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Group Classification of Two-Dimensional Steady Viscous Gas Dynamics Equations with Arbitrary State Equations

การจำแนกประเภทกลุ่มของสมการพลศาสตร์ของแก๊สความหนืดคงตัวสองมิติด้วยสมการสถานะใด

คณะผู้วิจัย

หัวหน้าโครงการ

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ABSTRACT

In the project group classification of steady viscous gas dynamics equations in the two-dimensional case (for plane and cylindrical symmetries) with arbitrary state equations is done. The group classification includes finding an equivalence group, kernel of all admitted groups and its extensions. After obtaining the admitted group an optimal system of subalgebras are constructed. For every subalgebra a representation of invariant solution is given.

บทคัดย่อ

ในโครงการนี้มีการจำแนกประเภทของสมการพลศาสตร์ของแก๊สความหนืดคงตัวในกรณีสองมิติ (สำหรับระนาบและสมมาตรแบบทรงกระบอก) ด้วยสมการสถานะใดๆ

การจำแนกประเภทกลุ่ม ประกอบด้วย การหากลุ่มสมมูล, ส่วนกลางของกลุ่มแอดมิทเทด และภาคขยายของส่วนกลาง หลังจากที่ได้กลุ่มแอดมิทเทด ก็จะได้ระบบเหมาะที่สุดของพีชคณิตย่อย สำหรับทุกๆ พีชคณิตย่อยเราจะมีตัวแทนของผลเฉลยยืนยง

CONTENTS

1 Introduction

1.1 Background

One of the important and difficult problems of computational aerodynamics is investigation of flows near the bodies. For obtaining complete information about a structure of flows under usual temperature and pressure one can use the viscous gas dynamics equations. Alongside with the complete viscous gas dynamics equations, the simplified equations are widely used. The simplified equations are obtained from complete by using various assumptions about character of flows for eliminating some terms (see, for example, [10, 19, 13]). The simplest of such models is the boundary layer model. Other approaches are based on the parabolized Navier-Stokes equations. For the parabolized Navier-Stokes equations it is assumed that they are valid in a wide range of flow parameters. Defining this range of applicability remains a unsolved theoretical problem.

While a multitude of numerical methods has been developed for constructing approximate solutions, there remains intense interest in finding exact solutions. Each exact solution has large value, first, as the exact description of a real process in frameworks of the given model, second, as a test for approvals and comparisons of various numerical methods, third, as a theoretical fact assisting to improve the used models. One of methods for constructing exact solutions is group analysis of differential equations. Group analysis side by side with constructing exact solutions provides a regular procedure for mathematical modelling by classifying differential equations with respect to arbitrary elements. At present, numerous differential equations are being studied by this method (see [11]).

It should be noted here that many of invariant solutions of the viscous gas dynamics equations have also been obtained by other methods [1, 2, 5, 6, 7, 8, 21, 20, 22]. The group classification of the viscous gas dynamics[1] equations was done in [4]. The group classification of two-dimensional steady viscous gas dynamics equations for an ideal gas was done in [14]. For some models of viscous gas dynamics equations, group analysis was used in [3].

This research project is devoted to application of group analysis for studying the two-dimensional steady viscous gas dynamics equations with arbitrary state equations:

$$u\tau_x + v\tau_y - \tau(u_x + v_y + \nu\frac{u}{x}) = 0, \tag{1}$$

$$uu_x + vu_y + \tau p_x = \tau\left((\lambda+\mu)(u_x + v_y + \nu\frac{u}{x})_x + \lambda_x(u_x + v_y + \nu\frac{u}{x}) + \right.$$
$$\left. + 2\mu_x u_x + \mu_y(u_y + v_x) + \mu(u_{xx} + u_{yy} + \nu\frac{u_x}{x}) - \mu\nu\frac{u}{x^2}\right),$$

$$uv_x + vv_y + \tau p_y = \tau\left((\lambda+\mu)(u_x + v_y + \nu\frac{u}{x})_y + \lambda_y(u_x + v_y + \nu\frac{u}{x}) + \right.$$
$$\left. + \mu_x(u_y + v_x) + 2\mu_y v_y + \mu(v_{xx} + v_{yy} + \nu\frac{v_x}{x})\right),$$

$$up_x + vp_y + A(p,\rho)(u_x + v_y + \nu\frac{u}{x}) = B(p,\rho)\left(\kappa(T_{xx} + T_{yy} + \nu\frac{T_x}{x}) + \kappa_x T_x + \kappa_y T_y + \right.$$

[1] Where the first $\lambda = \lambda(T)$ and the second $\mu = \mu(T)$ coefficients of viscosity are related by the equation $\lambda = -2\mu/3$, and $\kappa = \kappa(T)$.

$$+\mu\left(2(u_x^2+v_y^2+\nu\frac{u^2}{x^2})+(u_y+v_x)^2\right)\Bigg),$$

where $\nu=0$ corresponds to the plane flows and $\nu=1$ to the axi-symmetrical flows. The case of ideal gas $T=R^{-1}p\tau$ where the first $\lambda=\lambda(T)$ and the second $\mu=\mu(T)$ coefficients of viscosity are related by the equation $\lambda=-2\mu/3$. and $\kappa=\kappa(T)$) has been studied in [14]. Here we study the gas dynamics equations with arbitrary state equations.

1.2 Objectives

The objectives of the project are:

- To find admitted group of the two-dimensional steady viscous gas dynamics equations.
- To do group classification of these equations.
- To construct optimal system of subalgebras.
- To construct invariant exact solutions of the complete Navier-Stokes equations and simplified equations of viscous gas in order to compare solutions of these models.

1.3 Benefits

Expected benefits of this research project include the following. Exact solutions are good tests for comparisons of various numerical methods. By comparison of solutions of different simplifications with solutions of complete the Navier-Stokes equations can be useful for a determination of scopes of simplifications of complete equations.

The immediate beneficiaries of this project will be to those who use the Navier-Stokes equations for modelling of processes in different kind of technology: superconductor, aerodynamics and geodynamics. Longer term, results of this project will be used by researchers doing theoretical investigations of the Navier-Stokes equations and other scientific studies.

1.4 Scope and limitations

The present research deals only with the two-dimensional steady viscous gas dynamics equations.

2 Methodology

The application of group analysis implies some steps[2]. The first step is a group classification with respect to arbitrary elements. An admitted group is found at this step. The next step is a construction of an optimal system of subalgebras. Then one can attempt to find an invariant or partially invariant solution for each subalgebra of the optimal system.

[2]For more details the reader is referred to the attached article ([16], Appendix B), section 2.

3 Results

All necessary calculations were carried on a computer using the symbolic manipulation program REDUCE [9]. The calculations showed that the group of equivalence transformations corresponds to the Lie algebra with the generators

$$X_1^e = \partial_y,\ X_2^e = \partial_p,\ X_3^e = x\partial_x + y\partial_y + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa,$$

$$X_4^e = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2\tau\partial_\tau + 2\kappa\partial_\kappa,\ X_5^e = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

In the case $\nu = 0$ there is two more generators

$$X_6^e = \partial_x,\ X_7^e = y\partial_x - x\partial_y + v\partial_u - u\partial_v,$$

which correspond to shift and rotation.

Remark. If instead of the functions $A(p,\tau), B(p,\tau)$ one considers the internal energy $\varepsilon(p,\tau)$, then the operators $X_2^e. X_4^e$, and X_5^e are changed to

$$X_2^e = \partial_p - \tau\partial_\varepsilon,\ X_4^e = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2\tau\partial_\tau + 2\kappa\partial_\kappa + 2\varepsilon\partial_\varepsilon,$$

$$X_5^e = -\tau\partial_\tau + p\partial_p + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

and there is one more generator $X_8^e = \partial_\varepsilon$.

