การจำแนกกรุปของระบบสมการเชิงอนุพันธ์สามัญอันดับที่สองแบบที่มีสองสมการเชิงเส้นและแบบที่มีสองสมการไม่เชิงเส้นอิสระ

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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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GROUP CLASSIFICATION OF SYSTEMS OF TWO SECOND-ORDER LINEAR AND SYSTEMS OF TWO SECOND-ORDER AUTONOMOUS NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

Giovanna Fae Oguis

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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จุดมุ่งหมายของวิทยานิพนธ์นี้คือต้องการสร้างกรุปแบบบริบูรณ์ของระบบสมการเชิงอนุพันธ์สามัญอันดับที่สองแบบที่มีสองสมการเชิงเส้นและแบบที่มีสองสมการไม่เชิงเส้นอิสระที่อยู่ในรูป $\mathbf{y}''=\mathbf{F}(\mathbf{y})$ ในวิทยานิพนธ์ฉบับนี้ได้นำเสนองานวิจัยที่เกี่ยวข้องกับการจำแนกกรุปของระบบสมการเชิงอนุพันธ์สามัญไม่เชิงเส้นที่อยู่ในรูป $\mathbf{y}''=\mathbf{F}(x,\mathbf{y})$ ก่อนที่จะศึกษาการจำแนกกรุปของระบบสมการเชิงอนุพันธ์สามัญไม่เชิงเส้นอันดับสองแบบที่มีสองสมการไม่เชิงเส้นอิสระ และผลเฉลยที่ได้จากการศึกษานี้สามารถนำไปประยุกต์ใช้ในการจำแนกกรุปของระบบสมการเชิงเส้น

หลักกลวิธี 2 ขั้นตอนของโอฟเซียนิคอร์ฟได้ถูกนำมาใช้สำหรับการจำแนกกรุป ซึ่งแนวคิดดังกล่าวจะเกี่ยวข้องกับการทำให้สมการกำหนดง่ายขึ้นด้วยการใช้การแปลงสมมูล จากนั้นหาผลเฉลยจากกรณีที่ถูกลดรูปต่าง ๆ ของตัวก่อกำเนิด ซึ่งวิธีนี้ช่วยให้สามารถศึกษาพีชคณิตลีทั้งหมดที่เป็นไปได้สมบูรณ์ครบถ้วนทุกกรณี

รายการตัวแทนคลาสทั้งหมดของระบบสมการเชิงอนุพันธ์สามัญอันดับที่สองแบบที่มีสองสมการเชิงเส้นและแบบที่มีสองสมการไม่เชิงเส้นอิสระที่อยู่ในรูป $\mathbf{y}''=\mathbf{F}(\mathbf{y})$ ได้ถูกนำเสนอในช่วงท้ายของวิทยานิพนธ์นี้อีกด้วย

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา________

ปีการศึกษา 2559 ลายมือชื่ออาจารย์ที่ปรึกษา________

GIOVANNA FAE OGUIS : GROUP CLASSIFICATION OF SYSTEMS OF TWO SECOND-ORDER LINEAR AND SYSTEMS OF TWO SECOND-ORDER AUTONOMOUS NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS
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GROUP CLASSIFICATION/ LINEAR SYSTEMS/ AUTONOMOUS NONLINEAR SYSTEMS/ SECOND-ORDER/ ORDINARY DIFFERENTIAL EQUATIONS/ ADMITTED LIE GROUP/ ADMITTED LIE ALGEBRA

The purpose of this research is to provide a complete group classification of systems of two linear second-order ordinary differential equations, and the group classification of systems of two autonomous nonlinear second-order ordinary differential equations of the form $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$. Prior to the classification of systems of two autonomous nonlinear second-order ordinary differential equations, a preliminary study on nonlinear systems of the form $\mathbf{y}'' = \mathbf{F}(x, \mathbf{y})$ is presented. The preliminary study on nonlinear systems is also applicable for the group classification of linear systems.

Ovsiannikov's 2-step technique was mainly used to obtain the group classification. This approach involves simplifying the determining equations through exploiting equivalence transformations and then solving for the reduced cases of the generators. This allows one to study all possible admitted Lie algebras without omission.

The complete list of representative classes of systems of two linear second-order ordinary differential equations and nonlinear autonomous systems of two second-order ordinary differential equations of the form $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$ are given at the end.

School of Mathematics — Student's Signature ______

Academic Year 2016 — Advisor's Signature ______

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CHAPTER I

INTRODUCTION

Systems of second-order ordinary differential equations arise in various real-world applications and have been widely studied in many fields of science. They possess many interesting features including symmetry properties. The presence of symmetries allows the reduction of order of these differential equations, or even makes it possible to find general solutions by quadratures.

Group classification studies, dating more than a century back, were first initiated by the founder of symmetry analysis, Sophus Lie (1883, 1891, 1884, 1881). These studies were long forgotten until Ovsiannikov (1958, 1978) revived the work around five decades ago. Lie's works put emphasis on tackling the group classification in two ways: 1) the direct way and 2) the indirect way also known as the algebraic approach. The direct way involves directly finding solutions of the determining equations and allows one to study all possible admitted Lie algebras without omission. On the other hand the indirect way involves solving the determining equations up to finding relations between constants defining admitted generators. The algebraic approach takes into account the algebraic properties of an admitted Lie group and the knowledge of the algebraic structure of admitted Lie algebras in order to allow group classification (Mahomed and Leach, 1989; Gonzalez-Lopez et al., 1992; Popovych et al., 2010; Grigoriev et al., 2013). In one of Lie's works (Lie, 1883), he gave a complete group classification of a single second-order ordinary differential equation of the form $y'' = f(x, y)$. Later on Ovsiannikov (2004) did this group classification in a different way. The method he used, now also known as the direct approach, involved a two-step technique where the determining equations were first simplified through exploiting equivalence transformations and later on solved for the reduced cases of the generators. The same technique was used in a study (Phauk, 2013) to classify a more general case of equations of the form $y'' = P_3(x, y; y')$, where $P_3(x, y; y')$ is a polynomial of a third degree with respect to the first-order derivative y'. Sometimes it is difficult to select or tease out equivalent cases with respect to equivalence transformations. As similarly observed in the classification of a general scalar second-order ordinary differential equation of the form $y'' = f(x, y; y')$, the application of the direct technique gives rise to overwhelming difficulties. In this thesis, both the direct and indirect techniques are employed, but mainly utilizing the direct method.

Apart from dealing with classification problems there is a significant amount of research that deals with the dimension and structure of symmetry algebras of linearizable ordinary differential equations (Gorringe and Leach, 1988; Mahomed and Leach, 1989, 1990; Wafo Soh and Mahomed, 2000; Ibragimov, 1996; Boyko et al., 2012). This is also of importance since some nonlinear equations appear in disguised forms.

Published works (Wafo Soh, 2010; Meleshko, 2011; Boyko et al., 2012; Campoamor-Stursberg, 2011, 2012) show results on systems of two second-order ordinary differential equations with constant coefficients of the form

$$\mathbf{y}'' = \mathbf{M}\mathbf{y}, \tag{1.1}$$

where $\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}$ and $\mathbf{M}$ is a matrix with constant entries. However, these papers do not exhaust the set of all systems of linear second-order differential equations. In our study (Meleshko et al., 2014), we presented the complete group classification of these linear systems of two second-order ordinary differential equations with constant coefficients.

A study by Wafo Soh and Mahomed (2000) shows results of classification of systems of two second-order linear ordinary differential equations of the form

$$\mathbf{y}'' = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix} \mathbf{y}.$$

However, the list of all distinguished representatives of systems of two second-order linear differential equations was not obtained in this paper.

Despite all these extensive studies, it was surprising that the group classification of systems of two nonlinear second-order ordinary differential equations has not yet been exhausted. Even more surprising, both the group classification of systems of two linear second-order ordinary differential equations and the group classification of systems of two autonomous nonlinear second-order ordinary differential equations of the form

$$\mathbf{y}'' = \mathbf{F}(\mathbf{y}) \tag{1.2}$$

are not yet complete. Hence, this research considers the group classification of systems of two linear second-order ordinary differential equations and systems of two autonomous nonlinear of the form (1.2).

The systems studied here are generalizations of Lie's study (Lie, 1891).

The degenerate case, which is equivalent to the following

$$y'' = F(x, y, z), \qquad z'' = 0, \tag{1.3}$$

is omitted from this research. We call systems equivalent to (1.3) reducible systems, and irreducible otherwise.

This thesis is organized as follows. Chapter II introduces some background knowledge of Lie group analysis. Chapter III presents an algorithm in finding an admitted Lie group of a system of two linear second-order ordinary differential equations, followed by its classification. Chapter IV tackles the preliminary study of systems of two nonlinear second-order ordinary differential equations, and is followed by the subsequent group classification applied to autonomous systems of two second-order ordinary differential equations of the form (1.2) in Chapter V. Lastly, Chapter VI summarizes and concludes the results of the classifications.

CHAPTER II

GROUP ANALYSIS

In 1890, Sophus Lie, a Norwegian mathematician, introduced the theory of continuous transformation groups which are now known as Lie groups. Lie group analysis is a successful method for integration of linear and nonlinear differential equations by using their symmetries. Later, these methods were applied to many types of differential equations. An introduction to this method can be found in textbooks (cf. Ovsiannikov (1978); Olver (1986); Ibragimov (1999)). A collection of results by using this method is in the Handbooks of Lie Group Analysis (1994, 1995, 1996).

In this chapter, a review on some basic concepts of group analysis is given such as a one-parameter Lie group, the Lie algebra of a generator, and invariant solutions. Group classification is given in the last section.

In this thesis, the application of continuous groups to differential equations makes no use of the global aspects of Lie groups. Hence, we focus only on local Lie groups of transformations, and for brevity, such a group will be simply called a Lie group or a group.

2.1 Lie Groups of Transformations

Definition 1. A *group* G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

1. *Closure property:* For any element a and b of G, $\phi(a, b)$ is an element of G.

2. *Associative property:* For any element a, b, and c of G,

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c).$$

3. *Identity element:* There exists a unique identity element e of G such that for any element a of G,

$$\phi(a, e) = \phi(e, a) = a.$$

4. *Inverse element:* For any element a of G there exists a unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = e.$$

Definition 2. A *subgroup* of G is a group formed by a subset of elements of G with the same law of composition ϕ.

Definition 3. Let $\mathbf{z} = (z_1, z_2, \ldots, z_N)$ lie in the region $V \subset \mathbb{R}^N$. The set of transformations

$$\bar{\mathbf{z}} = \mathbf{g}(\mathbf{z}; a),$$

defined for each $\mathbf{z} \in V$, depending on parameter $a \in \triangle \subset \mathbb{R}$, with $\phi(a, b)$ defining a law of composition of parameters a and $b \in \triangle$, as above, forms a *group of transformations* on V if:

1. For each parameter $a \in \triangle$ the transformations are one-to-one onto V, in particular $\bar{\mathbf{z}}$ lies in V.

2. $\triangle$ with the law of composition ϕ forms a group G.

3. $\bar{\mathbf{z}} = \mathbf{z}$ when $a = e$, i.e.

$$\mathbf{g}(\mathbf{z}; e) = \mathbf{z}.$$

4. If $\bar{\mathbf{z}} = \mathbf{g}(\mathbf{z}; a)$ and $\bar{\bar{\mathbf{z}}} = \mathbf{g}(\bar{\mathbf{z}}; b)$, then

$$\bar{\bar{\mathbf{z}}} = \mathbf{g}(\mathbf{z}; \phi(a, b)).$$

2.1.1 One-Parameter Lie Group of Transformations

Definition 4. A group of transformations defines a *one-parameter Lie group of transformations* if in addition to axioms 1-4 of Definition 3:

5. a is a continuous parameter, i.e. $\triangle$ is an interval in $\mathbb{R}$. Without loss of generality $a = 0$ corresponds to the identity element e.

6. $\mathbf{g}$ is infinitely differentiable with respect to $\mathbf{z} \in V$ and an analytic function of $a \in \triangle$.

7. $\phi(a, b)$ is an analytic function of a and b, $a \in \triangle$ and $b \in \triangle$.

Due to the analyticity of the group operation ϕ, it is always possible to reparametrize the Lie group in such a way that the group operation becomes the ordinary sum in $\mathbb{R}$ (see proof in Bluman and Kumei, 1989).

2.2 Infinitesimal Transformations

Consider a one-parameter (a) Lie group of transformations

$$\bar{\mathbf{z}} = \mathbf{g}(\mathbf{z}; a) \tag{2.1}$$

with identity $a = 0$ and law of composition ϕ. Expanding equations (2.1) about $a = 0$, we have (for some neighborhood of $a = 0$)

$$\begin{aligned} \bar{\mathbf{z}} &= \mathbf{z} + a\left(\frac{\partial \mathbf{g}}{\partial a}(\mathbf{z}; a)\bigg|_{a=0} \right) + \frac{a^2}{2}\left(\frac{\partial^2 \mathbf{g}}{\partial a^2}(\mathbf{z}; a)\bigg|_{a=0} \right) + \cdots \\ &= \mathbf{z} + a\left(\frac{\partial \mathbf{g}}{\partial a}(\mathbf{z}; a)\bigg|_{a=0} \right) + O(a^2). \end{aligned} \tag{2.2}$$

Let

$$\xi(\mathbf{z}) = \left(\frac{\partial \mathbf{g}}{\partial a}(\mathbf{z}; a)\bigg|_{a=0} \right). \tag{2.3}$$

The transformation $\mathbf{z} = \mathbf{z} + a\xi(\mathbf{z})$ is called the *infinitesimal transformation* of the Lie group of transformations (2.1), and the components of $\xi(\mathbf{z})$ are called the infinitesimals of (2.1).

2.2.1 First Fundamental Lie Theorem

Theorem 1 (First Fundamental Lie Theorem)**.** The Lie group of transformations (2.1) corresponds to the solution of the initial value problem for the system of first order differential equations

$$\frac{d\bar{\mathbf{z}}}{da} = \xi(\mathbf{z}), \tag{2.4a}$$

with

$$\bar{\mathbf{z}} = \mathbf{z} \ \text{ when } \ a = 0. \tag{2.4b}$$

The tangent vector $\xi(\mathbf{z})$ is written in the form of the first order differential operator (the *symbol* in Lie's notation)

$$\mathbf{X} = \xi(\mathbf{z}) \cdot \nabla = \xi_1(\mathbf{z})\frac{\partial}{\partial z_1} + \cdots + \xi_N(\mathbf{z})\frac{\partial}{\partial z_N}$$

For any differentiable function $F(\mathbf{z})$,

$$\mathbf{X}F = \xi(\mathbf{z}) \cdot \nabla F = \xi_1(\mathbf{z})\frac{\partial F}{\partial z_1} + \cdots + \xi_N(\mathbf{z})\frac{\partial F}{\partial z_N}$$

and in particular,

$$\mathbf{X}\mathbf{z} = \xi(\mathbf{z}).$$

A one-parameter Lie group of transformations, which by Theorem 1 corresponds to its infinitesimal transformation, also corresponds to its infinitesimal operator. The latter allows to represent the solution of the differential equations (2.4a) with the initial conditions (2.4b) in terms of a Taylor series (*exponential map*)

$$\bar{\mathbf{z}} = exp(a\mathbf{X})\mathbf{z} = \mathbf{z} + a\mathbf{X}\mathbf{z} + \frac{a^2}{2}\mathbf{X}^2\mathbf{z} + \cdots = \sum_{k=0}^{\infty}\frac{a^k}{k!}\mathbf{X}^k\mathbf{z}$$

where $\mathbf{X}^k\mathbf{z} = \mathbf{X}(\mathbf{X}^{k-1}\mathbf{z})$, $\mathbf{X}^0\mathbf{z} = \mathbf{z}$.

2.3 Invariance of a Function

From here, we can introduce the concept of invariance of a function with respect to a Lie group of transformations, and prove the related invariant criterion.

Definition 5. An infinitely differentiable function $F(\mathbf{z})$ is said to be an *invariant function* (or simply an *invariant*) of the Lie group of transformations (2.1) if and only if for any group transformation (2.1), the condition

$$F(\bar{\mathbf{z}}) \equiv F(\mathbf{z})$$

holds true.

The invariance of the function is characterized in a very simple way by means of the infinitesimal generator of the group, as the following theorem shows.

Theorem 2. $F(\mathbf{z})$ is invariant under (2.1) if and only if

$$\mathbf{X}F(\mathbf{z}) = 0.$$

The invariance of a surface of $\mathbb{R}^N$ with respect to a Lie group can also be defined. A surface $F(\mathbf{z}) = 0$ is said to be an *invariant surface* with respect to the one-parameter Lie group (2.1) if $F(\bar{\mathbf{z}}) = 0$ when $F(\mathbf{z}) = 0$. As a consequence of Theorem 2, the following theorem immediately follows.

Theorem 3. A surface $F(\mathbf{z}) = 0$ is invariant under (2.1) if and only if

$$\mathbf{X}F(\mathbf{z}) = 0 \text{ when } F(\mathbf{z}) = 0.$$

A Lie group of transformations may depend as well on many parameters,

$$\bar{\mathbf{z}} = \mathbf{g}(\mathbf{z}; \mathbf{a}) \tag{2.5}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_r) \in \triangle \subset \mathbb{R}^r$. The *infinitesimal matrix* $\chi(\mathbf{z})$ is the $r \times N$ matrix with entries

$$\xi_{\alpha j}(\mathbf{z}) = \left.\frac{\partial \bar{z}_j}{\partial a_\alpha}\right|_{\mathbf{a}=0} = \left.\frac{\partial g_j(\mathbf{x}; \mathbf{a})}{\partial a_\alpha}\right|_{\mathbf{a}=0}$$

$(\alpha = 1, \ldots, r; j = 1, \ldots, N)$ may be constructed, and for each parameter a_α of the r-parameter Lie group of transformations (2.5), the infinitesimal generator $\mathbf{X}_\alpha$

$$\mathbf{X}_\alpha = \sum_{j=1}^{N} \xi_{\alpha j}(\mathbf{z}) \frac{\partial}{\partial z_j} \quad (\alpha = 1, \ldots, r)$$

is defined. The infinitesimal generator

$$\mathbf{X} = \sum_{\alpha=1}^{r} \sigma_\alpha \mathbf{X}_\alpha = \sum_{j=1}^{N} \xi_j(\mathbf{z}) \frac{\partial}{\partial z_j}, \quad \xi_j(\mathbf{z}) = \sum_{\alpha=1}^{r} \sigma_\alpha \xi_{\alpha j}(\mathbf{z})$$

where $\sigma_1, \ldots, \sigma_r$ are fixed real constants, in turn defines a one-parameter subgroup of an r-parameter Lie group of transformations.

Now for a given system of differential equations ε, the variable $\mathbf{z}$ is separated into two parts, $\mathbf{z} = (\mathbf{x}, \mathbf{u}) \in V \subset \mathbb{R}^n \times \mathbb{R}^m, N = n + m$. Here, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is the independent variable, $\mathbf{u} = (u^1, u^2, \ldots, u^m) \in \mathbb{R}^m$ is the dependent variable. The transformations (2.1) can be decomposed as

$$\begin{aligned} \bar{\mathbf{x}} &= \varphi(\mathbf{x}, \mathbf{u}; a), \\ \bar{\mathbf{u}} &= \psi(\mathbf{x}, \mathbf{u}; a). \end{aligned} \tag{2.6}$$

Also, let

$$\mathbf{u} = \mathbf{u_0}(\mathbf{x}) = (u_0^1(\mathbf{x}), u_0^2(\mathbf{x}), \ldots, u_0^m(\mathbf{x}))$$

be a solution of the equations ε. A Lie group of transformation of the form (2.6) admitted by ε has the two equivalent properties:

1. a transformation of the group maps any solution of ε into another solution of ε;

2. a transformation of the group leaves ε invariant, say, ε reads the same in terms of the variables $(\mathbf{x}, \mathbf{u})$ and in terms of the transformed variables $(\overline{\mathbf{x}}, \overline{\mathbf{u}})$.

The transformations (2.6) determine suitable transformations for the derivatives of the dependent variables $\mathbf{u}$ with respect to the independent variables $\mathbf{x}$. Let $\mathbf{u}^{(\mathbf{1})}$

denote the set of all $m \cdot n$ first order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$,

$$\mathbf{u}^{(1)} \equiv \left(\frac{\partial u^1}{\partial x_1}, \cdots, \frac{\partial u^1}{\partial x_n}, \cdots, \frac{\partial u^m}{\partial x_1}, \cdots, \frac{\partial u^m}{\partial x_n} \right)$$

and in general, let $\mathbf{u}^{(k)}$ denote the set of all kth-order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$. The transformations of the derivatives of the dependent variables lead to a natural extensions (prolongations) of the one-parameter Lie group of transformations (2.6). While the one-parameter Lie group of transformations (2.6) acts on the space $(\mathbf{x}, \mathbf{u})$, the extended group acts on the space $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)})$, and more in general, on the jet space $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)})$. Since all the information about a Lie group of transformations is contained in its infinitesimal generator, we need to compute its prolongations:

1. the first prolongation

$$\mathbf{X}^{(1)} = \mathbf{X} + \sum_{j=1}^{m} \sum_{i=1}^{n} \eta^j_{[i]}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) \frac{\partial}{\partial u^j_i}, \quad u^j_i = \frac{\partial u^j}{\partial x_i}$$

with

$$\eta^j_{[i]}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) = \frac{D\eta^j}{Dx_i} - \frac{D\xi_j}{Dx_i} \frac{\partial u^j}{\partial x_j}$$

2. the general kth-order prolongation recursively defined by

$$\mathbf{X}^{(k)} = \mathbf{X}^{(k-1)} + \sum_{j=1}^{m} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \eta^j_{[i_1,\ldots,i_k]} \frac{\partial}{\partial u^j_{i_1,\ldots,i_k}}, \quad u^j_{i_1,\ldots,i_k} = \frac{\partial^k u^j}{\partial x_{i_1} \ldots \partial x_{i_k}}$$

with

$$\eta^j_{[i_1,\ldots,i_k]} = \frac{D\eta^j_{[i_1,\ldots,i_{k-1}]}}{Dx_{ik}} - u^j_{i_1,\ldots,i_{k-1}j} \frac{D\xi_j}{Dx_{ik}}$$

Note that the Lie derivative $\frac{D}{Dx_i}$ is defined as

$$\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \frac{\partial u^j}{\partial x_i} \frac{\partial}{\partial u^j} + \frac{\partial^2 u^j}{\partial x_i \partial x_j} \frac{\partial}{\partial u^j_j} + \ldots,$$

and the Einstein convention of summation over repeated indices is used (and this notation is adopted all throughout the manuscript).

Remarkably, the search for one-parameter Lie groups of transformations leaving differential equations invariant leads usually to r-parameter Lie groups of transformations. Let

$$\mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}) = 0 \tag{2.7}$$

($\mathbf{F} = (F_1, \ldots, F_q)$) be a system of q differential equations of order k, with independent variables $\mathbf{x} \in \mathbb{R}^n$ and dependent variables $\mathbf{u} \in \mathbb{R}^m$. Suppose the system is written in normal form, i.e., it is solved with respect to some partial derivatives of order k_v for $v = 1, \ldots, q$:

$$F_v(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}) \equiv u^{j_v}_{i_1, \ldots, i_{k_v}} - f_v(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}) = 0. \tag{2.8}$$

The equations (2.8) can be considered as characterizing a submanifold in the kth-order *jet space*. One says that the one-parameter Lie group of transformations (2.6) leaves the system (2.8) *invariant* (*is admitted by* (2.8)) if and only if its kth prolongation leaves the submanifold of the jet space defined by (2.8) invariant.

2.4 Algorithm of Finding Lie Groups Admitted by Differential Equations

The following theorem, which is a consequence of Theorem 3, leads directly to the algorithm for the computation of the infinitesimals admitted by a given differential system.

Theorem 4 (Infinitesimal Criterion for differential equations)**.** Let

$$X = \xi_i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x_i} + \eta^j(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^j}$$

be the infinitesimal generator corresponding to (2.6) and $X^{(k)}$ the kth prolonged infinitesimal generator. The group (2.6) is admitted by the system (2.8) if and only if

$$X^{(k)}\mathbf{F}(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)}) = 0 \text{ whenever } \mathbf{F}(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)}) = 0. \tag{2.9}$$

If the differential system is in polynomial form in the derivatives, then the invariance condition (2.9) are polynomials in the derivative components, with coefficients expressed in linear combinations of the unknown ξ_i, η^j and their partial derivatives. Using (2.8) to eliminate the derivatives $u^{j_v}_{i_1,\ldots,i_{k_v}}$, the equations can be split with respect to the components of the remaining derivatives of $\mathbf{u}$ that can be arbitrarily varied (also called *parametric derivatives*). By equating the coefficients of these partial derivatives to zero, one obtains an overdetermined system of linear differential equations for the infinitesimals (also called system of *determining equations*), whose integration leads to the infinitesimals of the group. The infinitesimals involve arbitrary constants (and in some cases arbitrary functions) and hence, we have *de facto* r-parameter Lie groups (infinite-parameter Lie groups if arbitrary functions are involved). Note that the general solution of the determining equations generates a principal Lie algebra LS of the system ε. The set of transformations, which is finitely generated by one-parameter Lie groups corresponding to the generators $X \in LS$ is called a principal Lie group admitted by the system ε.

In this thesis, we limit ourselves to dealing with Lie groups of transformations admitted by differential equations with infinitesimals depending on the independent and dependent variables only. These are called *local Lie point symmetries*. Symmetries where the infinitesimals may depend on first (respectively, higher) order derivatives of the dependent variables with respect to the independent variables are called contact (respectively, generalized) symmetries, and symmetries with infinitesimals depending also on integrals of dependent variables are called nonlocal symmetries.

2.5 Lie Algebras of a Generator

Definition 6 (Lie Algebra)**.** The infinitesimal generators of an r-parameter Lie group, being solutions of a linear system of partial differential equations, span an r-dimensional vector space; by introducing an operation of commutation between two infinitesimal generators,

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha,$$

which is bilinear, antisymmetric and satisfies the Jacobi identity, say,

$$[[X_\alpha, X_\beta]X_\gamma] + [[X_\beta, X_\gamma], X_\alpha] + [[X_\gamma, X_\alpha], X_\beta] = 0,$$

the vector space of infinitesimal generators gains the structure of a Lie algebra.

It is worth to emphasize that the commutator of two infinitesimal generators is invariant with respect to any invertible change of variables, and commutes with the operation of prolongation.

Definition 7. A vector space L of generators is a Lie algebra if the commutator $[X_1, X_2]$ of any two generators $X_1 \in L$ and $X_2 \in L$ belongs to L.

Lemma 5. A commutator is invariant with respect to any change of variables.

For the proof of this, consider the change of variables $\tilde{z} = q(z)$. As the generators are invariant with respect to this operation, it follows that $X = X' = X(q_i)\partial_{\bar{z}_i}$ and $Y = Y' = Y(q_i)\partial_{\bar{z}_i}$. Hence.

$$[X', Y'] = (X'(Y(q_i)) - Y'(X(q_i)))\,\partial_{\bar{z}_i}$$
$$= (X(Y(q_i)) - Y(X(q_i)))\,\partial_{\bar{z}_i} = [X, Y](q_i)\partial_{\bar{z}_i} = [X, Y]'.$$

Theorem 6. If a system ε admits generators X and Y, then it admits their commutator $[X, Y]$.

This theorem means that the vector space of all admitted generators is a Lie algebra (admitted by the system ε). This algebra is called the principal algebra. To construct exact solutions, one uses subalgebras of the admitted algebra.

Definition 8 (Subalgebra)**.** A vector subspace $L' \subset L$ of Lie algebra L is called a subalgebra if it is a Lie algebra,i.e., for arbitrary vectors X_α and X_β from L', their commutator $[X_\alpha, X_\beta]$ belongs to L'.

Definition 9 (Ideal)**.** Let $I \subset L$ be a subspace of Lie algebra L such that $[X, Y] \in I, \quad \forall X \in I$ and $\forall Y \in L$ holds. The subspace I is called an ideal.

Definition 10 (Similar Lie Algebras)**.** Two Lie algebras L' and L'' are similar if there exists a change of variables that transforms one into the other.

Hence, if Lie algebras L' and L'' are similar, then the generators $X = \zeta^\beta(z)\partial_{z_\beta} \in L'$ and $\hat{X} = \hat{\zeta}^\beta(\bar{z})\partial_{\bar{z}_\beta} \in L''$ of these algebras are related by the formula

$$\bar{\zeta}^\beta(\bar{z}) = X(q^\beta(z))\big|_{z=q^{-1}(\bar{z})}.$$

A linear one-to-one map f of a Lie algebra L onto a Lie algebra K is called an isomorphism (algebra L and K are said to be isomorphic) if

$$f([X_\mu, X_\nu]_L) = [f(X_\mu), f(X_\nu)]_K,$$

where the indices L and K are used to denote the commutator in the corresponding algebra. An isomorphism of L onto itself is termed an automorphism. Therefore the set of all subalgebras can be classified with respect to automorphisms.

If L is an r-dimensional vector space of infinitesimal generators closed under the operation of commutation, i.e., L is an r-dimensional Lie algebra, and $\{X_1, \ldots, X_r\}$ is a basis, then

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} C^\gamma_{\alpha\beta} X_\gamma$$

with constant coefficients $C^{\gamma}_{\alpha\beta}$ known as *structure constants*; they transform like the components of a tensor under the changes of bases.

Notice that two Lie algebras are isomorphic if they have the same structure constants in an appropriately chosen basis.

For a given Lie algebra L_r with basis $\{X_1, X_2, ..., X_r\}$, any $X \in L$ is written as

$$X = x_\mu X_\mu.$$

Hence, elements of L_r are represented by vectors $x = (x_1, ..., x_r)$. Let L^A_r be the Lie algebra spanned by the following operators,

$$E_\mu = c^\lambda_{\mu\nu} x_\nu \frac{\partial}{\partial x_\lambda}, \quad \mu = 1, ..., r,$$

with the commutator defined as in Definition 6. The algebra L^A_r generates the group G^A of linear transformations of $\{x_\mu\}$. These transformations determine automorphisms of the Lie algebra L_r known as inner automorphisms. This set is denoted by $\mathrm{Int}(L_r)$. Accordingly, G^A is called the group of inner automorphisms of L_r, or the adjoint group of G. Any subalgebra $L_s \subset L_r$ is transformed into a similar subalgebra by an element of $\mathrm{Int}(L_r)$. Similarity is an equivalence relation; the collection of similar subalgebras of the same dimension compose a class.

Definition 11 (Optimal System)**.** A set of representatives from all classes is called an optimal system of subalgebras.

Thus, an optimal system of subalgebras of a Lie algebra L with inner automorphisms $A = \mathrm{Int}(L)$ is a collection of subalgebras $\Theta_A(L)$ such that

(1) No two elements of this collection can be transformed into each other by an inner automorphism of the Lie algebra L.

(2) Every subalgebra of the Lie algebra L can be transformed into one of the subalgebras of the set $\Theta_A(L)$ by an inner automorphism.

2.6 Use of Lie Symmetries of Differential Equations

The knowledge of Lie groups of transformations admitted by a given system of differential equations can be used to

1. lower the order or eventually reduce the equation to quadrature, in the case of ordinary differential equations; and

2. determine particular solutions, called *invariant and partially-invariant solutions*, or generate new solutions, once a special solution is known, in the case of ordinary or partial differential equations.

