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ที่มีฟังก์ชันแหล่งต้นทาง

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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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PRELIMINARY GROUP CLASSIFICATION OF THE FULL BOLTZMANN EQUATION WITH A SOURCE FUNCTION

Adisak Karnbanjong

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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PRELIMINARY GROUP CLASSIFICATION OF THE FULL BOLTZMANN EQUATION WITH A SOURCE FUNCTION

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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สมการโบลต์ซมันน์แบบฉบับเป็นสมการอินทิโกร-ดิฟเฟอเรนเชียลซึ่งอธิบายการวิวัฒน์เวลาของแรริไฟด์แก๊สด้วยพจน์ของฟังก์ชันการแจกแจงระดับโมเลกุล เพื่อให้การอธิบายทางกายภาพของบางปรากฏการณ์ด้วยสมการโบลต์ซมันน์ได้ครอบคลุมมากขึ้น บางครั้งมีความจำเป็นที่ต้องเพิ่มเติมพจน์เข้าไว้ในสมการโบลต์ซมันน์แบบฉบับนั้นด้วย วิทยานิพนธ์ฉบับนี้กล่าวถึงการประยุกต์ใช้การวิเคราะห์กรุปเบื้องต้นเพื่อหาผลเฉลยของสมการโบลต์ซมันน์ที่มีฟังก์ชันแหล่งต้นทาง โดยการใช้กรุปลี L_{11} ที่ยอมรับโดยสมการโบลต์ซมันน์แบบฉบับ

ส่วนแรกของวิทยานิพนธ์เกี่ยวข้องกับการอธิบายกลยุทธ์ในการสร้างสมการกำหนดของสมการไม่เฉพาะที่ไม่เอกพันธุ์ด้วยการใช้กรุปลีที่ยอมรับโดยสมการไม่เฉพาะที่เอกพันธุ์ ที่สมนัยกัน

ส่วนที่สองของวิทยานิพนธ์เป็นการประยุกต์ใช้กลยุทธ์ซึ่งถูกพัฒนาขึ้นเพื่อหาผลเฉลยของสมการโบลต์ซมันน์ที่มีฟังก์ชันแหล่งต้นทาง การหาผลเฉลยของสมการกำหนดที่มีฟังก์ชันแหล่งต้นทางสำหรับแต่ละพีชคณิตย่อยของระบบที่เหมาะสมที่สุดของพีชคณิตย่อยของพีชคณิตลี L_{11} ทำให้ได้การจำแนกกรุปเบื้องต้นตามรูปแบบฟังก์ชันแหล่งต้นทาง

ส่วนที่สามของวิทยานิพนธ์แสดงตัวแทนของผลเฉลยยืนยงของสมการโบลต์ซมันน์ที่มีฟังก์ชันแหล่งต้นทาง และนำเสนอสมการโบลต์ซมันน์ซึ่งถูกลดรูป โดยนำเสนอเฉพาะบางตัวแทนของผลเฉลยยืนยงซึ่งสามารถเขียนพจน์อินทิกรัลของการชนในรูปของตัวแปรใหม่ได้

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา Adisak Karnbanjong

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THE BOLTZMANN EQUATION/PRELIMINARY GROUP CLASSIFICATION/ ADMITTED LIE GROUPS/GROUP CLASSIFICATION/ INVARIANT SOLUTIONS

The classical Boltzmann equation is an integro-differential equation which describes the time evolution of rarefied gas in terms of a molecular distribution function. For some realistic physical situations, there is the need to include additional terms into the classical Boltzmann equation. This thesis is devoted to applying preliminary group classification to the Boltzmann equation with a source function by using the Lie group L_{11} admitted by the classical Boltzmann equation.

The first part of the thesis describes a strategy for deriving the determining equation of a non-homogeneous nonlocal equation using a known Lie group admitted by the corresponding homogeneous nonlocal equation.

The second part of the thesis is devoted to applying the developed strategy to the Boltzmann equation with a source. Solving the determining equation for the source function for each subalgebra of the optimal system of subalgebras of the Lie algebra L_{11}, one obtains a preliminary group classification with respect to the source function.

The third part of the thesis provides representations of invariant solutions of the Boltzmann equation with a source. The reduced equations are also shown for some representations of invariant solutions for which the collision integral can be written in the new variables.

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CHAPTER I

INTRODUCTION

The classical Boltzmann equation plays a central role in the kinetic theory of rarefied gases. For many physical processes, there is the need to include additional terms into the classical Boltzmann equation for describing some realistic physical situations. This chapter is devoted to review the Boltzmann equation, and the group analysis method.

1.1 The full Boltzmann equation

The Boltzmann equation is an integro-differential kinetic equation which describes the time evolution of rarefied gas in terms of a distribution function of molecules of the gas which interact through binary elastic collision. The distribution function $f(\mathbf{x}, \mathbf{v}, t)$ depends on seven independent variables: space variable $\mathbf{x} \in \mathbb{R}^3$, molecular velocity $\mathbf{v} \in \mathbb{R}^3$, and time $t \geq 0$. The function provides the probability distribution of molecules in the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ at time t. The Boltzmann equation has the following form

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = J(f, f), \tag{1.1}$$

with the collision integral

$$J(f, f) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2} B(g, \theta_1)(f^* f_1^* - f f_1)\, d\mathbf{n}\, d\mathbf{w}, \tag{1.2}$$

where

- $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$;
- $\mathbf{v} = (u, v, w),\ \mathbf{w} = (u_1, v_1, w_1) \in \mathbb{R}^3$ are the pre-collision velocities of two particles having the post-collision velocities: $\mathbf{v}^*,\ \mathbf{w}^*$, respectively;

- $\mathbf{g} = \mathbf{v} - \mathbf{w}$ is the relative velocity of the colliding particles, and $g = \|\mathbf{g}\|_2$;
- $d\mathbf{w} = du_1 dv_1 dw_1$ is a volume element in $\mathbb{R}^3$;
- $f = f(\mathbf{x}, \mathbf{v}, t),\ f_1 = f(\mathbf{x}, \mathbf{w}, t),\ f^* = f(\mathbf{x}, \mathbf{v}^*, t),\ f_1^* = f(\mathbf{x}, \mathbf{w}^*, t)$;
- $\mathbf{n}$ is a unit vector varying on the unit sphere $\mathbf{S}^2 = \{\mathbf{n} \in \mathbb{R}^3 |\ \|\mathbf{n}\|_2 = 1\}$, such as

$$\mathbf{n} \equiv (n_1, n_2, n_3) = (\sin\theta_1 \cos\epsilon, \sin\theta_1 \sin\epsilon, \cos\theta_1),$$

  where $\epsilon,\ \theta_1$ are the angles identifying a point on the sphere in spherical coordinates, and the angle θ_1 which is called the *scattering angle* is the angle between the vectors $\mathbf{g}$ and $\mathbf{n}$, i.e., $\cos\theta_1 = \frac{\mathbf{g}\cdot\mathbf{n}}{g}$;
- $d\mathbf{n}$ is a surface element of unit sphere in $\mathbb{R}^3$, i.e.,

$$d\mathbf{n} = \sin\theta_1 d\theta_1 d\epsilon, \quad \theta_1 \in [0, \pi]\,,\ \epsilon \in [0, 2\pi]\,.$$

Because of the elastic collision of two particles of the gas, the velocities $\mathbf{v}^*$ and $\mathbf{w}^*$ are determined by following formulae

$$\mathbf{v}^* = \frac{1}{2}(\mathbf{v} + \mathbf{w} + g\mathbf{n}),\ \mathbf{w}^* = \frac{1}{2}(\mathbf{v} + \mathbf{w} - g\mathbf{n}). \tag{1.3}$$

The function B is called the *collision scattering function*, defined by

$$B = g\sigma\left(g, \cos\theta_1\right), \tag{1.4}$$

where the function $\sigma : \mathbb{R}^+ \times [-1, 1] \to \mathbb{R}^+$ is called a *differential cross-section.*

Determination of the scattering function depends on the potential of intermolecular interaction of a gas. One of several types of the scattering function considered in the Boltzmann equation is a description of the particle interaction by inverse power intermolecular potential $U(r) \propto r^{-(\nu-1)},\ (\nu > 2)$. The scattering function has the form

$$B = b_\gamma\left(\cos\theta_1\right) g^\gamma,\ \gamma = \frac{\nu - 5}{\nu - 1}, \tag{1.5}$$

where $b_\gamma(\cos\theta_1)$ is a known function.

Molecules of a gas with $\nu = 5$, i.e., $\gamma = 0$, are called *Maxwell molecules* with

$$B = b_0(\cos\theta_1) \tag{1.6}$$

independent of g.

1.2 Exact solutions of an equation with source function

As the Boltzmann equation is widely used in rarefied gas dynamics, constructing exact solutions of this equation is of immense interest. In the case of no collision integral, i.e., $J(f,f) = 0$, the equation is known as *free motion equation* and has the general solution in the following form,

$$f(\mathbf{x},\mathbf{v},t) = F(\mathbf{v},\mathbf{x} - t\mathbf{v}). \tag{1.7}$$

A well-known exact solution of the Boltzmann equation for which the collision integral vanishes is the *Maxwell-Boltzmann distribution function*

$$f(\upsilon) = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\upsilon^2 e^{-\frac{m\upsilon^2}{2kT}}, \tag{1.8}$$

where $\upsilon = \|\mathbf{v}\|_2$, m is the molecular mass, k is the Boltzmann's constant, and T is the constant kinetic temperature. However, the presence of the complicated collision integral is one of the main difficulties for finding solutions of the Boltzmann equation. Researchers have tried to find solutions of the Boltzmann equation with simplified collision integral.

In 1975, Bobylev found a particular solution of the spatially homogeneous Boltzmann equation with Maxwell molecules by applying the Fourier transform, and representing the solution in a special form (Bobylev, 1975). Later, in 1976, Krook and Wu obtained the same solution by using the moment generating function (Krook and Wu, 1976). This solution is now called the *BKW solution*. The solution has the following form:

$$f(\upsilon,t) = (2\pi\tau)^{-3/2}e^{-\frac{\upsilon^2}{2\tau}}\left(1+\frac{1-\tau}{\tau}\left(\frac{\upsilon^2}{2\tau}-\frac{3}{2}\right)\right),$$
$$\tau = \tau(t) = 1-\theta e^{-\lambda t},\ \lambda = (\pi/2)\int_{-1}^{1} d\mu g(\mu)(1-\mu^2),\ t \geq 0, \tag{1.9}$$

where $f(\upsilon,t) \geq 0$ for $0 \leq \theta \leq 2/5$, (for more details, see (Bobylev, 1984)).

In order to model some physical situations by the Boltzmann equation, one needs to include additional terms into the classical equation such as removal events (e.g., chemical reactions), interactions with a background host medium, and presence of an external source (Spiga, 1984; Boffi and Spiga, 1982a; Boffi and Spiga, 1982b).

In 1984, Spiga studied the spatially homogeneous Boltzmann equation with Maxwell molecules including removal (Spiga, 1984),

$$f_t(\mathbf{v},t) - J(f,f) = -C_R n(t) f(\mathbf{v},t), \tag{1.10}$$

where a particular solution of Equation (1.10) was sought in the form of the BKW solution. However, the authors of (Santos and Brey, 1985) showed that using the equivalence transformation

$$\tau = \frac{1}{n_0 C_R}\ln(1+n_0 C_R t),\ \tilde{f} = \frac{n_0}{n(t)}f, \tag{1.11}$$

one can reduce Equation (1.10) to the classical Boltzmann equation: to Eq. (1.10) with $C_R = 0$.

In 1984, Nonenmacher (Nonenmacher, 1984) studied the Boltzmann equation for the spatially homogeneous case with a source function,

$$f_t(\mathbf{v},t) = C(f,f) + Q(\mathbf{v},t). \tag{1.12}$$

Using the moment generating function, he obtained the nonlinear second-order partial differential equation

$$(xu_t)_x - u^2 + u(0)(xu)_x = g, \tag{1.13}$$

where $u = u(x,t)$ is the moment generating function for the distribution function

$f(v,t)$, $g = g(x,t)$ relates to the moment generating function for source function $q(v,t)$ and $u(0) = u(0,t)$. Using the classical group analysis method, some forms of invariant solutions were found. However, because of the presence of a nonlocal term, $u(0)$, in Equation (1.13), the classical group analysis method cannot be applied to the equation. The results of Nonenmacher were corrected in (Grigoriev and Meleshko, 2012; Grigoriev et al., 2014), where the authors obtained the determining equation by using the method developed for equations with nonlocal terms, and the complete group classification of (1.13) was obtained.

The determining equation of the Fourier image of the spatially homogeneous and isotropic Boltzmann equation with a source,

$$\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)ds = q \tag{1.14}$$

was studied in (Grigoriev et al., 2015; Suriyawichitseranee et al., 2015; Long et al., 2017). Equation (1.14) with $q = q(x,t)$ and $q = q(x,t,\varphi)$ was analyzed in (Grigoriev et al., 2015) and (Long et al., 2017), respectively, using the analysis which was applied in (Grigoriev and Meleshko, 2012; Grigoriev et al., 2014). Later, in (Suriyawichitseranee et al., 2015), it was shown that the group classification obtained in (Grigoriev et al., 2015) is complete. The method used in (Grigoriev and Meleshko, 2012; Grigoriev et al., 2014) was also applied to other types of equations in (Long et al., 2017). Among them the following equations were considered:

the Burgers equation with a source term,

$$u_t - uu_x + u_{xx} = f, \tag{1.15}$$

and a nonlinear heat equation with delay in a source term,

$$u_t(x,t) - (u^k(x,t)u_x(x,t))_x = f(x,t,u(x,t),u(x,t-\tau)),\ (k \neq -\frac{4}{3}, 0). \tag{1.16}$$

This thesis is devoted to the full Boltzmann equation with a source function

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - J(f, f) = q, \tag{1.17}$$

where the source function q depends on both the independent and the dependent variables, i.e., $q = q(x, y, z, u, v, w, t, f)$.

1.3 Investigations of the Boltzmann equation by the group analysis method

One of the powerful general methods for finding exact solutions of differential equations (Ovsiannikov, 1978; Olver, 1986) and equations with nonlocal terms (Grigoriev and Meleshko, 1986; Grigoriev et al., 2010) is the group analysis method (Lie, 1883; Lie, 1891). The main concept for constructing solutions by this method is the concept of an admitted Lie group. Solutions invariant with respect to an admitted Lie group allow for a reduction of the number of independent and dependent variables. This means that the problem of finding invariant solutions is simpler than that of finding the general solution. The direct relation between a Lie group and its Lie algebra allows one to use the power of linear algebra. In particular, the algebraic structure of the Lie algebra corresponding to an admitted Lie group[1] introduces an algebraic structure into the set of invariant solutions. For finding an admitted Lie group, one needs to solve determining equations. The general solution of the determining equations generates a *principal Lie algebra* of an equation being studied.

The first application of group analysis to integro-differential kinetic equations was in (Taranov, 1976), where the Vlasov equations of collisionless plasma were reduced to a system of equations for moments, and the classical group analysis method was applied to a subsystem which consists of finite number of partial differential equations. Then the number of studied equations was extended to infinity. The method

[1]This Lie algebra is also called admitted.

(Taranov, 1976) was applied in (Bunimovich and Krasnoslobodtsev, 1982; Bunimovich and Krasnoslobodtsev, 1983) to the *Bhatnagar-Gross-Krook (BGK)* equation, which is a simplified model of the Boltzmann equation. In particular, in (Bunimovich and Krasnoslobodtsev, 1982; Bunimovich and Krasnoslobodtsev, 1983) it was found that the BGK equation admits the Lie algebra L_{11} spanned by the basis generators:

$$\begin{aligned}
&X_1 = \partial_x,\\
&X_2 = \partial_y,\\
&X_3 = \partial_z,\\
&X_4 = t\partial_x + \partial_u,\\
&X_5 = t\partial_y + \partial_v,\\
&X_6 = t\partial_z + \partial_w,\\
&X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v,\\
&X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,\\
&X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u,\\
&X_{10} = \partial_t,\\
&X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z - f\partial_f.
\end{aligned} \tag{1.18}$$

The authors (Bunimovich and Krasnoslobodtsev, 1982; Bunimovich and Krasnoslobodtsev, 1983) stated that this Lie algebra is also admitted by the Boltzmann equation (1.17). Later, using a representation of the admitted transformations it was directly verified in (Grigoriev and Meleshko, 1986) that the generators (1.18) are admitted by the Boltzmann equation (1.17).

In (Grigoriev and Meleshko, 1986; Grigoriev and Meleshko, 1987), a direct method for applying group analysis to integro-differential equations was developed. In (Bobylev and Dorodnitsyn, 2009) this method was applied to the Boltzmann equation. Furthermore, this method was applied to other integro-differential equations such as the kinetic equations of collisionless plasma (Kovalev et al., 1992; Grigoriev et al., 2010)[2], viscoelastic equations (Meleshko, 1988; Özer, 2003; Zhou and Meleshko, 2015), and population balance equation (Lin et al., 2016).

1.4 Preliminary group classification

Equations appearing in science usually include some arbitrary elements: constants and functions of the independent and dependent variables. Hence, the admitted Lie group also depends on these elements. A transformation that preserves the equations, while only changing their arbitrary elements is called an *equivalence transformation*. Among all equivalence transformations, a Lie group of equivalence transformations[3] plays a special role, since there is an algorithm available for its computation. This algorithm essentially consists of solving determining equations (Ovsiannikov, 1978; Meleshko, 1994; Grigoriev et al., 2010). Since equivalent equations possess similar group properties, this leads to the problem of group classification, which consists of finding all principal Lie groups admitted by the studied equation. The part of these groups, which is admitted for all arbitrary elements, is called the *kernel* of the group. Another part depends on the specification of arbitrary elements. This part contains nonequivalent extensions of the kernel. Equations defining this part are called *classifying equations*. A Lie group of equivalence transformations allows one to choose a simple representation of the arbitrary elements of a physical problem.

Solving determining equations in general is a nontrivial task: the presence of ar-

[2] See also references therein.

[3] Recently the authors of (Dos Santos Cardoso-Bihlo et al., 2011) proposed a method for finding discrete equivalence transformations for a system of partial differential equations.

bitrary elements leads to cumbersome compatibility analysis. One alternative method for solving determining equations consists of an algebraic approach. This approach takes into account the algebraic properties of an admitted Lie algebra, thus allowing for a significant simplification of the group classification. For example, the group classification of a single second-order ordinary differential equation, done by the founder of the group analysis method, S. Lie (Lie, 1883; Lie, 1891), still has no solution without using the algebraic structure of admitted Lie groups. Recently the algebraic approach was applied for group classification in (Dos Santos Cardoso-Bihlo et al., 2011; Bihlo et al., 2012; Popovych and Bihlo, 2012; Popovych et al., 2004; Popovych et al., 2010; Chirkunov, 2012; Kasatkin, 2012; Mkhize et al., 2015; Grigoriev et al., 2014; Suriyawichitseranee et al., 2015).

Besides complete group classification, there is a method proposed in (Akhatov et al., 1991) which is called *preliminary group classification*. Further development of this method is given in (Ibragimov et al., 1991; Dos Santos Cardoso-Bihlo et al., 2011). The main idea of preliminary group classification is based on the study of only those extensions of the kernel that are induced by the transformations from the corresponding equivalence Lie group. The problem of finding inequivalent cases of such extensions of symmetry is then reduced to the classification of inequivalent subgroups of the equivalence Lie group. In particular, if a Lie group is finite-parameter, then one can use an optimal systems of its subgroups.

For some classes of differential equations the results of preliminary group classification and complete group classification coincide. However, there are also examples where a preliminary group classification cannot reach the complete group classification. One of these examples is the nonlinear heat equation with a source term $q(u)$:

$$u_t - (u^\sigma u_x)_x = q(u). \tag{1.19}$$

The complete equivalence Lie group of this equation is defined by the generators

$$X_1^e = \partial_t,\ \ X_2^e = \partial_x,\ \ X_3^e = 2t\partial_t + x\partial_x - 2q\partial_q,\ \ X_4^e = \sigma t\partial_t - u\partial_u - (\sigma+1)q\partial_q.$$

It is found in (Dorodnitsyn, 1982) that for $q = u$, Equation (1.19) admits the generator

$$Y = e^{-\sigma t}(\partial_t + u\partial_u).$$

Because the generator Y does not belong to $Span(X_1^e, X_2^e, X_3^e, X_4^e)$, preliminary group classification cannot separate out Equation (1.19) with $q = u$.

The basic idea of this thesis is as follows. Consider a homogeneous equation

$$\Phi(f) = 0, \tag{1.20}$$

where Φ is a differential or integro-differential operator, and f is the solution sought. As a result of group analysis of Equation (1.20), there are many situations where are of interest to classify an inhomogeneous equation of the form

$$\Phi(f) = q, \tag{1.21}$$

where q is an arbitrary function[4]. As mentioned, the preliminary group classification method exploits an equivalence Lie group. Using an infinitesimal approach, the equivalence Lie group is found by solving the determining equations. In this thesis we applied a particular equivalence Lie group for which there is no necessity to solve the determining equations. The basic idea is to exploit group properties of the homogeneous equation (1.20) such as admitted Lie algebra, an optimal system of subalgebras of the Lie algebra, and representations of invariant solutions. Details of the approach are described in Chapter III.

[4]In many equations of mathematical physics the function q plays the role of a source.

1.5 The studied equation

The research considered in the thesis is related with the full Boltzmann equation with a source function

$$f_t + uf_x + vf_y + wf_z - J(f,f) = q, \tag{1.22}$$

with the collision integral

$$J(f,f) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(f^* f_1^* - ff_1)\sin\theta_1\, d\theta_1\, d\epsilon\, du_1\, dv_1\, dw_1, \tag{1.23}$$

where $q = q(x, y, z, u, v, w, t, f)$. The problem is to use the method of preliminary group classification for group classification of equation (1.22) with respect to the source function q using the Lie group L_{11}.

The objective of the research is to classify the full Boltzmann equation with a source function (1.22) by the group analysis method, find representations of invariant solutions, and find the reduced Boltzmann equation.

CHAPTER II

THE GROUP ANALYSIS METHOD

This chapter is devoted to the background knowledge of Lie Group analysis: the main definitions, admitted Lie groups of partial differential equation and Integro-differential equation, etc.

2.1 The main definitions

2.1.1 Local Lie group of transformations

Let V be an open set in $Z = \mathbb{R}^N$, Δ be a symmetric interval in $\mathbb{R}$. Assume that the point transformations involving a parameter a:

$$\bar{z}_i = g^i(\mathbf{z}; a), \ (i = 1, 2, \ldots, N) \tag{2.1}$$

are invertible. Here $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in V \subset Z$, $\bar{\mathbf{z}} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N)$, and the parameter $a \in \Delta$.

Definition 2.1.1. A set of transformations (2.1) is called a *local one-parameter Lie group*, denoted by G^1, if it has the following properties:

1. $g^i(\mathbf{z}; 0) = z_i$ for all $\mathbf{z} \in V$;

2. $g^i(g^i(\mathbf{z}; a); b) = g^i(\mathbf{z}; a + b)$ for all $a, b, a + b \in \Delta$, $\mathbf{z} \in V$;

3. if $a \in \Delta$ and $g^i(\mathbf{z}; a) = z_i$ for all $\mathbf{z} \in V$, then $a = 0$;

for all $i = 1, 2, \ldots, N$.

Definition 2.1.2. A Lie group[1] G^1 is called a *continuous* Lie group of the class C^k if the functions $g^i(\mathbf{z}; a)$, $(i = 1, 2, \ldots, N)$ belong to the class $C^k(V)$, the class of functions which are k times continuously differentiable on V.

Remark. In applied group analysis the functions g^i, $(i = 1, 2, \ldots, N)$ are all assumed to be sufficiently many times continuously differentiable with respect to z_i, $(i = 1, 2, \ldots, N)$.

2.1.2 Generator of a one-parameter group

The expansion of the functions $g^i(\mathbf{z}; a)$ into the Taylor series with respect to the parameter a in a neighborhood of $a = 0$, and the first property of the group G^1, i.e., $g^i(\mathbf{z}; 0) = z_i$, yield the *infinitesimal transformations* of the group G^1 (2.1):

$$\bar{z}_i \approx z_i + \zeta^i(\mathbf{z})a, \ \ (i = 1, 2, \ldots, N), \tag{2.2}$$

where

$$\zeta^i(\mathbf{z}) = \frac{\partial g^i(\mathbf{z}; a)}{\partial a}\bigg|_{a=0}. \tag{2.3}$$

The vector

$$\boldsymbol{\zeta}(\mathbf{z}) = (\zeta^1(\mathbf{z}), \zeta^2(\mathbf{z}), \ldots, \zeta^N(\mathbf{z})) \tag{2.4}$$

with the components (2.3) is the tangent vector at the point $\mathbf{z}$ to the curve described by the transformed point $\bar{\mathbf{z}}$. Hence, the tangent vector is called the *tangent vector field* of the group G^1. The tangent vector field (2.4) is also written as a first-order differential operator

$$X = \zeta^1(\mathbf{z})\partial_{z_1} + \zeta^2(\mathbf{z})\partial_{z_2} + \cdots + \zeta^N(\mathbf{z})\partial_{z_N} \equiv \zeta^i(\mathbf{z})\partial_{z_i}, \tag{2.5}$$

which is called the *infinitesimal generator*[2] of the group G^1, and the functions ζ^i are called the *coefficients* of the generator X. Here the notation $\zeta^i(\mathbf{z})\partial_{z_i}$ is the *Einstein*

[1] In short, a local Lie group of transformations will be also called a Lie group or a group.
[2] For brevity the infinitesimal generator will also called the generator.

summation notation to express the summation over the index i for all possible values of the index i.

2.1.3 Construction of a group with a given generator

Let an infinitesimal transformations (2.2), or the infinitesimal generator (2.5) be given. A one-parameter Lie group of transformations (2.1) is defined by the solution of the Cauchy problem for the first-order ordinary differential equations:

$$\frac{d\bar{z}_i}{da} = \zeta^i(\bar{\mathbf{z}}), \ \bar{z}_i|_{a=0} = z_i, \ (i = 1 \ldots, N) \tag{2.6}$$

which are known as the *Lie equations*.

Theorem 2.1.1 (Lie Theorem). *Let $\boldsymbol{\zeta} \in C^k(V)$ with $\boldsymbol{\zeta}(\boldsymbol{z}_0) \neq 0$ for some $\boldsymbol{z}_0 \in V$. The solution of the Cauchy problem (2.6) generates a local Lie group of transformations with the infinitesimal generator (2.5).*

The Lie theorem establishes a one-to-one correspondence between the Lie group of transformations G^1 (2.1) and the infinitesimal generator (2.5).

2.1.4 Prolongation of a generator

The space $Z = \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ consists of the space of n independent variables $\mathbf{x} = (x_1, \ldots, x_n) \equiv (x_i) \in \mathbb{R}^n$, and the space of m dependent variables $\mathbf{u} = (u^1, \ldots, u^m) \equiv (u^j) \in \mathbb{R}^m$. The space Z is prolonged by introducing the variables

$$\mathbf{p} = (p^j_{\boldsymbol{\alpha}}), \tag{2.7}$$

where

$$p^j_{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|} u^j}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|} u^j}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, \ |\boldsymbol{\alpha}| > 0. \tag{2.8}$$

Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, and $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Introduce the notations $\boldsymbol{\alpha}, k \equiv (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1}, \ldots, \alpha_n)$, and

$$p^j_{\boldsymbol{\alpha},k} = \frac{\partial}{\partial x_k}\left(\frac{\partial^{|\boldsymbol{\alpha}|} u^j}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}\right) = \frac{\partial^{|\boldsymbol{\alpha}|+1} u^j}{\partial x_1^{\alpha_1} \ldots \partial x_{k-1}^{\alpha_{k-1}} \partial x_k^{\alpha_k+1} \partial x_{k+1}^{\alpha_{k+1}} \ldots \partial x_n^{\alpha_n}}. \tag{2.9}$$

Remark. In the case of $|\boldsymbol{\alpha}| = 0$, i.e., $\boldsymbol{\alpha} = (0, 0, \ldots, 0)$, we set $p^j_{\boldsymbol{\alpha}} = u^j$.

The space J^l of the variables:

$$\mathbf{x} = (x_i),\ \mathbf{u} = (u^j),\ \mathbf{p} = (p^j_{\boldsymbol{\alpha}}),$$
$$(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m;\ |\boldsymbol{\alpha}| \leq l) \tag{2.10}$$

is called an l^{th} *prolongation of the space* Z. For $l = 0$ we set $J^0 = Z$.

Let the generator

$$X = \xi^i(\mathbf{x}, \mathbf{u})\partial_{x_i} + \eta^j(\mathbf{x}, \mathbf{u})\partial_{u^j} \tag{2.11}$$

be an infinitesimal generator of a one-parameter Lie group of transformations G^1,

$$\bar{x}_i = f^i(\mathbf{x}, \mathbf{u}; a),\ \bar{u}^j = \varphi^j(\mathbf{x}, \mathbf{u}; a),$$
$$(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m). \tag{2.12}$$

Definition 2.1.3. The generator

$$\underset{l}{X} = X + \sum_{j,\boldsymbol{\alpha}} \zeta^j_{\boldsymbol{\alpha}} \partial_{p^j_{\boldsymbol{\alpha}}} \tag{2.13}$$

with the coefficients

$$\zeta^j_{\boldsymbol{\alpha},k} = D_k(\eta^j_{\boldsymbol{\alpha}}) - \sum_i p^j_{\boldsymbol{\alpha},i} D_k(\xi^i) \tag{2.14}$$

is called the l^{th} *prolongation of the generator* X (2.11).

Here the operators D_k are the total derivatives with respect to x_k, defined by

$$D_k = \frac{\partial}{\partial x_k} + u^j_k \frac{\partial}{\partial u^j} + u^j_{ik} \frac{\partial}{\partial u^j_i} + u^j_{i_1 i_2 k} \frac{\partial}{\partial u^j_{i_1 i_2}} + \cdots,\ (k = 1, 2, \ldots, n). \tag{2.15}$$

The generator $\underset{l}{X}$ induces the local Lie group of transformations on the space J^l

$$\bar{\mathbf{x}} = \boldsymbol{f}(\mathbf{x}, \mathbf{u}; a), \;\; \bar{\mathbf{u}} = \boldsymbol{\varphi}(\mathbf{x}, \mathbf{u}; a), \;\; \bar{\mathbf{p}} = \boldsymbol{\psi}(\mathbf{x}, \mathbf{u}, \mathbf{p}; a) \tag{2.16}$$

with the prolonged generator $\underset{l}{X}$. This group is called the l^{th} *prolongation of the group* G^1 (2.12), denoted by $\underset{l}{G}^1$.

Given a group of transformations G^1 (2.12) with the generator X (2.11), the 1^{st} prolongation of the generator X is

$$\underset{1}{X} = X + \zeta^j_k \partial_{u^j_k} \tag{2.17}$$

with the coefficients

$$\zeta^j_k = D_k(\eta^j) - u^j_i D_k(\xi^i). \tag{2.18}$$

The 2^{nd} prolongation of the generator X is

$$\underset{2}{X} = \underset{1}{X} + \zeta^j_{k_1 k_2} \partial_{u^j_{k_1 k_2}} \tag{2.19}$$

with the coefficients

$$\begin{aligned} \zeta^j_{k_1 k_2} &= D_{k_2}(\zeta^j_{k_1}) - u^j_{k_1 i} D_{k_2}(\xi^i) \\ &= D_{k_2} D_{k_1}(\eta^j) - u^j_{k_2 i} D_{k_1}(\xi^i) - u^j_i D_{k_2} D_{k_1}(\xi^i) - u^j_{k_1 i} D_{k_2}(\xi^i). \end{aligned} \tag{2.20}$$

By repetition, the l^{th} prolongation of the generator X is

$$\underset{l}{X} = \underset{l-1}{X} + \zeta^j_{k_1 \cdots k_l} \partial_{u^j_{k_1 \cdots k_l}} \tag{2.21}$$

with the coefficients

$$\zeta^j_{k_1 \cdots k_l} = D_{k_l}(\zeta^j_{k_1 \cdots k_{l-1}}) - u^j_{k_1 \cdots k_{l-1} i} D_{k_l}(\xi^i). \tag{2.22}$$

2.2 Admitted Lie group of a partial differential equation

2.2.1 Invariant manifolds

Consider a system of s differential equations of l^{th} order

$$F^k(\mathbf{x},\mathbf{u},\mathbf{p}) = 0,\ (k = 1,2,\ldots,s), \tag{2.23}$$

where $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$ is the vector of independent variables, $\mathbf{u} = (u^1,\ldots,u^m) \in \mathbb{R}^m$ is the vector of dependent variables, $\mathbf{p}$ is the vector of partial derivatives of u^j with respect to x_i, $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,m)$, $n + m = N$, and $s < N$.
Let M be the manifold defined by the system (2.23),

$$M = \{(\mathbf{x},\mathbf{u},\mathbf{p}) \in J^l | F^k(\mathbf{x},\mathbf{u},\mathbf{p}) = 0,\ (k = 1,2,\ldots,s)\}. \tag{2.24}$$

If the rank of the Jacobi matrix is equal to s, i.e.,

$$rank\left(\frac{\partial(F^k)}{\partial(\mathbf{x},\mathbf{u})}\right) = s \tag{2.25}$$

at all point $\mathbf{z} = (\mathbf{x},\mathbf{u},\mathbf{p})$ satisfying Equations (2.23), then the locus of solutions $\mathbf{z}$ of Equations (2.23) is an $(N-s)$-dimensional manifold $M \subset \mathbb{R}^N$.

Definition 2.2.1. The system of equations (2.23) is said to be *invariant* with respect to a Lie group of transformations G^1

$$\bar{\mathbf{x}} = \boldsymbol{f}(\mathbf{x},\mathbf{u};a),\ \bar{\mathbf{u}} = \boldsymbol{\varphi}(\mathbf{x},\mathbf{u};a) \tag{2.26}$$

if each solution $\mathbf{u} = \mathbf{u}_0(\mathbf{x})$ of the system of equations (2.23) is mapped to a solution $\bar{\mathbf{u}} = \mathbf{u}_a(\bar{\mathbf{x}})$ of the same system of equations in new variables, i.e.,

$$F^k(\bar{\mathbf{x}},\bar{\mathbf{u}},\bar{\mathbf{p}}) = 0,\ (k = 1,2,\ldots,s). \tag{2.27}$$

Geometrically, the invariance of (2.23) means that each point $\mathbf{z}$ on the surface M (the manifold defined by (2.23)) is moved by G^1 along the surface M, i.e., $\mathbf{z} \in M$

implies that $\bar{\mathbf{z}} \in M$. The manifold M is also called an *invariant manifold* with respect to G^1.

Theorem 2.2.1. *The system of equations (2.23) is invariant with respect to a Lie group of transformations G^1 with an infinitesimal generator X if and only if*

$$XF^k(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})|_M = 0, \ (k = 1, 2, \dots, s), \tag{2.28}$$

where the symbol $|_M$ means that the equations XF^k are considered on the manifold M.

2.2.2 Definition of an admitted Lie group

Definition 2.2.2. A local Lie group of transformations G^1 (2.12) with a generator X (2.11) is *admitted* by the system (2.23) (or the system (2.23) *admits* the Lie group G^1) if the manifold (2.24) is an invariant manifold with respect to the prolonged group $\underset{l}{G}^1$. The generator X of the admitted Lie group G^1 is also called *admitted* by the system (2.23).

Theorem 2.2.2. *The system of equations (2.23) is invariant under a Lie group of transformations G^1 with an infinitesimal generator X if and only if*

$$\underset{l}{X}F^k(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})|_M = 0, \ (k = 1, 2, \dots, s), \tag{2.29}$$

where $\underset{l}{X}$ is the prolonged infinitesimal generator of X, and the symbol $|_M$ means that the equations XF^k are considered on the manifold M.

Equation (2.29) is called the *determining equation* and denoted by (DE).

Consider the system of equations (2.23), denoted by (S). The algorithm of finding an admitted Lie group is as follows.

Step 1 Give the form of the admitted generator (2.11) with the unknown coefficients $\xi^i(\mathbf{x}, \mathbf{u}), \ \eta^j(\mathbf{x}, \mathbf{u})$.

Step 2 Construct the prolonged infinitesimal operator $\underset{\iota}{X}$ with the coefficients defined by the prolongation formulae (2.14).

Step 3 Apply the prolonged operator $\underset{\iota}{X}$ to each equation of the system (S).

Step 4 Solve the determining equations

$$(DS): \qquad \underset{\iota}{X}F^k|_{(S)} = 0, \ (k = 1, 2, \ldots, s). \tag{2.30}$$

which are linear homogeneous differential equation for the unknown coefficients $\xi^i(\mathbf{x}, \mathbf{u}), \ \eta^j(\mathbf{x}, \mathbf{u})$.

As the coefficients of the generator X do not depend on the derivatives $p^j_{\boldsymbol{\alpha}}$, splitting the determining equations with respect to the parametric derivatives one obtain an overdetermined system. The general solution of the determining equations generates a *principal Lie algebra*, denoted by LS, of the system (S). The set of transformations, which is finitely generated by one-parameter Lie groups corresponding to the generators $X \in LS$ is called the *principal Lie group* admitted by the system (S), denoted by GS.

The above approach for finding an admitted Lie group which is based on the manifolds is called the *geometrical approach*. In this approach there is no necessity to have the existence of a solution of the system (S). Other approaches require existence of a solution of system (S).

2.3 Admitted Lie group of an integro-differential equation

2.3.1 Lie-Bäcklund operator

This subsection gives definitions of a Lie-Bäcklund operator and a canonical Lie-Bäcklund operator, and their properties (Grigoriev et al., 2010).

Definition 2.3.1. A locally analytic function (i.e., locally expandable in a Taylor series with respect to all arguments) of a finite number of variables $\mathbf{x}, \mathbf{u}, \mathbf{p}$ is called a *differential function*. The highest order of derivatives appearing in the differential function is called the *order* of this function. The set of all differential functions of all finite orders is denoted by $\mathscr{A}$.

Definition 2.3.2. Let $\xi^i, \eta^j \in \mathscr{A}$ be differential functions depending on any finite number of variables $\mathbf{x}, \mathbf{u}, \mathbf{p}$. A differential operator

$$X = \xi^i \partial_{x_i} + \eta^j \partial_{u^j} + \zeta^j_k \partial_{u^j_k} + \zeta^j_{k_1 k_2} \partial_{u^j_{k_1 k_2}} + \cdots , \tag{2.31}$$

where

$$\zeta^j_k = D_k(\eta^j - \xi^i u^j_i) + \xi^i u^j_{ki},$$

$$\zeta^j_{k_1 k_2} = D_{k_1} D_{k_2}(\eta^j - \xi^i u^j_i) + \xi^i u^j_{k_1 k_2 i}, \tag{2.32}$$

$$\vdots$$

is called a *Lie-Bäcklund operator*. The Lie-Bäcklund operator (2.31) is often written in abbreviated form

$$X = \xi^i \partial_{x_i} + \eta^j \partial_{u^j}, \tag{2.33}$$

where the prolongation given by (2.31)-(2.32).

Remark. The operator (2.31) is formally an infinite sum. However, it truncates when acting on any differential function.

The set of all Lie-Bäcklund operators is an infinite dimensional Lie algebra, denoted by $L_{\mathscr{B}}$. This Lie algebra is endowed with the following properties:

1. The total differentiation D_i is a Lie-Bäcklund operator, i.e., $D_i \in L_{\mathscr{B}}$. Furthermore,

$$X_* = \xi^i_* D_i \in L_{\mathscr{B}} \tag{2.34}$$

for any $\xi^i_* \in \mathscr{A}$.

2. Let L_* be the set of all Lie-Bäcklund operators of the form (2.34). Then L_* is an

ideal of $L_{\mathscr{B}}$, i.e., $[X, X_*] \in L_*$ for any $X \in L_{\mathscr{B}}$. Indeed,

$$[X, X_*] = (X(\xi_*^i) - X_*(\xi^i))D_i \in L_*.$$

3. Two operations X_1, $X_2 \in L_{\mathscr{B}}$ are said to be *equivalent* (i.e., $X_1 \sim X_2$) if $X_1 - X_2 \in L_*$.

Definition 2.3.3. The operators of the form

$$X = \eta^j \partial_{u^j} + \zeta_k^j \partial_{u_k^j} + \zeta_{k_1k_2}^j \partial_{u_{k_1k_2}^j} + \cdots , \tag{2.35}$$

are called *canonical Lie-Bäcklund operators.*

Remark. Any Lie-Bäcklund operator X is equivalent to a canonical Lie-Bäcklund operator $\tilde{X}$. Namely, $X \sim \tilde{X}$, where $\tilde{X} = X - \xi^i D_i$. The canonical Lie-Bäcklund operator and its coefficients have formulae as follows.

$$\begin{aligned} \tilde{X} &= X - \xi^i D_i \\ &= \xi^i \partial_{x_i} + \eta^j \partial_{u^j} + \zeta_k^j \partial_{u_k^j} + \zeta_{k_1k_2}^j \partial_{u_{k_1k_2}^j} + \cdots \\ &\quad - \xi^i (\partial_{x_i} + u_i^j \partial_{u^j} + u_{ki}^j \partial_{u_k^j} + u_{k_1k_2i}^j \partial_{u_{k_1k_2}^j} + \cdots) \\ &= (\eta^j - \xi^i u_i^j)\partial_{u^j} + (\zeta_k^j - \xi^i u_{ki}^j)\partial_{u_k^j} + (\zeta_{k_1k_2}^j - \xi^i u_{k_1k_2i}^j)\partial_{u_{k_1k_2}^j} + \cdots \\ &= \bar{\eta}^j \partial_{u^j} + D_k(\bar{\eta}^j)\partial_{u_k^j} + D_{k_1}D_{k_2}(\bar{\eta}^j)\partial_{u_{k_1k_2}^j} + \cdots , \end{aligned} \tag{2.36}$$

where $\bar{\eta}^j = \eta^j - \xi^i u_i^j$.

2.3.2 Definition of an admitted Lie group

Consider an abstract system of integro-differential equations:

$$\Phi(\mathbf{x}, \mathbf{u}) = 0. \tag{2.37}$$

Let G^1 be a Lie group of transformations with one-parameter a:

$$\bar{\mathbf{x}} = \boldsymbol{f}(\mathbf{x}, \mathbf{u}; a), \;\; \bar{\mathbf{u}} = \boldsymbol{\varphi}(\mathbf{x}, \mathbf{u}; a) \tag{2.38}$$

with the infinitesimal generator

$$X = \xi^i(\mathbf{x}, \mathbf{u})\partial_{x_i} + \eta^j(\mathbf{x}, \mathbf{u})\partial_{u^j}. \tag{2.39}$$

This group of transformations (2.38) maps a solution $\mathbf{u} = \mathbf{u}_0(\mathbf{x})$ of Equation (2.37) into a solution $\bar{\mathbf{u}} = \mathbf{u}_a(\bar{\mathbf{x}})$ of the same equations in new variables, i.e.,

$$\Phi(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = 0. \tag{2.40}$$

The transformed function $\mathbf{u}_a(\bar{\mathbf{x}})$ is determined by the formula

$$\mathbf{u}_a(\bar{\mathbf{x}}) = \boldsymbol{\varphi}(\mathbf{x}, \mathbf{u}_0(\mathbf{x}); a), \tag{2.41}$$

where $\mathbf{x} = \boldsymbol{F}(\bar{\mathbf{x}}; a)$ is the solution of the equation

$$\bar{\mathbf{x}} = \boldsymbol{f}(\mathbf{x}, \mathbf{u}_0(\mathbf{x}); a). \tag{2.42}$$

Differentiating the equation $\Phi(\bar{\mathbf{x}}, \mathbf{u}_a(\bar{\mathbf{x}})) = 0$ with respect to the parameter a, and setting $a = 0$, we obtain the equation

$$\left(\frac{\partial}{\partial a}\Phi(\bar{\mathbf{x}}, \mathbf{u}_a(\bar{\mathbf{x}}))\right)\Big|_{a=0} = 0. \tag{2.43}$$

This equation coincide with the equation

$$(\tilde{X}\Phi)(\mathbf{x}, \mathbf{u}_0(\mathbf{x})) = 0, \tag{2.44}$$

where $\tilde{X}$ is the prolongation of the canonical Lie-Bäcklund operator which is equivalent to the generator X (2.39):

$$\tilde{X} = \bar{\eta}^j \frac{\partial}{\partial u^j} + D_k(\bar{\eta}^j)\frac{\partial}{\partial u^j_k} + D_{k_1}D_{k_2}(\bar{\eta}^j)\frac{\partial}{\partial u^j_{k_1k_2}} + \cdots, \tag{2.45}$$

where $\bar{\eta}^j = \eta^j - \xi^i u_i^j$. Equation (2.44) can be constructed without requiring the property that the Lie group should transform a solution into a solution.