The kernel of the fundamental Lie algebra is made up of the generator

$$X_1 = \partial_y$$

if $\nu = 1$ and

$$X_1 = \partial_y,\ X_2 = \partial_x,\ X_3 = y\partial_x - x\partial_y + v\partial_u - u\partial_v$$

if $\nu = 0$. An extension of the kernel of the principal Lie algebra occurs by specializing the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. There are three types of the generators[3] admitted by system (1). The final results of the group classification are presented in the following table

	λ	μ	T	κ	A	B	z
a	$e^{\beta p}\Lambda(z)$	$e^{\beta p}M(z)$	$e^{\delta p}\Theta(z)$	$e^{(-\delta+\alpha+\beta)p}K(z)$	$A(z)$	$B(z)$	$\tau e^{-\alpha p}$
b	$p^{\beta+1}\Lambda(z)$	$p^{\beta+1}M(z)$	$p^{\delta}\Theta(z)$	$p^{-\delta+\alpha+\beta+2}K(z)$	$p\hat{A}(z)$	$B(z)$	$\tau p^{-\alpha}$
c	$\tau^{\beta}\Lambda(p)$	$\tau^{\beta}M(p)$	$\tau^{\delta}\Theta(p)$	$\tau^{-\delta+\beta+1}K(p)$	$A(p)$	$B(p)$	p

In this table the first column means the type of the extension of the algebra $\{X\}$ or $\{X, Y\}$: the types a, b, or c, respectively.

Thus, there are three kinds of extensions of the admitted by equations (1) group, which depend on the specifications of the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$.

For all these extensions[4]:

a) optimal systems of subalgebras are constructed;

b) representations of all invariant solutions are given;

[3]For more details the reader is referred to the attached article ([16], Appendix B), section 3.

[4]For more details the reader is referred to the attached article ([16], Appendix B), sections 4 and 5.

c) some invariant solutions are obtained;

Along with the results directly related with the objectives of the project the problem of compatibility of overdetermined system for double waves where the system consists of $2n-1$ quasilinear equations was studied[5]. Here n is the number of the independent variables. This problem appears if partially invariant solution of the double wave type is considered.

4 Conclusion

Thermodynamic state equations supplement the basic equations of fluid dynamics and thermodynamics by characterizing the specific fluid of interest. Many special real gas equations exist for specific fluids. The most commonly used thermal equation of state is the thermally perfect gas equation[6], where $p = R\rho T$. The thermally and calorically perfect gas ($\varepsilon = c_v T$) is a polytropic gas.

The general form of the thermal equation of state for real gases is [12]

$$p\tau = RTf(\tau, T),$$

where $f(\tau, T)$ is a gas compressibility factor. The equations of state $(f(\tau, T), \varepsilon(\tau, T))$, the coefficients of viscosity and heat conductivity can be obtained from experimental data, derived from kinetic theory or from an appropriate real gas equation of state. The latter approach is usually used in fluid dynamics. In our study the equations of state are obtained from the requirement to have additional symmetry properties. Additional symmetries allows constructing more exact solutions.

The obtained in the project results shows that the classification of the function $A(p, \tau)$ is similar to the inviscid gas dynamics equations [18] (Table 1). There is only one difference: the model 7 ([18], Table 1) with the projective generator is absent in our study. The latter is because of firstly, a presence of viscosity, and secondly, steadiness of studied flows. Classifications of the first $\lambda(p, \tau)$ and the second $\mu(p, \tau)$ coefficients of viscosity, and the coefficient of heat conductivity $\kappa(p, \tau)$ are related with the classification of the function $A(p, \tau)$. If one uses an additional symmetry for constructing invariant or partially invariant solution, then these coefficients must have the found special representations.

[5] For more details the reader is referred to the attached article ([15], Appendix C).

[6] Group classification of equations (1) for gases with $\varepsilon = \varepsilon(T)$ was done in [14].

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APPENDIX A

Short CURRICULUM VITAE

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Degrees:

- D.Sc. (1991) Differential Equations, Institute of Mathematics and Mechanics, Russian Academy of Sciences, Ekaterinburg (Sverdlovsk), Russia
- Ph.D. (1981) Applied Mathematics (Mechanics of Solids), Lavrentiev Institute of Hydrodynamics, Novosibirsk, Russia
- M.Sc. (1976) Applied Mathematics, Department of Mathematics and Mechanics, Novosibirsk State University, Novosibirsk, Russia

Positions:

- Professor of Mathematics (1996 / to date), Suranaree University of Technology, Thailand,
- Junior (1979-1982), Senior (1982-1991), Leading (1991 / to date) Researcher, Institute of Theoretical and Applied Mechanics, Siberian Branch of Russian Academy of Sciences, Novosibirsk

Academic Experience:

- Professor, Novosibirsk State University, Novosibirsk, Russia, 1991 - 1996
- Deputy Dean (1991-1996) of the Department of Mathematics and Mechanics, Novosibirsk State University, Novosibirsk, Russia
- Associate Professor of Mathematics (1983-1991), Novosibirsk State University, Novosibirsk
- Assistant Professor of Mathematics (1982-1983), Novosibirsk State University, Novosibirsk
- Lecturer (1980-1982), Novosibirsk Institute of Civil Engineers, Novosibirsk

Research Interests:

- **Principal Research Interests:**
  Methods for constructing exact solutions of PDE:
  - Group Analysis of Differential Equations,
  - Method of Differential Constraints
- **Other Research Interests:**
  - Continuum Mechanics
  - Symbolic (analytical) manipulations on computer
  - Mathematical and Numerical modeling
  - Software
- **Current Research Interests:**
  - Application of group analysis to Partial differential equations (PDE) Integro-differential and Functional differential equations,
  - Partially invariant solutions of PDE,
  - Multiple waves

External Professional Activities:

- Thailand Research Fund's Royal Golden Jubilee Ph.D. Grantee, 1998, 1999, 2001
- Referee for Journal of Applied Mathematics and Mechanics, and Nonlinear Dynamics
- Reviewer for Mathematical Reviews (in the field of Partial Differential Equations)
- Member (1991-1996) of the Science Council of the Institute of Theoretical and Applied Mechanics, Novosibirsk
- Member (1992-1996) of the Science Council for awarding Candidate of Science degree (PhD) at the Institute of Theoretical and Applied Mechanics, Novosibirsk

Publications: List of Publications includes more than 90 titles.

APPENDIX B

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

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Group classification of two-dimensional steady viscous gas dynamics equations with arbitrary state equations

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Abstract
This paper is devoted to the group classification of steady viscous gas dynamics equations in the two-dimensional case (with plane or cylindrical symmetry) with arbitrary state equations. Representations of all invariant solutions are given.

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Mathematics Subject Classification: 58J70, 76M60

1. Introduction

The analytic study of the properties of partial differential equations plays an important role in applied mathematics and mathematical physics. One of the methods for studying the properties of differential equations is group analysis. The modern state of group analysis is reviewed in [1]. Group analysis besides constructing exact solutions provides a regular procedure for mathematical modelling by classifying differential equations with respect to arbitrary elements. The application of group analysis implies some steps. The first step is a group classification with respect to arbitrary elements. An admitted group is found at this step. The next step is a construction of an optimal system of subalgebras. Then one can attempt to find an invariant or partially invariant solution for each subalgebra of the optimal system.

We should note here that many invariant solutions of the viscous gas dynamics equations have also been obtained by other methods [2–10]. The group classification of the viscous gas dynamics[1] equations was done in [11]. The group classification of two-dimensional steady viscous gas dynamics equations for an ideal gas was done in [12]. For some models of viscous gas dynamics equations, group analysis was used in [13]. Unsteady spherically symmetric viscous gas dynamics equations were studied in [14].

[1] Here the first, $\lambda = \lambda(T)$, and second, $\mu = \mu(T)$, coefficients of viscosity are related by the equation $\lambda = -2\mu/3$, and $\kappa = \kappa(T)$.