2.6.1 Invariant Solutions of Partial Differential Equations

The function $\mathbf{u} = \mathbf{u}_0(\mathbf{x})$ with components $u^j = u_0^j(\mathbf{x}) \quad (j = 1, \ldots, m)$, is said to be an *invariant solution* of (2.7) if $u^j = u_0^j(\mathbf{x})$ is an invariant surface of (2.6), and is a solution of (2.7), i.e., a solution is invariant if and only if

$$\begin{gathered} X(u^j - u_0^j(\mathbf{x})) = 0 \text{ for } u^j = u_0^j(\mathbf{x}) \quad (j = 1, \ldots, m) \\ \mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}) = 0. \end{gathered} \tag{2.10}$$

The first equations of (2.10), called the *invariant surface conditions*, have the form

$$\xi_1(\mathbf{x}, \mathbf{u}) \frac{\partial u^j}{\partial x_1} + \cdots + \xi_n(\mathbf{x}, \mathbf{u}) \frac{\partial u^j}{\partial x_n} = \eta^j(\mathbf{x}, \mathbf{u}) \quad (j = 1, \ldots, m)$$

and are solved by introducing the corresponding characteristic equations:

$$\frac{dx_1}{\xi_1(\mathbf{x}, \mathbf{u})} = \cdots = \frac{dx_n}{\xi_n(\mathbf{x}, \mathbf{u})} = \frac{du^1}{\eta_1(\mathbf{x}, \mathbf{u})} = \cdots = \frac{du^m}{\eta_m(\mathbf{x}, \mathbf{u})}.$$

This allows to express the solution $\mathbf{u} = \mathbf{u}_0(\mathbf{x})$ as

$$u^j = \psi^j(I_1(\mathbf{x}, \mathbf{u}), \ldots, I_{n-1}(\mathbf{x}, \mathbf{u})) \ (j = 1, \ldots, m).$$

By substituting this into the second equation of (2.10), a reduced system of differential equations involving $(n-1)$ independent variables (called *similarity variables*) is obtained. The name *similarity variables* is due to the fact that the scaling invariance, i.e., the invariance under the similarity transformations, was one of the first examples where this procedure has been used systematically.

2.7 Group Classification

Many differential equations involve arbitrary elements, constants, parameters or functions, which need to be determined. Mainly, these arbitrary elements are determined experimentally. However, the Lie group analysis has shown to be a versatile tool in specifying the forms of these elements systematically. The group classification problem consists of finding all principal Lie groups admitted by a system of partial differential equations. Part of these groups is admitted for all arbitrary elements. This part is called the kernel of admitted Lie groups. Another part depends on the specification of the arbitrary elements. This part contains nonequivalent extensions of the kernel. In this thesis, the system of two linear second-order ordinary differential equations and the system of two autonomous nonlinear second-order ordinary differential equations without the first-order derivatives are the chosen functions for classification.

The first problem of group classification is constructing transformations which change arbitrary elements, while preserving the differential structure of the equations themselves. These transformations are called equivalence transformations. The group classification is regarded with respect to such transformations.

At the stage where one studies for the specific cases of arbitrary elements, it is important to emphasize that there are several methods for solving the determining equations: i.e., using 1) the direct approach and/or 2) the algebraic approach. The direct method involves utilizing equivalence transformations to obtain generators of simple equations, which later on are substituted into the determining equations in order to find extensions of the generators. On the other hand, the algebraic approach involves solving the determining equations up to finding relations between constants defining admitted generators. This takes into account the algebraic properties of an admitted Lie group and the knowledge of the algebraic structure of the admitted Lie algebras. In this thesis, the direct method is mainly implemented.

2.7.1 Equivalence Lie Group

Consider a system of differential equations:

$$F^k(x,u,p,\phi)=0, \quad (k=1,\dots,s), \tag{2.11}$$

where $\phi : V \to \mathbb{R}^t$ are arbitrary elements of system (2.11) and $(x,u)\in V\subset\mathbb{R}^{n+m}$.

A nondegenerate change of dependent and independent variables that transforms a system of differential equations (2.11) to a system of equations of the same class or the same structure is called an equivalence transformation.

In order to find a Lie group of equivalence transformations, one must construct a transformation of the space $\mathbb{R}^{n+m+t}(x,u,\phi)$ that preserves the equations whilst only changing their representative $\phi=\phi(x,u)$. For this purpose, a one parameter Lie group of transformations of the space $\mathbb{R}^{n+m+t}$ with the group parameter a is used. Suppose that the following transformations compose a Lie group of equivalence transformations:

$$\bar{x}=f^x(x,u,\phi;a), \quad \bar{u}=f^u(x,u,\phi;a), \quad \bar{\phi}=f^\phi(x,u,\phi;a). \tag{2.12}$$

So the infinitesimal generator of this group (2.12) has the form

$$X^e=\xi^{x_i}\partial_{x_i}+\zeta^{u^j}\partial_{u^j}+\zeta^{\phi^k}\partial_{\phi^k}$$

with the coefficients

$$\xi^{x_i} = \left.\frac{\partial f^{x_i}(x,u,\phi;a)}{\partial a}\right|_{a=0},$$
$$\zeta^{u^j} = \left.\frac{\partial f^{u^j}(x,u,\phi;a)}{\partial a}\right|_{a=0},$$
$$\zeta^{\phi^k} = \left.\frac{\partial f^{\phi^k}(x,u,\phi;a)}{\partial a}\right|_{a=0},$$

where $i = 1,\dots,n;\quad j = 1,\dots,m;$ and $k = 1,\dots,t$. The main requirement for the Lie group of equivalence transformations is that any solution $u_0(x)$ of the system (2.11) with the functions $\phi(x,u)$ is transformed by (2.12) into a solution $u = u_a(\bar{x})$ of the system (2.11) of the same equations F^k but with other transformed functions $\phi_a(x,u)$. The functions $\phi_a(x,u)$ are defined as follows. Solving the relations

$$\bar{x} = f^x(x,u,\phi(x,u);a), \quad \bar{u} = f^u(x,u,\phi(x,u);a)$$

for (x,u), one obtains

$$x = g^x(\bar{x},\bar{u};a), \quad u = g^u(\bar{x},\bar{u};a). \tag{2.13}$$

The transformed function is

$$\phi_a(\bar{x},\bar{u}) = f^\phi(x,u,\phi(x,u);a),$$

where instead of (x,u), one has to substitute the expressions (2.13). Because of the definition of the function $\phi_a(\bar{x},\bar{u})$, the identity with respect to x and u follows:

$$(\phi_a \circ (f^x, f^u))(x,u,\phi(x,u);a) = f^\phi(x,u,\phi(x,u);a).$$

The transformed solution $T_a(u) = u_a(x)$ is obtained by solving the relations

$$\bar{x} = f^x(x,u_0(x),\phi(x,u_0(x));a)$$

for x and substituting the solution $x = \phi^x(\bar{x};a)$ into

$$u_a(\bar{x}) = f^u(x,u_0(x),\phi(x,u_0(x));a).$$

As for the function ϕ_a, the following identity with respect to x follows:

$$(u_a \circ f^x)(x, u_0(x), \phi(x, u_0(x)); a) = f^u(x, u_0(x), \phi(x, u_0(x)); a). \qquad (2.14)$$

Formulae for transformations of the partial derivatives $\bar{p}_a = f^p(x, u, p, \phi, \ldots, a)$ are obtained by differentiating (2.14) with respect to $\bar{x}$.

Lemma 7. The transformations $T_a(u)$, as constructed above, form a group.

The proof of this lemma follows from the property of a Lie group of transformations and the sequence of the equalities

$$\bar{x} = f^x(x, u_0(x), \phi(x, u_0(x)); a), \quad u_a(\bar{x}) = f^u(x, u_0(x), \phi(x, u_0(x)); a)$$
$$\tilde{x} = f^x(\bar{x}, u_a(\bar{x}), \phi_a(\bar{x}, u_a(\bar{x})); b), \quad u_b(\tilde{x}) = f^u(\bar{x}, u_a(\bar{x}), \phi_a(\bar{x}, u_a(\bar{x})); b)$$
$$(u_b \circ f^x)(\bar{x}, u_a(\bar{x}), \phi_a(\bar{x}, u_a(\bar{x})); b) = f^u(\bar{x}, u_a(\bar{x}), \phi_a(\bar{x}, u_a(\bar{x})); b)$$
$$f^u(f^x(x, u_0(x), \phi(x, u_0(x)); a), f^u(x, u_0(x), \phi(x, u_0(x)); a),$$
$$f^\phi(x, u_0(x), \phi(x, u_0(x)); a); b) = f^u(x, u_0(x), \phi(x, u_0(x)); a + b)$$
$$= (u_{a+b} \circ f^x)(x, u_0(x), \phi(x, u_0(x)); a + b).$$

Since the transformed function $u_a(\bar{x})$ is a solution of system (2.11) and along with the transformed arbitrary elements $\phi_a(\bar{x}, \bar{u})$, the equations

$$F^k(\bar{x}, u_a\bar{x}, \bar{p}_a(\bar{x}), \phi_a(\bar{x}, u_a(\bar{x}))) = 0, \quad (k = 1, \ldots, s)$$

are satisfied for any arbitrary $\bar{x}$. By one-to-one correspondence between x and $\bar{x}$, it follows that

$$F^k(f^x(z(x); a), f^u(z(x); a), f^p(z_p(x); a), f^\phi(z(x))) = 0, \quad (k = 1, \ldots, s)$$

where $z(x) = (x, u_0(x), \phi(x, u_0(x)))$ and $z_p(x) = (x, u_0(x), \phi(x, u_0(x)), p_0(x), ...)$. After differentiating these equations with respect to the group parameter a evaluated at 0, one obtains an algorithm for finding equivalence transformations (2.12).

The difference in the algorithms for obtaining an admitted Lie group and equivalence group is only in the prolongation formulae of the infinitesimal generator. Hence, after differentiating these equations with respect to the group parameter a, the determining equations

$$\tilde{X}^e F^k(x,u,p,\phi)\Big|_{\varepsilon} = 0 \quad (k = 1,\ldots,s) \tag{2.15}$$

are obtained. The prolonged operator for the equivalence Lie group is

$$\tilde{X}^e = X^e + \zeta^{u_x}\partial_{u_x} + \zeta^{\phi_x}\partial_{\phi_x} + \zeta^{\phi_u}\partial_{\phi_u} + \ldots$$

where the coordinates related to the dependent functions are

$$\zeta^{u_\lambda} = D^e_\lambda \zeta^u - u_x D^e_\lambda \xi^x, \quad D^e_\lambda = \partial_\lambda + u_\lambda \partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi,$$

where λ takes the values x_i, $(i = 1,\ldots,n)$, and the coordinates related to the arbitrary elements are

$$\zeta^{\phi_\gamma} = \tilde{D}^e_\gamma \zeta^\phi - \phi_x \tilde{D}^e_\gamma \xi^x - \phi_u \tilde{D}^e_\gamma \zeta^u, \quad \tilde{D}^e_\gamma = \partial_\gamma + \phi_\gamma \partial_\phi,$$

where γ takes the values x_i and u^j $(i = 1,\ldots,n, \quad j = 1,\ldots,m)$. The sign $|_\varepsilon$ means that the equations $\tilde{X}^e F^k(x,u,p,\phi)$ are considered on any solution $u_0(x)$ of system (2.11). The solution of the determining equations (2.15) gives the coefficients of the infinitesimal generator. The set of transformations, which is finitely generated by one-parameter Lie groups corresponding to the generators X^e, is called an equivalence group. This group is denoted by GS^e.

Theorem 8. The kernel of the principal Lie groups is included in the equivalence group GS^e.

The kernel and the equivalence group GS^e are considered in the same approach.

Remark 1. In some cases, additional requirements are included for arbitrary elements. For example, it is supposed that the arbitrary elements ϕ^u do not depend on the independent variables, i.e. $\dfrac{\partial \phi^u}{\partial x_k} = 0$. These conditions have to be appended to the original system of differential equations (2.11). These lead to additional determining equations.

CHAPTER III

APPLICATION OF GROUP ANALYSIS TO LINEAR SYSTEMS

The general form of a system of two linear second-order ordinary differential equations is

$$\mathbf{y}'' = B(x)\mathbf{y}' + A(x)\mathbf{y} + f(x), \tag{3.1}$$

where $A(x)$ and $B(x)$ are 2×2 matrices and $f(x)$ is a vector. In studying symmetries, it is convenient to rewrite equations in their simplest equivalent form. Hence, a simpler equivalent form of (3.1) is sought first before proceeding to the group classification.

Using a particular solution $\mathbf{y}_p(x)$ and the change $\mathbf{y} = \widetilde{\mathbf{y}} + \mathbf{y}_p$, without loss of generality, it can be assumed that $f(x) = 0$. Applying the change $\mathbf{y} = C(x)\widetilde{\mathbf{y}}$, where $C = C(x)$ is a nonsingular matrix, system (3.1) becomes

$$\widetilde{\mathbf{y}}'' = \widetilde{B}(x)\widetilde{\mathbf{y}}' + \widetilde{A}(x)\widetilde{\mathbf{y}}, \tag{3.2}$$

where $\widetilde{B} = C^{-1}(BC - 2C')$ and $\widetilde{A} = C^{-1}(AC + BC' - C'')$. If one chooses the matrix $C(x)$ such that $C' = \frac{1}{2}BC$, then $\widetilde{B} = 0$ and $\widetilde{A} = C^{-1}\left(A + \frac{1}{4}B^2 - \frac{1}{2}B'\right)C$. The existence of the nonsingular matrix $C(x)$ is guaranteed by the existence of the solution of the Cauchy problem

$$\begin{cases} C' = \frac{1}{2}BC \\ C(0) = I_2, \end{cases}$$

where I_2 is the unit 2×2 matrix. Notice that if the matrices A and B are constant, then the matrix $\widetilde{A}$ in (3.2) is constant only for commuting matrices A

and B. The complete study of noncommutative constant matrices A and B was done in (Meleshko et al., 2014). Without loss of generality up to equivalence transformations in the class of systems of the form (3.1), it suffices to study the systems of the form

$$\widetilde{\mathbf{y}}'' = \widetilde{A}(x)\widetilde{\mathbf{y}}. \tag{3.3}$$

Note that the above process of simplification of the 2×2 systems of the form (3.1) to systems of the form (3.3) can be extended to any $n \times n$ linear system.

Therefore, the classical group analysis method, which is described in detail in the succeeding sections, is applied to the system of equations

$$\mathbf{y}'' = \mathbf{A}\mathbf{y}, \tag{3.4}$$

where $\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}$. Another similar notation is also used in this thesis, i.e.,

$$\mathbf{y}'' = \mathbf{F}(\mathbf{x}, \mathbf{y}), \tag{3.5}$$

where $\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}$ and $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} F(x,y,z) \\ G(x,y,z) \end{pmatrix}$ where

$$\begin{cases} F(x,y,z) = a_{11}(x)y + a_{12}(x)z \\ G(x,y,z) = a_{21}(x)y + a_{22}(x)z. \end{cases} \tag{3.6}$$

Before finding the admitted Lie algebras of the linear system, it is essential to compute the equivalence transformations of the given system.

Notice also that every system of linear equations (3.4) admits the following generators:

$$y\partial_y + z\partial_z, \tag{3.7}$$

$$\zeta_1(x)\partial_y + \zeta_2(x)\partial_z, \tag{3.8}$$

where (3.7) is the homogeneity symmetry, and $\zeta_1(x)$ and $\zeta_2(x)$ are solutions of the equations (3.4), i.e.,

$$\zeta_1'' = a_{11}(x)\zeta_1 + a_{12}(x)\zeta_2, \qquad \zeta_2'' = a_{21}(x)\zeta_1 + a_{22}(x)\zeta_2.$$

Thus, for the classification problem, one needs to study systems of linear equations (3.4) which admit generators apart from (3.7) and (3.8).

3.1 Equivalence Transformations of (3.4)

Consider the linear system (3.4). Equivalence transformations of the studied system of equations are considered in this section. The arbitrary elements are the functions $a_{ij}(x)$, where the indices i and j run over the values 1 to 2 (For this chapter, i, $j = 1, 2$ is applied to all texts.). The generator of the equivalence Lie group is assumed to be in the form

$$X^e = \xi\partial_x + \eta^y\partial_y + \eta^z\partial_z + \zeta^{a_{ij}}\partial_{a_{ij}},$$

where the coefficients ξ, η^y, η^z, and $\zeta^{a_{ij}}$'s depend on the variables x, y, z, and a_{ij}'s. Note here that the summation with respect to repeated indices is assumed over i, $j = 1, 2$. The prolonged operator is

$$\widetilde{X^e} = X^e + \eta^{y'}\partial_{y'} + \eta^{z'}\partial_{z'} + \eta^{y''}\partial_{y''} + \eta^{z''}\partial_{z''} + \zeta^{a_{ij_x}}\partial_{a_{ij_x}} + \zeta^{a_{ij_y}}\partial_{a_{ij_y}} + \zeta^{a_{ij_z}}\partial_{a_{ij_z}}.$$

Note that the conditions $\dfrac{\partial a_{ij}}{\partial y} = 0$ and $\dfrac{\partial a_{ij}}{\partial z} = 0$ are appended to the original system. The coefficients of the prolonged generator are

$$\begin{aligned}
\eta^{y'} &= D_x^e\eta^y - y'D_x^e\xi, & \eta^{y''} &= D_x^e\eta^{y'} - y''D_x^e\xi,\\
\eta^{z'} &= D_x^e\eta^z - z'D_x^e\xi, & \eta^{z''} &= D_x^e\eta^{z'} - z''D_x^e\xi,\\
\zeta^{a_{ij_x}} &= \widetilde{D}_x^e\zeta^{a_{ij}} - a_{ij}'\widetilde{D}_x^e\xi, & \zeta^{a_{ij_y}} &= \widetilde{D}_y^e\zeta^{a_{ij}} - a_{ij}'\widetilde{D}_y^e\xi,\\
\zeta^{a_{ij_z}} &= \widetilde{D}_z^e\zeta^{a_{ij}} - a_{ij}'\widetilde{D}_z^e\xi.
\end{aligned}$$

Here, the operators D_x^e, $\widetilde{D}_x^e$, $\widetilde{D}_y^e$ and $\widetilde{D}_z^e$ are

$$\begin{aligned}
D_x^e &= \partial_x + y'\partial_y + z'\partial_z + y''\partial_{y'} + z''\partial_{z'} + a'_{ij}\partial_{a_{ij}} + a''_{ij}\partial_{a'_{ij}},\\
\widetilde{D}_x^e &= \partial_x + a'_{ij}\partial_{a_{ij}},\\
\widetilde{D}_y^e &= \partial_y,\\
\widetilde{D}_z^e &= \partial_z.
\end{aligned}$$

The determining equations of the equivalence Lie group become

$$\begin{aligned}
\left(\eta^{y''} - \zeta^{a_{11}}y - \zeta^{a_{12}}z - a_{11}\eta^y - a_{12}\eta^z\right)|_{\mathbf{y}''=\mathbf{A}\mathbf{y}} = 0,\\
\left(\eta^{z''} - \zeta^{a_{21}}y - \zeta^{a_{22}}z - a_{21}\eta^y - a_{22}\eta^z\right)|_{\mathbf{y}''=\mathbf{A}\mathbf{y}} = 0.
\end{aligned}$$

After substitutions of $\eta^{y'}$, $\eta^{y''}$, $\eta^{z'}$, $\eta^{z''}$, $\zeta^{a_{ij_x}}$, $\zeta^{a_{ij_y}}$, and $\zeta^{a_{ij_z}}$ and the transition onto the manifold $\mathbf{y}'' = \mathbf{A}\mathbf{y}$, the determining equations are split with respect to the variables $\mathbf{y}'$, a'_{ij}'s, and a''_{ij}'s. Initial analysis of the split determining equations leads to conditions that $\zeta^{a_{ij}}$'s do not depend on y and z, η^y and η^z do not depend on the a_{ij}'s, and ξ do not depend on y, z and a_{ij}'s. From here, it follows that $\xi = \xi(x)$. As a result, the remaining determining equations are as follows:

$$\begin{aligned}
&\eta^y_{yy} = 0, \quad \eta^y_{zz} = 0, \quad \eta^y_{yz} = 0, \quad \eta^y_{xz} = 0, \quad 2\eta^y_{xy} - \xi'' = 0,\\
&\eta^z_{yy} = 0, \quad \eta^z_{zz} = 0, \quad \eta^z_{yz} = 0, \quad \eta^z_{xy} = 0, \quad 2\eta^z_{xz} - \xi'' = 0,
\end{aligned} \tag{3.9a}$$

$$\begin{aligned}
&\eta^y_{xx} + \eta^y_y a_{11}y + \eta^y_y a_{12}z + \eta^y_z a_{21}y + \eta^y_z a_{22}z - 2\xi' a_{11}y - 2\xi' a_{12}z\\
&\quad - a_{11}\eta^y - a_{12}\eta^z - \zeta^{a_{11}}y - \zeta^{a_12}z = 0,
\end{aligned} \tag{3.9b}$$

$$\begin{aligned}
&\eta^z_{xx} + \eta^z_y a_{11}y + \eta^z_y a_{12}z + \eta^z_z a_{21}y + \eta^z_z a_{22}z - 2\xi' a_{21}y - 2\xi' a_{22}z\\
&\quad - a_{21}\eta^y - a_{22}\eta^z - \zeta^{a_{21}}y - \zeta^{a_22}z = 0.
\end{aligned} \tag{3.9c}$$

Solving equations (3.9a), it follows that

$$\begin{aligned}
\eta^y &= \tfrac{1}{2}\xi' y + k_1 y + k_2 z + \zeta_1(x),\\
\eta^z &= \tfrac{1}{2}\xi' z + k_3 z + k_4 y + \zeta_2(x),
\end{aligned}$$

where k_l's ($l = 1, \ldots, 4$) are constant. Substituting these into equations (3.9b) and (3.9c), and splitting these equations further with respect to y and z, the following solutions are obtained:

$$\begin{aligned}
\zeta^{a_{11}} &= \tfrac{1}{2}\xi''' - 2\xi' a_{11} - a_{12}k_4 + a_{21}k_2,\\
\zeta^{a_{12}} &= -2\xi' a_{12} + (a_{22} - a_{11})k_2 + a_{12}(k_1 - k_3),\\
\zeta^{a_{21}} &= -2\xi' a_{21} - (a_{22} - a_{11})k_4 - a_{21}(k_1 - k_3),\\
\zeta^{a_{22}} &= \tfrac{1}{2}\xi''' - 2\xi' a_{22} + a_{12}k_4 - a_{21}k_2.
\end{aligned}$$

Note also that $\zeta_1(x)$ and $\zeta_2(x)$ are solutions of the linear system (3.4), i.e.,

$$\zeta_1'' = a_{11}(x)\zeta_1 + a_{12}(x)\zeta_2, \qquad \zeta_2'' = a_{21}(x)\zeta_1 + a_{22}(x)\zeta_2.$$

From the above calculations*, it is shown that the equivalence Lie group of system (3.4) is defined by the following generators:

$$\begin{aligned}
X_1^e &: z\partial_y + a_{21}\partial_{a_{11}} + (a_{22} - a_{11})\partial_{a_{12}} - a_{21}\partial_{a_{22}}\\
X_2^e &: y\partial_z - a_{12}\partial_{a_{11}} + (a_{11} - a_{22})\partial_{a_{21}} - a_{12}\partial_{a_{22}}\\
X_3^e &: y\partial_y + z\partial_z\\
X_4^e &: y\partial_y - z\partial_z + 2(a_{12}\partial_{a_{12}} - a_{21}\partial_{a_{21}})\\
X_5^e &: 2\xi\partial_x + \xi'(y\partial_y + z\partial_z) + (\xi''' - 4\xi' a_{11})\partial_{a_{11}}\\
&\quad -4\xi' a_{12}\partial_{a_{12}} - 4\xi' a_{21}\partial_{a_{21}} + (\xi''' - 4\xi' a_{22})\partial_{a_{22}}
\end{aligned}$$

where $\xi = \xi(x)$ is an arbitrary function.

The transformations corresponding to the generators X_1^e, X_2^e, X_3^e and X_4^e define the linear changes of dependent variables $\tilde{\mathbf{y}} = P\mathbf{y}$ with a constant non-singular matrix P. The transformations corresponding to X_5^e are $\tilde{x} = \varphi(x), \tilde{y} = y\psi(x), \tilde{z} = z\psi(x)$ where the functions $\varphi(t)$ and $\psi(t)$ satisfy the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}.$$

*Computations were solved manually and were verified using the symbolic manipulation program REDUCE (Free CSL version 07-Oct-10).

Now that the equivalence transformations are obtained, then we are more than equipped to begin finding the admitted Lie algebras of the linear system (3.4).

3.2 Admitted Lie Group of the Linear System (3.4)

Admitted generators are sought in this form

$$X = \xi(x,y,z)\frac{\partial}{\partial x} + \eta^y(x,y,z)\frac{\partial}{\partial y} + \eta^z(x,y,z)\frac{\partial}{\partial z}. \tag{3.10}$$

The prolonged operator for this equation is

$$\widetilde{X} = X + \eta^{y'}\partial_{y'} + \eta^{z'}\partial_{z'} + \eta^{y''}\partial_{y''} + \eta^{z''}\partial_{z''} \tag{3.11}$$

with the coefficients

$$\eta^{y'} = D_x\eta^y - y'D_x\xi, \quad \eta^{y''} = D_x\eta^{y'} - y''D_x\xi,$$
$$\eta^{z'} = D_x\eta^z - z'D_x\xi, \quad \eta^{z''} = D_x\eta^{z'} - z''D_x\xi,$$

where

$$D_x = \partial_x + y'\partial_y + z'\partial_z + y''\partial_{y'} + z''\partial_{z'}.$$

According to the Lie algorithm (Ovsiannikov, 1978), X is admitted by the system (3.4) if it satisfies the associated determining equations, i.e., the generator (3.10) is admitted by the equations (3.4) if and only if

$$[\widetilde{X}(\mathbf{y}'' - \mathbf{A}\mathbf{y})]|_{[\mathbf{y}''=\mathbf{A}\mathbf{y}]} = 0.$$

The latter equations become

$$[\eta^{y''} - a_{11}(x)\eta^y - a_{12}(x)\eta^z - \xi(a'_{11}(x)y + a'_{12}z)]\big|_{\mathbf{y}''=\mathbf{A}\mathbf{y}} = 0,$$
$$[\eta^{z''} - a_{21}(x)\eta^y - a_{22}(x)\eta^z - \xi(a'_{21}(x)y + a'_{22}z)]\big|_{\mathbf{y}''=\mathbf{A}\mathbf{y}} = 0.$$

After substituting the coefficients $\eta^{y''}$, $\eta^{z''}$ and the differential equations $\mathbf{y}'' = \mathbf{A}\mathbf{y}$, and splitting with respect to the parametric derivatives y' and z', the first part of the determining equations are as follows:

$$\begin{aligned} &\xi_{yy}=0, && \xi_{zz}=0, && \xi_{yz}=0, \\ &\eta^y_{yy}=2\xi_{xy}, && \eta^y_{yz}=\xi_{xz}, && \eta^y_{zz}=0, \\ &\eta^z_{yy}=0, && \eta^z_{yz}=\xi_{xy}, && \eta^z_{zz}=2\xi_{xz}. \end{aligned} \tag{3.12}$$

The general solution of the first three (3) equations of (3.12) is

$$\xi=\xi_1(x)y+\xi_2(x)z+\xi_0(x). \tag{3.13}$$

Substituting equation (3.13) into the last six (6) equations of (3.12), the general solutions of η^y and η^z are obtained as follows

$$\begin{aligned} \eta^y &= 2\xi_1'(x)y+\xi_2'(x)yz+\eta_1(x)+\eta_{11}(x)y+\eta_{12}(x)z, \\ \eta^z &= 2\xi_2'(x)z+\xi_1'(x)yz+\eta_2(x)+\eta_{21}(x)y+\eta_{22}(x)z. \end{aligned} \tag{3.14}$$

Substituting the general solutions of ξ, η^y and η^z into the remaining unlisted determining equations, one obtains the following:

$$3\xi_1''y+\xi_2''z-\xi_0''+2\eta_{11}'-3a_{11}\xi_1y-3a_{12}\xi_1z-a_{21}\xi_2y-a_{22}\xi_2z=0, \tag{3.15}$$

$$2\xi_2''y+\eta_{12}'-a_{11}\xi_2y-a_{12}\xi_2z=0, \tag{3.16}$$

$$2\xi_1''z+\eta_{21}'-a_{21}\xi_1y-a_{22}\xi_1z=0, \tag{3.17}$$

$$3\xi_2''z+\xi_1''y-\xi_0''+2\eta_{22}'-a_{11}\xi_1y-a_{12}\xi_1z-3a_{21}\xi_2y-3a_{22}\xi_2z=0, \tag{3.18}$$

$$\begin{aligned} &-a_{11}'zy\xi_2-a_{11}'y^2\xi_1-a_{11}'y\xi_0-a_{12}'z^2\xi_2-a_{12}'zy\xi_1-a_{12}'z\xi_0 \\ &\quad -2\xi_0'za_{12}-2\xi_0'ya_{11}+\xi_1'''y^2-\xi_1'zya_{12}-\xi_1'y^2a_{11}+\xi_2'''zy-2\xi_2'z^2a_{12} \\ &\quad -2\xi_2'zya_{11}+\xi_2'zya_{22}+\xi_2'y^2a_{21}+\eta_1''+\eta_{11}''y+\eta_{12}''z-za_{11}\eta_{12}+za_{12}\eta_{11} \\ &\quad -za_{12}\eta_{22}+za_{22}\eta_{12}-ya_{12}\eta_{21}+ya_{21}\eta_{12}-a_{11}\eta_1-a_{12}\eta_2=0, \\ &-a_{21}'zy\xi_2-a_{21}'y^2\xi_1-a_{21}'y\xi_0-a_{22}'z^2\xi_2-a_{22}'zy\xi_1-a_{22}'z\xi_0-2\xi_0'za_{22} \\ &\quad -2\xi_0'ya_{21}+\xi_1'''zy+\xi_1'z^2a_{12}+\xi_1'zya_{11}-2\xi_1'zya_{22}-2\xi_1'y^2a_{21}+\xi_2'''z^2 \\ &\quad -\xi_2'z^2a_{22}-\xi_2'zya_{21}+\eta_2''+\eta_{21}''y+\eta_{22}''z+za_{12}\eta_{21}-za_{21}\eta_{12}+ya_{11}\eta_{21} \\ &\quad -ya_{21}\eta_{11}+ya_{21}\eta_{22}-ya_{22}\eta_{21}-a_{21}\eta_1-a_{22}\eta_2=0. \end{aligned} \tag{3.19}$$

Equations (3.16) and (3.17) can be split with respect to y and z. Hence, one obtains the following:

$$\xi_1 = \xi_2 = 0, \quad \eta_{12} = c_1, \quad \eta_{21} = c_2, \tag{3.20}$$

where c_1 and c_2 are constant. Substituting equations (3.20) into equations (3.15) and (3.18), one obtains the relations

$$\eta_{11} = \frac{1}{2}\xi_0' + c_3, \quad \eta_{22} = \frac{1}{2}\xi_0' + c_4, \tag{3.21}$$

where c_3 and c_4 are constant. Substituting equations (3.20) and (3.21) into equations (3.19), collecting terms, renaming $\xi_0(x)$ as $\xi(x)$, and keeping in mind that $F = a_{11}y + a_{12}z$ and $G = a_{21}y + a_{22}z$, the remaining determining equations are of the form

$$\begin{aligned} F_y(y(\xi' + k_1) + zk_2 + \eta_1) + F_z(z(\xi' + k_4) + yk_3 + \eta_2) + 2F_x\xi = \\ \xi'''y + \eta_1'' + F(k_1 - 3\xi') + Gk_2 \end{aligned} \tag{3.22}$$

$$\begin{aligned} G_y(y(\xi' + k_1) + zk_2 + \eta_1) + G_z(z(\xi' + k_4) + yk_3 + \eta_2) + 2G_x\xi = \\ \xi'''z + \eta_2'' + G(k_4 - 3\xi') + Fk_3. \end{aligned} \tag{3.23}$$

The admitted generator for this has the form

$$X = 2\xi(x)\partial_x + (y\xi' + yk_1 + zk_2 + \eta_1(x))\partial_y + (z\xi' + zk_4 + yk_3 + \eta_2(x))\partial_z \tag{3.24}$$

where $k_l, (l = 1, ..., 4)$ are constant, and ξ, η_1 and η_2 are some functions of x. From here, the determining equations (3.22) and (3.23) are analyzed through separating them into 2 cases:

1. there exists a generator with $\xi \neq 0$ in the admitted Lie algebra; and

2. $\xi = 0$ for all generators of the admitted Lie algebra.

