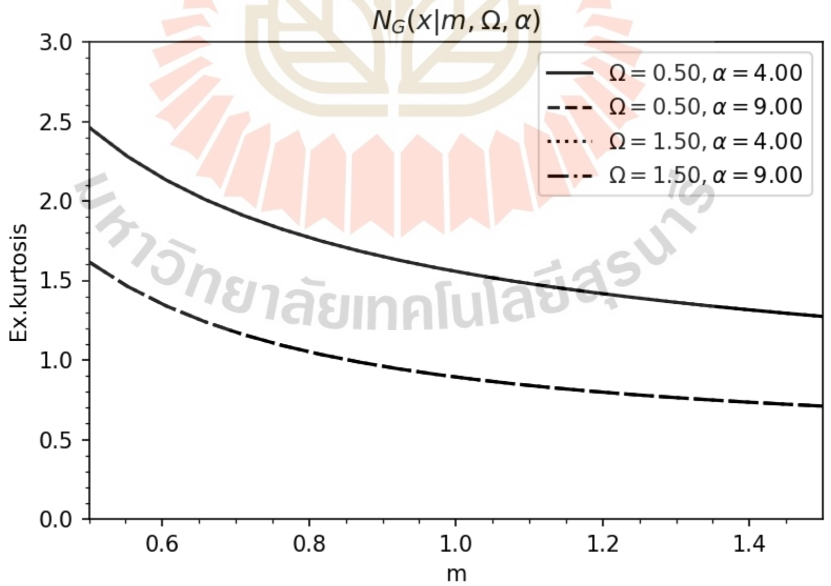

Figure 4.91 Graph of excess kurtosis value with $\Omega = 0.5, 1.5, \alpha = 4, 9$ and varying m of the Nakagami distribution with modulated gamma distribution parameter.

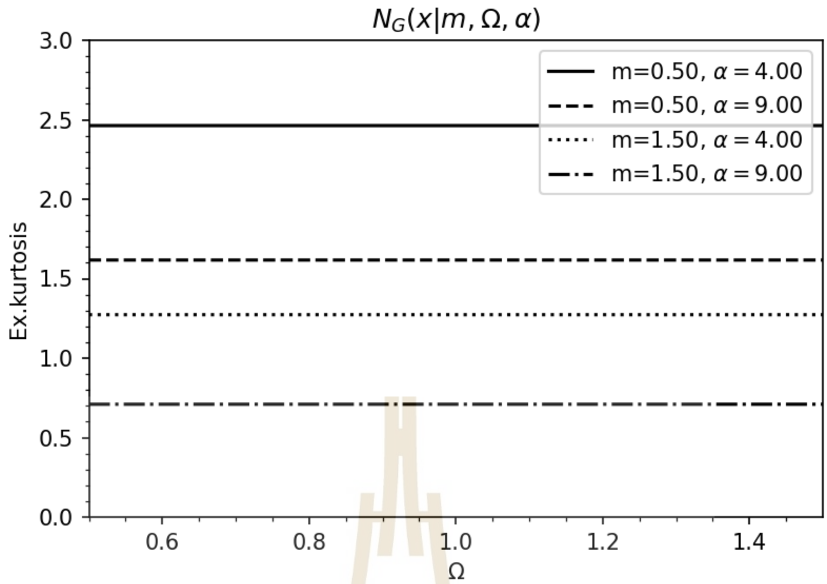

Figure 4.92 Graph of excess kurtosis value with $m = 0.5, 1.5, \alpha = 4, 9$ and varying Ω of the Nakagami distribution with modulated gamma distribution parameter.

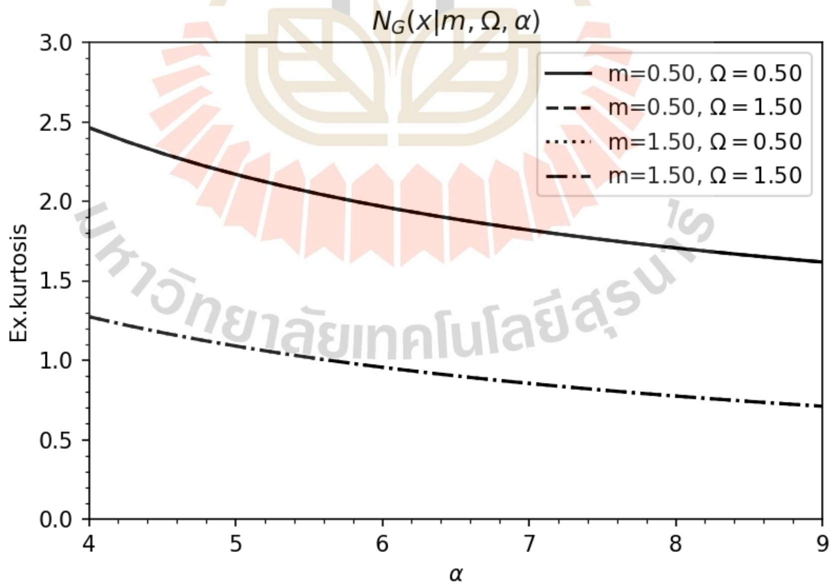

Figure 4.93 Graph of excess kurtosis value with $m = 0.5, 1.5, \Omega = 0.5, 1.5$ and varying α of the Nakagami distribution with modulated gamma distribution parameter.