This paper is devoted to the application of group analysis for studying the viscous gas dynamics equations with arbitrary state equations.

2. The group analysis algorithm

Let us first review the notations and techniques used in group analysis.

Let an lth-order system of differential equations

$$(S)\colon\ F^k(x, u, p) = 0 \qquad (k = 1, 2, \ldots, s)$$

be given. Here $x = (x_i)$, $(i = 1, 2, \ldots, n)$, are the independent variables, $u = (u^j)$ $(j = 1, 2, \ldots, m)$ are the dependent variables, $p = (p^k_\alpha)$ are the derivatives up to lth-order and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index with $|\alpha| \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n \leqslant l$.

2.1. Admitted Lie group of transformations

One of the main objects in group analysis is the local one-parameter Lie group G^1 of the transformations:

$$x_i' = f^{x_i}(x, u; a) \qquad u^{j\prime} = f^{u^j}(x, u; a) \qquad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m). \tag{1}$$

There is a one-to-one correspondence between groups G^1 and infinitesimal generators

$$X = \xi^i(x, u)\partial_{x_i} + \zeta^j(x, u)\partial_{u^j}$$

where

$$\xi^i(x, u) = \left(\frac{\mathrm{d}f^{x_i}}{\mathrm{d}a}\right)_{|a=0} \qquad \zeta^j(x, u) = \left(\frac{\mathrm{d}f^{u^j}}{\mathrm{d}a}\right)_{|a=0}.$$

The operator

$$\underset{l}{X} = X + \sum_{j,\alpha} \zeta^j_\alpha \partial_{p^j_\alpha}$$

with coefficients

$$\zeta^j_{\alpha,k} = D_k\zeta^j_\alpha - \sum_i p^j_{\alpha,i} D_k\xi^i. \tag{2}$$

is called the lth prolongation of a generator X. Here

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j,\alpha} p^j_{\alpha,k}\frac{\partial}{\partial p^j_\alpha}$$

are the operators of total differentiation with respect to x_k $(k = 1, 2, \ldots, n)$.

The algorithm for finding a local one-parameter Lie group (1) admitted by the system of differential equations (S) consists of the following four steps.

In the first step, the form of the generator

$$X = \xi^i(x, u)\partial_{x_i} + \zeta^j(x, u)\partial_{u^j}$$

is given, with unknown coefficients $\xi^i(x, u)$, $\zeta^j(x, u)$. In the second step the prolonged operator $\underset{l}{X}$ is applied to every equation of the system (S). In the next step the coefficients of the prolonged operator are substituted by using formulae (2). The equations obtained must be considered on the manifold (S). As a result one obtains the system of differential equations

$$DS\colon\ \underset{l}{X}F^k(x, u, p)_{|(S)} = 0 \qquad (k = 1, 2, \ldots, s). \tag{3}$$

This system of equations is called the system of determining equations and is an overdetermined system of linear homogeneous differential equations in the unknown coordinates $\xi^i(x, u), \zeta^j(x, u)$. The general solution of the determining equations DS generates a full group GS of the system (S). The feature of the admitted group is that under the action of any transformation of this group, every solution $u = U(x)$ of the system (S) is transformed into a solution $u = U_a(x)$ of the same system (S). Therefore, the admitted group allows construction of new solutions from known solutions. Note that the set of admitted generators generate a Lie algebra, which is called admitted by the system (S).

2.2. *Equivalence transformations*

Most systems of partially differential equations have arbitrary elements: arbitrary functions or arbitrary constants. These arbitrary elements can be separated into classes with respect to a group of equivalence transformations. An equivalence transformation is a nondegenerate change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to a system of equations of the same class. These transformations allow us to use the simplest representation of the given equations. Note that the admitted group depends on specialization of the arbitrary elements. The group classification problem consists in searching for an admitted group of transformations, which is admitted for all arbitrary elements of the system and all specializations of the arbitrary elements. The specialization of the arbitrary elements can extend the admitted group. For the calculation of equivalence transformations, we follow the approach developed in [15, 16], which consists of the following.

Suppose, the system of differential equation

$$F^k(x, u, p, \phi) = 0 \qquad (k = 1, 2, \ldots, s) \tag{4}$$

has arbitrary elements $\phi = (\phi^1, \phi^2, \ldots, \phi^t)$, which are functions (or constants) $\phi = \phi(x, u)$. A specific value of the arbitrary elements represents a concrete system of differential equations.

The problem of finding an equivalent transformation consists of constructing a transformation of the space $R^{n+m+t}(x, u, \phi)$ which preserves the equations by only changing their representative $\phi = \phi(x, u)$. For this purpose, we consider the one-parameter group of transformations of the space R^{n+m+t}:

$$x' = f^x(x, u, \phi; a) \qquad u' = f^u(x, u, \phi; a) \qquad \phi' = f^\phi(x, u, \phi; a). \tag{5}$$

A generator of this group has the form

$$X^e = \xi^x \partial_x + \zeta^u \partial_u + \zeta^\phi \partial_\phi \tag{6}$$

with the coordinates[2]:

$$\xi^i = \xi^i(x, u, \phi) \qquad \zeta^{u^j} = \zeta^{u^j}(x, u, \phi) \qquad \zeta^{\phi^k} = \zeta^{\phi^k}(x, u, \phi)$$
$$(i = 1, \ldots, n;\ j = 1, \ldots, m;\ k = 1, \ldots, t).$$

We use the main feature of the Lie group that any solution $u_0(x)$ of system (4) with functions $\phi(x, u)$ is transformed by (5) into another solution $u = u_a(x')$ of system (4), but with different (transformed) functions $\phi_a(x, u)$, which are defined in the following way. Solving the relations

$$x' = f^x(x, u, \phi(x, u); a) \qquad u' = f^u(x, u, \phi(x, u); a)$$

[2] Later the author discovered that similar assumptions about the coefficients of the operator were used in [17] for one class of ordinary differential equations with one nonessential restriction $\zeta^{\phi^k} = \zeta^{\phi^k}(x, \phi)$.

with respect to (x, u), we obtain

$$x = g^x(x', u'; a) \qquad u = g^u(x', u'; a). \tag{7}$$

Then the transformed function is

$$\phi_a(x', u') = f^\phi(x, u, \phi(x, u); a) \tag{8}$$

where instead of (x, u) we have to substitute their expressions (7). The transformed solution $u_a(x)$ is obtained by solving the relations

$$x' = f^x(x, u_0(x), \phi(x, u_0(x)); a)$$

with respect to (x):

$$x = \psi^x(x'; a)$$

and substituting into

$$u_a(x') = f^u(x, u_0(x), \phi(x, u_0(x)); a). \tag{9}$$

The formulae for the transformations of the partial derivatives p_a and the derivatives of the functions ϕ are obtained by differentiating (8) and (9) with respect to x' and u'.

The method for finding a group of equivalence transformations is similar to the algorithm for finding an admitted group of transformations. The difference only consists of the prolongation of the infinitesimal generator X^e. In agreement with the construction of the functions $u_a(x')$ and $\phi_a(x', u')$, the prolonged operator

$$\tilde{X}^e = X^e + \zeta^{u_x}\partial_{u_x} + \zeta^{\phi_x}\partial_{\phi_x} + \zeta^{\phi_u}\partial_{\phi_u} + \cdots$$

has the following coordinates

$$\zeta^{u_\lambda} = D^e_\lambda \zeta^u - u_x D^e_\lambda \xi^x \qquad (\lambda = x_1, x_2, \dots, x_n)$$

with $D^e_\lambda = \partial_\lambda + u_\lambda \partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi$ and

$$\zeta^{\phi_\lambda} = \tilde{D}^e_\lambda \zeta^\phi - \phi_x \tilde{D}^e_\lambda \xi^x - \phi_u \tilde{D}^e_\lambda \zeta^u \qquad (\lambda = u^1, u^2, \dots, u^m, x_1, x_2, \dots, x_n)$$

with $\tilde{D}^e_\lambda = \partial_\lambda + \phi_\lambda \partial_\phi$.