3.2.1 Case $\xi \neq 0$

Consider the generator (3.24) for which $\xi \neq 0$ in the admitted Lie algebra. Using the equivalence transformation

$$y_1 = y + \phi(x),\ z_1 = z + \psi(x),$$

the generator X becomes

$$X = 2\xi(x)\partial_x + (y_1\xi' - \xi'\phi + 2\xi\phi' + y_1k_1 - \phi k_1 + z_1k_2 - \psi k_2 + \eta_1(x))\partial_{y_1}$$
$$+(z_1\xi' - \xi'\psi + 2\xi\psi' + z_1k_4 - \psi k_4 + y_1k_3 - \phi k_3 + \eta_2(x))\partial_{z_1}.$$

One can choose the functions $\phi(x)$ and $\psi(x)$ such that

$$2\xi\phi' - \xi'\phi - \phi k_1 - \psi k_2 + \eta_1(x) = 0,$$
$$2\xi\psi' - \xi'\psi - \psi k_4 - \phi k_3 + \eta_2(x) = 0.$$

The generator X is then reduced to

$$X = 2\xi\partial_x + (y_1\xi' + y_1k_1 + z_1k_2)\partial_{y_1} + (z_1\xi' + z_1k_4 + y_1k_3)\partial_{z_1}.$$

Using the equivalence transformation

$$x_2 = \alpha(x),\ y_2 = y_1\beta(x),\ z_2 = z_1\beta(x),$$

where

$$\alpha''\beta = 2\alpha'\beta',\ (\alpha'\beta \neq 0),$$

the generator X is reduced further to

$$X = 2\alpha'\xi\partial_{x_2} + ((2\xi\beta'/\beta + \xi' + k_1)y_2 + z_2k_2)\partial_{y_2} + ((2\xi\beta'/\beta + \xi' + k_4)z_2 + y_2k_3)\partial_{z_2}.$$

Choosing $\beta(x)$ such that $2\xi\beta'/\beta + \xi' = 0$, the generator X is reduced to

$$X = 2\alpha'\xi\partial_{x_2} + (k_1y_2 + k_2z_2)\partial_{y_2} + (k_4z_2 + k_3y_2)\partial_{z_2}.$$

Notice that in this case

$$\frac{d(\alpha'\xi)}{dx_2} = 0,$$

i.e.,

$$\frac{d(\alpha'\xi)}{dx_2} = \frac{(\alpha'\xi)'}{\alpha'} = \xi' + \frac{\alpha''\xi}{\alpha'} = -2\xi\frac{\beta'}{\beta} + 2\xi\frac{\beta'}{\beta} = 0.$$

Thus, the generator X becomes

$$X = k\partial_{x_2} + (k_1 y_2 + k_2 z_2)\partial_{y_2} + (k_4 z_2 + k_3 y_2)\partial_{z_2},$$

where $k = 2\alpha'\xi \neq 0$ is a constant. Rewriting, the generator X follows the form

$$X = \partial_x + (k_1 y + k_2 z)\partial_y + (k_3 y + k_4 z)\partial_z, \tag{3.25}$$

for which the determining equations are

$$F_y(k_1 y + k_2 z) + F_z(k_3 y + k_4 z) + F_x = k_1 F + k_2 G, \tag{3.26}$$

$$G_y(k_1 y + k_2 z) + G_z(k_3 y + k_4 z) + G_x = k_3 F + k_4 G \tag{3.27}$$

or simply

$$(A\mathbf{y}) \cdot \nabla\mathbf{F} + \mathbf{F}_x = A\mathbf{F}, \tag{3.28}$$

where $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$, $\nabla = \begin{pmatrix} \partial_y \\ \partial_z \end{pmatrix}$, and "$\cdot$" denotes the dot product.

Further simplifications are related to the simplification of the matrix A. Using the equivalence transformation $\widetilde{\mathbf{y}} = P\mathbf{y}$, where $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ is a nonsingular constant matrix, equations (3.4) become $\widetilde{\mathbf{y}} = \widetilde{\mathbf{F}}(x, \widetilde{\mathbf{y}})$, where

$$\widetilde{\mathbf{F}}(x, \widetilde{\mathbf{y}}) = P\mathbf{F}(x, P^{-1}\widetilde{\mathbf{y}}).$$

The partial derivatives with respect to the variables $\mathbf{y}$ are changed as follows

$$\mathbf{b} \cdot \nabla = (P\mathbf{b}) \cdot \widetilde{\nabla}.$$

With this, equations (3.28) are changed as follows

$$\begin{aligned}
&(AP^{-1}\widetilde{\mathbf{y}}) \cdot \widetilde{\nabla}\left(P^{-1}\widetilde{\mathbf{F}}\right) + P^{-1}\widetilde{\mathbf{F}}_x - AP^{-1}\widetilde{\mathbf{F}} \\
&= P^{-1}((PAP^{-1}\widetilde{\mathbf{y}}) \cdot \widetilde{\nabla}\widetilde{\mathbf{F}} + \widetilde{\mathbf{F}}_x - PAP^{-1}\widetilde{\mathbf{F}}) \\
&= P^{-1}((\widetilde{A}\widetilde{\mathbf{y}}) \cdot \widetilde{\nabla}\widetilde{\mathbf{F}} + \widetilde{\mathbf{F}}_x - \widetilde{A}\widetilde{\mathbf{F}}) = 0.
\end{aligned}$$

This means that the change $\widetilde{\mathbf{y}} = P\mathbf{y}$ reduces equations (3.28) to the same form with the matrix A changed. The generator (3.25) is also changed to the same form with the matrix A changed:

$$X = \partial_x + (\widetilde{A}\widetilde{\mathbf{y}})\widetilde{\nabla}. \tag{3.29}$$

Using this change, the matrix A can be represented in its Jordan form. For a real-valued 2×2 matrix A, the real-valued Jordan matrix is of the following three types:

$$J_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad J_2 = \begin{pmatrix} a & c \\ -c & a \end{pmatrix} \quad J_3 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \tag{3.30}$$

where a, b, c are real numbers and $c > 0$. Also, c can be reduced to 1 using a dilation of x.

3.2.1.1 Case $A = J_1$

In this case, the determining equations (3.28) become

$$aa_{11}y + ba_{12}z + a'_{11}y + a'_{12}z - aa_{11}y - aa_{12}z = 0,$$
$$aa_{21}y + ba_{22}z + a'_{21}y + a'_{22}z - ba_{21}y - ba_{22}z = 0.$$

Splitting these equations with respect to y and z, the following conditions are satisfied

$$a'_{11} = 0, \quad a'_{12} = (a-b)a_{12},$$
$$a'_{22} = 0, \quad a'_{21} = (b-a)a_{21}.$$

These conditions give the form of F and G as

$$F(x,y,z) = c_1 y + c_2 e^{\alpha x} z,$$
$$G(x,y,z) = c_3 e^{-\alpha x} y + c_4 z,$$

where $\alpha = a - b$, and $c_i's\,(i = 1,2,3,4)$ are constant. Note that if $c_2 = c_3 = 0$, then the system of equations is a linear system with constant coefficients, which is not in the scope of this research as this has already been studied (Wafo Soh, 2010; Meleshko, 2011). This is also true if $\alpha = 0$. Hence, without loss of generality, one can assume that $\alpha c_2 \neq 0$. Using a dilation of x and then z, one can assume that $\alpha = c_2 = 1$. Thus,

$$F(x, y, z) = c_1 y + e^x z,$$

$$G(x, y, z) = c_3 e^{-x} y + c_4 z.$$

Since for $c_3 = 0$ the system of equations are reduced to the case where $G = 0$, then one can also assume that $c_3 \neq 0$. From (3.29) with $A = J_1$, one obtains

$$X = \partial_x + ay\partial_y + (a-1)z\partial_z.$$

Disregarding the trivial generator, the additional nontrivial generator

$$\partial_x - z\partial_z$$

is found.

3.2.1.2 Case $A = J_2$

In this case, the determining equations (3.28) become

$$(ay + cz)a_{11} + (-cy + az)a_{12} + a_{11}'y + a_{12}'z - aa_{11}y - aa_{12}z - ca_{21}y - ca_{22}z = 0,$$

$$(ay + cz)a_{21} + (-cy + az)a_{22} + a_{21}'y + a_{22}'z + ca_{11}y + ca_{12}z - aa_{21}y - aa_{22}z = 0.$$

Splitting these equations with respect to y and z, the following conditions are satisfied

$$a_{11}' = c(a_{12} + a_{21}), \quad a_{12}' = c(a_{22} - a_{11}),$$

$$a_{22}' = -c(a_{12} + a_{21}), \quad a_{21}' = c(a_{22} - a_{11}).$$

These give the following relations

$$a_{22} = -a_{11} + 2c_1, \quad a_{21} = a_{12} + 2c_2,$$

which lead to finding the solution of the following first order system of equations

$$a_{11}' = c(2a_{21} + 2c_2), \quad a_{12}' = c(-2a_{11} + 2c_1).$$

The general solution of these equations is

$$a_{11} = c_0 \sin(2cx) + c_3 \cos(2cx) + c_1,$$

$$a_{12} = c_0 \cos(2cx) - c_3 \sin(2cx) - c_2,$$

which give the general form of F and G as

$$F(x, y, z) = (c_0 \sin(2cx) + c_3 \cos(2cx) + c_1)y + (c_0 \cos(2cx) - c_3 \sin(2cx) - c_2)z,$$

$$G(x, y, z) = (c_0 \cos(2cx) - c_3 \sin(2cx) + c_2)y + (-c_0 \sin(2cx) - c_3 \cos(2cx) + c_1)z,$$

where $c_i's\,(i = 0, 1, 2, 3)$ are constant. Notice that if $c_3 \neq 0$, then the change $\widetilde{\mathbf{y}} = P\mathbf{y}$ with the matrix

$$P = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}$$

and the angle θ satisfying the equation $c_3\tau^4 - 4c_0\tau^3 - 6c_3\tau^2 + 4c_0\tau + c_3 = 0$, with $\tau = \tan(\theta)$, reduces the functions F and G to the form

$$F(x, y, z) = (c_0 \sin(2cx) + c_1)y + (c_0 \cos(2cx) - c_2)z,$$

$$G(x, y, z) = (c_0 \cos(2cx) + c_2)y + (-c_0 \sin(2cx) + c_1)z.$$

Hence, without loss of generality, one can choose $c_3 = 0$. Note also that if $c_0 = 0$, the system is reduced to a system of linear equations with constant coefficients, which is omitted in this study. Hence, one has to consider that $c_0 \neq 0$. Without loss of generality, one can also set that $c_0 = 2c = 1$. Thus, the system (3.2.1.2) is reduced to

$$F(x, y, z) = (\sin(x) + c_1)y + (\cos(x) - c_2)z,$$

$$G(x, y, z) = (\cos(x) + c_2)y + (-\sin(x) + c_1)z.$$

From (3.29) with $A = J_2$, the form of X is

$$2\partial_x + (2ay + z)\partial_y + (2az - y)\partial_z.$$

Disregarding the trivial generator, the additional generator

$$2\partial_x + z\partial_y - y\partial_z$$

is obtained in this case.

3.2.1.3 Case $A = J_3$

In this case, the determining equations (3.28) become

$$(ay+z)a_{11} + aa_{12}z + a'_{11}y + a'_{12}z - aa_{11}y - aa_{12}z - a_{21}y - a_{22}z = 0,$$
$$(ay+z)a_{21} + aa_{22}z + a'_{21}y + a'_{22}z - aa_{21}y - aa_{22}z = 0.$$

Splitting these equations with respect to y and z, the following conditions are satisfied

$$a'_{11} = a_{21}, \quad a'_{12} = a_{22} - a_{11},$$
$$a'_{22} = -a_{21}, \qquad a'_{21} = 0,$$

which give us the form of F and G:

$$F(x,y,z) = (c_3x + c_1)y + (-c_3x^2 + (c_4 - c_1)x + c_2)z,$$
$$G(x,y,z) = c_3y + (-c_3x + c_4)z,$$

where $c_i's\,(i = 1,2,3,4)$ are constant. Notice that for $c_3 = 0$, one has $G = c_4z$. Using an equivalence transformation, $G = 0$. This case is omitted in this study. Hence, one has to assume that $c_3 \neq 0$. Without loss of generality, set $c_3 = 1$. Hence,

$$F(x,y,z) = (x + c_1)y + (-x^2 + (c_4 - c_1)x + c_2)z,$$
$$G(x,y,z) = y + (-x + c_4)z.$$

From (3.29) with $A = J_3$, one obtains

$$X = \partial_x + (ay+z)\partial_y + az\partial_z.$$

Disregarding the trivial generator, the additional nontrivial generator

$$\partial_x + z\partial_y$$

is obtained.

3.2.2 Case $\xi = 0$

Consider all generators (3.24) of the admitted Lie algebra for which $\xi = 0$. For this case, the determining equations (3.22) and (3.23) are reduced to

$$F_y(k_1 y + k_2 z + \eta_1) + F_z(k_3 y + k_4 z + \eta_2) = \eta_1'' + k_1 F + k_2 G, \tag{3.31}$$

$$G_y(k_1 y + k_2 z + \eta_1) + G_z(k_3 y + k_4 z + \eta_2) = \eta_2'' + k_3 F + k_4 G \tag{3.32}$$

or simply

$$(A\mathbf{y} + k) \cdot \nabla \mathbf{F} = A\mathbf{F} + k'',$$

where $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$, $k = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \end{pmatrix}$, $\nabla = \begin{pmatrix} \partial_y \\ \partial_z \end{pmatrix}$. The admitted generator is rewritten as

$$X = (k_1\, y + k_2\, z + \eta_1(x))\, \partial_y + (k_3\, y + k_4\, z + \eta_2(x))\, \partial_z.$$

Substituting the functions (3.6) into the determining equations (3.31) and (3.32) and splitting with respect to y and z, one has

$$\begin{aligned} a_{21}k_2 - a_{12}k_3 &= 0, \\ (a_{11} - a_{22})k_2 + (k_4 - k_1)a_{12} &= 0, \\ (k_1 - k_4)a_{21} + (a_{22} - a_{11})k_3 &= 0, \end{aligned} \tag{3.33}$$

$$a_{11}\eta_1 + a_{12}\eta_2 = \eta_1'', \quad a_{21}\eta_1 + a_{22}\eta_2 = \eta_2''. \tag{3.34}$$

Equations (3.34) define the trivial set of generators. The nontrivial generators

$$X = (yk_1 + zk_2)\partial_y + (yk_3 + zk_4)\partial_z \tag{3.35}$$

are defined by the equations (3.33). Similar to the case where one admitted generator has $\xi \neq 0$, equations (3.33) are simplified by using the Jordan form of the matrix A.

3.2.2.1 Case $A = J_1$

For this case, equations (3.33) become

$$(b-a)a_{12} = 0, \quad (a-b)a_{21} = 0.$$

Since for $b = a$ the generator (3.35) is also trivial, one has to assume that $b \neq a$. The last condition gives

$$a_{12} = 0, \quad a_{21} = 0.$$

In this case, the linear system of equations (3.4) is reduced to the degenerate case with $G = 0$. Hence, no additional nontrivial generators are found.

3.2.2.2 Case $A = J_2$

For this case, equations (3.33) become

$$a_{11} - a_{22} = 0, \quad a_{12} + a_{21} = 0.$$

Here one has to assume that $a_{12} \neq 0$, else it is reduced to a degenerate form. Using the equivalence transformation of the form

$$\widetilde{x} = \varphi(x), \widetilde{y} = y\psi(x), \widetilde{z} = z\psi(x)$$

where $\dfrac{\varphi''}{\varphi'} = 2\dfrac{\psi'}{\psi}$, one can reduce $a_{12} = 1$. Also in this case one also has to assume that $a'_{11} \neq 0$, else it is equivalent to a degenerate case. Hence,

$$F(x,y,z) = a_{11}y + z,$$

$$G(x,y,z) = -y + a_{11}z.$$

The form of X is

$$(ay + cz)\partial_y + (az - cy)\partial_z.$$

Excluding the trivial generator $y\partial_y + z\partial_z$, the nontrivial generator

$$z\partial_y - y\partial_z$$

is found.

3.2.2.3 Case $A = J_3$

For this case, equations (3.33) become

$$a_{11} - a_{22} = 0, \quad a_{21} = 0.$$

In this case, the linear system of equations (3.4) is reduced to the degenerate case with $G = 0$. Hence, no additional nontrivial generators are found.

All in all, four cases of linear systems of equations which are not equivalent to the linear systems with constant coefficients and the degenerate case are found. The complete representative classes is summarized in Table 6.1.

CHAPTER IV

PRELIMINARY STUDY OF NONLINEAR SYSTEMS

This chapter focuses on the preliminary study of systems of two nonlinear second-order ordinary differential equations of the form (Moyo et al., 2013; Meleshko and Moyo, 2015)

$$\mathbf{y}'' = \mathbf{F}(x, \mathbf{y}), \tag{4.1}$$

where

$$\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F(x,y,z) \\ G(x,y,z) \end{pmatrix}.$$

The classical group analysis is applied to the system of equations (4.1). For finding group classes of the system of the form (4.1) in this chapter and the succeeding chapters, the case of systems of two linear second-order ordinary differential equations in Chapter III and the degenerate case (1.3) are omitted. We call systems that are equivalent to these cases as reducible systems, and irreducible otherwise.

4.1 Equivalence Transformations

Equivalence transformations of the studied system of equations are considered in this section. Consider the nonlinear system (4.1). The arbitrary elements are the functions $F(x, y, z)$ and $G(x, y, z)$. The generator of the equivalence Lie group is assumed to be of the form

$$X^e = \xi\partial_x + \eta^y\partial_y + \eta^z\partial_z + \zeta^F\partial_F + \zeta^G\partial_G,$$

where the coefficients ξ, η^y, η^z, ζ^F and ζ^G depend on the variables x, y, z, F and G. The prolonged operator is

$$\widetilde{X^e} = X^e + \eta^{y'}\partial_{y'} + \eta^{z'}\partial_{z'} + \eta^{y''}\partial_{y''} + \eta^{z''}\partial_{z''}$$
$$+ \zeta^{F_x}\partial_{F_x} + \zeta^{F_y}\partial_{F_y} + \zeta^{F_z}\partial_{F_z} + \zeta^{G_x}\partial_{G_x} + \zeta^{G_y}\partial_{G_y} + \zeta^{G_z}\partial_{G_z}.$$

The coefficients of the prolonged generator are

$$\eta^{y'} = D_x^e\eta^y - y'D_x^e\xi, \quad \eta^{y''} = D_x^e\eta^{y'} - y''D_x^e\xi,$$
$$\eta^{z'} = D_x^e\eta^z - z'D_x^e\xi, \quad \eta^{z''} = D_x^e\eta^{z'} - z''D_x^e\xi,$$
$$\zeta^{F_x} = \widetilde{D}_x^e\zeta^F - F_x\widetilde{D}_x^e\xi - F_y\widetilde{D}_x^e\eta^y - F_z\widetilde{D}_x^e\eta^z,$$
$$\zeta^{F_y} = \widetilde{D}_y^e\zeta^F - F_x\widetilde{D}_y^e\xi - F_y\widetilde{D}_y^e\eta^y - F_z\widetilde{D}_y^e\eta^z,$$
$$\zeta^{F_z} = \widetilde{D}_z^e\zeta^F - F_x\widetilde{D}_z^e\xi - F_y\widetilde{D}_z^e\eta^y - F_z\widetilde{D}_z^e\eta^z,$$
$$\zeta^{G_x} = \widetilde{D}_x^e\zeta^G - G_x\widetilde{D}_x^e\xi - G_y\widetilde{D}_x^e\eta^y - G_z\widetilde{D}_x^e\eta^z,$$
$$\zeta^{G_y} = \widetilde{D}_y^e\zeta^G - G_x\widetilde{D}_y^e\xi - G_y\widetilde{D}_y^e\eta^y - G_z\widetilde{D}_y^e\eta^z,$$
$$\zeta^{G_z} = \widetilde{D}_z^e\zeta^G - G_x\widetilde{D}_z^e\xi - G_y\widetilde{D}_z^e\eta^y - G_z\widetilde{D}_z^e\eta^z.$$

Here, the operators D_x^e, $\widetilde{D}_x^e$, $\widetilde{D}_y^e$ and $\widetilde{D}_z^e$ are

$$D_x^e = \partial_x + y'\partial_y + z'\partial_z + y''\partial_{y'} + z''\partial_{z'} + (F_x + y'F_y + z'F_z)\partial_F$$
$$+(G_x + y'G_y + z'G_z)\partial_G + (F_{xx} + y''F_y + z''F_z + y'F_{xy} + z'F_{xz})\partial_{F_x}$$
$$+(F_{xy} + y'F_{yy} + z'F_{yz})\partial_{F_y} + (F_{xz} + y'F_{yz} + z'F_{zz})\partial_{F_z}$$
$$+(G_{xx} + y''G_y + z''G_z + y'G_{xy} + z'G_{xz})\partial_{G_x} + (G_{xy} + y'G_{yy} + z'G_{yz})\partial_{G_y}$$
$$+(G_{xz} + y'G_{yz} + z'G_{zz})\partial_{G_z},$$
$$\widetilde{D}_x^e = \partial_x + F_x\partial_F + G_x\partial_G,$$
$$\widetilde{D}_y^e = \partial_y + F_y\partial_F + G_y\partial_G,$$
$$\widetilde{D}_z^e = \partial_z + F_z\partial_F + G_z\partial_G.$$

The determining equations of the equivalence Lie group become

$$\eta^{y''} - \zeta^F|_{\mathbf{y''=F}} = 0$$
$$\eta^{z''} - \zeta^G|_{\mathbf{y''=F}} = 0.$$

After substitutions of $\eta^{y''}$ and $\eta^{z''}$ and the transition onto the manifold $\mathbf{y}'' = \mathbf{F}$, the equation is split with respect to the variables $\mathbf{y}'$, F_x, F_y, F_z, F_{xx}, F_{xy}, F_{xz}, F_{yz}, F_{yy}, F_{zz}, G_x, G_y, G_z, G_{xx}, G_{xy}, G_{xz}, G_{yz}, G_{yy} and G_{zz}.

Initial analysis of the split determining equations yields that ξ, η^y, η^z do not depend on F and G. As a result, the remaining determining equations are as follows

$$\xi_{yz} = 0, \quad \xi_{yy} = 0, \quad \xi_{zz} = 0, \tag{4.2}$$

$$\begin{aligned} &\eta^y_{zz} = 0, \quad \eta^y_{yz} - \xi_{xz} = 0, \quad \eta^y_{yy} - 2\xi_{xy} = 0, \\ &\eta^z_{yy} = 0, \quad \eta^z_{yz} - \xi_{xy} = 0, \quad \eta^z_{zz} - 2\xi_{xz} = 0, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\eta^y_{xz} - \xi_z F = 0, \quad 2\eta^y_{xy} - \xi_{xx} - 3\xi_y F - \xi_z G = 0, \\ &\eta^z_{xy} - \xi_y G = 0, \quad 2\eta^z_{xz} - \xi_{xx} - \xi_y F - 3\xi_z G = 0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\eta^y_{xx} + \eta^y_y F + \eta^y_z G - 2\xi_x F - \zeta^F = 0, \\ &\eta^z_{xx} + \eta^z_y F + \eta^z_z G - 2\xi_x G - \zeta^G = 0. \end{aligned} \tag{4.5}$$

The general solution of equations (4.2) is

$$\xi = \xi_0(x) + \xi_1(x)y + \xi_2(x)z, \tag{4.6}$$

where $\xi_n(x)$ $(n = 0, 1, 2)$ are arbitrary functions of its arguments. Substituting this to remaining determining equations and solving equations (4.3), one finds that

$$\begin{aligned} \eta^y &= \xi_1' y^2 + \xi_2' yz + \eta^y_0(x) + \eta^y_1(x)y + \eta^y_2(x)z, \\ \eta^z &= \xi_1' yz + \xi_2' z^2 + \eta^z_0(x) + \eta^z_1(x)y + \eta^z_2(x)z, \end{aligned} \tag{4.7}$$

$\eta^y_n(x)$ and $\eta^z_n(x)$ $(n = 0, 1, 2)$ are arbitrary functions of its arguments. Substituting (4.6) and (4.7) into equations (4.4), and keeping in mind that F and G are arbitrary, one obtains that $\xi_1 = 0$, $\xi_2 = 0$, and η^y_2 and η^z_1 are constant. In addition, $\eta^y_1 = \dfrac{1}{2}\xi_{0_x} + \eta^y_{10}$ and $\eta^z_2 = \dfrac{1}{2}\xi_{0_x} + \eta^z_{20}$, where η^y_{10} and η^z_{20} are constant. Substituting all these to equations (4.5), one finds that

$$\begin{aligned} \zeta^F &= \frac{1}{2}\left(2\eta_0^{y\prime\prime} + \xi_0^{\prime\prime\prime}y - 3\xi_0^{\prime}F + 2\eta_{10}^{y}F + 2\eta_2^{y}G\right) \\ \zeta^G &= \frac{1}{2}\left(2\eta_0^{z\prime\prime} + \xi_0^{\prime\prime\prime}z - 3\xi_0^{\prime}G + 2\eta_{20}^{z}G + 2\eta_1^{z}F\right). \end{aligned} \tag{4.8}$$

Finally from the above calculations*, the equivalence Lie group is defined by the following generators:

$$\begin{aligned} X_1^e &= y\partial_y + F\partial_F, & X_2^e &= z\partial_y + G\partial_F, \\ X_3^e &= y\partial_z + F\partial_G, & X_4^e &= z\partial_z + G\partial_G, \\ X_5^e &= \phi_1(x)\partial_y + \phi_1^{\prime\prime}(x)\partial_F, & X_6^e &= \phi_2(x)\partial_z + \phi_2^{\prime\prime}(x)\partial_G, \end{aligned}$$

$$X_7^e = 2\xi(x)\partial_x + \xi'(x)y\partial_y + \xi'(x)z\partial_z + (\xi'''(x)y - 3\xi'(x)F)\partial_F + (\xi'''(x)z - 3\xi'(x)G)\partial_G.$$

Hence, the system (4.1) has the following equivalence transformations corresponding to the above equivalence Lie group:

1. a linear change of the dependent variables $\widetilde{\mathbf{y}} = P\mathbf{y}$ with constant nonsingular 2×2 matrix P;
2. the change $\widetilde{y} = y + \phi(x)$ and $\widetilde{z} = z + \psi(x)$; and
3. the transformation related with the change $\widetilde{x} = \phi(x)$, $\widetilde{y} = y\psi(x)$, $\widetilde{z} = z\psi(x)$, where the functions $\phi(x)$ and $\psi(x)$ satisfy the condition $\dfrac{\phi''}{\phi'} = 2\dfrac{\psi'}{\psi}$.

4.2 Determining equations

Admitted generators are sought in this form

$$X = \xi(x, y, z)\frac{\partial}{\partial x} + \eta^y(x, y, z)\frac{\partial}{\partial y} + \eta^z(x, y, z)\frac{\partial}{\partial z}. \tag{4.9}$$

The prolonged operator for this equation is

$$\widetilde{X} = X + \eta^{y'}\partial_{y'} + \eta^{z'}\partial_{z'} + \eta^{y''}\partial_{y''} + \eta^{z''}\partial_{z''} \tag{4.10}$$

*Computations were implemented with the aid of the symbolic manipulation program REDUCE (Free CSL version 07-Oct-10).

with the coefficients

$$\eta^{y'} = D_x\eta^y - y'D_x\xi, \quad \eta^{y''} = D_x\eta^{y'} - y''D_x\xi,$$
$$\eta^{z'} = D_x\eta^z - z'D_x\xi, \quad \eta^{z''} = D_x\eta^{z'} - z''D_x\xi,$$

where

$$D_x = \partial_x + y'\partial_y + z'\partial_z + y''\partial_{y'} + z''\partial_{z'}.$$

According to the Lie algorithm (Ovsiannikov, 1978), X is admitted by the system (4.1) if it satisfies the associated determining equations, i.e., the generator (4.9) is admitted by the equations (4.1) if and only if

$$[\widetilde{X}(\mathbf{y}'' - \mathbf{F})]|_{[\mathbf{y}''=\mathbf{F}]} = 0.$$

The previous equations become

$$[\eta^{y''} - F_x\xi - F_y\eta^y - F_z\eta^z]\big|_{\mathbf{y}''=\mathbf{F}} = 0,$$
$$[\eta^{z''} - G_x\xi - G_y\eta^y - G_z\eta^z]\big|_{\mathbf{y}''=\mathbf{F}} = 0.$$

After substituting the coefficients $\eta^{y''}$, $\eta^{z''}$ and the differential equations $\mathbf{y}'' = \mathbf{F}$, and splitting with respect to the parametric derivatives y' and z', the determining equations are as follows:

$$\xi_{yz} = 0, \quad \xi_{yy} = 0, \quad \xi_{zz} = 0, \tag{4.11}$$

$$\begin{aligned} &\eta^y_{zz} = 0, \quad \eta^y_{yz} - \xi_{xz} = 0, \quad \eta^y_{yy} - 2\xi_{xy} = 0, \\ &\eta^z_{yy} = 0, \quad \eta^z_{yz} - \xi_{xy} = 0, \quad \eta^z_{zz} - 2\xi_{xz} = 0, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\eta^y_{xz} - \xi_z F = 0, \quad 2\eta^y_{xy} - \xi_{xx} - 3\xi_y F - \xi_z G = 0, \\ &\eta^z_{xy} - \xi_y G = 0, \quad 2\eta^z_{xz} - \xi_{xx} - \xi_y F - 3\xi_z G = 0, \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\eta^y_{xx} + \eta^y_y F + \eta^y_z G - 2\xi_x F - F_x\xi - F_y\eta^y - F_x\eta^z = 0, \\ &\eta^z_{xx} + \eta^z_y F + \eta^z_z G - 2\xi_x G - G_x\xi - G_y\eta^y - G_x\eta^z = 0. \end{aligned} \tag{4.14}$$

Solving equations (4.11) and (4.12), one obtains the general solution for ξ, η^y and η^z :

$$\xi = \xi_0(x) + \xi_1(x)y + \xi_2(x)z, \tag{4.15}$$

$$\begin{aligned} \eta^y &= \xi_1' y^2 + \xi_2' yz + \eta_0^y(x) + \eta_1^y(x)y + \eta_2^y(x)z, \\ \eta^z &= \xi_1' yz + \xi_2' z^2 + \eta_0^z(x) + \eta_1^z(x)y + \eta_2^z(x)z, \end{aligned} \tag{4.16}$$

where $\xi_n(x)$, $\eta_n^y(x)$ and $\eta_n^z(x)$ $(n = 0, 1, 2)$ are arbitrary functions of its arguments. Differentiating the equations (4.13) with respect to y and z, one obtains the following determining equations

$$3\xi_1(F_y - G_z) + \xi_2 G_y = 0, \quad \xi_1 G_y = 0,$$

$$\xi_1 F_z + 3\xi_2(F_y - G_z) = 0, \quad \xi_2 F_z = 0.$$

From these equations, one can conclude that $\xi_1^2 + \xi_2^2 \neq 0$ only for the case where

$$F_y - G_z = 0, \quad G_y = 0, \quad F_z = 0. \tag{4.17}$$

Solving the conditions (4.17), one obtains the general solution

$$F(x, y, z) = a(x)y + b(x), \quad G(x, y, z) = a(x)z + c(x).$$

Using a particular solution and equivalence transformations, equations (4.1) are reduced to the trivial case of the free particle equation, which is omitted in this study. Hence, we consider the case only when the conditions (4.17) are not satisfied, implying that

$$\xi_1 = 0, \quad \xi_2 = 0.$$

Substituting all the conditions into equations (4.14), it follows that the determining equations in matrix form for irreducible systems of the form (4.1) are given by

$$2\xi \mathbf{F}_x + 3\xi' \mathbf{F} + (((A + \xi' E)\mathbf{y} + \zeta) \cdot \nabla)\mathbf{F} - A\mathbf{F} = \xi''' \mathbf{y} + \zeta'', \tag{4.18}$$

where the matrix $A = (a_{ij})$ is constant and $\zeta(x) = (\zeta_1, \zeta_2)^t$ is a vector. The associated infinitesimal generator has the form (Moyo et al., 2013)

$$X = 2\xi(x)\partial_x + ((A + \xi' E)\mathbf{y} + \zeta(x)) \cdot \nabla.$$

Similar to the case of linear systems, when the equivalence transformation (1) with linear change $\widetilde{\mathbf{y}} = P\mathbf{y}$ is applied to equations (4.1), equations (4.18) and its associated infinitesimal generator are reduced to the same form with the matrix A and the vector ζ changed.

