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Handbook for
Mathematical Olympiads Basics 1:
Number Theory

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Preface

This Handbook is no way original, but a compilation of many problems and books to enable bright Thai math students to take advantage of the internet. The few weeks of instruction in Olympics Camp, I hope, will be just the spark to help the enterprising student to "boldly go where no one has gone before," in life and in math.

Enjoy!

CHAPTER 1

Strategies

1. The Pigeon Hole Principle

Pigeon Hole Principle If $kn+1$ objects ($k \leq 1$) are distributed among n boxes, one of the boxes will contain at least $k+1$ objects.

PROBLEM 1. *Given a set of $n+1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.*

PROBLEM 2. *Prove that there exist integers a, b, c not all zero and each of absolute value less than one million, such that*

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$$

PROBLEM 3. *Given any set of ten natural numbers between* 1 *and* 99 *inclusive (decimal notation), prove that there are two disjoint nonempty subsets of the set with equal sums of their elements.*

PROBLEM 4. *Let* A *be any set of* 20 *distinct integers chosen from the arithmetic progression* $1, 4, 7, \ldots, 100$. *Prove that there must be two distinct integers in* A *whose sum is* 104.

PROBLEM 5. *Let X be any real number. Prove that among the numbers*

$$X, 2X, \ldots, (n-1)X$$

there is one that differs from an integer by at most $1/n$.

PROBLEM 6. *Prove that no seven positive integers, not exceeding 24, can have sums of all subsets different.*

CHAPTER 2

Number Theory

1. Warmups

PROPOSITION 1. *The following will be used:*

(1) *If a number a divides a product bc and is prime to one factor b, it must divide the other factor c.*

(2) *If a prime number a divides a product $bcd\ldots$, it must divide one of the factors of that product.*

(3) *If a is prime to each of the numbers b and $c4$, it is prime to the product bc.*

(4) *If a and b are prime to each other, every positive integral power of a is prime to every positive integral power of b.*

(5) *If a is prime to b, the fractions $\frac{a}{b}$ and $\frac{a^n}{b^m}$ are in their lowest terms, n and m being any positive integers. Also if $\frac{a}{b}$ and $\frac{c}{d}$ are any two equal fractions, and $\frac{a}{b}$ is in its lowest terms, then c and d must be equimultiples of a and b respectively.*

PROPOSITION 2. *The number of primes is infinite.*

PROPOSITION 3. *No rational algebraical formula can represent prime numbers only.*

PROPOSITION 4. *A number can be resolved into prime factors in only one way.*

PROPOSITION 5. *To find the number of divisors of a composite number.*

PROPOSITION 6. *To find the number of ways in which a composite number can be resolved into two factors.*

PROPOSITION 7. *To find the number of ways in which a composite number can be resolved into two factors which are prime to each other.*

PROPOSITION 8. *To find the sum of the divisors of a number.*

EXAMPLE 1. *Consider the number* 216000.

EXAMPLE 2. *If* n *is odd, show that* $n(n^2-1)$ *is divisible by* 24.

EXAMPLE 3. *Find the highest power of* 3 *which is contained in* 100!.

PROPOSITION 9. *To find the highest power of a prime number* a *which is contained in* $n!$.

PROPOSITION 10. *To prove that the product of* r *consecutive integers is divisible by* $r!$.

PROPOSITION 11. *If p is a prime number,the coefficient of every term in the expansion of $(a+b)^p$, except the first and last, is divisible by p.*

PROPOSITION 12. *If p is a prime number, to prove that*

$$(a+b+c+d+\cdots)^p = a^p + b^p + c^p + d^p + \cdots + M(p).$$

PROPOSITION 13. *(Fermat's Little Theorem) If p is a prime number and N is prime to p, then $N^{p-1}-1$ is a multiple of p.*

COROLLARY 1. *Since p is prime, $p-1$ is an even number except when $p=2$. Therefore*

$$(N^{\frac{p-1}{2}}+1)(N^{\frac{p-1}{2}}-1)=M(p).$$

REMARK 1. $N^p - N = M(p)$ whether N is prime to p or not.

EXAMPLE 4. *Show that $n^7 - n$ is divisible by 42.*

EXAMPLE 5. *Prove that every square number is of the form $5n$ or $5n+1$.*

EXAMPLE 6. *If x and y are positive integers, and if $x - y$ is even, shew that $x^2 - y^2$ is divisible by 4.*

EXAMPLE 7. *If $4x - y$ is a multiple of 3, show that $4x^2 + 7xy - 2y^2$ is divisible by 9.*

EXAMPLE 8. *In how many ways can the number 7056 be resolved into two factors?*

EXAMPLE 9. *Prove that* $2^{4n} - 1$ *is divisible by* 15.