Definition 2.3.4. A one-parameter Lie group G^1 of transformations (2.38) is a *Lie group admitted by (2.37)* if G^1 satisfies the equation (2.44) for any solution $u_0(x)$ of (2.37). Equations (2.44) are called the *determining equations*.

2.4 Lie algebra of generators

Let X_1, X_2 be two generators defined by

$$X_k = \xi_k^i(\mathbf{z})\partial_{z_i}, \ \ k = 1, 2. \tag{2.46}$$

Definition 2.4.1. The *commutator* of the generators X_1 and X_2, denoted by $[X_1, X_2]$, is a generator defined by

$$[X_1, X_2] = X_1X_2 - X_2X_1, \tag{2.47}$$

or equivalently

$$[X_1, X_2] = \left(X_1(\xi_2^i(\mathbf{z})) - X_2(\xi_1^i(\mathbf{z}))\right)\partial_{z_i}. \tag{2.48}$$

Let X_1, X_2, X_3 be any generators, and α, β be constant. The operation of commutation satisfies the following properties:

bilinearity :

$$[\alpha X_1 + \beta X_2, X_3] = \alpha[X_1, X_3] + \beta[X_2, X_3], \tag{2.49}$$

$$[X_1, \alpha X_2 + \beta X_3] = \alpha[X_1, X_2] + \beta[X_1, X_3]; \tag{2.50}$$

skew-symmetry :

$$[X_1, X_2] = -[X_2, X_1]; \tag{2.51}$$

the Jacobi identity :

$$[X_1,[X_2,X_3]]+[X_2,[X_3,X_1]]+[X_3,[X_1,X_2]]=0. \qquad (2.52)$$

Definition 2.4.2. A vector space L of generators is called a *Lie algebra* if it is closed under the commutator, i.e., $[X_1,X_2]\in L$ for any $X_1,X_2\in L$.

Remark. The Lie algebra is denoted by the same letter L, and its dimension is the dimension of the vector space L. An r-dimensional Lie algebra is denoted by L_r

Definition 2.4.3. Let L_r be a Lie algebra spanned by X_i, $(i=1,2,\dots,r)$. A subspace L_s of the vector space L_r is called a *subalgebra* of L_r if $[X,Y]\in L_s$ for any $X,Y\in L_s$. Furthermore, L_s is called an *ideal* of L_r if $[X,Y]\in L_s$ whenever $X\in L_s,\ Y\in L_r$.

Definition 2.4.4. Two Lie algebras of generators L and L' are *similar* if there exists a change of variable that transforms one into the other.

Remark. If L and L' are similar Lie algebras, then we obtain that the generators $X=\zeta^\beta(\mathbf{z})\partial_{z_\beta}\in L$ and $\hat{X}=\hat{\zeta}^\beta(\hat{\mathbf{z}})\partial_{\hat{z}_\beta}\in L'$ of these algebras are related by the formula

$$\hat{\zeta}^\beta(\hat{\mathbf{z}})=X(q^\beta(\mathbf{z}))\Big|_{\mathbf{z}=q^{-1}(\hat{\mathbf{z}})}.$$

A linear one-to-one mapping F of a Lie algebra L onto a Lie algebra L' is called an *isomorphism* (algebra L and L' are said to be *isomorphic*) if for any $X,Y\in L$, the equality

$$F([X,Y])=[F(X),F(Y)]'$$

holds, where the symbol $[\ ,\ ]$ and $[\ ,\ ]'$ are the commutators in L and L', respectively. An isomorphism of Lie algebra L onto itself is called an *automorphism* of L.

2.5 Classification of subalgebras and optimal system of subalgebras

Consider an $r-$dimensional Lie algebra L_r. Let X_1, X_2,...,X_r be a basis of the vector space L_r. In particular, $[X_i, X_j] \in L$, hence

$$[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma,\ (\alpha, \beta = 1, 2, \ldots, r). \tag{2.53}$$

The constant coefficients $c^\gamma_{\alpha\beta}$ are called the *structure constants* of the Lie algebra L_r.

Using relations (2.53), we can write them in the form of a table of commutators whose entry at the intersection of the X_α row with the X_β column is $[X_\alpha, X_\beta]$.

If $X = x^\alpha X_\alpha$ and $Y = y^\beta X_\beta$ belong to L_r, then

$$[X, Y] = x^\alpha y^\beta [X_\alpha, X_\beta] = x^\alpha y^\beta c^\gamma_{\alpha\beta} X_\gamma.$$

Therefore, for the coordinates $\mathbf{x} = (x^1, x^2, \ldots, x^r)$ and $\mathbf{y} = (y^1, y^2, \ldots, y^r)$ of the generators X and Y, we can define an operation of commutation $[\mathbf{x}, \mathbf{y}]$:

$$[\mathbf{x}, \mathbf{y}]^\gamma = x^\alpha y^\beta c^\gamma_{\alpha\beta}, \quad (\gamma = 1, 2, \ldots, r).$$

With this operation the vector space $R^r(\mathbf{x})$ becomes an isomorphic Lie algebra. This Lie algebra is also denoted by L_r.

An automorphism $A_y(t)$ of R^r is defined by

$$\hat{\mathbf{x}} = A_y(t)\mathbf{x},$$

where $\hat{\mathbf{x}} = (\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^r)$ is the solution of the equations

$$\frac{d}{dt}\hat{x}^\gamma = \hat{x}^\alpha y^\beta c^\gamma_{\alpha\beta}, \quad \hat{x}^\gamma_{|t=0} = x^\gamma.$$

The set of all automorphisms $A_y(t)$ is called the *set of inner automorphisms* of the Lie algebra L_r, denoted by $\mathrm{Int}(L_r)$. Any subalgebra $L_s \subset L_r$ is transformed by $A_y(t)$ into a similar subalgebra. Similar subalgebras of the same dimension compose an *equivalence class*, and select a representative of each class.

Definition 2.5.1. A set of the representatives of each of these classes is called an *optimal system of subalgebras*.

Thus, an optimal system of subalgebras of a Lie algebra L with inner automorphisms $A = \text{Int}(L)$ is a set of subalgebras $\Theta_A(L)$ such that

(1) there are no two elements of this set which can be transformed into each other by inner automorphisms of the Lie algebra L;

(2) any subalgebra of the Lie algebra L can be transformed into one of the subalgebras of the set $\Theta_A(L)$.

In group analysis the problem of finding all inner automorphisms is reduced to the problem for finding inner automorphisms A_k for the canonical basis vectors $y = e^k$, $(k = 1, 2, \ldots, r)$ in R^r:

$$\frac{d}{dt}\hat{x}^\gamma = \hat{x}^\alpha c^\gamma_{\alpha k}, \quad \hat{x}^\gamma_{|t=0} = x^\gamma, \quad (\gamma = 1, 2, \ldots, r).$$

The inner automorphisms A_k corresponds to the Lie group of transformations with the generator

$$x^\alpha c^\gamma_{\alpha k} \partial_{x^\gamma}.$$

2.6 Invariant solutions

Definition 2.6.1. Let a system (S) of differential equations admits a group G, and let H be a subgroup of G. A solution $\mathbf{u} = \mathbf{U}(\mathbf{x})$ of system (S) is called an $H-$invariant solution if the manifold $\mathbf{u} = \mathbf{U}(\mathbf{x})$ is an invariant manifold with respect to any transformation of the group H.

Let a system (S) of differential equations admit a group G, and let H_r be its $r-$parameter subgroup generated by

$$X_k = \xi_k^i(\mathbf{x}, \mathbf{u})\partial_{x_i} + \eta_k^j(\mathbf{x}, \mathbf{u})\partial_{u^j}, \ (k = 1, 2, \dots, r). \tag{2.54}$$

Let $r_* = rank\|\xi_k^i, \eta_k^j\|$. Hence, H_r has $m + n - r_*$ functionally independent invariants $J_1(\mathbf{x}, \mathbf{u}), \dots, J_{m+n-r_*}(\mathbf{x}, \mathbf{u})$. Suppose that the Jacobian of J with respect to $\mathbf{u}$ is of rank m, without loss of generality we can choose the first m invariants such that

$$rank\left\|\frac{\partial J_\alpha(\mathbf{x}, \mathbf{u})}{\partial u^j}\right\| = m, \tag{2.55}$$

where $\alpha, j = 1, \dots, m$. Hence, setting

$$\begin{gathered} \mu^\alpha = J_\beta(\mathbf{x}, \mathbf{u}), \ \lambda^\beta = J_{m+\nu}(\mathbf{x}, \mathbf{u}), \\ (\alpha = 1, 2, \dots, m; \ \nu = 1, 2, \dots, n - r_*), \end{gathered} \tag{2.56}$$

we can write the invariant solution of system (S) in the form

$$\mu^\alpha = \Phi^\alpha(\lambda^1, \dots, \lambda^{n-r_*}), \ (\alpha = 1, 2, \dots, m). \tag{2.57}$$

The functions Φ^α are determined by a system of differential equations, denoted by S/H_r. The system S/H_r is called a reduced system, the number $n - r_*$ is the number of independent variables in factor system S/H_r and it is called a rank of the invariant solution.

Remark. In most applications, we can choose invariants (2.56) such that the invariants λ^β $(\nu = 1, \dots, n - r_*)$ don't depend on the dependent variables u^j. This facilitates the calculation of invariant solutions. The condition (2.55) guarantees that equations (2.57) can be solved for u^j. Substituting them into original system (S), we can obtain reduced system S/H_r.

The algorithm of obtaining invariant solutions of differential equations consists of the following steps.

Step 1 Find the admitted Lie group of the differential equations by solving the determining equation. All admitted Lie groups form a Lie algebra.

Step 2 Choose a subalgebra and find invariants of the subalgebra.

Step 3 Construct a representation of an invariant.

Step 4 Substitute the representation into the original system of differential equations, to obtain equations with reduced number of independent variable.

CHAPTER III

PRELIMINARY GROUP CLASSIFICATION OF THE FULL BOLTZMANN EQUATION WITH A SOURCE

The purpose of this chapter is to give a preliminary group classification of the full Boltzmann equation with a source function of the form

$$f_t + uf_x + vf_y + wf_z - J(f, f) = q(x, y, z, u, v, w, t, f). \tag{3.1}$$

Solving the problem of preliminary group classification is divided into three parts. The first part provides the derivation and simplification of the determining equation of an equation with nonlocal operator in general form. The next part is about finding the determining equation of Equation (3.1). Finally, how to obtain source functions is illustrated by examples.

3.1 Derivation of determining equations for a source function (general case)

Consider an equation with nonlocal operator

$$\Phi(\mathbf{x}, f) = 0, \tag{3.2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is the vector of the independent variables, f is the dependent variable, and Φ is a nonlocal operator acting on f.

Let Lie group of transformations G^1 with scalar parameter a:

$$\bar{\mathbf{x}} = \mathbf{g}(\mathbf{x}, f; a), \ \ \bar{f} = \psi(\mathbf{x}, f; a), \tag{3.3}$$

be admitted by Equation (3.2) with the generator

$$X = \xi^i(\mathbf{x}, f)\partial_{x_i} + \eta(\mathbf{x}, f)\partial_f. \tag{3.4}$$

The latter property provides that

$$\tilde{X}(\Phi) + \xi^i D_i(\Phi) = h_X \Phi, \tag{3.5}$$

where h_X is a function of the independent and dependent variables and their derivatives, D_i are the total derivatives with respect to x_i, $(i = 1, 2 \ldots, n)$, and $\tilde{X}$ is the prolongation of the canonical Lie-Bäcklund operator which is equivalent to the generator X:

$$\tilde{X} = \tilde{\eta}\partial_f + D_k(\tilde{\eta})\partial_{f_{x_k}} + \cdots . \tag{3.6}$$

Here $\tilde{\eta} = \eta - \xi^i f_{x_i}$, and the actions of the derivatives ∂_f and $\partial_{f_{x_k}}$ are considered in terms of the *Fréchet derivatives*.

For the preliminary group classification of the equation

$$\Phi(\mathbf{x}, f) = q \tag{3.7}$$

we use the Lie algebra admitted by the homogeneous equation (3.2). Here q is an arbitrary function of the independent and dependent variables, i.e., $q = q(\mathbf{x}, f)$.

First, we define the determining equation for a Lie algebra admitted by Equation (3.7), and then this determining equation is simplified by using the property that the admitted generator belongs to a Lie algebra admitted by Equation (3.2).

According to the definition for a Lie group G to be admitted, the group of transformations (3.3) maps a solution $f_0(\mathbf{x})$ of Equation (3.7) into a solution $f_a(\bar{\mathbf{x}})$ of the same equation in the new variables:

$$\Phi(\bar{\mathbf{x}}, \bar{f}) = q(\bar{\mathbf{x}}, \bar{f}). \tag{3.8}$$

The transformed solution $f_a(\bar{\mathbf{x}})$ is defined by the formula

$$f_a(\bar{\mathbf{x}}) = \psi(\mathbf{x}, f_0(\mathbf{x}); a), \tag{3.9}$$

where $\mathbf{x} = \mathbf{G}(\bar{\mathbf{x}}; a)$ is the solution of the equation

$$\bar{\mathbf{x}} = \mathbf{g}(\mathbf{x}, f_0(\mathbf{x}); a). \tag{3.10}$$

Differentiating the equation

$$\Phi(\bar{\mathbf{x}}, f_a(\bar{\mathbf{x}})) = q(\bar{\mathbf{x}}, f_a(\bar{\mathbf{x}})) \tag{3.11}$$

with respect to the parameter a, and setting $a = 0$, we obtain the relation

$$\left(\frac{\partial}{\partial a}(\Phi(\bar{\mathbf{x}}, f_a(\bar{\mathbf{x}})) - q(\bar{\mathbf{x}}, f_a(\bar{\mathbf{x}})))\right)_{|a=0} = 0 \tag{3.12}$$

which is called the *determining equation*.

The determining equation (3.12) is derived under the assumption that the solution of Equation (3.10):

$$\mathbf{x} = \mathbf{G}(\bar{\mathbf{x}}; a), \tag{3.13}$$

is defined on a set sufficient for substituting $\mathbf{G}(\bar{\mathbf{x}}; a)$ into (3.11).

In general, because of the localness of the inverse function theorem, Equation (3.10) provides a local solution whereas for nonlocal Equation (3.7) one needs that the solution (3.13) is nonlocal.

However, notice that (3.12) coincides with the equation

$$\tilde{X}(\Phi - q)(\mathbf{x}, f_0(\mathbf{x})) = 0. \tag{3.14}$$

As $f_0(\mathbf{x})$ is a solution of Equation (3.7), the latter equation can be written as

$$(\tilde{X}(\Phi - q))_{|(3.7)} = 0, \tag{3.15}$$

which is further used for the group classification.

For the preliminary group classification studied in the thesis the generator X is assumed to belong to a Lie algebra admitted by the homogeneous equation. Hence, the generator X satisfies the property (3.5). Using this property, we obtain that

$$\begin{aligned}\tilde{X}(\Phi - q) &= \tilde{X}(\Phi) - \tilde{X}(q)\\ &= h_X\Phi - \xi^i D_i(\Phi) - (X^p(q) - \xi^i D_i(q))\\ &= h_X\Phi - X^p(q) - \xi^i D_i(\Phi - q),\end{aligned} \tag{3.16}$$

where X^p is the prolongation of the generator X: $X^p = \tilde{X} + \xi^i D_i$. As the function q does not depend on derivatives, then $X^p(q) = X(q)$. Considering Equation (3.16) on any solution of Equation (3.7), we obtain the determining equation,

$$\tilde{h}_X q - X(q) = 0, \tag{3.17}$$

where $\tilde{h}_X = (h_X)_{|(3.7)}$.

Thus, we come to the following algorithm for a preliminary group classification of non-homogeneous equation (3.7) with the help of a Lie group admitted by the homogeneous equation (3.2).

Let L_n be a finite-dimensional Lie algebra and its basis generators $X_1, X_2, \dots, X_n$.

Step 1 Take a subalgebra L_k, $(k \leq n)$ from an optimal system of subalgebras of L_n, with the basis generators $Y_i = c_i^j X_j$, $(i = 1, 2, \dots, k)$.

Step 2 Compute the prolongation $\tilde{Y}_i$ of the canonical Lie-Bäcklund operator corresponding to the generator Y_i $(i = 1, 2, \dots, k)$.

Step 3 Solve the overdetermined system of k partial differential equations

$$\tilde{h}_{Y_i} q - Y_i(q) = 0, \ (i = 1, 2, \ldots, k), \tag{3.18}$$

where $\tilde{h}_{Y_i} = (h_{Y_i})_{|(3.7)}$. The found solution provides a function $q(\mathbf{x}, f)$ such that Equation (3.7) admits the Lie algebra L_k.

Here k is the dimension of subalgebra L_k.

As the Lie algebra L_n is determined by its basis generators $X_1, X_2, \ldots, X_n$, the function h_{Y_i} in Equations (3.18) can be also written through the functions h_{X_j}, $(j = 1, 2, \ldots, n)$. In fact, because of the linearity of the Lie-Bäcklund operator,

$$\tilde{Y}_i = c_i^j \tilde{X}_j. \tag{3.19}$$

As

$$\begin{aligned}
Y_i &= c_i^1 X_1 + \cdots + c_i^n X_n \equiv c_i^j X_j \\
&= c_i^1(\xi^1_{X_1}\partial_{x_1} + \cdots + \xi^n_{X_1}\partial_{x_n}) + \cdots + c_i^n(\xi^1_{X_n}\partial_{x_1} + \cdots + \xi^n_{X_n}\partial_{x_n}) \\
&= (c_i^1\xi^1_{X_1} + \cdots + c_i^n\xi^1_{X_n})\partial_{x_1} + \cdots + (c_i^1\xi^n_{X_1} + \cdots + c_i^n\xi^n_{X_n})\partial_{x_n} \\
&\equiv c_i^j\xi^1_{X_j}\partial_{x_1} + \cdots + c_i^j\xi^n_{X_j}\partial_{x_n} \equiv c_i^j\xi^k_{X_j}\partial_{x_k},
\end{aligned}$$

then $\xi^k_{Y_i} = c_i^j \xi^k_{X_j}$. Hence,

$$\begin{aligned}
h_{Y_i}\Phi &= \tilde{Y}_i(\Phi) + \xi^k_{Y_i} D_k(\Phi) \\
&= c_i^j \tilde{X}_j(\Phi) + c_i^j \xi^k_{X_j} D_k(\Phi) \\
&= c_i^j(\tilde{X}_j(\Phi) + \xi^k_{X_j} D_k(\Phi)) \\
&= c_i^j h_{X_j}\Phi.
\end{aligned} \tag{3.20}$$

Thus, the determining equation (3.18) becomes

$$c_i^j(\tilde{h}_{X_j}q - X_j(q)) = 0,\ (i = 1, 2, \ldots, k). \tag{3.21}$$

3.2 Determining equation for a source function of the Boltzmann equation

Let Φ be an integro-differential operator acting on f

$$\Phi(f) = f_t + uf_x + vf_y + wf_z - J(f, f), \tag{3.22}$$

where $J(f, f)$ is the collision integral.

As mentioned in the Introduction, the Boltzmann equation $\Phi(f) = 0$ admits the Lie algebra L_{11} with the basis generators (1.18). The Lie algebra L_{11} is isomorphic to the Lie algebra L_{11}^g admitted by the gas dynamics equations (Grigoriev et al., 1999; Grigoriev and Meleshko, 2001). In (Ovsiannikov, 1999), an optimal system of subalgebras of L_{11}^g is constructed (see Appendix C). Because of the isomorphism of L_{11} and L_{11}^g, we can apply the method described above for solving the preliminary group classification problem of the Boltzmann equation with a source,

$$\Phi(f) = q, \tag{3.23}$$

where $q = q(x, y, z, u, v, w, t, f)$.

First, we provide definitions, equations, and their variables which are useful for the next four subsections.

Let $f = f_0(x, y, z, u, v, w, t)$ be a function; the transformed function is denoted as $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$. The collision integral for the transformed function has the form

$$J(\bar{f}, \bar{f}) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(\bar{g}, \theta_1)(\bar{f}^*\bar{f}_1^* - \bar{f}\bar{f}_1)d\mathbf{n}d\mathbf{w}, \tag{3.24}$$

where

$$\bar{f} = \bar{f}(\bar{\mathbf{x}}, \bar{\mathbf{v}}, \bar{t}),\ \bar{f}_1 = \bar{f}(\bar{\mathbf{x}}, \mathbf{w}, \bar{t}),\ \bar{f}^* = \bar{f}(\bar{\mathbf{x}}, \bar{\mathbf{v}}^*, \bar{t}),\ \bar{f}_1^* = \bar{f}(\bar{\mathbf{x}}, \bar{\mathbf{w}}^*, \bar{t}),$$

$$\bar{\mathbf{x}} = (\bar{x}, \bar{y}, \bar{z}),\ \bar{\mathbf{v}} = (\bar{u}, \bar{v}, \bar{w}),\ \mathbf{w} = (u_1, v_1, w_1),\ \bar{\mathbf{v}}^* = \frac{1}{2}(\bar{\mathbf{v}} + \mathbf{w} + \bar{g}\mathbf{n}),$$

$$\bar{\mathbf{w}}^* = \frac{1}{2}(\bar{\mathbf{v}} + \mathbf{w} - \bar{g}\mathbf{n}),\ \bar{\mathbf{g}} = \bar{\mathbf{v}} - \mathbf{w},\ \bar{g} = \|\bar{\mathbf{g}}\|_2. \tag{3.25}$$

In this section we show that Equation (3.21) for the Boltzmann equation (3.23) is

$$c_i^{11}(2q) + c_i^j(X_j(q)) = 0,\ (i = 1, \ldots, k). \tag{3.26}$$

3.2.1 Derivation of the function h_{X_1}

The group of transformations corresponding to the generator X_1 is

$$\bar{x} = x + a,\ \bar{y} = y,\ \bar{z} = z,\ \bar{u} = u,\ \bar{v} = v,\ \bar{w} = w,\ \bar{t} = t,\ \bar{f} = f. \tag{3.27}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x} - a, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}). \tag{3.28}$$

It follows that

$$\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0t}(\bar{x} - a, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}),$$

$$\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0x}(\bar{x} - a, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}),$$

$$\bar{f}_{\bar{y}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0y}(\bar{x} - a, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}),$$

$$\bar{f}_{\bar{z}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0z}(\bar{x} - a, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}), \tag{3.29}$$

and

$$\bar{f}^* = f_0(\bar{x} - a, \bar{y}, \bar{z}, \frac{1}{2}(\bar{u} + u_1 + \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3), \bar{t}),$$

$$\bar{f}_1^* = f_0(\bar{x} - a, \bar{y}, \bar{z}, \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3), \bar{t}),$$

$$\bar{f}_1 = f_0(\bar{x} - a, \bar{y}, \bar{z}, u_1, v_1, w_1, \bar{t}). \tag{3.30}$$

Using (3.27), one finds that

$$\bar{g} = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2}. \tag{3.31}$$

Substituting (3.28), (3.30), (3.31) into (3.24), the collision term becomes

$$(J(\bar{f}, \bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (J(f_0, f_0))(\bar{x} - a, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}). \tag{3.32}$$

Then

$$\begin{aligned}(\Phi(\bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= (f_{0t} + uf_{0x} + vf_{0y} + wf_{0z} \\ &\quad - J(f_0, f_0))(\bar{x} - a, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) \\ &= (\Phi(f_0))(\bar{x} - a, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}).\end{aligned} \tag{3.33}$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_1(\Phi) = -D_x(\Phi). \tag{3.34}$$

As $X_1 = \partial_x$ then

$$h_{X_1}\Phi = \tilde{X}_1(\Phi) + D_x(\Phi) = -D_x(\Phi) + D_x(\Phi) = 0. \tag{3.35}$$

This means that

$$\tilde{h}_{X_1} = 0. \tag{3.36}$$

Similarly, we derive that (see Appendix B),

$$\tilde{h}_{X_j} = 0, \ (j = 2, 3, 10). \tag{3.37}$$

3.2.2 Derivation of the function h_{X_4}

The group of transformations corresponding to the generator X_4 is

$$\bar{x} = x + at, \ \bar{y} = y, \ \bar{z} = z, \ \bar{u} = u + a, \ \bar{v} = v, \ \bar{w} = w, \ \bar{t} = t, \ \bar{f} = f. \tag{3.38}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t}). \tag{3.39}$$

It follows that

$$\begin{aligned}
&\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (-af_{0x} + f_{0t})(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t}),\\
&\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0x}(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t}),\\
&\bar{f}_{\bar{y}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0y}(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t}),\\
&\bar{f}_{\bar{z}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0z}(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t}),
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
\bar{f}^* &= f_0(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \frac{1}{2}(\bar{u} + u_1 + \bar{g}n_1) - a, \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3), \bar{t})\\
&= f_0(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \frac{1}{2}((\bar{u} - a) + (u_1 - a) + \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2),\\
&\frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3), \bar{t}),\\
\bar{f}_1^* &= f_0(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1) - a, \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3), \bar{t})
\end{aligned}$$

$$= f_0(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \frac{1}{2}((\bar{u} - a) + (u_1 - a) - \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2),$$

$$\frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3), \bar{t}),$$

$$\bar{f}_1 = f_0(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, u_1 - a, v_1, w_1, \bar{t}). \quad (3.41)$$

Using (3.38), one finds that

$$\bar{g} = \sqrt{(u + a - u_1)^2 + (v - v_1)^2 + (w - w_1)^2}$$

$$= \sqrt{(u - (u_1 - a))^2 + (\bar{v} - v_1)^2 + (\bar{w} - w_1)^2}. \quad (3.42)$$

Substituting (3.39), (3.41), (3.42) into (3.24), and using the change of variables: $\tilde{u}_1 = u_1 - a$, the collision term becomes

$$(J(\bar{f}, \bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (J(f_0, f_0))(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t}). \quad (3.43)$$

Then

$$(\Phi(\bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (-af_{0x} + f_{0t} + (u + a)f_{0x} + vf_{0y} + wf_{0z}$$

$$- J(f_0, f_0))(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t})$$

$$= (f_{0t} + uf_{0x} + vf_{0y} + wf_{0z} - J(f_0, f_0))(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t})$$

$$= (\Phi(f_0))(\bar{x} - a\bar{t}, \bar{y}, \bar{z}, \bar{u} - a, \bar{v}, \bar{w}, \bar{t}). \quad (3.44)$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_4(\Phi) = -(tD_x(\Phi) + D_u(\Phi)). \quad (3.45)$$

As $X_4 = t\partial_x + \partial_u$ then

$$h_{X_4}\Phi = \tilde{X}_4(\Phi) + (tD_x(\Phi) + D_u(\Phi))$$

$$= -(tD_x(\Phi) + D_u(\Phi)) + (tD_x(\Phi) + D_u(\Phi)) = 0. \tag{3.46}$$

This means that

$$\tilde{h}_{X_4} = 0. \tag{3.47}$$

Similarly, we derive that (see Appendix B),

$$\tilde{h}_{X_j} = 0, \ (j = 5, 6). \tag{3.48}$$

3.2.3 Derivation of the function h_{X_7}

The group of transformations corresponding to the generator X_7 is

$$\bar{x} = x, \ \bar{y} = y\cos(a) - z\sin(a), \ \bar{z} = y\sin(a) + z\cos(a),$$

$$\bar{u} = u, \ \bar{v} = v\cos(a) - w\sin(a), \ \bar{w} = v\sin(a) + w\cos(a), \ \bar{t} = t, \ \bar{f} = f. \tag{3.49}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), \bar{u},$$

$$\bar{v}\cos(a) + \bar{w}\sin(a), \bar{w}\cos(a) - \bar{v}\sin(a), \bar{t}). \tag{3.50}$$

It follows that

$$\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0t}(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), \bar{u},$$

$$\bar{v}\cos(a) + \bar{w}\sin(a), \bar{w}\cos(a) - \bar{v}\sin(a), \bar{t}),$$

$$\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0x}(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), \bar{u},$$

$$\bar{v}\cos(a) + \bar{w}\sin(a), \bar{w}\cos(a) - \bar{v}\sin(a), \bar{t}),$$

$$\bar{f}_{\bar{y}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (f_{0y}\cos(a) - f_{0z}\sin(a))(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a)$$

$$- \bar{y}\sin(a), \bar{u}, \bar{v}\cos(a) + \bar{w}\sin(a), \bar{w}\cos(a) - \bar{v}\sin(a), \bar{t}),$$

$$\bar{f}_{\bar{z}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (f_{0y}\sin(a) + f_{0z}\cos(a))(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a)$$

$$- \bar{y}\sin(a), \bar{u}, \bar{v}\cos(a) + \bar{w}\sin(a), \bar{w}\cos(a) - \bar{v}\sin(a), \bar{t}), \qquad (3.51)$$

and

$$\bar{f}_1 = f_0(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), u_1, v_1\cos(a) + w_1\sin(a),$$

$$w_1\cos(a) - v_1\sin(a), \bar{t})$$

$$= f_0(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), \tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \bar{t}),$$

$$\bar{f}^* = f_0(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), \frac{1}{2}(\bar{u} + u_1 + \bar{g}n_1),$$

$$\frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2)\cos(a) + \frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3)\sin(a),$$

$$\frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3)\cos(a) - \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2)\sin(a), \bar{t})$$

$$= f_0(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), \frac{1}{2}(\bar{u} + \tilde{u}_1 + \bar{g}\tilde{n}_1),$$

$$\frac{1}{2}(\bar{v}\cos(a) + \bar{w}\sin(a) + \tilde{v}_1 + \bar{g}\tilde{n}_2), \frac{1}{2}(\bar{w}\cos(a) - \bar{v}\sin(a) + \tilde{w}_1 + \bar{g}\tilde{n}_3), \bar{t}),$$

$$\bar{f}_1^* = f_0(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1),$$

$$\frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2)\cos(a) + \frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3)\sin(a),$$

$$\frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3)\cos(a) - \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2)\sin(a), \bar{t})$$

$$= f_0(\bar{x}, \bar{y}\cos(a) + \bar{z}\sin(a), \bar{z}\cos(a) - \bar{y}\sin(a), \frac{1}{2}(\bar{u} + \tilde{u}_1 - \bar{g}\tilde{n}_1), \frac{1}{2}(\bar{v}\cos(a)$$

$$+ \bar{w}\sin(a) + \tilde{v}_1 - \bar{g}\tilde{n}_2), \frac{1}{2}(\bar{w}\cos(a) - \bar{v}\sin(a) + \tilde{w}_1 - \bar{g}\tilde{n}_3), \bar{t}), \qquad (3.52)$$

where

$$\tilde{u}_1 = u_1,\ \tilde{v}_1 = v_1\cos(a) + w_1\sin(a),\ \tilde{w}_1 = w_1\cos(a) - v_1\sin(a),$$

$$\tilde{n}_1 = n_1,\ \tilde{n}_2 = \cos(a)n_2 + \sin(a)n_3,\ \tilde{n}_3 = \cos(a)n_3 - \sin(a)n_2. \quad (3.53)$$

Using (3.49), one finds that

$$\bar{g} = \sqrt{(u-u_1)^2 + (v\cos(a) - w\sin(a) - v_1)^2 + (v\sin(a) + w\cos(a) - w_1)^2}$$
$$= \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (\tilde{v}-\tilde{v}_1)^2 + (\tilde{w}-\tilde{w}_1)^2}. \quad (3.54)$$

Substituting (3.50), (3.52), (3.54) into (3.24), and using the change of variables: $\tilde{u}_1, \tilde{v}_1, \tilde{w}_1$ defined by (3.53), the collision term becomes

$$(J(\bar{f},\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (J(f_0,f_0))(\bar{x},\bar{y}\cos(a)+\bar{z}\sin(a),\bar{z}\cos(a)-\bar{y}\sin(a),\bar{u},$$
$$\bar{v}\cos(a)+\bar{w}\sin(a),\bar{w}\cos(a)-\bar{v}\sin(a),\bar{t}). \quad (3.55)$$

Here we have used the properties that $\|(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3)\|_2 = 1$, and $|\frac{\partial(u_1,v_1,w_1)}{\partial(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1)}| = 1$.

Hence,

$$(\Phi(\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (f_{0t} + uf_{0x} + (v\cos(a) - w\sin(a))(f_{0y}\cos(a) - f_{0z}\sin(a))$$
$$+ (v\sin(a) + w\cos(a))(f_{0y}\sin(a) + f_{0z}\cos(a)) - J(f_0,f_0))(\bar{x},\bar{y}\cos(a)$$
$$+ \bar{z}\sin(a),\bar{z}\cos(a)-\bar{y}\sin(a),\bar{u},\bar{v}\cos(a)+\bar{w}\sin(a),\bar{w}\cos(a)-\bar{v}\sin(a),\bar{t})$$
$$= (f_{0t} + uf_{0x} + vf_{0y} + wf_{0z} - J(f_0,f_0))(\bar{x},\bar{y}\cos(a)+\bar{z}\sin(a),$$
$$\bar{z}\cos(a)-\bar{y}\sin(a),\bar{u},\bar{v}\cos(a)+\bar{w}\sin(a),\bar{w}\cos(a)-\bar{v}\sin(a),\bar{t})$$
$$= (\Phi(f_0))(\bar{x},\bar{y}\cos(a)+\bar{z}\sin(a),\bar{z}\cos(a)-\bar{y}\sin(a),\bar{u},\bar{v}\cos(a)+\bar{w}\sin(a),$$
$$\bar{w}\cos(a)-\bar{v}\sin(a),\bar{t}). \quad (3.56)$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_7(\Phi) = -yD_z(\Phi) + zD_y(\Phi) - vD_w(\Phi) + wD_v(\Phi). \tag{3.57}$$

As $X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v$, then

$$\begin{aligned} h_{X_7}\Phi &= \tilde{X}_7(\Phi) + (yD_z(\Phi) - zD_y(\Phi) + vD_w(\Phi) - wD_v(\Phi)) \\ &= (-yD_z(\Phi) + zD_y(\Phi) - vD_w(\Phi) + wD_v(\Phi)) \\ &\quad + (yD_z(\Phi) - zD_y(\Phi) + vD_w(\Phi) - wD_v(\Phi)) = 0. \end{aligned} \tag{3.58}$$

This means that

$$\tilde{h}_{X_7} = 0. \tag{3.59}$$

Similarly, we derive that (see Appendix B),

$$\tilde{h}_{X_j} = 0, \ (j = 8, 9). \tag{3.60}$$

3.2.4 Derivation of the function $h_{X_{11}}$

The group of transformations corresponding to the generator X_{11} is

$$\bar{x} = xe^a, \ \bar{y} = ye^a, \ \bar{z} = ze^a, \ \bar{u} = u, \ \bar{v} = v, \ \bar{w} = w, \ \bar{t} = te^a, \ \bar{f} = fe^{-a}. \tag{3.61}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = e^{-a} f_0(\bar{x}e^{-a}, \bar{y}e^{-a}, \bar{z}e^{-a}, \bar{u}, \bar{v}, \bar{w}, \bar{t}e^{-a}). \tag{3.62}$$

It follows that

$$\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = e^{-2a} f_{0t}(\bar{x}e^{-a}, \bar{y}e^{-a}, \bar{z}e^{-a}, \bar{u}, \bar{v}, \bar{w}, \bar{t}e^{-a}),$$

$$\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = e^{-2a} f_{0x}(\bar{x}e^{-a}, \bar{y}e^{-a}, \bar{z}e^{-a}, \bar{u}, \bar{v}, \bar{w}, \bar{t}e^{-a}),$$

$$\bar{f}_{\bar{y}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = e^{-2a}f_{0y}(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},\bar{u},\bar{v},\bar{w},\bar{t}e^{-a}),$$

$$\bar{f}_{\bar{z}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = e^{-2a}f_{0z}(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},\bar{u},\bar{v},\bar{w},\bar{t}e^{-a}), \tag{3.63}$$

and

$$\bar{f}^* = e^{-a}f_0(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},\frac{1}{2}(\bar{u}+u_1+\bar{g}n_1),\frac{1}{2}(\bar{v}+v_1+\bar{g}n_2),$$
$$\frac{1}{2}(\bar{w}+w_1+\bar{g}n_3),\bar{t}e^{-a}),$$

$$\bar{f}_1^* = e^{-a}f_0(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},\frac{1}{2}(\bar{u}+u_1-\bar{g}n_1),\frac{1}{2}(\bar{v}+v_1-\bar{g}n_2),$$
$$\frac{1}{2}(\bar{w}+w_1-\bar{g}n_3),\bar{t}e^{-a}),$$

$$\bar{f}_1 = e^{-a}f_0(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},u_1,v_1,w_1,\bar{t}e^{-a}). \tag{3.64}$$

Using (3.61), one finds that

$$\bar{g} = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}. \tag{3.65}$$

Substituting (3.62), (3.64), (3.65) into (3.24), the collision term becomes

$$(J(\bar{f},\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = e^{-2a}(J(f_0,f_0))(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},$$
$$\bar{u},\bar{v},\bar{w},\bar{t}e^{-a}). \tag{3.66}$$

Then

$$(\Phi(\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (e^{-2a}f_{0t}+ue^{-2a}f_{0x}+ve^{-2a}f_{0y}+we^{-2a}f_{0z}$$
$$-e^{-2a}J(f_0,f_0))(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},\bar{u},\bar{v},\bar{w},\bar{t}e^{-a})$$
$$= e^{-2a}(f_{0t}+uf_{0x}+vf_{0y}+wf_{0z}-J(f_0,f_0))(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},\bar{u},\bar{v},\bar{w},\bar{t}e^{-a})$$
$$= e^{-2a}(\Phi(f_0))(\bar{x}e^{-a},\bar{y}e^{-a},\bar{z}e^{-a},\bar{u},\bar{v},\bar{w},\bar{t}e^{-a}). \tag{3.67}$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_{11}(\Phi) = -2\Phi - xD_x(\Phi) - yD_y(\Phi) - zD_z(\Phi) - tD_t(\Phi). \tag{3.68}$$

As $X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z - f\partial_f$, then

$$\begin{aligned} h_{X_{11}}\Phi &= \tilde{X}_{11}(\Phi) + (xD_x(\Phi) + yD_y(\Phi) + zD_z(\Phi) + tD_t(\Phi)) \\ &= (-2\Phi - xD_x(\Phi) - yD_y(\Phi) - zD_z(\Phi) - tD_t(\Phi)(\Phi)) \\ &+ (xD_x(\Phi) + yD_y(\Phi) + zD_z(\Phi) + tD_t(\Phi)) = -2\Phi. \end{aligned} \tag{3.69}$$

This means that

$$\tilde{h}_{X_{11}} = -2. \tag{3.70}$$

3.3 Illustrative examples of obtaining the source function

This section gives examples which illustrate the application of the above strategy. Complete results of the preliminary group classification are presented in Appendix D.

For solving differential equations corresponding to a subalgebra of the Lie algebra L_{11} one needs to find their integrals. Among these subalgebras there are subalgebras containing the generators of the rotations X_7, X_8, and X_9. In some of these cases it is appropriate to write the determining equations (3.26) in the cylindrical or spherical coordinate systems. It should also be mentioned that the necessity of the changing the coordinate system depends on the complexity of the system of determining equations. As different coordinate systems are used we provide illustrative examples in all of these systems.

3.3.1 Illustrative examples in the Cartesian coordinate system

This section provides two examples of obtaining the source function for 1-dimensional and 2-dimensional subalgebras.

Example 3.3.1. (A source function corresponding to the subalgebra 1.8 : $\{X_{11}\}$.) Substituting

$$c_1^{11} = 1,\ c_1^j = 0,\ j \neq 11$$

into (3.26), we obtain the partial differential equation

$$2q - fq_f + tq_t + xq_x + yq_y + zq_z = 0. \tag{3.71}$$

The solution of Equation (3.71) can be obtained by the method of characteristics. The characteristic system of Equation (3.71) is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{0} = \frac{dv}{0} = \frac{dw}{0} = \frac{dt}{t} = \frac{df}{-f} = \frac{dq}{-2q}. \tag{3.72}$$

The independent first integrals are

$$\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, u, v, w, ft, qt^2. \tag{3.73}$$

Hence, the general solution of Equation (3.71) has the form

$$q = t^{-2}\Psi(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, u, v, w, ft), \tag{3.74}$$

where Ψ is an arbitrary function of 7 variables.

Example 3.3.2. (A source function corresponding to the subalgebra 2.13 : $\{X_4, X_{11}\}$.) For this subalgebra,

$$c_1^4 = 1,\ c_1^j = 0,\ j \neq 4;\ c_2^{11} = 1,\ c_2^j = 0,\ j \neq 11.$$

Substituting these coefficients into (3.26), we obtain the system of two partial differential equations:

$$q_u + tq_x = 0, \tag{3.75}$$

$$2q - fq_f + tq_t + xq_x + yq_y + zq_z = 0. \tag{3.76}$$

The characteristic system of Equation (3.75) is

$$\frac{dx}{t} = \frac{dy}{0} = \frac{dz}{0} = \frac{du}{1} = \frac{dv}{0} = \frac{dw}{0} = \frac{dt}{0} = \frac{df}{0} = \frac{dq}{0}. \tag{3.77}$$

Then, the independent first integrals are

$$y, z, u - \frac{x}{t}, v, w, t, f, q. \tag{3.78}$$

Hence, the general solution of Equation (3.75) is

$$q = \Psi_1(y, z, \tilde{u}, v, w, t, f),\ \tilde{u} = u - \frac{x}{t} \tag{3.79}$$

where Ψ_1 is an arbitrary function of 7 variables. Substituting the latter representation into Equation (3.76), we derive

$$2\Psi_1 - f\Psi_{1f} + t\Psi_{1t} + y\Psi_{1y} + z\Psi_{1z} = 0. \tag{3.80}$$

The characteristic system of the latter equation is

$$\frac{dy}{y} = \frac{dz}{z} = \frac{d\tilde{u}}{0} = \frac{dv}{0} = \frac{dw}{0} = \frac{dt}{t} = \frac{df}{-f} = \frac{d\Psi_1}{-2\Psi_1}. \tag{3.81}$$

which has the independent first integrals:

$$\frac{y}{t}, \frac{z}{t}, \tilde{u}, v, w, ft, \Psi_1 t^2. \tag{3.82}$$

Hence, the general solution of Equation (3.80) is

$$\Psi_1 = t^{-2}\Psi(\frac{y}{t}, \frac{z}{t}, \tilde{u}, v, w, ft), \tag{3.83}$$

where Ψ is an arbitrary function of 6 variables. Thus, the general solution of system of equations (3.75)-(3.76) has the form

$$q = t^{-2}\Psi(\frac{y}{t}, \frac{z}{t}, u - \frac{x}{t}, v, w, ft). \tag{3.84}$$

3.3.2 Illustrative examples in the cylindrical coordinate system

Here we give two examples of solving a system of partial differential equations where the cylindrical coordinate system is used.

Example 3.3.3. (A source function corresponding to the subalgebra 1.2 : $\{\beta X_4 + X_7\},\ \beta \neq 0.$)

Substituting

$$c_1^4 = \beta,\ c_1^7 = 1,\ c_1^j = 0,\ j \neq 4, 7$$

into (3.26), we obtain the equation

$$\beta q_u + \beta t q_x - w q_v + v q_w - z q_y + y q_z = 0, \tag{3.85}$$

where $q = q(x, y, z, u, v, w, t, f)$. Using change of variables in cylindrical coordinates, Equation (3.85) becomes

$$\beta q_u + \beta t q_x + q_\theta = 0, \tag{3.86}$$

where $q = q(x, r, \theta, u, V, W, t, f)$. Thus, the general solution of Equation (3.85) in the cylindrical coordinate system has the form

$$q = \Psi(t, r, \beta\theta - \frac{x}{t}, u - \frac{x}{t}, V, W, f), \tag{3.87}$$

where Ψ is an arbitrary function of 7 variables.