An equivalence group GS^e of transformations is generated by $G^1(X^e)$.

Remark 1. In some cases one may have additional requirements for the arbitrary elements. For example, the arbitrary elements ϕ^μ may be supposed to be independent of the independent variables $\frac{\partial \phi^\mu}{\partial x_k} = 0$. When studying the equivalence group, such conditions have to be added to the original system of differential equations (4), leading to additional determining equations.

Remark 2. Note that in the case of the Navier–Stokes equations, kinematic viscosity is the arbitrary element and these equations can be transformed to equations (14) by scaling the independent and dependent variables.

2.3. *Invariant and partially invariant solutions*

For each subgroup of the admitted group GS, one can try to find an invariant or partially invariant solution. Let $H \subset GS$ be a group admitted by the system of equations (S). Assume that $X_1, \dots, X_r$ is a basis of the Lie algebra L^r which corresponds to the group H. An invariant or partially invariant solution with respect to the group H is called an H-solution. The method [18] for constructing H-solutions with respect to the group H requires us to find a universal invariant of this group: a set of all functionally independent invariants. For this purpose one needs to solve the overdetermined linear system of differential equations:

$$X_i \phi(x, u) = 0 \qquad (i = 1, 2, \dots, r). \tag{10}$$

Because $X_1, \ldots, X_r$ generate a Lie algebra, system (10) is complete. Its general solution can be expressed through the $m+n-r_*$ invariants

$$J = (J^1(x,u), J^2(x,u), \ldots, J^{m+n-r_*}(x,u))$$

where r_* is the total rank of the matrix composed of the coefficients of the generators $X_i\ (i = 1, 2, \ldots, r)$. If the rank of the Jacobi matrix $\frac{\partial(J^1,\ldots,J^{m+n-r_*})}{\partial(u_1,\ldots,u_m)}$ is equal to q, then without loss of generality, one can choose the first $q \leqslant m$ invariants $J^1, \ldots, J^q$ such that the rank of the Jacobi matrix $\frac{\partial(J^1,\ldots,J^q)}{\partial(u_1,\ldots,u_m)}$ is equal to q and the remaining $k = m+n-r_*-q$ invariants $J^{q+1}, J^{q+2}, \ldots, J^{m+n-r_*}$ only depend on the independent variables x. H-solutions are characterized by two integers: the rank $\sigma = \delta + n - r_* \geqslant 0$ and the defect $\delta \geqslant 0$; thus one uses the notation $H(\sigma, \delta)$-solution. The rank and defect must satisfy the inequalities

$$k \leqslant \sigma < n \qquad \max\{r_* - n, m - q, 0\} \leqslant \delta \leqslant \min\{r_* - 1, m - 1\}.$$

To construct a representation of $H(\sigma, \delta)$-solutions, one needs to separate the universal invariant into two parts: $J = (\bar{J}, \bar{\bar{J}})$, where $l = m - \delta$ and

$$\bar{J} = (J^1, \ldots, J^l) \qquad \bar{\bar{J}} = (J^{l+1}, J^{l+2}, \ldots, J^{m+n-r_*}).$$

This means that one can choose the number l such that $1 \leqslant l \leqslant q \leqslant m$. The rank and defect of the $H(\sigma, \delta)$-solution are $\delta = m - l$, $\sigma = m + n - r_* - l = \delta + n - r_*$. A solution is called invariant if $\delta = 0$, otherwise it is called a partially invariant solution. From the first l invariants $J^1, J^2, \ldots, J^l$ one can define the l dependent functions

$$u^i = \phi^i(\bar{J}, u^{l+1}, u^{l+2}, \ldots, u^m, x) \qquad (i = 1, \ldots, l). \tag{11}$$

The functions $u^{l+1}, u^{l+2}, \ldots, u^m$ are called superfluous. The representation of the $H(\sigma, \delta)$-solution is obtained by assuming that the first part of the universal invariant is a function of the second part:

$$\bar{J} = \Psi(\bar{\bar{J}}) \tag{12}$$

and substituting (12) into (11). Thus, the representation of an invariant or partially invariant solution is

$$u^i = \Phi^i(\bar{\bar{J}}, u^{l+1}, u^{l+2}, \ldots, u^m, x) \qquad (i = 1, \ldots, l) \tag{13}$$

where $\Phi^i = \phi^i(\Psi(\bar{\bar{J}}), u^{l+1}, u^{l+2}, \ldots, u^m, x)$.

If $\delta \neq 0$, then either $\sigma = k$ or $\sigma > k$. In the first case ($\sigma = k$) the partially invariant solution is called regular, otherwise it is called irregular [19]. The number $\sigma - k$ is called the measure of irregularity.

After constructing the representation of an invariant or partially invariant solution one needs to substitute it into the original system of equations. The system of equations in the functions Ψ^i and the superfluous functions thus obtained is called the reduced system. This system is overdetermined and requires analysis of compatibility. Usually the compatibility analysis is easier for invariant solutions than for the partially invariant ones.

If H' is a subgroup of H, then it may be possible that a partially invariant $H(\sigma, \delta)$-solution is a partially invariant $H'(\sigma', \delta')$-solution. In this case $\delta' \leqslant \delta, \sigma' \geqslant \sigma$ [18]. A solution is called reducible to a $H'(\sigma', \delta')$-solution if there exists $H' \subset H$ such that $\delta' < \delta, \sigma' = \sigma$. In particular, a solution is called reducible to an invariant solution if there exists $H' \subset H$ with $\delta' = 0$. Thus, a natural problem is to reduce a partially invariant $H(\sigma, \delta)$-solution to an invariant $H'(\sigma, 0)$-solution.

3. Viscous gas dynamics equations

The viscous gas dynamics equations govern the three-dimensional motion of a compressible, thermal conductive, Newtonian viscous gas flow

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \tau\,\mathrm{div}(P) \qquad \frac{\mathrm{d}\tau}{\mathrm{d}t} - \tau\,\mathrm{div}(v) = 0 \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \tau P{:}D + \tau\,\mathrm{div}(\kappa\nabla T).$$

Here $\tau = 1/\rho$ is the specific volume, ρ is the density, v is the velocity, P is the stress tensor, $D = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^*\right)$ is the rate-of-strain tensor, ε is the internal energy, T is the temperature and κ is the coefficient of heat conductivity. The Stokes axioms for a viscous gas give

$$P = (-p + \lambda\,\mathrm{div}(v))I + 2\mu D$$

where p is the pressure, λ and μ are the first and second coefficients of viscosity, respectively. These coefficients of viscosity are related to the coefficient of bulk viscosity k by the expression

$$k = \lambda + \tfrac{2}{3}\mu.$$

In general, it is believed that k is negligible except in the study of the structure of shock waves and in the absorption and attenuation of acoustic waves.