EXAMPLE 10. *Prove that* $n(n+1)(n+6)$ *is a multiple of* 6.

EXAMPLE 11. *If* n *is even, shew that* $n(n^2+20)$ *is divisible by* 48.

EXAMPLE 12. *If a number is both square and cube, shew that it is of the form* $7n$ *or* $7n+1$.

EXAMPLE 13. *If* n *is a prime number greater than* 5, *shew that* n^4-1 *is divisible by* 240.

2. Greatest common divisor

PROBLEM 7. *Find all functions f which satisfy the three conditions:*

(1) $f(x, x) = x$,
(2) $f(x, y) = f(y, x)$,
(3) $f(x, y) = f(x, x + y)$.

PROPOSITION 14. ***(Division Algorithm)*** *If a and b are arbitrary integers, $b > 0$, there are unique inters q and r such that*

$$a = qb + r,$$

when $0 \leq r < b$.

PROPOSITION 15. *Given positive integers a and b, there are integers s and t such that*

$$sa + tb = \gcd(a, b).$$

PROBLEM 8. *Find integers x and y such that*

$$754x + 221y = \gcd(754, 221).$$

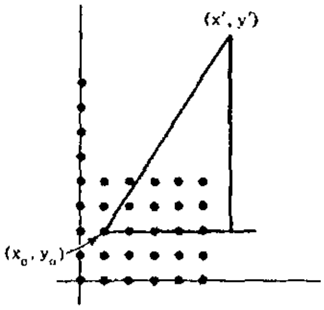

Figure 3.1.

FIGURE 1. Lattice points in plane

PROPOSITION 16. *The equation* $ax + by = c$*,* a*,* b*,* c *integers, has a solution in integers* x *and* y *if and only if* $\gcd(a, b)$ *divides* c*. Moreover, if* (x_0, y_0) *is an integer solution, then for each integer* k*, the values,*

$$\begin{aligned} x' &= x_0 + bk/d, \\ y' &= y_0 - ak/d, \end{aligned}$$

where $d = \gcd(a, b)$ *are also a solution and all integer solutions are of this form.*

PROBLEM 9. *Prove that the fraction* $(21n+4)/(14n+3)$ *is irreducible for every natural number* n.

PROBLEM 10. *The measure of a given angle is* $180°/n$, *where* n *is a positive integer not divisible by* 3. *Prove that the angle can be trisected by Euclidean means (straightedge and compass).*

PROBLEM 11. *If* $\gcd(a,b) = 1$, *prove that:*

(1) $\gcd(a-b, a+b) \leq 2,$

(2) $\gcd(a-b, a+b, ab) = 1,$

(3) $\gcd(a^2-ab+b^2, a+b) \leq 3.$

PROBLEM 12. *When Mr. Smith cashed a check for x dollars and y cents, he received instead y dollars and x cents, and found that he and two cents more than twice the proper amount. For how much was the check written?*

PROBLEM 13. *Find the smallest positive integer a for which*

$$1001x + 770y = 1,000,000 + a$$

is possible, and show that it has then 100 *solutions in positive integers.*

3. Modular Arithmetic

3.1. Review.

REMARK 2. *If a is any number, then any other number N may be expressed in the form $N = aq + r$, where q is the integral quotient when N is divided by a, and r is a remainder less than a. The number a is sometimes called the* modulus*; and to any given modulus a there are a different* forms *of a number N, each form corresponding to a different value of r.*

DEFINITION 1. *If b, c are two integers, which when divided by a leave the same remainder, they are said to be* **congruent** *with repsect to the modulus a. In this case, $b - c$ is a multiple of a and following the notation of Gauss we shall sometimes express this as follows:*

$$b \equiv c \pmod{a},$$

or $b - c \equiv 0 \pmod{a}$. Either of these formulae is called a **congruence**.

PROPOSITION 17. *If b, c are congruent with respect to modulus a, then pb and pc are congruent, p being any integer.*

PROPOSITION 18. *If a is prime to b, and the quantities*

$$a, 2a, 3a, \ldots, (b-1)a$$

are divided by b, the remainders are all different.

COROLLARY 2. *If a is prime to b, and c is any number, the b terms of the arithmetic progression*

$$c, c+a, c+2b, \ldots, c+(b-1)a,$$

when divided by b will leave the same remainders as the terms of the series

$$c, c+1, c+2, \ldots, c+(b-1),$$

although not necessarily in this order; and therefore the remainders will be $0, 1, 2, \ldots, b-1$.