4.2.10 Nakagami distribution with modulated generalized inverse Gaussian parameters

The graphs of parameters of Nakagami distribution with modulated generalized inverse Gaussian parameters are presented as follows:

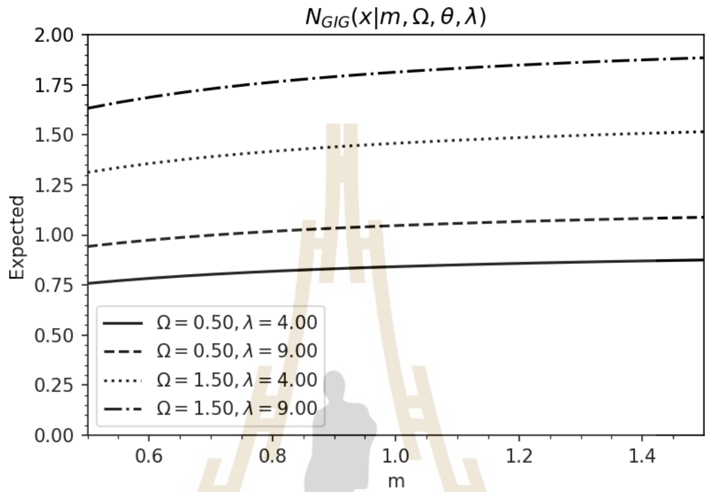

Figure 4.94 Graph of expectation value with $\Omega = 0.5, 1.5, \lambda = 4, 9$ and varying m of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

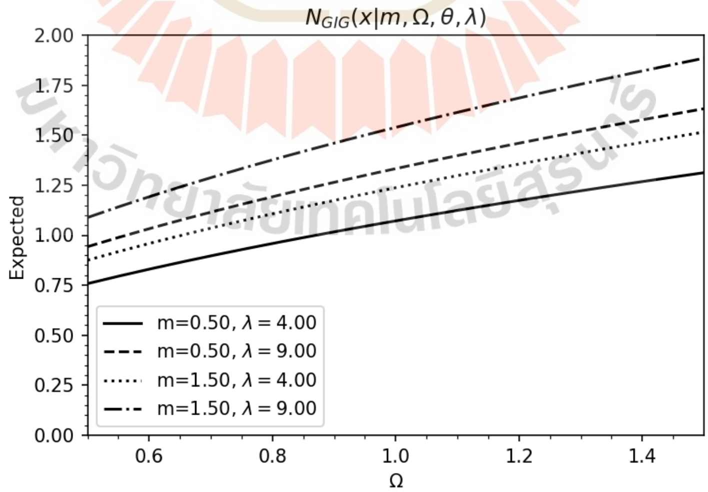

Figure 4.95 Graph of expectation value with $m = 0.5, 1.5, \lambda = 4, 9$ and varying Ω of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

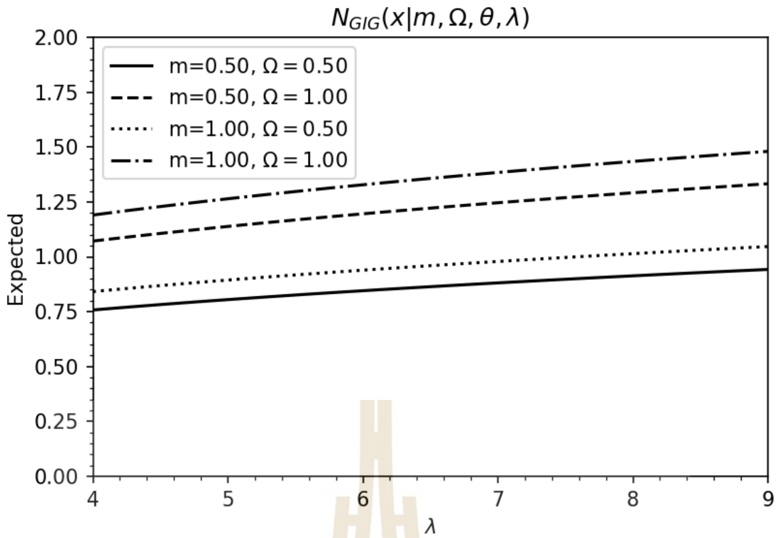

Figure 4.96 Graph of expectation value with $m = 0.5, 1.0, \Omega = 0.5, 1.0$ and varying λ of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

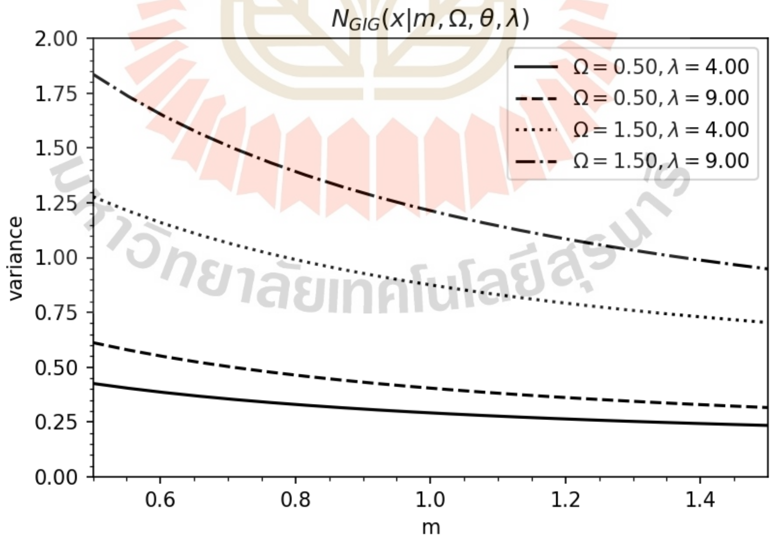

Figure 4.97 Graph of variance value with $\Omega = 0.5, 1.5, \lambda = 4, 9$ and varying m of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