Example 3.3.4. (A source function corresponding to the subalgebra 2.2 : $\{\alpha X_4 + X_7, \beta X_4 + X_{11}\}$.)

For this subalgebra,

$$c_1^4 = \alpha,\ c_1^7 = 1,\ c_1^j = 0,\ j \neq 4, 7;\ c_2^4 = \beta,\ c_2^{11} = 1,\ c_2^j = 0,\ j \neq 4, 11.$$

Substituting these coefficients into (3.26), we obtain the system of two partial differential equations:

$$\alpha q_u - w q_v + v q_w + \alpha t q_x - z q_y + y q_z = 0, \tag{3.88}$$

$$2q - f q_f + t q_t + \beta q_u + (x + \beta t) q_x + y q_y + z q_z = 0, \tag{3.89}$$

where $q = q(x, y, z, u, v, w, t, f)$. Using cylindrical coordinates, Equations (3.88)-(3.89) become

$$\alpha q_u + \alpha t q_x + q_\theta = 0, \tag{3.90}$$

$$2q - f q_f + t q_t + \beta q_u + (x + \beta t) q_x + r q_r = 0, \tag{3.91}$$

where $q = q(x, r, \theta, u, V, W, t, f)$. The general solution of Equation (3.90) is

$$q = \Psi_1(t, r, \hat{p}, \hat{u}, V, W, f), \tag{3.92}$$

where $\hat{u} = u - \alpha\theta, \hat{p} = x - \alpha t\theta$, and Ψ_1 is an arbitrary function of 7 variables. Substituting q into Equation (3.91), it becomes

$$2\Psi_1 - f\Psi_{1f} + (\hat{p} + \beta t)\Psi_{1\hat{p}} + t\Psi_{1t} + \beta\Psi_{1\hat{u}} + r\Psi_{1r} = 0. \tag{3.93}$$

The general solution of the latter equation is

$$\Psi_1 = t^{-2}\Psi(\frac{r}{t}, \frac{\hat{p}}{t} - \beta \ln t, \hat{u} - \beta \ln t, V, W, ft), \tag{3.94}$$

where Ψ is an arbitrary function of 6 variables. Thus, the general solution of system of equations (3.88)-(3.89) in the cylindrical coordinate system has the form

$$q = t^{-2}\Psi(\frac{r}{t}, \frac{x}{t} - \alpha\theta - \beta \ln t, u - \alpha\theta - \beta \ln t, V, W, ft). \qquad (3.95)$$

3.3.3 Illustrative examples in the spherical coordinate system

In this section we give two examples of solving a system of partial differential equations corresponding to the generators of the rotations X_7, X_8, and X_9. For the first one we use a change of variables to the spherical coordinate system. For the second one, there is no necessity of changing variables.

Example 3.3.5. (A source function corresponding to the subalgebra 3.8 : $\{X_7, X_8, X_9\}$.)

Substituting the coefficients

$$c_1^7 = 1,\ c_1^j = 0,\ j \neq 7;\ c_2^8 = 1,\ c_2^j = 0,\ j \neq 8;\ c_3^9 = 1,\ c_3^j = 0,\ j \neq 9$$

into (3.26), we obtain the system of three partial differential equations:

$$-wq_v + vq_w - zq_y + yq_z = 0, \qquad (3.96)$$

$$wq_u - uq_w + zq_x - xq_z = 0, \qquad (3.97)$$

$$-vq_u + uq_v - yq_x + xq_y = 0, \qquad (3.98)$$

where $q = q(x, y, z, u, v, w, t, f)$. Using a change of variables to the spherical coordinate system, Equations (3.96)-(3.98) become

$$-\sin(\varphi)q_\theta - \cos(\varphi)\cot(\theta)q_\varphi - \frac{\cos(\varphi)}{\sin(\theta)}(Wq_V - Vq_W) = 0, \qquad (3.99)$$

$$\cos(\varphi)q_\theta - \sin(\varphi)\cot(\theta)q_\varphi - \frac{\sin(\varphi)}{\sin(\theta)}(Wq_V - Vq_W) = 0, \qquad (3.100)$$

$$q_\varphi = 0, \qquad (3.101)$$

where $q = q(r, \theta, \varphi, U, V, W, t, f)$. Because of Equation (3.101), q does not depend on φ, i.e., $q = q(r, \theta, U, V, W, t, f)$. Equations (3.99)-(3.100) become

$$\sin(\theta)\sin(\varphi)q_\theta + \cos(\varphi)(Wq_V - Vq_W) = 0, \tag{3.102}$$

$$\sin(\theta)\cos(\varphi)q_\theta - \sin(\varphi)(Wq_V - Vq_W) = 0. \tag{3.103}$$

The latter system of equations can be rewritten in the form

$$q_\theta = 0, \tag{3.104}$$

$$Wq_V - Vq_W = 0. \tag{3.105}$$

Thus, the general solution of system of equations (3.96)-(3.98) in the spherical coordinate system has the form

$$q = \Psi(t, r, U, \sqrt{V^2 + W^2}, f), \tag{3.106}$$

where Ψ is an arbitrary function of 5 variables.

Example 3.3.6. (A source function corresponding to the subalgebra 6.10 : $\{X_1, X_2, X_3, X_7, X_8, X_9\}$.)
Substituting the coefficients

$$c_1^1 = 1,\ c_1^j = 0,\ j \neq 1;\ c_2^2 = 1,\ c_2^j = 0,\ j \neq 2;\ c_3^3 = 1,\ c_3^j = 1,\ j \neq 3;$$

$$c_4^7 = 1,\ c_4^j = 0,\ j \neq 7;\ c_5^8 = 1,\ c_5^j = 0,\ j \neq 8;\ c_6^9 = 1,\ c_6^j = 0,\ j \neq 9,$$

into (3.26), we obtain the system of six partial differential equations:

$$q_x = 0, \tag{3.107}$$

$$q_y = 0, \tag{3.108}$$

$$q_z = 0, \tag{3.109}$$

$$-wq_v + vq_w - zq_y + yq_z = 0, \tag{3.110}$$

$$wq_u - uq_w + zq_x - xq_z = 0, \tag{3.111}$$

$$-vq_u + uq_v - yq_x + xq_y = 0, \tag{3.112}$$

where $q = q(x, y, z, u, v, w, t, f)$. Because of Equations (3.107)-(3.109), q does not depend on x, y and z. Hence, Equations (3.110)-(3.112) become

$$-wq_v + vq_w = 0, \tag{3.113}$$

$$wq_u - uq_w = 0, \tag{3.114}$$

$$-vq_u + uq_v = 0, \tag{3.115}$$

where $q = q(u, v, w, t, f)$. Notice that these equations are linearly dependent. The general solution of Equation (3.113) is

$$q = \Psi_1(u, \hat{v}, t, f), \hat{v} = \sqrt{v^2 + w^2} \tag{3.116}$$

where Ψ_1 is an arbitrary function of 4 variables. Substituting the latter solution into Equation (3.114), it becomes

$$\Psi_{1u} - \frac{u}{\hat{v}}\Psi_{1\hat{v}} = 0. \tag{3.117}$$

Thus, the general solution of system of equations (3.107)-(3.112) in the spherical coordinate system has the form

$$q = \Psi(\sqrt{u^2 + v^2 + w^2}, t, f), \tag{3.118}$$

where Ψ is an arbitrary function of 3 variables.

CHAPTER IV

INVARIANT SOLUTIONS

This chapter is devoted to obtaining representations of invariant solutions of the Boltzmann equation with a source function. Complete results of the representations of invariant solutions are presented in Appendix E.

Besides discussing representations of invariant solutions, this chapter is also devoted to finding reduced equations. It should be noted that for some of representations of invariant solutions, the problem of obtaining a reduced equation is extremely difficult. In this chapter we give several simple examples of obtaining reduced equations. For some subalgebras, results of the reduced Boltzmann equations are presented in Appendix F.

4.1 Representations of invariant solutions

In the previous chapter we obtained source functions q_k for all subalgebras L_k of the optimal system of the Lie algebra L_{11} such that the Boltzmann equation with the source function q_k admits the Lie algebra L_k.

Let L be any subalgebra of L_k. As L_k is a subalgebra of L_{11}, then L is a subalgebra of L_{11}. According to the definition of an optimal system of subalgebras, L is equivalent to one of subalgebras of the optimal system of subalgebras of the Lie algebra L_{11}, say $\tilde{L}$. Hence, invariant solutions with respect to these subalgebras L and $\tilde{L}$ are equivalent.

As the set of representations of invariant solutions of all subalgebras from the optimal system of subalgebras will be found, the solutions invariant with respect to the subalgebra L and the subalgebra $\tilde{L}$ are also equivalent.

In this chapter we will construct representations of invariant solutions for all subalgebras of the optimal system of subalgebras. This study will provide representations of all possible invariant solutions of the Boltzmann equation of the form (3.23).

4.2 Illustrative examples of obtaining the representation of invariant solutions

In this section we give examples which illustrate the method of finding a representation of an invariant solution in the Cartesian, cylindrical, and spherical coordinate systems.

4.2.1 Illustrative examples in the Cartesian coordinate system

This section provides two examples of obtaining a representation of invariant solutions for 1-dimensional and 2-dimensional subalgebras of the Lie algebra L_{11}.

Example 4.2.1. (A representation of invariant solutions corresponding to the subalgebra 1.8 : $\{X_{11}\}$.)
For this subalgebra, finding invariants one needs to solve the equation

$$X_{11}(J) = 0,$$

i.e.,

$$tJ_t + xJ_x + yJ_y + zJ_z - fJ_f = 0, \tag{4.1}$$

where $J = J(x, y, z, u, v, w, t, f)$. A solution of Equation (4.1) can be obtained by the method of characteristics. The characteristic system of Equation (4.1) is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{0} = \frac{dv}{0} = \frac{dw}{0} = \frac{dt}{t} = \frac{df}{-f}. \tag{4.2}$$

The independent invariants are

$$J_1 = \frac{x}{t}, J_2 = \frac{y}{t}, J_3 = \frac{z}{t}, J_4 = u, J_5 = v, J_6 = w, J_7 = ft. \tag{4.3}$$

The general solution of the latter equation is

$$J = \Omega_1(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, u, v, w, ft), \tag{4.4}$$

where Ω_1 is an arbitrary function of 7 variables. Thus, a representation of invariant solutions for this subalgebra has the form

$$f = t^{-1}\Omega(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, u, v, w), \tag{4.5}$$

where Ω is an arbitrary function of 6 variables.

Example 4.2.2. (A representation of invariant solutions corresponding to the subalgebra 2.13 : $\{X_4, X_{11}\}$.)
For this subalgebra, finding invariants one needs to solve the system of equations

$$X_4(J) = 0, \ X_{11}(J) = 0,$$

i.e.,

$$tJ_x + J_u = 0, \tag{4.6}$$

$$tJ_t + xJ_x + yJ_y + zJ_z - fJ_f = 0, \tag{4.7}$$

where $J = J(x, y, z, u, v, w, t, f)$. The general solution of Equation (4.6) is

$$J = J(y, z, \hat{u}, v, w, t, f), \ \hat{u} = u - \frac{x}{t} \tag{4.8}$$

where J is an arbitrary function of 7 variables. Substituting the latter representation into Equation (4.7), we derive

$$tJ_t + yJ_y + zJ_z - fJ_f = 0. \tag{4.9}$$

The general solution of the latter equation is

$$J = \Omega_1(\frac{y}{t}, \frac{z}{t}, u - \frac{x}{t}, v, w, ft). \quad (4.10)$$

where Ω_1 is an arbitrary function of 6 variables. Thus, a representation of invariant solutions for this subalgebra has the form

$$f = t^{-1}\Omega(\frac{y}{t}, \frac{z}{t}, u - \frac{x}{t}, v, w), \quad (4.11)$$

where Ω is an arbitrary function of 5 variables.

4.2.2 Illustrative examples in the cylindrical coordinate system

Here we give two examples of solving a system of partial differential equations, where the cylindrical coordinate system is used.

Example 4.2.3. (A representation of invariant solution corresponding to the subalgebra 1.2 : $\{\beta X_4 + X_7\},\ \beta \neq 0.$)

In the cylindrical coordinate system, the generator of this subalgebra is

$$\beta X_{4c} + X_{7c}, \quad (4.12)$$

where $X_{4c} = t\partial_x + \partial_u,\ X_{7c} = \partial_\theta$. For this subalgebra, finding invariants one needs to solve the equation

$$(\beta X_{4c} + X_{7c})(J) = 0, \quad (4.13)$$

i.e.,

$$\beta(tJ_x + J_u) + J_\theta = 0, \quad (4.14)$$

where $J = J(x, r, \theta, u, V, W, t, f)$. The independent invariants are

$$J_1 = t, J_2 = r, J_3 = \beta\theta - \frac{x}{t}, J_4 = u - \frac{x}{t}, J_5 = V, J_6 = W, J_7 = f. \quad (4.15)$$

The general solution of Equation (4.14) is

$$J = \Omega_1(t, r, \beta\theta - \frac{x}{t}, u - \frac{x}{t}, V, W, f), \tag{4.16}$$

where Ω_1 is an arbitrary function of 7 independent variables. Thus, a representation of invariant solutions in the cylindrical coordinate system for this subalgebra is

$$f = \Omega(t, r, \beta\theta - \frac{x}{t}, u - \frac{x}{t}, V, W), \tag{4.17}$$

where Ω is an arbitrary function of 6 independent variables.

Example 4.2.4. (A representation of invariant solution corresponding to the subalgebra 2.2 : $\{\alpha X_4 + X_7, \beta X_4 + X_{11}\}$.)

In the cylindrical coordinate system, the generators of this subalgebra are

$$\alpha X_{4c} + X_{7c},\ \ \beta X_{4c} + X_{11c} \tag{4.18}$$

where $X_{11c} = t\partial_t + x\partial_x + r\partial_r - f\partial_f$. For this subalgebra, finding invariants one needs to solve the system of equations

$$(\alpha X_{4c} + X_{7c})(J) = 0,\ (\beta X_{4c} + X_{11c})(J) = 0 \tag{4.19}$$

i.e.,

$$\alpha(tJ_x + J_u) + J_\theta = 0, \tag{4.20}$$

$$\beta(tJ_x + J_u) + tJ_t + xJ_x + rJ_r - fJ_f = 0. \tag{4.21}$$

where $J = J(x, r, \theta, u, V, W, t, f)$. The general solution of Equation (4.20) is

$$J = \Omega_1(t, r, \hat{p}, \hat{u}, V, W, f), \tag{4.22}$$

where $\hat{u} = u - \alpha\theta$, $\hat{p} = x - \alpha t\theta$, and Ω_1 is an arbitrary function of 7 variables.

Substituting the latter solution into Equation (4.21), it becomes

$$(\hat{p}+\beta t)\Omega_{1\hat{p}}+\beta\Omega_{1\hat{u}}+t\Omega_{1t}+r\Omega_{1r}-f\Omega_{1f}=0. \tag{4.23}$$

The general solution of the latter equation is

$$J=\Omega_2(\frac{r}{t},\frac{\hat{p}}{t}-\beta\ln t,\hat{u}-\beta\ln t,V,W,ft), \tag{4.24}$$

where Ω_2 is an arbitrary function of 6 variables. Thus, a representation of invariant solutions in the cylindrical coordinate system for this subalgebra is

$$f=t^{-1}\Omega(\frac{r}{t},\frac{x}{t}-\alpha\theta-\beta\ln t,u-\alpha\theta-\beta\ln t,V,W), \tag{4.25}$$

where Ω is an arbitrary function of 5 variables.

4.2.3 Illustrative example in the spherical coordinate system

In this section we give an example of obtaining a representation of invariant solutions in the spherical coordinate system.

Example 4.2.5. (A representation of invariant solutions corresponding to the subalgebra 3.8 : $\{X_7,X_8,X_9\}$.)
In the spherical coordinate system, the generators of this subalgebra are

$$X_{7s},\ X_{8s},\ X_{9s}, \tag{4.26}$$

where $X_{7s}=-\sin(\varphi)\partial_\theta-\cos(\varphi)\cot(\theta)\partial_\varphi-\frac{\cos(\varphi)}{\sin(\theta)}(W\partial_V-V\partial_W)$, $X_{8s}=\cos(\varphi)\partial_\theta-\sin(\varphi)\cot(\theta)\partial_\varphi-\frac{\sin(\varphi)}{\sin(\theta)}(W\partial_V-V\partial_W)$, $X_{9s}=\partial_\varphi$. For this subalgebra, finding invariants one needs to solve the system of equations

$$X_{7s}(J)=0,\ X_{8s}(J)=0,\ X_{9s}(J)=0, \tag{4.27}$$

i.e.,

$$-\sin(\varphi)J_\theta - \cos(\varphi)\cot(\theta)J_\varphi - \frac{\cos(\varphi)}{\sin(\theta)}(WJ_V - VJ_W) = 0, \tag{4.28}$$

$$\cos(\varphi)J_\theta - \sin(\varphi)\cot(\theta)J_\varphi - \frac{\sin(\varphi)}{\sin(\theta)}(WJ_V - VJ_W) = 0, \tag{4.29}$$

$$J_\varphi = 0, \tag{4.30}$$

where $J = J(r, \theta, \varphi, U, V, W, t, f)$. Because of Equation (4.30), J does not depend on φ, i.e., $J = J(r, \theta, U, V, W, t, f)$. Equations (4.28)-(4.29) become

$$\sin(\theta)\sin(\varphi)J_\theta + \cos(\varphi)(WJ_V - VJ_W) = 0, \tag{4.31}$$

$$\sin(\theta)\cos(\varphi)J_\theta - sin(\varphi)(WJ_V - VJ_W) = 0. \tag{4.32}$$

By the same reason as Example 3.14, the general solution of the system of equations (4.28)-(4.30) is

$$J = \Omega_1(t, r, U, \sqrt{V^2 + W^2}, f), \tag{4.33}$$

where Ω_1 is an arbitrary function of 5 variables. Thus, a representation of invariant solutions in the spherical coordinate system for this subalgebra is

$$f = \Omega(t, r, U, \sqrt{V^2 + W^2}), \tag{4.34}$$

where Ω is an arbitrary function of 4 variables.

4.3 Illustrative examples of obtaining the reduced Boltzmann equation

According to Appendix A, the full Boltzmann equations with a source function in the cylindrical, and in the spherical coordinate systems are provided. Substituting the source function and the representation of an invariant solution from the two previous sections into the Boltzmann equation, we obtain the reduced Boltzmann equation with source function. In this section we give some examples for obtaining the reduced Boltzmann equation with a source function in Cartesian, cylindrical, and spherical coordinate systems.

4.3.1 Illustrative examples in the Cartesian coordinate system

Example 4.3.1. (The reduced Boltzmann equation corresponding to the subalgebra $1.8 : \{X_{11}\}$.)

Recall the source function and the representation of invariant solutions corresponding to subalgebra $\{X_{11}\}$:

$$q = t^{-2}\Psi(p_1, p_2, p_3, u, v, w, ft), \tag{4.35}$$

$$f = t^{-1}\Omega(p_1, p_2, p_3, u, v, w), \tag{4.36}$$

where $p_1 = \frac{x}{t},\ p_2 = \frac{y}{t},\ p_3 = \frac{z}{t}$.

Then

$$f_t = t^{-2}(-\Omega - xt^{-1}\Omega_{p_1} - yt^{-1}\Omega_{p_2} - zt^{-1}\Omega_{p_3}),$$

$$f_x = t^{-2}\Omega_{p_1},\ f_y = t^{-2}\Omega_{p_2},\ f_z = t^{-2}\Omega_{p_3}. \tag{4.37}$$

Hence, the differential part of the Boltzmann equation (1.22) becomes

$$f_t + uf_x + vf_y + wf_z = t^{-2}(-\Omega + (u - p_1)\Omega_{p_1} + (v - p_2)\Omega_{p_2} + (w - p_3)\Omega_{p_3}). \tag{4.38}$$

As $f = t^{-1}\Omega(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, u, v, w)$, then

$$f_1 = t^{-1}\Omega(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, u_1, v_1, w_1),$$

$$f^* = t^{-1}\Omega(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, \frac{1}{2}(u + u_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$$

$$f_1^* = t^{-1}\Omega(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, \frac{1}{2}(u + u_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3)), \tag{4.39}$$

and the source function (4.35) becomes

$$q = t^{-2}\Psi(p_1, p_2, p_3, u, v, w, \Omega). \tag{4.40}$$

Substituting (4.39) into the collision integral of the Boltzmann equation (1.22), we obtain

$$J(f, f) = t^{-2} \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1) \sin\theta_1 \, d\theta_1 \, d\epsilon \, du_1 \, dv_1 \, dw_1, \tag{4.41}$$

where

$$\Omega = \Omega(p_1, p_2, p_3, u, v, w), \;\; \Omega_1 = \Omega(p_1, p_2, p_3, u_1, v_1, w_1),$$

$$\Omega^* = \Omega(p_1, p_2, p_3, \frac{1}{2}(u + u_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$$

$$\Omega_1^* = \Omega(p_1, p_2, p_3, \frac{1}{2}(u + u_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$$

$$g = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2}, \;\; p_1 = \frac{x}{t}, \;\; p_2 = \frac{y}{t}, \;\; p_3 = \frac{z}{t}. \tag{4.42}$$

Substituting (4.38), (4.41), and (4.40) into the Boltzmann equation (1.22), and multiplying the obtained equation by t^2, the reduced Boltzmann equation is

$$-\Omega + (u - p_1)\Omega_{p_1} + (v - p_2)\Omega_{p_2} + (w - p_3\Omega_{p_3}) - J(\Omega, \Omega)$$

$$= \Psi(p_1, p_2, p_3, u, v, w, \Omega), \tag{4.43}$$

where

$$J(\Omega, \Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1) \sin\theta_1 \, d\theta_1 \, d\epsilon \, du_1 \, dv_1 \, dw_1. \tag{4.44}$$

Example 4.3.2. (The reduced Boltzmann equation corresponding to the subalgebra 2.13 : $\{X_4, X_{11}\}$.)

Recall the source function and the representation of invariant solutions corresponding to subalgebra $\{X_4, X_{11}\}$:

$$q = t^{-2}\Psi(p_1, p_2, \tilde{u}, v, w, ft), \tag{4.45}$$

$$f = t^{-1}\Omega(p_1, p_2, \tilde{u}, v, w), \tag{4.46}$$

where $p_1 = \frac{y}{t},\ p_2 = \frac{z}{t},\ \tilde{u} = u - \frac{x}{t}$.

Then

$$f_t = t^{-2}(-\Omega - yt^{-1}\Omega_{p_1} - zt^{-1}\Omega_{p_2} + xt^{-1}\Omega_{\tilde{u}}),$$

$$f_x = -t^{-2}\Omega_{\tilde{u}},\ f_y = t^{-2}\Omega_{p_1},\ f_z = t^{-2}\Omega_{p_2}. \tag{4.47}$$

Hence, the differential part of the Boltzmann equation (1.22) becomes

$$f_t + uf_x + vf_y + wf_z = t^{-2}(-\Omega + (v - p_1)\Omega_{p_1} + (w - p_2)\Omega_{p_2} - \tilde{u}\Omega_{\tilde{u}}). \tag{4.48}$$

As $f = t^{-1}\Omega(\frac{y}{t}, \frac{z}{t}, u - \frac{x}{t}, v, w)$, then

$$f_1 = t^{-1}\Omega(\frac{y}{t}, \frac{z}{t}, u_1 - \frac{x}{t}, v_1, w_1),$$

$$f^* = t^{-1}\Omega(\frac{y}{t}, \frac{z}{t}, \frac{1}{2}(u + u_1 + gn_1) - \frac{x}{t}, \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3))$$

$$= t^{-1}\Omega(\frac{y}{t}, \frac{z}{t}, \frac{1}{2}((u - \frac{x}{t}) + (u_1 - \frac{x}{t}) + gn_1), \frac{1}{2}(v + v_1 + gn_2),$$

$$\frac{1}{2}(w + w_1 + gn_3)),$$

$$f_1^* = t^{-1}\Omega(\frac{y}{t}, \frac{z}{t}, \frac{1}{2}(u + u_1 - gn_1) - \frac{x}{t}, \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$$

$$= t^{-1}\Omega(\frac{y}{t}, \frac{z}{t}, \frac{1}{2}((u - \frac{x}{t}) + (u_1 - \frac{x}{t}) - gn_1), \frac{1}{2}(v + v_1 - gn_2),$$

$$\frac{1}{2}(w + w_1 - gn_3)), \tag{4.49}$$

and the source function (4.45) becomes

$$q = t^{-2}\Psi(p_1, p_2, \tilde{u}, v, w, \Omega). \tag{4.50}$$

Substituting (4.49) into the collision integral of the Boltzmann equation (1.22), and integrating by change of variable: $\tilde{u}_1 = u_1 - \frac{x}{t}$, we obtain

$$J(f, f) = t^{-2} \int_{\mathbb{R}^3} \int_{\mathbf{S}^2} B(g, \theta_1)(\Omega^* \Omega_1^* - \Omega\Omega_1) \sin\theta_1 \, d\theta_1 \, d\epsilon \, d\tilde{u}_1 \, dv_1 \, dw_1, \tag{4.51}$$

where

$$\Omega = \Omega(p_1, p_2, \tilde{u}, v, w), \;\; \Omega_1 = \Omega(p_1, p_2, \tilde{u}_1, v_1, w_1),$$

$$\Omega^* = \Omega(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$$

$$\Omega_1^* = \Omega(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3)),$$

$$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (v - v_1)^2 + (w - w_1)^2}, \;\; p_1 = \frac{y}{t}, \;\; p_2 = \frac{z}{t}, \;\; \tilde{u} = u - \frac{x}{t}. \tag{4.52}$$

Substituting (4.48), (4.51), and (4.50) into the Boltzmann equation (1.22), and multiplying the obtained equation by t^2, the reduced Boltzmann equation is

$$-\Omega + (v - p_1)\Omega_{p_1} + (w - p_2)\Omega_{p_2} - \tilde{u}\Omega_{\tilde{u}} - J(\Omega, \Omega) = \Psi(p_1, p_2, \tilde{u}, v, w, \Omega), \tag{4.53}$$

where

$$J(\Omega, \Omega) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2} B(g, \theta_1)(\phi^* \phi_1^* - \phi\phi_1) \sin\theta_1 \, d\theta_1 \, d\epsilon \, d\tilde{u}_1 \, dv_1 \, dw_1. \tag{4.54}$$

4.3.2 Illustrative examples in the cylindrical coordinate system

Example 4.3.3. (The reduced Boltzmann equation corresponding to the subalgebra $1.2 : \{\beta X_4 + X_7\},\ \beta \neq 0.$)

Recall the source function and the representation of invariant solutions corresponding to subalgebra $\{\beta X_4 + X_7\}$:

$$q = \Psi(t, r, p, \tilde{u}, V, W, f), \tag{4.55}$$

$$f = \Omega(t, r, p, \tilde{u}, V, W), \tag{4.56}$$

where $p = \beta\theta - \frac{x}{r},\ \tilde{u} = u - \frac{x}{t}$.

Then

$$f_t = \Omega_t + xt^{-2}\Omega_p + xt^{-2}\Omega_{\tilde{u}},\ f_x = -t^{-1}\Omega_p - t^{-1}\Omega_{\tilde{u}},$$

$$f_r = \Omega_r,\ f_\theta = \beta\Omega_p,\ f_V = \Omega_V,\ f_W = \Omega_W. \tag{4.57}$$

Hence, the differential part of the Boltzmann equation (A.14) becomes

$$f_t + uf_x + Vf_r + \frac{W}{r}f_\theta + \frac{W^2}{r}f_V - \frac{WV}{r}f_W = \Omega_t + (\frac{\beta W}{r} - \frac{\tilde{u}}{t})\Omega_p + V\Omega_r - \frac{\tilde{u}}{t}\Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W. \tag{4.58}$$

As $f = \Omega(t, r, \beta\theta - \frac{x}{r}, u - \frac{x}{t}, V, W)$, then

$$f_1 = \Omega(t, r, \beta\theta - \frac{x}{r}, u_1 - \frac{x}{t}, V_1, W_1),$$

$$f^* = \Omega(t, r, \beta\theta - \frac{x}{r}, \frac{1}{2}(u + u_1 + g_c n_{1c}) - \frac{x}{t}, \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c}))$$

$$= \Omega(t, r, \beta\theta - \frac{x}{r}, \frac{1}{2}((u - \frac{x}{t}) + (u_1 - \frac{x}{t}) + g_c n_{1c}), \frac{1}{2}(V + V_1 + g_c n_{2c}),$$

$$\frac{1}{2}(W + W_1 + g_c n_{3c})),$$

$$f_1^* = \Omega(t, r, \beta\theta - \frac{x}{r}, \frac{1}{2}(u + u_1 - g_c n_{1c}) - \frac{x}{t}, \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c}))$$

$$= \Omega(t, r, \beta\theta - \frac{x}{r}, \frac{1}{2}((u - \frac{x}{t}) + (u_1 - \frac{x}{t}) - g_c n_{1c}), \frac{1}{2}(V + V_1 - g_c n_{2c}),$$

$$\frac{1}{2}(W + W_1 - g_c n_{3c})), \tag{4.59}$$

and the source function (4.55) becomes

$$q = \Psi(t, r, p, \tilde{u}, V, W, \Omega). \tag{4.60}$$

Substituting (4.59) into the collision integral of the Boltzmann equation (A.14), and integrating by change of variable: $\tilde{u}_1 = u_1 - \frac{x}{t}$, we obtain

$$J(f, f) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1) \sin\theta_1 \, d\theta_1 \, d\epsilon \, d\tilde{u}_1 \, dV_1 \, dW_1, \tag{4.61}$$

where

$$\Omega = \Omega(t, r, p, \tilde{u}, V, W), \ \ \Omega_1 = \Omega(t, r, p, \tilde{u}_1, V_1, W_1),$$

$$\Omega^* = \Omega(t, r, p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + g_c n_{1c}), \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$$

$$\Omega_1^* = \Omega(t, r, p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - g_c n_{1c}), \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})),$$

$$g_c = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (V - V_1)^2 + (W - W_1)^2}, \ p = \beta\theta - \frac{x}{r}, \ \tilde{u} = u - \frac{x}{t}. \tag{4.62}$$

Substituting (4.58), (4.61), and (4.60) into the Boltzmann equation in the cylindrical coordinate system (A.14), the reduced Boltzmann equation is

$$\Omega_t + (\frac{\beta W}{r} - \frac{\tilde{u}}{t})\Omega_p + V\Omega_r - \frac{\tilde{u}}{t}\Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega, \Omega)$$

$$= \Psi(t, r, p, \tilde{u}, V, W, \Omega), \tag{4.63}$$

where

$$J(\Omega,\Omega)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\sin\theta_1\,d\theta_1\,d\epsilon\,d\tilde{u}_1\,dV_1\,dW_1. \tag{4.64}$$

Example 4.3.4. (The reduced Boltzmann equation corresponding to the subalgebra 2.2 : $\{\alpha X_4+X_7,\beta X_4+X_{11}\}$.)

Recall the source function and the representation of invariant solutions corresponding to subalgebra $\{\beta X_4+X_7\}$:

$$q=t^{-2}\Psi(p_1,p_2,\tilde{u},V,W,ft), \tag{4.65}$$

$$f=t^{-1}\Omega(p_1,p_2,\tilde{u},V,W), \tag{4.66}$$

where $p_1=\frac{r}{t},\ p_2=\frac{x}{t}-\alpha\theta-\beta\ln t,\ \tilde{u}=u-\alpha\theta-\beta\ln t.$

Then

$$f_t=t^{-2}(-\Omega-rt^{-1}\Omega_{p_1}+(-xt^{-1}-\beta)\Omega_{p_2}-\beta\Omega_{\tilde{u}}),\, f_x=t^{-2}\Omega_{p_2},$$

$$f_r=t^{-2}\Omega_{p_1},\, f_\theta=t^{-2}(-\alpha t\Omega_{p_2}-\alpha t\Omega_{\tilde{u}}),\, f_V=t^{-2}(t\Omega_V),\, f_W=t^{-2}(t\Omega_W).$$

Thus, the differential part of the Boltzmann equation (A.14) becomes

$$f_t+uf_x+Vf_r+\frac{W}{r}f_\theta+\frac{W^2}{r}f_V-\frac{WV}{r}f_W=t^{-2}(-\Omega+(V-p_1)\Omega_{p_1}$$

$$+(\tilde{u}-p_2-\beta-\frac{\alpha W}{p_1})\Omega_{p_2}-(\frac{\alpha W}{p_1}+\beta)\Omega_{\tilde{u}}+\frac{W^2}{p_1}\Omega_V-\frac{VW}{p_1}\Omega_W). \tag{4.67}$$

As $f=t^{-1}\Omega(\frac{r}{t},\frac{x}{t}-\alpha\theta-\beta\ln t,u-\alpha\theta-\beta\ln t,V,W)$, then

$$f_1=t^{-1}\Omega(\frac{r}{t},\frac{x}{t}-\alpha\theta-\beta\ln t,u_1-\alpha\theta-\beta\ln t,V_1,W_1),$$

$$f^*=t^{-1}\Omega(\frac{r}{t},\frac{x}{t}-\alpha\theta-\beta\ln t,\frac{1}{2}(u+u_1+g_cn_{1c})-\alpha\theta-\beta\ln t$$

$$,\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c}))$$

$$= t^{-1}\Omega(\frac{r}{t}, \frac{x}{t} - \alpha\theta - \beta\ln t, \frac{1}{2}((u - \alpha\theta - \beta\ln t) + (u_1 - \alpha\theta - \beta\ln t) + g_c n_{1c})$$

$$, \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$$

$$f_1^* = t^{-1}\Omega(\frac{r}{t}, \frac{x}{t} - \alpha\theta - \beta\ln t, \frac{1}{2}(u + u_1 - g_c n_{1c}) - \alpha\theta - \beta\ln t$$

$$, \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c}))$$

$$= t^{-1}\Omega(\frac{r}{t}, \frac{x}{t} - \alpha\theta - \beta\ln t, \frac{1}{2}((u - \alpha\theta - \beta\ln t) + (u_1 - \alpha\theta - \beta\ln t) - g_c n_{1c})$$

$$, \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})), \tag{4.68}$$

and the source function (4.65) becomes

$$q = t^{-2}\Psi(p_1, p_2, \tilde{u}, V, W, \Omega). \tag{4.69}$$

Substituting (4.68) into the collision integral of the Boltzmann equation (A.14), and integrating by change of variable: $\tilde{u}_1 = u_1 - \alpha\theta - \beta\ln t$, we obtain

$$J(f,f) = t^{-2}\int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\sin\theta_1\, d\theta_1\, d\epsilon\, d\tilde{u}_1\, dV_1\, dW_1, \tag{4.70}$$

where

$$\Omega = \Omega(p_1, p_2, \tilde{u}, V, W),\ \ \Omega_1 = \Omega(p_1, p_2, \tilde{u}_1, V_1, W_1),$$

$$\Omega^* = \Omega(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + g_c n_{1c}), \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$$

$$\Omega_1^* = \Omega(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - g_c n_{1c}), \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})),$$

$$g_c = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (V - V_1)^2 + (W - W_1)^2},$$

$$p_1 = \frac{r}{t},\ p_2 = \frac{x}{t} - \alpha\theta - \beta\ln t,\ \tilde{u} = u - \alpha\theta - \beta\ln t. \tag{4.71}$$

Substituting (4.67), (4.70), and (4.69) into the Boltzmann equation in the cylindrical coordinate system (A.14), and multiplying the obtained equation by t^2, the reduced Boltzmann equation is

$$-\Omega + (V - p_1)\Omega_{p_1} + (\tilde{u} - p_2 - \beta - \frac{\alpha W}{p_1})\Omega_{p_2} - (\frac{\alpha W}{p_1} + \beta)\Omega_{\tilde{u}} + \frac{W^2}{p_1}\Omega_V$$

$$- \frac{VW}{p_1}\Omega_W - J(\Omega, \Omega) = \Psi(p_1, p_2, \tilde{u}, V, W, \Omega), \qquad (4.72)$$

where

$$J(\Omega, \Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\sin\theta_1\, d\theta_1\, d\epsilon\, d\tilde{u}_1\, dV_1\, dW_1.$$

4.3.3 Illustrative example in the spherical coordinate system

Example 4.3.5. (The reduced Boltzmann equation corresponding to the subalgebra $3.8 : \{X_7, X_8, X_9\}$.)

Recall the source function and the representation of invariant solutions corresponding to subalgebra $\{X_7, X_8, X_9\}$:

$$q = \Psi(t, r, U, \tilde{V}, f), \qquad (4.73)$$

$$f = \Omega(t, r, U, \tilde{V}), \qquad (4.74)$$

where $\tilde{V} = \sqrt{V^2 + W^2}$.

Then

$$f_t = \Omega_t,\ f_r = \Omega_r,\ f_\varphi = 0,\ f_\theta = 0,\ f_U = \Omega_U,$$

$$f_V = \frac{V}{\sqrt{V^2 + W^2}}\Omega_{\tilde{V}},\ f_W = \frac{W}{\sqrt{V^2 + W^2}}\Omega_{\tilde{V}}. \qquad (4.75)$$

Thus, the differential part of the Boltzmann equation (A.29) becomes

$$f_t + Uf_r + \frac{W}{r\sin\theta}f_\varphi + \frac{V}{r}f_\theta + \frac{V^2 + W^2}{r}f_U + \frac{W^2\cot\theta - UV}{r}f_V$$

$$-\frac{W(U+V\cot\theta)}{r}f_W = \Omega_t + U\Omega_r + \frac{\tilde{V}^2}{r}\Omega_U - \frac{U\tilde{V}}{r}\Omega_{\tilde{V}}. \tag{4.76}$$

As $f = \Omega(t, r, U, \sqrt{V^2+W^2})$, then

$$f_1 = \Omega(t, r, U_1, \sqrt{V_1^2+W_1^2}),$$

$$f^* = \Omega(t, r, \frac{1}{2}(U+U_1+g_s n_{1s}), \sqrt{(\frac{1}{2}(V+V_1+g_s n_{2s}))^2 + (\frac{1}{2}(W+W_1+g_s n_{3s}))^2}),$$

$$f_1^* = \Omega(t, r, \frac{1}{2}(U+U_1-g_s n_{1s}),$$
$$\sqrt{(\frac{1}{2}(V+V_1-g_s n_{2s}))^2 + (\frac{1}{2}(W+W_1-g_s n_{3s}))^2}), \tag{4.77}$$

and the source function (4.73) becomes

$$q = \Psi(t, r, U, \tilde{V}, \Omega). \tag{4.78}$$

Substituting (4.77) into the collision integral of the Boltzmann equation (A.29), we obtain

$$J(f,f) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_s,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\sin\theta_1\, d\theta_1\, d\epsilon\, dU_1\, dV_1\, dW_1, \tag{4.79}$$

where

$$\Omega = \Omega(t, r, U, \sqrt{V^2+W^2}, \;\; \Omega_1 = \Omega(t, r, U_1, \sqrt{V_1^2+W_1^2}),$$

$$\Omega^* = \Omega(t, r, \frac{1}{2}(U+U_1+g_s n_{1s}), \frac{1}{2}\sqrt{(V+V_1+g_s n_{2s})^2 + (W+W_1+g_s n_{3s})^2},$$

$$\Omega_1^* = \Omega(t, r, \frac{1}{2}(U+U_1-g_s n_{1s}), \frac{1}{2}\sqrt{(V+V_1-g_s n_{2s})^2 + (W+W_1-g_s n_{3s})^2},$$

$$g_s = \sqrt{(U-U_1)^2 + (V-V_1)^2 + (W-W_1)^2}. \tag{4.80}$$

Substituting (4.76), (4.79), and (4.78) into the Boltzmann equation in the spherical coordinate system (A.29), the reduced Boltzmann equation is

$$\Omega_t + U\Omega_r + \frac{\tilde{V}^2}{r}\Omega_U - \frac{U\tilde{V}}{r}\Omega_{\tilde{V}} - J(\Omega,\Omega) = \Psi(t,r,U,\tilde{V},\Omega), \tag{4.81}$$

where

$$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_s,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\sin\theta_1\, d\theta_1\, d\epsilon\, dU_1\, dV_1\, dW_1. \tag{4.82}$$

Remark. Consider the term g_s in (4.80). It is difficult to write $(V-V_1)^2+(W-W_1)^2$ in the new variable $\tilde{V}=\sqrt{V^2+W^2}$ (or $\tilde{V}=V^2+W^2$).

CHAPTER V

CONCLUSION

This thesis is devoted to group classification of the full Boltzmann equation with a source term

$$f_t + uf_x + vf_y + wf_z - J(f,f) = q, \tag{5.1}$$

where $q = q(x, y, z, u, v, w, t, f)$. The main idea of the applied method is to use the Lie algebra L_{11} admitted by the equation without the source term.

The thesis is separated into three parts. The first part of the thesis is devoted to the general study of deriving determining equation of the Lie group admitted by nonlocal equation $\Phi(f) = q$ using the group properties of the homogeneous equation $\Phi(f) = 0$: a strategy for constructing the determining equation for the source function q is derived.

The developed strategy is applied to Equation (5.1) in the next part. The determining equation for the function $q(x, y, z, u, v, w, t, f)$ for each subalgebra of the optimal system of subalgebras of the Lie algebra L_{11} were studied. Obtaining the source function is illustrated by some examples. The complete results of the preliminary group classification are presented in Appendix C.

The third part of the thesis provides representations of invariant solutions of Equation (5.1). Complete results of the representations of invariant solutions are given in Appendix D. The reduced equations are also considered in this part. It should be noted that for some representations of invariant solutions the collision integral is difficult to write in new variables. A few illustrative examples of reduced equations are given. More results on the reduced Boltzmann equation are shown in Appendix E.

REFERENCES

REFERENCES

Akhatov, I. S., Gazizov, R. K., and Ibragimov, N. H. (1991). Nonlocal symmetries. Heuristic approach. **J. Math. Sci.** 55 (1).

Bihlo, A., Dos Santos Cardoso-Bihlo, E., and Popovych, R. O. (2012). Complete group classification of a class of nonlinear wave equations. **J. Math. Phys.** 53 (123515): 32.

Bobylev, A. V. (1975). On exact solutions of the Boltzmann equation. **Dokl. Akad. Nauk SSSR.** 225 (6): 1296–1299.

Bobylev, A. V. (1984). Exact solutions of the nonlinear Boltzmann equation and the theory of relaxation of a Maxwellian gas. **Teor. and matem. physika.** 60 (20): 280–310.

Bobylev, A. V. and Dorodnitsyn, V. A. (2009). Symmetries of evolution equations with non-local operators and applications to the Boltzmann equation. **Discrete Contin. Dyn. Syst.** 24 (1): 35–57.

Boffi, V. C. and Spiga, G. (1982a). Global solution to a nonlinear integral evolution problem in particle transport theory. **J. Math. Phys.** 23 (11): 2299–2303.

Boffi, V. C. and Spiga, G. (1982b). Nonlinear diffusion of test particles in the presence of an external conservative force. **Phys Fluids.** 25: 1987–1992.

Bunimovich, A. I. and Krasnoslobodtsev, A. V. (1982). Invariant–group solutions of kinetic equations. **Mechan. Jzydkosti i Gasa.** (4): 135–140.

Bunimovich, A. I. and Krasnoslobodtsev, A. V. (1983). On some invariant transformations of kinetic equations. **Vestnik Moscow State Univ., Ser. 1., Matemat. Mechan.** (4): 69–72.

Chirkunov, Y. A. (2012). Generalized equivalence transformations and group classification of systems of differential equations. **Journal of Applied Mechanics and Technical Physics.** 53 (2): 147–155.

Dorodnitsyn, V. A. (1982). On invariant solutions of a nonlinear heat equation with a source or a sink. **Zh. Vychisl. Mat. Mat. Fiz.** 22 (6): 1393–1400.