The systems of two nonlinear second-order ordinary differential equations are equivalent to one of the following ten (10) types listed below in Table 4.1 (See also (Moyo et al., 2013)). Looking closely at these systems, there is a necessity to conduct an initial study where the systems of two equations do not depend on x. This is the focus of the next chapter.

Table 4.1 Ten nonequivalent types of nonlinear systems. For all the cases, h_1, h_2, f and g are arbitrary functions of their arguments.

	F and G	Relations and Conditions	Admitted Generator
1.	$F = e^{ax} f(u,v)$, $G = e^{bx} g(u,v)$	$u = ye^{-ax},\ v = ze^{-bx}$, $a,\ b$ are constant	$\partial_x + ay\partial_y + bz\partial_z$
2.	$F = e^{ax}(\cos(cx) f(u,v) + \sin(cx) g(u,v))$, $G = e^{ax}(-\sin(cx) f(u,v) + \cos(cx) g(u,v))$	$u = e^{-ax}(y\cos(cx) - z\sin(cx))$, $v = e^{-ax}(y\sin(cx) + z\cos(cx))$, $a,\ c \neq 0$ are constant	$\partial_x + (ay + cz)\partial_y + (-cy + az)\partial_z$
3.	$F = e^{ax}(f(u,v) + xg(u,v))$, $G = e^{ax} g(u,v)$	$u = e^{-ax}(y - zx),\ v = ze^{-ax}$, a is constant	$\partial_x + (ay + z)\partial_y + az\partial_z$
4.	$F = (y + h_1(x)) f(x,v) - h_1''$, $G = (z + h_2(x)) g(x,v) - h_2''$	$v = (z + h_2(x))(y + h_1(x))^{\alpha}$, $\alpha \neq 0$ is constant	$(ay + h_1)\partial_y + (bz + h_2)\partial_z$
5.	$F = (y + h_1(x)) f(x,v) - h_1''(x)$, $G = h_2''(x)\ln(y + h_1(x)) + g(x,v)$	$v = z - h_2(x)\ln(y + h_1(x))$	$(ay + h_1)\partial_y + h_2\partial_z$
6.	$F = \dfrac{h_1''(x)}{h_1(x)} y + f(x,v)$, $G = \dfrac{h_2''(x)}{h_1(x)} y + g(x,v)$	$v = z - \dfrac{h_2(x)}{h_1(x)} y,\ h_1(x) \neq 0$	$h_1\partial_y + h_2\partial_z$

Table 4.1 Ten nonequivalent types of nonlinear systems. For all the cases, h_1, h_2, f and g are arbitrary functions of their arguments. (Continued)

	F and G	Relations and Conditions	Admitted Generator
7.	$F = e^{au}(\cos(cu)f(x,v) + \sin(cu)g(x,v))$,	$y = ve^{au}\sin(cu),\ z = e^{au}\cos(cu)$,	
	$G = e^{au}(-\sin(cu)f(x,v) + \cos(cu)g(x,v))$	$a,\ c \neq 0$ are constant	$(ay + cz + h_1)\partial_y + (-cy + az + h_2)\partial_z$
8.	$F = \dfrac{y}{z + h_1(x)}f(x,v) + g(x,v)$,	$v = z + h_1(x)$	
	$G = -h_1''(x) + f(x,v)$		$(z + h_1)\partial_y$
9.	$F = \dfrac{h_2''(x)}{2}u^2 + uf(x,v) + g(x,v)$,	$u = \dfrac{z + h_1(x)}{h_2(x)},\ v = y - \dfrac{(z + h_1(x))^2}{h_2(x)}$,	
	$G = -h_1''(x) + h_2''(x)u + f(x,v)$	$h_2(x) \neq 0$	$(z + h_1)\partial_y + h_2\partial_z$
10.	$F = e^u(uf(x,v) + g(x,v))$,	$y = uve^u,\ z = ve^u$	
	$G = e^u f(x,v)$		$(ay + z + h_1)\partial_y + (az + h_2)\partial_z$

CHAPTER V

APPLICATION OF GROUP ANALYSIS TO AUTONOMOUS NONLINEAR SYSTEMS WITHOUT FIRST DERIVATIVES

This chapter focuses on systems of two nonlinear second-order ordinary differential equations (4.1) where F and G do not depend on x, i.e., of the form

$$\mathbf{y}'' = \mathbf{F}(\mathbf{y}), \tag{5.1}$$

where

$$\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F(y,z) \\ G(y,z) \end{pmatrix}.$$

The classical group analysis method is applied to the system of equations (5.1).

5.1 Equivalence Transformations

The process of finding the equivalence Lie group of the nonlinear system (5.1) is similar to finding the equivalence Lie group of the nonlinear system (4.1) with the difference that the arbitrary elements for system (5.1) are the functions $F(y,z)$ and $G(y,z)$. In addition, the conditions

$$F_x = 0, \quad G_x = 0$$

are included for analysis.

Calculations* show that the equivalence Lie group is defined by the following generators:

$$X_1^e = y\partial_y + F\partial_F, \qquad X_2^e = z\partial_y + G\partial_F,$$
$$X_3^e = y\partial_z + F\partial_G, \qquad X_4^e = z\partial_z + G\partial_G,$$
$$X_5^e = \partial_y + \partial_z, \qquad X_6^e = \partial_y - \partial_z,$$
$$X_7^e = x\partial_x - 2(F\partial_F + G\partial_G), \quad X_8^e = \partial_x.$$

Hence, the system (5.1) has similar equivalence transformations as the system (4.1):

1. a linear change of the dependent variables $\widetilde{\mathbf{y}} = P\mathbf{y}$ with constant nonsingular 2×2 matrix P;
2. the change $\widetilde{y} = y + \phi(x)$ and $\widetilde{z} = z + \psi(x)$; and
3. the transformation related with the change $\widetilde{x} = \phi(x)$, $\widetilde{y} = y\psi(x)$, $\widetilde{z} = z\psi(x)$, where the functions $\phi(x)$ and $\psi(x)$ satisfy the condition $\dfrac{\phi''}{\phi'} = 2\dfrac{\psi'}{\psi}$.

5.2 Determining Equations

Since for autonomous systems, $\mathbf{F}_x = 0$, then the determining equations (4.18) of irreducible systems have the form

$$3\xi'\mathbf{F} + (((A + \xi'E)\mathbf{y} + \zeta) \cdot \nabla)\mathbf{F} - A\mathbf{F} - \xi'''\mathbf{y} - \zeta'' = 0 \tag{5.2}$$

and with an admitted generator of the form

$$X = 2\xi(x)\partial_x + ((A + \xi'E)\mathbf{y} + \zeta(x)) \cdot \nabla. \tag{5.3}$$

This also implies that the generator ∂_x is admitted by system (5.1).

*Computations were implemented with the aid of the symbolic manipulation program REDUCE (Free CSL version 07-Oct-10).

Differentiating the determining equations (5.2) with respect to x, equations (5.2) become

$$3\xi''\mathbf{F} + ((\xi''\mathbf{y} + \zeta') \cdot \nabla)\mathbf{F} - \xi^{(4)}\mathbf{y} - \zeta''' = 0. \tag{5.4}$$

From here, the group classification study is reduced into two cases, namely,

1. the case with at least one admitted generator with $\xi'' \neq 0$; and
2. the case where all admitted generators have $\xi'' = 0$.

The group classification of the two (2) cases are explained in detail in the succeeding sections.

5.2.1 Case $\xi'' \neq 0$

For the case of systems admitting at least one generator with $\xi'' \neq 0$, consider the differentiated determining equations (5.4) with respect to x and divide them by ξ''. The determining equations become

$$3\mathbf{F} + \left(\left(\mathbf{y} + \frac{\zeta'}{\xi''}\right) \cdot \nabla\right)\mathbf{F} - \frac{\xi^{(4)}}{\xi''}\mathbf{y} - \frac{\zeta'''}{\xi''} = 0. \tag{5.5}$$

Fixing x, and shifting y and z, equations (5.5) are reduced to

$$3\mathbf{F} + (\mathbf{y} \cdot \nabla)\mathbf{F} - a\mathbf{y} - \mathbf{b} = 0$$

where vector $\mathbf{b} = (b, c)^t$, and a, b, c are constant.

The general solution of these equations is

$$\begin{aligned} F &= \frac{b}{3} + \frac{ay}{4} + y^{-3}f(u), \\ G &= \frac{c}{3} + \frac{az}{4} + z^{-3}g(u), \end{aligned} \tag{5.6}$$

where $u = \dfrac{z}{y}$ and $f'g' \neq 0$. It is easy to see that if $f'g' = 0$, system (5.1) is equivalent to the linear case, which was already studied in Chapter III, and hence, is excluded for this chapter. The functions F and G in (5.6) are then substituted into the determining equations (5.2). The determining equations are then solved directly in order to find generators admitted by equations (5.1). The first part of the determining equations is given as follows:

$$\xi''' - a\xi' = 0, \tag{5.7}$$

$$\begin{aligned} (\zeta_1 u - \zeta_2)f' + 3\zeta_1 f &= 0, \\ (u^2\zeta_1 - u\zeta_2)g' + 3\zeta_2 g &= 0, \end{aligned} \tag{5.8}$$

$$\begin{aligned} 12\zeta_1'' - 12b\xi' - 3a\zeta_1 + 4a_{11}b + 4a_{12}c &= 0, \\ 12\zeta_2'' - 12c\xi' - 3a\zeta_2 + 4a_{21}b + 4a_{22}c &= 0, \end{aligned} \tag{5.9}$$

$$\begin{aligned} (a_{11}u^4 + a_{12}u^5 - a_{21}u^3 - a_{22}u^4)f' + (4a_{11}u^3 + 3a_{12}u^4)f + a_{12}g &= 0, \\ (a_{11}u^2 + a_{12}u^3 - a_{21}u - a_{22}u^2)g' + a_{21}u^4 f + (3a_{21} + 4a_{22}u)g &= 0, \end{aligned} \tag{5.10}$$

where a_{ij}'s are constant.

5.2.1.1 General solution of ξ

From equation (5.7), it can be seen that the general solution of ξ depends on three values of a, i.e., $a = 0$, $a = -p^2$ and $a = p^2$, where $p \neq 0$.

5.2.1.1.1 Case $a = 0$. For this case, the general solution of ξ is

$$\xi = \xi_2 x^2 + \xi_1 x + \xi_0,$$

where $\xi_2 \neq 0$, ξ_1, ξ_0 are constant.

5.2.1.1.2 Case $a = -p^2$ For this case, the general solution of ξ is

$$\xi = \xi_1 \cos(px) + \xi_2 \sin(px) + \xi_0,$$

where $\xi_2 \neq 0$, $\xi_1 \neq 0$, ξ_0 are constant.

5.2.1.1.3 Case $a = p^2$ Lastly, the general solution of ξ for this case is

$$\xi = \xi_1 e^{-px} + \xi_2 e^{px} + \xi_0,$$

where $\xi_2 \neq 0$, $\xi_1 \neq 0$, ξ_0 are constant.

5.2.1.2 General solution of ζ and representations of f and g

Subsequently the determining equations (5.8) lead to the study of two cases, i.e., (1) there exists a generator with $\zeta_1 \neq 0$ and (2) all generators have $\zeta_1 = 0$.

5.2.1.2.1 Case $\zeta_1 \neq 0$. Suppose that there exists a generator with $\zeta_1 \neq 0$. Dividing by ζ_1 and differentiating with respect to x the equations (5.8), one obtains $\zeta_2 = k\zeta_1$, where k is a constant. Substituting ζ_2 back into equations (5.8), one obtains $f = f_0(u-k)^{-3}$ and $g = g_0 u^3 (u-k)^{-3}$. Also, differentiating equations (5.9) with respect to x and taking its linear combination, it follows that $c = kb$. At this point, equations (5.6) have the following form

$$F = \frac{b}{3} + \frac{ay}{4} + f_0(z - yk)^{-3},$$
$$G = \frac{kb}{3} + \frac{az}{4} + g_0(z - yk)^{-3}.$$

Utilizing equivalence transformations, one can verify that this is a reducible case, and is, therefore, excluded.

5.2.1.2.2 Case $\zeta_1 = 0$. Consider that all generators have $\zeta_1 = 0$. From equations (5.8), it follows that $\zeta_2 = 0$. Differentiating equations (5.9) with respect to x, it immediately follows that $b = c = 0$. Observe in equations (5.10) that if the functions f and g are arbitrary, then we obtain the conditions that all of the a_{ij}'s are equal to zero. Hence, we obtain the following table of classification (see Table 5.1) .

Table 5.1 Group classification of nonlinear systems of the form $\mathbf{y}'' = \mathbf{F}\mathbf{y}$ admitting at least one generator with $\xi'' \neq 0$. Here, f and g are arbitrary functions of $u = \frac{z}{y}$, and $f'g' \neq 0$.

F	G	Extension of Kernel
$y^{-3}f(u)$	$z^{-3}g(u)$	Y_2, Y_3
$\frac{-p^2y}{4} + y^{-3}f(u)$	$\frac{-p^2z}{4} + z^{-3}g(u)$	Y_7, Y_8
$\frac{p^2y}{4} + y^{-3}f(u)$	$\frac{p^2z}{4} + z^{-3}g(u)$	Y_9, Y_{10}

In order to obtain additional extensions of the generator, one must find the general solutions of f and g. Hence, from equations (5.10), the equivalence transformation $\widetilde{\mathbf{y}} = P\mathbf{y}$, where P is a constant nonsingular 2×2 matrix, is utilized. Similar to the linear case, the constant matrix A is reduced to one of the real-valued Jordan forms (3.30). The general solutions for f and g (excluding reducible cases) are listed as follows:

Jordan form	f	g
J_1	$f_0u^{-\frac{4\gamma}{\gamma-1}}$	$f_1u^{-\frac{4}{\gamma-1}}$
J_2	$(g_0y - g_1z)\tau(y,z)$	$(g_0z + g_1y)\tau(y,z)$
J_3	$e^{\frac{\kappa}{u}}(f_0u^{-4} + h_1u^{-3})$	$f_0e^{\frac{\kappa}{u}}$

where $\tau(y,z) = e^{-4\alpha \arctan\frac{z}{y}}(y^2+z^2)^{-2}$, and $f_0f_1 \neq 0$, $g_0^2 + g_1^2 \neq 0$, $\gamma \neq 0, 1$ $\alpha \neq 0, 1$, $\kappa \neq 0$, and h_1 are constant.

5.2.1.3 Extension of the kernel of the admitted Lie algebras

Combining the solutions of functions f and g, ξ and ζ, and excluding reducible systems, the table of classification for nonlinear systems of the form $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$ which admit at least one generator with $\xi'' \neq 0$ is listed on Table 5.2.

Table 5.2 Group classification of nonlinear systems of the form $\mathbf{y}'' = \mathbf{F}\mathbf{y}$ admitting at least one generator with $\xi'' \neq 0$ excluding arbitrary functions f and g which depend on $u = zy^{-1}$.

	Cases	F and G	Extension of Kernel
1.	$a = 0$ and J_1	$F = y^{-3} f_0 u^{-\frac{4\gamma}{\gamma-1}}$	
		$G = z^{-3} f_1 u^{-\frac{4}{\gamma-1}}$	$Y_2,\ Y_3,\ Y_4$
2.	$a = -p^2$ and J_1	$F = \frac{-p^2 y}{4} + y^{-3} f_0 u^{-\frac{4\gamma}{\gamma-1}}$	
		$G = \frac{-p^2 z}{4} + z^{-3} f_1 u^{-\frac{4}{\gamma-1}}$	$Y_7,\ Y_8,\ Y_4$
3.	$a = p^2$ and J_1	$F = \frac{p^2 y}{4} + y^{-3} f_0 u^{-\frac{4\gamma}{\gamma-1}}$	
		$G = \frac{p^2 z}{4} + z^{-3} f_1 u^{-\frac{4}{\gamma-1}}$	$Y_9,\ Y_{10},\ Y_4$
4.	$a = 0$ and J_2	$F = y^{-3}(g_0 y - g_1 z)\tau(y,z)$	
		$G = z^{-3}(g_0 z + g_1 y)\tau(y,z)$	$Y_2,\ Y_3,\ Y_5$
5.	$a = -p^2$ and J_2	$F = \frac{-p^2 y}{4} + y^{-3}(g_0 y - g_1 z)\tau(y,z)$	
		$G = \frac{-p^2 z}{4} + z^{-3}(g_0 z + g_1 y)\tau(y,z)$	$Y_7,\ Y_8,\ Y_5$
6.	$a = p^2$ and J_2	$F = \frac{p^2 y}{4} + y^{-3}(g_0 y - g_1 z)\tau(y,z)$	
		$G = \frac{p^2 z}{4} + z^{-3}(g_0 z + g_1 y)\tau(y,z)$	$Y_9,\ Y_{10},\ Y_5$

Table 5.2 Group classification of nonlinear systems of the form $\mathbf{y}'' = \mathbf{F}\mathbf{y}$ admitting at least one generator with $\xi'' \neq 0$ excluding arbitrary functions f and g which depend on $u = zy^{-1}$. (Continued)

	Cases	F and G	Extension of Kernel
7.	$a = 0$ and J_3	$F = y^{-3}e^{\frac{\kappa}{u}}(f_0u^{-4} + h_1u^{-3})$	
		$G = z^{-3}f_0e^{\frac{\kappa}{u}}$	Y_2, Y_3, Y_6
8.	$a = -p^2$ and J_3	$F = \frac{-p^2y}{4} + y^{-3}e^{\frac{\kappa}{u}}(f_0u^{-4} + h_1u^{-3})$	
		$G = \frac{-p^2z}{4} + z^{-3}f_0e^{\frac{\kappa}{u}}$	Y_7, Y_8, Y_6
9.	$a = p^2$ and J_3	$F = \frac{p^2y}{4} + y^{-3}e^{\frac{\kappa}{u}}(f_0u^{-4} + h_1u^{-3})$	
		$G = \frac{p^2z}{4} + z^{-3}f_0e^{\frac{\kappa}{u}}$	Y_9, Y_{10}, Y_6

The kernel of the admitted Lie algebras consists of the generator $X_1 = \partial_x$, which is omitted on the list. The extension of the kernel is listed as follows:

$$Y_2 = 2x\partial_x + y\partial_y + z\partial_z, \qquad Y_7 = 2\cos(px)\partial_x - p\sin(px)(y\partial_y + z\partial_z),$$
$$Y_3 = x(x\partial_x + y\partial_y + z\partial_z), \qquad Y_8 = 2\sin(px)\partial_x + p\cos(px)(y\partial_y + z\partial_z),$$
$$Y_4 = \gamma y\partial_y + z\partial_z, \qquad Y_9 = e^{-px}(2\partial_x - p(y\partial_y + z\partial_z)),$$
$$Y_5 = (\alpha y + z)\partial_y + (\alpha z - y)\partial_z, \quad Y_{10} = e^{px}(2\partial_x + p(y\partial_y + z\partial_z)),$$
$$Y_6 = (\kappa y + 4z)\partial_y + \kappa z\partial_z.$$

The Lie algebras Y_2, Y_3, Y_7, Y_8, Y_9, and Y_{10} are associated with the coefficient ξ and the Lie algebras Y_4, Y_5, and Y_6 are related to the type of Jordan form of matrix A. Also, using different equivalence transformations, p is reduced to 2 and κ is reduced to 1.

5.2.2 Case $\xi'' = 0$

For the case of systems where all admitted generators have $\xi'' = 0$, return first to the analysis of the determining equations (5.2). Since $\xi'' = 0$, it follows that $\xi = \xi_0 + \xi_1 x$, where ξ_1 and ξ_0 are constant. This property of the coefficient forces ζ to become constant.

5.2.2.1 Claim: ζ is constant

Proof. Substituting $\xi'' = 0$ into determining equations (5.2), these equations reduce to the form

$$\begin{aligned} \zeta_1'' &= F_y\zeta_1 + F_z\zeta_2 + q_1, \\ \zeta_2'' &= G_y\zeta_1 + G_z\zeta_2 + q_2, \end{aligned} \tag{5.11}$$

where q_1 and q_2 are functions of y and z. Differentiating these equations with respect to x, one obtains

$$\zeta_1''' = F_y\zeta_1' + F_z\zeta_2',$$
$$\zeta_2''' = G_y\zeta_1' + G_z\zeta_2'.$$

Differentiating the latter equations with respect to y and z, one obtains the conditions

$$F_{yy}\zeta_1' + F_{yz}\zeta_2' = 0,$$
$$F_{yz}\zeta_1' + F_{zz}\zeta_2' = 0,$$

$$G_{yy}\zeta_1' + G_{yz}\zeta_2' = 0,$$
$$G_{yz}\zeta_1' + G_{zz}\zeta_2' = 0.$$

From here, one can study two cases: (1) $F_{zz} \neq 0$ and (2) $F_{zz} = 0$.

5.2.2.1.1 Case $F_{zz} \neq 0$. For this case, one has

$$\zeta_2' = -\frac{F_{yz}}{F_{zz}}\zeta_1', \quad F_{yy} - \frac{F_{yz}^2}{F_{zz}} = 0, \quad G_{yy} - G_{yz}\frac{F_{yz}}{F_{zz}} = 0, \quad G_{yz} - G_{zz}\frac{F_{yz}}{F_{zz}} = 0.$$

Thus,

$$\frac{F_{yz}}{F_{zz}} = k,$$

and

$$F_{yy} - kF_{yz} = 0, \quad G_{yy} - kG_{yz} = 0, \quad G_{yz} - kG_{zz} = 0$$

or

$$(F_y - kF_z)_y = 0, \quad (F_y - kF_z)_z = 0, \quad (G_y - kG_z)_y = 0, \quad (G_y - kG_z)_z = 0.$$

One obtains the general solution for F and G. If

$$F_y - kF_z = k_1, \quad G_y - kG_z = k_2,$$

then

$$F = \Phi(z + ky) + k_1 y,$$

$$G = \Psi(z + ky) + k_2 y.$$

Using the change of variables

$$\bar{y} = y, \quad \bar{z} = z + ky,$$

system (5.1) becomes

$$y'' = \Phi(\bar{z}) + k_1 y, \qquad \bar{z}'' = (k\Phi(\bar{z}) + \Psi(\bar{z})) + (kk_1 + k_2)y.$$

Thus, one has

$$y'' = k_1 y + f(z), \qquad z'' = k_2 y + g(z), \qquad \text{where } k_2 f'' \neq 0.$$

Substituting the general solution of F and G into equations (5.11), one has

$$\begin{aligned} \zeta_1'' &= k_1\zeta_1 + f'\zeta_2 + q_1, \\ \zeta_2'' &= k_2\zeta_1 + g'\zeta_2 + q_2, \end{aligned} \tag{5.12}$$

Differentiating the first equation of (5.12) by x and then z, one obtains

$$f''\zeta_2' = 0.$$

Since $f'' \neq 0$, then $\zeta_2' = 0$. Differentiating the second equation of (5.12) by x, one has

$$0 = k_2\zeta_1' \Rightarrow \zeta_1' = 0.$$

Hence, for this case ζ is constant.

5.2.2.1.2 Case $F_{zz} = 0$. Let $F_{zz} = 0$, then by symmetry $G_{yy} = 0$. Hence, one has the conditions

$$F_{yy}\zeta_1' + F_{yz}\zeta_2' = 0, \;\; F_{yz}\zeta_1' = 0, \;\; G_{yz}\zeta_2' = 0, \;\; G_{yz}\zeta_1' + G_{zz}\zeta_2' = 0.$$

If $F_{yz} \neq 0$, then from the first equation, one obtains $\zeta_1' = 0, \;\; \zeta_2' = 0$. Hence,

$$F_{yz} = 0, \;\; F_{zz} = 0, \;\; G_{yy} = 0, \;\; G_{yz} = 0$$

and

$$F_{yy}\zeta_1' = 0, \;\; G_{zz}\zeta_2' = 0.$$

Thus, the general solution of this is

$$y'' = k_1 z + f(y), \qquad z'' = k_2 y + g(z), \qquad \text{where } k_1 k_2 (f''^2 + g''^2) \neq 0.$$

Substituting this into (5.12), one has

$$\begin{aligned} \zeta_1'' &= f'\zeta_1 + k_1\zeta_2 + q_1, \\ \zeta_2'' &= k_2\zeta_1 + g'\zeta_2 + q_2. \end{aligned} \tag{5.13}$$

Differentiating the first equation of (5.13) with respect to x and y, and because $f'' \neq 0$, one has $\zeta_1' = 0$. Similarly, differentiating the second equation of (5.13) with respect to x and z, and because $g'' \neq 0$, then $\zeta_2' = 0$.

Hence, for both cases $F_{zz} \neq 0$ and $F_{zz} = 0$, one obtains that ζ is a constant.

□

The determining equations (5.2) are then reduced to

$$3\xi_1 \mathbf{F} + (((A + \xi_1 E)\mathbf{y} + \mathbf{k}) \cdot \nabla)\mathbf{F} - A\mathbf{F} = 0 \tag{5.14}$$

with the following admitted generator

$$X = 2(\xi_0 + \xi_1 x)\partial_x + (A\mathbf{y} + \mathbf{k}) \cdot \nabla \tag{5.15}$$

where ξ_0, ξ_1, the matrix A and the vector $\mathbf{k}$ are constant. By rewriting (5.15), the generator can be represented as

$$X = c_i X_i \qquad (i = 1, \ldots, 8) \tag{5.16}$$

where c_i's are constant. Corresponding to the constants c_i's, the basis operators of the Lie algebra are as follows:

$$\begin{aligned} &X_1 = \partial_x \quad X_2 = x\partial_x \quad X_3 = \partial_y \quad X_4 = \partial_z \\ &X_5 = y\partial_y \quad X_6 = z\partial_z \quad X_7 = z\partial_y \quad X_8 = y\partial_z. \end{aligned} \tag{5.17}$$

From here, the one-dimensional optimal system of one parameter subgroups of the main group of system (5.1) with $\xi'' = 0$ is constructed. Note that the action of equivalence transformations coincides with the action of group automorphisms. For the direct approach, sometimes it is difficult to select out equivalent cases with respect to equivalence transformations. Fortunately, if the algebraic structure of the admitted Lie algebra is known, then using the algebraic approach aids in simplifying the group classification problem. Thus, for finding the group classification of systems of two nonlinear second-order ordinary differential equations of the form (5.1) with all admitted generators satisfying $\xi'' = 0$, the one-dimensional optimal system of one parameter subgroups is utilized and is then proceeded by the direct approach. The commutators of the basis operators are

$$\begin{array}{llllll} [X_1, X_2] & = & X_1, & [X_5, X_7] & = & -X_7, \\ [X_3, X_5] & = & X_3, & [X_5, X_8] & = & X_8, \\ [X_3, X_8] & = & X_4, & [X_6, X_7] & = & X_7, \\ [X_4, X_6] & = & X_4, & [X_6, X_8] & = & -X_8, \\ [X_4, X_7] & = & X_3, & [X_7, X_8] & = & X_6 - X_5. \end{array} \tag{5.18}$$

The following inner automorphisms A_i $(i = 1, ..., 8)$ of the above Lie algebra are found without difficulties:

$$
\begin{aligned}
&A_1 : \hat{c}_1 = c_1 - a_1 c_2, \\
&A_2 : \hat{c}_1 = e^{a_2} c_1, \\
&A_3 : \hat{c}_3 = c_3 - a_3 c_5, \quad \hat{c}_4 = c_4 - a_3 c_8, \\
&A_4 : \hat{c}_3 = c_3 - a_4 c_7, \quad \hat{c}_4 = c_4 - a_4 c_6, \\
&A_5 : \hat{c}_3 = e^{a_5} c_3, \quad \hat{c}_7 = e^{a_5} c_7 \quad \hat{c}_8 = e^{-a_5} c_8, \\
&A_6 : \hat{c}_4 = e^{a_6} c_4, \quad \hat{c}_7 = e^{-a_6} c_7 \quad \hat{c}_8 = e^{a_6} c_8, \\
&A_7 : \hat{c}_3 = c_3 + a_7 c_4,\ \hat{c}_5 = c_5 + a_7 c_8,\ \hat{c}_6 = c_6 - a_7 c_8, \\
&\qquad\qquad \hat{c}_7 = c_7 - a_7^2 c_8 + a_7 c_6 - a_7 c_5, \\
&A_8 : \hat{c}_4 = c_4 + a_8 c_3,\ \hat{c}_5 = c_5 - a_8 c_7,\ \hat{c}_6 = c_6 + a_8 c_7, \\
&\qquad\qquad \hat{c}_8 = c_8 - a_8^2 c_7 - a_8 c_6 + a_8 c_5.
\end{aligned}
\tag{5.19}
$$

Note that $a_i\,(i = 1, ..., 8)$ are the parameters on which the transformations of the group depend on. Apart from these automorphisms, the following involutions hold:

$$
\begin{aligned}
&E_1 : \bar{z} = -z| \quad \bar{c_4} = -c_4,\ \bar{c_7} = -c_7,\ \bar{c_8} = -c_8; \\
&E_2 : \bar{y} = -y| \quad \bar{c_3} = -c_3,\ \bar{c_7} = -c_7,\ \bar{c_8} = -c_8; \\
&E_3 : \bar{x} = -x| \quad \bar{c_1} = -c_1; \\
&E_4 : \bar{y} = z,\ \bar{z} = y| \quad \bar{c_3} = c_4,\ \bar{c_4} = c_3,\ \bar{c_5} = c_6,\ \bar{c_5} = c_5,\ \bar{c_7} = c_8,\ \bar{c_8} = c_7.
\end{aligned}
$$

We study the way in which the coefficients of equation (5.16) are changed under the action of inner automorphisms of the group above. Here and further on, only changeable coordinates of the generator are presented. Looking closely at the commutators, the Lie algebra L_8, which is composed of the generators $X_i\ \ (i = 1, ..., 8)$, can be split into 2 subalgebras $L_2 \oplus L_6 = \{X_1, X_2\} \oplus \{X_3, X_4, X_5, X_6, X_7, X_8\}$. Note also that L_6 can be decomposed further to $L_4 \oplus I_2 = \{X_5, X_6, X_7, X_8\} \oplus \{X_3, X_4\}$, where L_4 makes up a 4-dimensional subalgebra and I_2 is ideal.

5.2.2.2 One-dimensional optimal system of subalgebras of the Lie algebra $L_4 = \{X_5, X_6, X_7, X_8\}$

Consider the 4-dimensional subalgebra $L_4 = \{X_5, X_6, X_7, X_8\}$. Cross-referencing the results found here with the study of Patera and Winternitz (1977), there is a need to show the classification of this 4-dimensional Lie algebra due to some misprint in their paper.