PROPOSITION 19. *(Fermat's Little Theorem again) If p is a prime number and N prime to p, then $N^{p-1} - 1$ is a multiple of p.*

DEFINITION 2. *Let the symbol $\phi(a)$ denote the number of integers less than a number a and prime to it.*

PROPOSITION 20. *To shew that if the numbers $a, b, c, d, \ldots$ are prime to each other,*

$$\phi(abcd\cdots) = \phi(a)\cdot\phi(b)\cdot\phi(c)\cdots$$

PROPOSITION 21. *To find the number of positive integers less than a given number, and prime to it.*

EXAMPLE 14. *Shew that the sum of all the integers which are less than N and prime to it is $\frac{1}{2}N\phi(N)$.*

COROLLARY 3. *From the last article, it follows that the number of integers which are less than N and* not *prime to it is:*

$$N - N(1 - \frac{1}{a})(1 - \frac{1}{b})(1 - \frac{1}{c})(1 - \frac{1}{d})\cdots$$

PROPOSITION 22. *(Wilson's Theorem) If p is a prime number, $1 + (p - 1)!$ is divisible by p.*

COROLLARY 4. *If $2p+1$ is a prime number, $(p!)^2+(-1)^p$ is divisible by $2p+1$.*

EXAMPLE 15. *(Induction) If p is a prime number, $x^p - x$ is divisible by p.*

EXAMPLE 16. *(Induction) Prove that* $5^{2n+2} - 24n - 25$ *is divisible by* 576.

EXAMPLE 17. *Show that* $10^n + 3 \cdot 4^{n+2} + 5$ *is divisible by* 9.

EXAMPLE 18. *Show that* $a^{4b+1} - a$ *is divisible by* 30.

EXAMPLE 19. *Find all solutions to the following linear congruences:*

(1) $2x \equiv 5 \pmod 7$

(2) $17x \equiv 14 \pmod{21}$

(3) $15x \equiv 9 \pmod{25}$

3.2. Modular Techniques.

PROBLEM 14. *Prove that any subset of 55 numbers chosen from the set* $\{1, 2, 3, 4, \ldots, 100\}$ *must contain two numbers differing by* 9.

Modular Arithmetic

PROBLEM 15. *Let* $N = 22 \times 31 + 11 \times 17 + 13 \times 19$. *Determine (a) the parity of* N*; (b) the unit digit of* N*; (c) the remainder when* N *is divided by* 7.

PROBLEM 16. *What are the last two digits of* 3^{1234}*?*

PROBLEM 17. *Show that some positive multiple of* 21 *has* 241 *as its final three digits.*

PROBLEM 18. *Prove that for any set of n integers, there is a subset of them whose sum is divisible by n.*

PROBLEM 19. *Prove that if* $2n + 1$ *and* $3n + 1$ *are both perfect squares, then* n *is divisible by* 40.

PROBLEM 20. *Prove that the expressions*

$$2x + 3y, 9x + 5y$$

are divisible by 17 *for the same set of integral values of* x *and* y.

THEOREM 1. *(Chinese Remainder Theorem) If m and n are relatively prime integers greater than one, and a and b are arbitrary integers, there exists an integer x such that*

$$\begin{aligned} x &\equiv a \pmod{m}, \\ x &\equiv b \pmod{n}. \end{aligned}$$

More generally, if $m_1, m_2, \ldots, m_k$ are pairwise relatively prime numbers greater than one, and $a_1, a_2, \ldots, a_k$ are arbitrary integers, there exists an integer x such that

$$x \equiv a_i \pmod{m_i},$$

for $i = 1, 2, \ldots, k$.

PROBLEM 21. *Do there exist $1,000,000$ consecutive integers each of which contains a repeated prime factor?*

PROBLEM 22. *Prove that any subset of* 55 *numbers chosen from the set* $\{1, 2, 3, \ldots, 100\}$ *must contain numbers differing by* 10, 12, *and* 13, *but does not contain a pair differing by* 11.

PROBLEM 23. (1) *Show that* $2^{2x+1} + 1$ *is divisible by* 3.

(2) *Show that* $4^{2x+1} + 2^{3x+1} + 1$ *is divisible by* 7.

(3) *If* $n > 0$, *prove that* 12 *divides* $n^4 - 4n^3 + 5n^2 - 2n$.

(4) *Prove that* $(2903)^n - (803)^n - (464)^n + (261)^n$ *is divisible by* 1897.

Problem 24. (1) *Prove that no prime three more than a multiple of four is a sum of two squares. (Hint: Work modulo 4.)*

(2) *Prove that the sequence (in base-10 notation)*

$$11, 111, 1111, 11111, \ldots$$

contains no squares.

(3) *Prove that the difference of the squares of any two odd numbers is exactly divisible by* 8.

(4) *Prove that* $2^{70} + 3^{70}$ *is divisible by* 13.