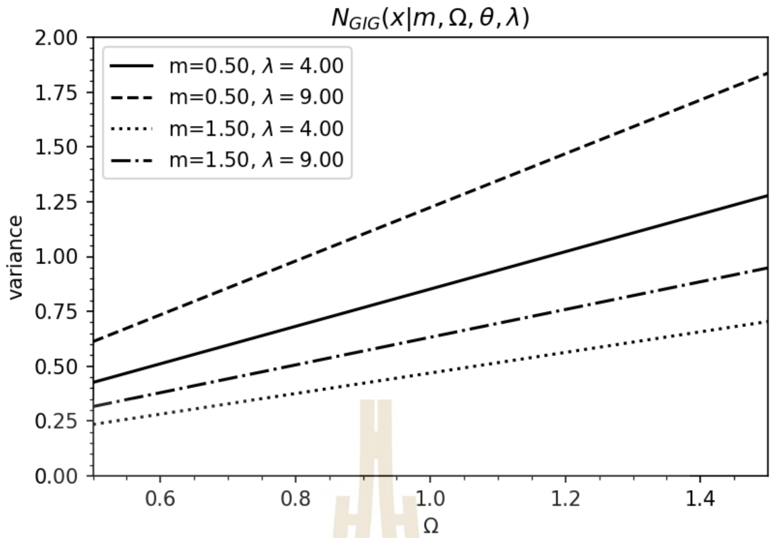

Figure 4.98 Graph of variance value with $m = 0.5, 1, 5, \lambda = 4, 9$ and varying Ω of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

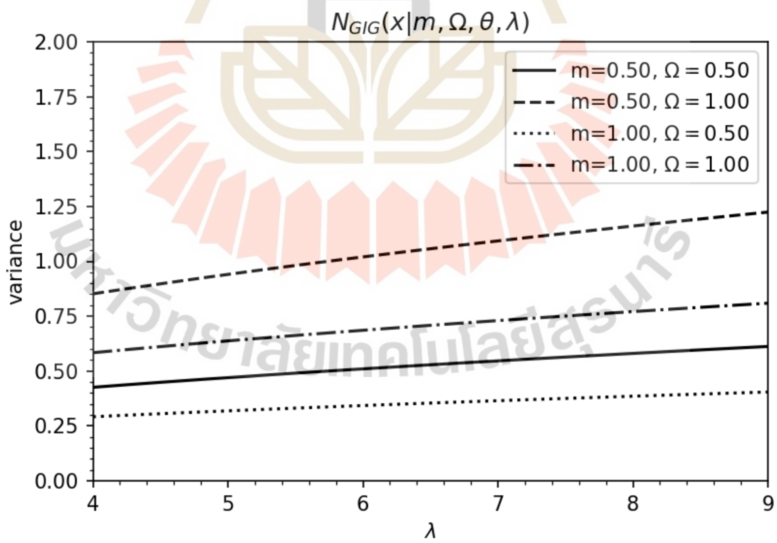

Figure 4.99 Graph of variance value with $m = 0.5, 1.0, \Omega = 0.5, 1.0$ and varying λ of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

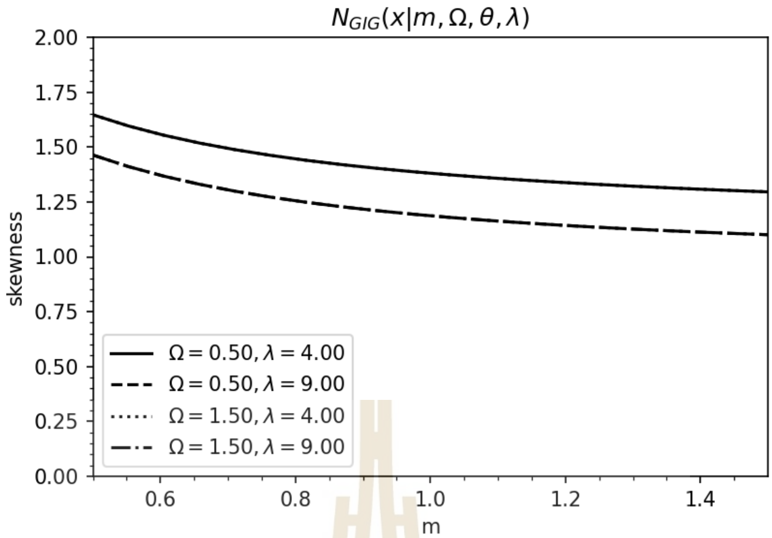

Figure 4.100 Graph of skewness value with $\Omega = 0.5, 1.5, \lambda = 4, 9$ and varying of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

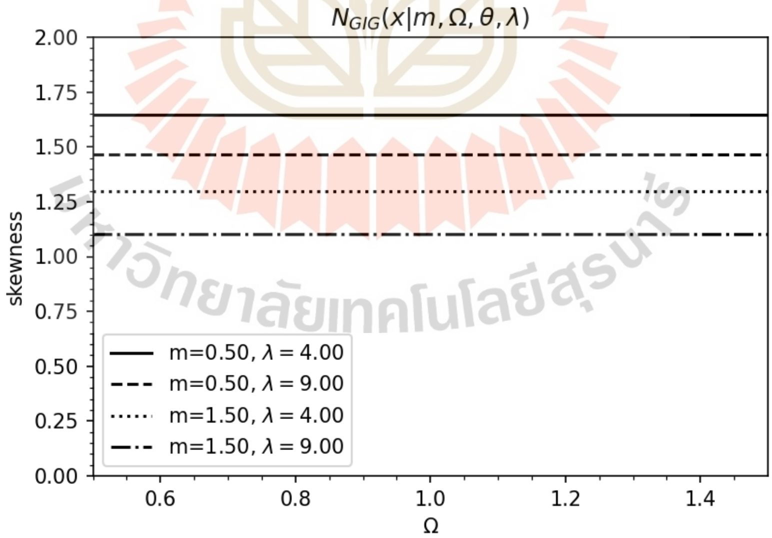

Figure 4.101 Graph of skewness value with $m = 0.5, 1.5, \lambda = 4, 9$ and varying Ω of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