Suppose that the operator X of a one parameter subgroup has the form

$$X = c_5X_5 + c_6X_6 + c_7X_7 + c_8X_8 = c_5y\partial_y + c_6z\partial_z + c_7z\partial_y + c_8y\partial_z \tag{5.20}$$

or

$$X = \left[\begin{pmatrix} c_5 & c_7 \\ c_8 & c_6 \end{pmatrix}\mathbf{y}\right] \cdot \nabla.$$

For this, automorphisms A_5 up to A_8 are utilized in order to find the one-dimensional optimal system of subalgebras of this Lie algebra. From the automorphisms A_5 and A_6, one can find the invariant $\bar{c_7}\bar{c_8} = c_7c_8$, which leads one to consider the following cases:

$$(a) \quad c_7c_8 > 0$$

$$(b) \quad c_7c_8 < 0$$

$$(c) \quad c_7c_8 = 0.$$

Utilizing the invariant of A_7 and A_8, which is $(\bar{c_5} - \bar{c_6})^2 + 4\bar{c_7}\bar{c_8} = (c_5 - c_6)^2 + 4c_7c_8$, one can obtain relations between c_5, c_6, c_7 and c_8. It can be verified that the coefficients of equation (5.20) satisfy only the following cases:

$$(a)\ c_5 - c_6 \neq 0, \quad c_7 = 0, \quad c_8 = 0;$$

$$(b)\ c_5 - c_6 = 0, \quad c_7 = 1, \quad c_8 = 0;$$

$$(c)\ c_5 - c_6 = 0, \quad c_7 = -1, \quad c_8 = 1.$$

The involutions are also utilized. Hence, the following one-dimensional optimal system of subalgebras of the Lie algebra L_4 is obtained:

$$\begin{aligned}
&1.\quad X_5 + \alpha X_6 \ \text{ where } \ -1 \le \alpha \le 1\\
&2.\quad \alpha(X_5 + X_6) + X_8 - X_7 \ \text{ where } \alpha \ge 0\\
&3.\quad \beta(X_5 + X_6) + X_7 \text{ where } \beta = 0, 1\\
&4.\quad 0.
\end{aligned} \tag{5.21}$$

Note that the 0 element is considered on this list (Ovsiannikov, 1993). There is a necessity to include this element on the list as when the direct sum $L_4 \oplus I_2$ is applied, more subalgebras of the Lie algebra L_6 may appear on the list.

Remark 1: The one-dimensional optimal system of subalgebras (5.21) closely resembles Patera and Winternitz's (1977) dimension 1 of Algebra $A_{3,8} \oplus A_1$.

Remark 2: As the action of the above automorphisms coincides with the action of the equivalence transformations, it is possible to get the optimal system of one-dimensional subalgebras of the Lie algebra L_4 using the latter. From the determining equations (5.14) admitting the generator (5.20) and the utilization of the equivalence transformation $\widetilde{y} = Py$, where P is a nonsingular 2×2 matrix with constant entries, the matrix of coefficients of (5.20)

$$\begin{pmatrix} c_5 & c_7 \\ c_8 & c_6 \end{pmatrix}$$

is reduced to one of its real-valued Jordan forms (3.30). Looking closely at (5.21), subalgebra 1. coincides with Jordan matrix J_1, subalgebra 2. coincides with Jordan matrix J_2, and subalgebra 3. coincides with Jordan matrix J_3.

5.2.2.3 One-dimensional subalgebras of the Lie algebra $L_6 = \{X_3, X_4, X_5, X_6, X_7, X_8\}$

After obtaining the one-dimensional optimal system (5.21) of subalgebras of the Lie algebra $L_4 = \{X_5, X_6, X_7, X_8\}$, the next step is to combine L_4 with the ideal $I_2 = \{X_3, X_4\}$. Here, again Ovsiannikov's two-step method (Ovsiannikov, 1993) is applied. Hence, for the study of the one-dimensional subalgebras of the Lie algebra L_6, the study is reduced to analyzing the following elements:

$$\begin{aligned}
&1.\quad c_3X_3 + c_4X_4 + X_5 + \alpha X_6 \quad \text{where } -1 \le \alpha \le 1\\
&2.\quad c_3X_3 + c_4X_4 + \alpha(X_5 + X_6) + X_8 - X_7 \quad \text{where } \alpha \ge 0\\
&3.\quad c_3X_3 + c_4X_4 + \beta(X_5 + X_6) + X_7 \quad \text{where } \beta = 0, 1\\
&4.\quad c_3X_3 + c_4X_4.
\end{aligned} \tag{5.22}$$

Using automorphisms A_3, A_4 and the involutions, the list of one-dimensional subalgebras of the Lie algebra $L_6 = \{X_3, X_4, X_5, X_6, X_7, X_8\}$ is obtained as follows:

$$\begin{aligned}
&1.\quad X_5 + \alpha X_6 \quad \text{where } -1 \le \alpha \le 1\\
&2.\quad X_4 + X_5\\
&3.\quad X_8 - X_7\\
&4.\quad \beta X_3 + \alpha(X_5 + X_6) + X_8 - X_7 \quad \text{where } \beta = -1, 0, 1,\ \alpha > 0\\
&5.\quad \beta X_4 + X_7 \quad \text{where } \beta = 0, 1\\
&6.\quad X_5 + X_6 + X_7\\
&7.\quad X_3\\
&8.\quad 0.
\end{aligned} \tag{5.23}$$

Again, it is necessary to study the element 0 of the subalgebras of the Lie algebra L_6 as this may generate additional elements when L_6 is combined with L_2.

5.2.2.4 One-dimensional subalgebras of the Lie algebra $L_8 = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8\}$

Combining L_6 with L_2 and keeping in mind that for autonomous systems X_1 is already admitted, the following elements comprise the list of one-dimensional subalgebras of the Lie algebra L_8:

1. $\gamma X_2 + X_5 + \alpha X_6$ where $-1 \leq \alpha \leq 1$
2. $\gamma X_2 + X_4 + X_5$
3. $\gamma X_2 + X_8 - X_7$
4. $\gamma X_2 + \beta X_3 + \alpha(X_5 + X_6) + X_8 - X_7$ where $\beta = -1, 0, 1,\ \alpha > 0$
5. $\gamma X_2 + \beta X_4 + X_7$ where $\beta = 0, 1$
6. $\gamma X_2 + X_5 + X_6 + X_7$
7. $\gamma X_2 + X_3$
8. X_2.

(5.24)

Using this list of subalgebras, the solutions of F and G are sought after. These functions are substituted into the determining equations (5.2), which are solved completely in order to find all other generators admitting equations (5.1).

5.2.2.5 Representations of systems of two nonlinear second-order ordinary differential equations with all generators having $\xi'' = 0$

From (5.16), $c_i\,(i = 1, \ldots, 8)$ are the coefficients of the generator chosen from the above list of subalgebras (5.24). Only one subalgebra is presented in this Chapter as computations for the other subalgebras are done in a similar way. See Appendix A for obtaining representations of systems admitting the other remaining subalgebras.

5.2.2.5.1 Subalgebra 1. with the generator $\gamma X_2 + X_5 + \alpha X_6$ where $-1 \leq \alpha \leq 1$. For this case, the determining equations (5.14) become

$$yF_y + \alpha z F_z - (2\gamma - 1)F = 0,$$

$$yG_y + \alpha z G_z - (2\gamma - \alpha)G = 0.$$

The general solution of these equations is

$$F(y,z) = f(u)y^{1-2\gamma} \quad \text{and} \quad G(y,z) = g(u)y^{\alpha-2\gamma}, \tag{5.25}$$

where $u = y^\alpha/z$ and $gf' \neq 0$. If $gf' = 0$, then system (5.1) is equivalent to a reducible case. Substituting these functions to the determining equations (5.2), the following initial determining equations are obtained

$$\begin{aligned}
&y^{2\alpha}a_{12}(\alpha uf' + (1-2\gamma)f - ug) \\
&+y^{\alpha+1}u((\alpha a_{11} + (\alpha-1)\xi_1 - a_{22})uf' - 2(\gamma a_{11} + (\gamma-2)\xi_1)f) \\
&+y^\alpha\zeta_1 u(\alpha uf' + (1-2\gamma)f) - y^2 a_{21}u^3 f' - y\zeta_2 u^3 f' = 0, \\
&y^{2\alpha}a_{12}(\alpha ug' + (\alpha-2\gamma)g) \\
&+y^{\alpha+1}u((\alpha a_{11} + (\alpha-1)\xi_1 - a_{22})ug' - ((\alpha-2\gamma)a_{11} + (\alpha-2\gamma+3)\xi_1 - a_{22})g) \\
&+y^\alpha u\zeta_1(\alpha ug' + (\alpha-2\gamma)g) - y^2 a_{21}u(u^2g' + f) - y\zeta_2 u^3 g' = 0.
\end{aligned} \tag{5.26}$$

Determining equations (5.26) can be split with respect to y, where the powers of y depend on the values of α. Thus, upon further analysis, the study is continued based on the following cases: (1) $\alpha = 0$, (2) $\alpha = \frac{1}{2}$, (3) $\alpha = 1$ and (4) $\alpha \neq 0, \frac{1}{2}, 1$.

1. **Case** $\alpha = 0$

   After splitting equations (5.26) with respect to y, it can be verified that $a_{21} = 0$ and one is left with the following determining equations

$$((1-2\gamma)(\zeta_1 u + a_{12}))f - a_{12}ug = 0, \tag{5.27a}$$

$$2(\gamma a_{11} + (\gamma-2)\xi_1)f + (a_{22} + \xi_1 + \zeta_2 u)uf' = 0, \tag{5.27b}$$

$$\gamma(a_{12} + \zeta_1 u)g = 0, \tag{5.27c}$$

$$(2\gamma a_{11} + (2\gamma-3)\xi_1 + a_{22})g + (a_{22} + \xi_1 + \zeta_2 u)ug' = 0. \tag{5.27d}$$

From equation (5.27c), $\gamma \neq 0$. Notice that if $\gamma = 0$, G becomes a function solely of z and hence, this case is reducible. Thus, it follows that $a_{12} = 0$ and $\zeta_1 = 0$. These conditions also satisfy equation (5.27a). Dividing equation (5.27b) by f' and u, and differentiating it with respect to u 2 times, one can study the following cases: a. $\left(\frac{f}{uf'}\right)'' \neq 0$ and b. $\left(\frac{f}{uf'}\right)'' = 0$.

1.a Case $\left(\frac{f}{uf'}\right)'' \neq 0$

For this case, it follows that $a_{11} = \xi_1 \frac{(2-\gamma)}{\gamma}$. Consequently, $\zeta_2 = 0$ and $a_{22} = -\xi_1$. These conditions also satisfy equation (5.27d). These conditions give no other extensions of the generator apart from the studied subalgebra.

1.b Case $\left(\frac{f}{uf'}\right)'' = 0$

For this case, it follows that $\frac{f}{uf'} = \kappa u + \beta$. Furthermore, the general solution of this depends on β, i.e., whether i. $\beta \neq 0$ or ii. $\beta = 0$.

1.b.i Case $\beta \neq 0$

For this case, the general solution for f (with a possible shift) is $f_0 \left(\frac{1}{u}\right)^{\beta}$, $f_0 \neq 0$. Substituting this into the determining equations (5.27b), one gets $a_{22} = \frac{(2\gamma - \beta - 4)\xi_1 + 2\gamma a_{11}}{\beta}$ and $\zeta_2 = 0$. Consequently, the general solution for g is $g_0 \left(\frac{1}{u}\right)^{\beta+1}$, $g_0 \neq 0$. From here, the extension

$$\beta X_5 + 2\gamma X_6$$

is obtained along with the studied subalgebra.

1.b.ii Case $\beta = 0$

For this case, it follows that $\kappa \neq 0$. Hence, the general solution for f is $f_0 e^{\kappa/u}$, $f_0 \neq 0$. Substituting this into equation (5.27b), one obtains $\zeta_2 = \dfrac{2(\gamma a_{11} + (\gamma - 2)\xi_1)}{\kappa}$ and $a_{22} = -\xi_1$. Consequently, the general solution for g is $g_0 e^{\kappa/u}$, $g_0 \neq 0$. The extension

$$2\gamma X_4 + \kappa X_5$$

is obtained aside from the studied subalgebra.

2. **Case** $\alpha = \frac{1}{2}$

After splitting equations (5.26) with respect to y, it follows that $a_{21} = 0$. Also, since $(1-4\gamma)g + ug' = 0$ leads to a reducible case it then follows that $\zeta_1 = 0$. The remaining determining equations are

$$2a_{12}(1-2\gamma)f - 2a_{12}ug + (a_{12} - 2\zeta_2 u^2)uf' = 0, \tag{5.28a}$$

$$4((2-\gamma)\xi_1 - \gamma a_{11})f + (a_{11} - 2a_{22} - \xi_1)uf' = 0, \tag{5.28b}$$

$$(1-4\gamma)a_{12}g + (a_{12} - 2\zeta_2 u^2)ug' = 0, \tag{5.28c}$$

$$((1-4\gamma)a_{11} + (7-4\gamma)\xi_1 - 2a_{22})g + (a_{11} - 2a_{22} - \xi_1)ug' = 0. \tag{5.28d}$$

Dividing equation (5.28d) by g (as it is nonzero) and differentiating it with respect to u, one is left to study the following cases: a. $\left(\dfrac{ug'}{g}\right)' \neq 0$ and b. $\left(\dfrac{ug'}{g}\right)' = 0$.

2.a Case $\left(\dfrac{ug'}{g}\right)' \neq 0$

For this case, it follows that $a_{11} = 2a_{22} + \xi_1$. If $\gamma = 0$ then $\xi_1 = 0$, but if $\gamma \neq 0$ then $a_{22} = \xi_1 \left(\dfrac{1-\gamma}{\gamma}\right)$. These conditions also satisfy equation (5.28b). From equation (5.28c), the following cases are studied: i. there exists a generator with $a_{12} \neq 0$, and ii. all generators have $a_{12} = 0$.

2.a.i Case $a_{12} \neq 0$

If there exists a generator with $a_{12} \neq 0$, then g satisfies the form $(1-4\gamma)g + (1-\beta u^2)ug' = 0$. Notice that $\beta = 0$ leads to a reducible case. Hence, $\beta \neq 0$. Without loss of generality, one can assume that $\beta = 1$. Then the general solution of g is $g_0\left(1 - \frac{1}{u^2}\right)^{\widetilde{\gamma}}$, where $g_0 \neq 0$ and $\widetilde{\gamma} = \frac{1-4\gamma}{2} \neq 0$ (if $\widetilde{\gamma} = 0$, the case is reducible). Substituting this into equation (5.28c), we obtain that $\zeta_2 = \frac{a_{12}}{2}$. From equation (5.28a), it follows that $f = \phi(u)\left(1 - \frac{1}{u^2}\right)^{\widetilde{\gamma}+(1/2)}$, where $\phi = f_0 - 2g_0\left(\frac{1}{(u^2-1)^{(1/2)}}\right)$. Here, the extension

$$X_4 + 2X_7$$

is obtained besides the studied subalgebra.

2.a.ii Case $a_{12} = 0$

i. For the case where all generators have $a_{12} = 0$, it follows that $\zeta_2 = 0$. All remaining equations are satisfied, and no other extensions are obtained.

2.b Case $\left(\frac{ug'}{g}\right)' = 0$

For this case, the general solution is $g = g_0 u^{\kappa}$, where $g_0 \neq 0$. Substituting this into equation (5.28c), further analysis leads one to obtain that $a_{12} = 0$ and $\zeta_2 = 0$. These conditions also satisfy (5.28a). From equation (5.28b), the form of f satisfies $(\kappa+1)f - uf' = 0$. The general solution is $f = f_0 u^{\kappa+1}$, where $f_0 \neq 0$, $\kappa \neq -1$. Moreover, this leads to $a_{22} = (\kappa - 4\gamma + 1)(a_{11} - \xi_1) + 8\xi_1$. Here, the extension

$$(\kappa+1)X_2 + 2X_6$$

is obtained apart from the studied subalgebra.

3. **Case** $\alpha = 1$

The determining equations after splitting equations (5.26) with respect to y

are as follows

$$(1-2\gamma)\zeta_1 f + (\zeta_1 - \zeta_2 u)uf' = 0, \tag{5.29a}$$

$$\begin{aligned}((1-2\gamma)a_{12} + ((4-2\gamma)\xi_1 u - 2\gamma a_{11}u))f - a_{12}ug \\ +((a_{11}-a_{22})u + a_{12} - a_{21}u^2)uf' = 0,\end{aligned} \tag{5.29b}$$

$$(1-2\gamma)\zeta_1 g + (\zeta_1 - \zeta_2 u)ug' = 0, \tag{5.29c}$$

$$\begin{aligned}-a_{21}uf + g((1-2\gamma)a_{11}u + (1-2\gamma)a_{12} + (4-2\gamma)\xi_1 u - a_{22}u) \\ +g'u((a_{11}-a_{22})u + a_{12} - a_{21}u^2) = 0.\end{aligned} \tag{5.29d}$$

From equations (5.29a) and (5.29c), one can study the following 2 cases: a. $fg' - gf' = 0$, and b. $fg' - gf' \neq 0$.

3.a Case $fg' - gf' = 0$

For this case, we obtain the relation $g = g_0 f$ where g_0 is a constant. Using equivalence transformations, one can show that the second equation can be reduced to zero, i.e., $G = 0$, which is equivalent to a reducible case.

3.b Case $fg' - gf' \neq 0$

It follows from equations (5.29a) and (5.29c) that $\zeta_1 = \zeta_2 = 0$. From here, one can assume that $g = \phi(u)f$ (as f is nonzero), where $\phi' \neq 0$. If it is assumed further that $\phi = \psi(u) + 1/u$, then the remaining determining equations (5.29b) and (5.29d) are reduced as follows:

$$\begin{aligned}(2(-\gamma a_{11}u + (2-\gamma)\xi_1 u) - (\gamma + \psi u)a_{12})f \\ +((a_{11}-a_{22})u + a_{12} - a_{21}u^2)uf' = 0,\end{aligned} \tag{5.30a}$$

$$((a_{11}-a_{22})u + a_{12} - a_{21}u^2)\psi' + a_{12}\psi^2 + (a_{11} + 2a_{12}u^{-1} - a_{22})\psi = 0. \tag{5.30b}$$

These equations lead one to study the following two cases:

i. there exists at least one generator with $a_{12} \neq 0$, and

ii. where all generators have $a_{12} = 0$.

3.b.i Case $a_{12} \neq 0$

For the case where there exists at least one generator with $a_{12} \neq 0$, it follows that $\psi(u) = -\dfrac{\kappa u^2 + \lambda u + \beta}{u(\beta - \psi_0 u)}$, where $\beta \neq 0$, $\psi_0 \neq 0, \lambda$, κ are constant. Without loss of generality, it is assumed further that $\beta = 1$. Consequently, we obtain $a_{11} = \lambda a_{12} + a_{22}$ and $a_{21} = -\kappa a_{12}$. Substituting this into determining equations (5.30a), the solution for f appears, which depends on the following three cases: A. $4\kappa - \lambda^2 > 0$, B. $4\kappa - \lambda^2 < 0$, and C. $4\kappa - \lambda^2 = 0$.

3.b.i.A Case $4\kappa - \lambda^2 > 0$

For this case, it is assumed that $4\kappa - \lambda^2 = p^2$, $p \neq 0$. The solution for f is

$$f_0 \frac{(1-\psi_0 u)u^{2\gamma-1}}{(\kappa u^2 + \lambda u + 1)^\gamma} e^{\left(\frac{(2\lambda\gamma - 4\mu)}{p} \arctan\left(\frac{\lambda + 2\kappa u}{p}\right)\right)}$$

where μ is constant.

3.b.i.B Case $4\kappa - \lambda^2 < 0$

For this case, it is assumed that $4\kappa - \lambda^2 = -p^2$, $p \neq 0$. The solution for f is

$$f_0 \frac{(1-\psi_0 u)u^{2\gamma-1}}{(\kappa u^2 + \lambda u + 1)^\gamma} \left(\frac{2\kappa u + \lambda - p}{2\kappa u + \lambda + p}\right)^{\frac{\lambda\gamma - 2\mu}{p}}$$

where μ is constant.

3.b.i.C Case $4\kappa - \lambda^2 = 0$

For this case, it follows that

$$f = f_0 \frac{(1-\psi_0 u)u^{2\gamma-1}}{(\kappa u^2 + \lambda u + 1)^\gamma} e^{\left(-\frac{4(\gamma + \mu u)}{\lambda u + 2}\right)}$$

where μ is constant.

All three cases yield the same results for a_{22}, ξ_1 and the extension of the generator. If $\gamma \neq 0$, then $a_{22} = \dfrac{(2-\gamma)\xi_1 - \mu a_{12}}{\gamma}$. If $\gamma = 0$, then $\xi_1 = \dfrac{\mu a_{12}}{2}$. The extension

$$\mu X_2 + \lambda X_5 + X_7 - \kappa X_8$$

is obtained apart from the studied subalgebra.

3.b.ii Case $a_{12} = 0$

For the case where all generators have $a_{12} = 0$, the determining equations (5.30) are reduced to

$$2(-\gamma a_{11} u + (2-\gamma)\xi_1 u) f + ((a_{11} - a_{22})u - a_{21}u^2) u f' = 0, \quad (5.31a)$$

$$((a_{11} - a_{22})u - a_{21}u^2)\psi' + (a_{11} - a_{22})\psi = 0. \quad (5.31b)$$

Dividing equation (5.31a) by uf' and differentiating this equation with respect to u twice, this leads to the study of the following sub-cases: A. $\left(\dfrac{f}{uf'}\right)'' \neq 0$, and B. $\left(\dfrac{f}{uf'}\right)'' = 0$.

3.b.ii.A Case $\left(\dfrac{f}{uf'}\right)'' \neq 0$

If $\left(\dfrac{f}{uf'}\right)'' \neq 0$, then it follows that if $\gamma \neq 0$ then $a_{11} = \xi_1 \dfrac{2-\gamma}{\gamma}$, $a_{22} = \xi_1 \dfrac{2-\gamma}{\gamma}$ and $a_{21} = 0$. If $\gamma = 0$ then $\xi_1 = 0$, $a_{22} = a_{11}$ and $a_{21} = 0$. For both cases, no extensions are obtained apart from the studied subalgebra.

3.b.ii.B Case $\left(\dfrac{f}{uf'}\right)'' = 0$

If $\left(\dfrac{f}{f'u}\right)'' = 0$, then the general solution for f is $f_0 \left(\dfrac{u}{1+u}\right)^{\kappa}$, where $\kappa \neq 0$ (else it is reducible) and $f_0 \neq 0$. Substituting this into equation (5.31a), one obtains that $a_{21} = 2\left(\dfrac{-\gamma a_{11} + (2-\gamma)\xi_1}{\kappa}\right)$ and $a_{22} = \dfrac{(\kappa - 2\gamma)a_{11} + (4-2\gamma)\xi_1}{\kappa}$.

Substituting this into equation (5.31b), one finds that ψ satisfies $\psi'(u^2+u)+\psi=0$. The general solution of this is $\psi = g_0\left(\frac{u+1}{u}\right)$. The extension

$$\kappa X_2 + 2(X_6 + X_8)$$

is obtained aside from the studied subalgebra.

4. **Case** $\alpha \neq 0, \frac{1}{2}, 1$

For the case where $\alpha \neq 0, \frac{1}{2}, 1$, the determining equations (5.26) are split with respect to y. Since $f' \neq 0$, it follows that $\zeta_2 = 0$ and $a_{21} = 0$. Notice also that since $\alpha u g' + (\alpha - 2\gamma)g = 0$ leads to a reducible case, then $\zeta_1 = 0$ and $a_{12} = 0$. Substituting these conditions, the determining equations (5.26) become

$$(\alpha a_{11} + (\alpha-1)\xi_1 - a_{22})uf' + (-2\gamma a_{11} + (4-2\gamma)\xi_1)f = 0, \qquad (5.32a)$$

$$(\alpha a_{11}+(\alpha-1)\xi_1-a_{22})ug'+((\alpha-2\gamma)a_{11}+(\alpha-2\gamma+3)\xi_1-a_{22})g=0. \quad (5.32b)$$

Dividing equation (5.32a) by f (as it is nonzero) and differentiating with respect to u, the following cases are studied: a. $\left(\frac{uf'}{f}\right)' \neq 0$ and b. $\left(\frac{uf'}{f}\right)' = 0$.

4.a **Case** $\left(\frac{uf'}{f}\right)' \neq 0$

For this case, it follows from equation (5.32a) that $a_{22} = \alpha a_{11} + (\alpha-1)\xi_1$. Consequently, if $\gamma \neq 0$ then $a_{11} = \frac{2-\gamma}{\gamma}\xi_1$, and if $\gamma = 0$ then $\xi_1 = 0$. These conditions also satisfy equation (5.32b). No other extensions of the generator is found other than the studied subalgebra.

4.b **Case** $\left(\frac{uf'}{f}\right)' = 0$

For this case, the general solution for f is $f_0 u^\kappa$, where $f_0\kappa \neq 0$. Substituting this function into the determining equations (5.32a), it follows that $a_{22} = \dfrac{(\kappa\alpha - 2\gamma)a_{11} + (\kappa\alpha - \kappa - 2\gamma + 4)\xi_1}{\kappa}$. From (5.32b), extensions of the generator can only be found if g satisfies the condition $g'u - g(\kappa - 1) = 0$, where the general solution for g is $g_0 u^{\kappa-1}$, $g_0 \neq 0$. Another extension of the generator apart from the studied subalgebra is

$$\kappa X_2 + 2X_6.$$

The complete representative classes for the autonomous system with all admitted generators having $\xi'' = 0$ is listed in Tables 6.3 and 6.4.

CHAPTER VI

CONCLUSIONS

In this thesis, the complete group classification of systems of two linear second-ordinary differential equations and the group classification of systems of two nonlinear second-ordinary differential equations of the form $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$ were studied. A preliminary study of two nonlinear second-order differential equations of the form $\mathbf{y}'' = \mathbf{F}(x, \mathbf{y})$ was also done before classification of the latter.

The group classification process is done both directly and indirectly (algebraic approach). For the direct approach, all possible Lie algebras were found with the aid of the equivalence transformations applied to the determining equations. As for the algebraic approach, the study was reduced to the analysis of relations between constants of the generator with its corresponding basis operators.

The results of the group classification of the linear system are presented in Table 6.1, where the first column presents the form of the functions F and G and the second column lists the admitted generator apart from the trivial ones (3.7 and 3.8) obtained in Chapter III.

For the autonomous nonlinear system, the analysis of determining equations were separated into classes depending on the coefficient of generator ξ, i.e., whether it involves 1) at least one admitted generator with $\xi'' \neq 0$ or 2) all admitted generators have $\xi'' = 0$. Note also that the kernel of the generator contains ∂_x. The results of the group classification of the autonomous nonlinear system is presented in Tables 6.2, 6.3 and 6.4. Table 6.2 provides the group classification of the autonomous nonlinear system which admits at least one generator containing

$\xi'' = 0$, and Tables 6.3 and 6.4 present the group classification of the autonomous nonlinear system for which all admitted generators have $\xi'' = 0$. The first two columns of Tables 6.2 and 6.3 show the forms of F and G, which admit the Lie algebras listed on the fourth column. The third column of these tables show the conditions and relations of F and G. Table 6.4 gives additional extensions of the generators in Table 6.3 along with its representative classes. The generators found in the Tables are defined as follows:

$$
\begin{aligned}
&Y_2 = 2x\partial_x + y\partial_y + z\partial_z, && X_2 = x\partial_x,\\
&Y_3 = x(x\partial_x + y\partial_y + z\partial_z), && X_3 = \partial_y,\\
&Y_4 = (\gamma - 4)y\partial_y + \gamma z\partial_z, && X_4 = \partial_z,\\
&Y_5 = (\alpha y - 4z)\partial_y + (\alpha z + 4y)\partial_z, && X_5 = y\partial_y,\\
&Y_6 = (y + 4z)\partial_y + z\partial_z, && X_6 = z\partial_z,\\
&Y_7 = \cos 2x\partial_x - \sin 2x(y\partial_y + z\partial_z), && X_7 = z\partial_y,\\
&Y_8 = \sin 2x\partial_x + \cos 2x(y\partial_y + z\partial_z), && X_8 = y\partial_z,\\
&Y_9 = e^{-2x}(\partial_x - (y\partial_y + z\partial_z)), &&\\
&Y_{10} = e^{2x}(\partial_x + y\partial_y + z\partial_z). &&
\end{aligned}
$$

It is highly likely that the same methods shown in this thesis are applicable to finding the group classification of systems of two nonlinear second-order ordinary differential equations, which will be next goal for further studies. As seen in the preliminary study of nonlinear systems in Chapter IV, simplified classes of the functions F and G are not yet known. In addition, it is also believed that this can be extended to systems in more general cases. Another recommendation for further studies is to find applications of these classes in the real world.

Table 6.1 Group classification of linear systems. Here, c_i $(i = 1, \dots, 4)$ are constant and $c_3 c' \neq 0$.

	F and G	Admitted Generator
1.	$F(x,y,z) = c_1 y + e^x z,$	
	$G(x,y,z) = c_3 e^{-x} y + c_4 z$	$\partial_x - z\partial_z$
2.	$F(x,y,z) = (\sin(x) + c_1)y + (\cos(x) - c_2)z,$	
	$G(x,y,z) = (\cos(x) + c_2)y + (-\sin(x) + c_1)z$	$2\partial_x + z\partial_y - y\partial_z$
3.	$F(x,y,z) = (x + c_1)y + (-x^2 + (c_4 - c_1)x + c_2)z,$	
	$G(x,y,z) = y + (-x + c_4)z$	$\partial_x + z\partial_y$
4.	$F(x,y,z) = c(x)y + z$	
	$G(x,y,z) = -y + c(x)z$	$z\partial_y - y\partial_z$

Table 6.2 Group classification of nonlinear systems of the form $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$ admitting at least one generator with $\xi'' \neq 0$. Here $\tau(y,z) = e^{\alpha \arctan \frac{z}{y}}(y^2+z^2)^{-2}$, and $f_0 f_1 \neq 0$, $g_0^2 + g_1^2 \neq 0$, $\gamma \neq 0, 4$, $\alpha \neq 0$, -4, and h_1 are constant.

F	G	κ	Extension of Kernel
$\kappa y + y^{-3} f(u)$	$\kappa z + z^{-3} g(u)$	0	Y_2, Y_3
		-1	Y_7, Y_8
		1	Y_9, Y_{10}
$\kappa y + f_0 y^{-3}(\frac{z}{y})^{\gamma-4}$	$\kappa z + f_1 z^{-3}(\frac{z}{y})^{\gamma}$	0	Y_2, Y_3, Y_4
		-1	Y_7, Y_8, Y_4
		1	Y_9, Y_{10}, Y_4
$\kappa y + (g_0 y - g_1 z)\tau(y,z)$	$\kappa z + (g_0 z + g_1 y)\tau(y,z)$	0	Y_2, Y_3, Y_5
		-1	Y_7, Y_8, Y_5
		1	Y_9, Y_{10}, Y_5
$\kappa y + e^{\frac{y}{z}} z^{-4}(f_0 y + h_1 z)$	$\kappa z + f_0 z^{-3} e^{\frac{y}{z}}$	0	Y_2, Y_3, Y_6
		-1	Y_7, Y_8, Y_6
		1	Y_9, Y_{10}, Y_6

Table 6.3 Group classification of systems admitting all generators with $\xi'' = 0$. Here we have $\theta_1(u,v) = (\cos(u)f(v)+\sin(u)g(v))$, $\theta_2(u,v) = \sin(u)f(v) - \cos(u)g(v)$, $\chi_1(\alpha) = \dfrac{\alpha}{\alpha^2+1}$ and $\chi_2(\alpha) = \dfrac{1}{\alpha^2+1}$.