A viscous gas is a two-parametric medium. As the main thermodynamic variables, we choose pressure p and specific volume τ: entropy η, internal energy ε and temperature T are functions of pressure and specific volume

$$\eta = \eta(p,\tau) \qquad \varepsilon = \varepsilon(p,\tau) \qquad T = T(p,\tau).$$

The first and second thermodynamic laws require these functions to satisfy the equations

$$\eta_p = \frac{\varepsilon_p}{T} \qquad \eta_\tau = \frac{\varepsilon_\tau + p}{T} \qquad 3\lambda + 2\mu \geqslant 0 \quad \mu \geqslant 0 \quad \kappa \geqslant 0.$$

For simplicity of classification we study the case which corresponds to an essentially viscous and heat conductive gas

$$\mu \neq 0 \qquad \kappa \neq 0.$$

Thus, the viscous gas dynamics equations we study are

$$\begin{aligned}
&\frac{\mathrm{d}v}{\mathrm{d}t} + \tau\nabla p = \tau((\lambda+\mu)\nabla(\mathrm{div}(v)) + (\mathrm{div}(v))\nabla\lambda + \mu\Delta v + 2D(\nabla\mu))\\
&\frac{\mathrm{d}\tau}{\mathrm{d}t} - \tau\,\mathrm{div}(v) = 0\\
&\frac{\mathrm{d}p}{\mathrm{d}t} + A(p,\tau)\mathrm{div}(v) = B(p,\tau)(\lambda(\mathrm{div}(v))^2 + 2\mu D{:}D + (\nabla\kappa)(\nabla T) + \kappa\Delta T)
\end{aligned} \qquad (14)$$

with functions

$$A = \frac{\tau(\varepsilon_\tau + p)}{\varepsilon_p} \qquad B = \frac{\tau}{\varepsilon_p}.$$

Note that the internal energy and entropy can be expressed through the functions $A = A(p,\tau)$, $B = B(p,\tau)$ by the formulae

$$\varepsilon_p = \frac{\tau}{B} \qquad \varepsilon_\tau = \frac{A}{B} - p \qquad \eta_p = \frac{\tau}{BT} \qquad \eta_\tau = \frac{A}{BT}.$$

The conditions $\varepsilon_{p\tau} = \varepsilon_{\tau p}$, $\eta_{p\tau} = \eta_{\tau p}$ lead to the restrictions

$$\tau B_\tau + BA_p - AB_p = B^2 + B \qquad \tau T_\tau = AT_p - TB. \qquad (15)$$

In the case of an ideal gas (i.e. the gas that obeys the Clapeyron equation $T = R^{-1}p\tau$) $B = B(\tau p)$, $A = p(1 + B(\tau p))$ with an arbitrary function $B(\tau p)$. For a polytropic gas $\varepsilon = (\gamma - 1)^{-1}\tau p$ and once more this simplifies the functions A and B: $B = (\gamma - 1)$, $A = \gamma p$. Here R is the gas constant and γ is a polytropic exponent. If τ and μ are constants, then system (14) is split into two parts: the Navier–Stokes equations and energy equation[3].

[3] Sometimes in the literature equations (14) are called the full Navier–Stokes equations.

3.1. *Two-dimensional steady viscous gas dynamics equations*

In this paper we study the two-dimensional steady viscous gas dynamics equations

$$
\begin{aligned}
&u\tau_x + v\tau_y - \tau\left(u_x + v_y + \nu\frac{u}{x}\right) = 0\\
&uu_x + vu_y + \tau p_x = \tau\Bigg((\lambda+\mu)\left(u_x + v_y + \nu\frac{u}{x}\right)_x + \lambda_x\left(u_x + v_y + \nu\frac{u}{x}\right)\\
&\qquad + 2\mu_x u_x + \mu_y(u_y + v_x) + \mu\left(u_{xx} + u_{yy} + \nu\frac{u_x}{x}\right) - \mu\nu\frac{u}{x^2}\Bigg)\\
&uv_x + vv_y + \tau p_y = \tau\Bigg((\lambda+\mu)\left(u_x + v_y + \nu\frac{u}{x}\right)_y + \lambda_y\left(u_x + v_y + \nu\frac{u}{x}\right)\\
&\qquad + \mu_x(u_y + v_x) + 2\mu_y v_y + \mu\left(v_{xx} + v_{yy} + \nu\frac{v_x}{x}\right)\Bigg)\\
&up_x + vp_y + A(p,\rho)\left(u_x + v_y + \nu\frac{u}{x}\right) = B(p,\rho)\Bigg(\kappa\left(T_{xx} + T_{yy} + \nu\frac{T_x}{x}\right) + \kappa_x T_x + \kappa_y T_y\\
&\qquad + \mu\left(2\left(u_x^2 + v_y^2 + \nu\frac{u^2}{x^2}\right) + (u_y + v_x)^2\right)\Bigg)
\end{aligned}
\tag{16}
$$

where $\nu = 0$ corresponds to the plane flows and $\nu = 1$ to the axisymmetrical flows. The case of ideal gas $T = R^{-1}p\tau$ where the first, $\lambda = \lambda(T)$, and second, $\mu = \mu(T)$, coefficients of viscosity are related by the equation $\lambda = -2\mu/3$ and $\kappa = \kappa(T)$ has been studied in [12]. Here we study the gas dynamics equations with arbitrary state equations.

Since the arbitrary elements satisfy restrictions (15) and $A = A(p,\tau)$, $B = B(p,\tau)$, $\lambda = \lambda(p,\tau)$, $\mu = \mu(p,\tau)$, $\kappa = \kappa(p,\tau)$, $T = T(p,\tau)$, hence for calculating the equivalence group of transformations we have to append the equations

$$
\begin{array}{llll}
A_x = 0 & A_y = 0 & A_u = 0 & A_v = 0\\
B_x = 0 & B_y = 0 & B_u = 0 & B_v = 0\\
\lambda_x = 0 & \lambda_y = 0 & \lambda_u = 0 & \lambda_v = 0\\
\mu_x = 0 & \mu_y = 0 & \mu_u = 0 & \mu_v = 0\\
\kappa_x = 0 & \kappa_y = 0 & \kappa_u = 0 & \kappa_v = 0\\
T_x = 0 & T_y = 0 & T_u = 0 & T_v = 0
\end{array}
\tag{17}
$$

to equations (16). All coefficients of the infinitesimal generator of the equivalence group are dependent on all independent, dependent variables and arbitrary elements

$$x, y, u, v, \tau, p, A, B, \lambda, \mu, \kappa, T.$$

All necessary calculations were carried out on a computer using the symbolic manipulation program REDUCE [20]. The calculations showed that the group of equivalence transformations of equations (16), (17) corresponds to the Lie algebra with generators

$$X_1^e = \partial_y \qquad X_2^e = \partial_p \qquad X_3^e = x\partial_x + y\partial_y + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa$$

$$X_4^e = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2\tau\partial_\tau + 2\kappa\partial_\kappa \qquad X_5^e = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

In the case $\nu = 0$ there are two more generators

$$X_6^e = \partial_x \qquad X_7^e = y\partial_x - x\partial_y + v\partial_u - u\partial_v$$

which correspond to shift and rotation.

Remark. If instead of the functions $A(p,\tau)$ and $B(p,\tau)$, one considers the internal energy $\varepsilon(p,\tau)$, then the operators X_2^ε, X_4^ε and X_5^ε are changed to

$$X_2^\varepsilon = \partial_p - \tau\partial_\varepsilon \qquad X_4^\varepsilon = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2\tau\partial_\tau + 2\kappa\partial_\kappa + 2\varepsilon\partial_\varepsilon$$

$$X_5^\varepsilon = -\tau\partial_\tau + p\partial_p + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa$$

and there is one more generator $X_8^\varepsilon = \partial_\varepsilon$.

3.2. Admitted group

For finding the admitted group we look for the generator

$$X = \zeta^x\partial_x + \zeta^y\partial_y + \zeta^u\partial_u + \zeta^v\partial_v + \zeta^\tau\partial_\tau + \zeta^p\partial_p$$

with the coefficients depending on x, y, u, v, τ, p. Calculations lead to the following result.

The kernel of the fundamental Lie algebra is made up of the generator

$$X_1 = \partial_y$$

if $\nu = 1$ and

$$X_1 = \partial_y \qquad X_2 = \partial_x \qquad X_3 = y\partial_x - x\partial_y + v\partial_u - u\partial_v$$

if $\nu = 0$. An extension of the kernel of the principal Lie algebra occurs by specializing the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. Note that the functions $A = A(p,\tau), B = B(p,\tau), T = T(p,\tau)$ have to satisfy equations (15). There are three types of generators admitted by system (16). Further, α, β and δ are arbitrary constants.