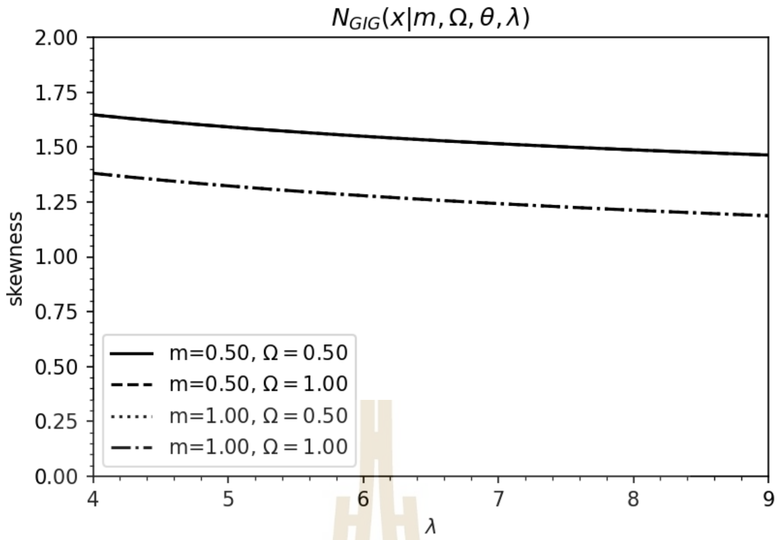

Figure 4.102 Graph of skewness value with $m = 0.5, 1.0, \Omega = 0.5, 1.0$ and varying λ of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

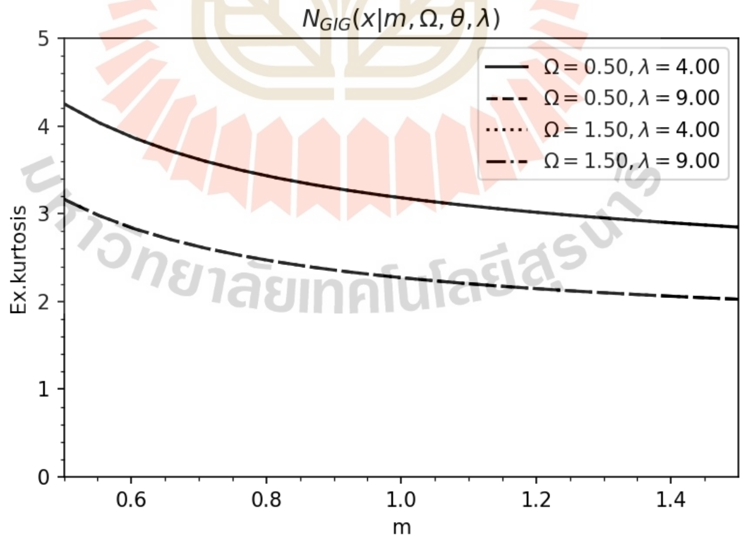

Figure 4.103 Graph of excess kurtosis value with $\Omega = 0.5, 1.5, \lambda = 4, 9$ and varying m of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

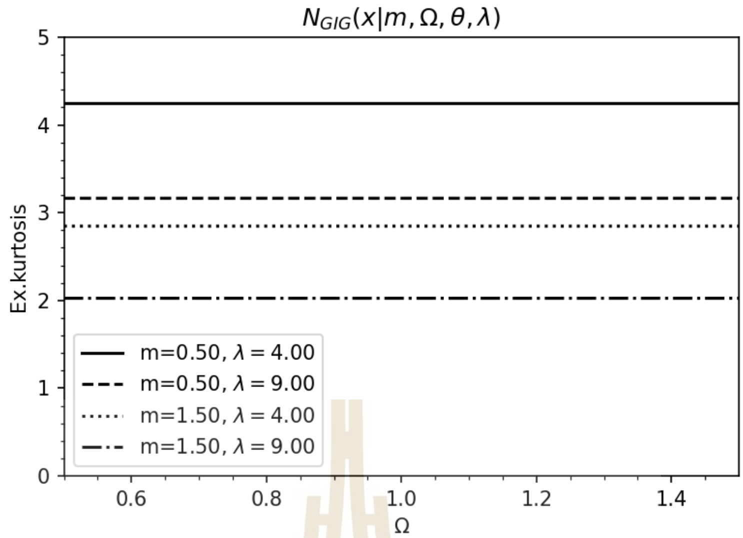

Figure 4.104 Graph of excess kurtosis value with $m = 0.5, 1.5, \lambda = 4, 9$ and varying Ω of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