F	G	Relations	Extension of Kernel
$f(u)y^{(1-2\gamma)}$	$g(u)y^{(\alpha-2\gamma)}$	$u = \frac{y^\alpha}{z}$, $-1 \le \alpha \le 1$, $f'g \neq 0$	$\gamma X_2 + X_5 + \alpha X_6$
$f(u)y^{(1-2\gamma)}$	$g(u)y^{(-2\gamma)}$	$u = ye^{-z}$, $f'g \neq 0$	$\gamma X_2 + X_4 + X_5$
$e^{-2\gamma u}\theta_1(u,v)$	$-e^{-2\gamma u}\theta_2(u,v)$	$y = v\cos(u),\ z = v\sin(u)$, $f^2+g^2 \neq 0$	$\gamma X_2 - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u,v)$	$e^{(\alpha-2\gamma)u}\theta_2(u,v)$	$y = ve^{\alpha u}\cos(u) + \chi_1(\alpha)$	
		$z = ve^{\alpha u}\sin(u) - \chi_2(\alpha)$, $\alpha > 0$, $f^2+g^2 \neq 0$	$\gamma X_2 - X_3 + \alpha(X_5+X_6) - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u,v)$	$e^{(\alpha-2\gamma)u}\theta_2(u,v)$	$y = ve^{\alpha u}\cos(u) - \chi_1(\alpha)$	
		$z = ve^{\alpha u}\sin(u) + \chi_2(\alpha)$, $\alpha > 0$, $f^2+g^2 \neq 0$	$\gamma X_2 + X_3 + \alpha(X_5+X_6) - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u,v)$	$e^{(\alpha-2\gamma)u}\theta_2(u,v)$	$y = ve^{\alpha u}\cos(u)$	
		$z = ve^{\alpha u}\sin(u)$, $\alpha > 0$, $f^2+g^2 \neq 0$	$\gamma X_2 + \alpha(X_5+X_6) - X_7 + X_8$
$(g(v)u + f(v))e^{(-2\gamma u)}$	$g(v)e^{(-2\gamma u)}$	$y = uv,\ z = v$, $g \neq 0$	$\gamma X_2 + X_7$
$(g(u)z + f(u))e^{(-2\gamma z)}$	$g(u)e^{(-2\gamma z)}$	$u = z^2 - 2y$, $g' \neq 0$	$\gamma X_2 + X_4 + X_7$
$((y/z)g(u) + f(u))e^{((1-2\gamma)(y/z))}$	$g(u)e^{((1-2\gamma)(y/z))}$	$u = ze^{-y/z}$, $g \neq 0$	$\gamma X_2 + X_5 + X_6 + X_7$
$f(z)e^{-2\gamma y}$	$g(z)e^{-2\gamma y}$	$f'g \neq 0$	$\gamma X_2 + X_3$

Table 6.4 Group classification of systems admitting all generators with $\xi'' = 0$. Here, f_0, g_0, ϕ_0, ϕ_1, α, β, κ, μ_0, λ and γ are constant.

Subalgebra 1. $\gamma X_2 + X_5$

F	G	Relations	Additional Extension of Kernel
$f_0 z^{\beta} y^{1+\tilde{\gamma}}$	$g_0 z^{\beta+1} y^{\tilde{\gamma}}$	$\tilde{\gamma} = -2\gamma \neq 0,\ \beta \neq 0,\ f_0 g_0 \neq 0$	$\beta X_5 - \tilde{\gamma} X_6{}^*$
$f_0 y^{1+\tilde{\gamma}} e^{\kappa z}$	$g_0 y^{\tilde{\gamma}} e^{\kappa z}$	$\tilde{\gamma} = -2\gamma \neq 0,\ \kappa \neq 0,\ f_0 g_0 \neq 0$	$\kappa X_5 - \tilde{\gamma} X_4$

Subalgebra 1. $\gamma X_2 + X_5 + \frac{1}{2} X_6$

F	G	Relations	Additional Extension of Kernel
$\phi(y,z)(y - z^2)^{\tilde{\gamma}}$	$g_0 (y - z^2)^{\tilde{\gamma}}$	$\tilde{\gamma} = \frac{1-4\gamma}{2} \neq 0,\ g_0 \neq 0$ $\phi = f_0 (y - z^2)^{1/2} + 2 g_0 z$	$X_4 + 2X_7$
$f_0 z^{-\tilde{\kappa}} y^{\tilde{\gamma}+1}$	$g_0 z^{-\tilde{\kappa}+1} y^{\tilde{\gamma}}$	$f_0 g_0 \neq 0,\ \tilde{\kappa} = \kappa + 1 \neq 0,\ \tilde{\gamma} = \frac{\tilde{\kappa} - 4\gamma}{2} \neq 0$	$\tilde{\kappa} X_2 + 2X_6{}^*$

Subalgebra 1. $\gamma X_2 + X_5 + X_6$

F	G	Relations	Additional Extension of Kernel
$f_0 \frac{z - \alpha y}{(z^2 + \lambda y z + \kappa y^2)^{\gamma}} \psi_i(y,z)$	$-f_0 \frac{(\kappa y + (\lambda + \alpha) z)}{(z^2 + \lambda y z + \kappa y^2)^{\gamma}} \psi_i(y,z)$	$i = 1,2,3,\ \alpha \neq 0,\ f_0 \neq 0$	$\mu X_2 + \lambda X_5 + X_7 - \kappa X_8$

Here, $\psi_1(y,z) = e^{\frac{2\lambda\gamma - 4\mu}{p} \arctan \frac{\lambda z + 2\kappa y}{pz}}$ with $4\kappa - \lambda^2 = p^2$, $p \neq 0$;

$\psi_2(y,z) = \left(\frac{2\kappa y + (\lambda + p) z}{2\kappa y + (\lambda - p) z}\right)^{\frac{2\mu - \lambda\gamma}{p}}$ with $4\kappa - \lambda^2 = -p^2$, $p \neq 0$; and

$\psi_3(y,z) = e^{-\frac{4(\mu y + \gamma z)}{\lambda y + 2z}}$ with $4\kappa - \lambda^2 = 0$

*Can be merged together

Table 6.4 Group classification of systems admitting all generators with $\xi''=0$. Here, f_0, g_0, ϕ_0, ϕ_1, α, β, κ, μ_0, λ and γ are constant. (Continued)

Subalgebra 1. $\gamma X_2 + X_5 + X_6$ (Continued)

F	G	Relations	Additional Extension of Kernel
$f_0\left(\frac{y}{y+z}\right)^{\kappa} y^{1-2\gamma}$	$\left(g_0 - f_0\frac{y}{y+z}\right)\left(\frac{y}{y+z}\right)^{\kappa-1} y^{1-2\gamma}$	$f_0\neq 0,\ \gamma\neq 0,\ \kappa\neq 0$	$\kappa X_2 + 2(X_6+X_8)$

Subalgebra 1. $\gamma X_2 + X_5 + \alpha X_6,\ -1\leq\alpha\leq 1$

F	G	Relations	Additional Extension of Kernel
$f_0 z^{-\kappa} y^{\tilde{\gamma}+1}$	$g_0 z^{1-\kappa} y^{\tilde{\gamma}}$	$f_0 g_0\neq 0,\ \tilde{\gamma}=\alpha\kappa-2\gamma,\ \alpha\neq 0,\ 1/2,\ 1,\ \kappa\neq 0$	$\kappa X_2 + 2X_6{}^*$

Subalgebra 2. $\gamma X_2 + X_4 + X_5$

F	G	Relations	Additional Extension of Kernel
$f_0 y^{(\kappa+1)} e^{-\alpha z}$	$g_0 y^{(\kappa)} e^{-\alpha z}$	$f_0 g_0\neq 0,\ \gamma=\frac{\alpha-\kappa}{2},\ \kappa\alpha\neq 0$	$\alpha X_5 + \kappa X_4$

Subalgebra 3. $-X_7 + X_8$

F	G	Relations	Additional Extension of Kernel
$(f_0\cos(u)+g_0\sin(u))v^{\kappa}$	$(f_0\sin(u)-g_0\cos(u))v^{\kappa}$	$f_0^2+g_0^2\neq 0,\ \kappa\neq 1,$ $u=\arctan(z/y),\ v^2=y^2+z^2$	$\frac{1-\kappa}{2}X_2 + X_5 + X_6$

Subalgebra 3. $\gamma X_2 - X_7 + X_8,\ \gamma\neq 0$

F	G	Relations	Additional Extension of Kernel
$e^{\tilde{\gamma}u}(f_0\cos(u)+g_0\sin(u))v^{\kappa}$	$e^{\tilde{\gamma}u}(f_0\sin(u)-g_0\cos(u))v^{\kappa}$	$f_0^2+g_0^2\neq 0,\ \kappa\neq 1,$ $\tilde{\gamma}=-2\gamma\neq 0,\ u=\arctan(z/y),\ v^2=y^2+z^2$	$\tilde{\gamma}(X_5+X_6)+(1-\kappa)(X_8-X_7)$

Table 6.4 Group classification of systems admitting all generators with $\xi'' = 0$. Here, f_0, g_0, ϕ_0, ϕ_1, α, β, κ, μ_0, λ and γ are constant. (Continued)

Subalgebra 4. $\gamma X_2 - X_3 + \alpha(X_5 + X_6) - X_7 + X_8,\ \alpha > 0$

F	G	Relations	Additional Extension of Kernel
$e^{(\alpha-2\gamma)u}(f_0\cos(u) + g_0\sin(u))v^{\kappa}$	$e^{(\alpha-2\gamma)u}(f_0\sin(u) - g_0\cos(u))v^{\kappa}$	$f_0^2 + g_0^2 \neq 0,\ \kappa \neq 1,$ $u = \arctan\left(\frac{z+\chi_2(\alpha)}{y-\chi_1(\alpha)}\right),$ $v^2 = e^{-2\alpha u}((y-\chi_1(\alpha))^2 + (z+\chi_2(\alpha))^2),$ $\chi_1(\alpha) = \frac{\alpha}{\alpha^2+1},\ \chi_2(\alpha) = \frac{1}{\alpha^2+1}$	$\frac{1-\kappa}{2}X_2 + X_5 + X_6 - \chi_1 X_3 + \chi_2 X_4$

Subalgebra 4. $\gamma X_2 + X_3 + \alpha(X_5 + X_6) - X_7 + X_8,\ \alpha > 0$

F	G	Relations	Additional Extension of Kernel
$e^{(\alpha-2\gamma)u}(f_0\cos(u) + g_0\sin(u))v^{\kappa}$	$e^{(\alpha-2\gamma)u}(f_0\sin(u) - g_0\cos(u))v^{\kappa}$	$f_0^2 + g_0^2 \neq 0,\ \kappa \neq 1,$ $u = \arctan\left(\frac{z-\chi_2(\alpha)}{y+\chi_1(\alpha)}\right),$ $v^2 = e^{-2\alpha u}((y+\chi_1(\alpha))^2 + (z-\chi_2(\alpha))^2),$ $\chi_1(\alpha) = \frac{\alpha}{\alpha^2+1},\ \chi_2(\alpha) = \frac{1}{\alpha^2+1}$	$\frac{1-\kappa}{2}X_2 + X_5 + X_6 + \chi_1 X_3 - \chi_2 X_4$

Subalgebra 4. $\gamma X_2 + \alpha(X_5 + X_6) - X_7 + X_8,\ \alpha > 0$

F	G	Relations	Additional Extension of Kernel
$e^{(\alpha-2\gamma)u}(f_0\cos(u) + g_0\sin(u))v^{\kappa}$	$e^{(\alpha-2\gamma)u}(f_0\sin(u) - g_0\cos(u))v^{\kappa}$	$f_0^2 + g_0^2 \neq 0,\ \kappa \neq 1,$ $u = \arctan(z/y),\ v^2 = e^{-2\alpha u}(y^2 + z^2)$	$\frac{1-\kappa}{2}X_2 + X_5 + X_6$

Table 6.4 Group classification of systems admitting all generators with $\xi'' = 0$. Here, f_0, g_0, ϕ_0, ϕ_1, α, β, κ, μ_0, λ and γ are constant. (Continued)

Subalgebra 5. $\gamma X_2 + X_7$			
F	G	Relations	Additional Extension of Kernel
$g_0 z^{\beta-1} e^{-y/z}(y + \kappa\tilde{\gamma} z)$	$g_0 z^{\beta} e^{-y/z}$	$g_0 \neq 0,\ \tilde{\gamma} = 2\gamma$	$X_5 + \tilde{\gamma} X_6 + (\beta - 1) X_7$
Subalgebra 5. $\gamma X_2 + X_4 + X_7$			
F	G	Relations	Additional Extension of Kernel
$(g_0 z + f_0) e^{(\beta u - 2\gamma z)}$	$g_0 e^{(\beta u - 2\gamma z)}$	$g_0 \neq 0,\ u = z^2 - 2y,\ \beta \neq 0$	$\beta X_2 + X_3$
Subalgebra 5. $X_4 + X_7$			
F	G	Relations	Additional Extension of Kernel
$(g_0 z + f_0 u^{\frac{1}{2}}) u^{\kappa}$	$g_0 u^{\kappa}$	$g_0 \neq 0,\ u = z^2 - 2y,\ \kappa = \dfrac{1 - 2\tilde{\kappa}}{2} \neq 0$	$\tilde{\kappa} X_2 + 2X_5 + X_6$
Subalgebra 6. $\gamma X_2 + X_5 + X_6 + X_7$			
F	G	Relations	Additional Extension of Kernel
$(g_0 y + f_0 z) z^{\kappa - 1} e^{-\tilde{\gamma}(y/z)}$	$g_0 z^{\kappa} e^{-\tilde{\gamma}(y/z)}$	$g_0 \neq 0,\ \tilde{\gamma} = 2\gamma + \kappa - 1 \neq 0$	$\tilde{\gamma} X_2 + 2X_7$
Subalgebra 7. $\gamma X_2 + X_3,\ \gamma \neq 0$			
F	G	Relations	Additional Extension of Kernel
$(f_0 z^{\beta-1} e^{\kappa z - \tilde{\gamma} y})(\kappa z + \tilde{\gamma}\phi_1)$	$g_0 z^{\beta} e^{\kappa z - \tilde{\gamma} y}$	$\tilde{\gamma} = 2\gamma,\ f_0 = g_0/\tilde{\gamma} \neq 0$	$(\beta - 1) X_3 + \kappa X_7 + \tilde{\gamma} X_6$
$(g_0 e^{\beta z + \kappa z^2 - \tilde{\gamma} y})\phi(z)$	$g_0 e^{\beta z + \kappa z^2 - \tilde{\gamma} y}$	$\tilde{\gamma} = 2\gamma,\ g_0 \neq 0,$ $\phi = \phi_0 z + \phi_1,\ \phi_0 \neq 0,\ \kappa = (\tilde{\gamma}\phi_0)/2$	$\beta X_3 + 2\kappa X_7 + \tilde{\gamma} X_4$

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APPENDIX

APPENDIX

COMPUTATIONS FOR EXTENSIONS OF OTHER SUBALGEBRAS IN CHAPTER V

Subalgebra 2. with generator $\gamma X_2 + X_4 + X_5$.

For this case, the determining equations (5.14) become

$$yF_y + F_z + (2\gamma - 1)F = 0,$$
$$yG_y + G_z + 2\gamma G = 0.$$

The general solution of these equations is

$$F(y,z) = f(u)y^{1-2\gamma}, \quad G(y,z) = g(u)y^{-2\gamma}, \tag{A.1}$$

where $u = ye^{-z}$ and $gf' \neq 0$ (or else it is a degenerate case). This solution is substituted into the determining equations (5.2), which are split with respect to z after substituting $y = ue^z$. The determining equations are

$$\zeta_1(f'u + (1-2\gamma)f) - a_{12}g = 0, \tag{A.2a}$$

$$f'u(a_{11} - \zeta_2 + \xi_1) + f(2\xi_1(2-\gamma) - 2\gamma a_{11}) = 0, \tag{A.2b}$$

$$a_{21}f' = 0, \tag{A.2c}$$

$$(\xi_1 + a_{22})f' = 0, \tag{A.2d}$$

$$a_{12}(f'u + (1-2\gamma)f = 0, \tag{A.2e}$$

$$\zeta_1(g'u - 2\gamma g) = 0, \tag{A.2f}$$

$$g'u(a_{11} - \zeta_2 + \xi_1) + g(\xi_1(3-2\gamma) - 2\gamma a_{11} - a_{22}) = 0, \tag{A.2g}$$

$$a_{21}(f - g'u) = 0, \tag{A.2h}$$

$$a_{12}(g'u - 2\gamma g) = 0, \tag{A.2i}$$

$$(\xi_1 + a_{22})g' = 0. \tag{A.2j}$$

Since $f' \neq 0$, equations (A.2c) and (A.2d) give the conditions $a_{21} = 0$ and $\xi_1 = -a_{22}$, respectively. These conditions also satisfy equations (A.2h) and (A.2j). Also, since $g'u - 2\gamma g = 0$ makes the system (5.1) equivalent to a degenerate case then it follows from equations (A.2f) and (A.2i) that $\zeta_1 = 0$ and $a_{12} = 0$, respectively. These conditions also satisfy equations (A.2a) and (A.2e). Equations (A.2b) and (A.2g) are reduced as follows:

$$f'u(a_{11} - a_{22} - \zeta_2) - 2f(\gamma(a_{11} - a_{22}) + 2a_{22}) = 0$$

$$g'u(a_{11} - a_{22} - \zeta_2) - 2g(\gamma(a_{11} - a_{22}) + 2a_{22}) = 0.$$

From here, there is a need to study 2 cases: 1. $f'g - g'f \neq 0$, and 2. $f'g - g'f = 0$.

2.1 **Case** $f'g - g'f \neq 0$.

For this case, we get the condition $\zeta_2 = a_{11} - a_{22}$. If $\gamma = 0$, then $a_{22} = 0$. If $\gamma \neq 0$, then $a_{11} = \dfrac{\gamma - 2}{\gamma} a_{22}$. This gives no other generators apart from the studied subalgebra.

2.2 **Case** $f'g - g'f = 0$.

For this case, we obtain functions $g = g_0 u^\alpha$ and $f = f_1 g$, where $\alpha \neq 0$ (else degenerate case), $g_0 \neq 0$ and $f_1 \neq 0$ are constant. This gives us $\zeta_2 = \frac{1}{\alpha}((\alpha - 2\gamma)a_{11} + (2\gamma - \alpha - 4)a_{22})$. These result to an additional extension of the generator,

$$\kappa X_4 + \alpha X_5,$$

aside from the studied subalgebra. Note that $\gamma = \dfrac{\alpha - \kappa}{2}$, $\kappa \neq 0$ is constant.

Subalgebra 3. with generator $\gamma X_2 + X_8 - X_7$.

For this case, the determining equations (5.14) become

$$-zF_y + yF_z + 2\gamma F + G = 0,$$
$$-zG_y + yG_z - F + 2\gamma G = 0$$

for which the general solution is

$$\begin{aligned} F(y,z) &= e^{-2\gamma u}(\cos(u)f(v) + \sin(u)g(v)), \\ G(y,z) &= e^{-2\gamma u}(\sin(u)f(v) - \cos(u)g(v)), \end{aligned} \tag{A.3}$$

where $u = \arctan(z/y)$, $v^2 = y^2 + z^2$ and $f^2 + g^2 \neq 0$ (else degenerate). These functions are substituted into the determining equations (5.2), which are split with respect to y and z after substitution of these variables with respect to u and v. The determining equations are as follows

$$a_{11}v(f'v - f) + a_{12}gv + a_{21}v(-2f\gamma + g) + \xi_1 v(f'v + 3f) + f'v\zeta_1 - 2f\gamma\zeta_2 + g\zeta_2 = 0, \tag{A.4a}$$

$$2a_{11}v(f\gamma - g) + a_{12}v(f'v - f) + a_{21}v(f'v - f) + 2a_{22}v(-f\gamma + g) + f'v\zeta_2 + 2f\gamma\zeta_1 - g\zeta_1 = 0, \tag{A.4b}$$

$$a_{12}(2f\gamma - g) - a_{21}g + a_{22}(f'v - f) + \xi_1(f'v + 3f) = 0, \tag{A.4c}$$

$$f'v\zeta_1 - 2f\gamma\zeta_2 + g\zeta_2 = 0, \tag{A.4d}$$

$$f'v\zeta_2 + 2f\gamma\zeta_1 - g\zeta_1 = 0, \tag{A.4e}$$

$$a_{11}g'v^2 - 2a_{21}\gamma gv - a_{22}gv + \xi_1 v(g'v + 3g) - g'v\zeta_1 + f\zeta_2 + 2\gamma g\zeta_2 = 0, \tag{A.4f}$$

$$2a_{11}\gamma gv + a_{12}v(g'v + g) + a_{21}v(g'v + g) - 2a_{22}\gamma gv - g'v\zeta_2 - f\zeta_1 - 2\gamma g\zeta_1 = 0, \tag{A.4g}$$

$$-a_{11}g + 2a_{12}\gamma g + a_{22}g'v + \xi_1(g'v + 3g) = 0, \tag{A.4h}$$

$$g'v\zeta_1 - f\zeta_2 - 2\gamma g\zeta_2 = 0, \tag{A.4i}$$

$$g'v\zeta_2 + f\zeta_1 + 2\gamma g\zeta_1 = 0. \tag{A.4j}$$

Equations (A.4d), (A.4e), (A.4i) and (A.4j) force $\zeta_1 = 0$ and $\zeta_2 = 0$. If not, these lead to $f = 0$ and $g = 0$, which lead to a degenerate case. The remaining determining equations are simplified as follows

$$(a_{11} + \xi_1)f'v - (2\gamma a_{21} + a_{11} - 3\xi_1)f + \ a_{12} + a_{21})g = 0, \tag{A.5a}$$

$$(a_{12} + a_{21})f'v - (2a_{22}\gamma - 2a_{11}\gamma + a_{21} + a_{12})f + 2\ a_{22} - a_{11})g = 0, \tag{A.5b}$$

$$(a_{22} + \xi_1)f'v - (a_{22} - 2\gamma a_{12} - 3\xi_1)f - \ a_{12} + a_{21})g = 0, \tag{A.5c}$$

$$(a_{11} + \xi_1)g'v - (2\gamma a_{21} + a_{22} - 3\xi_1)g = 0, \tag{A.5d}$$

$$(a_{12} + a_{21})g'v - (2a_{22}\gamma - 2a_{11}\gamma - a_{21} - a_{12})g = 0, \tag{A.5e}$$

$$(a_{22} + \xi_1)g'v - (a_{11} - 2\gamma a_{12} - 3\xi_1)g = 0. \tag{A.5f}$$

From Equations (A.5d), (A.5e) and (A.5f) (since g is nonzero), we obtain that $a_{21} = -a_{12}$ and $a_{22} = a_{11}$. From here, we need to separate whether 1. $\gamma = 0$, or 2. $\gamma \neq 0$.

3.1 **Case** $\gamma \neq 0$.

If $\gamma \neq 0$, we obtain $a_{11} = a_{22}$. From here, we need to study the following cases: a. $f'g - g'f = 0$, and b. $f'g - g'f \neq 0$.

3.1.a **Case** $f'g - g'f = 0$.

If $f'g - g'f = 0$, then we obtain that $f = f_0 g$, where $f_0 \neq 0$ is constant. Substituting this into the remaining determining equation, we obtain that either $\left(\frac{g'v + 3g}{2\gamma g}\right)' \neq 0$ or $\left(\frac{g'v + 3g}{2\gamma g}\right)' = 0$. If it is not satisfied, we obtain that $a_{21} = \frac{-2a_{22}}{\gamma}$. No extensions of the generator are obtained. If it is satisfied, then we get that $g = g_0 v^{2\kappa\gamma - 3}$, where $g_0 \neq 0$ is constant. It will also follow that $a_{21} = \kappa\xi_1 + \left(\kappa - \frac{2}{\gamma}\right) a_{22}$. Here, we obtain another extension apart from the studied subalgebra, which is

$$2X_2 + X_5 + X_6 + \kappa(X_8 - X_7)$$

.

3.1.b **Case** $f'g - g'f \neq 0$.

If $f'g - g'f \neq 0$, we obtain that $a_{22} = -\xi_1$. Substituting this into the remaining determining equation, we obtain that $\xi_1 = \frac{\gamma a_{21}}{2}$. No extensions of the generator are obtained.

3.2 **Case** $\gamma = 0$.

If $\gamma = 0$, we also obtain $a_{11} = a_{22}$. From here,we need to study the following cases: a. $f'v - f = 0$, and b. $f'v - f \neq 0$.

3.2.a **Case** $f'v - f = 0$.

If $f'v - f = 0$, then $f = f_0 v$, where f_0 is constant. Substituting this into the determining equations, we obtain that $\xi_1 = 0$. The remaining determining equation leads us to study the following cases: i. $g'v - g = 0$, and ii. $g'v - g \neq 0$.

3.2.a.i **Case** $g'v - g = 0$.

If $g'v - g = 0$, the $g = g_0 v$, where g_0 is constant. Aside from the studied subalgebra, we obtain the extension

$$X_5 + X_6.$$

3.2.a.ii **Case** $g'v - g \neq 0$.

If $g'v - g \neq 0$, then $a_{22} = 0$. No extensions are obtained.

3.2.b **Case** $f'v - f \neq 0$.

If $f'v - f = 0 \neq 0$, then we need to study the following cases: i. $f'g - g'f = 0$, and ii. $f'g - g'f \neq 0$.

3.2.b.i **Case** $f'g - g'f = 0$.

If $f'g - g'f = 0$, we need to study further if $\left(\frac{-(f'v-3f)}{f'v-f}\right)' = 0$ or $\left(\frac{-(f'v-3f)}{f'v-f}\right)' \neq 0$. If it is satisfied, we obtain that $f = f_0 v^{\kappa}$, where $\kappa \neq 1$. Substituting this into the determining equation, we obtain that $a_{22} = \frac{-(\kappa+3)\xi_1}{\kappa - 1}$. Another extension is obtained aside from the studied subalgebra, i.e., we obtain

$$\frac{1-\kappa}{2}X_2 + X_5 + X_6.$$

If it is not satisfied, we obtain that $\xi_1 = 0$. No extensions are obtained.

3.2.b.ii **Case** $f'g - g'f \neq 0$.

If $f'g - g'f \neq 0$, then it follows that $\xi_1 = 0$. No extensions are obtained.

Subalgebra 4. with generator $\gamma X_2 + \alpha(X_5 + X_6) + X_8 - X_7$, $\alpha > 0$**.**

For this case, the determining equations (5.14) become

$$(\alpha y - z)F_y + (\alpha z + y)F_z + (2\gamma - \alpha)F + G = 0,$$
$$(\alpha y - z)G_y + (\alpha z + y)G_z - F + (2\gamma - \alpha)G = 0$$

for which the general solution is

$$\begin{aligned} F(y,z) &= e^{(\alpha-2\gamma)u}(\cos(u)f(v) + \sin(u)g(v)), \\ G(y,z) &= e^{(\alpha-2\gamma)u}(\sin(u)f(v) - \cos(u)g(v)), \end{aligned} \tag{A.6}$$

where $u = \arctan(z/y)$ and $v^2 = e^{-2\alpha u}(y^2+z^2)$. These functions are substituted into the determining equations (5.2), which are split with respect to u after substitutions of y and z. The following determining equations are obtained:

$$f'v(-\alpha\zeta_2 + \zeta_1) + f\zeta_2(\alpha - 2\gamma) + g\zeta_2 = 0, \tag{A.7a}$$

$$f'v(\alpha\zeta_1+\zeta_2)+g'v(-\alpha\zeta_2+\zeta_1)+f(-\alpha\zeta_1+2\gamma\zeta_1-\zeta_2)$$
$$+g(\alpha\zeta_2-2\gamma\zeta_2-\zeta_1)=0, \tag{A.7b}$$

$$f'v(\alpha\zeta_2-\zeta_1)+4g'v(\alpha\zeta_1+\zeta_2)+f(-\alpha\zeta_2+2\gamma\zeta_2+4\zeta_1)$$
$$+g(-4\alpha\zeta_1+8\gamma\zeta_1-\zeta_2)=0, \tag{A.7c}$$

$$f'v(-\alpha a_{21}+a_{11}+\xi_1)+f(\alpha a_{21}-2\gamma a_{21}-a_{11}+3\xi_1)+g(a_{12}+a_{21})=0, \tag{A.7d}$$

$$f'v(\alpha a_{11}-\alpha a_{22}+a_{12}+a_{21})+g'v(-\alpha a_{21}+a_{11}+\xi_1)$$
$$+f(-\alpha a_{11}+\alpha a_{22}+2\gamma a_{11}-2\gamma a_{22}-a_{12}-a_{21})$$
$$+g(\alpha a_{21}-2\gamma a_{21}-2a_{11}+a_{22}+3\xi_1)=0, \tag{A.7e}$$

$$f'v(4\alpha a_{12}+3\alpha a_{21}-3a_{11}+4a_{22}+\xi_1)+4g'v(\alpha a_{11}-\alpha a_{22}+a_{12}+a_{21})$$
$$+f(-4\alpha a_{12}-3\alpha a_{21}+8\gamma a_{12}+6\gamma a_{21}+3a_{11}-4a_{22}+3\xi_1)$$
$$+g(-4\alpha a_{11}+4\alpha a_{22}+8\gamma a_{11}-8\gamma a_{22}-3a_{12}-3a_{21})=0, \tag{A.7f}$$

$$f'v(\alpha a_{11}-\alpha a_{22}+a_{12}+a_{21})+g'v(-2\alpha a_{12}-\alpha a_{21}+a_{11}-2a_{22}-\xi_1)$$
$$+f(-\alpha a_{11}+\alpha a_{22}+2\gamma a_{11}-2\gamma a_{22}-a_{12}-a_{21})$$
$$+g(2\alpha a_{12}+\alpha a_{21}-4\gamma a_{12}-2\gamma a_{21}+a_{22}-3\xi_1)=0, \tag{A.7g}$$

$$g'v(\alpha\zeta_2-\zeta_1)+f\zeta_2+g\zeta_2(-\alpha+2\gamma)=0, \tag{A.7h}$$

$$f'v(\alpha\zeta_2-\zeta_1)+g'v(\alpha\zeta_1+\zeta_2)+f(-\alpha\zeta_2+2\gamma\zeta_2+\zeta_1)$$
$$+g(-\alpha\zeta_1+2\gamma\zeta_1-\zeta_2)=0, \tag{A.7i}$$

$$4f'v(\alpha\zeta_1+\zeta_2)+g'v(-\alpha\zeta_2+\zeta_1)+f(-4\alpha\zeta_1+8\gamma\zeta_1-\zeta_2)$$
$$+g(\alpha\zeta_2-2\gamma\zeta_2-4\zeta_1)=0, \tag{A.7j}$$

$$g'v(\alpha a_{21}-a_{11}-\xi_1)+g(-\alpha a_{21}+2\gamma a_{21}+a_{22}-3\xi_1)=0, \tag{A.7k}$$

$$f'v(\alpha a_{21}-a_{11}-\xi_1)+g'v(\alpha a_{11}-\alpha a_{22}+a_{12}+a_{21})$$
$$+f(-\alpha a_{21}+2\gamma a_{21}+a_{11}-3\xi_1)+g(-\alpha a_{11}+\alpha a_{22}+2\gamma a_{11}-2\gamma a_{22})=0, \tag{A.7l}$$

$$4f'v(\alpha a_{11} - \alpha a_{22} + a_{12} + a_{21}) + g'v(-4\alpha a_{12} - 3\alpha a_{21} + 3a_{11} - 4a_{22} - \xi_1)$$
$$+4f(-\alpha a_{11} + \alpha a_{22} + 2\gamma a_{11} - 2\gamma a_{22} - a_{12} - a_{21})$$
$$+g(4\alpha a_{12} + 3\alpha a_{21} - 8\gamma a_{12} - 6\gamma a_{21} - 4a_{11} + 5a_{22} - 3\xi_1) = 0, \quad \text{(A.7m)}$$

$$f'v(2\alpha a_{12} + \alpha a_{21} - a_{11} + 2a_{22} + \xi_1) + g'v(\alpha a_{11} - \alpha a_{22} + a_{12} + a_{21})$$
$$+f(-2\alpha a_{12} - \alpha a_{21} + 4\gamma a_{12} + 2\gamma a_{21} + a_{11} - 2a_{22} + 3\xi_1) \quad \text{(A.7n)}$$
$$+g(-\alpha a_{11} + \alpha a_{22} + 2\gamma a_{11} - 2\gamma a_{22} - 2a_{12} - 2a_{21}) = 0.$$