Type (a). If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{array}{ll} \alpha\tau A_\tau + A_p = 0 & \alpha\tau B_\tau + B_p = 0 \\ \alpha\tau\mu_\tau + \mu_p = \beta\mu & \alpha\tau\lambda_\tau + \lambda_p = \beta\lambda \\ \alpha\tau T_\tau + T_p = \delta T & \alpha\tau\kappa_\tau + \kappa_p = (-\delta+\alpha+\beta)\kappa \end{array} \tag{18}$$

then there is one more admitted generator:

$$Y_a = \alpha(u\partial_u + v\partial_v) + 2\alpha\tau\partial_\tau + 2\partial_p + (\alpha+2\beta)(x\partial_x + y\partial_y).$$

The general solution of equations (18) is

$$\begin{array}{lll} A = A(\tau e^{-\alpha p}) & B = B(\tau e^{-\alpha p}) & \mu = e^{\beta p}M(\tau e^{-\alpha p}) \\ \lambda = e^{\beta p}\Lambda(\tau e^{-\alpha p}) & T = e^{\delta p}\Theta(\tau e^{-\alpha p}) & \kappa = e^{(-\delta+\alpha+\beta)p}K(\tau e^{-\alpha p}) \end{array} \tag{19}$$

where the functions $A(z)$, $B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau e^{-\alpha p}$):

$$-\alpha zBA' + zB'(1+\alpha A) = B^2 + B \qquad (1+\alpha A)z\Theta' = (\delta A - B)\Theta. \tag{20}$$

The internal energy is represented by the formula

$$\varepsilon = e^{\alpha p}(\varphi(z) - zp) + \psi(p) \qquad \psi'(p) = Ce^{\alpha p}$$

where the function $\varphi(z)$ and constant C can be accounted arbitrarily and they are related to the functions $A(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{A(z)}{B(z)} \qquad C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - \alpha\varphi(z).$$

In this case the function $\Theta(z)$ has to satisfy the equation

$$(C - z + \alpha\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z).$$

Type (b). If the functions $A(\tau,p)$, $B(\tau,p)$, $\lambda(\tau,p)$, $\mu(\tau,p)$, $\kappa(\tau,p)$, $T(\tau,p)$ satisfy the equations

$$\begin{array}{ll} \alpha\tau A_\tau + pA_p = A & \alpha\tau B_\tau + pB_p = 0 \\ \alpha\tau\mu_\tau + p\mu_p = (\beta+1)\mu & \alpha\tau\lambda_\tau + p\lambda_p = (\beta+1)\lambda \\ \alpha\tau T_\tau + pT_p = \delta T & \alpha\tau\kappa_\tau + pT_p = (-\delta+2+\alpha+\beta)\kappa \end{array} \tag{21}$$

then there is an extension by the generator

$$Y_b = (1+\alpha)(u\partial_u + v\partial_v) + 2\alpha\tau\partial_\tau + 2p\partial_p + (\alpha+2\beta+1)(x\partial_x + y\partial_y).$$

The general solution of equations (21) is

$$\begin{array}{lll} A = p\hat{A}(\tau p^{-\alpha}) & B = B(\tau p^{-\alpha}) & \mu = p^{\beta+1}M(\tau p^{-\alpha}) \\ \lambda = p^{\beta+1}\Lambda(\tau p^{-\alpha}) & T = p^{\delta}\Theta(\tau p^{-\alpha}) & \kappa = p^{-\delta+\alpha+\beta+2}K(\tau p^{-\alpha}) \end{array} \tag{22}$$

where the functions $\hat{A}(z)$, $B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau p^{-\alpha}$):

$$-\alpha zB\hat{A}' + zB'(1+\alpha\hat{A}) = B^2 + B - B\hat{A} \qquad (1+\alpha\hat{A})z\Theta' = (\delta\hat{A} - B)\Theta. \tag{23}$$

The internal energy is represented by the formula

$$\varepsilon = p^{(\alpha+1)}(\varphi(z) - z) + \psi(p) \qquad \psi'(p) = Cp^{\alpha}$$

where the function $\varphi(z)$ and constant C are arbitrary and they are related to the functions $\hat{A}(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{\hat{A}(z)}{B(z)} \qquad C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - (\alpha+1)\varphi(z).$$

The function $\Theta(z)$ is represented through the function $\varphi(z)$ by the formula

$$(C - z + (\alpha+1)\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z).$$

Note that an ideal gas belongs to this type if $\delta = \alpha + 1$ and the function $\varphi(z)$ satisfies the equation

$$\delta(z\varphi' - \varphi) = C.$$

Type (c). If the functions $A(\tau,p)$, $B(\tau,p)$, $\lambda(\tau,p)$, $\mu(\tau,p)$, $\kappa(\tau,p)$, $T(\tau,p)$ satisfy the equations

$$\begin{array}{llll} A_\tau = 0 & B_\tau = 0 & \tau\mu_\tau = \beta\mu & \tau\lambda_\tau = \beta\lambda \\ \tau T_\tau = \delta T & \tau\kappa_\tau = (-\delta+1+\beta)\kappa & & \end{array} \tag{24}$$

then there is one more admitted generator:

$$Y_c = u\partial_u + v\partial_v + 2\tau\partial_\tau + (1+2\beta)(x\partial_x + y\partial_y).$$

The general solution of equations (24) is

$$\begin{array}{llll} A = A(p) & B = B(p) & \mu = \tau^{\beta}M(p) & \lambda = \tau^{\beta}\Lambda(p) \\ T = \tau^{\delta}\Theta(p) & \kappa = \tau^{-\delta+\beta+1}K(p) & & \end{array} \tag{25}$$

where the functions $A(p)$, $B(p)$ and $\Theta(p)$ satisfy the equations

$$BA' - AB' = B^2 + B \qquad A\Theta' = (\delta + B)\Theta. \tag{26}$$

The internal energy is represented by the formula

$$\varepsilon = \tau\varphi(p) - \tau p$$

	λ	μ	T	κ	A	B	z	Condition
a	$e^{\beta p}\Lambda(z)$	$e^{\beta p}M(z)$	$e^{\delta p}\Theta(z)$	$e^{(-\delta+\alpha+\beta)p}K(z)$	$A(z)$	$B(z)$	$\tau e^{-\alpha p}$	(20)
b	$p^{\beta+1}\Lambda(z)$	$p^{\beta+1}M(z)$	$p^{\delta}\Theta(z)$	$p^{-\delta+\alpha+\beta+2}K(z)$	$p\hat{A}(z)$	$B(z)$	$\tau p^{-\alpha}$	(23)
c	$\tau^{\beta}\Lambda(p)$	$\tau^{\beta}M(p)$	$\tau^{\delta}\Theta(p)$	$\tau^{-\delta+\beta+1}K(p)$	$A(p)$	$B(p)$	p	(26)

where $\varphi(p)$ is an arbitrary function and is related to the functions $A(p)$ and $B(p)$ by the formula

$$\varphi(p) = \frac{A(p)}{B(p)}.$$

In this case the function $\Theta(p)$ is related to the function $\varphi(z)$ by the formula

$$\varphi(p)\Theta'(p) = (1 - \delta + \delta\varphi'(p))\Theta(p).$$

Note that if $\delta = 1$ and $\varphi = Cp$, then the gas is ideal.

The final results of the group classification are presented in table 1.

In this table the first column means the type of extension of the algebra $\{X\}$ or $\{X, Y\}$: the type a, b, or c, respectively. The last column means conditions for the state functions.

Thus, there are three kinds of extensions of the groups admitted by equations (16), which depend on the specifications of the functions $A = A(p,\tau)$, $B = B(p,\tau)$, $\lambda = \lambda(p,\tau)$, $\mu = \mu(p,\tau)$, $\kappa = \kappa(p,\tau)$, $T = T(p,\tau)$. These extensions can be one dimensional and two dimensional[4].

The one-dimensional extensions are with the generators $\{Y_a\}$, $\{Y_b\}$ or $\{Y_c\}$.