Dos Santos Cardoso-Bihlo, E., Bihlo, A., and Popovych, R. O. (2011). Enhanced preliminary group classification of a class of generalized diffusion equations. **Commun. Nonlinear Sci. Numer. Simul.** 16: 3622–3638.

Grigoriev, Y. N., Ibragimov, N. H., Kovalev, V. F., and Meleshko, S. V. (2010). **Symmetries of Integro-differential Equations and their Applications in Mechanics and Plasma Physics.** (Vol. 806). Springer, Berlin / Heidelberg.

Grigoriev, Y. N. and Meleshko, S. V. (1986). Investigation of invariant solutions of the Boltzmann kinetic equation and its models. **Preprint of Institute of Theoretical and Applied Mechanics of Akad.** Nauk SSSR. Sib. Otdelenie.

Grigoriev, Y. N. and Meleshko, S. V. (1987). Group analysis of the integro–differential Boltzman equation. **Dokl. Akad. Nauk SSSR.** 297 (2): 323–327.

Grigoriev, Y. N. and Meleshko, S. V. (2001). Group classification and representations of invariant solutions of the full Boltzmann equation. **AIP Conference Proceedings.** 585: 109–116.

Grigoriev, Y. N. and Meleshko, S. V. (2012). On group classification of the spatially homogeneous and isotropic Boltzmann equation with sources. **Int. J. of Non-Lin. Mech.** 47: 1014–1019.

Grigoriev, Y. N., Meleshko, S. V., and Sattayatham, P. (1999). Classification of invariant solutions of the Boltzmann equation. **J. of Physics A: Math. and General.** 32: 337–343.

Grigoriev, Y. N., Meleshko, S. V., and Suriyawichitseranee, A. (2014). On group classification of the spatially homogeneous and isotropic Boltzmann equation with sources II. **Int. J. Non-Lin. Mech.** 61: 15–18.

Grigoriev, Y. N., Meleshko, S. V., and Suriyawichitseranee, A. (2015). Application of group analysis to the spatially homogeneous and isotropic Boltzmann equation with source using its Fourier image. **J. Phys.: Conf. Ser.** 621 (012006).

Ibragimov, N. H., Torrisi, M., and Valenti, A. (1991). Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$. **J. Math. Phys.** 32: 2988–2995.

Kasatkin, A. A. (2012). Symmetry properties for systems of two ordinary fractional differential equations. **Ufimsk. Mat. Zh.** 4 (1): 71–81.

Kovalev, V. F., Krivenko, S. V., and Pustovalov, V. V. (1992). Group symmetry of the kinetic equations of the collisionless plasma. **JETP Lett.** 55(4): 256–259.

Krook, M. and Wu, T. T. (1976). Formation of Maxwellian tails. **Phys. Rev. Lett.** 36 (19): 1107–1109.

Lie, S. (1883). Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten. III. **Archiv for Matematik og Naturvidenskab.** 8 (4): 371–427. Reprinted in Lie's Gesammelte Abhandlungen, 1924, 5, paper XIY, pp. 362–427.

Lie, S. (1891). **Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen.** B.G.Teubner, Leipzig. Bearbeitet und herausgegeben von Dr. G.Scheffers.

Lin, F., Flood, A. E., and Meleshko, S. V. (2016). Exact solutions of population balance equation. **Commun. Nonlinear Sci. Numer. Simul.** 36: 378–390.

Long, F. S., Karnbanjong, A., Suriyawichitseranee, A., Grigoriev, Y. N., and Meleshko, S. V. (2017). Application of a Lie group admitted by a homogeneous equation for group classification of a corresponding inhomogeneous equation. **Commun. Nonlinear Sci. Numer. Simulat.** 48: 350–360.

Meleshko, S. V. (1988). Group properties of equations of motions of a viscoelastic medium. **Model. Mekh.** 2 (19): 114–126.

Meleshko, S. V. (1994). Homogeneous autonomous systems with three independent variables. **Prikl. Matem. Mech.** 58(5): 97–102. English transl. in **J. Appl. Maths. Mechs.** 1994, 58 (5).

Mkhize, T. G., Moyo, S., and Meleshko, S. V. (2015). Complete group classification of systems of two linear second-order ordinary differential equations. Algebraic approach. **Mathematical Methods in the Applied Sciences.** 38: 1824–1837.

Nonenmacher, T. F. (1984). Application of the similarity method to the nonlinear Boltzmann equation. **J. Appl. Math. Physics (ZAMP).** 35 (5): 680–691.

Olver, P. J. (1986). **Applications of Lie groups to differential equations.** Springer-Verlag, New York.

Ovsiannikov, L. V. (1978). **Group Analysis of Differential Equations.** Nauka, Moscow. English translation, Ames, W.F., Ed., published by Academic Press, New York, 1982.

Ovsiannikov, L. V. (1994). The “PODMODELI”program. Gas dynamics. **J. Appl. Maths. Mechs.** 58 (4): 601–627.

Ovsiannikov, L. V. (1999). Some results of the implementation of the “PODMODELI”program for the gas dynamics equations. **J. Appl. Maths Mechs.** 63 (3): 349–358.

Özer, T. (2003). Symmetry group classification of two-dimensional elastodynamics problem in nonlocal elasticity. **Int. J. Eng. Sci.** 41 (18): 2193–2211.

Popovych, R. O. and Bihlo, A. (2012). Symmetry preserving parameterization schemes. **J. Math. Phys.** 53 (073102): 36.

Popovych, R. O., Ivanova, N. M., and Eshraghi, H. (2004). Group classification of (1+1)-dimensional Schrödinger equations with potentials and power nonlinearities. **J. Math. Phys.** 45 (8): 3049–3057.

Popovych, R. O., Kunzinger, M., and Eshraghi, H. (2010). Admissible transformations and normalized classes of nonlinear Schrödinger equations. **Acta Appl. Math.** 109: 315–359.

Santos, A. and Brey, J. J. (1985). Comments on "a generalized BKW solution of the nonlinear Boltzmann equation with removal". **Physics of Fluids.** 29 (5): 1750.

Spiga, G. (1984). A generalized BKW solution of the nonlinear Boltzmann equation with removal. **Phys. Fluids.** 27: 2599–2600.

Suriyawichitseranee, A., Grigoriev, Y. N., and Meleshko, S. V. (2015). Group analysis of the Fourier transform of the spatially homogeneous and isotropic Boltzmann equation with a source term. **Comm. Nonlin. Sci. Num. Simul.** 20: 719–730.

Taranov, V. B. (1976). On symmetry of one–dimensional high frequency motion of noncollision plasma. **Journ. Techn. Physiki.** 46 (6): 1271–1277.

Zhou, L. Q. and Meleshko, S. V. (2015). Group analysis of integro-differential equations describing stress relaxation behavior of one-dimensional viscoelastic materials. **International Journal of Non-Linear Mechanics.** 77: 223–231.

APPENDICES

APPENDIX A

CHANGE OF VARIABLES TO CYLINDRICAL AND SPHERICAL COORDINATES WITH VELOCITIES

In this appendix, we provide change of variables to cylindrical and spherical coordinates with velocities. Solving some systems of partial differential equations requires such a change of variables. The differential terms in the Cartesian coordinates are changed to the cylindrical coordinates and in the spherical coordinates as well. The full Boltzmann equations in cylindrical and in spherical coordinates are also provided for reducing the equations. Recall the full Boltzmann equation and the admitted Lie group of the equation. The full Boltzmann equation is

$$f_t + uf_x + vf_y + wf_z = J(f, f), \tag{A.1}$$

with the collision term

$$J(f, f) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g, \theta_1)(f^* f_1^* - ff_1) \sin\theta_1 \, d\theta_1 \, d\epsilon \, du_1 \, dv_1 \, dw_1, \tag{A.2}$$

where

$$f = f(x, y, z, u, v, w, t), \qquad f_1 = f(x, y, z, u_1, v_1, w_1, t),$$

$$f^* = f(x, y, z, u^*, v^*, w^*, t), \qquad f_1^* = f(x, y, z, u_1^*, v_1^*, w_1^*, t),$$

and

$$u^* = \frac{1}{2}(u + u_1 + gn_1), \;\; v^* = \frac{1}{2}(v + v_1 + gn_2), \;\; w^* = \frac{1}{2}(w + w_1 + gn_3),$$

$$u_1^* = \frac{1}{2}(u + u_1 - gn_1),\ v_1^* = \frac{1}{2}(v + v_1 - gn_2),\ w_1^* = \frac{1}{2}(w + w_1 - gn_3),$$

$$\mathbf{g} = \mathbf{v} - \mathbf{w} = (u - u_1, v - v_1, w - w_1),\ g = \|\mathbf{g}\|_2,\ \mathbf{n} = (n_1, n_2, n_3), \|\mathbf{n}\|_2 = 1.$$

The admitted Lie algebra L_{11} of the Boltzmann equation (A.1) is

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = \partial_z,\ X_4 = t\partial_x + \partial_u,\ X_5 = t\partial_y + \partial_v,$$

$$X_6 = t\partial_z + \partial_w,\ X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v,\ X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,$$

$$X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u,\ X_{10} = \partial_t,\ X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z - f\partial_f. \quad \text{(A.3)}$$

A.1 Change of variables to cylindrical coordinates

Consider the following change of variables (Ovsiannikov, 1994)

$$z = r\sin\theta,\ y = r\cos\theta,\ v = V\cos\theta - W\sin\theta,\ w = V\sin\theta + W\cos\theta \quad \text{(A.4)}$$

where $\theta \in [0, 2\pi)$, $r > 0$. Then

$$r = \sqrt{y^2 + z^2},\ \theta = \arctan(\frac{z}{y}),\ V = v\cos\theta + w\sin\theta,\ W = -v\sin\theta + w\cos\theta. \quad \text{(A.5)}$$

Let q be a function in Cartesian coordinates with velocities and Q be a function in cylindrical coordinates with velocities relating with q as

$$q(x, y, z, u, v, w, t) = Q(x, r(y, z), \theta(y, z), u, V(v, w, \theta(y, z)),$$

$$W(v, w, \theta(y, z)), t, f). \quad \text{(A.6)}$$

Using the change of variables rule, some differential relations between the two systems are

$$q_x = Q_x,\ q_u = Q_u,\ q_t = Q_t,\ q_f = Q_f,\ yq_z - zq_y + vq_w - wq_v = Q_\theta,$$

$$yq_y + zq_z = rQ_r,\ vq_v + wq_w = VQ_V + WQ_W,\ wq_v - vq_w = WQ_V - VQ_W. \quad \text{(A.7)}$$

A.1.1 The corresponding basis generators

Let X_{ic} be a basis generator in cylindrical coordinates with velocities corresponding the basis generator X_i, $i = 1, 2, \ldots, 11$. Using the change of variable to cylindrical coordinates, we obtain

$$X_{1c} = \partial_x,\ X_{2c} = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta - \frac{W\sin\theta}{r}\partial_V + \frac{V\sin\theta}{r}\partial_W,$$

$$X_{3c} = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta + \frac{W\cos\theta}{r}\partial_V - \frac{V\cos\theta}{r}\partial_W,\ X_{4c} = t\partial_x + \partial_u,$$

$$X_{5c} = t\cos\theta\partial_r - \frac{t\sin\theta}{r}\partial_\theta + (\cos\theta - \frac{tW\sin\theta}{r})\partial_V + (\frac{tV\sin\theta}{r} - \sin\theta)\partial_W,$$

$$X_{6c} = t\sin\theta\partial_r + \frac{t\cos\theta}{r}\partial_\theta + (\sin\theta + \frac{tW\cos\theta}{r})\partial_V + (\cos\theta - \frac{tV\cos\theta}{r})\partial_W,\ X_{7c} = \partial_\theta,$$

$$X_{8c} = r\sin\theta\partial_x - x\sin\theta\partial_r - \frac{x\cos\theta}{r}\partial_\theta + (V\sin\theta + W\cos\theta)\partial_u$$

$$+ (-u\sin\theta - \frac{xW\cos\theta}{r})\partial_V + (\frac{xV\cos\theta}{r} - u\cos\theta)\partial_W,$$

$$X_{9c} = -r\cos\theta\partial_x + x\cos\theta\partial_r - \frac{x\sin\theta}{r}\partial_\theta + (-V\cos\theta + W\sin\theta)\partial_u$$

$$+ (u\cos\theta - \frac{xW\sin\theta}{r})\partial_V - u\sin\theta\partial_W,$$

$$X_{10c} = \partial_t,\ X_{11c} = t\partial_t + x\partial_x + r\partial_r - f\partial_f. \quad \text{(A.8)}$$

A.1.2 The full Boltzmann equation in cylindrical coordinates

Let f be a function in Cartesian coordinates with velocities and F be a function in cylindrical coordinates with velocities relating with f as

$$f(x, y, z, u, v, w, t) = F(x, r(y, z), \theta(y, z), u, V(v, w, \theta(y, z)), W(v, w, \theta(y, z)), t). \quad \text{(A.9)}$$

Then

$$f_t = F_t,\ f_x = F_x,$$

$$f_y = F_r r_y + F_\theta \theta_y + F_V(-v\sin\theta + w\cos\theta)\theta_y + F_W(-v\cos\theta - w\sin\theta)\theta_y,$$

$$f_z = F_r r_z + F_\theta \theta_z + F_V(-v\sin\theta + w\cos\theta)\theta_z + F_W(-v\cos\theta - w\sin\theta)\theta_z.$$

It follows that

$$f_t + uf_x + vf_y + wf_z = F_t + uF_x + VF_r + \frac{W}{r}F_\theta + \frac{W^2}{r}F_V - \frac{WV}{r}F_W. \qquad \text{(A.10)}$$

According to the change of variables (A.5), we get

$$V_1 = v_1\cos\theta + w_1\sin\theta,\ W_1 = -v_1\sin\theta + w_1\cos\theta,$$

$$u^* = \frac{1}{2}(u + u_1 + gn_1) = \frac{1}{2}(u + u_1 + g_c n_{1c}),$$

$$V^* = v^*\cos\theta + w^*\sin\theta = \frac{1}{2}(v + v_1 + gn_2)\cos\theta + \frac{1}{2}(w + w_1 + gn_3)\sin\theta$$

$$= \frac{1}{2}(v\cos\theta + w\sin\theta + v_1\cos\theta + w_1\sin\theta + g(n_2\cos\theta + n_3\sin\theta))$$

$$= \frac{1}{2}(V + V_1 + g_c n_{2c}),$$

$$W^* = -v^*\sin\theta + w^*\cos\theta = -\frac{1}{2}(v + v_1 + gn_2)\sin\theta + \frac{1}{2}(w + w_1 + gn_3)\cos\theta$$

$$= \frac{1}{2}(-v\sin\theta + w\cos\theta - v_1\sin\theta + w_1\cos\theta + g(-n_2\sin\theta + n_3\cos\theta))$$

$$= \frac{1}{2}(W + W_1 + g_c n_{3c}), \qquad \text{(A.11)}$$

where $n_{1c} = n_1,\ n_{2c} = n_2\cos\theta + n_3\sin\theta,\ n_{3c} = -n_2\sin\theta + n_3\cos\theta$. By the same way for finding u^*, V^*, W^*, we obtain

$$u_1^* = \frac{1}{2}(u + u_1 - g_c n_{1c}),$$

$$V_1^* = \frac{1}{2}(V + V_1 - g_c n_{2c}),$$

$$W_1^* = \frac{1}{2}(W + W_1 - g_c n_{3c}). \tag{A.12}$$

The variables g_c and $\mathbf{g}_c \cdot \mathbf{n}_c$ are as follows.

$$\begin{aligned}
g_c &:= ((u-u_1)^2 + (V-V_1)^2 + (W-W_1)^2)^{1/2} \\
&= ((u-u_1)^2 + ((v\cos\theta + w\sin\theta) - (v_1\cos\theta + w_1\sin\theta))^2 + ((-v\sin\theta + w\cos\theta) \\
&\quad - (-v_1\sin\theta + w_1\cos\theta))^2)^{1/2} \\
&= ((u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2)^{1/2} = g, \\
\mathbf{g}_c \cdot \mathbf{n}_c &:= (u-u_1)n_{1c} + (V-V_1)n_{2c} + (W-W_1)n_{3c} \\
&= (u-u_1)n_1 + ((v\cos\theta + w\sin\theta) - (v_1\cos\theta + w_1\sin\theta))(n_2\cos\theta + n_3\sin\theta) \\
&\quad + ((-v\sin\theta + w\cos\theta) - (-v_1\sin\theta + w_1\cos\theta))(-n_2\sin\theta + n_3\cos\theta) \\
&= (u-u_1)n_1 + (v-v_1)n_2 + (w-w_1)n_3 = \mathbf{g}\cdot\mathbf{n}.
\end{aligned} \tag{A.13}$$

The Jacobian for integration by change of variables is $|\frac{\partial(u_1,v_1,w_1)}{\partial(u_1,V_1,W_1)}| = 1$. Therefore, the full Boltzmann equation in the cylindrical coordinate system with velocities is

$$F_t + uF_x + VF_r + \frac{W}{r}F_\theta + \frac{W^2}{r}F_V - \frac{WV}{r}F_W = J(F,F), \tag{A.14}$$

with the collision integral

$$J(F,F) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(F^*F_1^* - FF_1)\sin\theta_1\, d\theta_1\, d\epsilon\, du_1\, dV_1\, dW_1, \tag{A.15}$$

where

$$F = F(x,r,\theta,u,V,W,t),\ \ F_1 = F(x,r,\theta,u_1,V_1,W_1,t),$$

$$F^* = F(x, r, \theta, u^*, V^*, W^*, t), \;\; F_1^* = F(x, r, \theta, u_1^*, V_1^*, W_1^*, t),$$

and

$$u^* = \frac{1}{2}(u + u_1 + g_c n_{1c}), \;\; V^* = \frac{1}{2}(V + V_1 + g_c n_{2c}), \;\; W^* = \frac{1}{2}(W + W_1 + g_c n_{3c}),$$

$$u_1^* = \frac{1}{2}(u + u_1 - g_c n_{1c}), \;\; V_1^* = \frac{1}{2}(V + V_1 - g_c n_{2c}), \;\; W_1^* = \frac{1}{2}(W + W_1 - g_c n_{3c}),$$

$$\mathbf{g}_c = ((u - u_1), (V - V_1), (W - W_1)), \;\; g_c = \|\mathbf{g}_c\|_2,$$

$$\mathbf{n}_c = (n_{1c}, n_{2c}, n_{3c}) = (\cos\theta_1, \cos(\epsilon - \theta)\sin\theta_1, \sin(\epsilon - \theta)\sin\theta_1).$$

Remark. $g_c = g$ and $\mathbf{g}_c \cdot \mathbf{n}_c = \mathbf{g} \cdot \mathbf{n}$.

Remark. For the cylindrical coordinates with velocities, the unit vector $\mathbf{n}$ in the Cartesian coordinates is defined by

$$\mathbf{n} = (n_1, n_2, n_3) := (\cos\theta_1, \sin\theta_1\cos\epsilon, \sin\theta_1\sin\epsilon), \tag{A.16}$$

as shown in Figure A.1. It follows that we can use the trigonometric identities:

$$\begin{aligned}
n_{1c} &= n_1 = \cos\theta_1, \\
n_{2c} &= n_2\cos\theta + n_3\sin\theta = \sin\theta_1\cos\epsilon\cos\theta + \sin\theta_1\sin\epsilon\sin\theta \\
&= \cos(\epsilon - \theta)\sin\theta_1, \\
n_{3c} &= -n_2\sin\theta + n_3\cos\theta = -\sin\theta_1\cos\epsilon\sin\theta + \sin\theta_1\sin\epsilon\cos\theta \\
&= \sin(\epsilon - \theta)\sin\theta_1.
\end{aligned} \tag{A.17}$$

so that we can integrate in the collision integral by change of variable: $\tilde{\epsilon} = \epsilon - \theta$.

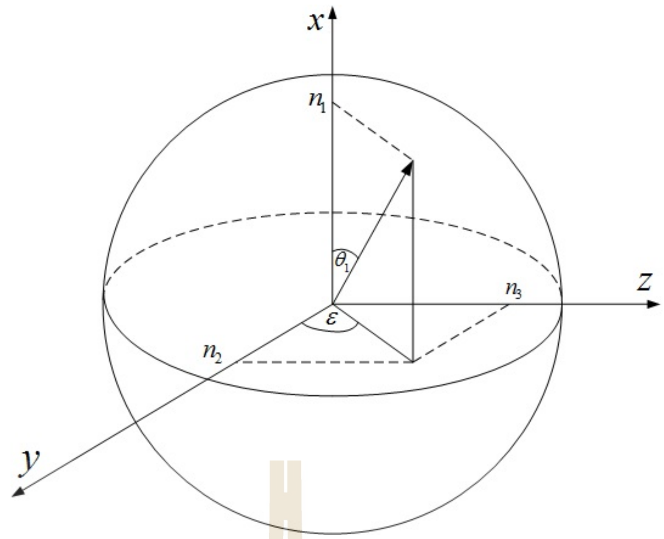

Figure A.1: The unit vector in Cartesian coordinates for integration in cylindrical coordinates.

A.2 Change of variables to spherical coordinates

Consider the following change of variables (Ovsiannikov, 1994)

$$x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta, u = U\sin\theta\cos\varphi + V\cos\theta\cos\varphi - W\sin\varphi,$$

$$v = U\sin\theta\sin\varphi + V\cos\theta\sin\varphi + W\cos\varphi, w = U\cos\theta - V\sin\theta, \tag{A.18}$$

where $\varphi \in [0,\pi)$, $\theta \in [0, 2\pi)$, $r > 0$. Then

$$r = \sqrt{x^2+y^2+z^2},\ \varphi = \arctan(\frac{y}{x}),\ \theta = \arctan(\frac{\sqrt{x^2+y^2}}{z}),$$

$$U = u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta,\ V = u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta,$$

$$W = -u\sin\varphi + v\cos\varphi. \tag{A.19}$$

Let q be a function in Cartesian coordinates with velocities and Q be a function in spherical coordinates with velocities relating with q as

$$q(x,y,z,u,v,w,t) = Q(r(x,y,z), \varphi(x,y), \theta(x,y,z), U(u,v,w,\varphi(x,y),\theta(x,y,z))$$

$$, V(u,v,w,\varphi(x,y),\theta(x,y,z)), W(u,v,\varphi(x,y),t,f). \tag{A.20}$$

Using change rule, some differential relations between the two systems are

$$q_t = Q_t,\ q_f = Q_f,\ xq_x + yq_y + zq_z = rQ_r,\ xq_y - yq_x + uq_v - vq_u = Q_\varphi,$$

$$yq_z - zq_y + vq_w - wq_v = -\sin(\varphi)Q_\theta - \cos(\varphi)\cot(\theta)Q_\varphi - \frac{\cos(\varphi)}{\sin(\theta)}(WQ_V - VQ_W)$$

$$zq_x - xq_z + wq_u - uq_w = \cos(\varphi)Q_\theta - \sin(\varphi)\cot(\theta)Q_\varphi - \frac{\sin(\varphi)}{\sin(\theta)}(WQ_V - VQ_W). \tag{A.21}$$

A.2.1 The corresponding basis generators

Let X_{is} be a basis generator in spherical coordinates with velocities corresponding the basis generator X_i, $i = 1, 2, \ldots, 11$. Using the change of variable, we obtain the generators in spherical coordinates. The generators which are used in this thesis are

$$X_{7s} = -\sin(\varphi)\partial_\theta - \cos(\varphi)\cot(\theta)\partial_\varphi - \frac{\cos(\varphi)}{\sin(\theta)}(W\partial_V - V\partial_W),$$

$$X_{8s} = \cos(\varphi)\partial_\theta - \sin(\varphi)\cot(\theta)\partial_\varphi - \frac{\sin(\varphi)}{\sin(\theta)}(W\partial_V - V\partial_W),$$

$$X_{9s} = \partial_\varphi,\ X_{10s} = \partial_t,\ X_{11s} = t\partial_t + r\partial_r - f\partial_f. \tag{A.22}$$

A.2.2 The full Boltzmann equation in spherical coordinates

Let f be a function in Cartesian coordinates with velocities and F be a function in spherical coordinates with velocities relating with f as

$$f(x,y,z,u,v,w,t) = F(r(x,y,z),\varphi(x,y),\theta(x,y,z),U(u,v,w,\varphi(x,y),\theta(x,y,z)),V(u,v,w,\varphi(x,y),\theta(x,y,z)),W(u,v,\varphi(x,y)),t). \tag{A.23}$$

Then

$$f_t = F_t,\ \ f_z = F_r r_z + F_\theta \theta_z + F_U U_z + F_V V_z$$

$$f_x = F_r r_x + F_\varphi \varphi_x + F_\theta \theta_x + F_U U_x + F_V V_x + F_W W_x,$$

$$f_y = F_r r_y + F_\varphi \varphi_y + F_\theta \theta_y + F_U U_y + F_V V_y + F_W W_y.$$

It follows that

$$f_t + uf_x + vf_y + wf_z = F_t + UF_r + \frac{W}{r\sin\theta}F_\varphi + \frac{V}{r}F_\theta + \frac{V^2+W^2}{r}F_U$$
$$+ \frac{W^2\cot\theta - UV}{r}F_V - \frac{W(U+V\cot\theta)}{r}F_W. \qquad \text{(A.24)}$$

According to the change of variables (A.19), we get

$$U_1 = u_1\sin\theta\cos\varphi + v_1\sin\theta\sin\varphi + w_1\cos\theta,$$

$$V_1 = u_1\cos\theta\cos\varphi + v_1\cos\theta\sin\varphi - w_1\sin\theta,$$

$$W_1 = -u_1\sin\varphi + v_1\cos\varphi,$$

$$U^* = u^*\sin\theta\cos\varphi + v^*\sin\theta\sin\varphi + w^*\cos\theta$$

$$= \frac{1}{2}(u+u_1+gn_1)\sin\theta\cos\varphi + \frac{1}{2}(v+v_1+gn_2)\sin\theta\sin\varphi + \frac{1}{2}(w+w_1+gn_3)\cos\theta$$

$$= \frac{1}{2}(u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta + u_1\sin\theta\cos\varphi + v_1\sin\theta\sin\varphi + w_1\cos\theta$$

$$+ g(n_1\sin\theta\cos\varphi + n_2\sin\theta\sin\varphi + n_3\cos\theta))$$

$$= \frac{1}{2}(U + U_1 + g_s n_{1s}),$$

$$V^* = u^*\cos\theta\cos\varphi + v^*\cos\theta\sin\varphi - w^*\sin\theta$$

$$= \frac{1}{2}(u+u_1+gn_1)\cos\theta\cos\varphi + \frac{1}{2}(v+v_1+gn_2)\cos\theta\sin\varphi - \frac{1}{2}(w+w_1+gn_3)\sin\theta$$

$$= \frac{1}{2}(u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta + u_1\cos\theta\cos\varphi + v_1\cos\theta\sin\varphi - w_1\sin\theta$$

$$+ g(n_1\cos\theta\cos\varphi + n_2\cos\theta\sin\varphi - n_3\sin\theta))$$

$$= \frac{1}{2}(V + V_1 + g_s n_{2s}),$$

$$W^* = -u^*\sin\varphi + v^*\cos\varphi$$

$$= -\frac{1}{2}(u + u_1 + gn_1)\sin\varphi + \frac{1}{2}(v + v_1 + gn_2)\cos\varphi$$

$$= \frac{1}{2}(-u\sin\varphi + v\cos\varphi - u_1\sin\varphi + v_1\cos\varphi + g(-n_1\sin\varphi + n_2\cos\varphi))$$

$$= \frac{1}{2}(W + W_1 + g_s n_{3s}), \tag{A.25}$$

where

$$n_{1s} = n_1\sin\theta\cos\varphi + n_2\sin\theta\sin\varphi + n_3\cos\theta,$$

$$n_{2s} = n_1\cos\theta\cos\varphi + n_2\cos\theta\sin\varphi - n_3\sin\theta,$$

$$n_{3s} = -n_1\sin\varphi + n_2\cos\varphi. \tag{A.26}$$

By the same way for finding U^*, V^*, W^*, we obtain

$$U_1^* = \frac{1}{2}(U + U_1 - g_s n_{1s}),$$

$$V_1^* = \frac{1}{2}(V + V_1 - g_s n_{2s}),$$

$$W_1^* = \frac{1}{2}(W + W_1 - g_s n_{3s}). \tag{A.27}$$

The variables g_s and $\mathbf{g}_s \cdot \mathbf{n}_s$ are as follows.

$$g_s := ((U - U_1)^2 + (V - V_1)^2 + (W - W_1)^2)^{1/2}$$

$$= (((u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta) - (u_1\sin\theta\cos\varphi + v_1\sin\theta\sin\varphi$$

$$+ w_1\cos\theta))^2 + ((u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta) - (u_1\cos\theta\cos\varphi$$

$$+ v_1\cos\theta\sin\varphi - w_1\sin\theta))^2 + ((-u\sin\varphi + v\cos\varphi) - (-u_1\sin\varphi + v_1\cos\varphi))^2)^{1/2}$$

$$= ((u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2)^{1/2} = g,$$

$$\mathbf{g}_s \cdot \mathbf{n}_s := (U-U_1)n_{1s} + (V-V_1)n_{2s} + (W-W_1)n_{3s}$$

$$= ((u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta) - (u_1\sin\theta\cos\varphi + v_1\sin\theta\sin\varphi$$

$$+ w_1\cos\theta))(n_1\sin\theta\cos\varphi + n_2\sin\theta\sin\varphi + n_3\cos\theta)$$

$$+ ((u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta) - (u_1\cos\theta\cos\varphi + v_1\cos\theta\sin\varphi$$

$$- w_1\sin\theta))(n_1\cos\theta\cos\varphi + n_2\cos\theta\sin\varphi - n_3\sin\theta)$$

$$+ ((-u\sin\varphi + v\cos\varphi) - (-u_1\sin\varphi + v_1\cos\varphi))(-n_1\sin\varphi + n_2\cos\varphi)$$

$$= (u-u_1)n_1 + (v-v_1)n_2 + (w-w_1)n_3 = \mathbf{g}\cdot\mathbf{n}. \tag{A.28}$$

The Jacobian for integration by change of variables is $|\frac{\partial(u_1,v_1,w_1)}{\partial(U_1,V_1,W_1)}| = 1$. Therefore, the full Boltzmann equation in spherical coordinates is

$$F_t + UF_r + \frac{W}{r\sin\theta}F_\varphi + \frac{V}{r}F_\theta + \frac{V^2+W^2}{r}F_U + \frac{W^2\cot\theta - UV}{r}F_V$$

$$- \frac{W(U+V\cot\theta)}{r}F_W = J(F,F), \tag{A.29}$$

with the collision term

$$J(F,F) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_s,\theta_1)(F^*F_1^* - FF_1)\sin\theta_1\,d\theta_1\,d\epsilon\,dU_1\,dV_1\,dW_1, \tag{A.30}$$

where

$$F = F(r, \varphi, \theta, U, V, W, t),\ \ F_1 = F(r, \varphi, \theta, U_1, V_1, W_1, t),$$

$$F^* = F(r, \varphi, \theta, U^*, V^*, W^*, t),\ \ F_1^* = F(r, \varphi, \theta, U_1^*, V_1^*, W_1^*, t),$$

and

$$U^* = \frac{1}{2}(U + U_1 + g_s n_{1s}),\ \ V^* = \frac{1}{2}(V + V_1 + g_s n_{2s}),\ \ W^* = \frac{1}{2}(W + W_1 + g_s n_{3s}),$$

$$U_1^* = \frac{1}{2}(U + U_1 - g_s n_{1s}),\ \ V_1^* = \frac{1}{2}(V + V_1 - g_s n_{2s}),\ \ W_1^* = \frac{1}{2}(W + W_1 - g_s n_{3s}),$$

$$\mathbf{g}_s = ((U - U_1), (V - V_1), (W - W_1)),\ \ g_s = \|\mathbf{g}_s\|_2,\ \ \mathbf{n}_s = (n_{1s}, n_{2s}, n_{3s}),$$

$$n_{1s} = \cos(\epsilon - \varphi)\sin\theta\sin\theta_1 + \cos\theta_1\cos\theta,$$

$$n_{2s} = \cos(\epsilon - \varphi)\cos\theta\sin\theta_1 - \cos\theta_1\sin\theta,\ \ n_{3s} = \sin(\epsilon - \varphi)\sin\theta_1.$$

Remark. $g_s = g$ and $\mathbf{g}_s \cdot \mathbf{n}_s = \mathbf{g} \cdot \mathbf{n}$.

Remark. For the spherical coordinates with velocities, the unit vector $\mathbf{n}$ in the Cartesian coordinates is defined by

$$\mathbf{n} = (n_1, n_2, n_3) := (\sin\theta_1\cos\epsilon, \sin\theta_1\sin\epsilon, \cos\theta_1), \tag{A.31}$$

as shown in Figure A.2. It follows that we can use the trigonometric identities:

$$\begin{aligned} n_{1s} &= n_1\sin\theta\cos\varphi + n_2\sin\theta\sin\varphi + n_3\cos\theta \\ &= \sin\theta_1\cos\epsilon\sin\theta\cos\varphi + \sin\theta_1\sin\epsilon\sin\theta\sin\varphi + \cos\theta_1\cos\theta \\ &= \cos(\epsilon - \varphi)\sin\theta\sin\theta_1 + \cos\theta_1\cos\theta, \end{aligned}$$

$$n_{2s} = n_1\cos\theta\cos\varphi + n_2\cos\theta\sin\varphi - n_3\sin\theta$$

$$= \sin\theta_1 \cos\epsilon \cos\theta \cos\varphi + \sin\theta_1 \sin\epsilon \cos\theta \sin\varphi - \cos\theta_1 \sin\theta$$

$$= \cos(\epsilon - \varphi) \cos\theta \sin\theta_1 - \cos\theta_1 \sin\theta,$$

$$n_{3s} = -n_1 \sin\varphi + n_2 \cos\varphi = -\sin\theta_1 \cos\epsilon \sin\varphi + \sin\theta_1 \sin\epsilon \cos\varphi$$

$$= \sin(\epsilon - \varphi) \sin\theta_1, \tag{A.32}$$

such that we can integrate in the collision integral by change of variable: $\tilde{\epsilon} = \epsilon - \varphi$.

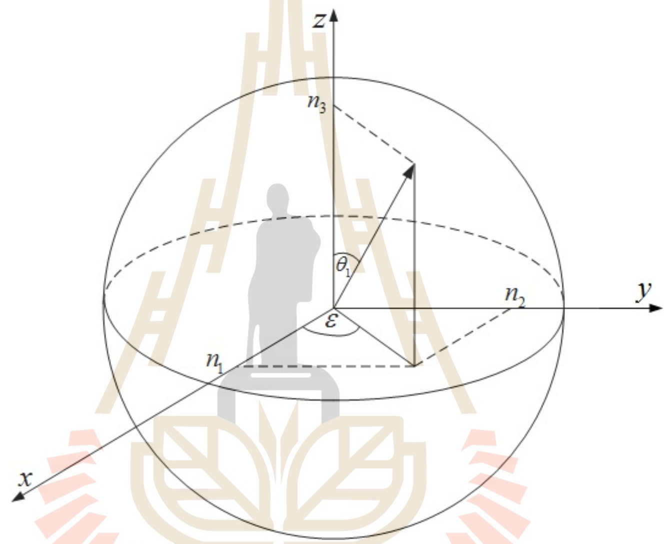

Figure A.2: The unit vector in Cartesian coordinates for integration in spherical coordinates.

APPENDIX B

DERIVATION OF THE REMAINING FUNCTIONS h_{X_j}

B.1 Derivation of the function h_{X_2}

The group of transformations corresponding to the generator X_2 is

$$\bar{x} = x,\ \bar{y} = y + a,\ \bar{z} = z,\ \bar{u} = u,\ \bar{v} = v,\ \bar{w} = w,\ \bar{t} = t,\ \bar{f} = f. \tag{B.1}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x}, \bar{y} - a, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}). \tag{B.2}$$

It follows that

$$\begin{aligned}
\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0t}(\bar{x}, \bar{y} - a, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}),\\
\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0x}(\bar{x}, \bar{y} - a, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}),\\
\bar{f}_{\bar{y}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0y}(\bar{x}, \bar{y} - a, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}),\\
\bar{f}_{\bar{z}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0z}(\bar{x}, \bar{y} - a, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}),
\end{aligned} \tag{B.3}$$

and

$$\bar{f}^* = f_0(\bar{x}, \bar{y} - a, \bar{z}, \frac{1}{2}(\bar{u} + u_1 + \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3), \bar{t}),$$

$$\bar{f}_1^* = f_0(\bar{x}, \bar{y} - a, \bar{z}, \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3), \bar{t}),$$

$$\bar{f}_1 = f_0(\bar{x}, \bar{y} - a, \bar{z}, u_1, v_1, w_1, \bar{t}), \tag{B.4}$$

Using (B.1), one finds that

$$\bar{g} = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2}. \tag{B.5}$$

Then

$$\begin{aligned}(\Phi(\bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= (f_{0t} + u f_{0x} + v f_{0y} + w f_{0z} \\ &\quad - J(f_0, f_0))(\bar{x}, \bar{y} - a, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) \\ &= (\Phi(f_0))(\bar{x}, \bar{y} - a, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}).\end{aligned} \tag{B.6}$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_2(\Phi) = -D_y(\Phi). \tag{B.7}$$

As $X_2 = \partial_y$ then

$$h_{X_2}\Phi = \tilde{X}_2(\Phi) + D_y(\Phi) = -D_y(\Phi) + D_y(\Phi) = 0. \tag{B.8}$$

This means that

$$\tilde{h}_{X_2} = 0. \tag{B.9}$$

B.2 Derivation of the function h_{X_3}

The group of transformations corresponding to the generator X_3 is

$$\bar{x} = x,\ \bar{y} = y,\ \bar{z} = z + a,\ \bar{u} = u,\ \bar{v} = v,\ \bar{w} = w,\ \bar{t} = t,\ \bar{f} = f. \tag{B.10}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x}, \bar{y}, \bar{z} - a, \bar{u}, \bar{v}, \bar{w}, \bar{t}). \tag{B.11}$$

It follows that

$$\begin{aligned}
\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0t}(\bar{x}, \bar{y}, \bar{z} - a, \bar{u}, \bar{v}, \bar{w}, \bar{t}), \\
\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0x}(\bar{x}, \bar{y}, \bar{z} - a, \bar{u}, \bar{v}, \bar{w}, \bar{t}), \\
\bar{f}_{\bar{y}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0y}(\bar{x}, \bar{y}, \bar{z} - a, \bar{u}, \bar{v}, \bar{w}, \bar{t}), \\
\bar{f}_{\bar{z}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0z}(\bar{x}, \bar{y}, \bar{z} - a, \bar{u}, \bar{v}, \bar{w}, \bar{t}),
\end{aligned} \tag{B.12}$$

and

$$\begin{aligned}
\bar{f}^* &= f_0(\bar{x}, \bar{y}, \bar{z} - a, \frac{1}{2}(\bar{u} + u_1 + \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3), \bar{t}), \\
\bar{f}_1^* &= f_0(\bar{x}, \bar{y}, \bar{z} - a, \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3), \bar{t}), \\
\bar{f}_1 &= f_0(\bar{x}, \bar{y}, \bar{z} - a, u_1, v_1, w_1, \bar{t}).
\end{aligned} \tag{B.13}$$

Using (B.10), one finds that

$$\bar{g} = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2}. \tag{B.14}$$

Substituting (B.11), (B.13), (B.14) into (3.24), the collision term becomes

$$(J(\bar{f}, \bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (J(f_0, f_0))(\bar{x}, \bar{y}, \bar{z} - a, \bar{u}, \bar{v}, \bar{w}, \bar{t}). \tag{B.15}$$

Then

$$(\Phi(\bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (f_{0t} + u f_{0x} + v f_{0y} + w f_{0z}$$

$$- J(f_0, f_0))(\bar{x}, \bar{y}, \bar{z} - a, \bar{u}, \bar{v}, \bar{w}, \bar{t})$$

$$= (\Phi(f_0))(\bar{x}, \bar{y}, \bar{z} - a, \bar{u}, \bar{v}, \bar{w}, \bar{t}). \tag{B.16}$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_3(\Phi) = -D_z(\Phi). \tag{B.17}$$

As $X_3 = \partial_z$ then

$$h_{X_3}\Phi = \tilde{X}_3(\Phi) + D_z(\Phi) = -D_z(\Phi) + D_z(\Phi) = 0. \tag{B.18}$$

This means that

$$\tilde{h}_{X_3} = 0. \tag{B.19}$$

B.3 Derivation of the function h_{X_5}

The group of transformations corresponding to the generator X_5 is

$$\bar{x} = x,\ \bar{y} = y + at,\ \bar{z} = z,\ \bar{u} = u,\ \bar{v} = v + a,\ \bar{w} = w,\ \bar{t} = t,\ \bar{f} = f. \tag{B.20}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x}, \bar{y} - a\bar{t}, \bar{z}, \bar{u}, \bar{v} - a, \bar{w}, \bar{t}). \tag{B.21}$$

It follows that

$$\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (-af_{0y} + f_{0t})(\bar{x}, \bar{y} - a\bar{t}, \bar{z}, \bar{u}, \bar{v} - a, \bar{w}, \bar{t}),$$

$$\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0x}(\bar{x}, \bar{y} - a\bar{t}, \bar{z}, \bar{u}, \bar{v} - a, \bar{w}, \bar{t}),$$

$$\bar{f}_{\bar{y}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0y}(\bar{x}, \bar{y} - a\bar{t}, \bar{z}, \bar{u}, \bar{v} - a, \bar{w}, \bar{t}),$$

$$\bar{f}_{\bar{z}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = f_{0z}(\bar{x},\bar{y}-a\bar{t},\bar{z},\bar{u},\bar{v}-a,\bar{w},\bar{t}), \tag{B.22}$$

and

$$\bar{f}^* = f_0(\bar{x},\bar{y}-a\bar{t},\bar{z},\frac{1}{2}(\bar{u}+u_1+\bar{g}n_1),\frac{1}{2}(\bar{v}+v_1+\bar{g}n_2)-a,\frac{1}{2}(\bar{w}+w_1+\bar{g}n_3),\bar{t})$$

$$= f_0(\bar{x},\bar{y}-a\bar{t},\bar{z},\frac{1}{2}(\bar{u}+u_1+\bar{g}n_1),\frac{1}{2}((\bar{v}-a)+(v_1-a)+\bar{g}n_2),$$

$$\frac{1}{2}(\bar{w}+w_1+\bar{g}n_3),\bar{t}),$$

$$\bar{f}_1^* = f_0(\bar{x},\bar{y}-a\bar{t},\bar{z},\frac{1}{2}(\bar{u}+u_1-\bar{g}n_1),\frac{1}{2}(\bar{v}+v_1-\bar{g}n_2)-a,\frac{1}{2}(\bar{w}+w_1-\bar{g}n_3),\bar{t})$$

$$= f_0(\bar{x},\bar{y}-a\bar{t},\bar{z},\frac{1}{2}(\bar{u}+u_1-\bar{g}n_1),\frac{1}{2}((\bar{v}-a)+(v_1-a)-\bar{g}n_2),$$

$$\frac{1}{2}(\bar{w}+w_1-\bar{g}n_3),\bar{t}),$$

$$\bar{f}_1 = f_0(\bar{x},\bar{y}-a\bar{t},\bar{z},u_1,v_1-a,w_1,\bar{t}). \tag{B.23}$$

Using (B.20), one finds that

$$\bar{g} = \sqrt{(u-u_1)^2+(v+a-v_1)^2+(w-w_1)^2}$$

$$= \sqrt{(u-u_1)^2+(\bar{v}-(v_1-a))^2+(\bar{w}-w_1)^2}. \tag{B.24}$$

Substituting (B.21), (B.23), (B.24) into (3.24) and using the change of variables: $\tilde{v}_1 = v_1 - a$, the collision term becomes

$$(J(\bar{f},\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (J(f_0,f_0))(\bar{x},\bar{y}-a\bar{t},\bar{z},\bar{u},\bar{v}-a,\bar{w},\bar{t}). \tag{B.25}$$