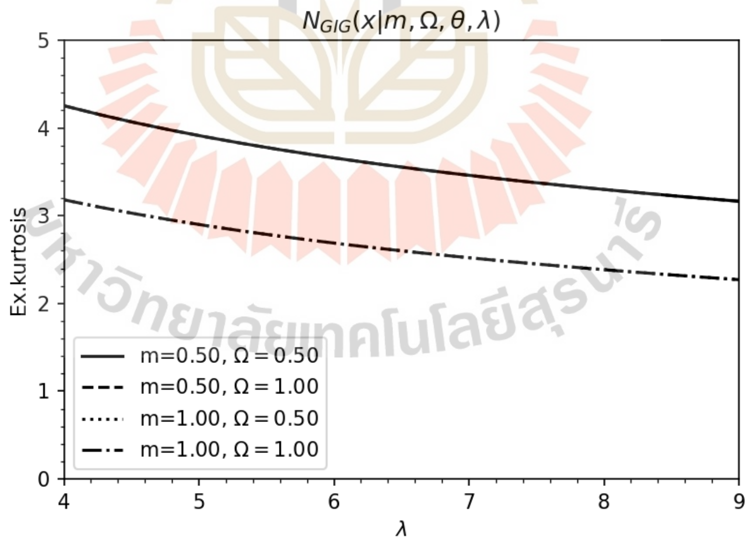

Figure 4.105 Graph of excess kurtosis value with $m = 0.5, 1.0, \Omega = 0.5, 1.0$ and varying λ of the Nakagami distribution with modulated generalized inverse Gaussian parameters.

The following table briefly presents how the observed distributions relate to the medical ultrasound signal model.

Table 4.1 The summarized analysis of statistical distributions appearing in medical ultrasound images.

Distribution	high density of random scatterers	coherent component	Specific density of random scatterers	intensity parameter modulated by gamma distribution	intensity & coherent parameters modulated by gamma distribution	intensity & coherent parameters modulated by IG distribution	intensity & coherent parameters modulated by GIG distribution
Rayleigh	✓						
Rician	✓	✓					
K			✓				
HK		✓	✓	✓			
GK		✓	✓		✓		
RiIG		✓	✓			✓	
Nakagami	✓	✓					
NG		✓	✓			✓	
NGIG		✓	✓				✓

Remark: The words mentioned in table 4.1 mean

- K means K-distribution;
- HK means Homodyned K-distribution;
- GK means Generalized K-distribution;

- IG means inverse Gaussian distribution;
- GIG means generalized inverse Gaussian distribution;
- RiIG means Rician distribution with modulated inverse Gaussian distribution parameters;
- NG means Nakagami distribution with modulated gamma distribution parameter;
- NGIG means Nakagami distribution with modulated generalized inverse Gaussian parameters.

CHAPTER V

CONCLUSION

In this thesis, we performed an analysis of the statistical distributions appearing in medical ultrasound images. For the case of high density of random scatterers, Rayleigh, Rician and Nakagami distributions were proposed for modelling the ultrasound wave. However, just Rician and Nakagami distributions are able to explain in the part of coherent component. If one want to specific density of random scatterers, K-distribution and Homodyned K-distributions can perform modelling an ultrasound wave, whereas Homodyned K-distributions is extended to be able to explain in the part of coherent component. However, some distributions modulated parameters with some other distributions are proposed for modelling, i.e. Homodyned K-distributions, Generalized K-distribution, Rician distribution with modulated inverse Gaussian distribution parameters, Nakagami distribution with modulated gamma distribution parameter and Nakagami distribution with modulated generalized inverse Gaussian parameters. The mentioned distributions are able to describe an ultrasound wave model in the case of specific density of random scatterers with coherent component.

The graphs of the relation between number parameters, σ, ε, α, λ, m or Ω and distribution parameters mean, variance, skewness and kurtosis of the considered distributions (presented in Chapter IV) perform that we can adjust the number parameters of some distribution to make its structure similar to another one's structure.

In our research, Nakagami distribution with modulated generalized inverse

Gaussian parameters is the most generalized distribution which relates to medical ultrasound image. The mentioned distribution is able to explain physical phenomenon with the specification of the density of random scatterers and coherent component. Also its parameters are modulated by more generalized distribution, generalized inverse Gaussian distribution, which is an extend concept of gamma distribution and inverse Gaussian distribution.

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APPENDIX

APPENDIX A

APPLICATION OF PYTHON IN MEDICAL IMAGE PROCESSING

This chapter presents some Python commands using in this thesis.

A.1 Graph of distributions by Python

A.1.1 Loading library

In order to use some commands of Python for plotting graph, these libraries should be loaded, matplotlib.pyplot, cycler, numpy, scipy.stats, scipy.integrate. The code is as follows.

```
import mathplotlib.pyplot as plt
from cycler import cycler
import numpy as np
from scipy.stats import geninvgauss as gig
from scipy.integrate import quad
from scipy.stats import nakagami as nak
```

A.1.2 Configuration of graph plotting

In order to configure the line style in graph plotting the command is as follows.

```
plt.rc('axes', prop_cycle=(cycler(linestyle=['-', '–', ':', '-.']) +
cycler(color=['k','k','k','k'])))
```

A.1.3 Function defining

The pdf functions of the Nakagami distribution, Generalized inverse Gaussian distribution and Nakagami distribution with modulated generalized inverse Gaussian parameters are defined as follows.

- **Nakagami distribution**

```
def N(x, m, ome):
omesqrt = np.sqrt(ome)
return nak.pdf(x, m, scale=omesqrt)
```

- **Generalized inverse Gaussian distribution**

```
def GIG(w,lam):
mu = np.sqrt(lam)
b = np.divide(lam, mu)
return gig.pdf(w, -.5, b, scale=mu) #\theta=-.5
```

- **Nakagami distribution with modulated generalized inverse Gaussian parameters**

```
def f(w, A, m, ome, lam):
return np.multiply(N(A, m, np.multiply(ome, w)), GIG(w, lam))
def NGIG(A, m, ome, lam):
return quad(f, 0, np.inf, args=(A, m, ome, lam))[0]
```