The two-dimensional extensions are with the generators $\{Y_a, Y_b\}$, $\{Y_a, Y_c\}$ or $\{Y_b, Y_c\}$.

The group with the extension $\{Y_a, Y_b\}$ is admitted by equations (16) if

$$A = A_0\tau^{\alpha} \qquad B = -1 \qquad \mu = \mu_0\tau^{\beta+\alpha} \qquad \lambda = \lambda_0\tau^{\beta+\alpha}$$
$$\kappa = \kappa_0\tau^{\beta+2\alpha} \qquad T = T_0\tau \qquad \alpha \neq 0.$$

In this case the internal energy is $\varepsilon = -\left(\tau p + A_0\int \tau^{\alpha}\,d\tau\right)$. Instead of the operators Y_a and Y_b, one can use their linear combinations:

$$\hat{Y}_a = \partial_p \qquad \hat{Y}_b = (1+\alpha)(u\partial_u + v\partial_v) + 2\tau\partial_\tau + (\alpha + 2\beta + 1)(x\partial_x + y\partial_y).$$

The group with the extension of type $\{Y_a, Y_c\}$ is admitted by equations (16) if

$$A = A_0 \qquad B = -1 \qquad \mu = \mu_0\tau^{\beta}e^{\alpha p} \qquad \lambda = \lambda_0\tau^{\beta}e^{\alpha p}$$
$$\kappa = \kappa_0\tau^{\beta - A_0\sigma}e^{(\alpha-\sigma)p} \qquad T = T_0\tau^{1+A_0\sigma}e^{\sigma p}.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0\tau)$ and by taking linear combinations of the operators Y_a and Y_c, one obtains another basis of the generators:

$$\hat{Y}_a = \partial_p + \alpha(x\partial_x + y\partial_y) \qquad \hat{Y}_c = u\partial_u + v\partial_v + 2\tau\partial_\tau + (2\beta+1)(x\partial_x + y\partial_y).$$

There is a third type of extensions $\{Y_b, Y_c\}$ if

$$A = \gamma p \qquad B = \gamma - 1 \qquad \mu = \mu_0\tau^{\beta}p^{1+\alpha} \qquad \lambda = \lambda_0\tau^{\beta}p^{1+\alpha}$$
$$\kappa = \kappa_0\tau^{\gamma(1-\alpha)+\beta}p^{\alpha-\delta+2} \qquad T = T_0\tau^{\gamma(\delta-1)+1}p^{\delta} \qquad \gamma \neq 1.$$

The internal energy in this case is

$$\varepsilon = \frac{\tau p}{\gamma - 1}$$

[4] There is no three-dimensional extension because of incompatibility of the system of differential equations for the functions $A = A(p,\tau)$, $B = B(p,\tau)$, $\lambda = \lambda(p,\tau)$, $\mu = \mu(p,\tau)$, $\kappa = \kappa(p,\tau)$, $T = T(p,\tau)$.

and linear combinations of the operators Y_b and Y_c are

$$\hat{Y}_b = u\partial_u + v\partial_v + 2p\partial_p + (2\alpha+1)(x\partial_x + y\partial_y) \qquad \hat{Y}_c = \tau\partial_\tau - p\partial_p + (\beta-\alpha)(x\partial_x + y\partial_y).$$

Note that a polytropic gas belongs to the last case of gases, where γ is a polytropic exponent.

In the formulae above $A_0, \mu_0, \lambda_0, \kappa_0, T_0, \alpha, \beta, \gamma, \delta, \sigma$ are arbitrary constants: the commutators are

$$[\hat{Y}_a, \hat{Y}_b] = 0 \qquad [\hat{Y}_a, \hat{Y}_c] = 0 \qquad [\hat{Y}_b, \hat{Y}_c] = 0.$$

Remark. By direct checking one can set for the general unsteady three-dimensional gas flow the same models of types (a), (b) and (c), described by equations (19), (20), (22), (23), (25) and (26), with the following generalized generators:

$$Y_a = \alpha v\partial_v + 2\alpha\tau\partial_\tau + 2\partial_p + (\alpha+2\beta)x\partial_x + 2\beta t\partial_t$$
$$Y_b = (1+\alpha)v\partial_v + 2\alpha\tau\partial_\tau + 2p\partial_p + (\alpha+2\beta+1)x\partial_x + 2\beta t\partial_t$$
$$Y_c = v\partial_v + 2\tau\partial_\tau + (1+2\beta)x\partial_x + 2\beta t\partial_t.$$

The kernel includes the Galilean group with generators

$$X_i = \partial_{x_i} \qquad X_{3+i} = t\partial_{x_i} + \partial_{v_i} \qquad Y_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i} + v_i\partial_{v_j} - v_j\partial_{v_i}$$
$$X_{10} = \partial_t \qquad (i, j = 1, 2, 3 \quad i < j).$$

It has to be mentioned that the group classification of the viscous gas dynamics equations in the case of an ideal gas with the first, $\lambda = \lambda(T)$, and second, $\mu = \mu(T)$, coefficients of viscosity related by the equation $\lambda = -2\mu/3$, and $\kappa = \kappa(T)$ was done in [11]. Two-dimensional steady viscous gas dynamics equations and their simplifications (parabolized models) for ideal gas were studied in [12]. The group classification of spherically symmetric flows with arbitrary state equations was considered in [14].

4. Optimal system of subalgebras

In this section two groups are studied. One is the group with generators

$$L_4 = \{X_1, X_2, X_3, Y\}.$$

The other is the group with generators

$$L_2 = \{X_2, Y\}.$$

Here Y is one of the generators: $Y = Y_a$ (with the parameter z, which is used later $z = \alpha+2\beta$), $Y = Y_b$ ($z = \alpha+2\beta+1$) or $Y = Y_c$ ($z = 2\beta+1$). These groups correspond to the plane ($\nu = 0$) case and axisymmetrical ($\nu = 1$) case with one extension, respectively. The classifications of subalgebras of the algebras L_4 and L_2 are given in this section.

The classification subdivides a set of H-solutions into equivalent (similar) classes. Any two H-solutions f_1 and f_2 are elements of the same equivalence class if there exists a transformation $T_a \in GS$ such that $f_2 = T_a f_1$. Otherwise f_1, f_2 belong to different classes and they are called essentially different H-solutions. The classification of H-solutions is related to the optimal system of subalgebras Θ of the admitted algebra L. To obtain the optimal system of subalgebras Θ, we use the algorithm developed in [21, 22]. Let us consider the algebra

$L_4 = \{X_1, X_2, X_3, Y\}$. The table of commutators is

	X_1	X_2	X_3	Y
X_1	0	0	X_2	zX_1
X_2	0	0	$-X_1$	zX_2
X_3	$-X_2$	X_1	0	0
Y	$-zX_1$	$-zX_2$	0	0

Automorphisms are recovered by the table of commutators and consist of the automorphisms

$$A_1: \quad x_1' = x_1 + zy_1a_1 \qquad x_2' = x_2 - x_3a_1$$
$$A_2: \quad x_1' = x_1 + x_3a_2 \qquad x_2' = x_2 + zy_1a_2$$
$$A_3: \quad x_1' = x_1\cos a_3 - x_2\sin a_3 \qquad x_2' = x_1\sin a_3 + x_2\cos a_3$$
$$A_4: \quad x_1' = x_1e^{za_4} \qquad x_2' = x_2e^{za_4}.$$

Here x_i $(i = 1, 2, 3)$ and y_1 are coordinates of the operator $Z = x_1X_1 + x_2X_2 + x_3X_3 + y_1Y$ before the transformation and x_i' $(i = 1, 2, 3)$ and y_1' are coordinates of the operator Z' after action of the automorphism, a_i are parameters of the automorphisms. In the expressions for automorphisms only transformed coordinates are presented. There is also one involution

$$E: \quad x_1' = -x_1 \qquad x_2' = -x_2$$

which corresponds to the change of variables $x \to -x, y \to -y, u \to -u$ and $v \to -v$ without change of equations (16).