Then

$$(\Phi(\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (-af_{0y}+f_{0t}+uf_{0x}+(v+a)f_{0y}+wf_{0z}$$

$$-J(f_0,f_0))(\bar{x},\bar{y}-a\bar{t},\bar{z},\bar{u},\bar{v}-a,\bar{w},\bar{t})$$

$$= (f_{0t} + uf_{0x} + vf_{0y} + wf_{0z} - J(f_0, f_0))(\bar{x}, \bar{y} - a\bar{t}, \bar{z}, \bar{u}, \bar{v} - a, \bar{w}, \bar{t})$$

$$= (\Phi(f_0))(\bar{x}, \bar{y} - a\bar{t}, \bar{z}, \bar{u}, \bar{v} - a, \bar{w}, \bar{t}). \quad \text{(B.26)}$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_5(\Phi) = -(tD_y(\Phi) + D_v(\Phi)). \quad \text{(B.27)}$$

As $X_5 = t\partial_y + \partial_v$ then

$$h_{X_5}\Phi = \tilde{X}_5(\Phi) + (tD_y(\Phi) + D_v(\Phi))$$

$$= -(tD_y(\Phi) + D_v(\Phi)) + (tD_y(\Phi) + D_v(\Phi)) = 0. \quad \text{(B.28)}$$

This means that

$$\tilde{h}_{X_5} = 0. \quad \text{(B.29)}$$

B.4 Derivation of the function h_{X_6}

The group of transformations corresponding to the generator X_6 is

$$\bar{x} = x,\ \bar{y} = y,\ \bar{z} = z + at,\ \bar{u} = u,\ \bar{v} = v,\ \bar{w} = w + a,\ \bar{t} = t,\ \bar{f} = f. \quad \text{(B.30)}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x}, \bar{y}, \bar{z} - a\bar{t}, \bar{u}, \bar{v}, \bar{w} - a, \bar{t}). \quad \text{(B.31)}$$

It follows that

$$\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (-af_{0z} + f_{0t})(\bar{x}, \bar{y}, \bar{z} - a\bar{t}, \bar{u}, \bar{v}, \bar{w} - a, \bar{t}),$$

$$\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0x}(\bar{x}, \bar{y}, \bar{z} - a\bar{t}, \bar{u}, \bar{v}, \bar{w} - a, \bar{t}),$$

$$\bar{f}_{\bar{y}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = f_{0y}(\bar{x},\bar{y},\bar{z}-a\bar{t},\bar{u},\bar{v},\bar{w}-a,\bar{t}),$$

$$\bar{f}_{\bar{z}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = f_{0z}(\bar{x},\bar{y},\bar{z}-a\bar{t},\bar{u},\bar{v},\bar{w}-a,\bar{t}), \tag{B.32}$$

and

$$\begin{aligned}
\bar{f}^* &= f_0(\bar{x},\bar{y},\bar{z}-a\bar{t},\frac{1}{2}(\bar{u}+u_1+\bar{g}n_1),\frac{1}{2}(\bar{v}+v_1+\bar{g}n_2),\frac{1}{2}(\bar{w}+w_1+\bar{g}n_3)-a,\bar{t}) \\
&= f_0(\bar{x},\bar{y},\bar{z}-a\bar{t},\frac{1}{2}(\bar{u}+u_1+\bar{g}n_1),\frac{1}{2}(\bar{v}+v_1+\bar{g}n_2), \\
&\frac{1}{2}((\bar{w}-a)+(w_1-a)+\bar{g}n_3),\bar{t}), \\
\bar{f}_1^* &= f_0(\bar{x},\bar{y},\bar{z}-a\bar{t},\frac{1}{2}(\bar{u}+u_1-\bar{g}n_1),\frac{1}{2}(\bar{v}+v_1-\bar{g}n_2),\frac{1}{2}(\bar{w}+w_1-\bar{g}n_3)-a,\bar{t}) \\
&= f_0(\bar{x},\bar{y},\bar{z}-a\bar{t},\frac{1}{2}(\bar{u}+u_1-\bar{g}n_1),\frac{1}{2}(\bar{v}+v_1-\bar{g}n_2), \\
&\frac{1}{2}((\bar{w}-a)+(w_1-a)-\bar{g}n_3),\bar{t}), \\
\bar{f}_1 &= f_0(\bar{x},\bar{y},\bar{z}-a\bar{t},u_1,v_1,w_1-a,\bar{t}).
\end{aligned} \tag{B.33}$$

Using (B.30), one finds that

$$\begin{aligned}
\bar{g} &= \sqrt{(u-u_1)^2+(v-v_1)^2+(w+a-w_1)^2} \\
&= \sqrt{(u-u_1)^2+(\bar{v}-v_1)^2+(\bar{w}-(w_1-a))^2}.
\end{aligned} \tag{B.34}$$

Substituting (B.31), (B.33), (B.34) into (3.24) and using the change of variables: $\tilde{w}_1 = w_1 - a$, the collision term becomes

$$(J(\bar{f},\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (J(f_0,f_0))(\bar{x}-a\bar{t},\bar{y},\bar{z},\bar{u}-a,\bar{v},\bar{w},\bar{t}). \tag{B.35}$$

Then

$$(\Phi(\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (-af_{0z}+f_{0t}+uf_{0x}+vf_{0y}+(w+a)f_{0z}$$

$$- J(f_0, f_0))(\bar{x}, \bar{y}, \bar{z} - a\bar{t}, \bar{u}, \bar{v}, \bar{w} - a, \bar{t})$$

$$= (f_{0t} + uf_{0x} + vf_{0y} + wf_{0z} - J(f_0, f_0))(\bar{x}, \bar{y}, \bar{z} - a\bar{t}, \bar{u}, \bar{v}, \bar{w} - a, \bar{t})$$

$$= (\Phi(f_0))(\bar{x}, \bar{y}, \bar{z} - a\bar{t}, \bar{u}, \bar{v}, \bar{w} - a, \bar{t}). \tag{B.36}$$

Differentiating the above equation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_6(\Phi) = -(tD_z(\Phi) + D_w(\Phi)). \tag{B.37}$$

As $X_6 = t\partial_z + \partial_w$ then

$$h_{X_6}\Phi = \tilde{X}_6(\Phi) + (tD_z(\Phi) + D_w(\Phi))$$

$$= -(tD_z(\Phi) + D_w(\Phi)) + (tD_z(\Phi) + D_w(\Phi)) = 0. \tag{B.38}$$

This means that

$$\tilde{h}_{X_6} = 0. \tag{B.39}$$

B.5 Derivation of the function h_{X_8}

The group of transformations corresponding to the generator X_8 is

$$\bar{x} = z\sin(a) + x\cos(a), \ \ \bar{y} = y, \ \ \bar{z} = z\cos(a) - x\sin(a),$$

$$\bar{u} = u\cos(a) + w\sin(a), \ \ \bar{v} = v, \ \ \bar{w} = w\cos(a) - u\sin(a), \ \ \bar{t} = t, \ \ \bar{f} = f. \tag{B.40}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x}\cos(a) - \bar{z}\sin(a), \bar{y}, \bar{z}\cos(a) + \bar{x}\sin(a),$$

$$\bar{u}\cos(a) - \bar{w}\sin(a), \bar{v}, \bar{w}\cos(a) + \bar{u}\sin(a), \bar{t}). \tag{B.41}$$

It follows that

$$\bar{f}_{\bar{t}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = f_{0t}(\bar{x}\cos(a)-\bar{z}\sin(a),\bar{y},\bar{z}\cos(a)+\bar{x}\sin(a),$$

$$\bar{u}\cos(a)-\bar{w}\sin(a),\bar{v},\bar{w}\cos(a)+\bar{u}\sin(a),\bar{t}),$$

$$\bar{f}_{\bar{x}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (f_{0x}\cos(a)+f_{0z}\sin(a))(\bar{x}\cos(a)-\bar{z}\sin(a),\bar{y},$$

$$\bar{z}\cos(a)+\bar{x}\sin(a),\bar{u}\cos(a)-\bar{w}\sin(a),\bar{v},\bar{w}\cos(a)+\bar{u}\sin(a),\bar{t}),$$

$$\bar{f}_{\bar{y}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = f_{0y}(\bar{x}\cos(a)-\bar{z}\sin(a),\bar{y},\bar{z}\cos(a)+\bar{x}\sin(a),$$

$$\bar{u}\cos(a)-\bar{w}\sin(a),\bar{v},\bar{w}\cos(a)+\bar{u}\sin(a),\bar{t}),$$

$$\bar{f}_{\bar{z}}(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (-f_{0x}\sin(a)+f_{0z}\cos(a))(\bar{x}\cos(a)-\bar{z}\sin(a),\bar{y},\bar{z}\cos(a)$$

$$+\bar{x}\sin(a),\bar{u}\cos(a)-\bar{w}\sin(a),\bar{v},\bar{w}\cos(a)+\bar{u}\sin(a),\bar{t}), \qquad \text{(B.42)}$$

and

$$\bar{f}_1 = f_0(\bar{x}\cos(a)-\bar{z}\sin(a),\bar{y},\bar{z}\cos(a)+\bar{x}\sin(a),u_1\cos(a)-w_1\sin(a),v_1,$$

$$w_1\cos(a)+u_1\sin(a),\bar{t})$$

$$= f_0(\bar{x},\bar{y}\cos(a)+\bar{z}\sin(a),\bar{z}\cos(a)-\bar{y}\sin(a),\tilde{u}_1,\tilde{v}_1,\tilde{w}_1,\bar{t}),$$

$$\bar{f}^* = f_0(\bar{x}\cos(a)-\bar{z}\sin(a),\bar{y},\bar{z}\cos(a)+\bar{x}\sin(a),\frac{1}{2}(\bar{u}+u_1+\bar{g}n_1)\cos(a)$$

$$-\frac{1}{2}(\bar{w}+w_1+\bar{g}n_3)\sin(a),\frac{1}{2}(\bar{v}+v_1+\bar{g}n_2),\frac{1}{2}(\bar{w}+w_1+\bar{g}n_3)\cos(a)$$

$$+\frac{1}{2}(\bar{u}+u_1+\bar{g}n_1)\sin(a),\bar{t})$$

$$= f_0(\bar{x}\cos(a)-\bar{z}\sin(a),\bar{y},\bar{z}\cos(a)+\bar{x}\sin(a),\frac{1}{2}(\bar{u}\cos(a)-\bar{w}\sin(a)$$

$$+\tilde{u}_1+\bar{g}\tilde{n}_1),\frac{1}{2}(\bar{v}+\tilde{v}_1+\bar{g}\tilde{n}_2),\frac{1}{2}(\bar{w}\cos(a)+\bar{u}\sin(a)+\tilde{w}_1+\bar{g}\tilde{n}_3),\bar{t})$$

$$\bar{f}_1^* = f_0(\bar{x}\cos(a) - \bar{z}\sin(a), \bar{y}, \bar{z}\cos(a) + \bar{x}\sin(a), \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1)\cos(a)$$

$$- \frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3)\sin(a), \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3)\cos(a)$$

$$+ \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1)\sin(a), \bar{t})$$

$$= f_0(\bar{x}\cos(a) - \bar{z}\sin(a), \bar{y}, \bar{z}\cos(a) + \bar{x}\sin(a), \frac{1}{2}(\bar{u}\cos(a) - \bar{w}\sin(a) + \tilde{u}_1$$

$$- \bar{g}\tilde{n}_1), \frac{1}{2}(\bar{v} + \tilde{v}_1 - \bar{g}\tilde{n}_2), \frac{1}{2}(\bar{w}\cos(a) + \bar{u}\sin(a) + \tilde{w}_1 - \bar{g}\tilde{n}_3), \bar{t}), \qquad \text{(B.43)}$$

where

$$\tilde{u}_1 = u_1\cos(a) - w_1\sin(a), \ \ \tilde{v}_1 = v_1, \ \ \tilde{w}_1 = w_1\cos(a) + u_1\sin(a),$$

$$\tilde{n}_1 = n_1\cos(a) - n_3\sin(a), \ \ \tilde{n}_2 = n_2, \ \ \tilde{n}_3 = n_3\cos(a) + n_1\sin(a). \qquad \text{(B.44)}$$

Using (B.40), one finds that

$$\bar{g} = \sqrt{(u\cos(a) + w\sin(a) - u_1)^2 + (v - v_1)^2 + (w\cos(a) - u\sin(a) - w_1)^2}$$

$$= \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (\tilde{v} - \tilde{v}_1)^2 + (\tilde{w} - \tilde{w}_1)^2}. \qquad \text{(B.45)}$$

Substituting (B.41), (B.43), (B.45) into (3.24) and using the change of variables: $\tilde{u}_1, \tilde{v}_1, \tilde{w}_1$ defined by (B.44), the collision term becomes

$$(J(\bar{f}, \bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (J(f_0, f_0))(\bar{x}\cos(a) - \bar{z}\sin(a), \bar{y}, \bar{z}\cos(a) + \bar{x}\sin(a)$$

$$, \bar{u}\cos(a) - \bar{w}\sin(a), \bar{v}, \bar{w}\cos(a) + \bar{u}\sin(a), \bar{t}). \qquad \text{(B.46)}$$

Here we have used the properties that $\|(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)\|_2 = 1$, and $|\frac{\partial(u_1, v_1, w_1)}{\partial(\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)}| = 1$.

Hence,

$$(\Phi(\bar{f}))(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (f_{0t} + (u\cos(a) + w\sin(a))(f_{0x}\cos(a) + f_{0z}\sin(a))$$

$$+ vf_{0y} + (w\cos(a) - u\sin(a))(-f_{0x}\sin(a) + f_{0z}\cos(a))$$

$$- J(f_0, f_0))(\bar{x}\cos(a) - \bar{z}\sin(a), \bar{y}, \bar{z}\cos(a) + \bar{x}\sin(a), \bar{u}\cos(a) - \bar{w}\sin(a),$$

$$\bar{v}, \bar{w}\cos(a) + \bar{u}\sin(a), \bar{t})$$

$$= (f_{0t} + uf_{0x} + vf_{0y} + wf_{0z} - J(f_0, f_0)(\bar{x}\cos(a) - \bar{z}\sin(a), \bar{y},$$

$$\bar{z}\cos(a) + \bar{x}\sin(a), \bar{u}\cos(a) - \bar{w}\sin(a), \bar{v}, \bar{w}\cos(a) + \bar{u}\sin(a), \bar{t})$$

$$= (\Phi(f_0))(\bar{x}\cos(a) - \bar{z}\sin(a), \bar{y}, \bar{z}\cos(a) + \bar{x}\sin(a), \bar{u}\cos(a) - \bar{w}\sin(a),$$

$$\bar{v}, \bar{w}\cos(a) + \bar{u}\sin(a), \bar{t}). \tag{B.47}$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_8(\Phi) = -zD_x(\Phi) + xD_z(\Phi) - wD_v(\Phi) + vD_w(\Phi). \tag{B.48}$$

As $X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w$ then

$$h_{X_8}\Phi = \tilde{X}_8(\Phi) + (zD_x(\Phi) - xD_z(\Phi) + wD_u(\Phi) - uD_w(\Phi))$$

$$= (-zD_x(\Phi) + xD_z(\Phi) - wD_v(\Phi) + vD_w(\Phi))$$

$$+ (zD_x(\Phi) - xD_z(\Phi) + wD_u(\Phi) - uD_w(\Phi)) = 0. \tag{B.49}$$

This means that

$$\tilde{h}_{X_8} = 0. \tag{B.50}$$

B.6 Derivation of the function h_{X_9}

The group of transformations corresponding to the generator X_9 is

$$\bar{x} = x\cos(a) - y\sin(a),\ \bar{y} = y\cos(a) + x\sin(a),\ \bar{z} = z,$$

$$\bar{u} = u\cos(a) - v\sin(a),\ \ \bar{v} = v\cos(a) + u\sin(a),\ \ \bar{w} = w,\ \ \bar{t} = t,\ \ \bar{f} = f. \tag{B.51}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z}, \bar{u}\cos(a) + \bar{v}\sin(a),$$
$$\bar{v}\cos(a) - \bar{u}\sin(a), \bar{w}, \bar{t}). \tag{B.52}$$

It follows that

$$\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0t}(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z},$$
$$\bar{u}\cos(a) + \bar{v}\sin(a), \bar{v}\cos(a) - \bar{u}\sin(a), \bar{w}, \bar{t}),$$
$$\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (f_{0x}\cos(a) - f_{0y}\sin(a))(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a)$$
$$- \bar{x}\sin(a), \bar{z}, \bar{u}\cos(a) + \bar{v}\sin(a), \bar{v}\cos(a) - \bar{u}\sin(a), \bar{w}, \bar{t}),$$
$$\bar{f}_{\bar{y}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = (f_{0x}\sin(a) + f_{0y}\cos(a))(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a)$$
$$- \bar{x}\sin(a), \bar{z}, \bar{u}\cos(a) + \bar{v}\sin(a), \bar{v}\cos(a) - \bar{u}\sin(a), \bar{w}, \bar{t}),$$
$$\bar{f}_{\bar{z}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_{0z}(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z}, \bar{u}\cos(a)$$
$$+ \bar{v}\sin(a), \bar{v}\cos(a) - \bar{u}\sin(a), \bar{w}, \bar{t}), \tag{B.53}$$

and

$$\bar{f}_1 = f_0(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z}, u_1\cos(a) + v_1\sin(a),$$
$$v_1\cos(a) - u_1\sin(a), w_1, \bar{t})$$
$$= f_0(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z}, \tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \bar{t}),$$

$$\bar{f}^* = f_0(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z}, \frac{1}{2}(\bar{u} + u_1 + \bar{g}n_1)\cos(a)$$

$$+ \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2)\sin(a), \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2)\cos(a) - \frac{1}{2}(\bar{u} + u_1 + \bar{g}n_1)\sin(a),$$

$$\frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3), \bar{t})$$

$$= f_0(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z}, \frac{1}{2}(\bar{u}\cos(a) + \bar{v}\sin(a) + \tilde{u}_1 + \bar{g}\tilde{n}_1),$$

$$\frac{1}{2}(\bar{v}\cos(a) - \bar{u}\sin(a) + \tilde{v}_1 + \bar{g}\tilde{n}_2), \frac{1}{2}(\bar{w} + \tilde{w}_1 + \bar{g}\tilde{n}_3), \bar{t}),$$

$$\bar{f}_1^* = f_0(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z}, \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1)\cos(a)$$

$$+ \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2)\sin(a), \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2)\cos(a) - \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1)\sin(a),$$

$$\frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3), \bar{t})$$

$$= f_0(\bar{x}\cos(a) + \bar{y}\sin(a), \bar{y}\cos(a) - \bar{x}\sin(a), \bar{z}, \frac{1}{2}(\bar{u}\cos(a) + \bar{v}\sin(a) + \tilde{u}_1 - \bar{g}\tilde{n}_1),$$

$$\frac{1}{2}(\bar{v}\cos(a) - \bar{u}\sin(a) + \tilde{v}_1 - \bar{g}\tilde{n}_2), \frac{1}{2}(\bar{w} + \tilde{w}_1 - \bar{g}\tilde{n}_3), \bar{t}), \tag{B.54}$$

where

$$\tilde{u}_1 = u_1\cos(a) + v_1\sin(a),\ \tilde{v}_1 = v_1\cos(a) - u_1\sin(a),\ \tilde{w}_1 = w_1,$$

$$\tilde{n}_1 = n_1\cos(a) + n_2\sin(a),\ \tilde{n}_2 = n_2\cos(a) - n_1\sin(a),\ \tilde{n}_3 = n_3. \tag{B.55}$$

Using (B.51), one finds that

$$\bar{g} = \sqrt{(u\cos(a) - v\sin(a) - u_1)^2 + (v\cos(a) + u\sin(a) - v_1)^2 + (w - w_1)^2}$$

$$= \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (\tilde{v} - \tilde{v}_1)^2 + (\tilde{w} - \tilde{w}_1)^2}. \tag{B.56}$$

Substituting (B.52), (B.54), (B.56) into (3.24) and using the change of variables: $\tilde{u}_1, \tilde{v}_1, \tilde{w}_1$ defined by (B.55), the collision term becomes

$$(J(\bar{f},\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (J(f_0,f_0))(\bar{x}\cos(a)+\bar{y}\sin(a),\bar{y}\cos(a)-\bar{x}\sin(a),\bar{z},$$
$$\bar{u}\cos(a)+\bar{v}\sin(a),\bar{v}\cos(a)-\bar{u}\sin(a),\bar{w},\bar{t}). \qquad \text{(B.57)}$$

Here we have used the properties that $|(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3)| = 1$, and $|\frac{\partial(u_1,v_1,w_1)}{\partial(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1)}| = 1$.

Hence,

$$(\Phi(\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (f_{0t} + (u\cos(a)-v\sin(a))(f_{0x}\cos(a)-f_{0y}\sin(a))$$
$$+ (v\cos(a)+u\sin(a))(f_{0x}\sin(a)+f_{0y}\cos(a)) + wf_{0z} - J(f_0,f_0))(\bar{x}\cos(a)$$
$$+ \bar{y}\sin(a),\bar{y}\cos(a)-\bar{x}\sin(a),\bar{z},\bar{u}\cos(a)+\bar{v}\sin(a),\bar{v}\cos(a)-\bar{u}\sin(a),\bar{w},\bar{t})$$
$$= (f_{0t} + uf_{0x} + vf_{0y} + wf_{0z} - J(f_0,f_0))(\bar{x}\cos(a)+\bar{y}\sin(a),$$
$$\bar{y}\cos(a)-\bar{x}\sin(a),\bar{z},\bar{u}\cos(a)+\bar{v}\sin(a),\bar{v}\cos(a)-\bar{u}\sin(a),\bar{w},\bar{t})$$
$$= (\Phi(f_0))(\bar{x}\cos(a)+\bar{y}\sin(a),\bar{y}\cos(a)-\bar{x}\sin(a),\bar{z},\bar{u}\cos(a)+\bar{v}\sin(a),$$
$$\bar{v}\cos(a)-\bar{u}\sin(a),\bar{w},\bar{t}). \qquad \text{(B.58)}$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_9(\Phi) = -xD_y(\Phi) + yD_x(\Phi) - uD_v(\Phi) + vD_u(\Phi). \qquad \text{(B.59)}$$

As $X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u$ then

$$h_{X_9}\Phi = \tilde{X}_9(\Phi) + (xD_y(\Phi) - yD_x(\Phi) + uD_v(\Phi) - vD_u(\Phi))$$
$$= (-xD_y(\Phi) + yD_x(\Phi) - uD_v(\Phi) + vD_u(\Phi))$$
$$+ (xD_y(\Phi) - yD_x(\Phi) + uD_v(\Phi) - vD_u(\Phi)) = 0. \qquad \text{(B.60)}$$

This means that

$$\tilde{h}_{X_9} = 0. \tag{B.61}$$

B.7 Derivation of the function $h_{X_{10}}$

The group of transformations corresponding to the generator X_{10} is

$$\bar{x} = x,\ \bar{y} = y,\ \bar{z} = z,\ \bar{u} = u,\ \bar{v} = v,\ \bar{w} = w,\ \bar{t} = t + a,\ \bar{f} = f. \tag{B.62}$$

This group of transformations maps a function $f = f_0(x, y, z, u, v, w, t)$ to the function $\bar{f} = f_a(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t})$, where the transformed function is determined by the formula

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) = f_0(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t} - a). \tag{B.63}$$

It follows that

$$\begin{aligned}
\bar{f}_{\bar{t}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0t}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t} - a), \\
\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0x}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t} - a), \\
\bar{f}_{\bar{y}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0y}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t} - a), \\
\bar{f}_{\bar{z}}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t}) &= f_{0z}(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}, \bar{t} - a),
\end{aligned} \tag{B.64}$$

and

$$\begin{aligned}
\bar{f}^* &= f_0(\bar{x}, \bar{y}, \bar{z}, \frac{1}{2}(\bar{u} + u_1 + \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 + \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 + \bar{g}n_3), \bar{t} - a), \\
\bar{f}_1^* &= f_0(\bar{x}, \bar{y}, \bar{z}, \frac{1}{2}(\bar{u} + u_1 - \bar{g}n_1), \frac{1}{2}(\bar{v} + v_1 - \bar{g}n_2), \frac{1}{2}(\bar{w} + w_1 - \bar{g}n_3), \bar{t} - a), \\
\bar{f}_1 &= f_0(\bar{x}, \bar{y}, \bar{z}, u_1, v_1, w_1, \bar{t} - a).
\end{aligned} \tag{B.65}$$

Using (B.62), one finds that

$$\bar{g} = \sqrt{(u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2}. \tag{B.66}$$

Substituting (B.63), (B.65), (B.66) into (3.24), the collision term becomes

$$(J(\bar{f},\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) = (J(f_0,f_0))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}-a). \tag{B.67}$$

Then

$$\begin{aligned}(\Phi(\bar{f}))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}) &= (f_{0t} + uf_{0x} + vf_{0y} + wf_{0z} - J(f_0,f_0))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\\ &\qquad \bar{w},\bar{t}-a)\\ &= (\Phi(f_0))(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w},\bar{t}-a). \end{aligned} \tag{B.68}$$

Differentiating the latter relation with respect to a and setting $a = 0$, we derive

$$\tilde{X}_{10}(\Phi) = -D_t(\Phi). \tag{B.69}$$

As $X_{10} = \partial_t$ then

$$h_{X_{10}}\Phi = \tilde{X}_{10}(\Phi) + D_t(\Phi) = -D_t(\Phi) + D_t(\Phi) = 0. \tag{B.70}$$

This means that

$$\tilde{h}_{X_{10}} = 0. \tag{B.71}$$

APPENDIX C

AN OPTIMAL SYSTEM OF SUBALGEBRAS OF L_{11}

An optimal system of subalgebras of Lie algebras L_{11} obtained in (Ovsiannikov, 1994) is shown in this Appendix. There are 13, 27, 47, 50, 37, 25, 14, 5, 2, 2, and 1 subalgebras of $k-$dimensional subalgebras, $k = 1, 2, \dots, 11$, respectively, in the optimal system.

Table C.1: An optimal system of subalgebras of L_{11}.

No.	Subalgebra
1.1	$\beta 4 + 7 + \alpha 11,\ \alpha \neq 0$
1.2	$\beta 4 + 7,\ \beta \neq 0$
1.3	7
1.4	$1 + 7$
1.5	$\beta 4 + 7 + \beta 10,\ \beta \neq 0$
1.6	$7 + 10$
1.7	$\beta 4 + 11,\ \beta \neq 0$
1.8	11
1.9	$4 + 10$
1.10	10
1.11	$3 + 4$
1.12	4
1.13	1
2.1	$10, 7 + \alpha 11,\ \alpha \neq 0$
2.2	$\alpha 4 + 7, \beta 4 + 11$
2.3	$4, 7 + \alpha 11,\ \alpha \neq 0$
2.4	$1, \beta 4 + 7 + \alpha 11,\ \alpha \neq 0$
2.5	$7, 10$
2.6	$1 + 7, 10$
2.7	$\alpha 1 + 7, 4 + 10$
2.8	$4, 7$
2.9	$1, \beta 4 + 7$
2.10	$4, 1 + 7$
2.11	$1, \beta 4 + 7 + 10$
2.12	$10, 11$
2.13	$4, 11$
2.14	$4, \alpha 5 + 11,\ \alpha \neq 0$
2.15	$1, \beta 4 + \alpha 5 + 11,\ \alpha \neq 0$
2.16	$1, \beta 4 + 11$
2.17	$1, 10$
2.18	$3, 4 + \alpha 6 + 10,\ \alpha \neq 0$
2.19	$1, 4 + 10$
2.20	$\alpha 1 + \sigma 3 + 5, \beta 1 + \tau 2 + 6,$ $\alpha^2 + \beta^2 + (\sigma + \tau)^2 = 1$
2.21	$3 + 5, 2 - 6$
2.22	$5, 6$
2.23	$\alpha 1 + 2, 3 + 4$
2.24	$\alpha 1 + 2, 4$
2.25	$1, 3 + 4$
2.26	$1, 4$
2.27	$2, 3$
3.1	$7, 10, 11$
3.2	$1, 10, \beta 4 + 7 + \alpha 11$
3.3	$4, 7, 11$

Table C.1: An optimal system of subalgebras of L_{11} (Continued).

No.	Subalgebra
3.4	$1, \alpha 4 + 7, \beta 4 + 11$
3.5	$5, 6, \beta 4 + 7 + \alpha 11,\ \alpha \neq 0$
3.6	$1, 4, 7 + \alpha 11,\ \alpha \neq 0$
3.7	$2, 3, \beta 4 + 7 + \alpha 11,\ \alpha \neq 0$
3.8	$7, 8, 9$
3.9	$1, \alpha 4 + 7, 4 + 10$
3.10	$5, 6, \beta 4 + 7$
3.11	$1, 4, 7$
3.12	$2, 3, \beta 4 + 7,\ \beta \neq 0$
3.13	$2, 3, 7$
3.14	$5, 6, 1 + \alpha 4 + 7$
3.15	$3 + 5, 2 - 6, \alpha 1 + \beta 4 + 7$
3.16	$2, 3, 1 + 7$
3.17	$1, 4, 7 + 10$
3.18	$2, 3, \beta 4 + 7 + \beta 10,\ \beta \neq 0$
3.19	$2, 3, 7 + 10$
3.20	$1, 10, \beta 4 + 11$
3.21	$5, 6, \beta 4 + 11$
3.22	$1, \alpha 4 + 6, \beta 4 + \sigma 5 + 11$
3.23	$1, 4, 11$
3.25	$2, 3, \beta 4 + \sigma 5 + 11,\ \sigma \neq 0$
3.26	$2, 3, \beta 4 + 11$
3.27	$3, \alpha 1 + \beta 2 + 6, 4 + 10$
3.28	$1, 2 + 4, 10$
3.29	$1, 4, 10$
3.30	$2, 3, 4 + \sigma 6 + 10,\ \sigma \neq 0$
3.31	$2, 3, 4 + 10$
3.32	$2, 3, 6 + 10$
3.33	$2, 3, 10$
3.34	$-\delta 2 + \beta 3 + 4, \delta 1 + \sigma 2 - \alpha 3 + 5,$ $-\beta 1 + \alpha 2 + \tau 3 + 6,$ $\alpha^2 + \beta^2 + \delta^2 + (\sigma + \tau)^2 = 1$
3.35	$4, 3 + 5, 2 - 6$
3.36	$1 + 4, 5, 6$
3.37	$4, 5, 6$
3.38	$\alpha 1 + 3, \beta 1 + 5, \sigma 1 + \tau 2 + 6,$ $\beta^2 + \sigma^2 + \tau^2 = 1$
3.39	$\alpha 1 + 3, 5, 6$
3.40	$1, 3 + 5, \tau 2 + 6,\ \tau \neq -1$
3.41	$1, 3 + 5, 2 - 6$
3.42	$1, 5, 6$
3.43	$\beta 1 + 3, 2, 4$
3.44	$2, 3, 4$
3.45	$1, 2, 3 + 4$
3.46	$1, 2, 4$
3.47	$1, 2, 3$
4.1	$7, 8, 9, 11$
4.2	$1, \alpha 4 + 7, 10, 11$
4.3	$2, 3, 10, 7 + \alpha 11$
4.4	$1, 4, 10, 7 + \alpha 11,\ \alpha \neq 0$
4.5	$5, 6, \alpha 4 + 7, \beta 4 + 11$
4.6	$1, 4, 7, 11$
4.7	$2, 3, \alpha 4 + 7, \beta 4 + 11$
4.8	$4, 5, 6, 7 + \alpha 11,\ \alpha \neq 0$
4.9	$1, 5, 6, \beta 4 + 7 + \alpha 11$
4.10	$2, 3, 4, 7 + \alpha 11$
4.11	$1, 2, 3, \beta 4 + 7 + \alpha 11,\ \alpha \neq 0$
4.12	$1, 2, 3, \beta 4 + 7$
4.13	$7, 8, 9, 10$
4.14	$2, 3, 7, 10$
4.15	$2, 3, 1 + 7, 10$
4.16	$2, 3, \alpha 1 + 7, 4 + 10$
4.17	$4, 5, 6, 7$
4.18	$4, 5, 6, 1 + 7$
4.19	$4, 3 + 5, 2 - 6, \alpha 1 + 7$
4.20	$1, 3 + 5, 2 - 6, \alpha 4 + 7$
4.21	$2, 3, 4, 1 + 7$
4.22	$1, 2, 3, \beta 4 + 7 + 10$

Table C.1: An optimal system of subalgebras of L_{11} (Continued).

No.	Subalgebra
4.23	$1, 4, 10, 11$
4.24	$2, 3, 10, \alpha 6 + 11,\ \alpha \neq 0$
4.25	$2, 3, 10, 11$
4.26	$4, 5, 6, 11$
4.27	$1, \alpha 4 + 5, 6, \beta 4 + 11,\ \alpha \neq 0$
4.28	$1, 5, 6, \beta 4 + 11$
4.29	$1, 4, 6, \alpha 5 + 11$
4.30	$2, 3, \alpha 4 + 6, \beta 4 + \sigma 5 + 11$
4.31	$2, 3, 4, \alpha 5 + \beta 6 + 11,\ \alpha^2 + \beta^2 \neq 0$
4.32	$2, 3, 4, 11$
4.33	$1, 2, 3, \beta 4 + 11,\ \beta \neq 0$
4.34	$1, 2, 3, 11$
4.35	$2, 3, \alpha 1 + 5, 4 + \beta 6 + 10$
4.36	$2, 3, \alpha 1 + 5, 6 + 10$
4.37	$2, 3, 1 + 5, 10$
4.38	$2, 3, 5, 10$
4.39	$1, 2, 3, 4 + 10$
4.40	$1, 2, 3, 10$
4.41	$1, \sigma 2 + \tau 3 + 4, \alpha 3 + 5, \beta 2 + 6,$ $\sigma^2 + \tau^2 + (\alpha + \beta)^2 = 1$
4.42	$1, 4, 3 + 5, 2 - 6$
4.43	$1, 4, 5, 6$
4.44	$2, \alpha 1 + 3, 1 + 5, 6,\ \alpha \neq 0$
4.45	$2, 3, 1 + 5, 6$
4.46	$1 + \beta 3, 2, 5, 6$
4.47	$2, 3, 5, 6$
4.48	$1, 2, 3 + 5, 6$
4.49	$1, 2, 5, 6$
4.50	$1, 2, 3, 4$
5.1	$7, 8, 9, 10, 11$
5.2	$1, 4, 7, 10, 11$
5.3	$2, 3, 7, 10, 11$
5.4	$1, 2, 3, 10, \beta 4 + 7 + \alpha 11$
5.5	$4, 5, 6, 7, 11$
5.6	$2, 3, 4, 7, 11$
5.7	$1, 5, 6, \alpha 4 + 7, \beta 4 + 11$
5.8	$1, 2, 3, \alpha 4 + 7, \beta 4 + 11$

No.	Subalgebra
5.9	$1, 4, 5, 6, 7 + \alpha 11$
5.10	$2, 3, 5, 6, \beta 4 + 7 + \alpha 11,\ \alpha \neq 0$
5.11	$1, 2, 3, 4, 7 + \alpha 11,\ \alpha \neq 0$
5.12	$1, 2, 3, \alpha 4 + 7, 4 + 10$
5.13	$2, 3, 5, 6, \beta 4 + 7,\ \beta \neq 0$
5.14	$2, 3, 5, 6, 7$
5.15	$1, 2, 3, 4, 7$
5.16	$1, 4, 3 + 5, 2 - 6, 7$
5.17	$2, 3, 5, 6, 1 + 7$
5.18	$2, 3, 5, 6, \beta 4 + 7 + \beta 10,\ \beta \neq 0$
5.19	$2, 3, 5, 6, 7 + 10$
5.20	$1, 2, 3, 4, 7 + 10$
5.21	$2, 3, 5, 10, \beta 6 + 11$
5.22	$1, 2, 3, 10, 4 + \beta 11$
5.23	$1, 2, 3, 10, 11$
5.24	$1, 4, 5, 6, 11$
5.25	$2, 3, \alpha 4 + 5, 6, \beta 4 + 11,\ \alpha \neq 0$
5.26	$2, 3, 5, 6, \beta 4 + 11$
5.27	$2, 3, 4, 6, \beta 5 + 11$
5.28	$1, 2, 3, 6, \beta 4 + 11,\ \beta \neq 0$
5.29	$1, 2, 3, 4, 11$
5.30	$2, 3, \alpha 1 + 5, 6, 4 + 10,\ \alpha \neq 0$
5.31	$2, 3, 5, 6, 4 + 10$
5.32	$2, 3, 1 + 5, 6, 10$
5.33	$2, 3, 5, 6, 10$
5.34	$1, 2, 3, 6, 4 + 10$
5.35	$2, 3, 4, 5, 6$
5.36	$2, 3, 4, 5, 1 + 6$
5.37	$1, 2, 3, 5, 6$
6.1	$1, 2, 3, 7, 10, 11$
6.2	$2, 3, 5, 6, 10, 7 + \alpha 11,\ \alpha \neq 0$
6.3	$1, 2, 3, 4, 10, 7 + \alpha 11$
6.4	$1, 4, 5, 6, 7, 11$
6.5	$1, 2, 3, 4, 7, 11$
6.6	$2, 3, 5, 6, \alpha 4 + 7, \beta 4 + 11$
6.7	$2, 3, 4, 5, 6, 7 + \alpha 11,\ \alpha \neq 0$
6.8	$1, 2, 3, 5, 6, \beta 4 + 7 + \alpha 11$

Table C.1: An optimal system of subalgebras of L_{11} (Continued).

No.	Subalgebra
6.9	$4, 5, 6, 7, 8, 9$
6.10	$1, 2, 3, 7, 8, 9$
6.11	$2, 3, 5, 6, 1 + 7, 10$
6.12	$2, 3, 5, 6, \alpha 1 + 7, 4 + 10$
6.13	$2, 3, 5, 6, 7, 10$
6.14	$2, 3, 4, 5, 6, 1 + 7$
6.15	$2, 3, 4, 5, 6, 7$
6.16	$1, 2, 3, 5, 6, 7 + 10$
6.17	$2, 3, 5, 6, 10, 11$
6.18	$1, 2, 3, 4, 10, \alpha 6 + 11,\ \alpha \neq 0$
6.19	$1, 2, 3, 4, 10, 11$
6.20	$1, 2, 3, 5, 6, \alpha 4 + 11$
6.21	$2, 3, 4, 5, 6, 11$
6.22	$1, 2, 3, 5, 6, 10$
6.23	$1, 2, 3, 5, 6, 4 + 10$
6.24	$1, 2, 3, 4, 5, 6$
6.25	$1, 2, 3, 5, 6, \beta 4 + 7$
7.1	$1, 2, 3, 7, 8, 9, 11$
7.2	$4, 5, 6, 7, 8, 9, 11$
7.3	$2, 3, 5, 6, 7, 10, 11$
7.4	$1, 2, 3, 4, 7, 10, 11$
7.5	$1, 2, 3, 5, 6, 10, \beta 4 + 7 + \alpha 11$
7.6	$2, 3, 4, 5, 6, 7, 11$
7.7	$1, 2, 3, 5, 6, \alpha 4 + 7, \beta 4 + 11$
7.8	$1, 2, 3, 4, 5, 6, 7 + \alpha 11,\ \alpha \neq 0$
7.9	$1, 2, 3, 7, 8, 9, 10$
7.10	$1, 2, 3, 5, 6, \alpha 4 + 7, 4 + 10$
7.11	$1, 2, 3, 4, 5, 6, 7 + 10$
7.12	$1, 2, 3, 5, 6, 10, \alpha 4 + 11$
7.13	$1, 2, 3, 4, 5, 6, 11$
7.14	$1, 2, 3, 4, 5, 6, 10$
8.1	$1, 2, 3, 7, 8, 9, 10, 11$
8.2	$1, 2, 3, 5, 6, \alpha 4 + 7, 10, \beta 4 + 11$
8.3	$1, 2, 3, 4, 5, 6, 7 + \alpha 11, 10$
8.4	$1, 2, 3, 4, 5, 6, 7, 11$
8.5	$1, 2, 3, 4, 5, 6, 10, 11$

No.	Subalgebra
9.1	$1, 2, 3, 4, 5, 6, 7, 10, 11$
9.2	$1, 2, 3, 4, 5, 6, 7, 8, 9$
10.1	$1, 2, 3, 4, 5, 6, 7, 8, 9, 11$
10.2	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$
11.1	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

APPENDIX D

GROUP CLASSIFICATION

Complete results of the preliminary group classification of the Boltzmann equation with a source function are shown in this Appendix. Numbers in the first column of Table D.1 coincide with the numbers in the first column of Table C.1. The superscripts ^{c}, and ^{s} which are next to this subalgebra number in the first column indicate that the source function q is presented in the cylindrical coordinate system, or the spherical coordinate system, respectively. Here Ψ_k is an arbitrary function of k independent variables, and C is constant.

Table D.1: Group classification.