From here, as $g \neq 0$, we can assume that $f = \phi g$, where $\phi = \phi(v) \neq 0$. Substituting this into the determining equations (A.7), these equations are reduced to the following

$$g'\phi v(-\alpha\zeta_2 + \zeta_1) + g(-\phi'\alpha v\zeta_2 + \phi' v\zeta_1 + \alpha\phi\zeta_2 - 2\gamma\phi\zeta_2 + \zeta_2) = 0, \quad \text{(A.8a)}$$

$$g'v(\alpha\phi\zeta_1 - \alpha\zeta_2 + \phi\zeta_2 + \zeta_1) + g(\phi'\alpha v\zeta_1 + \phi' v\zeta_2 - \alpha\phi\zeta_1 + \alpha\zeta_2 + 2\gamma\phi\zeta_1 - 2\gamma\zeta_2 - \phi\zeta_2 - \zeta_1) = 0, \quad \text{(A.8b)}$$

$$g'v(\alpha\phi\zeta_2 + 4\alpha\zeta_1 - \phi\zeta_1 + 4\zeta_2)$$
$$+g(\phi'\alpha v\zeta_2 - \phi' v\zeta_1 - \alpha\phi\zeta_2 - 4\alpha\zeta_1 + 2\gamma\phi\zeta_2 + 8\gamma\zeta_1 + 4\phi\zeta_1 - \zeta_2) = 0, \quad \text{(A.8c)}$$

$$g'\phi v(-\alpha a_{21} + a_{11} + \xi_1)$$
$$+g(-\phi'\alpha a_{21} v + \phi' a_{11} v + \phi'\xi_1 v + \alpha a_{21}\phi - 2\gamma a_{21}\phi - a_{11}\phi + a_{12} + a_{21} + 3\xi_1\phi) = 0, \quad \text{(A.8d)}$$

$$g'v(\alpha a_{11}\phi - \alpha a_{21} - \alpha a_{22}\phi + a_{11} + a_{12}\phi + a_{21}\phi + \xi_1)$$
$$+g(\phi'\alpha a_{11} v - \phi'\alpha a_{22} v + \phi' a_{12} v + \phi' a_{21} v - \alpha a_{11}\phi$$
$$+\alpha a_{21} + \alpha a_{22}\phi + 2\gamma a_{11}\phi - 2\gamma a_{21} - 2\gamma a_{22}\phi - 2a_{11} - a_{12}\phi - a_{21}\phi + a_{22} + 3\xi_1) = 0, \quad \text{(A.8e)}$$

$$g'v(4\alpha a_{11} + 4\alpha a_{12}\phi + 3\alpha a_{21}\phi - 4\alpha a_{22} - 3a_{11}\phi + 4a_{12} + 4a_{21} + 4a_{22}\phi + \xi_1\phi)$$
$$+g(4\phi'\alpha a_{12} v + 3\phi'\alpha a_{21} v - 3\phi' a_{11} v + 4\phi' a_{22} v + \phi'\xi_1 v - 4\alpha a_{11}$$
$$-4\alpha a_{12}\phi - 3\alpha a_{21}\phi + 4\alpha a_{22} + 8\gamma a_{11} + 8\gamma a_{12}\phi + 6\gamma a_{21}\phi$$
$$-8\gamma a_{22} + 3a_{11}\phi - 3a_{12} - 3a_{21} - 4a_{22}\phi + 3\xi_1\phi) = 0, \quad \text{(A.8f)}$$

$$\begin{aligned} &g'v(\alpha a_{11}\phi - 2\alpha a_{12} - \alpha a_{21} - \alpha a_{22}\phi + a_{11} + a_{12}\phi + a_{21}\phi - 2a_{22} - \xi_1) \\ &+g(\phi'\alpha a_{11}v - \phi'\alpha a_{22}v + \phi' a_{12}v + \phi' a_{21}v \\ &-\alpha a_{11}\phi + 2\alpha a_{12} + \alpha a_{21} + \alpha a_{22}\phi + 2\gamma a_{11}\phi \\ &-4\gamma a_{12} - 2\gamma a_{21} - 2\gamma a_{22}\phi - a_{12}\phi - a_{21}\phi + a_{22} - 3\xi_1) = 0, \end{aligned} \tag{A.8g}$$

$$g'v(\alpha\zeta_2 - \zeta_1) + g\zeta_2(-\alpha + 2\gamma + \phi) = 0, \tag{A.8h}$$

$$\begin{aligned} &g'v(\alpha\phi\zeta_2 + \alpha\zeta_1 - \phi\zeta_1 + \zeta_2) \\ &+g(\phi'\alpha v\zeta_2 - \phi' v\zeta_1 - \alpha\phi\zeta_2 - \alpha\zeta_1 + 2\gamma\phi\zeta_2 + 2\gamma\zeta_1 + \phi\zeta_1 - \zeta_2) = 0, \end{aligned} \tag{A.8i}$$

$$\begin{aligned} &g'v(4\alpha\phi\zeta_1 - \alpha\zeta_2 + 4\phi\zeta_2 + \zeta_1) \\ &+g(4\phi'\alpha v\zeta_1 + 4\phi' v\zeta_2 - 4\alpha\phi\zeta_1 + \alpha\zeta_2 + 8\gamma\phi\zeta_1 - 2\gamma\zeta_2 - \phi\zeta_2 - 4\zeta_1) = 0, \end{aligned} \tag{A.8j}$$

$$g'v(\alpha a_{21} - a_{11} - \xi_1) + g(-\alpha a_{21} + 2\gamma a_{21} + a_{22} - 3\xi_1) = 0, \tag{A.8k}$$

$$\begin{aligned} &g'v(\alpha a_{11} + \alpha a_{21}\phi - \alpha a_{22} - a_{11}\phi + a_{12} + a_{21} - \xi_1\phi) \\ &+g(\phi'\alpha a_{21}v - \phi' a_{11}v - \phi'\xi_1 v - \alpha a_{11} \\ &-\alpha a_{21}\phi + \alpha a_{22} + 2\gamma a_{11} + 2\gamma a_{21}\phi - 2\gamma a_{22} + a_{11}\phi - 3\xi_1\phi) = 0, \end{aligned} \tag{A.8l}$$

$$\begin{aligned} &g'v(4\alpha a_{11}\phi - 4\alpha a_{12} - 3\alpha a_{21} - 4\alpha a_{22}\phi + 3a_{11} + 4a_{12}\phi + 4a_{21}\phi - 4a_{22} - \xi_1) \\ &+g(4\phi'\alpha a_{11}v - 4\phi'\alpha a_{22}v + 4\phi' a_{12}v + 4\phi' a_{21}v \\ &-4\alpha a_{11}\phi + 4\alpha a_{12} + 3\alpha a_{21} + 4\alpha a_{22}\phi + 8\gamma a_{11}\phi - 8\gamma a_{12} \\ &-6\gamma a_{21} - 8\gamma a_{22}\phi - 4a_{11} - 4a_{12}\phi - 4a_{21}\phi + 5a_{22} - 3\xi_1) = 0, \end{aligned} \tag{A.8m}$$

$$\begin{aligned} &g'v(\alpha a_{11} + 2\alpha a_{12}\phi + \alpha a_{21}\phi - \alpha a_{22} - a_{11}\phi + a_{12} + a_{21} + 2a_{22}\phi + \xi_1\phi) \\ &+g(2\phi'\alpha a_{12}v + \phi'\alpha a_{21}v - \phi' a_{11}v + 2\phi' a_{22}v + \phi'\xi_1 v \\ &-\alpha a_{11} - 2\alpha a_{12}\phi - \alpha a_{21}\phi + \alpha a_{22} + 2\gamma a_{11} + 4\gamma a_{12}\phi + 2\gamma a_{21}\phi \\ &-2\gamma a_{22} + a_{11}\phi - 2a_{12} - 2a_{21} - 2a_{22}\phi + 3\xi_1\phi) = 0. \end{aligned} \tag{A.8n}$$

Based on equation (A.8k), we can further study the following two cases:
1. $\left(\dfrac{g'v}{g}\right)' = 0$, and 2. $\left(\dfrac{g'v}{g}\right)' \neq 0$.

4a.1 **Case** $\left(\dfrac{g'v}{g}\right)' = 0.$

If $\left(\frac{g'v}{g}\right)' = 0$, then $g = g_0 v^{\kappa}$, where $g_0 \neq 0$, κ are constant. Substituting this into equation (A.8k), we obtain that $a_{22} = (\alpha(1-\kappa)-2\gamma)a_{21}+\kappa a_{11}+(\kappa+3)\xi_1$. From here, we need to study the following cases: a. $\kappa = 0$, and b. $\kappa \neq 0$.

4a.1.a **Case** $\kappa = 0$.

If $\kappa = 0$, then we need to study if $\phi' = 0$ or $\phi' \neq 0$. For $\phi' = 0$, we obtain that $a_{12} = -a_{21}$, $a_{11} = (\alpha - 2\gamma)a_{21} + 3\xi_1$ and $\zeta_1 = \zeta_2 = 0$. Aside from the studied subalgebra, we obtain the extension

$$X_2 + 2(X_5 + X_6).$$

If $\phi' \neq 0$, we obtain that $\zeta_1 = \zeta_2 = 0$. Consequently, we obtain that $a_{12} = -a_{21}$, $a_{11} = \frac{(2\alpha - \gamma)a_{21}}{2}$ and $\xi_1 = \frac{\gamma a_{21}}{2}$. No additional extensions are obtained.

4a.1.b **Case** $\kappa \neq 0$.

Similarly for $\kappa \neq 0$, we need to study if $\phi' = 0$ or $\phi' \neq 0$. We note that for $\phi' = 0$, $\kappa \neq 1$ as this is equivalent to the studied case subalgebra 3. For $\phi' = 0$, we obtain that $\zeta_1 = \zeta_2 = 0$, $a_{12} = -a_{21}$, and $a_{11} = \frac{(\alpha(\kappa-1) + 2\gamma)a_{21} - (\kappa+3)\xi_1}{\kappa - 1}$. Aside from the studied subalgebra, we obtain the extension

$$\frac{1-\kappa}{2}X_2 + X_5 + X_6.$$

If $\phi' \neq 0$, we obtain that $\zeta_2 = 0$. Consequently, we obtain that $a_{11} = \frac{(2\alpha - \gamma)a_{21}}{2}$, $a_{12} = -a_{21}$, $\xi_1 = \frac{\gamma a_{21}}{2}$ and $\zeta_1 = 0$. No additional extensions are obtained.

4a.2 **Case** $\left(\frac{g'v}{g}\right)' \neq 0$.

If $\left(\frac{g'v}{g}\right)' \neq 0$, then from equation (A.8k), it follows that $a_{11} = -\xi_1 + \alpha a_{21}$ and $a_{22} = (\alpha - 2\gamma)a_{21} + 3\xi_1$. From equation (A.8d), we can consider 2 cases: a. $\phi' = 0$ or b. $\phi' \neq 0$.

4a.2.a **Case** $\phi' = 0$.

For this case, $\phi = constant$. It follows from equation (A.8d) that $a_{12} = 2(\gamma a_{21} - 2\xi_1)\phi - a_{21}$. Substituting these into the remaining equations, equation (A.8m) gives the condition that $\xi_1 = \frac{\gamma a_{21}}{2}$. These also satisfies equations (A.8e), (A.8f), (A.8g), (A.8l) and (A.8n). From equation (A.8h), $\zeta_1 = \alpha\zeta_2$. Consequently from the same equation, one must study the following 2 cases: i. $\phi = \alpha - 2\gamma$ or ii. $\phi \neq \alpha - 2\gamma$.

4a.2.a.i **Case** $\phi = \alpha - 2\gamma$.

For this case, equation (A.8a) gives the condition that $\zeta_2 = 0$. No extensions are obtained in this case.

4a.2.a.ii **Case** $\phi \neq \alpha - 2\gamma$.

For this case, equation (A.8h) gives the condition that $\zeta_2 = 0$. No extensions are obtained in this case.

4a.2.b **Case** $\phi' \neq 0$.

For this case, from equation (A.8d), it follows that $\xi_1 = \frac{\gamma a_{21}}{2}$ and $a_{12} = -a_{21}$. These conditions satisfy equations (A.8e), (A.8f), (A.8g), (A.8h), (A.8l), (A.8m) and (A.8n) From here, we substitute equation (A.8h) into the other equations. From equation (A.8a), we can consider 2 cases: i. $\left(\frac{\phi^2+1}{\phi' v}\right)' = 0$ or ii. $\left(\frac{\phi^2+1}{\phi' v}\right)' \neq 0$.

4a.2.b.i **Case** $\left(\frac{\phi^2+1}{\phi' v}\right)' = 0$.

For this case, $\phi = \tan(\kappa \ln(v))$, where $\kappa \neq 0$. Substituting this into the remaining determining equations and from equation (A.8a), one obtains $\zeta_1 = \dfrac{\zeta_2(\alpha\kappa - 1)}{\kappa}$. Consequently from what remains of equation (A.8a), one must study the following cases:

A. $g'v + \kappa g(\tan(\kappa \ln(v)) - \alpha + 2\gamma) \neq 0$ or

B. $g'v + \kappa g(\tan(\kappa \ln(v)) - \alpha + 2\gamma) = 0$.

4a.2.b.i.A **Case** $g'v + \kappa g(\tan(\kappa \ln(v)) - \alpha + 2\gamma) \neq 0$.

For this case, it follows that $\zeta_2 = 0$. No additional extensions are obtained.

4a.2.b.i.B **Case** $g'v + \kappa g(\tan(\kappa \ln(v)) - \alpha + 2\gamma) = 0$.

For this case, it follows that $g = \dfrac{v^{\kappa(\alpha-2\gamma)}}{(\tan^2(\kappa \log(v)) + 1)^{1/2}}$. Substituting into the remaining determining equations, it leads to $\zeta_2 = 0$. No additional extensions are obtained.

4a.2.b.ii **Case** $\left(\dfrac{\phi^2+1}{\phi' v}\right)' \neq 0$.

For this case, it follows from equation (A.8a) that $\zeta_2 = 0$ and consequently, $\zeta_1 = 0$. All remaining equations are satisfied and no additional extensions are obtained.

Subalgebra 4. with generator $\gamma X_2 - X_3 + \alpha(X_5 + X_6) + X_8 - X_7$, $\alpha > 0$**.**

For this case, the determining equations (5.14) become

$$(\alpha y - z - 1)F_y + (\alpha z + y)F_z + (2\gamma - \alpha)F + G = 0,$$
$$(\alpha y - z - 1)G_y + (\alpha z + y)G_z - F + (2\gamma - \alpha)G = 0$$

for which the general solution is

$$\begin{aligned} F(y,z) &= e^{(\alpha-2\gamma)u}(\cos(u)f(v) + \sin(u)g(v)), \\ G(y,z) &= e^{(\alpha-2\gamma)u}(\sin(u)f(v) - \cos(u)g(v)), \end{aligned} \tag{A.9}$$

where $u = \arctan\left(\dfrac{z + \chi_2(\alpha)}{y - \chi_1(\alpha)}\right)$ and $v^2 = e^{-2\alpha u}((y - \chi_1(\alpha))^2 + (z + \chi_2(\alpha))^2)$, with $\chi_1(\alpha) = \dfrac{\alpha}{\alpha^2+1}$ and $\chi_2(\alpha) = \dfrac{1}{\alpha^2+1}$. These functions are substituted into the determining equations (5.2), which are split with respect to u after substitution of y and z. The following determining equations are obtained:

$$f'v(-\alpha a_{21}+a_{11}+\xi_1)+f(\alpha a_{21}-2\gamma a_{21}-a_{11}+3\xi_1)+g(a_{12}+a_{21})=0, \tag{A.10a}$$

$$\begin{gathered}f'v(\alpha^2 a_{21}-\alpha a_{22}-\alpha\xi_1+a_{12}+a_{21})+f(-\alpha^2 a_{21}+2\alpha\gamma a_{21}+\alpha a_{22}\\ -3\alpha\xi_1+2\gamma a_{11}-2\gamma a_{22}-a_{12}-a_{21})+g(-\alpha a_{12}-\alpha a_{21}-2a_{11}+2a_{22})=0,\end{gathered} \tag{A.10b}$$

$$\begin{gathered}f'v(-\alpha^3\zeta_2+\alpha^2 a_{21}+\alpha^2\zeta_1-\alpha\zeta_2+a_{21}+\zeta_1)\\ +f(\alpha^3\zeta_2-2\alpha^2\gamma\zeta_2-\alpha^2 a_{21}+2\alpha\gamma a_{21}+\alpha a_{11}-7\alpha\xi_1+\alpha\zeta_2+2\gamma a_{11}\\ +2\gamma\xi_1-2\gamma\zeta_2-a_{12}-a_{21})+g(\alpha^2\zeta_2-2\alpha a_{12}-\alpha a_{21}-2a_{11}+a_{22}-\xi_1+\zeta_2)=0,\end{gathered} \tag{A.10c}$$

$$\begin{gathered}f'v(-\alpha^4\zeta_2+\alpha^3\zeta_1+\alpha^2 a_{22}+\alpha^2\xi_1-\alpha^2\zeta_2+\alpha\zeta_1+a_{22}+\xi_1)\\ +f(\alpha^4\zeta_2-2\alpha^3\gamma\zeta_2+\alpha^2 a_{11}-\alpha^2 a_{22}\\ -4\alpha^2\xi_1+\alpha^2\zeta_2+2\alpha\gamma a_{22}+2\alpha\gamma\xi_1-2\alpha\gamma\zeta_2-\alpha a_{12}+2\gamma a_{12}-a_{22}+3\xi_1)\\ +g(\alpha^3\zeta_2-\alpha^2 a_{12}-\alpha a_{22}-\alpha\xi_1+\alpha\zeta_2-a_{12}-a_{21})=0,\end{gathered} \tag{A.10d}$$

$$\begin{gathered}f'v\zeta_2(\alpha^4+2\alpha^2+1)+f(-\alpha^4\zeta_2+2\alpha^3\gamma\zeta_2-\alpha^3\zeta_1+2\alpha^2\gamma\zeta_1-\alpha^2 a_{11}+3\alpha^2\xi_1-\alpha^2\zeta_2\\ +2\alpha\gamma\zeta_2+\alpha a_{12}+\alpha a_{21}-\alpha\zeta_1+2\gamma\zeta_1-a_{22}+3\xi_1)\\ +g(-\alpha^3\zeta_2+\alpha^2 a_{12}-\alpha^2\zeta_1+\alpha a_{11}-\alpha a_{22}-\alpha\zeta_2-a_{21}-\zeta_1)=0,\end{gathered} \tag{A.10e}$$

$$g'v(-\alpha a_{21}+a_{11}+\xi_1)+g(\alpha a_{21}-2\gamma a_{21}-a_{22}+3\xi_1)=0, \tag{A.10f}$$

$$\begin{gathered}g'v(\alpha^2 a_{12}+\alpha^2 a_{21}+a_{12}+a_{21})+g(-\alpha^2 a_{12}-\alpha^2 a_{21}+2\alpha\gamma a_{12}\\ +2\alpha\gamma a_{21}-2\alpha a_{11}+2\alpha a_{22}+2\gamma a_{11}-2\gamma a_{22}+a_{12}+a_{21})=0,\end{gathered} \tag{A.10g}$$

$$g'v(\alpha a_{12}+a_{22}+\xi_1)+g(-\alpha a_{12}+2\gamma a_{12}-a_{11}+3\xi_1)=0, \tag{A.10h}$$

$$g'v(\alpha^4 a_{21} + \alpha^4 \zeta_1 + 2\alpha^2 a_{21} + 2\alpha^2 \zeta_1 + a_{21} + \zeta_1)$$
$$+f(\alpha^3 \zeta_1 + \alpha^2 a_{11} + \alpha^2 \xi_1 - \alpha^2 \zeta_2 - \alpha a_{12} - \alpha a_{21} + \alpha \zeta_1 + a_{22} + \xi_1 - \zeta_2)$$
$$+g(-\alpha^4 a_{21} - \alpha^4 \zeta_1 + 2\alpha^3 \gamma a_{21} + 2\alpha^3 \gamma \zeta_1 - \alpha^3 a_{11} + \alpha^3 a_{22} - 4\alpha^3 \xi_1 + \alpha^3 \zeta_2$$
$$+2\alpha^2 \gamma a_{11} + 2\alpha^2 \gamma \xi_1 - 2\alpha^2 \gamma \zeta_2 - \alpha^2 a_{21} - \alpha^2 \zeta_1 + 2\alpha \gamma a_{21} + 2\alpha \gamma \zeta_1 -$$
$$2\alpha a_{11} + 2\alpha a_{22} - 4\alpha \xi_1 + \alpha \zeta_2 + 2\gamma a_{11} + 2\gamma \xi_1 - 2\gamma \zeta_2 + a_{12} + a_{21}) = 0, \tag{A.10i}$$

$$g'v(\alpha^3 a_{21} + \alpha^3 \zeta_1 + \alpha^2 \zeta_2 + \alpha a_{21} + \alpha \zeta_1 + \zeta_2)$$
$$+f(\alpha^2 \zeta_1 + \alpha a_{11} + \alpha \xi_1 - a_{12} + \zeta_1) + g(-\alpha^3 a_{21} - \alpha^3 \zeta_1 + 2\alpha^2 \gamma a_{21}$$
$$+2\alpha^2 \gamma \zeta_1 - \alpha^2 a_{11} + \alpha^2 a_{22} - 4\alpha^2 \xi_1 + 2\alpha \gamma a_{11} + 2\alpha \gamma \xi_1 - \alpha \zeta_1 + 2\gamma \zeta_1 - a_{11} + 3\xi_1) = 0. \tag{A.10j}$$

As g is nonzero, we can assume that $f = \phi(v)g$, where $\phi \neq 0$. The determining equations are reduced as follows.

$$(\phi' g + \phi g')v(-\alpha a_{21} + a_{11} + \xi_1) + \phi g(\alpha a_{21} - 2\gamma a_{21} - a_{11} + 3\xi_1) + g(a_{12} + a_{21}) = 0, \tag{A.11a}$$

$$(\phi' g + \phi g')v(\alpha^2 a_{21} - \alpha a_{22} - \alpha \xi_1 + a_{12} + a_{21}) + \phi g(-\alpha^2 a_{21} + 2\alpha \gamma a_{21} + \alpha a_{22}$$
$$-3\alpha \xi_1 + 2\gamma a_{11} - 2\gamma a_{22} - a_{12} - a_{21}) + g(-\alpha a_{12} - \alpha a_{21} - 2a_{11} + 2a_{22}) = 0, \tag{A.11b}$$

$$(\phi' g + \phi g')v(-\alpha^3 \zeta_2 + \alpha^2 a_{21} + \alpha^2 \zeta_1 - \alpha \zeta_2 + a_{21} + \zeta_1)$$
$$+\phi g(\alpha^3 \zeta_2 - 2\alpha^2 \gamma \zeta_2 - \alpha^2 a_{21} + 2\alpha \gamma a_{21} + \alpha a_{11} - 7\alpha \xi_1 + \alpha \zeta_2 + 2\gamma a_{11}$$
$$+2\gamma \xi_1 - 2\gamma \zeta_2 - a_{12} - a_{21}) + g(\alpha^2 \zeta_2 - 2\alpha a_{12} - \alpha a_{21} - 2a_{11} + a_{22} - \xi_1 + \zeta_2) = 0, \tag{A.11c}$$

$$(\phi' g + \phi g')v(-\alpha^4 \zeta_2 + \alpha^3 \zeta_1 + \alpha^2 a_{22} + \alpha^2 \xi_1 - \alpha^2 \zeta_2 + \alpha \zeta_1 + a_{22} + \xi_1)$$
$$+\phi g(\alpha^4 \zeta_2 - 2\alpha^3 \gamma \zeta_2 + \alpha^2 a_{11} - \alpha^2 a_{22}$$
$$-4\alpha^2 \xi_1 + \alpha^2 \zeta_2 + 2\alpha \gamma a_{22} + 2\alpha \gamma \xi_1 - 2\alpha \gamma \zeta_2 - \alpha a_{12} + 2\gamma a_{12} - a_{22} + 3\xi_1)$$
$$+g(\alpha^3 \zeta_2 - \alpha^2 a_{12} - \alpha a_{22} - \alpha \xi_1 + \alpha \zeta_2 - a_{12} - a_{21}) = 0, \tag{A.11d}$$

$$\begin{aligned}(\phi' g + \phi g')v\zeta_2(\alpha^4 + 2\alpha^2 + 1)\\ + f(-\alpha^4\zeta_2 + 2\alpha^3\gamma\zeta_2 - \alpha^3\zeta_1 + 2\alpha^2\gamma\zeta_1 - \alpha^2 a_{11} + 3\alpha^2\xi_1 - \alpha^2\zeta_2\\ + 2\alpha\gamma\zeta_2 + \alpha a_{12} + \alpha a_{21} - \alpha\zeta_1 + 2\gamma\zeta_1 - a_{22} + 3\xi_1)\\ + g(-\alpha^3\zeta_2 + \alpha^2 a_{12} - \alpha^2\zeta_1 + \alpha a_{11} - \alpha a_{22} - \alpha\zeta_2 - a_{21} - \zeta_1) = 0,\end{aligned} \tag{A.11e}$$

$$g'v(-\alpha a_{21} + a_{11} + \xi_1) + g(\alpha a_{21} - 2\gamma a_{21} - a_{22} + 3\xi_1) = 0, \tag{A.11f}$$

$$\begin{aligned}g'v(\alpha^2 a_{12} + \alpha^2 a_{21} + a_{12} + a_{21}) + g(-\alpha^2 a_{12} - \alpha^2 a_{21} + 2\alpha\gamma a_{12}\\ + 2\alpha\gamma a_{21} - 2\alpha a_{11} + 2\alpha a_{22} + 2\gamma a_{11} - 2\gamma a_{22} + a_{12} + a_{21}) = 0,\end{aligned} \tag{A.11g}$$

$$g'v(\alpha a_{12} + a_{22} + \xi_1) + g(-\alpha a_{12} + 2\gamma a_{12} - a_{11} + 3\xi_1) = 0, \tag{A.11h}$$

$$\begin{aligned}g'v(\alpha^4 a_{21} + \alpha^4\zeta_1 + 2\alpha^2 a_{21} + 2\alpha^2\zeta_1 + a_{21} + \zeta_1)\\ + \phi g(\alpha^3\zeta_1 + \alpha^2 a_{11} + \alpha^2\xi_1 - \alpha^2\zeta_2 - \alpha a_{12} - \alpha a_{21} + \alpha\zeta_1 + a_{22} + \xi_1 - \zeta_2)\\ + g(-\alpha^4 a_{21} - \alpha^4\zeta_1 + 2\alpha^3\gamma a_{21} + 2\alpha^3\gamma\zeta_1 - \alpha^3 a_{11} + \alpha^3 a_{22} - 4\alpha^3\xi_1 + \alpha^3\zeta_2\\ + 2\alpha^2\gamma a_{11} + 2\alpha^2\gamma\xi_1 - 2\alpha^2\gamma\zeta_2 - \alpha^2 a_{21} - \alpha^2\zeta_1 + 2\alpha\gamma a_{21} + 2\alpha\gamma\zeta_1 -\\ 2\alpha a_{11} + 2\alpha a_{22} - 4\alpha\xi_1 + \alpha\zeta_2 + 2\gamma a_{11} + 2\gamma\xi_1 - 2\gamma\zeta_2 + a_{12} + a_{21}) = 0,\end{aligned} \tag{A.11i}$$

$$\begin{aligned}g'v(\alpha^3 a_{21} + \alpha^3\zeta_1 + \alpha^2\zeta_2 + \alpha a_{21} + \alpha\zeta_1 + \zeta_2)\\ + \phi g(\alpha^2\zeta_1 + \alpha a_{11} + \alpha\xi_1 - a_{12} + \zeta_1) + g(-\alpha^3 a_{21} - \alpha^3\zeta_1 + 2\alpha^2\gamma a_{21}\\ + 2\alpha^2\gamma\zeta_1 - \alpha^2 a_{11} + \alpha^2 a_{22} - 4\alpha^2\xi_1 + 2\alpha\gamma a_{11} + 2\alpha\gamma\xi_1 - \alpha\zeta_1 + 2\gamma\zeta_1 - a_{11} + 3\xi_1) = 0.\end{aligned} \tag{A.11j}$$

From equations (A.11f)-(A.11h), we need to study the following cases:
1. $\left(\frac{g'v}{g}\right)' = 0$, and 2. $\left(\frac{g'v}{g}\right)' \neq 0$.

4b.1 **Case** $\left(\frac{g'v}{g}\right)' = 0$.

If $\left(\frac{g'v}{g}\right)' = 0$, then $g = g_0 v^\kappa$, where $g_0 \neq 0$, κ are constant. Substituting this into the determining equations, from the determining equations one obtains the relations $a_{11} = a_{22}$, $a_{12} = a_{21}$, $\zeta_1 = -(\alpha\zeta_2 + a_{21})$ and $a_{22} = \frac{1}{4}((\alpha^2+1)(\kappa+3)\zeta_2) + 2(2\alpha - \gamma)a_{21}$, under the assumption that ϕ is constant. Another extension is obtained, i.e.,

$$\frac{1-\kappa}{2}X_2 + X_5 + X_6 + \chi_2 X_4 - \chi_1 X_3.$$

If $\phi' \neq 0$, then we obtain that $\xi_1 = \frac{\gamma a_{21}}{2}$. Substituting into the determining equations, we obtain that $\zeta_1 = -a_{21}$ and $\zeta_2 = 0$. No additional extensions of the generator are obtained other than the studied subalgebra.

4b.2 **Case** $\left(\frac{g'v}{g}\right)' \neq 0.$

If $\left(\frac{g'v}{g}\right)' \neq 0$, then we get that $a_{22} = -(\alpha a_{12}+\xi_1)$ and $a_{11} = (2\gamma-\alpha)a_{12}+3\xi_1$. Substituting these and analyzing the determining equations, it gives us that $\xi = -\frac{\gamma a_{21}}{2}$ and $a_{12} = -a_{21}$. Consequently, $\zeta_2 = 0$ and $\zeta_1 = -a_{21}$. No additional extensions of the generator are obtained other than the studied subalgebra.