A.1.4 Initiate variables

Variables of the considered distribution can be defined as follows.

```
A = np.linspace(0.01, 10, 100)
mu = np.array([1])
ome = np.array([2])
lam = np.array([1, 2, 3])
car = np.vstack(np.meshgrid(mu, ome, lam)).reshape(3,-1).T
y_pdf = np.zeros((car.shape[0], A.size))
```

A.1.5 Assign values to variables

To assign values calculated from the considered distribution, we can code as follows.

```
for i, j in zip(car, np.arange(car.shape[0])):
for k, l in zip(A, np.arange(A.size)):
y_pdf[j][l] = NGIG(k, i[0], i[1], i[2])
```

A.1.6 Graph display

To display graph of each distribution, the code is as follows.

```
plt.figure(dpi=150)
for i, j in zip(car, np.arange(car.shape[0])):
plt.plot(A, np.round(y_pdf[j], 6),
label=r"m=, $\Omega={}$,$\lambda={ }$".format(i[0], i[1], i[2]))
plt.title(r"GRAPH TITLE")
plt.ylabel("Probability density")
plt.xlabel("Amplitude $A$")
plt.ylim(0, 0.8)
plt.xlim(A.min(), A.max())
plt.minorticks_on()
plt.legend();
plt.savefig('PixName.png')
```

A.2 Graph of parameters of distributions by Python

A.2.1 Moment functions defining

In order to define moment functions of the distribution, the code is as follows.

```
def f0(A, m, ome, the, lam):
return NGIG(A, m, ome, the, lam)
def g0(A, m, ome, the, lam, c, k):
return np.multiply(np.power(np.subtract(A, c), k), f0(A, m, ome, the, lam))
def mo(m, ome, the, lam, c, k):
return quad(g0, 0, 100, args=(m, ome, the, lam, c, k))[0]
```

A.2.2 Expected functions defining

In order to define expected functions of the distribution, the code is as follows.

```
def expe(m, ome, lam):
return mo(m, ome, -.5, lam, 0, 1)
```

A.2.3 Initiate parameters

Parameters m, Ω and λ of the considered distribution can be defined as follows.

```
x_m = np.linspace(0.5, 1.5, 20)
ome1 = np.array([0.5, 1.5])
lam1 = np.array([4, 9])
car1 = np.array(np.meshgrid(ome1, lam1)).T.reshape(-1,2)
```

```
x_ome = np.linspace(0.5, 1.5, 20)
m2 = np.array([0.5, 1.5])
lam2 = np.array([4, 9])
car2 = np.array(np.meshgrid(m2, lam2)).T.reshape(-1,2)

x_lam = np.linspace(4, 9, 20)
m3 = np.array([0.5, 1])
ome3 = np.array([0.5, 1])
car3 = np.array(np.meshgrid(m3, ome3)).T.reshape(-1,2)

y_m_exp = np.zeros((car1.shape[0], x_m.size))
y_ome_exp = np.zeros((car2.shape[0], x_ome.size))
y_lam_exp = np.zeros((car3.shape[0], x_lam.size))
```

A.2.4 Expected calculation

In order to calculate parameters m, Ω and λ of the considered distribution, the code is as follow.

1. m:

```
for i, j in zip(car1, np.arange(car1.shape[0])):
for k, l in zip(x_m, np.arange(x_m.size)):
y_m_exp[j][l] = expe(k, i[0], i[1])
```

2. Ω:

```
for i, j in zip(car2, np.arange(car2.shape[0])):
for k, l in zip(x_ome, np.arange(x_ome.size)):
y_ome_exp[j][l] = expe(i[0], k, i[1])
```

3. λ:

```
for i, j in zip(car3, np.arange(car3.shape[0])):
for k, l in zip(x_lam, np.arange(x_lam.size)):
y_lam_exp[j][l] = expe(i[0], i[1], sk )
```

A.2.5 Expected value graph display

To display graph of expected value of each distribution, the code is as follows.

1. m:

```
plt.figure(dpi=150)
for i, j in zip(car1, np.arange(car1.shape[0])):
plt.plot(x_m, np.round(y_m_exp[j], 6),
label=r"$\Omega={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'Expected')
plt.xlabel(r'm')
plt.ylim(0, 2)
plt.xlim(x_m.min(), x_m.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

2. Ω:

```
plt.figure(dpi=150)
for i, j in zip(car2, np.arange(car2.shape[0])):
plt.plot(x_ome, np.round(y_ome_exp[j], 6),
label=r"$m={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
```

```
plt.title(r"GRAPH TITLE")
plt.ylabel(r'Expected')
plt.xlabel(r'\Omega')
plt.ylim(0, 2)
plt.xlim(x_ome.min(), x_ome.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

3. λ:

```
plt.figure(dpi=150)
for i, j in zip(car3, np.arange(car3.shape[0])):
plt.plot(x_lam, np.round(y_lam_exp[j], 6),
label=r"$m={:.2f}, \Omega={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'Expected')
plt.xlabel(r'\lambda')
plt.ylim(0, 2)
plt.xlim(x_lam.min(), x_lam.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

A.2.6 k-th central moment function defining

For k-th central moment of a distribution with parameter m, Ω or λ, one can find by:

1. m:

```
def mo_me_m(m, ome, lam, k, m_n, car_n):
return mo(m, ome, -.5, lam, y_m_exp[car_n][m_n], k)
```

2. Ω:

```
def mo_me_ome(m, ome, lam, k, ome_n, car_n):
return mo(m, ome, -.5, lam, y_ome_exp[car_n][ome_n], k)
```

3. λ:

```
def mo_me_lam(m, ome, lam, k, lam_n, car_n):
return mo(m, ome, -.5, lam, y_lam_exp[car_n][lam_n], k)
```

A.2.7 Variance function defining

For defining variance function of each parameter m, Ω and λ, one can find by:

1. m:

```
def var_m(m, ome, lam, m_n, car_n):
return mo_me_m(m, ome, lam, 2, m_n, car_n)
```

2. Ω:

```
def var_ome(m, ome, lam, ome_n, car_n):
return mo_me_ome(m, ome, lam, 2, ome_n, car_n)
```

3. λ:

```
def var_lam(m, ome, lam, lam_n, car_n):
return mo_me_lam(m, ome, lam, 2, lam_n, car_n)
```

A.2.8 Variance calculation

In order to calculate variance, then the code is as follows.

1. m:

```
y_m_var = np.zeros((car1.shape[0], x_m.size))
for i, j in zip(car1, np.arange(car1.shape[0])):
for k, l in zip(x_m, np.arange(x_m.size)):
y_m_var[j][l] = var_m(k, i[0], i[1], l, j)
```

2. Ω:

```
y_ome_var = np.zeros((car1.shape[0], x_ome.size))
for i, j in zip(car2, np.arange(car2.shape[0])):
for k, l in zip(x_ome, np.arange(x_ome.size)):
y_ome_var[j][l] = var_ome(i[0], k, i[1], l, j)
```

3. λ:

```
y_lam_var = np.zeros((car1.shape[0], x_lam.size))
for i, j in zip(car3, np.arange(car3.shape[0])):
for k, l in zip(x_lam, np.arange(x_lam.size)):
y_lam_var[j][l] = var_lam(i[0], i[1], k, l, j)
```

A.2.9 Variance value graph display

To display graph of Variance value of each distribution, the code is as follows.

1. m:

```
plt.figure(dpi=150)
for i, j in zip(car1, np.arange(car1.shape[0])):
plt.plot(x_m, np.round(y_m_var[j], 6),
label=r"$\Omega={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'variance')
plt.xlabel(r'm')
plt.ylim(0, 2)
plt.xlim(x_m.min(), x_m.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

2. Ω:

```
plt.figure(dpi=150)
for i, j in zip(car2, np.arange(car2.shape[0])):
plt.plot(x_ome, np.round(y_ome_var[j], 6),
label=r"$m={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'variance')
plt.xlabel(r'\Omega')
plt.ylim(0, 2)
```

```
plt.xlim(x_ome.min(), x_ome.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

3. λ:

```
plt.figure(dpi=150)
for i, j in zip(car3, np.arange(car3.shape[0])):
plt.plot(x_lam, np.round(y_lam_var[j], 6),
label=r"$m={:.2f}, \Omega={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'variance')
plt.xlabel(r'\lambda')
plt.ylim(0, 2)
plt.xlim(x_lam.min(), x_lam.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

A.2.10 Standardized moment function defining

For Standardized moment of a distribution with parameter m, Ω or λ, one can find by:

1. m:

```
std_m = np.sqrt(y_m_var.copy())
def std_mo_m(m, ome, lam, k, m_n, car_n):
return np.divide(mo_me_m(m, ome, lam, k, m_n, car_n),
np.power(std_m[car_n][m_n], k))
```

2. Ω:

```
std_ome = np.sqrt(y_ome_var.copy())
def std_mo_ome(m, ome, lam, k, ome_n, car_n):
return np.divide(mo_me_ome(m, ome, lam, k, ome_n, car_n),
np.power(std_ome[car_n][ome_n], k))
```

3. λ:

```
std_lam = np.sqrt(y_lam_var.copy())
def std_mo_lam(m, ome, lam, k, lam_n, car_n):
return np.divide(mo_me_lam(m, ome, lam, k, lam_n, car_n),
np.power(std_lam[car_n][lam_n], k))
```

A.2.11 Skewness function defining

For Skewness of a distribution with parameter m, Ω or λ, one can find by:

1. m:

```
def skew_m(m, ome, lam, m_n, car_n):
return std_mo_m(m, ome, lam, 3, m_n, car_n)
```

2. Ω:

```
def skew_ome(m, ome, lam, ome_n, car_n):
return std_mo_ome(m, ome, lam, 3, ome_n, car_n)
```

3. λ:

```
def skew_lam(m, ome, lam, lam_n, car_n):
return std_mo_lam(m, ome, lam, 3, lam_n, car_n)
```

A.2.12 Skewness calculation

In order to calculate Skewness, then the code is as follows.

1. m:

```
y_m_skew = np.zeros((car1.shape[0], x_m.size))
for i, j in zip(car1, np.arange(car1.shape[0])):
for k, l in zip(x_m, np.arange(x_m.size)):
y_m_skew[j][l] = skew_m(k, i[0], i[1], l, j)
```

2. Ω:

```
y_ome_skew = np.zeros((car1.shape[0], x_ome.size))
for i, j in zip(car2, np.arange(car2.shape[0])):
for k, l in zip(x_ome, np.arange(x_ome.size)):
y_ome_skew[j][l] = skew_ome(i[0], k, i[1], l, j)
```

3. λ:

```
y_lam_skew = np.zeros((car1.shape[0], x_lam.size))
for i, j in zip(car3, np.arange(car3.shape[0])):
for k, l in zip(x_lam, np.arange(x_lam.size)):
y_lam_skew[j][l] = skew_lam(i[0], i[1], k, l, j)
```

A.2.13 Skewness value graph display

To display graph of Skewness value of each distribution, the code is as follows.