No.	Source function q	Subalgebra
1.1^c	$t^{-2}\Psi_7(\frac{x}{t}-\frac{\beta}{\alpha}\ln t, \frac{r}{t}, \theta-\frac{1}{\alpha}\ln t, u-\frac{\beta}{\alpha}\ln t, V, W, ft)$	$\beta 4+7+\alpha 11,\ \alpha\neq 0$
1.2^c	$\Psi_7(t, r, \beta\theta-\frac{x}{t}, u-\frac{x}{t}, V, W, f)$	$\beta 4+7,\ \beta\neq 0$
1.3^c	$\Psi_7(t, x, r, u, V, W, f)$	7
1.4^c	$\Psi_7(t, r, x-\theta, u, V, W, f)$	$1+7$
1.5^c	$\Psi_7(t^2-2x, r, t-\beta\theta, u-t, V, W, f)$	$\beta 4+7+\beta 10,\ \beta\neq 0$
1.6^c	$\Psi_7(x, r, t-\theta, u, V, W, f)$	$7+10$
1.7	$t^{-2}\Psi_7(\frac{y}{t}, \frac{z}{t}, \frac{x}{t}-\beta\ln t, u-\beta\ln t, v, w, ft)$	$\beta 4+11,\ \beta\neq 0$
1.8	$t^{-2}\Psi_7(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, u, v, w, ft)$	11
1.9	$\Psi_7(t^2-2x, y, z, u-t, v, w, f)$	$4+10$
1.10	$\Psi_7(x, y, z, u, v, w, f)$	10
1.11	$\Psi_7(t, x-tz, y, u-z, v, w, f)$	$3+4$
1.12	$\Psi_7(t, y, z, u-\frac{x}{t}, v, w, f)$	4
1.13	$\Psi_7(t, y, z, u, v, w, f)$	1
2.1^c	$x^{-2}\Psi_6(\frac{r}{x}, \alpha\theta-\ln x, u, V, W, fx)$	$10, 7+\alpha 11,\ \alpha\neq 0$
2.2^c	$t^{-2}\Psi_6(\frac{r}{t}, \frac{x}{t}-\alpha\theta-\beta\ln t, u-\alpha\theta-\beta\ln t, V, W, ft)$	$\alpha 4+7, \beta 4+11$
2.3^c	$t^{-2}\Psi_6(\frac{r}{t}, \alpha\theta-\ln t, u-\frac{x}{t}, V, W, ft)$	$4, 7+\alpha 11,\ \alpha\neq 0$
2.4^c	$t^{-2}\Psi_6(\frac{r}{t}, \alpha\theta-\ln t, u-\frac{\beta}{\alpha}\ln t, V, W, ft)$	$1, \beta 4+7+\alpha 11,\ \alpha\neq 0$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
2.5^c	$\Psi_6(r, x, u, V, W, f)$	$7, 10$
2.6^c	$\Psi_6(r, x-\theta, u, V, W, f)$	$1+7, 10$
2.7^c	$\Psi_6(r, 2(x-\alpha\theta)-t^2, u-t, V, W, f)$	$\alpha 1+7, 4+10$
2.8^c	$\Psi_6(t, r, u-\frac{x}{t}, V, W, f)$	$4, 7$
2.9^c	$\Psi_6(t, r, u-\beta\theta, V, W, f)$	$1, \beta 4+7$
2.10^c	$\Psi_6(t, r, u-\frac{x}{t}+\frac{\theta}{t}, V, W, f)$	$4, 1+7$
2.11^c	$\Psi_6(r, \theta-t, u-\beta t, V, W, f)$	$1, \beta 4+7+10$
2.12	$x^{-2}\Psi_6(\frac{y}{x}, \frac{z}{x}, u, v, w, fx)$	$10, 11$
2.13	$t^{-2}\Psi_6(\frac{y}{t}, \frac{z}{t}, u-\frac{x}{t}, v, w, ft)$	$4, 11$
2.14	$t^{-2}\Psi_6(\frac{y}{t}-\alpha\ln t, \frac{z}{t}, u-\frac{x}{t}, v-\alpha\ln t, w, ft)$	$4, \alpha 5+11,\ \alpha\neq 0$
2.15	$t^{-2}\Psi_6(\frac{z}{t}, \frac{y}{t}-\alpha\ln t, u-\beta\ln t, v-\alpha\ln t, w, ft)$	$1, \beta 4+\alpha 5+11,\ \alpha\neq 0$
2.16	$t^{-2}\Psi_6(\frac{y}{t}, \frac{z}{t}, u-\beta\ln t, v, w, ft)$	$1, \beta 4+11$
2.17	$\Psi_6(y, z, u, v, w, f)$	$1, 10$
2.18	$\Psi_6(2x-t^2, y, u-t, v, w-\alpha t, f)$	$3, 4+\alpha 6+10,\ \alpha\neq 0$
2.19	$\Psi_6(y, z, u-t, v, w, f)$	$1, 4+10$
2.20	$\Psi_6(t, \alpha(ty-\tau z)+\beta(tz-\sigma y)+x(\sigma\tau-t^2), u, v+\frac{\beta y-\tau x}{\alpha\tau-\beta t}, w+\frac{tx-\alpha y}{\alpha\tau-\beta t}, f)$	$\alpha 1+\sigma 3+5, \beta 1+\tau 2+6,$ $\alpha^2+\beta^2+(\sigma+\tau)^2=1$
2.21	$\Psi_6(t, x, u, v-\frac{z+ty}{t^2+1}, w+\frac{y-tz}{t^2+1}, f)$	$3+5, 2-6$
2.22	$\Psi_6(t, x, u, v-\frac{y}{t}, w-\frac{z}{t}, f)$	$5, 6$
2.23	$\Psi_6(t, x-\alpha y-tz, u-z, v, w, f)$	$\alpha 1+2, 3+4$
2.24	$\Psi_6(t, z, u+\frac{\alpha y-x}{t}, v, w, f)$	$\alpha 1+2, 4$
2.25	$\Psi_6(t, y, u-z, v, w, f)$	$1, 3+4$
2.26	$\Psi_6(t, y, z, v, w, f)$	$1, 4$
2.27	$\Psi_6(t, x, u, v, w, f)$	$2, 3$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
3.1^c	$x^{-2}\Psi_5(\frac{r}{x}, u, V, W, fx)$	$7, 10, 11$
3.2^c	$e^{-2\alpha\theta}\Psi_5(re^{-\alpha\theta}, u-\beta\theta, V, W, fe^{\alpha\theta})$	$1, 10, \beta 4+7+\alpha 11$
3.3^c	$t^{-2}\Psi_5(\frac{r}{t}, u-\frac{x}{t}, V, W, ft)$	$4, 7, 11$
3.4^c	$t^{-2}\Psi_5(\frac{r}{t}, u-\alpha\theta-\beta\ln t, V, W, ft)$	$1, \alpha 4+7, \beta 4+11$
3.5	$t^{-2}\Psi_5(\frac{x}{t}-\frac{\beta}{\alpha}\ln t, u-\frac{\beta}{\alpha}\ln t, \arctan(\frac{w-\frac{z}{t}}{v-\frac{y}{t}})-\frac{1}{\alpha}\ln t, \sqrt{(v-\frac{y}{t})^2+(w-\frac{z}{t})^2}, ft)$	$5, 6, \beta 4+7+\alpha 11,\ \alpha\neq 0$
3.6^c	$t^{-2}\Psi_5(\frac{r}{t}, \alpha\theta-\ln t, V, W, ft)$	$1, 4, 7+\alpha 11,\ \alpha\neq 0$
3.7	$t^{-2}\Psi_5(\frac{x}{t}-\frac{\beta}{\alpha}\ln t, u-\frac{\beta}{\alpha}\ln t, \arctan(\frac{w}{v})-\frac{1}{\alpha}\ln t, \sqrt{v^2+w^2}, ft)$	$2, 3, \beta 4+7+\alpha 11,\ \alpha\neq 0$
3.8^s	$\Psi_5(t, r, U, V^2+W^2, f)$	$7, 8, 9$
3.9^c	$\Psi_5(r, u-t-\alpha\theta, V, W, f)$	$1, \alpha 4+7, 4+10$
3.10	$\Psi_5(t, \frac{x}{t}-\beta\arctan(\frac{w-\frac{z}{t}}{v-\frac{y}{t}}), u-\beta\arctan(\frac{w-\frac{z}{t}}{v-\frac{y}{t}}), \sqrt{(v-\frac{y}{t})^2+(w-\frac{z}{t})^2}, f)$	$5, 6, \beta 4+7$
3.11^c	$\Psi_5(t, r, V, W, f)$	$1, 4, 7$
3.12	$\Psi_5(t, u-\frac{x}{t}, \arctan(\frac{w}{v})-\frac{x}{\beta t}, \sqrt{v^2+w^2}, f)$	$2, 3, \beta 4+7,\ \beta\neq 0$
3.13	$\Psi_5(t, x, u, \sqrt{v^2+w^2}, f)$	$2, 3, 7$
3.14	$\Psi_5(t, u-\frac{\alpha x}{1+\alpha t}, \arctan(\frac{w-\frac{z}{t}}{v-\frac{y}{t}})-\frac{x}{1+\alpha t}, \sqrt{(v-\frac{y}{t})^2+(w-\frac{z}{t})^2}, f)$	$5, 6, 1+\alpha 4+7$
3.15	$\Psi_5(t, u+\beta\arctan(\frac{y-tv+w}{z-tw-v}), x+(\alpha+\beta t)\arctan(\frac{y-tv+w}{z-tw-v}),$ $\sqrt{(y-tv+w)^2+(z-tw-v)^2}, f)$	$3+5, 2-6, \alpha 1+\beta 4+7$
3.16	$\Psi_5(t, u, \arctan(\frac{w}{v})-x, \sqrt{v^2+w^2}, f)$	$2, 3, 1+7$
3.17^c	$\Psi_5(t-\theta, r, V, W, f)$	$1, 4, 7+10$
3.18	$\Psi_5(2x-t^2, u-t, \arctan(\frac{w}{v})-\frac{t}{\beta}, \sqrt{v^2+w^2}, f)$	$2, 3, \beta 4+7+\beta 10,\ \beta\neq 0$
3.19	$\Psi_5(x, u, \arctan(\frac{w}{v})-t, \sqrt{v^2+w^2}, f)$	$2, 3, 7+10$
3.20	$y^{-2}\Psi_5(\frac{y}{z}, u-\beta\ln y, v, w, fy)$	$1, 10, \beta 4+11$
3.21	$t^{-2}\Psi_5(\frac{x}{t}-\beta\ln t, u-\beta\ln t, v-\frac{y}{t}, w-\frac{z}{t}, ft)$	$5, 6, \beta 4+11$
3.22	$t^{-2}\Psi_5(\frac{y}{t}-\sigma\ln t, u-\frac{\alpha z}{t}-\beta\ln t, v-\sigma\ln t, w-\frac{z}{t}, ft)$	$1, \alpha 4+6, \beta 4+\sigma 5+11$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
3.23	$t^{-2}\Psi_5(\frac{y}{t}, \frac{z}{t} - \sigma \ln t, v, w - \sigma \ln t, ft)$	$1, 4, \sigma 6 + 11,\ \sigma \neq 0$
3.24	$t^{-2}\Psi_5(\frac{y}{t}, \frac{z}{t}, v, w, ft)$	$1, 4, 11$
3.25	$t^{-2}\Psi_5(\frac{x}{t} - \beta \ln t, u - \beta \ln t, v - \sigma \ln t, w, ft)$	$2, 3, \beta 4 + \sigma 5 + 11,\ \sigma \neq 0$
3.26	$t^{-2}\Psi_5(\frac{x}{t} - \beta \ln t, u - \beta \ln t, v, w, ft),$	$2, 3, \beta 4 + 11$
3.27	$\Psi_5(t^2 - 2x + 2\alpha w, y - \beta w, u - t, v, f)$	$3, \alpha 1 + \beta 2 + 6, 4 + 10$
3.28	$\Psi_5(z, u - y, v, w, f)$	$1, 2 + 4, 10$
3.29	$\Psi_5(y, z, v, w, f)$	$1, 4, 10$
3.30	$\Psi_5(t^2 - 2x, u - t, v, w - \sigma t, f)$	$2, 3, 4 + \sigma 6 + 10,\ \sigma \neq 0$
3.31	$\Psi_5(t^2 - 2x, u - t, v, w, f)$	$2, 3, 4 + 10$
3.32	$\Psi_5(x, u, v, w - t, f)$	$2, 3, 6 + 10$
3.33	$\Psi_5(x, u, v, w, f)$	$2, 3, 10$
3.34	$\Psi_5(t, x - tu - \delta v + \beta w, y - tv + \delta u - \sigma v - \alpha w,$ $z - tw - \beta u + \alpha v - \tau w, f)$	$-\delta 2 + \beta 3 + 4, \delta 1 + \sigma 2 - \alpha 3 + 5, -\beta 1 + \alpha 2 + \tau 3 + 6,$ $\alpha^2 + \beta^2 + \delta^2 + (\sigma + \tau)^2 = 1$
3.35	$\Psi_5(t, x - tu, tw - z + v, y - tz + (t^2 + 1)w, f)$	$4, 3 + 5, 2 - 6$
3.36	$\Psi_5(t, u(t + 1) - x, y - tv, z - tw, f)$	$1 + 4, 5, 6$
3.37	$\Psi_5(t, u - \frac{x}{t}, v - \frac{y}{t}, w - \frac{z}{t}, f)$	$4, 5, 6$
3.38	$\Psi_5(t, \tau w + tv - y, w(\sigma - \alpha t) + \beta v - x + \alpha z, u, f)$	$\alpha 1 + 3, \beta 1 + 5, \sigma 1 + \tau 2 + 6,\ \beta^2 + \sigma^2 + \tau^2 = 1$
3.39	$\Psi_5(t, y - tv, x - \alpha(z - tw), u, f)$	$\alpha 1 + 3, 5, 6$
3.40	$\Psi_5(t, w(t^2 - \tau) + y - tz, \tau w - y + tv, u, f)$	$1, 3 + 5, \tau 2 + 6,\ \tau \neq -1$
3.41	$\Psi_5(t, w(t^2 + 1) + y - tz, w + y - tv, u, f)$	$1, 3 + 5, 2 - 6$
3.42	$\Psi_5(t, u, v - \frac{y}{t}, w - \frac{z}{t}, f)$	$1, 5, 6$
3.43	$\Psi_5(t, tu - x + \beta z, v, w, f)$	$\beta 1 + 3, 2, 4$
3.44	$\Psi_5(t, u - \frac{x}{t}, v, w, f)$	$2, 3, 4$
3.45	$\Psi_5(t, u - z, v, w, f)$	$1, 2, 3 + 4$
3.46	$\Psi_5(t, z, v, w, f)$	$1, 2, 4$
3.47	$\Psi_5(t, u, v, w, f)$	$1, 2, 3$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
4.1^s	$r^{-2}\Psi_4(\frac{t}{r}, U, V^2+W^2, fr)$	$7,8,9,11$
4.2^c	$r^{-2}\Psi_4(u-\alpha\theta, V, W, fr)$	$1, \alpha 4+7, 10, 11$
4.3	$e^{2\alpha\arctan(\frac{v}{w})}\Psi_4(xe^{\alpha\arctan(\frac{v}{w})}, \sqrt{v^2+w^2}, u, fe^{-\alpha\arctan(\frac{v}{w})})$	$2,3,10,7+\alpha 11$
4.4^c	$r^{-2}\Psi_4(\alpha\theta-\ln r, V, W, fr)$	$1,4,10,7+\alpha 11,\ \alpha\neq 0$
4.5	$t^{-2}\Psi_4(\frac{\sqrt{(y-tv)^2+(z-tw)^2}}{t}, t^{-\beta}e^{\frac{x-\alpha t\arctan(\frac{z-tw}{y-tv})}{t}}, t^{-\beta}e^{u-\alpha\arctan(\frac{z-tw}{y-tv})}, ft)$	$5,6,\alpha 4+7,\beta 4+11$
4.6^c	$r^{-2}\Psi_4(\frac{t}{r}, V, W, fr)$	$1,4,7,11$
4.7	$t^{-2}\Psi_4(t^{-\beta}e^{\frac{x+\alpha t\arctan(\frac{v}{w})}{t}}, t^{-\beta}e^{u+\alpha\arctan(\frac{v}{w})}, \sqrt{v^2+w^2}, ft)$	$2,3,\alpha 4+7,\beta 4+11$
4.8	$t^{-2}\Psi_4(\frac{x-tu}{t}, \frac{\sqrt{(y-tv)^2+(z-tw)^2}}{t}, te^{-\alpha\arctan(\frac{z-tw}{y-tv})}, ft)$	$4,5,6,7+\alpha 11,\ \alpha\neq 0$
4.9	$e^{-2\alpha\arctan(\frac{z-tw}{y-tv})}\Psi_4(\sqrt{(y-tv)^2+(z-tw)^2}e^{-\alpha\arctan(\frac{z-tw}{y-tv})}, te^{-\alpha\arctan(\frac{z-tw}{y-tv})},$ $u-\beta\arctan(\frac{z-tw}{y-tv}), fe^{\alpha\arctan(\frac{z-tw}{y-tv})})$	$1,5,6,\beta 4+7+\alpha 11$
4.10	$e^{2\alpha\arctan(\frac{v}{w})}\Psi_4((x-tu)e^{\alpha\arctan(\frac{v}{w})}, te^{\alpha\arctan(\frac{v}{w})}, \sqrt{v^2+w^2}, fe^{-\alpha\arctan(\frac{v}{w})})$	$2,3,4,7+\alpha 11$
4.11	$t^{-2}\Psi_4(te^{\alpha\arctan(\frac{v}{w})}, t^{-\beta}e^{\alpha u}, \sqrt{v^2+w^2}, ft)$	$1,2,3,\beta 4+7+\alpha 11,\ \alpha\neq 0$
4.12	$\Psi_4(u+\beta\arctan(\frac{v}{w}), \sqrt{v^2+w^2}, t, f)$	$1,2,3,\beta 4+7$
4.13^s	$\Psi_4(r, U, V^2+W^2, f)$	$7,8,9,10$
4.14	$\Psi_4(v^2+w^2, x, u, f)$	$2,3,7,10$
4.15	$\Psi_4(x+\arctan(\frac{v}{w}), \sqrt{v^2+w^2}, u, f)$	$2,3,1+7,10$
4.16	$\Psi_4(2\alpha\arctan(\frac{v}{w})+2x-t^2, \sqrt{v^2+w^2}, t-u, f)$	$2,3,\alpha 1+7,4+10$
4.17	$\Psi_4(x-tu, (y-tv)^2+(z-tw)^2, t, f)$	$4,5,6,7$
4.18	$\Psi_4(x-tu-\arctan(\frac{z-tw}{y-tv}), \sqrt{(y-tv)^2+(z-tw)^2}, t, f)$	$4,5,6,1+7$
4.19	$\Psi_4(x-tu-\alpha\arctan(\frac{z-v-tw}{w+y-tv}), \sqrt{(w+y-tv)^2+(z-v-tw)^2}, t, f)$	$4,3+5,2-6,\alpha 1+7$
4.20	$\Psi_4(u-\alpha\arctan(\frac{z-v-tw}{w+y-tv}), \sqrt{(w+y-tv)^2+(z-v-tw)^2}, t, f)$	$1,3+5,2-6,\alpha 4+7$
4.21	$\Psi_4(x-tu+\arctan(\frac{v}{w}), \sqrt{v^2+w^2}, t, f)$	$2,3,4,1+7$
4.22	$\Psi_4(u-\beta t, t+\arctan(\frac{v}{w}), \sqrt{v^2+w^2}, f)$	$1,2,3,\beta 4+7+10$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
4.23	$y^{-2}\Psi_4(\frac{z}{y}, v, w, fy)$	$1, 4, 10, 11$
4.24	$x^{-2}\Psi_4(u, v, w - \alpha \ln x, fx)$	$2, 3, 10, \alpha 6 + 11,\ \alpha \neq 0$
4.25	$x^{-2}\Psi_4(u, v, w, fx)$	$2, 3, 10, 11$
4.26	$t^{-2}\Psi_4(u - \frac{x}{t}, v - \frac{y}{t}, w - \frac{z}{t}, ft)$	$4, 5, 6, 11$
4.27	$t^{-2}\Psi_4(\frac{y-tv}{t}, \frac{z-tw}{t}, t^{-\beta}e^{u-\alpha v}, ft)$	$1, \alpha 4 + 5, 6, \beta 4 + 11,\ \alpha \neq 0$
4.28	$t^{-2}\Psi_4(u - \beta \ln t, v - \frac{y}{t}, w - \frac{z}{t}, ft)$	$1, 5, 6, \beta 4 + 11$
4.29	$t^{-2}\Psi_4(\frac{y}{t} - \alpha \ln t, v - \alpha \ln t, w - \frac{z}{t}, ft)$	$1, 4, 6, \alpha 5 + 11$
4.30	$t^{-2}\Psi_4(t^{-\beta}e^{\frac{x-\alpha tw}{t}}, t^{-\beta}e^{u-\alpha w}, t^{-\sigma}e^{v}, ft)$	$2, 3, \alpha 4 + 6, \beta 4 + \sigma 5 + 11$
4.31	$t^{-2}\Psi_4(u - \frac{x}{t}, v - \alpha \ln t, w - \beta \ln t, ft)$	$2, 3, 4, \alpha 5 + \beta 6 + 11,\ \alpha^2 + \beta^2 \neq 0$
4.32	$t^{-2}\Psi_4(u - \frac{x}{t}, v, w, ft)$	$2, 3, 4, 11$
4.33	$t^{-2}\Psi_4(u - \beta \ln t, v, w, ft)$	$1, 2, 3, \beta 4 + 11,\ \beta \neq 0$
4.34	$t^{-2}\Psi_4(u, v, w, ft)$	$1, 2, 3, 11$
4.35	$\Psi_4(t^2 - 2x + 2\alpha v, u - t, w - \beta t, f)$	$2, 3, \alpha 1 + 5, 4 + \beta 6 + 10$
4.36	$\Psi_4(x - \alpha v, u, w - t, f)$	$2, 3, \alpha 1 + 5, 6 + 10$
4.37	$\Psi_4(u, v - x, w, f)$	$2, 3, 1 + 5, 10$
4.38	$\Psi_4(x, u, w, f)$	$2, 3, 5, 10$
4.39	$\Psi_4(u - t, v, w, f)$	$1, 2, 3, 4 + 10$
4.40	$\Psi_4(u, v, w, f)$	$1, 2, 3, 10$
4.41	$\Psi_4(y - \sigma u - tv - \beta w, z - \tau u - \alpha v - tw, t, f)$	$1, \sigma 2 + \tau 3 + 4, \alpha 3 + 5, \beta 2 + 6,$ $\sigma^2 + \tau^2 + (\alpha + \beta)^2 = 1$
4.42	$\Psi_4(y - tv + w, y - tz + w(t^2 + 1), t, f)$	$1, 4, 3 + 5, 2 - 6$
4.43	$\Psi_4(v - \frac{y}{t}, w - \frac{z}{t}, t, f)$	$1, 4, 5, 6$
4.44	$\Psi_4(v - x + \alpha(z - tw), u, t, f)$	$2, \alpha 1 + 3, 1 + 5, 6,\ \alpha \neq 0$
4.45	$\Psi_4(u, v - x, t, f)$	$2, 3, 1 + 5, 6$
4.46	$\Psi_4(t, u, w + \frac{\beta x - z}{t}, f)$	$1 + \beta 3, 2, 5, 6$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
4.47	$\Psi_4(x,u,t,f)$	$2,3,5,6$
4.48	$\Psi_4(v-z+tw,u,t,f)$	$1,2,3+5,6$
4.49	$\Psi_4(u,w-\frac{z}{t},t,f)$	$1,2,5,6$
4.50	$\Psi_4(v,w,t,f)$	$1,2,3,4$
5.1^s	$r^{-2}\Psi_3(U,V^2+W^2,fr)$	$7,8,9,10,11$
5.2^c	$r^{-2}\Psi_3(V,W,fr)$	$1,4,7,10,11$
5.3	$x^{-2}\Psi_3(v^2+w^2,u,fx)$	$2,3,7,10,11$
5.4	$e^{2\alpha\arctan(\frac{v}{w})}\Psi_3(u+\beta\arctan(\frac{v}{w}),\sqrt{v^2+w^2},fe^{-\alpha\arctan(\frac{v}{w})})$	$1,2,3,10,\beta4+7+\alpha11$
5.5	$t^{-2}\Psi_3(\frac{x-tu}{t},\frac{(y-tv)^2+(z-tw)^2}{t^2},ft)$	$4,5,6,7,11$
5.6	$t^{-2}\Psi_3(\frac{x-tu}{t},v^2+w^2,ft)$	$2,3,4,7,11$
5.7	$t^{-2}\Psi_3(t^\beta e^{\alpha\arctan(\frac{z-tw}{y-tv})-u},\sqrt{(\frac{y-tv}{t})^2+(\frac{z-tw}{t})^2},ft)$	$1,5,6,\alpha4+7,\beta4+11$
5.8	$t^{-2}\Psi_3(t^{-\beta}e^{u+\alpha\arctan(\frac{v}{w})},\sqrt{v^2+w^2},ft)$	$1,2,3,\alpha4+7,\beta4+11$
5.9	$e^{-2\alpha\arctan(\frac{z-tw}{y-tv})}\Psi_3(\sqrt{(y-tv)^2+(z-tw)^2}e^{-\alpha\arctan(\frac{z-tw}{y-tv})},te^{-\alpha\arctan(\frac{z-tw}{y-tv})},$ $fe^{\alpha\arctan(\frac{z-tw}{y-tv})})$	$1,4,5,6,7+\alpha11$
5.10	$t^{-2}\Psi_3(\frac{x}{t}-\frac{\beta}{\alpha}\ln t,u-\frac{\beta}{\alpha}\ln t,ft)$	$2,3,5,6,\beta4+7+\alpha11,\ \alpha\neq 0$
5.11	$t^{-2}\Psi_3(te^{\alpha\arctan(\frac{v}{w})},\sqrt{v^2+w^2},ft)$	$1,2,3,4,7+\alpha11,\ \alpha\neq 0$
5.12	$\Psi_3(t-u-\alpha\arctan(\frac{v}{w}),\sqrt{v^2+w^2},f)$	$1,2,3,\alpha4+7,4+10$
5.13	$\Psi_3(u-\frac{x}{t},t,f)$	$2,3,5,6,\beta4+7,\ \beta\neq 0$
5.14	(4.47)	$2,3,5,6,7$
5.15	$\Psi_3(v^2+w^2,t,f)$	$1,2,3,4,7$
5.16	$\Psi_3((w+y-tv)^2+(v-z+tw)^2,t,f)$	$1,4,3+5,2-6,7$
5.17	$\Psi_3(u,t,f)$	$2,3,5,6,1+7$
5.18	$\Psi_3(t^2-2x,u-t,f)$	$2,3,5,6,\beta4+7+\beta10,\ \beta\neq 0$
5.19	$\Psi_3(x,u,f)$	$2,3,5,6,7+10$
5.20	$\Psi_3(t+\arctan(\frac{v}{w}),\sqrt{v^2+w^2},f)$	$1,2,3,4,7+10$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
5.21	$x^{-2}\Psi_3(u, w-\beta\ln x, fx)$	$2,3,5,10,\beta 6+11$
5.22	$e^{-2\beta u}\Psi_3(v, w, fe^{\beta u})$	$1,2,3,10,4+\beta 11$
5.23	$f^2\Psi_3(u, v, w)$	$1,2,3,10,11$
5.24	$t^{-2}\Psi_3(v-\frac{y}{t}, w-\frac{z}{t}, ft)$	$1,4,5,6,11$
5.25	$t^{-2}\Psi_3(u-\frac{x}{t}, v-\frac{x}{\alpha t}+\frac{\beta}{\alpha}\ln t, ft)$	$2,3,\alpha 4+5,6,\beta 4+11,\ \alpha\neq 0$
5.26	$t^{-2}\Psi_3(\frac{x}{t}-\beta\ln t, u-\beta\ln t, ft)$	$2,3,5,6,\beta 4+11$
5.27	$t^{-2}\Psi_3(u-\frac{x}{t}, v-\beta\ln t, ft)$	$2,3,4,6,\beta 5+11$
5.28	$t^{-2}\Psi_3(u-\beta\ln t, v, ft)$	$1,2,3,6,\beta 4+11,\ \beta\neq 0$
5.29	$t^{-2}\Psi_3(v, w, ft)$	$1,2,3,4,11$
5.30	$\Psi_3(2(x-\alpha v)-t^2, u-t, f)$	$2,3,\alpha 1+5,6,4+10,\ \alpha\neq 0$
5.31	$\Psi_3(t^2-2x, u-t, f)$	$2,3,5,6,4+10$
5.32	$\Psi_3(u, v-x, f)$	$2,3,1+5,6,10$
5.33	$\Psi_3(x, u, f)$	$2,3,5,6,10$
5.34	$\Psi_3(u-t, v, f)$	$1,2,3,6,4+10$
5.35	$\Psi_3(u-\frac{x}{t}, t, f)$	$2,3,4,5,6$
5.36	$\Psi_3(t, tu-x+w, f)$	$2,3,4,5,1+6$
5.37	$\Psi_3(u, t, f)$	$1,2,3,5,6$
6.1	$f^2\Psi_2(v^2+w^2, u)$	$1,2,3,7,10,11$
6.2	$x^{-2}\Psi_2(u, fx)$	$2,3,5,6,10,7+\alpha 11,\ \alpha\neq 0$
6.3	$e^{2\alpha\arctan(\frac{v}{w})}\Psi_2(\sqrt{v^2+w^2}, fe^{-\alpha\arctan(\frac{v}{w})})$	$1,2,3,4,10,7+\alpha 11$
6.4	$t^{-2}\Psi_2((v-\frac{y}{t})^2+(w-\frac{z}{t})^2, ft)$	$1,4,5,6,7,11$
6.5	$t^{-2}\Psi_2(v^2+w^2, ft)$	$1,2,3,4,7,11$
6.6	$t^{-2}\Psi_2(u-\frac{x}{t}, ft),\ \alpha\neq 0$	$2,3,5,6,\alpha 4+7,\beta 4+11$
	$(5.26),\ \alpha=0$	
6.7	$t^{-2}\Psi_2(\frac{x-tu}{t}, ft)$	$2,3,4,5,6,7+\alpha 11,\ \alpha\neq 0$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
6.8	$e^{-\frac{2\alpha u}{\beta}}\Psi_2(te^{-\frac{\alpha u}{\beta}}, fe^{\frac{\alpha u}{\beta}}),\ \beta \neq 0$	$1, 2, 3, 5, 6, \beta 4 + 7 + \alpha 11$
	$t^{-2}\Psi_2(u, ft),\ \beta = 0, \alpha \neq 0$	
	$(5.37),\ \beta = 0, \alpha = 0$	
6.9	$\Psi_3((u - \frac{x}{t})^2 + (v - \frac{y}{t})^2 + (w - \frac{z}{t})^2, t, f)$	$4, 5, 6, 7, 8, 9$
6.10	$\Psi_3(\sqrt{u^2 + v^2 + w^2}, t, f)$	$1, 2, 3, 7, 8, 9$
6.11	$\Psi_2(u, f)$	$2, 3, 5, 6, 1 + 7, 10$
6.12	$\Psi_2(u - t, f),\ \alpha \neq 0$	$2, 3, 5, 6, \alpha 1 + 7, 4 + 10$
	$(5.31),\ \alpha = 0$	
6.13	(5.33)	$2, 3, 5, 6, 7, 10$
6.14	$\Psi_2(t, f)$	$2, 3, 4, 5, 6, 1 + 7$
6.15	(5.35)	$2, 3, 4, 5, 6, 7$
6.16	$\Psi_2(u, f)$	$1, 2, 3, 5, 6, 7 + 10$
6.17	$x^{-2}\Psi_2(u, fx)$	$2, 3, 5, 6, 10, 11$
6.18	$e^{-\frac{2w}{\alpha}}\Psi_2(v, fe^{\frac{w}{\alpha}})$	$1, 2, 3, 4, 10, \alpha 6 + 11,\ \alpha \neq 0$
6.19	$f^2\Psi_2(v, w)$	$1, 2, 3, 4, 10, 11$
6.20	$t^{-2}\Psi_2(u - \alpha \ln t, ft)$	$1, 2, 3, 5, 6, \alpha 4 + 11$
6.21	$t^{-2}\Psi_2(\frac{x - tu}{t}, ft)$	$2, 3, 4, 5, 6, 11$
6.22	$\Psi_2(u, f)$	$1, 2, 3, 5, 6, 10$
6.23	$\Psi_2(u - t, f)$	$1, 2, 3, 5, 6, 4 + 10$
6.24	$\Psi_2(t, f)$	$1, 2, 3, 4, 5, 6$
6.25	$\Psi_2(t, f),\ \beta \neq 0$	$1, 2, 3, 5, 6, \beta 4 + 7$
	$(5.37),\ \beta = 0$	
7.1	$t^{-2}\Psi_2(u^2 + v^2 + w^2, ft)$	$1, 2, 3, 7, 8, 9, 11$
7.2	$t^{-2}\Psi_2((u - \frac{x}{t})^2 + (v - \frac{y}{t})^2 + (w - \frac{z}{t})^2, ft)$	$4, 5, 6, 7, 8, 9, 11$
7.3	(6.17)	$2, 3, 5, 6, 7, 10, 11$
7.4	$f^2\Psi_1(v^2 + w^2)$	$1, 2, 3, 4, 7, 10, 11$

Table D.1: Group classification (Continued).

No.	Source function q	Subalgebra
7.5	$e^{-\frac{2\alpha u}{\beta}}\Psi_1(fe^{\frac{\alpha u}{\beta}}),\ \beta \neq 0$	$1, 2, 3, 5, 6, 10, \beta 4 + 7 + \alpha 11$
	$f^2\Psi_1(u),\ \beta = 0, \alpha \neq 0$	
	(6.22), $\beta = 0, \alpha = 0$	
7.6	(6.21)	$2, 3, 4, 5, 6, 7, 11$
7.7	$t^{-2}\Psi_1(ft),\ \alpha \neq 0$	$1, 2, 3, 5, 6, \alpha 4 + 7, \beta 4 + 11$
	(6.20), $\alpha = 0$	
7.8	$t^{-2}\Psi_1(ft)$	$1, 2, 3, 4, 5, 6, 7 + \alpha 11,\ \alpha \neq 0$
7.9	$\Psi_2(u^2 + v^2 + w^2, f)$	$1, 2, 3, 7, 8, 9, 10$
7.10	$\Psi_1(f),\ \alpha \neq 0$	$1, 2, 3, 5, 6, \alpha 4 + 7, 4 + 10$
	(6.23), $\alpha = 0$	
7.11	$\Psi_1(f)$	$1, 2, 3, 4, 5, 6, 7 + 10$
7.12	$f^2\Psi_1(f^\alpha e^u)$	$1, 2, 3, 5, 6, 10, \alpha 4 + 11$
7.13	$t^{-2}\Psi_1(ft)$	$1, 2, 3, 4, 5, 6, 11$
7.14	$\Psi_1(f)$	$1, 2, 3, 4, 5, 6, 10$
8.1	$f^2\Psi_1(u^2 + v^2 + w^2)$	$1, 2, 3, 7, 8, 9, 10, 11$
8.2	$Cf^2,\ \alpha \neq 0$	$1, 2, 3, 5, 6, \alpha 4 + 7, 10, \beta 4 + 11$
	(7.12), $\alpha = 0$	
8.3	$Cf^2,\ \alpha \neq 0$	$1, 2, 3, 4, 5, 6, 7 + \alpha 11, 10$
	(7.14), $\alpha = 0$	
8.4	(7.13)	$1, 2, 3, 4, 5, 6, 7, 11$
8.5	Cf^2	$1, 2, 3, 4, 5, 6, 10, 11$
9.1	(8.5)	$1, 2, 3, 4, 5, 6, 7, 10, 11$
9.2	(6.24)	$1, 2, 3, 4, 5, 6, 7, 8, 9$
10.1	(7.13)	$1, 2, 3, 4, 5, 6, 7, 8, 9, 11$
10.2	(7.14)	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$
11.1	(8.5)	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

APPENDIX E

REPRESENTATIONS OF INVARIANT SOLUTIONS

Complete results of representations of invariant solutions with respect to subalgebras of the Lie algebra L_{11} are presented in this Appendix. Numbers in the first column of Table E.1 correspond to the numbers in the first column of Table C.1. The superscripts ^{c}, and ^{s} which are next to this subalgebra number in the first column indicate that the representation of invariant solution is presented in the cylindrical coordinate system, or the spherical coordinate system, respectively. Here Ω_k is an arbitrary function of k independent variables, and C is constant.

Table E.1: Representations of invariant solutions.

No.	Representation of invariant solution f of Equation (1.22)
1.1^c	$t^{-1}\Omega_6(\frac{x}{t} - \frac{\beta}{\alpha}\ln t, \frac{r}{t}, \theta - \frac{1}{\alpha}\ln t, u - \frac{\beta}{\alpha}\ln t, V, W)$
1.2^c	$\Omega_6(t, r, \beta\theta - \frac{x}{t}, u - \frac{x}{t}, V, W)$
1.3^c	$\Omega_6(t, x, r, u, V, W)$
1.4^c	$\Omega_6(t, r, x - \theta, u, V, W)$
1.5^c	$\Omega_6(t^2 - 2x, r, t - \beta\theta, u - t, V, W)$
1.6^c	$\Omega_6(x, r, t - \theta, u, V, W)$
1.7	$t^{-1}\Omega_6(\frac{y}{t}, \frac{z}{t}, \frac{x}{t} - \beta\ln t, u - \beta\ln t, v, w)$
1.8	$t^{-1}\Omega_6(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, u, v, w)$
1.9	$\Omega_6(t^2 - 2x, y, z, u - t, v, w)$
1.10	$\Omega_6(x, y, z, u, v, w)$
1.11	$\Omega_6(t, x - tz, y, u - z, v, w)$
1.12	$\Omega_6(t, y, z, u - \frac{x}{t}, v, w)$
1.13	$\Omega_6(t, y, z, u, v, w)$
2.1^c	$x^{-1}\Omega_5(\frac{r}{x}, \alpha\theta - \ln x, u, V, W)$
2.2^c	$t^{-1}\Omega_5(\frac{r}{t}, \frac{x}{t} - \alpha\theta - \beta\ln t, u - \alpha\theta - \beta\ln t, V, W)$
2.3^c	$t^{-1}\Omega_5(\frac{r}{t}, \alpha\theta - \ln t, u - \frac{x}{t}, V, W)$
2.4^c	$t^{-1}\Omega_5(\frac{r}{t}, \alpha\theta - \ln t, u - \frac{\beta}{\alpha}\ln t, V, W)$

Table E.1: Representations of invariant solutions (Continued).

No.	Representation of invariant solution f of Equation (1.22)
2.5^c	$\Omega_5(r, x, u, V, W)$
2.6^c	$\Omega_5(r, x-\theta, u, V, W)$
2.7^c	$\Omega_5(r, 2(x-\alpha\theta)-t^2, u-t, V, W)$
2.8^c	$\Omega_5(t, r, u-\frac{x}{t}, V, W)$
2.9^c	$\Omega_5(t, r, u-\beta\theta, V, W)$
2.10^c	$\Omega_5(t, r, u-\frac{x}{t}+\frac{\theta}{t}, V, W)$
2.11^c	$\Omega_5(r, \theta-t, u-\beta t, V, W)$
2.12	$x^{-1}\Omega_5(\frac{y}{x}, \frac{z}{x}, u, v, w)$
2.13	$t^{-1}\Omega_5(\frac{y}{t}, \frac{z}{t}, u-\frac{x}{t}, v, w)$
2.14	$t^{-1}\Omega_5(\frac{y}{t}-\alpha\ln t, \frac{z}{t}, u-\frac{x}{t}, v-\alpha\ln t, w)$
2.15	$t^{-1}\Omega_5(\frac{z}{t}, \frac{y}{t}-\alpha\ln t, u-\beta\ln t, v-\alpha\ln t, w)$
2.16	$t^{-1}\Omega_5(\frac{y}{t}, \frac{z}{t}, u-\beta\ln t, v, w)$
2.17	$\Omega_5(y, z, u, v, w)$
2.18	$\Omega_5(2x-t^2, y, u-t, v, w-\alpha t)$
2.19	$\Omega_5(y, z, u-t, v, w)$
2.20	$\Omega_5(t, \alpha(ty-\tau z)+\beta(tz-\sigma y)+x(\sigma\tau-t^2), u, v+\frac{\beta y-\tau x}{\alpha\tau-\beta t}, w+\frac{tx-\alpha y}{\alpha\tau-\beta t})$
2.21	$\Omega_5(t, x, u, v-\frac{z+ty}{t^2+1}, w+\frac{y-tz}{t^2+1})$
2.22	$\Omega_5(t, x, u, v-\frac{y}{t}, w-\frac{z}{t})$
2.23	$\Omega_5(t, x-\alpha y-tz, u-z, v, w)$
2.24	$\Omega_5(t, z, u+\frac{\alpha y-x}{t}, v, w)$
2.25	$\Omega_5(t, y, u-z, v, w)$
2.26	$\Omega_5(t, y, z, v, w)$
2.27	$\Omega_5(t, x, u, v, w)$
3.1^c	$x^{-1}\Omega_4(\frac{r}{x}, u, V, W)$
3.2^c	$e^{-\alpha\theta}\Omega_4(re^{-\alpha\theta}, u-\beta\theta, V, W)$
3.3^c	$t^{-1}\Omega_4(\frac{r}{t}, u-\frac{x}{t}, V, W)$
3.4^c	$t^{-1}\Omega_4(\frac{r}{t}, u-\alpha\theta-\beta\ln t, V, W)$
3.5	$t^{-1}\Omega_4(\frac{x}{t}-\frac{\beta}{\alpha}\ln t, u-\frac{\beta}{\alpha}\ln t, \arctan(\frac{w-\frac{z}{t}}{v-\frac{y}{t}})-\frac{1}{\alpha}\ln t, \sqrt{(v-\frac{y}{t})^2+(w-\frac{z}{t})^2})$
3.6^c	$t^{-1}\Omega_4(\frac{r}{t}, \alpha\theta-\ln t, V, W)$
3.7	$t^{-1}\Omega_4(\frac{x}{t}-\frac{\beta}{\alpha}\ln t, u-\frac{\beta}{\alpha}\ln t, \arctan(\frac{w}{v})-\frac{1}{\alpha}\ln t, \sqrt{v^2+w^2})$
3.8^s	$\Omega_4(t, r, U, V^2+W^2)$
3.9^c	$\Omega_4(r, u-t-\alpha\theta, V, W)$
3.10	$\Omega_4(t, \frac{x}{t}-\beta\arctan(\frac{w-\frac{z}{t}}{v-\frac{y}{t}}), u-\beta\arctan(\frac{w-\frac{z}{t}}{v-\frac{y}{t}}), \sqrt{(v-\frac{y}{t})^2+(w-\frac{z}{t})^2})$
3.11^c	$\Omega_4(t, r, V, W)$
3.12	$\Omega_4(t, u-\frac{x}{t}, \arctan(\frac{w}{v})-\frac{x}{\beta t}, \sqrt{v^2+w^2})$

Table E.1: Representations of invariant solutions (Continued).

No.	Representation of invariant solution f of Equation (1.22)
3.13	$\Omega_4(t, x, u, \sqrt{v^2+w^2})$
3.14	$\Omega_4(t, u - \frac{\alpha x}{1+\alpha t}, \arctan(\frac{w-\frac{z}{t}}{v-\frac{y}{t}}) - \frac{x}{1+\alpha t}, \sqrt{(v-\frac{y}{t})^2+(w-\frac{z}{t})^2})$
3.15	$\Omega_4(t, u + \beta\arctan(\frac{y-tv+w}{z-tw-v}), x + (\alpha+\beta t)\arctan(\frac{y-tv+w}{z-tw-v}),$ $\sqrt{(y-tv+w)^2+(z-tw-v)^2})$
3.16	$\Omega_4(t, u, \arctan(\frac{w}{v}) - x, \sqrt{v^2+w^2})$
3.17^c	$\Omega_4(t-\theta, r, V, W)$
3.18	$\Omega_4(2x-t^2, u-t, \arctan(\frac{w}{v}) - \frac{t}{\beta}, \sqrt{v^2+w^2})$
3.19	$\Omega_4(x, u, \arctan(\frac{w}{v}) - t, \sqrt{v^2+w^2})$
3.20	$y^{-1}\Omega_4(\frac{y}{z}, u - \beta\ln y, v, w)$
3.21	$t^{-1}\Omega_4(\frac{x}{t} - \beta\ln t, u - \beta\ln t, v - \frac{y}{t}, w - \frac{z}{t})$
3.22	$t^{-1}\Omega_4(\frac{y}{t} - \sigma\ln t, u - \frac{\alpha z}{t} - \beta\ln t, v - \sigma\ln t, w - \frac{z}{t})$
3.23	$t^{-1}\Omega_4(\frac{y}{t}, \frac{z}{t} - \sigma\ln t, v, w - \sigma\ln t)$
3.24	$t^{-1}\Omega_4(\frac{y}{t}, \frac{z}{t}, v, w)$
3.25	$t^{-1}\Omega_4(\frac{x}{t} - \beta\ln t, u - \beta\ln t, v - \sigma\ln t, w)$
3.26	$t^{-1}\Omega_4(\frac{x}{t} - \beta\ln t, u - \beta\ln t, v, w)$
3.27	$\Omega_4(y - \frac{\beta x}{\alpha} + \frac{\beta t^2}{2\alpha}, u-t, v, w - \frac{x}{\alpha} + \frac{t^2}{2\alpha}),\ \alpha \neq 0$
	$\Omega_4(t^2-2x, u-t, v, w - \frac{y}{\beta}),\ \alpha = 0,\ \beta \neq 0$
	$\Omega_4(t^2-2x, y, u-t, v),\ \alpha = 0,\ \beta = 0$
3.28	$\Omega_4(z, u-y, v, w)$
3.29	$\Omega_4(y, z, v, w)$
3.30	$\Omega_4(t^2-2x, u-t, v, w-\sigma t)$
3.31	$\Omega_4(t^2-2x, u-t, v, w)$
3.32	$\Omega_4(x, u, v, w-t)$
3.33	$\Omega_4(x, u, v, w)$
3.34	$\Omega_4(t, x - tu - \delta v + \beta w, y - tv + \delta u - \sigma v - \alpha w, z - tw - \beta u + \alpha v - \tau w)$
3.35	$\Omega_4(u - \frac{x}{t}, v - \frac{y}{t} - \frac{tz-y}{t(t^2+1)}, w - \frac{tz-y}{t^2+1}, t)$
3.36	$\Omega_4(u - \frac{x}{t+1}, v - \frac{y}{t}, w - \frac{z}{t}, t)$
3.37	$\Omega_4(u - \frac{x}{t}, v - \frac{y}{t}, w - \frac{z}{t}, t)$
3.38	$\Omega_4(u, v - \frac{y}{t} + \frac{\tau(x-\alpha z-\frac{\beta y}{t})}{\alpha t^2-\sigma t+\tau\beta}, w + \frac{t(x-\alpha z-\frac{\beta y}{t})}{\alpha t^2-\sigma t+\tau\beta}, t)$
3.39	$\Omega_4(u, v - \frac{y}{t}, w + \frac{x-\alpha z}{\alpha t}, t), \alpha \neq 0$
	$\Omega_4(x, u, v - \frac{y}{t}, t),\ \alpha = 0$
3.40	$\Omega_4(t, w(t^2-\tau) + y - tz, \tau w - y + tv, u)$
3.41	$\Omega_4(t, w(t^2+1) + y - tz, w + y - tv, u)$
3.42	$\Omega_4(t, u, v - \frac{y}{t}, w - \frac{z}{t})$
3.43	$\Omega_4(t, tu - x + \beta z, v, w)$
3.44	$\Omega_4(t, u - \frac{x}{t}, v, w)$

Table E.1: Representations of invariant solutions (Continued).