Subalgebra 4. with generator $\gamma X_2 + X_3 + \alpha(X_5+X_6) + X_8 - X_7$, $\alpha > 0$**.**

For this case, the determining equations (5.14) become

$$(\alpha y - z + 1)F_y + (\alpha z + y)F_z + (2\gamma - \alpha)F + G = 0,$$
$$(\alpha y - z + 1)G_y + (\alpha z + y)G_z - F + (2\gamma - \alpha)G = 0$$

for which the general solution is

$$F(y,z) = e^{(\alpha-2\gamma)u}(\cos(u)f(v) + \sin(u)g(v)),$$
$$G(y,z) = e^{(\alpha-2\gamma)u}(\sin(u)f(v) - \cos(u)g(v)), \qquad \text{(A.12)}$$

where $u = \arctan\left(\frac{z-\chi_2(\alpha)}{y+\chi_1(\alpha)}\right)$ and $v^2 = e^{-2\alpha u}((y+\chi_1(\alpha))^2 + (z-\chi_2(\alpha))^2)$, with $\chi_1(\alpha) = \frac{\alpha}{\alpha^2+1}$ and $\chi_2(\alpha) = \frac{1}{\alpha^2+1}$. These functions are substituted into the determining equations (5.2), which are split with respect to u after substitution of y and z. The following determining equations are obtained after initial analysis.

$$f'v(-\alpha a_{21} + a_{11} + \xi_1) + f(\alpha a_{21} - 2\gamma a_{21} - a_{11} + 3\xi_1) + g(a_{12} + a_{21}) = 0, \quad \text{(A.13a)}$$

$$f'v(\alpha^2 a_{21} - \alpha a_{22} - \alpha\xi_1 + a_{12} + a_{21})$$
$$+f(-\alpha^2 a_{21} + 2\alpha\gamma a_{21} + \alpha a_{22} - 3\alpha\xi_1 + 2\gamma a_{11} - 2\gamma a_{22} - a_{12} - a_{21}) \tag{A.13b}$$
$$+g(-\alpha a_{12} - \alpha a_{21} - 2a_{11} + 2a_{22}) = 0,$$

$$f'v(\alpha^3\zeta_2 + \alpha^2 a_{21} - \alpha^2\zeta_1 + \alpha\zeta_2 + a_{21} - \zeta_1) + f(-\alpha^3\zeta_2 + 2\alpha^2\gamma\zeta_2 - \alpha^2 a_{21} + 2\alpha\gamma a_{21}$$
$$+\alpha a_{11} - 7\alpha\xi_1 - \alpha\zeta_2 + 2\gamma a_{11} + 2\gamma\xi_1 + 2\gamma\zeta_2 - a_{12} - a_{21})$$
$$+g(-\alpha^2\zeta_2 - 2\alpha a_{12} - \alpha a_{21} - 2a_{11} + a_{22} - \xi_1 - \zeta_2) = 0, \tag{A.13c}$$

$$f'v(\alpha^4\zeta_2 - \alpha^3\zeta_1 + \alpha^2 a_{22} + \alpha^2\xi_1 + \alpha^2\zeta_2 - \alpha\zeta_1 + a_{22} + \xi_1)$$
$$+f(-\alpha^4\zeta_2 + 2\alpha^3\gamma\zeta_2 + \alpha^2 a_{11} - \alpha^2 a_{22} - 4\alpha^2\xi_1 - \alpha^2\zeta_2$$
$$+2\alpha\gamma a_{22} + 2\alpha\gamma\xi_1 + 2\alpha\gamma\zeta_2 - \alpha a_{12} + 2\gamma a_{12} - a_{22} + 3\xi_1) \tag{A.13d}$$
$$-g(\alpha^3\zeta_2 + \alpha^2 a_{12} + \alpha a_{22} + \alpha\xi_1 + \alpha\zeta_2 + a_{12} + a_{21}) = 0,$$

$$f'v\zeta_2(\alpha^4 + 2\alpha^2 + 1) + f(-\alpha^4\zeta_2 + 2\alpha^3\gamma\zeta_2 - \alpha^3\zeta_1$$
$$+2\alpha^2\gamma\zeta_1 + \alpha^2 a_{11} - 3\alpha^2\xi_1 - \alpha^2\zeta_2 + 2\alpha\gamma\zeta_2 - \alpha a_{12} - \alpha a_{21} - \alpha\zeta_1 + 2\gamma\zeta_1 + a_{22} - 3\xi_1)$$
$$+g(-\alpha^3\zeta_2 - \alpha^2 a_{12} - \alpha^2\zeta_1 - \alpha a_{11} + \alpha a_{22} - \alpha\zeta_2 + a_{21} - \zeta_1) = 0, \tag{A.13e}$$

$$g'v(-\alpha a_{21} + a_{11} + \xi_1) + g(\alpha a_{21} - 2\gamma a_{21} - a_{22} + 3\xi_1) = 0, \tag{A.13f}$$

$$g'v(-\alpha^3\zeta_2 + \alpha^2\zeta_1 - \alpha a_{22} - \alpha\xi_1 - \alpha\zeta_2 + a_{12} + \zeta_1)$$
$$+f(-\alpha^2\zeta_2 + \alpha a_{21} - a_{22} - \xi_1 - \zeta_2) \tag{A.13g}$$
$$+g(\alpha^3\zeta_2 - 2\alpha^2\gamma\zeta_2 + 4\alpha\xi_1 + \alpha\zeta_2 - 2\gamma a_{22} - 2\gamma\xi_1 - 2\gamma\zeta_2) = 0,$$

$$g'v(\alpha^3\zeta_2 + \alpha^2 a_{21} - \alpha^2\zeta_1 + \alpha\zeta_2 + a_{21} - \zeta_1) + f(\alpha^2\zeta_2 - \alpha a_{21} + a_{22} + \xi_1 + \zeta_2)$$
$$+g(-\alpha^3\zeta_2 + 2\alpha^2\gamma\zeta_2 - \alpha^2 a_{21} + 2\alpha\gamma a_{21} - \alpha a_{11}$$
$$+2\alpha a_{22} - 7\alpha\xi_1 - \alpha\zeta_2 + 2\gamma a_{11} + 2\gamma\xi_1 + 2\gamma\zeta_2 + a_{12} + a_{21}) = 0, \tag{A.13h}$$

$$g'v(\alpha^4\zeta_2 - \alpha^3\zeta_1 + \alpha^2 a_{22} + \alpha^2\xi_1 + \alpha^2\zeta_2 - \alpha\zeta_1 + a_{22} + \xi_1)$$
$$+f\alpha(\alpha^2\zeta_2 - \alpha a_{21} + a_{22} + \xi_1 + \zeta_2) + g(-\alpha^4\zeta_2 + 2\alpha^3\gamma\zeta_2 - 4\alpha^2\xi_1 - \alpha^2\zeta_2 + 2\alpha\gamma a_{22}$$
$$+2\alpha\gamma\xi_1 + 2\alpha\gamma\zeta_2 - \alpha a_{12} + 2\gamma a_{12} - a_{11} + 3\xi_1) = 0, \tag{A.13i}$$

$$g'v\zeta_2(\alpha^4+2\alpha^2+1)+f(\alpha^3\zeta_2-\alpha^2a_{21}+\alpha^2\zeta_1-\alpha a_{11}+\alpha a_{22}+\alpha\zeta_2+a_{12}+\zeta_1)$$
$$+g(-\alpha^4\zeta_2+2\alpha^3\gamma\zeta_2-\alpha^3\zeta_1+2\alpha^2\gamma\zeta_1+\alpha^2a_{22}$$
$$-3\alpha^2\xi_1-\alpha^2\zeta_2+2\alpha\gamma\zeta_2+\alpha a_{12}+\alpha a_{21}-\alpha\zeta_1+2\gamma\zeta_1+a_{11}-3\xi_1)=0. \tag{A.13j}$$

As g is nonzero, we can assume that $f=\phi(v)g$, where $\phi\neq 0$. The determining equations are reduced as follows.

$$(\phi'g+\phi g')v(-\alpha a_{21}+a_{11}+\xi_1)+\phi g(\alpha a_{21}-2\gamma a_{21}-a_{11}+3\xi_1)+g(a_{12}+a_{21})=0, \tag{A.14a}$$

$$(\phi'g+\phi g')v(\alpha^2a_{21}-\alpha a_{22}-\alpha\xi_1+a_{12}+a_{21})$$
$$+\phi g(-\alpha^2a_{21}+2\alpha\gamma a_{21}+\alpha a_{22}-3\alpha\xi_1+2\gamma a_{11}-2\gamma a_{22}-a_{12}-a_{21}) \tag{A.14b}$$
$$+g(-\alpha a_{12}-\alpha a_{21}-2a_{11}+2a_{22})=0,$$

$$(\phi'g+\phi g')v(\alpha^3\zeta_2+\alpha^2a_{21}-\alpha^2\zeta_1+\alpha\zeta_2+a_{21}-\zeta_1)$$
$$+\phi g(-\alpha^3\zeta_2+2\alpha^2\gamma\zeta_2-\alpha^2a_{21}+2\alpha\gamma a_{21}$$
$$+\alpha a_{11}-7\alpha\xi_1-\alpha\zeta_2+2\gamma a_{11}+2\gamma\xi_1+2\gamma\zeta_2-a_{12}-a_{21}) \tag{A.14c}$$
$$+g(-\alpha^2\zeta_2-2\alpha a_{12}-\alpha a_{21}-2a_{11}+a_{22}-\xi_1-\zeta_2)=0,$$

$$(\phi'g+\phi g')v(\alpha^4\zeta_2-\alpha^3\zeta_1+\alpha^2a_{22}+\alpha^2\xi_1+\alpha^2\zeta_2-\alpha\zeta_1+a_{22}+\xi_1)$$
$$+\phi g(-\alpha^4\zeta_2+2\alpha^3\gamma\zeta_2+\alpha^2a_{11}-\alpha^2a_{22}-4\alpha^2\xi_1-\alpha^2\zeta_2$$
$$+2\alpha\gamma a_{22}+2\alpha\gamma\xi_1+2\alpha\gamma\zeta_2-\alpha a_{12}+2\gamma a_{12}-a_{22}+3\xi_1) \tag{A.14d}$$
$$-g(\alpha^3\zeta_2+\alpha^2a_{12}+\alpha a_{22}+\alpha\xi_1+\alpha\zeta_2+a_{12}+a_{21})=0,$$

$$(\phi'g+\phi g')v\zeta_2(\alpha^4+2\alpha^2+1)+\phi g(-\alpha^4\zeta_2+2\alpha^3\gamma\zeta_2-\alpha^3\zeta_1$$
$$+2\alpha^2\gamma\zeta_1+\alpha^2a_{11}-3\alpha^2\xi_1-\alpha^2\zeta_2+2\alpha\gamma\zeta_2-\alpha a_{12}-\alpha a_{21}-\alpha\zeta_1+2\gamma\zeta_1+a_{22}-3\xi_1)$$
$$+g(-\alpha^3\zeta_2-\alpha^2a_{12}-\alpha^2\zeta_1-\alpha a_{11}+\alpha a_{22}-\alpha\zeta_2+a_{21}-\zeta_1)=0, \tag{A.14e}$$

$$g'v(-\alpha a_{21}+a_{11}+\xi_1)+g(\alpha a_{21}-2\gamma a_{21}-a_{22}+3\xi_1)=0, \tag{A.14f}$$

$$g'v(-\alpha^3\zeta_2+\alpha^2\zeta_1-\alpha a_{22}-\alpha\xi_1-\alpha\zeta_2+a_{12}+\zeta_1)$$
$$+\phi g(-\alpha^2\zeta_2+\alpha a_{21}-a_{22}-\xi_1-\zeta_2) \tag{A.14g}$$
$$+g(\alpha^3\zeta_2-2\alpha^2\gamma\zeta_2+4\alpha\xi_1+\alpha\zeta_2-2\gamma a_{22}-2\gamma\xi_1-2\gamma\zeta_2)=0,$$

$$g'v(\alpha^3\zeta_2+\alpha^2a_{21}-\alpha^2\zeta_1+\alpha\zeta_2+a_{21}-\zeta_1)+\phi g(\alpha^2\zeta_2-\alpha a_{21}+a_{22}+\xi_1+\zeta_2)$$
$$+g(-\alpha^3\zeta_2+2\alpha^2\gamma\zeta_2-\alpha^2a_{21}+2\alpha\gamma a_{21}-\alpha a_{11}$$
$$+2\alpha a_{22}-7\alpha\xi_1-\alpha\zeta_2+2\gamma a_{11}+2\gamma\xi_1+2\gamma\zeta_2+a_{12}+a_{21})=0, \tag{A.14h}$$

$$g'v(\alpha^4\zeta_2-\alpha^3\zeta_1+\alpha^2a_{22}+\alpha^2\xi_1+\alpha^2\zeta_2-\alpha\zeta_1+a_{22}+\xi_1)$$
$$+\phi g\alpha(\alpha^2\zeta_2-\alpha a_{21}+a_{22}+\xi_1+\zeta_2)+g(-\alpha^4\zeta_2+2\alpha^3\gamma\zeta_2-4\alpha^2\xi_1-\alpha^2\zeta_2+2\alpha\gamma a_{22}$$
$$+2\alpha\gamma\xi_1+2\alpha\gamma\zeta_2-\alpha a_{12}+2\gamma a_{12}-a_{11}+3\xi_1)=0, \tag{A.14i}$$

$$g'v\zeta_2(\alpha^4+2\alpha^2+1)+\phi g(\alpha^3\zeta_2-\alpha^2a_{21}+\alpha^2\zeta_1-\alpha a_{11}+\alpha a_{22}+\alpha\zeta_2+a_{12}+\zeta_1)$$
$$+g(-\alpha^4\zeta_2+2\alpha^3\gamma\zeta_2-\alpha^3\zeta_1+2\alpha^2\gamma\zeta_1+\alpha^2a_{22}$$
$$-3\alpha^2\xi_1-\alpha^2\zeta_2+2\alpha\gamma\zeta_2+\alpha a_{12}+\alpha a_{21}-\alpha\zeta_1+2\gamma\zeta_1+a_{11}-3\xi_1)=0. \tag{A.14j}$$

From the determining equations, we need to study the following cases: 1. $\left(\frac{g'v}{g}\right)'=0$, and 2. $\left(\frac{g'v}{g}\right)'\neq 0$.

4c.1 **Case** $\left(\frac{g'v}{g}\right)'=0$.

If $\left(\frac{g'v}{g}\right)'=0$, then $g=g_0v^\kappa$, where $g_0\neq 0$, κ are constant. substituting this into the determining equations, we obtain that $a_{22}=(\alpha(1-\kappa)-2\gamma)a_{21}+\kappa a_{11}+(\kappa+3)\xi_1$. From here, we need to study the following cases: a. $\phi'=0$, and b. $\phi'\neq 0$.

4c.1.a **Case** $\phi'=0$.

If $\phi'=0$, we obtain that $\zeta_1=a_{21}-\alpha\zeta_2$ and $a_{11}=\frac{1}{4}((2(2\alpha-\gamma))a_{21}-(\alpha^2+1)(\kappa+3)\zeta_2)$. Consequently, $\xi_1=\frac{1}{4}(2\gamma a_{21}+(\alpha^2+1)(\kappa-1)\zeta_2)$ and

$a_{12} = -a_{21}$. Aside from the studied subalgebra, we obtain the extension

$$\frac{1-\kappa}{2}X_2 + X_5 + X_6 + \chi_1 X_3 - \chi_2 X_4.$$

4c.1.b **Case** $\phi' \neq 0$.

If $\phi' \neq 0$, there are no extensions obtained.

4c.2 **Case** $\left(\frac{g'v}{g}\right)' \neq 0$.

If $\left(\frac{g'v}{g}\right)' \neq 0$, then we get that $a_{11} = \alpha a_{21} - \xi_1$ and $a_{22} = (\alpha - 2\gamma)a_{21} + 3\xi_1$. Substituting these and analyzing the determining equations, it gives us that $\xi = \frac{\gamma a_{21}}{2}$ and $a_{12} = -a_{21}$. Consequently, $\zeta_2 = 0$ and $\zeta_1 = a_{21}$. No additional extensions are obtained other than the studied subalgebra.

Subalgebra 5. with generator $\gamma X_2 + X_7$**.**

For this case, the determining equations (5.14) become

$$zF_y + 2\gamma F - G = 0,$$
$$zG_y + 2\gamma G = 0$$

for which the general solution is

$$F(y,z) = (ug(z) + f(z))e^{-2\gamma u}, \quad G(y,z) = g(z)e^{-2\gamma u}, \tag{A.15}$$

where $u = y/z$ and $\gamma g \neq 0$ ($\gamma g = 0$ makes system (5.1) equivalent to a reducible case). These functions are substituted into the determining equations (5.2) and are split with respect to y. Initial analysis results to $a_{21} = 0$, $\zeta_2 = 0$ and $a_{11} = a_{22}$. The remaining determining equations become

$$f'z^2(\xi_1 + a_{22}) - f((2\gamma a_{12} + a_{22} - 3\xi_1)z + 2\gamma\zeta_1) + g\zeta_1 = 0, \tag{A.16a}$$

$$g'z^2(\xi_1 + a_{22}) - g((2\gamma a_{12} + a_{22} - 3\xi_1)z + 2\gamma\zeta_1) = 0. \tag{A.16b}$$

Now suppose $f(z) = g(z)\phi(z)$ (since g is nonzero) where $\phi' \neq 0$. If $\phi' = 0$, using equivalence transformation, system (5.1) is equivalent to a reducible system. Substituting f into equation (A.16a) and obtaining linear combinations with (A.16b), equation (A.16a) is reduced to

$$\phi' z^2(\xi_1 + a_{22}) + \zeta_1 = 0. \tag{A.17}$$

Looking closely at equations (A.16b) and (A.17), there is a need to study the following 2 cases: 1. there exist at least one generator where $\xi_1 + a_{22} \neq 0$, and 2. all generators have $\xi_1 + a_{22} = 0$.

5a.1 **Case there exist at least one generator where $\xi_1 + a_{22} \neq 0$.**

For this case, we obtain the general solution $\phi = \phi_0 + \dfrac{\alpha}{2\gamma z}$ and $g = g_0 z^\beta e^{-\alpha/z}$. Substituting these functions to equations (A.16b) and (A.17), we obtain that $a_{12} = \dfrac{\beta a_{22} + \beta\xi_1 - a_{22} + 3\xi_1}{2\gamma}$ and $\zeta_1 = \dfrac{\alpha(a_{22} + \xi_1)}{2\gamma}$. One extension of the generator is found,i.e.,

$$X_5 + 2\gamma X_6 + (\beta - 1)X_7.$$

5a.2 **Case all generators have $\xi_1 + a_{22} = 0$.**

For this case, we have $\xi_1 = -a_{22}$. Consequently, $\zeta_1 = 0$ and $a_{12} = \dfrac{2\xi_1}{\gamma}$. No extensions of the studied subalgebra were found for this case.

Subalgebra 5. with generator $\gamma X_2 + X_4 + X_7$.

For this case, the determining equations (5.14) become

$$zF_y + F_z + 2\gamma F - G = 0,$$

$$zG_y + G_z + 2\gamma G = 0$$

for which the general solution is

$$F(y, z) = (zg(u) + f(u))e^{(-2\gamma z))}, \quad G(y, z) = g(u)e^{(-2\gamma z)}, \tag{A.18}$$

where $u = z^2 - 2y$ and $g' \neq 0$ ($g' = 0$ is equivalent to a degenerate case). These functions are substituted into the determining equations (5.2), which is split with respect to z after substitution of $2y = z^2 - u$. Initial analysis of the determining equations yields $a_{21} = 0$, and $a_{11} = 2a_{22} + \xi_1$. The remaining determining equations are

$$2f'((a_{22} + \xi_1)u - \zeta_1) - 2f(\gamma\zeta_2 + a_{22} - \xi_1) - g(a_{12} - \zeta_2) = 0, \quad \text{(A.19a)}$$

$$2f'(a_{12} - \zeta_2) + 2f\gamma(a_{22} + \xi_1) - 2g'((a_{22} + \xi_1)u - \zeta_1) + g(2\gamma\zeta_2 + a_{22} - 3\xi_1) = 0, \quad \text{(A.19b)}$$

$$g'(a_{12} - \zeta_2) + g\gamma(a_{22} + \xi_1) = 0, \quad \text{(A.19c)}$$

$$2g'((a_{22} + \xi_1)u - \zeta_1) - g(2\gamma\zeta_2 + a_{22} - 3\xi_1) = 0. \quad \text{(A.19d)}$$

Dividing equation (A.19d) with g' and differentiating with respect to u two times, one is left to study the following 2 cases: 1. $\left(\frac{g}{g'}\right)'' \neq 0$, and 2. $\left(\frac{g}{g'}\right)'' = 0$.

5b.1 **Case** $\left(\frac{g}{g'}\right)'' \neq 0$.

For this case, it follows from equation (A.19d) that $a_{22} = 3\xi_1 - 2\gamma\zeta_2$. Consequently, from equation (A.19d), $\xi_1 = \frac{1}{2}\gamma\zeta_2$ and $\zeta_1 = 0$. Substituting these into determining equations (A.19c), we get that $a_{12} = \zeta_2$. These conditions also satisfy equations (A.19a) and (A.19b). No other extensions were obtained in this case.

5b.2 **Case** $\left(\frac{g}{g'}\right)'' = 0$.

For the second case, the form of g satisfies $g = g'(\kappa u + \beta)$ for which the general solution will depend on κ, i.e., i. $\kappa \neq 0$ or ii. $\kappa = 0$.

5b.2.i **Case** $\kappa \neq 0$.

For $\kappa \neq 0$ (with possible shifting leads to $\beta = 0$), the general solution of g is $g_0 u^\kappa$. Substituting this function into determining equations (A.19c) and (A.19d), one obtains $a_{12} = \zeta_2$ and $\zeta_1 = 0$, respectively. Consequently from equation (A.19c), there is a need to study 2 separate cases of γ, i.e., A. $\gamma \neq 0$, or B. $\gamma = 0$.

5b.2.i.A **Case** $\gamma \neq 0$.

For this case, one has $a_{22} = -\xi_1$. Consequently from equation (A.19d), one obtains $\xi_1 = \frac{\gamma\zeta_2}{2}$. This condition satisfies equation (A.19b). Also, since $\gamma \neq 0$, from equation (A.19a), one obtains $\zeta_2 = 0$. From here, no extensions are obtained.

5b.2.i.B **Case** $\gamma = 0$.

For this case, as a consequence of equation (A.19d), either $\kappa = \frac{1}{2}$ or $\kappa \neq \frac{1}{2}$. (Observe later that these 2 cases can be generalized.) If $\kappa = \frac{1}{2}$, it follows that $\xi_1 = 0$. This also satisfies equation (A.19b). From equation (A.19a), either f satisfies $f'u - f \neq 0$ or $f'u - f = 0$. For $f'u - f \neq 0$, it follows that $a_{22} = 0$ and hence, no extensions are obtained. For $f'u - f = 0$, the general solution is $f = f_0 u$. Another extension of the generator is obtained here:

$$2X_5 + X_6.$$

If $\kappa \neq \frac{1}{2}$, it follows that $a_{22} = -\frac{(2\kappa + 3)\,\xi_1}{2\kappa - 1}$. This also satisfies equation (A.19b). Moreover, from equation (A.19a), either f satisfies $f'u - f\left(\kappa + \frac{1}{2}\right) \neq 0$ or $f'u - f\left(\kappa + \frac{1}{2}\right) = 0$. For $f'u - f\left(\kappa + \frac{1}{2}\right) \neq 0$, it follows that $\xi_1 = 0$ and hence, no extensions are obtained. For $f'u - f\left(\kappa + \frac{1}{2}\right) = 0$, the general solution is $f = f_0 u^{\kappa + \frac{1}{2}}$. Another extension of the generator is obtained here:

$$\left(\frac{1}{2} - \kappa\right) + 2X_5 + X_6.$$

5b.2.ii **Case** $\kappa = 0$.

When $\kappa = 0$, then $\beta \neq 0$ and the general solution for g is $g_0 e^{\beta u}$. Substituting this into the determining equations (A.19d), one obtains that $\zeta_1 = \frac{1}{2}(2\xi_1 - \gamma\zeta_2)$ and $a_{22} = -\xi_1$. From equation (A.19c), one has $a_{12} = \zeta_2$. These conditions also satisfy equation (A.19b). From (A.19a), one needs to study if $f' \neq \beta f$ or $f' = \beta f$.

For $f' \neq \beta f$, one obtains $\xi_1 = \frac{\gamma\zeta_2}{2}$. No extensions were found.

For $f' = \beta f$, the general solution for is $f = f_0 e^{\beta u}$. The extension

$$\beta X_2 + X_3$$

is obtained.

Subalgebra 6. with generator $\gamma X_2 + X_5 + X_6 + X_7$**.**

For this case, the determining equations (5.14) become

$$(y+z)F_y + zF_z + (2\gamma - 1)F - G = 0,$$

$$(y+z)G_y + zG_z + 2\gamma G - G = 0$$

for which the general solution is

$$F(y,z) = ((y/z)g(u) + f(u))e^{((1-2\gamma)(y/z))}, \quad G(y,z) = g(u)e^{((1-2\gamma)(y/z))}, \quad \text{(A.20)}$$

where $u = ze^{-y/z}$ and $g \neq 0$. These functions are again substituted into the determining equations (5.2), which is split with respect to z after substitution of $y = z(\ln z - \ln u)$. Initial analysis of the determining equations yields $a_{21} = 0$, $\zeta_2 = 0$, $\zeta_1 = 0$, and $a_{11} = a_{22}$. The remaining determining equations are

$$\begin{aligned} &f'u(-a_{12} + a_{22} + \xi_1) - f(2\gamma a_{12} - a_{12} + a_{22} - 3\xi_1) \\ &-g'\log(u)u(-a_{12} + a_{22} + \xi_1) + g\log(u)(2\gamma a_{12} - a_{12} + a_{22} - 3\xi_1) = 0, \end{aligned} \quad \text{(A.21a)}$$

$$g'u(-a_{12} + a_{22} + \xi_1) - g(2\gamma a_{12} - a_{12} + a_{22} - 3\xi_1) = 0. \quad \text{(A.21b)}$$

From here, we study 2 cases: 1. $\left(\frac{ug'}{g}\right)' \neq 0$ and 2. $\left(\frac{ug'}{g}\right)' = 0$.

6.1 **Case** $\left(\frac{ug'}{g}\right)' \neq 0$.

For this case, equation (A.21b) gives $a_{12} = a_{22} + \xi_1$. Consequently, we need to study 2 cases of γ, i.e., $\gamma = 0$ or $\gamma \neq 0$. If $\gamma = 0$, then $\xi_1 = 0$. If $\gamma \neq 0$, then $a_{22} = \frac{(2-\gamma)\xi_1}{\gamma}$. These conditions also satisfy equation (A.21a). Both cases of γ result to having no extension of the generator.

6.2 **Case** $\left(\frac{ug'}{g}\right)' = 0$.

For this case, the general solution of g is $g = g_0 u^k$. Substituting to equation (A.21b), one needs to separate 2 cases of κ, i.e., i. $\kappa = 1$ and ii. $\kappa \neq 1$. (Observe later that these 2 cases can be generalized.)

6.2.i **Case** $\kappa = 1$.

From equation (A.21b), then it follows that $a_{12} = \frac{2\xi_1}{\gamma}$ (notice that $\gamma \neq 0$ or else system (5.1) is equivalent to a degenerate case). Substituting this into equation (A.21a), f is again separated into 2 cases, A. $uf' - f \neq 0$, or B. $uf' - f = 0$.

6.2.i.A **Case** $uf' - f \neq 0$.

For this case, $a_{22} = \frac{(2-\gamma)\xi_1}{\gamma}$. No extensions were found.

6.2.i.B **Case** $uf' - f = 0$.

For this case, the general solution for f is $f = f_0 u$. The extension of the generator

$$\gamma X_2 + X_7$$

is obtained.

6.2.ii **Case** $\kappa \neq 1$.

For this case, from equation (A.21b), $a_{22}\frac{1}{\kappa - 1}\left((2\gamma + \kappa - 1)a_{12} - (\kappa + 3)\xi_1\right)$. Substituting this into equation

(A.21a), f is again separated into 2 cases, A. $uf' - \kappa f \neq 0$, or B. $uf' - \kappa f = 0$.

6.2.ii.A **Case** $uf' - \kappa f \neq 0$.

For this case, $\xi_1 = \dfrac{\gamma a_{12}}{2}$. No extensions were found.

6.2.ii.B **Case** $uf' - \kappa f = 0$.

For this case, the general solution for f is $f = f_0 u^k$. The extension of the generator

$$\tilde{\gamma} X_2 + X_7,$$

where $\tilde{\gamma} = 2\gamma + \kappa - 1$, is obtained.

Subalgebra 7. with generator $\gamma X_2 + X_3$.

For this case, the determining equations (5.14) become

$$F_y + 2\gamma F = 0,$$
$$G_y + 2\gamma G = 0$$

for which the general solution is

$$F(y,z) = f(z)e^{-2\gamma y} \quad G(y,z) = g(z)e^{-2\gamma y}, \tag{A.22}$$

where $gf' \neq 0$ and $\gamma \neq 0$. If one of them is zero, then system (5.1) is equivalent to a degenerate case. These functions are again substituted into the determining equations (5.2) and are split with respect to y. The determining equations after splitting with respect to y become

$$f'(\zeta_2 + (\xi_1 + a_{22})z) - f(2\gamma a_{12} z + 2\gamma\zeta_1 + a_{11} - 3\xi_1) - g a_{12} = 0, \tag{A.23a}$$

$$g'(\zeta_2 + (\xi_1 + a_{22})z) - g(2\gamma a_{12} z + 2\gamma\zeta_1 + a_{22} - 3\xi_1) - f a_{21} = 0, \tag{A.23b}$$

$$f' a_{21} - 2f\gamma(a_{11} + \xi_1) = 0, \tag{A.23c}$$

$$g'a_{21} - 2g\gamma(a_{11} + \xi_1) = 0. \tag{A.23d}$$

From equations (A.23c) and (A.23c), there is a need to study if $f'g - g'f = 0$ or $f'g - g'f \neq 0$. If $f'g - g'f = 0$ one has $f = f_0 g$, where f_0 is constant. Notice that this is equivalent to a degenerate case and hence, this case is omitted. Hence, $f'g - g'f \neq 0$. From equations (A.23c) and (A.23c), one gets that $a_{21} = 0$ and $a_{11} = -\xi_1$. Substituting this into equations (A.23a) and (A.23b), one obtains

$$f'(\zeta_2 + (\xi_1 + a_{22})z) - 2f(\gamma a_{12} z + \gamma\zeta_1 - 2\xi_1) - g a_{12} = 0, \tag{A.24a}$$

$$g'(\zeta_2 + (\xi_1 + a_{22})z) - g(2\gamma a_{12} z + 2\gamma\zeta_1 + a_{22} - 3\xi_1) = 0. \tag{A.24b}$$

We can suppose that $f(z) = g(z)\phi(z)$, where $\phi' \neq 0$ (as g is nonzero). Substituting this into equation (A.24a) and taking linear combinations with equation (A.24b), equation (A.24a) is transformed into

$$\phi'(\zeta_2 + (\xi_1 + a_{22})z) + (\xi_1 + a_{22})\phi - a_{12} = 0, \tag{A.25}$$

From here, there is a need to study the following 3 cases: 1. there exist at least one generator where $\xi_1 + a_{22} \neq 0$, 2. all generators have $\xi_1 + a_{22} = 0$ but $\zeta_2 \neq 0$ for at least one generator, and 3. all generators have $\xi_1 + a_{22} = 0$ and $\zeta_2 = 0$.

7.1 Case there exist at least one generator where $\xi_1 + a_{22} \neq 0$.

For this case, we obtain $\zeta_2 = 0$ (after possible shifting of z). Moreover, one obtains that the forms of ϕ and g satisfy the equations

$$z\phi' + \phi = \phi_0 \text{ and}$$
$$zg' - (2\gamma\phi_0 z + \beta)g = 0.$$

The general solution of this is $\phi = \phi_0 + \dfrac{\phi_1}{z}$ and $g = g_0 z^\beta e^{2\gamma\phi_0 z}$. Substituting these functions into equation (A.24b), we obtain that $a_{12} = \phi_0(\xi_1 + a_{22})$, and

$\zeta_1 = \dfrac{1}{2\gamma}\left(\beta(\xi_1 + a_{22}) - (a_{22} - 3\xi_1)\right)$. This gives one additional extension of the generator, $(\beta - 1)X_3 + 2\gamma X_6 + 2\gamma\phi_0 X_7$ or simply

$$(\beta - 1)X_3 + \tilde{\gamma} X_6 + \kappa X_7,$$

where $\tilde{\gamma} = 2\gamma \neq 0$ and $\kappa = 2\gamma\phi_0$ are constant.

7.2 Case all generators have $\xi_1 + a_{22} = 0$ but $\zeta_2 \neq 0$ for at least one generator.

For this case, it follows that $a_{22} = -\xi_1$. Moreover, the forms of ϕ and g satisfy the following equations

$$\phi' = \phi_0,$$
$$g' - (2\gamma\phi_0 z + \beta)g = 0,$$

for which the general solution is $\phi = \phi_0 z + \phi_1$ and $g = g_0 e^{\beta z + \gamma\phi_0 z^2}$, where $\phi_0 \neq 0$ (else, system (5.1) is equivalent to a degenerate case). Substituting these functions into equation (A.24b), one obtains that $a_{12} = \phi_0\zeta_2$ and $\zeta_1 = \dfrac{1}{2\gamma}\left(\beta\zeta_2 + 4\xi_1\right)$. It yields an additional extension of the generator,

$$\beta X_3 + 2\gamma X_4 + 2\gamma\phi_0 X_5$$

or simply

$$\beta X_3 + \tilde{\gamma} X_4 + 2\kappa X_7,$$

where $\tilde{\gamma} = 2\gamma \neq 0$ and $\kappa = \gamma\phi_0$ are constant.

7.3 Case all generators have $\xi_1 + a_{22} = 0$ and $\zeta_2 = 0$.

For this case, it follows that $a_{22} = -\xi_1$ and $\zeta_2 = 0$. Consequently, from equation (A.24b), $a_{12} = 0$ and $\zeta_1 = \dfrac{2\xi_1}{\gamma}$. These conditions also satisfy equation (A.25). No extensions were found here.

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