1. m:

```
plt.figure(dpi=150)
for i, j in zip(car1, np.arange(car1.shape[0])):
plt.plot(x_m, np.round(y_m_skew[j], 6),
```

```
label=r"$\Omega={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'skewness')
plt.xlabel(r'm')
plt.ylim(0, 2)
plt.xlim(x_m.min(), x_m.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

2. Ω:

```
plt.figure(dpi=150)
for i, j in zip(car2, np.arange(car2.shape[0])):
plt.plot(x_ome, np.round(y_ome_skew[j], 6),
label=r"$m={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'skewness')
plt.xlabel(r'\Omega')
plt.ylim(0, 2)
plt.xlim(x_ome.min(), x_ome.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

3. λ:

```
plt.figure(dpi=150)
for i, j in zip(car3, np.arange(car3.shape[0])):
```

```
plt.plot(x_lam, np.round(y_lam_skew[j], 6),
label=r"$m={:.2f}, \Omega={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'skewness')
plt.xlabel(r'\lambda')
plt.ylim(0, 2)
plt.xlim(x_lam.min(), x_lam.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

A.2.14 Kurtosis function defining

For Kurtosis of a distribution with parameter m, Ω or λ, one can find by:

1. m:

```
def kur_m(m, ome, lam, m_n, car_n):
return std_mo_m(m, ome, lam, 4, m_n, car_n)
```

2. Ω:

```
def kur_ome(m, ome, lam, ome_n, car_n):
return std_mo_ome(m, ome, lam, 4, ome_n, car_n)
```

3. λ:

```
def kur_lam(m, ome, lam, lam_n, car_n):
return std_mo_lam(m, ome, lam, 4, lam_n, car_n)
```

A.2.15 Kurtosis calculation

In order to calculate Kurtosis, then the code is as follows.

1. m:

```
y_m_kur = np.zeros((car1.shape[0], x_m.size))
for i, j in zip(car1, np.arange(car1.shape[0])):
for k, l in zip(x_m, np.arange(x_m.size)):
y_m_kur[j][l] = kur_m(k, i[0], i[1], l, j)
```

2. Ω:

```
y_ome_kur = np.zeros((car1.shape[0], x_ome.size))
for i, j in zip(car2, np.arange(car2.shape[0])):
for k, l in zip(x_ome, np.arange(x_ome.size)):
y_ome_kur[j][l] = kur_ome(i[0], k, i[1], l, j)
```

3. λ:

```
y_lam_kur = np.zeros((car1.shape[0], x_lam.size))
for i, j in zip(car3, np.arange(car3.shape[0])):
for k, l in zip(x_lam, np.arange(x_lam.size)):
y_lam_kur[j][l] = kur_lam(i[0], i[1], k, l, j)
```

A.2.16 Kurtosis value graph display

To display graph of Kurtosis value of each distribution, the code is as follows.

1. m:

```
plt.figure(dpi=150)
for i, j in zip(car1, np.arange(car1.shape[0])):
plt.plot(x_m, np.round(y_m_kur[j], 6),
```

```
label=r"$\Omega={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'kurtosis')
plt.xlabel(r'm')
plt.ylim(0, 2)
plt.xlim(x_m.min(), x_m.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

2. Ω:

```
plt.figure(dpi=150)
for i, j in zip(car2, np.arange(car2.shape[0])):
plt.plot(x_ome, np.round(y_ome_kur[j], 6),
label=r"$m={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'kurtosis')
plt.xlabel(r'\Omega')
plt.ylim(0, 2)
plt.xlim(x_ome.min(), x_ome.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

3. λ:

```
plt.figure(dpi=150)
for i, j in zip(car3, np.arange(car3.shape[0])):
plt.plot(x_lam, np.round(y_lam_kur[j], 6),
```

```
label=r"$m={:.2f}, \Omega={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'kurtosis')
plt.xlabel(r'\lambda')
plt.ylim(0, 2)
plt.xlim(x_lam.min(), x_lam.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

A.2.17 Excess kurtosis calculation

In order to calculate Excess kurtosis, then the code is as follows.

1. m:

```
y_m_ex_kur = y_m_kur - 3
```

2. Ω:

```
y_ome_ex_kur = y_ome_kur - 3
```

3. λ:

```
y_lam_ex_kur = y_lam_kur - 3
```

A.2.18 Excess kurtosis value graph display

To display graph of Excess kurtosis value of each distribution, the code is as follows.

1. m:

```
plt.figure(dpi=150)
for i, j in zip(car1, np.arange(car1.shape[0])):
```

```
plt.plot(x_m, np.round(y_m_ex_kur[j], 6),
label=r"$\Omega={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'Ex.kurtosis')
plt.xlabel(r'm')
plt.ylim(0, 2)
plt.xlim(x_m.min(), x_m.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

2. Ω:

```
plt.figure(dpi=150)
for i, j in zip(car2, np.arange(car2.shape[0])):
plt.plot(x_ome, np.round(y_ome_ex_kur[j], 6),
label=r"$m={:.2f}, \lambda={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'Ex.kurtosis')
plt.xlabel(r'\Omega')
plt.ylim(0, 2)
plt.xlim(x_ome.min(), x_ome.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

3. λ:

```
plt.figure(dpi=150)
for i, j in zip(car3, np.arange(car3.shape[0])):
```

```
plt.plot(x_lam, np.round(y_lam_ex_kur[j], 6),
label=r"$m={:.2f}, \Omega={:.2f}$".format(i[0], i[1]))
plt.title(r"GRAPH TITLE")
plt.ylabel(r'Ex.kurtosis')
plt.xlabel(r'\lambda')
plt.ylim(0, 2)
plt.xlim(x_lam.min(), x_lam.max())
plt.minorticks_on()
plt.legend();
plt.savefig('GRAPHNAME.png')
```

Remark The proposed code can be adjusted for finding expected value, variance, skewness and kurtosis for other distributions by replacing the distribution function.

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