No.	Representation of invariant solution f of Equation (1.22)
3.45	$\Omega_4(t, u-z, v, w)$
3.46	$\Omega_4(t, z, v, w)$
3.47	$\Omega_4(t, u, v, w)$
4.1^s	$r^{-1}\Omega_3(\frac{t}{r}, U, V^2+W^2)$
4.2^c	$r^{-1}\Omega_3(u-\alpha\theta, V, W)$
4.3	$e^{\alpha\arctan(\frac{v}{w})}\Omega_3(xe^{\alpha\arctan(\frac{v}{w})}, \sqrt{v^2+w^2}, u)$
4.4^c	$r^{-1}\Omega_3(\alpha\theta-\ln r, V, W)$
4.5	$t^{-1}\Omega_3(\frac{\sqrt{(y-tv)^2+(z-tw)^2}}{t}, t^{-\beta}e^{\frac{x-\alpha t\arctan(\frac{z-tw}{y-tv})}{t}}, t^{-\beta}e^{u-\alpha\arctan(\frac{z-tw}{y-tv})})$
4.6^c	$r^{-1}\Omega_3(\frac{t}{r}, V, W)$
4.7	$t^{-1}\Omega_3(t^{-\beta}e^{\frac{x+\alpha t\arctan(\frac{v}{w})}{t}}, t^{-\beta}e^{u+\alpha\arctan(\frac{v}{w})}, \sqrt{v^2+w^2})$
4.8	$t^{-1}\Omega_3(\frac{x-tu}{t}, \frac{\sqrt{(y-tv)^2+(z-tw)^2}}{t}, te^{-\alpha\arctan(\frac{z-tw}{y-tv})})$
4.9	$e^{-\alpha\arctan(\frac{z-tw}{y-tv})}\Omega_3(\sqrt{(y-tv)^2+(z-tw)^2}e^{-\alpha\arctan(\frac{z-tw}{y-tv})}, te^{-\alpha\arctan(\frac{z-tw}{y-tv})},$ $u-\beta\arctan(\frac{z-tw}{y-tv}))$
4.10	$e^{\alpha\arctan(\frac{v}{w})}\Omega_3((x-tu)e^{\alpha\arctan(\frac{v}{w})}, te^{\alpha\arctan(\frac{v}{w})}, \sqrt{v^2+w^2})$
4.11	$t^{-1}\Omega_3(te^{\alpha\arctan(\frac{v}{w})}, t^{-\beta}e^{\alpha u}, \sqrt{v^2+w^2})$
4.12	$\Omega_3(u+\beta\arctan(\frac{v}{w}), \sqrt{v^2+w^2}, t)$
4.13^s	$\Omega_3(r, U, V^2+W^2)$
4.14	$\Omega_3(x, u, v^2+w^2)$
4.15	$\Omega_3(x+\arctan(\frac{v}{w}), \sqrt{v^2+w^2}, u)$
4.16	$\Omega_3(2\alpha\arctan(\frac{v}{w})+2x-t^2, \sqrt{v^2+w^2}, u-t)$
4.17	$\Omega_3(u-\frac{x}{t}, (v-\frac{y}{t})^2+(w-\frac{z}{t})^2, t)$
4.18	$\Omega_3(x-tu-\arctan(\frac{z-tw}{y-tv}), \sqrt{(y-tv)^2+(z-tw)^2}, t)$
4.19	$\Omega_3(x-tu-\alpha\arctan(\frac{z-v-tw}{w+y-tv}), \sqrt{(w+y-tv)^2+(z-v-tw)^2}, t)$
4.20	$\Omega_3(u-\alpha\arctan(\frac{z-v-tw}{w+y-tv}), \sqrt{(w+y-tv)^2+(z-v-tw)^2}, t)$
4.21	$\Omega_3(x-tu+\arctan(\frac{v}{w}), \sqrt{v^2+w^2}, t)$
4.22	$\Omega_3(u-\beta t, t+\arctan(\frac{v}{w}), \sqrt{v^2+w^2})$
4.23	$y^{-1}\Omega_3(\frac{z}{y}, v, w)$
4.24	$x^{-1}\Omega_3(u, v, w-\alpha\ln x)$
4.25	$x^{-1}\Omega_3(u, v, w)$
4.26	$t^{-1}\Omega_3(u-\frac{x}{t}, v-\frac{y}{t}, w-\frac{z}{t})$
4.27	$t^{-1}\Omega_3(\frac{y-tv}{t}, \frac{z-tw}{t}, t^{-\beta}e^{u-\alpha v})$
4.28	$t^{-1}\Omega_3(u-\beta\ln t, v-\frac{y}{t}, w-\frac{z}{t})$

Table E.1: Representations of invariant solutions (Continued).

No.	Representation of invariant solution f of Equation (1.22)
4.29	$t^{-1}\Omega_3(\frac{y}{t}-\alpha\ln t, v-\alpha\ln t, w-\frac{z}{t})$
4.30	$t^{-1}\Omega_3(t^{-\beta}e^{\frac{x-\alpha tw}{t}}, t^{-\beta}e^{u-\alpha w}, t^{-\sigma}e^{v})$
4.31	$t^{-1}\Omega_3(u-\frac{x}{t}, v-\alpha\ln t, w-\beta\ln t)$
4.32	$t^{-1}\Omega_3(u-\frac{x}{t}, v, w)$
4.33	$t^{-1}\Omega_3(u-\beta\ln t, v, w)$
4.34	$t^{-1}\Omega_3(u, v, w)$
4.35	$\Omega_3(u-t, v-\frac{x}{\alpha}+\frac{t^2}{2\alpha}, w-\beta t),\ \alpha\neq 0$
	$\Omega_3(t^2-2x, u-t, w-\beta t),\ \alpha=0$
4.36	$\Omega_3(u, v-\frac{x}{\alpha}, w-t),\ \alpha\neq 0$
	$\Omega_3(x, u, w-t),\ \alpha=0$
4.37	$\Omega_3(u, v-x, w)$
4.38	$\Omega_3(x, u, w)$
4.39	$\Omega_3(u-t, v, w)$
4.40	$\Omega_3(u, v, w)$
4.41	$\Omega_3(y-\sigma u-tv-\beta w, z-\tau u-\alpha v-tw, t)$
4.42	$\Omega_3(y-tv+w, y-tz+w(t^2+1), t)$
4.43	$\Omega_3(v-\frac{y}{t}, w-\frac{z}{t}, t)$
4.44	$\Omega_3(v-x+\alpha(z-tw), u, t)$
4.45	$\Omega_3(u, v-x, t)$
4.46	$\Omega_3(t, u, w+\frac{\beta x-z}{t})$
4.47	$\Omega_3(x, u, t)$
4.48	$\Omega_3(v-z+tw, u, t)$
4.49	$\Omega_3(u, w-\frac{z}{t}, t)$
4.50	$\Omega_3(v, w, t)$
5.1^{s}	$r^{-1}\Omega_2(U, V^2+W^2)$
5.2^{c}	$r^{-1}\Omega_2(V, W)$
5.3	$x^{-1}\Omega_2(v^2+w^2, u)$
5.4	$e^{\alpha\arctan(\frac{v}{w})}\Omega_2(u+\beta\arctan(\frac{v}{w}), \sqrt{v^2+w^2})$
5.5	$t^{-1}\Omega_2(\frac{x-tu}{t}, \frac{(y-tv)^2+(z-tw)^2}{t^2})$
5.6	$t^{-1}\Omega_2(\frac{x-tu}{t}, v^2+w^2)$
5.7	$t^{-1}\Omega_2(t^{\beta}e^{\alpha\arctan(\frac{z-tw}{y-tv})-u}, \sqrt{(\frac{y-tv}{t})^2+(\frac{z-tw}{t})^2})$
5.8	$t^{-1}\Omega_2(t^{-\beta}e^{u+\alpha\arctan(\frac{v}{w})}, \sqrt{v^2+w^2})$
5.9	$e^{-\alpha\arctan(\frac{z-tw}{y-tv})}\Omega_2(\sqrt{(y-tv)^2+(z-tw)^2}e^{-\alpha\arctan(\frac{z-tw}{y-tv})}, te^{-\alpha\arctan(\frac{z-tw}{y-tv})})$
5.10	$t^{-1}\Omega_2(\frac{x}{t}-\frac{\beta}{\alpha}\ln t, u-\frac{\beta}{\alpha}\ln t)$

Table E.1: Representations of invariant solutions (Continued).

No.	Representation of invariant solution f of Equation (1.22)
5.11	$t^{-1}\Omega_2(te^{\alpha\arctan(\frac{v}{w})}, \sqrt{v^2+w^2})$
5.12	$\Omega_2(t-u-\alpha\arctan(\frac{v}{w}), \sqrt{v^2+w^2})$
5.13	$\Omega_2(u-\frac{x}{t}, t)$
5.14	(4.47)
5.15	$\Omega_2(v^2+w^2, t)$
5.16	$\Omega_2((w+y-tv)^2+(v-z+tw)^2, t)$
5.17	$\Omega_2(u, t)$
5.18	$\Omega_2(t^2-2x, u-t)$
5.19	$\Omega_2(x, u)$
5.20	$\Omega_2(t+\arctan(\frac{v}{w}), \sqrt{v^2+w^2})$
5.21	$x^{-1}\Omega_2(u, w-\beta\ln x)$
5.22	$e^{-\beta u}\Omega_2(v, w)$
5.23	*None*
5.24	$t^{-1}\Omega_2(v-\frac{y}{t}, w-\frac{z}{t})$
5.25	$t^{-1}\Omega_2(u-\frac{x}{t}, v-\frac{x}{\alpha t}+\frac{\beta}{\alpha}\ln t)$
5.26	$t^{-1}\Omega_2(\frac{x}{t}-\beta\ln t, u-\beta\ln t)$
5.27	$t^{-1}\Omega_2(u-\frac{x}{t}, v-\beta\ln t)$
5.28	$t^{-1}\Omega_2(u-\beta\ln t, v)$
5.29	$t^{-1}\Omega_2(v, w)$
5.30	$\Omega_2(2(x-\alpha v)-t^2, u-t)$
5.31	$\Omega_2(t^2-2x, u-t)$
5.32	$\Omega_2(u, v-x)$
5.33	$\Omega_2(x, u)$
5.34	$\Omega_2(u-t, v)$
5.35	$\Omega_2(u-\frac{x}{t}, t)$
5.36	$\Omega_2(t, tu-x+w)$
5.37	$\Omega_2(u, t)$
6.1	*None*
6.2	$x^{-1}\Omega_1(u)$
6.3	$e^{\alpha\arctan(\frac{v}{w})}\Omega_1(\sqrt{v^2+w^2})$
6.4	$t^{-1}\Omega_1((v-\frac{y}{t})^2+(w-\frac{z}{t})^2)$
6.5	$t^{-1}\Omega_1(v^2+w^2)$
6.6	$t^{-1}\Omega_1(u-\frac{x}{t}),\ \alpha\neq 0$
	(5.26), $\alpha = 0$
6.7	$t^{-1}\Omega_1(u-\frac{x}{t})$

Table E.1: Representations of invariant solutions (Continued).

No.	Representation of invariant solution f of Equation (1.22)
6.8	$e^{-\frac{\alpha u}{\beta}}\Omega_1(te^{-\frac{\alpha u}{\beta}}),\ \beta \neq 0$
	$t^{-1}\Omega_1(u),\ \beta = 0,\ \alpha \neq 0$
	$(5.37),\ \beta = 0,\ \alpha = 0$
6.9	$\Omega_2(\sqrt{(u-\frac{x}{t})^2+(v-\frac{y}{t})^2+(w-\frac{z}{t})^2}, t)$
6.10	$\Omega_2(\sqrt{u^2+v^2+w^2}, t)$
6.11	$\Omega_1(u)$
6.12	$\Omega_1(u-t),\ \alpha \neq 0$
	$(5.31),\ \alpha = 0$
6.13	(5.33)
6.14	$\Omega_1(t)$
6.15	(5.35)
6.16	$\Omega_1(u)$
6.17	$x^{-1}\Omega_1(u)$
6.18	$e^{-\frac{w}{\alpha}}\Omega_1(v)$
6.19	*None*
6.20	$t^{-1}\Omega_1(u-\alpha \ln t)$
6.21	$t^{-1}\Omega_1(u-\frac{x}{t})$
6.22	$\Omega_1(u)$
6.23	$\Omega_1(u-t)$
6.24	$\Omega_1(t)$
6.25	$\Omega_1(t),\ \beta \neq 0$
	$(5.37),\ \beta = 0$
7.1	$t^{-1}\Omega_1(\sqrt{u^2+v^2+w^2})$
7.2	$t^{-1}\Omega_1(\sqrt{(u-\frac{x}{t})^2+(v-\frac{y}{t})^2+(w-\frac{z}{t})^2})$
7.3	(6.17)
7.4	*None*
7.5	$Ce^{-\frac{\alpha u}{\beta}},\ \beta \neq 0$
	None, $\beta = 0,\ \alpha \neq 0$
	$(6.22),\ \beta = 0,\ \alpha = 0$
7.6	(6.21)
7.7	$Ct^{-1},\ \alpha \neq 0$
	$(6.20),\ \alpha = 0$
7.8	Ct^{-1}

Table E.1: Representations of invariant solutions (Continued).

No.	Representation of invariant solution f of Equation (1.22)
7.9	$\Omega_1(\sqrt{u^2+v^2+w^2})$
7.10	$C, \ \alpha \neq 0$
	$(6.23), \ \alpha = 0$
7.11	C
7.12	$Ce^{-\frac{u}{\alpha}}$
7.13	Ct^{-1}
7.14	C
8.1	$None$
8.2	$None, \ \alpha \neq 0$
	$(7.12), \ \alpha = 0$
8.3	$None, \ \alpha \neq 0$
	$(7.14), \ \alpha = 0$
8.4	(7.13)
8.5	$None$
9.1	(8.5)
9.2	(6.24)
10.1	(7.13)
10.2	(7.14)
11.1	(8.5)

APPENDIX F

THE REDUCED BOLTZMANN EQUATION WITH A SOURCE

Results of reduced Boltzmann equations for some representations of invariant solutions are presented in this Appendix. The numbers in the first column of Table F.1 correspond to the numbers in the first column of Table C.1. The superscript ^{c} is next to a subalgebra number in the first column indicates that the reduced equation is presented in the cylindrical coordinate system. Here Ω_k, Ψ_k are arbitrary functions of k independent variables, and C is constant.

Table F.1: The reduced Boltzmann equation with a source.