(5) *Prove that the sum of two odd squares cannot be a square.*

(6) *Determine all integral solutions of* $a^2 + b^2 + c^2 = a^2b^2$. *(Hint: Analyze modulo 4).*

PROBLEM 25. (1) *If* $x^3 + y^3 = z^3$ *has a solution in integers* x, y, z, *show that one of the three must be a multiple of 7.*

(2) *If* n *is a positive integer greater than 1 such that* $2^n + n^2$ *is prime, show that* $n \equiv 3 \pmod 6$.

(3) *Let x be an integer one less than a multiple of 24. Prove that if a and b are positive integers such that $ab = x$, then $a + b$ is a multiple of 24.*

(4) *Prove that if $n^2 + m$ and $n^2 - m$ are perfect squares, then m is divisible by 24.*

PROBLEM 26. *Let S be a set of primes such that $a, b \in S$ (a and b need not be distinct) implies that $ab + 4 \in S$. Show that S must be empty. (Hint: One approach is to work modulo 7).*

PROBLEM 27. *Prove that there are no integers x and y for which*

$$x^2 + 3xy - 2y^2 = 122.$$

(Hint: Use the quadratic equation to solve for x, then look at the discriminant modulo 17. Can it ever be a perfect square?)

PROBLEM 28. *Let a, b, c, d be fixed integers with d not divisible by 5. Assume that m is an integer for which*

$$am^3 + bm^2 + cm + d$$

is divisisible by 5. Prove that there exists an integer n for which $dn^3 + cn^2 + bn + a$ is also divisible by 5.

Problem 29. *Prove that* $(21n-3)/4$ *and* $(15n+2)/4$ *cannot both be integers for the same positive integer* n.

4. Unique Factorization

PROBLEM 30. *How many divisors of*

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

are there?

PROBLEM 31. *An integer* $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ *is a perfect square if and only if* a_i *is even for each* i, *a perfect cube if and only if each* a_i *is a multiple of three, and so forth.*

PROBLEM 32. *Let $a, b, \ldots, g$ be a finite number of positive integers. Suppose their unique factorizations are*

$$\begin{aligned} a &= p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \\ b &= p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}, \\ & \\ g &= p_1^{g_1} p_2^{g_2} \cdots p_k^{g_k}, \end{aligned}$$

where $a_1, a_2, \ldots, a_k, b_1, \ldots, b_k, \ldots, g_1, \ldots, g_k$ are nonnegative integers (some may be zero). Find their gcd. and lcm.

PROBLEM 33. *Use unique factorization to show that $\sqrt{2}$ is irrational.*

PROBLEM 34. *Find the smallest positive integer n such that $n/2$ is a perfect square, $n/3$ is a perfect cube, and $n/5$ is a perfect fifth power.*

PROBLEM 35. *Prove that there is one and only one natural number n such that $2^8 + 2^{11} + 2^n$ is a perfect square.*

PROBLEM 36. *Let n be a given positive integer. How many solutions are there in ordered positive-integer pairs (x, y) to the equation*

$$\frac{xy}{x+y} = n?$$

PROBLEM 37. *Let r and s be positive integers. Derive a formula for the number of ordered quadruples (a, b, c, d) of positive integers such that*

$$3^4 7^s = lcm(a, b, c) = lcm(a, b, d) = lcm(a, c, d) = lcm(b, c, d).$$

Problem 38. *Show that* 1000! *ends with* 249 *zeros.*

Problem 39. *Show that there are an infinite number of primes of the form* $6n-1$.

PROBLEM 40. *Find the smallest number with* 28 *divisors.*

PROBLEM 41. *Given distinct integers* a, b, c, d *such that*

$$(x-a)(x-b)(x-c)(x-d)-4=0$$

has an integral root r, *show that* $4r = a+b+c+d$.

Problem 42. (1) *Prove that* $\sqrt[3]{72}$ *is irrational.*

(2) *Prove that there is no set of integers* m, n, p *except* $0, 0, 0$ *for which* $m + n\sqrt{2} + p\sqrt{3} = 0$.

5. Positional Notation

PROBLEM 43. *Does* $\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 8x \rfloor + \lfloor 16x \rfloor + \lfloor 32x \rfloor = 12345$ *have a solution?*

PROBLEM 44. *When* 4444^{4444} *is written in decimal notation, the sum of its digits is* A*. Let* B *be the sum of the digits of* A*. Find the sum of the digits of* B*.*

Problem 45. *Prove that there does not exist an integer which is doubled when the initial digit is transferred to the end.*

Problem 46. (1) *Solve the following equation for the positive integers x and y:*

$$(360 + 3x)^2 = 492604.$$

(2) *If* $62ab427$ *is a multiple of* 99, *find* a *and* b.

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