No.	Reduced equations
1.1^c	$-\Omega + (\tilde{u} - p_1 - \frac{\beta}{\alpha})\Omega_{p_1} + (V - p_2)\Omega_{p_2} + (\frac{W}{p_2} - \frac{1}{\alpha})\Omega_{p_3} - \frac{\beta}{\alpha}\Omega_{\tilde{u}} + \frac{W^2}{p_2}\Omega_V - \frac{VW}{p_2}\Omega_W - J(\Omega,\Omega) = \Psi_7(p_1,p_2,p_3,\tilde{u},V,W,\Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dV_1\, dW_1$
	$\Omega = \Omega_6(p_1,p_2,p_3,\tilde{u},V,W),\ \Omega^* = \Omega_6(p_1,p_2,p_3,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$
	$\Omega_1 = \Omega_6(p_1,p_2,p_3,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_6(p_1,p_2,p_3,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$
	$g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p_1 = \frac{x}{t} - \frac{\beta}{\alpha}\ln t,\ p_2 = \frac{r}{t},\ p_3 = \theta - \frac{1}{\alpha}\ln t,\ \tilde{u} = u - \frac{\beta}{\alpha}\ln t.$
1.2^c	$\Omega_t + (\frac{\beta W}{r} - \frac{\tilde{u}}{t})\Omega_p + V\Omega_r - \frac{\tilde{u}}{t}\Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_7(t,r,p,\tilde{u},V,W,\Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dV_1\, dW_1$
	$\Omega = \Omega_6(t,r,p,\tilde{u},V,W),\ \Omega^* = \Omega_6(t,r,p,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$
	$\Omega_1 = \Omega_6(t,r,p,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_6(t,r,p,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$
	$g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p = \beta\theta - \frac{x}{r},\ \tilde{u} = u - \frac{x}{t}.$
1.3^c	$\Omega_t + u\Omega_x + V\Omega_r + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_7(t,x,r,u,V,W,\Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dV_1\, dW_1$
	$\Omega = \Omega_6(t,x,r,u,V,W),\ \Omega^* = \Omega_6(t,x,r,\frac{1}{2}(u+u_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$
	$\Omega_1 = \Omega_6(t,x,r,u_1,V_1,W_1),\ \Omega_1^* = \Omega_6(t,x,r,\frac{1}{2}(u+u_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$
	$g_c = \sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2}.$
1.4^c	$\Omega_t + (u - \frac{W}{r})\Omega_p + V\Omega_r + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_7(t,r,p,u,V,W,\Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dV_1\, dW_1$
	$\Omega = \Omega_6(t,r,p,u,V,W),\ \Omega^* = \Omega_6(t,r,p,\frac{1}{2}(u+u_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$
	$\Omega_1 = \Omega_6(t,r,p,u_1,V_1,W_1),\ \Omega_1^* = \Omega_6(t,r,p,\frac{1}{2}(u+u_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$
	$g_c = \sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2},\ p = x - \theta.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
1.5^c	$-2\tilde{u}\Omega_{p_1} + (1 - \frac{\beta W}{r})\Omega_{p_2} + V\Omega_r - \Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_7(r, p_1, p_2, \tilde{u}, V, W, \Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dV_1\, dW_1$
	$\Omega = \Omega_6(r, p_1, p_2, \tilde{u}, V, W),\ \Omega^* = \Omega_6(r, p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + g_c n_{1c}), \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$
	$\Omega_1 = \Omega_6(r, p_1, p_2, \tilde{u}_1, V_1, W_1),\ \Omega_1^* = \Omega_6(r, p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - g_c n_{1c}), \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})),$
	$g_c = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (V - V_1)^2 + (W - W_1)^2},\ p_1 = t^2 - 2x,\ p_2 = t - \beta\theta,\ \tilde{u} = u - t.$
1.6^c	$(1 - \frac{W}{r})\Omega_p + u\Omega_x + V\Omega_r + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_7(x, r, p, u, V, W, \Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dV_1\, dW_1$
	$\Omega = \Omega_6(x, r, p, u, V, W),\ \Omega^* = \Omega_6(x, r, p, \frac{1}{2}(u + u_1 + g_c n_{1c}), \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$
	$\Omega_1 = \Omega_6(x, r, p, u_1, V_1, W_1),\ \Omega_1^* = \Omega_6(x, r, p, \frac{1}{2}(u + u_1 - g_c n_{1c}), \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})),$
	$g_c = \sqrt{(u - u_1)^2 + (V - V_1)^2 + (W - W_1)^2},\ p = t - \theta.$
1.7	$-\Omega + (v - p_1)\Omega_{p_1} + (w - p_2)\Omega_{p_2} + (\tilde{u} - p_3 - \beta)\Omega_{p_3} - \beta\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_7(p_1, p_2, p_3, \tilde{u}, v, w, \Omega),$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$
	$\Omega = \Omega_6(p_1, p_2, p_3, \tilde{u}, v, w),\ \Omega^* = \Omega_6(p_1, p_2, p_3, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_6(p_1, p_2, p_3, \tilde{u}_1, v_1, w_1),\ \Omega_1^* = \Omega_6(p_1, p_2, p_3, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (v - v_1)^2 + (w - w_1)^2},\ p_1 = \frac{y}{t},\ p_2 = \frac{z}{t},\ p_3 = \frac{x}{t} - \beta\ln t,\ \tilde{u} = u - \beta\ln t.$
1.8	$-\Omega + (u - p_1)\Omega_{p_1} + (v - p_2)\Omega_{p_2} + (w - p_3\Omega_{p_3}) - J(\Omega,\Omega) = \Psi_7(p_1, p_2, p_3, u, v, w, \Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g, \theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$
	$\Omega = \Omega_6(p_1, p_2, p_3, u, v, w),\ \Omega^* = \Omega_6(p_1, p_2, p_3, \frac{1}{2}(u + u_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_6(p_1, p_2, p_3, u_1, v_1, w_1),\ \Omega_1^* = \Omega_6(p_1, p_2, p_3, \frac{1}{2}(u + u_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2},\ p_1 = \frac{x}{t},\ p_2 = \frac{y}{t},\ p_3 = \frac{z}{t}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
1.9	$-2\tilde{u}\Omega_p - \Omega_{\tilde{u}} + v\Omega_y + w\Omega_z - J(\Omega,\Omega) = \Psi_7(p,y,z,\tilde{u},v,w,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$ $\Omega = \Omega_6(p,y,z,\tilde{u},v,w),\ \Omega^* = \Omega_6(p,y,z,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3)),$ $\Omega_1 = \Omega_6(p,y,z,\tilde{u}_1,v_1,w_1),\ \Omega_1^* = \Omega_6(p,y,z,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3)),$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ p = t^2-2x,\ \tilde{u} = u-t.$
1.10	$u\Omega_x + v\Omega_y + w\Omega_z - J(\Omega,\Omega) = \Psi_7(x,y,z,u,v,w,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$ $\Omega = \Omega_6(x,y,z,u,v,w),\ \Omega^* = \Omega_6(x,y,z,\frac{1}{2}(u+u_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3)),$ $\Omega_1 = \Omega_6(x,y,z,u_1,v_1,w_1),\ \Omega_1^* = \Omega_6(x,y,z,\frac{1}{2}(u+u_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
1.11	$-w\Omega_{\tilde{u}} + (\tilde{u}-tw)\Omega_p + v\Omega_y + \Omega_t - J(\Omega,\Omega) = \Psi_7(p,y,\tilde{u},v,w,t,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$ $\Omega = \Omega_6(p,y,\tilde{u},v,w,t),\ \Omega^* = \Omega_6(p,y,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3),t),$ $\Omega_1 = \Omega_6(p,y,\tilde{u}_1,v_1,w_1,t),\ \Omega_1^* = \Omega_6(p,y,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3),t),$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ p = x-tz,\ \tilde{u} = u-z.$
1.12	$-t^{-1}\tilde{u}\Omega_{\tilde{u}} + \Omega_t + v\Omega_y + w\Omega_z - J(\Omega,\Omega) = \Psi_7(y,z,\tilde{u},v,w,t,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$ $\Omega = \Omega_6(y,z,\tilde{u},v,w,t),\ \Omega^* = \Omega_6(y,z,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3),t),$ $\Omega_1 = \Omega_6(y,z,\tilde{u}_1,v_1,w_1,t),\ \Omega_1^* = \Omega_6(y,z,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3),t),$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u-\frac{x}{t}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
1.13	$\Omega_t + v\Omega_y + w\Omega_z - J(\Omega,\Omega) = \Psi_7(y,z,u,v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_6(y,z,u,v,w,t),\ \Omega^* = \Omega_6(y,z,\frac{1}{2}(u+u_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3),t),$
	$\Omega_1 = \Omega_6(y,z,u_1,v_1,w_1,t),\ \Omega_1^* = \Omega_6(y,z,\frac{1}{2}(u+u_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3),t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
2.1^c	$-u\Omega + (V-p_1u)\Omega_{p_1} + (\frac{\alpha W}{p_1}-u)\Omega_{p_2} + \frac{W^2}{p_1}\Omega_V - \frac{VW}{p_1}\Omega_W - J(\Omega,\Omega) = \Psi_6(p_1,p_2,u,V,W,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dV_1\,dW_1$
	$\Omega = \Omega_5(p_1,p_2,u,V,W),\ \Omega^* = \Omega_5(p_1,p_2,\frac{1}{2}(u+u_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$
	$\Omega_1 = \Omega_5(p_1,p_2,u_1,V_1,W_1),\ \Omega_1^* = \Omega_5(p_1,p_2,\frac{1}{2}(u+u_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$
	$g_c = \sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2},\ p_1 = \frac{r}{x},\ p_2 = \alpha\theta - \ln x.$
2.2^c	$-\Omega + (V-p_1)\Omega_{p_1} + (\tilde{u}-p_2-\beta-\frac{\alpha W}{p_1})\Omega_{p_2} - (\frac{\alpha W}{p_1}+\beta)\Omega_{\tilde{u}} + \frac{W^2}{p_1}\Omega_V - \frac{VW}{p_1}\Omega_W - J(\Omega,\Omega) = \Psi_6(p_1,p_2,\tilde{u},V,W,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$
	$\Omega = \Omega_5(p_1,p_2,\tilde{u},V,W),\ \Omega^* = \Omega_5(p_1,p_2,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$
	$\Omega_1 = \Omega_5(p_1,p_2,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_5(p_1,p_2,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$
	$g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p_1 = \frac{r}{t},\ p_2 = \frac{x}{t} - \alpha\theta - \beta\ln t,\ \tilde{u} = u - \alpha\theta - \beta\ln t.$
2.3^c	$-\Omega + (V-p_1)\Omega_{p_1} + (\frac{\alpha W}{p_1}-1)\Omega_{p_2} - \tilde{u}\Omega_{\tilde{u}} + \frac{W^2}{p_1}\Omega_V - \frac{VW}{p_1}\Omega_W - J(\Omega,\Omega) = \Psi_6(p_1,p_2,\tilde{u},V,W,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$
	$\Omega = \Omega_5(p_1,p_2,\tilde{u},V,W),\ \Omega^* = \Omega_5(p_1,p_2,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$
	$\Omega_1 = \Omega_5(p_1,p_2,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_5(p_1,p_2,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$
	$g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p_1 = \frac{r}{t},\ p_2 = \alpha\theta - \ln t,\ \tilde{u} = u - \frac{x}{t}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
2.4^c	$-\Omega+(V-p_1)\Omega_{p_1}+(\frac{\alpha W}{p_1}-1)\Omega_{p_2}-\frac{\beta}{\alpha}\Omega_{\tilde{u}}+\frac{W^2}{p_1}\Omega_V-\frac{VW}{p_1}\Omega_W-J(\Omega,\Omega)=\Psi_6(p_1,p_2,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega)=\int_{\mathbb{R}^3}\int_{\mathbf{S}^2}B(g_c,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega=\Omega_5(p_1,p_2,\tilde{u},V,W),\ \Omega^*=\Omega_5(p_1,p_2,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1=\Omega_5(p_1,p_2,\tilde{u}_1,V_1,W_1),\ \Omega_1^*=\Omega_5(p_1,p_2,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c=\sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p_1=\frac{r}{t},\ p_2=\alpha\theta-\ln t,\ \tilde{u}=u-\frac{\beta}{\alpha}\ln t.$
2.5^c	$u\Omega_x+V\Omega_r+\frac{W^2}{r}\Omega_V-\frac{VW}{r}\Omega_W-J(\Omega,\Omega)=\Psi_6(r,x,u,V,W,\Omega)$ $J(\Omega,\Omega)=\int_{\mathbb{R}^3}\int_{\mathbf{S}^2}B(g_c,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\,d\mathbf{n}\,du_1\,dV_1\,dW_1$ $\Omega=\Omega_5(r,x,u,V,W),\ \Omega^*=\Omega_5(r,x,\frac{1}{2}(u+u_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1=\Omega_5(r,x,u_1,V_1,W_1),\ \Omega_1^*=\Omega_5(r,x,\frac{1}{2}(u+u_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c=\sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2}.$
2.6^c	$(u-\frac{W}{r})\Omega_p+V\Omega_r+\frac{W^2}{r}\Omega_V-\frac{VW}{r}\Omega_W-J(\Omega,\Omega)=\Psi_6(r,p,u,V,W,\Omega)$ $J(\Omega,\Omega)=\int_{\mathbb{R}^3}\int_{\mathbf{S}^2}B(g_c,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\,d\mathbf{n}\,du_1\,dV_1\,dW_1$ $\Omega=\Omega_5(r,p,u,V,W),\ \Omega^*=\Omega_5(r,p,\frac{1}{2}(u+u_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1=\Omega_5(r,p,u_1,V_1,W_1),\ \Omega_1^*=\Omega_5(r,p,\frac{1}{2}(u+u_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c=\sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2},\ p=x-\theta.$
2.7^c	$2(\tilde{u}-\frac{\alpha W}{r})\Omega_p+V\Omega_r-\Omega_{\tilde{u}}+\frac{W^2}{r}\Omega_V-\frac{VW}{r}\Omega_W-J(\Omega,\Omega)=\Psi_6(r,p,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega)=\int_{\mathbb{R}^3}\int_{\mathbf{S}^2}B(g_c,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega=\Omega_5(r,p,\tilde{u},V,W),\ \Omega^*=\Omega_5(r,p,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1=\Omega_5(r,p,\tilde{u}_1,V_1,W_1),\ \Omega_1^*=\Omega_5(r,p,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c=\sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p=2(x-\alpha\theta)-t^2,\ \tilde{u}=u-t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
2.8^c	$\Omega_t + V\Omega_r - \frac{\tilde{u}}{t}\Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_6(t,r,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega = \Omega_5(t,r,\tilde{u},V,W),\ \Omega^* = \Omega_5(t,r,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_5(t,r,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_5(t,r,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ \tilde{u} = u - \frac{x}{t}.$
2.9^c	$\Omega_t + V\Omega_r - \frac{\beta W}{r}\Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_6(t,r,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega = \Omega_5(t,r,\tilde{u},V,W),\ \Omega^* = \Omega_5(t,r,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_5(t,r,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_5(t,r,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ \tilde{u} = u - \beta\theta.$
2.10^c	$\Omega_t + V\Omega_r + t^{-1}(\frac{W}{r} - \tilde{u})\Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_6(t,r,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega = \Omega_5(t,r,\tilde{u},V,W),\ \Omega^* = \Omega_5(t,r,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_5(t,r,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_5(t,r,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ \tilde{u} = u - \frac{x}{t} + \frac{\theta}{t}.$
2.11^c	$(\frac{W}{r} - 1)\Omega_p + V\Omega_r - \beta\Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_6(r,p,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega = \Omega_5(r,p,\tilde{u},V,W),\ \Omega^* = \Omega_5(r,p,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_5(r,p,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_5(r,p,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p = \theta - t,\ \tilde{u} = u - \beta t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
2.12	$-u\Omega + (v - up_1)\Omega_{p_1} + (w - up_2)\Omega_{p_2} - J(\Omega,\Omega) = \Psi_6(p_1, p_2, u, v, w, \Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$
	$\Omega = \Omega_5(p_1, p_2, u, v, w),\ \Omega^* = \Omega_5(p_1, p_2, \frac{1}{2}(u + u_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_5(p_1, p_2, u_1, v_1, w_1),\ \Omega_1^* = \Omega_5(p_1, p_2, \frac{1}{2}(u + u_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2},\ p_1 = \frac{y}{x},\ p_2 = \frac{z}{x}.$
2.13	$-\Omega + (v - p_1)\Omega_{p_1} + (w - p_2)\Omega_{p_2} - \tilde{u}\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_6(p_1, p_2, \tilde{u}, v, w, \Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$
	$\Omega = \Omega_5(p_1, p_2, \tilde{u}, v, w),\ \Omega^* = \Omega_5(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_5(p_1, p_2, \tilde{u}_1, v_1, w_1),\ \Omega_1^* = \Omega_5(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (v - v_1)^2 + (w - w_1)^2},\ p_1 = \frac{y}{t},\ p_2 = \frac{z}{t},\ \tilde{u} = u - \frac{x}{t}.$
2.14	$-\Omega + (\tilde{v} - p_1 - \alpha)\Omega_{p_1} + (w - p_2)\Omega_{p_2} - \tilde{u}\Omega_{\tilde{u}} - \alpha\Omega_{\tilde{v}} - J(\Omega,\Omega) = \Psi_6(p_1, p_2, \tilde{u}, \tilde{v}, w, \Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, dw_1$
	$\Omega = \Omega_5(p_1, p_2, \tilde{u}, \tilde{v}, w),\ \Omega^* = \Omega_5(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_5(p_1, p_2, \tilde{u}_1, \tilde{v}_1, w_1),\ \Omega_1^* = \Omega_5(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (\tilde{v} - \tilde{v}_1)^2 + (w - w_1)^2},\ p_1 = \frac{y}{t} - \alpha\ln t,\ p_2 = \frac{z}{t},\ \tilde{u} = u - \frac{x}{t},\ \tilde{v} = v - \alpha\ln t.$
2.15	$-\Omega + (w - p_1)\Omega_{p_1} + (\tilde{v} - p_2 - \alpha)\Omega_{p_2} - \beta\Omega_{\tilde{u}} - \alpha\Omega_{\tilde{v}} - J(\Omega,\Omega) = \Psi_6(p_1, p_2, \tilde{u}, \tilde{v}, w, \Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, dw_1$
	$\Omega = \Omega_5(p_1, p_2, \tilde{u}, \tilde{v}, w),\ \Omega^* = \Omega_5(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_5(p_1, p_2, \tilde{u}_1, \tilde{v}_1, w_1),\ \Omega_1^* = \Omega_5(p_1, p_2, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (\tilde{v} - \tilde{v}_1)^2 + (w - w_1)^2},\ p_1 = \frac{z}{t},\ p_2 = \frac{y}{t} - \alpha\ln t,\ \tilde{u} = u - \beta\ln t,\ \tilde{v} = v - \alpha\ln t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
2.16	$-\Omega+(v-p_1)\Omega_{p_1}+(w-p_2)\Omega_{p_2}-\beta\Omega_{\tilde{u}}-J(\Omega,\Omega)=\Psi_6(p_1,p_2,\tilde{u},v,w,\Omega)$ $J(\Omega,\Omega)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2}B(g,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$ $\Omega=\Omega_5(p_1,p_2,\tilde{u},v,w),\ \Omega^*=\Omega_5(p_1,p_2,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3)),$ $\Omega_1=\Omega_5(p_1,p_2,\tilde{u}_1,v_1,w_1),\ \Omega_1^*=\Omega_5(p_1,p_2,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3))$ $g=\sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ p_1=\frac{y}{t},\ p_2=\frac{z}{t},\ \tilde{u}=u-\beta\ln t.$
2.17	$v\Omega_y+w\Omega_z-J(\Omega,\Omega)=\Psi_6(y,z,u,v,w,\Omega)$ $J(\Omega,\Omega)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2}B(g,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$ $\Omega=\Omega_5(y,z,u,v,w),\ \Omega^*=\Omega_5(y,z,\frac{1}{2}(u+u_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3)),$ $\Omega_1=\Omega_5(y,z,u_1,v_1,w_1),\ \Omega_1^*=\Omega_5(y,z,\frac{1}{2}(u+u_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3))$ $g=\sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
2.18	$2\tilde{u}\Omega_p+v\Omega_y-\Omega_{\tilde{u}}-\alpha\Omega_{\tilde{w}}-J(\Omega,\Omega)=\Psi_6(p,y,\tilde{u},v,\tilde{w},\Omega)$ $J(\Omega,\Omega)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2}B(g,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,d\tilde{w}_1$ $\Omega=\Omega_5(p,y,\tilde{u},v,\tilde{w}),\ \Omega^*=\Omega_5(p,y,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1=\Omega_5(p,y,\tilde{u}_1,v_1,\tilde{w}_1),\ \Omega_1^*=\Omega_5(p,y,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$ $g=\sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ p=2x-t^2,\ \tilde{u}=u-t,\ \tilde{w}=w-\alpha t.$
2.19	$v\Omega_y+w\Omega_z-\Omega_{\tilde{u}}-J(\Omega,\Omega)=\Psi_6(y,z,\tilde{u},v,w,\Omega)$ $J(\Omega,\Omega)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2}B(g,\theta_1)(\Omega^*\Omega_1^*-\Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$ $\Omega=\Omega_5(y,z,\tilde{u},v,w),\ \Omega^*=\Omega_5(y,z,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3)),$ $\Omega_1=\Omega_5(y,z,\tilde{u}_1,v_1,w_1),\ \Omega_1^*=\Omega_5(y,z,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3))$ $g=\sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u}=u-t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
2.20	$\Omega_t + (u(\sigma\tau - t^2) - \frac{\beta p}{\alpha\tau-\beta t} + (\alpha t - \beta\sigma)\tilde{v} + (\beta t - \alpha\tau)\tilde{w})\Omega_p + \frac{\beta\tilde{v}-\tau u}{\alpha\tau-\beta t}\Omega_{\tilde{v}} + \frac{tu-\alpha\tilde{v}}{\alpha\tau-\beta t}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_6(t,p,u,\tilde{v},\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,d\tilde{v}_1\,d\tilde{w}_1$ $\Omega = \Omega_5(t,p,u,\tilde{v},\tilde{w}),\ \Omega^* = \Omega_5(t,p,\frac{1}{2}(u+u_1+gn_1),\frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_5(t,p,u_1,\tilde{v}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_5(t,p,\frac{1}{2}(u+u_1-gn_1),\frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$ $g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ p = \alpha(ty-\tau z)+\beta(tz-\sigma y)+x(\sigma\tau-t^2),$ $\tilde{v} = v + \frac{\beta y-\tau x}{\alpha\tau-\beta t},\ \tilde{w} = w + \frac{tx-\alpha y}{\alpha\tau-\beta t}.$
2.21	$\Omega_t + u\Omega_x - \frac{1}{t^2+1}(t\tilde{v}+\tilde{w})\Omega_{\tilde{v}} + \frac{1}{t^2+1}(\tilde{v}-t\tilde{w})\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_6(x,u,\tilde{v},\tilde{w},t,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,d\tilde{v}_1\,d\tilde{w}_1$ $\Omega = \Omega_5(x,u,\tilde{v},\tilde{w},t),\ \Omega^* = \Omega_5(x,\frac{1}{2}(u+u_1+gn_1),\frac{1}{2}(\tilde{v}+\tilde{v}_1+g\tilde{n}_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1+g\tilde{n}_3),t),$ $\Omega_1 = \Omega_5(x,u_1,\tilde{v}_1,\tilde{w}_1,t),\ \Omega_1^* = \Omega_5(x,\frac{1}{2}(u+u_1-gn_1),\frac{1}{2}(\tilde{v}+\tilde{v}_1-g\tilde{n}_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1-g\tilde{n}_3),t)$ $g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{v} = v - \frac{z+ty}{t^2+1},\ \tilde{w} = w + \frac{y-tz}{t^2+1}.$
2.22	$\Omega_t + u\Omega_x - \tilde{v}t^{-1}\Omega_{\tilde{v}} - \tilde{w}t^{-1}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_6(x,u,\tilde{v},\tilde{w},t,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,d\tilde{v}_1\,d\tilde{w}_1$ $\Omega = \Omega_5(x,u,\tilde{v},\tilde{w},t),\ \Omega^* = \Omega_5(x,\frac{1}{2}(u+u_1+gn_1),\frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3),t),$ $\Omega_1 = \Omega_5(x,u_1,\tilde{v}_1,\tilde{w}_1,t),\ \Omega_1^* = \Omega_5(x,\frac{1}{2}(u+u_1-gn_1),\frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3),t)$ $g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{v} = v - \frac{y}{t},\ \tilde{w} = w - \frac{z}{t}.$
2.23	$\Omega_t + (\tilde{u} - \alpha v - wt)\Omega_p - w\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_6(p,\tilde{u},v,w,t,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$ $\Omega = \Omega_5(p,\tilde{u},v,w,t),\ \Omega^* = \Omega_5(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3),t),$ $\Omega_1 = \Omega_5(p,\tilde{u}_1,v_1,w_1,t),\ \Omega_1^* = \Omega_5(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3),t)$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ p = x-\alpha y-tz,\ \tilde{u} = u-z.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
2.24	$\Omega_t + w\Omega_z + t^{-1}(\alpha v - \tilde{u})\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_6(z,\tilde{u},v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_5(z,\tilde{u},v,w,t),\ \Omega^* = \Omega_5(z,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3),t),$
	$\Omega_1 = \Omega_5(z,\tilde{u}_1,v_1,w_1,t),\ \Omega_1^* = \Omega_5(z,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3),t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u + \frac{\alpha y - x}{t}.$
2.25	$\Omega_t + v\Omega_y - w\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_6(y,\tilde{u},v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_5(y,\tilde{u},v,w,t),\ \Omega^* = \Omega_5(y,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3),t),$
	$\Omega_1 = \Omega_5(y,\tilde{u}_1,v_1,w_1,t),\ \Omega_1^* = \Omega_5(y,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3),t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u - z.$
2.26	$\Omega_t + v\Omega_y + w\Omega_z - J(\Omega,\Omega) = \Psi_6(y,z,v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_5(y,z,v,w,t),\ \Omega^* = \Omega_5(y,z,\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3),t),$
	$\Omega_1 = \Omega_5(y,z,v_1,w_1,t),\ \Omega_1^* = \Omega_5(y,z,\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3),t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
2.27	$\Omega_t + u\Omega_x - J(\Omega,\Omega) = \Psi_6(x,u,v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_5(x,u,v,w,t),\ \Omega^* = \Omega_5(x,\frac{1}{2}(u+u_1+gn_1),\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3),t),$
	$\Omega_1 = \Omega_5(x,u_1,v_1,w_1,t),\ \Omega_1^* = \Omega_5(x,\frac{1}{2}(u+u_1-gn_1),\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3),t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
3.1^c	$-u\Omega + (V-up)\Omega_p + \frac{W^2}{p}\Omega_V - \frac{VW}{p}\Omega_W - J(\Omega,\Omega) = \Psi_5(p,u,V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dV_1\,dW_1$ $\Omega = \Omega_4(p,u,V,W),\ \Omega^* = \Omega_4(p,\frac{1}{2}(u+u_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_4(p,u_1,V_1,W_1),\ \Omega_1^* = \Omega_4(p,\frac{1}{2}(u+u_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2},\ p=\frac{r}{x}.$
3.2^c	$-\frac{\alpha W}{p}\Omega + (V-\alpha W)\Omega_p - \frac{\beta W}{p}\Omega_{\tilde{u}} + \frac{W^2}{p}\Omega_V - \frac{VW}{p}\Omega_W - J(\Omega,\Omega) = \Psi_5(p,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega = \Omega_4(p,\tilde{u},V,W),\ \Omega^* = \Omega_4(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_4(p,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_4(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p=re^{-\alpha\theta},\ \tilde{u}=u-\beta\theta.$
3.3^c	$-\Omega + (V-p)\Omega_p - \tilde{u}\Omega_{\tilde{u}} + \frac{W^2}{p}\Omega_V - \frac{VW}{p}\Omega_W - J(\Omega,\Omega) = \Psi_5(p,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega = \Omega_4(p,\tilde{u},V,W),\ \Omega^* = \Omega_4(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_4(p,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_4(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p=\frac{r}{t},\ \tilde{u}=u-\frac{x}{t}.$
3.4^c	$-\Omega + (V-p)\Omega_p - (\frac{\alpha W}{p}+\beta)\Omega_{\tilde{u}} + \frac{W^2}{p}\Omega_V - \frac{VW}{p}\Omega_W - J(\Omega,\Omega) = \Psi_5(p,\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega = \Omega_4(p,\tilde{u},V,W),\ \Omega^* = \Omega_4(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_4(p,\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_4(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ p=\frac{r}{t},\ \tilde{u}=u-\alpha\theta-\beta\ln t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
3.6^c	$-\Omega + (V - p_1)\Omega_{p_1} + (\frac{\alpha W}{p_1} - 1)\Omega_{p_2} + \frac{W^2}{p_1}\Omega_V - \frac{VW}{p_1}\Omega_W - J(\Omega,\Omega) = \Psi_5(p_1, p_2, V, W, \Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dV_1\, dW_1$
	$\Omega = \Omega_4(p_1, p_2, V, W),\ \Omega^* = \Omega_4(p_1, p_2, \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$
	$\Omega_1 = \Omega_4(p_1, p_2, V_1, W_1),\ \Omega_1^* = \Omega_4(p_1, p_2, \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})),$
	$g_c = \sqrt{(u - u_1)^2 + (V - V_1)^2 + (W - W_1)^2},\ p_1 = \frac{r}{t},\ p_2 = \alpha\theta - \ln t.$
3.9^c	$V\Omega_r - (\frac{\alpha W}{r} + 1)\Omega_{\tilde{u}} + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_5(r, \tilde{u}, V, W, \Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dV_1\, dW_1$
	$\Omega = \Omega_4(r, \tilde{u}, V, W),\ \Omega^* = \Omega_4(r, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + g_c n_{1c}), \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$
	$\Omega_1 = \Omega_4(r, \tilde{u}_1, V_1, W_1),\ \Omega_1^* = \Omega_4(r, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - g_c n_{1c}), \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})),$
	$g_c = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (V - V_1)^2 + (W - W_1)^2},\ \tilde{u} = u - \alpha\theta - t.$
3.11^c	$\Omega_t + V\Omega_r + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_5(t, r, V, W, \Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dV_1\, dW_1$
	$\Omega = \Omega_4(t, r, V, W),\ \Omega^* = \Omega_4(t, r, \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$
	$\Omega_1 = \Omega_4(t, r, V_1, W_1),\ \Omega_1^* = \Omega_4(t, r, \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})),$
	$g_c = \sqrt{(u - u_1)^2 + (V - V_1)^2 + (W - W_1)^2}.$
3.17^c	$(1 - \frac{W}{r})\Omega_p + V\Omega_r + \frac{W^2}{r}\Omega_V - \frac{VW}{r}\Omega_W - J(\Omega,\Omega) = \Psi_5(p, r, V, W, \Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dV_1\, dW_1$
	$\Omega = \Omega_4(p, r, V, W),\ \Omega^* = \Omega_4(p, r, \frac{1}{2}(V + V_1 + g_c n_{2c}), \frac{1}{2}(W + W_1 + g_c n_{3c})),$
	$\Omega_1 = \Omega_4(p, r, V_1, W_1),\ \Omega_1^* = \Omega_4(p, r, \frac{1}{2}(V + V_1 - g_c n_{2c}), \frac{1}{2}(W + W_1 - g_c n_{3c})),$
	$g_c = \sqrt{(u - u_1)^2 + (V - V_1)^2 + (W - W_1)^2},\ p = t - \theta.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
3.20	$-v\Omega + p(v - wp)\Omega_p - \beta v \Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_5(p, \tilde{u}, v, w, \Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$
	$\Omega = \Omega_4(p, \tilde{u}, v, w),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_4(p, \tilde{u}_1, v_1, w_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (v - v_1)^2 + (w - w_1)^2},\ p = \frac{y}{z},\ \tilde{u} = u - \beta \ln y.$
3.21	$-\Omega + (\tilde{u} - p - \beta)\Omega_p - \beta\Omega_{\tilde{u}} - \tilde{v}\Omega_{\tilde{v}} - \tilde{w}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(p, \tilde{u}, \tilde{v}, \tilde{w}, \Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, d\tilde{w}_1$
	$\Omega = \Omega_4(p, \tilde{u}, \tilde{v}, \tilde{w}),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 + gn_2), \frac{1}{2}(\tilde{w} + \tilde{w}_1 + gn_3)),$
	$\Omega_1 = \Omega_4(p, \tilde{u}_1, \tilde{v}_1, \tilde{w}_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 - gn_2), \frac{1}{2}(\tilde{w} + \tilde{w}_1 - gn_3))$
	$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (\tilde{v} - \tilde{v}_1)^2 + (\tilde{w} - \tilde{w}_1)^2},\ p = \frac{x}{t} - \beta \ln t,\ \tilde{u} = u - \beta \ln t,\ \tilde{v} = v - \frac{y}{t},\ \tilde{w} = w - \frac{z}{t}.$
3.22	$-\Omega + (\tilde{v} - p - \sigma)\Omega_p - (\alpha\tilde{w} + \beta)\Omega_{\tilde{u}} - \sigma\Omega_{\tilde{v}} - \tilde{w}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(p, \tilde{u}, \tilde{v}, \tilde{w}, \Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, d\tilde{w}_1$
	$\Omega = \Omega_4(p, \tilde{u}, \tilde{v}, \tilde{w}),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 + gn_2), \frac{1}{2}(\tilde{w} + \tilde{w}_1 + gn_3)),$
	$\Omega_1 = \Omega_4(p, \tilde{u}_1, \tilde{v}_1, \tilde{w}_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 - gn_2), \frac{1}{2}(\tilde{w} + \tilde{w}_1 - gn_3))$
	$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (\tilde{v} - \tilde{v}_1)^2 + (\tilde{w} - \tilde{w}_1)^2},\ p = \frac{y}{t} - \sigma \ln t,\ \tilde{u} = u - \frac{\alpha z}{t} - \beta \ln t,\ \tilde{v} = v - \sigma \ln t,\ \tilde{w} = w - \frac{z}{t}.$
3.23	$-\Omega + (v - p_1)\Omega_{p_1} + (\tilde{w} - p_2 - \sigma)\Omega_{p_2} - \sigma\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(p_1, p_2, v, \tilde{w}, \Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, d\tilde{w}_1$
	$\Omega = \Omega_4(p_1, p_2, v, \tilde{w}),\ \Omega^* = \Omega_4(p_1, p_2, \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(\tilde{w} + \tilde{w}_1 + gn_3))$
	$\Omega_1 = \Omega_4(p_1, p_2, v_1, \tilde{w}_1),\ \Omega_1^* = \Omega_4(p_1, p_2, \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(\tilde{w} + \tilde{w}_1 - gn_3))$
	$g = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (\tilde{w} - \tilde{w}_1)^2},\ p_1 = \frac{y}{t},\ p_2 = \frac{z}{t} - \sigma \ln t,\ \tilde{w} = w - \sigma \ln t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
3.24	$-\Omega + (v - p_1)\Omega_{p_1} + (w - p_2)\Omega_{p_2} - J(\Omega,\Omega) = \Psi_5(p_1, p_2, v, w, \Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$ $\Omega = \Omega_4(p_1, p_2, v, w),\ \Omega^* = \Omega_4(p_1, p_2, \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3))$ $\Omega_1 = \Omega_4(p_1, p_2, v_1, w_1),\ \Omega_1^* = \Omega_4(p_1, p_2, \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$ $g = \sqrt{(u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2},\ p_1 = \frac{y}{t},\ p_2 = \frac{z}{t}.$
3.25	$-\Omega + (\tilde{u} - p - \beta)\Omega_p - \beta\Omega_{\tilde{u}} - \sigma\Omega_{\tilde{v}} - J(\Omega,\Omega) = \Psi_5(p, \tilde{u}, \tilde{v}, w, \Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, dw_1$ $\Omega = \Omega_4(p, \tilde{u}, \tilde{v}, w),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$ $\Omega_1 = \Omega_4(p, \tilde{u}_1, \tilde{v}_1, w_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (\tilde{v}-\tilde{v}_1)^2 + (w-w_1)^2},\ p = \frac{x}{t} - \beta\ln t,\ \tilde{u} = u - \beta\ln t,\ \tilde{v} = v - \sigma\ln t.$
3.26	$-\Omega + (\tilde{u} - p - \beta)\Omega_p - \beta\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_5(p, \tilde{u}, v, w, \Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$ $\Omega = \Omega_4(p, \tilde{u}, v, w),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$ $\Omega_1 = \Omega_4(p, \tilde{u}_1, v_1, w_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (w-w_1)^2},\ p = \frac{x}{t} - \beta\ln t,\ \tilde{u} = u - \beta\ln t.$
3.27 ($\alpha \neq 0$)	$(v - \frac{\beta\tilde{u}}{\alpha})\Omega_p - \Omega_{\tilde{u}} - \frac{\tilde{u}}{\alpha}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(p, \tilde{u}, v, \tilde{w}, \Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, d\tilde{w}_1$ $\Omega = \Omega_4(p, \tilde{u}, v, \tilde{w}),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(\tilde{w} + \tilde{w}_1 + gn_3)),$ $\Omega_1 = \Omega_4(p, \tilde{u}_1, v_1, \tilde{w}_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(\tilde{w} + \tilde{w}_1 - gn_3))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (\tilde{w}-\tilde{w}_1)^2},\ p = y - \frac{\beta x}{\alpha} + \frac{\beta t^2}{2\alpha},\ \tilde{u} = u - t,\ \tilde{w} = w - \frac{x}{\alpha} + \frac{t^2}{2\alpha}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
3.27 ($\alpha = 0, \beta \neq 0$)	$-2\tilde{u}\Omega_p - \Omega_{\tilde{u}} - \frac{v}{\beta}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(p,\tilde{u},v,\tilde{w},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, d\tilde{w}_1$
	$\Omega = \Omega_4(p,\tilde{u},v,\tilde{w}),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$
	$\Omega_1 = \Omega_4(p,\tilde{u}_1,v_1,\tilde{w}_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (\tilde{w}-\tilde{w}_1)^2},\ p = t^2 - 2x,\ \tilde{u} = u - t,\ \tilde{w} = w - \frac{y}{\beta}.$
3.27 ($\alpha = 0, \beta = 0$)	$-2\tilde{u}\Omega_p - \Omega_{\tilde{u}} + v\Omega_y - J(\Omega,\Omega) = \Psi_5(p,y,\tilde{u},v,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, d\tilde{w}_1$
	$\Omega = \Omega_4(p,y,\tilde{u},v),\ \Omega^* = \Omega_4(p,y, \frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2)),$
	$\Omega_1 = \Omega_4(p,y,\tilde{u}_1,v_1),\ \Omega_1^* = \Omega_4(p,y, \frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (w-w_1)^2},\ p = t^2 - 2x,\ \tilde{u} = u - t.$
3.28	$-v\Omega_{\tilde{u}} + w\Omega_z - J(\Omega,\Omega) = \Psi_5(z,\tilde{u},v,w,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$
	$\Omega = \Omega_4(z,\tilde{u},v,w),\ \Omega^* = \Omega_4(z, \frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$
	$\Omega_1 = \Omega_4(z,\tilde{u}_1,v_1,w_1),\ \Omega_1^* = \Omega_4(z, \frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (w-w_1)^2},\ \tilde{u} = u - y.$
3.29	$v\Omega_y + w\Omega_z - J(\Omega,\Omega) = \Psi_5(y,z,v,w,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$
	$\Omega = \Omega_4(y,z,v,w),\ \Omega^* = \Omega_4(y,z, \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$
	$\Omega_1 = \Omega_4(y,z,v_1,w_1),\ \Omega_1^* = \Omega_4(y,z, \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$
	$g = \sqrt{(u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
3.30	$-2\tilde{u}\Omega_p - \Omega_{\tilde{u}} - \sigma\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(p,\tilde{u},v,\tilde{w},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, d\tilde{w}_1$
	$\Omega = \Omega_4(p,\tilde{u},v,\tilde{w}),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$
	$\Omega_1 = \Omega_4(p,\tilde{u}_1,v_1,\tilde{w}_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (\tilde{w}-\tilde{w}_1)^2},\ p = t^2 - 2x,\ \tilde{u} = u - t,\ \tilde{w} = w - \sigma t.$
3.31	$-2\tilde{u}\Omega_p - \Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_5(p,\tilde{u},v,w,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$
	$\Omega = \Omega_4(p,\tilde{u},v,w),\ \Omega^* = \Omega_4(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$
	$\Omega_1 = \Omega_4(p,\tilde{u}_1,v_1,w_1),\ \Omega_1^* = \Omega_4(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (w-w_1)^2},\ p = t^2 - 2x,\ \tilde{u} = u - t.$
3.32	$-\Omega_{\tilde{w}} + u\Omega_x - J(\Omega,\Omega) = \Psi_5(x,u,v,\tilde{w},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, d\tilde{w}_1$
	$\Omega = \Omega_4(x,u,v,\tilde{w}),\ \Omega^* = \Omega_4(x, \frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$
	$\Omega_1 = \Omega_4(x,u_1,v_1,\tilde{w}_1),\ \Omega_1^* = \Omega_4(x, \frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$
	$g = \sqrt{(u-u_1)^2 + (v-v_1)^2 + (\tilde{w}-\tilde{w}_1)^2},\ \tilde{w} = w - t.$
3.33	$u\Omega_x - J(\Omega,\Omega) = \Psi_5(x,u,v,w,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$
	$\Omega = \Omega_4(x,u,v,w),\ \Omega^* = \Omega_4(x, \frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$
	$\Omega_1 = \Omega_4(x,u_1,v_1,w_1),\ \Omega_1^* = \Omega_4(x, \frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$
	$g = \sqrt{(u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
3.35	$\Omega_t - \tilde{u}t^{-1}\Omega_{\tilde{u}} - \frac{\tilde{v}-t\tilde{w}}{t^2+1}\Omega_{\tilde{v}} - \frac{t\tilde{v}+\tilde{w}}{t^2+1}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(\tilde{u},\tilde{v},\tilde{w},t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,d\tilde{v}_1\,d\tilde{w}_1$
	$\Omega = \Omega_4(\tilde{u},\tilde{v},\tilde{w},t),\ \Omega^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3), t),$
	$\Omega_1 = \Omega_4(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1,t),\ \Omega_1^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3), t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{u} = u - \frac{x}{t},\ \tilde{v} = v - \frac{y}{t} - \frac{tz-y}{t(t^2+1)},\ \tilde{w} = w - \frac{tz-y}{t^2+1}.$
3.36	$\Omega_t - \frac{\tilde{u}}{t+1}\Omega_{\tilde{u}} - \frac{\tilde{v}}{t}\Omega_{\tilde{v}} - \frac{\tilde{w}}{t}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(\tilde{u},\tilde{v},\tilde{w},t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,d\tilde{v}_1\,d\tilde{w}_1$
	$\Omega = \Omega_4(\tilde{u},\tilde{v},\tilde{w},t),\ \Omega^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3), t),$
	$\Omega_1 = \Omega_4(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1,t),\ \Omega_1^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3), t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{u} = u - \frac{x}{t+1},\ \tilde{v} = v - \frac{y}{t},\ \tilde{w} = w - \frac{z}{t}.$
3.37	$\Omega_t - \tilde{u}t^{-1}\Omega_{\tilde{u}} - \tilde{v}t^{-1}\Omega_{\tilde{v}} - \tilde{w}t^{-1}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(\tilde{u},\tilde{v},\tilde{w},t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,d\tilde{v}_1\,d\tilde{w}_1$
	$\Omega = \Omega_4(\tilde{u},\tilde{v},\tilde{w},t),\ \Omega^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3), t),$
	$\Omega_1 = \Omega_4(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1,t),\ \Omega_1^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3), t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{u} = u - \frac{x}{t},\ \tilde{v} = v - \frac{y}{t},\ \tilde{w} = w - \frac{z}{t}.$
3.42	$\Omega_t - \tilde{v}t^{-1}\Omega_{\tilde{v}} - \tilde{w}t^{-1}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_5(u,\tilde{v},\tilde{w},t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,d\tilde{v}_1\,d\tilde{w}_1$
	$\Omega = \Omega_4(u,\tilde{v},\tilde{w},t),\ \Omega^* = \Omega_4(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3), t),$
	$\Omega_1 = \Omega_4(u_1,\tilde{v}_1,\tilde{w}_1,t),\ \Omega_1^* = \Omega_4(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3), t)$
	$g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{v} = v - \frac{y}{t},\ \tilde{w} = w - \frac{z}{t}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
3.44	$\Omega_t - \tilde{u}t^{-1}\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_5(\tilde{u},v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_4(\tilde{u},v,w,t),\ \Omega^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3), t),$
	$\Omega_1 = \Omega_4(\tilde{u}_1,v_1,w_1,t),\ \Omega_1^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3), t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u - \frac{x}{t}.$
3.45	$\Omega_t - w\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_5(\tilde{u},v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_4(\tilde{u},v,w,t),\ \Omega^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3), t),$
	$\Omega_1 = \Omega_4(\tilde{u}_1,v_1,w_1,t),\ \Omega_1^* = \Omega_4(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3), t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u - z.$
3.46	$\Omega_t + w\Omega_z - J(\Omega,\Omega) = \Psi_5(z,v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_4(z,v,w,t),\ \Omega^* = \Omega_4(z, \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3), t),$
	$\Omega_1 = \Omega_4(z,v_1,w_1,t),\ \Omega_1^* = \Omega_4(z, \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3), t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
3.47	$\Omega_t - J(\Omega,\Omega) = \Psi_5(u,v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_4(u,v,w,t),\ \Omega^* = \Omega_4(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3), t),$
	$\Omega_1 = \Omega_4(u_1,v_1,w_1,t),\ \Omega_1^* = \Omega_4(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3), t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
4.2^c	$-V\Omega - \alpha W\Omega_{\tilde{u}} + W^2\Omega_V - VW\Omega_W - J(\Omega,\Omega) = \Psi_4(\tilde{u},V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dV_1\,dW_1$ $\Omega = \Omega_3(\tilde{u},V,W),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1+g_cn_{1c}),\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_3(\tilde{u}_1,V_1,W_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1-g_cn_{1c}),\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(V-V_1)^2+(W-W_1)^2},\ \tilde{u} = u-\alpha\theta.$
4.4^c	$-V\Omega + (\alpha W - V)\Omega_p + W^2\Omega_V - VW\Omega_W - J(\Omega,\Omega) = \Psi_4(p,V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dV_1\,dW_1$ $\Omega = \Omega_3(p,V,W),\ \Omega^* = \Omega_3(p,\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_3(p,V_1,W_1),\ \Omega_1^* = \Omega_3(p,\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2},\ p = \alpha\theta - \ln r.$
4.6^c	$-V\Omega + (1-pV)\Omega_p + W^2\Omega_V - VW\Omega_W - J(\Omega,\Omega) = \Psi_4(p,V,W,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dV_1\,dW_1$ $\Omega = \Omega_3(p,V,W),\ \Omega^* = \Omega_3(p,\frac{1}{2}(V+V_1+g_cn_{2c}),\frac{1}{2}(W+W_1+g_cn_{3c})),$ $\Omega_1 = \Omega_3(p,V_1,W_1),\ \Omega_1^* = \Omega_3(p,\frac{1}{2}(V+V_1-g_cn_{2c}),\frac{1}{2}(W+W_1-g_cn_{3c})),$ $g_c = \sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2},\ p = \frac{t}{r}.$
4.23	$-v\Omega + (w-vp)\Omega_p - J(\Omega,\Omega) = \Psi_4(p,v,w,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$ $\Omega = \Omega_3(p,v,w),\ \Omega^* = \Omega_3(p,\frac{1}{2}(v+v_1+gn_2),\frac{1}{2}(w+w_1+gn_3)),$ $\Omega_1 = \Omega_3(p,v_1,w_1),\ \Omega_1^* = \Omega_3(p,\frac{1}{2}(v+v_1-gn_2),\frac{1}{2}(w+w_1-gn_3))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2},\ p = \frac{z}{y}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
4.24	$-u\Omega - \alpha u\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(u,v,\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, d\tilde{w}_1$ $\Omega = \Omega_3(u,v,\tilde{w}),\ \Omega^* = \Omega_3(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_3(u_1,v_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{w} = w - \alpha\ln x.$
4.25	$-u\Omega - J(\Omega,\Omega) = \Psi_4(u,v,w,\Omega)$ $J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$ $\Omega = \Omega_3(u,v,w),\ \Omega^* = \Omega_3(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$ $\Omega_1 = \Omega_3(u_1,v_1,w_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
4.26	$-\Omega - \tilde{u}\Omega_{\tilde{u}} - \tilde{v}\Omega_{\tilde{v}} - \tilde{w}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(\tilde{u},\tilde{v},\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, d\tilde{w}_1$ $\Omega = \Omega_3(\tilde{u},\tilde{v},\tilde{w}),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_3(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{u} = u - \frac{x}{t},\ \tilde{v} = v - \frac{y}{t},\ \tilde{w} = w - \frac{z}{t}.$
4.28	$-\Omega - \beta\Omega_{\tilde{u}} - \tilde{v}\Omega_{\tilde{v}} - \tilde{w}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(\tilde{u},\tilde{v},\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, d\tilde{w}_1$ $\Omega = \Omega_3(\tilde{u},\tilde{v},\tilde{w}),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_3(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{u} = u - \beta\ln t,\ \tilde{v} = v - \frac{y}{t},\ \tilde{w} = w - \frac{z}{t}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
4.29	$-\Omega + (\tilde{v} - p - \alpha)\Omega_p - \alpha\Omega_{\tilde{v}} - \tilde{w}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(p,\tilde{v},\tilde{w},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, d\tilde{v}_1\, d\tilde{w}_1$
	$\Omega = \Omega_3(p,\tilde{v},\tilde{w}),\ \Omega^* = \Omega_3(p, \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$
	$\Omega_1 = \Omega_3(p,\tilde{v}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(p, \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$
	$g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ p = \frac{y}{t} - \alpha\ln t,\ \tilde{v} = v - \alpha\ln t,\ \tilde{w} = w - \frac{z}{t}.$
4.31	$-\Omega - \tilde{u}\Omega_{\tilde{u}} - \alpha\Omega_{\tilde{v}} - \beta\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(\tilde{u},\tilde{v},\tilde{w},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, d\tilde{w}_1$
	$\Omega = \Omega_3(\tilde{u},\tilde{v},\tilde{w}),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$
	$\Omega_1 = \Omega_3(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{u} = u - \frac{x}{t},\ \tilde{v} = v - \alpha\ln t,\ \tilde{w} = w - \beta\ln t.$
4.32	$-\Omega - \tilde{u}\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_4(\tilde{u},v,w,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$
	$\Omega = \Omega_3(\tilde{u},v,w),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$
	$\Omega_1 = \Omega_3(\tilde{u}_1,v_1,w_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u - \frac{x}{t}.$
4.33	$-\Omega - \beta\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_4(\tilde{u},v,w,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$
	$\Omega = \Omega_3(\tilde{u},v,w),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$
	$\Omega_1 = \Omega_3(\tilde{u}_1,v_1,w_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u - \beta\ln t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
4.34	$-\Omega - J(\Omega,\Omega) = \Psi_4(u,v,w,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$ $\Omega = \Omega_3(u,v,w),\ \Omega^* = \Omega_3(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$ $\Omega_1 = \Omega_3(u_1,v_1,w_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
4.35 ($\alpha \neq 0$)	$-\Omega_{\tilde{u}} - \frac{\tilde{u}}{\alpha}\Omega_{\tilde{v}} - \beta\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(\tilde{u},\tilde{v},\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, d\tilde{v}_1\, d\tilde{w}_1$ $\Omega = \Omega_3(\tilde{u},\tilde{v},\tilde{w}),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_3(\tilde{u}_1,\tilde{v}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3)),$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{u} = u - t,\ \tilde{v} = v - \frac{x}{\alpha} + \frac{t^2}{2\alpha},\ \tilde{w} = w - \beta t.$
4.35 ($\alpha = 0$)	$-\Omega_{\tilde{u}} - 2\tilde{u}\Omega_p - \beta\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(p,\tilde{u},\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, d\tilde{w}_1$ $\Omega = \Omega_3(p,\tilde{u},\tilde{w}),\ \Omega^* = \Omega_3(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_3(p,\tilde{u}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3)),$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ p = t^2 - 2x,\ \tilde{u} = u - t,\ \tilde{w} = w - \beta t.$
4.36 ($\alpha \neq 0$)	$-\frac{u}{\alpha}\Omega_{\tilde{v}} - \Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(u,\tilde{v},\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, d\tilde{v}_1\, d\tilde{w}_1$ $\Omega = \Omega_3(u,\tilde{v},\tilde{w}),\ \Omega^* = \Omega_3(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_3(u_1,\tilde{v}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3)),$ $g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{v} = v - \frac{x}{\alpha},\ \tilde{w} = w - t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
4.36 ($\alpha = 0$)	$u\Omega_x - \Omega_{\tilde{w}} - J(\Omega, \Omega) = \Psi_4(x, u, \tilde{w}, \Omega)$
	$J(\Omega, \Omega) = \int\limits_{\mathbb{R}^3} \int\limits_{\mathbf{S}^2} B(g, \theta_1)(\Omega^* \Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, d\tilde{w}_1$
	$\Omega = \Omega_3(x, u, \tilde{w}),\ \Omega^* = \Omega_3(x, \frac{1}{2}(u + u_1 + gn_1), \frac{1}{2}(\tilde{w} + \tilde{w}_1 + gn_3)),$
	$\Omega_1 = \Omega_3(x, u_1, \tilde{w}_1),\ \Omega_1^* = \Omega_3(x, \frac{1}{2}(u + u_1 - gn_1), \frac{1}{2}(\tilde{w} + \tilde{w}_1 - gn_3)),$
	$g = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (\tilde{w} - \tilde{w}_1)^2},\ \tilde{w} = w - t.$
4.37	$-u\Omega_{\tilde{v}} - J(\Omega, \Omega) = \Psi_4(u, \tilde{v}, w, \Omega)$
	$J(\Omega, \Omega) = \int\limits_{\mathbb{R}^3} \int\limits_{\mathbf{S}^2} B(g, \theta_1)(\Omega^* \Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, d\tilde{v}_1\, dw_1$
	$\Omega = \Omega_3(u, \tilde{v}, w),\ \Omega^* = \Omega_3(\frac{1}{2}(u + u_1 + gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_3(u_1, \tilde{v}_1, w_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(u + u_1 - gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(u - u_1)^2 + (\tilde{v} - \tilde{v}_1)^2 + (w - w_1)^2},\ \tilde{v} = v - x.$
4.38	$u\Omega_x - J(\Omega, \Omega) = \Psi_4(x, u, w, \Omega)$
	$J(\Omega, \Omega) = \int\limits_{\mathbb{R}^3} \int\limits_{\mathbf{S}^2} B(g, \theta_1)(\Omega^* \Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$
	$\Omega = \Omega_3(x, u, w),\ \Omega^* = \Omega_3(x, \frac{1}{2}(u + u_1 + gn_1), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_3(x, u_1, w_1),\ \Omega_1^* = \Omega_3(x, \frac{1}{2}(u + u_1 - gn_1), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2}.$
4.39	$-\Omega_{\tilde{u}} - J(\Omega, \Omega) = \Psi_4(\tilde{u}, v, w, \Omega)$
	$J(\Omega, \Omega) = \int\limits_{\mathbb{R}^3} \int\limits_{\mathbf{S}^2} B(g, \theta_1)(\Omega^* \Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$
	$\Omega = \Omega_3(\tilde{u}, v, w),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_3(\tilde{u}_1, v_1, w_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (v - v_1)^2 + (w - w_1)^2},\ \tilde{u} = u - t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
4.40	$-J(\Omega,\Omega) = \Psi_4(u,v,w,\Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$
	$\Omega = \Omega_3(u,v,w),\ \Omega^* = \Omega_3(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3)),$
	$\Omega_1 = \Omega_3(u_1,v_1,w_1),\ \Omega_1^* = \Omega_3(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3))$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
4.43	$\Omega_t - t^{-1}\tilde{v}\Omega_{\tilde{v}} - t^{-1}\tilde{w}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(\tilde{v},\tilde{w},t,\Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, d\tilde{v}_1\, d\tilde{w}_1$
	$\Omega = \Omega_3(\tilde{v},\tilde{w},t),\ \Omega^* = \Omega_3(\frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3), t),$
	$\Omega_1 = \Omega_3(\tilde{v}_1,\tilde{w}_1,t),\ \Omega_1^* = \Omega_3(\frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3), t)$
	$g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{v} = v - \frac{y}{t},\ \tilde{w} = w - \frac{z}{t}.$
4.45	$\Omega_t - u\Omega_{\tilde{v}} - J(\Omega,\Omega) = \Psi_4(u,\tilde{v},t,\Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, d\tilde{v}_1\, dw_1$
	$\Omega = \Omega_3(u,\tilde{v},t),\ \Omega^* = \Omega_3(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2), t),$
	$\Omega_1 = \Omega_3(u_1,\tilde{v}_1,t),\ \Omega_1^* = \Omega_3(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2), t)$
	$g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(w-w_1)^2},\ \tilde{v} = v - x.$
4.46	$\Omega_t + t^{-1}(\beta u - \tilde{w})\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(t,u,\tilde{w},\Omega)$
	$J(\Omega,\Omega) = \int_{\mathbb{R}^3}\int_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, d\tilde{w}_1$
	$\Omega = \Omega_3(t,u,\tilde{w}),\ \Omega^* = \Omega_3(t, \frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$
	$\Omega_1 = \Omega_3(t,u_1,\tilde{w}_1),\ \Omega_1^* = \Omega_3(t, \frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3)),$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{w} = w + \frac{\beta x - z}{t}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
4.47	$\Omega_t + u\Omega_x - J(\Omega,\Omega) = \Psi_4(x,u,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_3(x,u,t),\ \Omega^* = \Omega_3(x, \frac{1}{2}(u+u_1+gn_1), t),$
	$\Omega_1 = \Omega_3(x,u_1,t),\ \Omega_1^* = \Omega_3(x, \frac{1}{2}(u+u_1-gn_1), t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
4.49	$\Omega_t - t^{-1}\tilde{w}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_4(u,\tilde{w},t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,d\tilde{w}_1$
	$\Omega = \Omega_3(u,\tilde{w},t),\ \Omega^* = \Omega_3(\frac{1}{2}(u+u_1+gn_1), \frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3), t),$
	$\Omega_1 = \Omega_3(u_1,\tilde{w}_1,t),\ \Omega_1^* = \Omega_3(\frac{1}{2}(u+u_1-gn_1), \frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3), t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{w} = w - \frac{z}{t}.$
4.50	$\Omega_t - J(\Omega,\Omega) = \Psi_4(v,w,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_3(v,w,t),\ \Omega^* = \Omega_3(\frac{1}{2}(v+v_1+gn_2), \frac{1}{2}(w+w_1+gn_3), t),$
	$\Omega_1 = \Omega_3(v_1,w_1,t),\ \Omega_1^* = \Omega_3(\frac{1}{2}(v+v_1-gn_2), \frac{1}{2}(w+w_1-gn_3), t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
5.2^c	$-V\Omega + W^2\Omega_V - VW\Omega_W - J(\Omega,\Omega) = \Psi_3(V,W,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g_c,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dV_1\,dW_1$
	$\Omega = \Omega_2(V,W),\ \Omega^* = \Omega_2(\frac{1}{2}(V+V_1+g_cn_{2c}), \frac{1}{2}(W+W_1+g_cn_{3c})),$
	$\Omega_1 = \Omega_2(V_1,W_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(V+V_1-g_cn_{2c}), \frac{1}{2}(W+W_1-g_cn_{3c})),$
	$g_c = \sqrt{(u-u_1)^2+(V-V_1)^2+(W-W_1)^2}.$
5.10	$-\Omega + (\tilde{u} - p - \frac{\beta}{\alpha})\Omega_p - \frac{\beta}{\alpha}\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_3(p,\tilde{u},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(p,\tilde{u}),\ \Omega^* = \Omega_2(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1)),$
	$\Omega_1 = \Omega_2(p,\tilde{u}_1),\ \Omega_1^* = \Omega_2(p, \frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ p = \frac{x}{t} - \frac{\beta}{\alpha}\ln t,\ \tilde{u} = u - \frac{\beta}{\alpha}\ln t.$
5.13	$\Omega_t - t^{-1}\tilde{u}\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_3(\tilde{u},t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(\tilde{u},t),\ \Omega^* = \Omega_2(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1), t),$
	$\Omega_1 = \Omega_2(\tilde{u}_1,t),\ \Omega_1^* = \Omega_2(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1), t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u - \frac{x}{t}.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
5.17	$\Omega_t - J(\Omega,\Omega) = \Psi_3(u,t,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$ $\Omega = \Omega_2(u,t),\ \Omega^* = \Omega_2(\frac{1}{2}(u+u_1+gn_1),t),$ $\Omega_1 = \Omega_2(u_1,t),\ \Omega_1^* = \Omega_2(\frac{1}{2}(u+u_1-gn_1),t)$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
5.18	$-2\tilde{u}\Omega_p - \Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_3(p,\tilde{u},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$ $\Omega = \Omega_2(p,\tilde{u}),\ \Omega^* = \Omega_2(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1)),$ $\Omega_1 = \Omega_2(p,u_1),\ \Omega_1^* = \Omega_2(p,\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ p = t^2-2x,\ \tilde{u} = u-t.$
5.19	$u\Omega_x - J(\Omega,\Omega) = \Psi_3(x,u,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$ $\Omega = \Omega_2(x,u),\ \Omega^* = \Omega_2(x,\frac{1}{2}(u+u_1+gn_1)),$ $\Omega_1 = \Omega_2(x,u_1),\ \Omega_1^* = \Omega_2(x,\frac{1}{2}(u+u_1-gn_1))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
5.21	$-u\Omega - \beta u\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_3(u,\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,d\tilde{w}_1$ $\Omega = \Omega_2(u,\tilde{w}),\ \Omega^* = \Omega_2(\frac{1}{2}(u+u_1+gn_1),\frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_2(u_1,\tilde{w}_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(u+u_1-gn_1),\frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{w} = w-\beta\ln x.$
5.24	$-\Omega - \tilde{v}\Omega_{\tilde{v}} - \tilde{w}\Omega_{\tilde{w}} - J(\Omega,\Omega) = \Psi_3(\tilde{v},\tilde{w},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,d\tilde{v}_1\,d\tilde{w}_1$ $\Omega = \Omega_2(\tilde{v},\tilde{w}),\ \Omega^* = \Omega_2(\frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1+gn_3)),$ $\Omega_1 = \Omega_2(\tilde{v}_1,\tilde{w}_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2),\frac{1}{2}(\tilde{w}+\tilde{w}_1-gn_3))$ $g = \sqrt{(u-u_1)^2+(\tilde{v}-\tilde{v}_1)^2+(\tilde{w}-\tilde{w}_1)^2},\ \tilde{v} = v-\frac{y}{t},\ \tilde{w} = w-\frac{z}{t}.$
5.25	$-\Omega - \tilde{u}\Omega_{\tilde{u}} + \frac{1}{\alpha}(\beta-\tilde{u})\Omega_{\tilde{v}} - J(\Omega,\Omega) = \Psi_3(\tilde{u},\tilde{v},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,d\tilde{v}_1\,dw_1$ $\Omega = \Omega_2(\tilde{u},\tilde{v}),\ \Omega^* = \Omega_2(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(\tilde{v}+\tilde{v}_1+gn_2)),$ $\Omega_1 = \Omega_2(\tilde{u}_1,\tilde{v}_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(\tilde{v}+\tilde{v}_1-gn_2))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(\tilde{v}-\tilde{v}_1)^2+(w-w_1)^2},\ \tilde{u} = u-\frac{x}{t},\ \tilde{v} = v-\frac{x}{\alpha t}+\frac{\beta}{\alpha}\ln t.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
5.26	$-\Omega + (\tilde{u} - p - \beta)\Omega_p - \beta\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_3(p,\tilde{u},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(p,\tilde{u}),\ \Omega^* = \Omega_2(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1)),$
	$\Omega_1 = \Omega_2(p,u_1),\ \Omega_1^* = \Omega_2(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (w-w_1)^2},\ p = \frac{x}{t} - \beta\ln t,\ \tilde{u} = u - \beta\ln t.$
5.27	$-\Omega - \tilde{u}\Omega_{\tilde{u}} - \beta\Omega_{\tilde{v}} - J(\Omega,\Omega) = \Psi_3(\tilde{u},\tilde{v},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,d\tilde{v}_1\,dw_1$
	$\Omega = \Omega_2(\tilde{u},\tilde{v}),\ \Omega^* = \Omega_2(\frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 + gn_2)),$
	$\Omega_1 = \Omega_2(\tilde{u}_1,\tilde{v}_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 - gn_2))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (\tilde{v}-\tilde{v}_1)^2 + (w-w_1)^2},\ \tilde{u} = u - \frac{x}{t},\ \tilde{v} = v - \beta\ln t.$
5.28	$-\Omega - \beta\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_3(\tilde{u},v,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(\tilde{u},v),\ \Omega^* = \Omega_2(\frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1), \frac{1}{2}(v + v_1 + gn_2)),$
	$\Omega_1 = \Omega_2(\tilde{u}_1,v_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1), \frac{1}{2}(v + v_1 - gn_2))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (w-w_1)^2},\ \tilde{u} = u - \beta\ln t.$
5.29	$-\Omega - J(\Omega,\Omega) = \Psi_3(v,w,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(v,w),\ \Omega^* = \Omega_2(\frac{1}{2}(v + v_1 + gn_2), \frac{1}{2}(w + w_1 + gn_3)),$
	$\Omega_1 = \Omega_2(v_1,w_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(v + v_1 - gn_2), \frac{1}{2}(w + w_1 - gn_3))$
	$g = \sqrt{(u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2}.$
5.31	$-2\tilde{u}\Omega_p - \Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_3(p,\tilde{u},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(p,\tilde{u}),\ \Omega^* = \Omega_2(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1)),$
	$\Omega_1 = \Omega_2(p,u_1),\ \Omega_1^* = \Omega_2(p, \frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2 + (v-v_1)^2 + (w-w_1)^2},\ p = t^2 - 2x,\ \tilde{u} = u - t.$
5.32	$-u\Omega_{\tilde{v}} - J(\Omega,\Omega) = \Psi_3(u,\tilde{v},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,d\tilde{v}_1\,dw_1$
	$\Omega = \Omega_2(u,\tilde{v}),\ \Omega^* = \Omega_2(\frac{1}{2}(u + u_1 + gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 + gn_2)),$
	$\Omega_1 = \Omega_2(u_1,\tilde{v}_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(u + u_1 - gn_1), \frac{1}{2}(\tilde{v} + \tilde{v}_1 - gn_2))$
	$g = \sqrt{(u-u_1)^2 + (\tilde{v}-\tilde{v}_1)^2 + (w-w_1)^2},\ \tilde{v} = v - x.$

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
5.33	$u\Omega_x - J(\Omega,\Omega) = \Psi_3(x,u,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(x,u),\ \Omega^* = \Omega_2(x,\frac{1}{2}(u+u_1+gn_1)),$
	$\Omega_1 = \Omega_2(x,u_1),\ \Omega_1^* = \Omega_2(x,\frac{1}{2}(u+u_1-gn_1))$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
5.34	$-\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_3(\tilde{u},v,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(\tilde{u},v),\ \Omega^* = \Omega_2(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),\frac{1}{2}(v+v_1+gn_2)),$
	$\Omega_1 = \Omega_2(\tilde{u}_1,v_1),\ \Omega_1^* = \Omega_2(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),\frac{1}{2}(v+v_1-gn_2))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u-t.$
5.35	$\Omega_t - t^{-1}\tilde{u}\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_3(\tilde{u},t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(\tilde{u},t),\ \Omega^* = \Omega_2(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1),t),$
	$\Omega_1 = \Omega_2(\tilde{u}_1,t),\ \Omega_1^* = \Omega_2(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1),t)$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u-\frac{x}{t}.$
5.37	$\Omega_t - J(\Omega,\Omega) = \Psi_3(u,t,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_2(u,t),\ \Omega^* = \Omega_2(\frac{1}{2}(u+u_1+gn_1),t),$
	$\Omega_1 = \Omega_2(u_1,t),\ \Omega_1^* = \Omega_2(\frac{1}{2}(u+u_1-gn_1),t)$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
6.2	$-u\Omega - J(\Omega,\Omega) = \Psi_2(u,\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,du_1\,dv_1\,dw_1$
	$\Omega = \Omega_1(u),\ \Omega^* = \Omega_1(\frac{1}{2}(u+u_1+gn_1)),$
	$\Omega_1 = \Omega_1(u_1),\ \Omega_1^* = \Omega_1(\frac{1}{2}(u+u_1-gn_1))$
	$g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
6.6 ($\alpha \neq 0$)	$-\Omega - \tilde{u}\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_2(\tilde{u},\Omega)$
	$J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\,d\mathbf{n}\,d\tilde{u}_1\,dv_1\,dw_1$
	$\Omega = \Omega_1(\tilde{u}),\ \Omega^* = \Omega_1(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1)),$
	$\Omega_1 = \Omega_1(\tilde{u}_1),\ \Omega_1^* = \Omega_1(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1))$
	$g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u-\frac{x}{t}.$
6.6 ($\alpha = 0$)	(5.26)

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
6.7	$-\Omega - \tilde{u}\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_2(\tilde{u},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$ $\Omega = \Omega_1(\tilde{u}),\ \Omega^* = \Omega_1(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1)),$ $\Omega_1 = \Omega_1(\tilde{u}_1),\ \Omega_1^* = \Omega_1(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u - \frac{x}{t}.$
6.8 ($\beta = 0,\ \alpha \neq 0$)	$-\Omega - J(\Omega,\Omega) = \Psi_2(u,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$ $\Omega = \Omega_1(u),\ \Omega^* = \Omega_1(\frac{1}{2}(u+u_1+gn_1)),$ $\Omega_1 = \Omega_1(u_1),\ \Omega_1^* = \Omega_1(\frac{1}{2}(u+u_1-gn_1))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
6.8 ($\beta = 0,\ \alpha = 0$)	(5.37)
6.11	$-J(\Omega,\Omega) = \Psi_2(u,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$ $\Omega = \Omega_1(u),\ \Omega^* = \Omega_1(\frac{1}{2}(u+u_1+gn_1)),$ $\Omega_1 = \Omega_1(u_1),\ \Omega_1^* = \Omega_1(\frac{1}{2}(u+u_1-gn_1))$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
6.12 ($\alpha \neq 0$)	$-\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_2(\tilde{u},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$ $\Omega = \Omega_1(\tilde{u}),\ \Omega^* = \Omega_1(\frac{1}{2}(\tilde{u}+\tilde{u}_1+gn_1)),$ $\Omega_1 = \Omega_1(\tilde{u}_1),\ \Omega_1^* = \Omega_1(\frac{1}{2}(\tilde{u}+\tilde{u}_1-gn_1))$ $g = \sqrt{(\tilde{u}-\tilde{u}_1)^2+(v-v_1)^2+(w-w_1)^2},\ \tilde{u} = u - t.$
6.12 ($\alpha = 0$)	(5.31)
6.13	(5.33)
6.14	$-J(\Omega,\Omega) = \Psi_2(t,\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, du_1\, dv_1\, dw_1$ $\Omega = \Omega_1(t),\ \Omega^* = \Omega_1(t),$ $\Omega_1 = \Omega_1(t),\ \Omega_1^* = \Omega_1(t)$ $g = \sqrt{(u-u_1)^2+(v-v_1)^2+(w-w_1)^2}.$
6.15	(5.35)
6.16	(6.11)
6.17	(6.2)

Table F.1: The reduced Boltzmann equation with a source (Continued).

No.	Reduced equations
6.20	$-\Omega - \alpha\Omega_{\tilde{u}} - J(\Omega,\Omega) = \Psi_2(\tilde{u},\Omega)$ $J(\Omega,\Omega) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbf{S}^2} B(g,\theta_1)(\Omega^*\Omega_1^* - \Omega\Omega_1)\, d\mathbf{n}\, d\tilde{u}_1\, dv_1\, dw_1$ $\Omega = \Omega_1(\tilde{u}),\ \Omega^* = \Omega_1(\frac{1}{2}(\tilde{u} + \tilde{u}_1 + gn_1)),$ $\Omega_1 = \Omega_1(\tilde{u}_1),\ \Omega_1^* = \Omega_1(\frac{1}{2}(\tilde{u} + \tilde{u}_1 - gn_1))$ $g = \sqrt{(\tilde{u} - \tilde{u}_1)^2 + (v - v_1)^2 + (w - w_1)^2},\ \tilde{u} = u - \alpha \ln t.$
6.21	(6.6), $\alpha \neq 0$
6.22	(6.11)
6.23	(6.12), $\alpha \neq 0$
6.24	(6.14)
6.25 ($\beta \neq 0$)	(6.14)
6.25 ($\beta = 0$)	(5.37)
7.3	(6.17)
7.5 ($\beta = 0,\ \alpha = 0$)	(6.22)
7.6	(6.21)
7.7 ($\alpha = 0$)	(6.20)
7.10 ($\alpha = 0$)	(6.23)
8.2 ($\alpha = 0$)	(7.12)
8.3 ($\alpha = 0$)	(7.14)
8.4	(7.13)
9.2	(6.14)
10.1	(7.13)
10.2	(7.14)

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