The Lie algebra L_4 has the following decomposition: $L_4 = N_2 \oplus J_2$, where $N_2 = \{X_3, Y\}$ is a subalgebra and $J_2 = \{X_1, X_2\}$ is an ideal. The Lie algebra N_2 is Abelian. Hence, its classification is trivial and consists of the subalgebras

$$\{X_3 + hY\}, \qquad \{Y\}, \qquad \{X_3, Y\}.$$

The optimal system of subalgebras of the algebra L_4 is obtained by gluing the ideal J_2 to the constructed subalgebras of the optimal system of subalgebras of the algebra N_2.

Because the number of independent variables is two, invariant solutions can be constructed only with respect to one- and two-dimensional subalgebras. These subalgebras of the optimal system are

$$\{X_3, Y\}, \qquad \{X_1, Y + hX_2\}, \qquad \{X_1, X_2\}, \qquad \{X_3 + qY\}, \qquad \{Y + hX_2\}, \qquad \{X_1\}$$

where $zh = 0$, and q and h are arbitrary constants.

The optimal system of subalgebras of the algebra $L_2 = \{X_2, Y\}$ consists of the subalgebras

$$\{X_2, Y\}, \qquad \{Y + hX_2\}, \qquad \{X_2\}$$

where $zh = 0$ and h is an arbitrary constant.

4.1. Representations of invariant solutions

The next step in the construction of invariant solutions consists of finding universal invariants. Note that the subalgebras from the optimal system of the algebra L_2 are those from the optimal system of the algebra L_4. Thus, it is enough to consider representations of invariant solutions of the algebra L_4. Before presenting the results, we give some remarks. In the case of two-dimensional subalgebras, the representations of invariant solutions are obtained by assuming that all invariants are constants. The subalgebra $\{X_1, X_2\}$ has no invariant solutions.

The invariant solution with respect to the subalgebra $\{X_1\}$ is a trivial one-dimensional steady solution of the viscous gas dynamics equations.

According to the theory of group analysis [18], after constructing the representations of invariant solutions one needs to substitute the representations of solutions into the original system of equations.

4.1.1. Subalgebra $\{X_3, Y\}$. For the operator X_3 it is convenient to use the cylindrical coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad u = U\cos\theta - V\sin\theta \qquad v = U\cos\theta + V\cos\theta.$$

In these coordinates there are the following relations:

$$X_3 = \partial_\theta \qquad x\partial_x + y\partial_y = r\partial_r \qquad u\partial_u + v\partial_v = U\partial_U + V\partial_V.$$

Note that if $z = 0$, then there are no invariant solutions. Hence, we have only to study the case $z \neq 0$.

In case (a) $z = \alpha + 2\beta \neq 0$ and the universal invariant consists of the invariants

$$Ur^{-\alpha/z},\ Vr^{-\alpha/z},\ \tau r^{-2\alpha/z},\ p - 2z^{-1}\ln r.$$

In case (b) $z = \alpha + 2\beta + 1 \neq 0$ and the universal invariant consists of the invariants

$$Ur^{-(1+\alpha)/z},\ Vr^{-(1+\alpha)/z},\ \tau r^{-2\alpha/z},\ pr^{-2/z}.$$

In case (c) $z = 2\beta + 1 \neq 0$ and the universal invariant consists of the invariants

$$Ur^{-1/z},\ Vr^{-1/z},\ \tau r^{-2/z},\ p.$$

4.1.2. Subalgebra $\{X_1, Y + hX_2\}$ $(zh = 0)$. If $z = 0$ and $h = 0$, then there are no invariant solutions.

In case (a) $z = \alpha + 2\beta$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$uy^{-\alpha/z},\ vy^{-\alpha/z},\ \tau y^{-2\alpha/z},\ p - 2z^{-1}\ln y.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$u\mathrm{e}^{-\alpha y/h},\ v\mathrm{e}^{-\alpha y/h},\ \tau\mathrm{e}^{-2\alpha y/h},\ p - 2h^{-1}y.$$

In case (b) $z = \alpha + 2\beta + 1$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$uy^{-(1+\alpha)/z},\ vy^{-(1+\alpha)/z},\ \tau y^{-2\alpha/z},\ py^{-2/z}.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$u\mathrm{e}^{-(1+\alpha)y/h},\ v\mathrm{e}^{-(1+\alpha)y/h},\ \tau\mathrm{e}^{-2\alpha y/h},\ p\mathrm{e}^{-2y/h}.$$

In case (c) $z = 2\beta + 1$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$uy^{-1/z},\ vy^{-1/z},\ \tau y^{-2/z},\ p.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$u\mathrm{e}^{-y/h},\ v\mathrm{e}^{-y/h},\ \tau\mathrm{e}^{-2y/h},\ p.$$

4.1.3. Subalgebra $\{X_3 + qY\}$. The operators $X_3 + qY$ in cylindrical coordinates are

$$X_3 + qY_a = \partial_\theta + q((\alpha + 2\beta)r\partial_r + \alpha(U\partial_U + V\partial_V) + 2\alpha\tau\partial_\tau + 2\partial_p)$$

$$X_3 + qY_b = \partial_\theta + q((\alpha + 2\beta + 1)r\partial_r + (1+\alpha)(U\partial_U + V\partial_V) + 2\alpha\tau\partial_\tau + 2p\partial_p)$$

$$X_3 + qY_c = \partial_\theta + q((2\beta + 1)r\partial_r + U\partial_U + V\partial_V + 2\tau\partial_\tau).$$

In case (a) the universal invariant consists of the invariants

$$\bar{J} = (Ue^{-\alpha q\theta}, Ve^{-\alpha q\theta}, \tau e^{-2\alpha q\theta}, p - 2q\theta) \qquad \bar{\bar{J}} = re^{-(\alpha+2\beta)q\theta}.$$

In case (b) the universal invariant consists of the invariants

$$\bar{J} = \left(Ue^{-(1+\alpha)q\theta}, Ve^{-(1+\alpha)q\theta}, \tau e^{-2\alpha q\theta}, pe^{-2q\theta}\right) \qquad \bar{\bar{J}} = re^{-(\alpha+2\beta+1)q\theta}.$$

In case (c) the universal invariant consists of the invariants

$$\bar{J} = (Ue^{-q\theta}, Ve^{-q\theta}, \tau e^{-2q\theta}, p) \qquad \bar{\bar{J}} = re^{-(2\beta+1)q\theta}.$$

4.1.4. Subalgebra $\{Y + hX_2\}$ $(zh = 0)$. Note that if $z = 0$ and $h = 0$, then there are no invariant solutions.

In case (a) $z = \alpha + 2\beta$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$\bar{J} = \left(uy^{-\alpha/z}, vy^{-\alpha/z}, \tau y^{-2\alpha/z}, p - 2z^{-1}\ln y\right) \qquad \bar{\bar{J}} = x/y.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$\bar{J} = \left(ue^{-\alpha y/h}, ve^{-\alpha y/h}, \tau e^{-2\alpha y/h}, p - 2h^{-1}y\right) \qquad \bar{\bar{J}} = x.$$

In case (b) $z = \alpha + 2\beta + 1$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$\bar{J} = \left(uy^{-(1+\alpha)/z}, vy^{-(1+\alpha)/z}, \tau y^{-2\alpha/z}, py^{-2/z}\right) \qquad \bar{\bar{J}} = x/y.$$

If $z = 2\beta + 1 = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$\bar{J} = \left(ue^{-(1+\alpha)y/h}, ve^{-(1+\alpha)y/h}, \tau e^{-2\alpha y/h}, pe^{-2y/h}\right) \qquad \bar{\bar{J}} = x.$$

In case (c) $z = 2\beta + 1$. If $z \neq 0$, then $h = 0$ and the universal invariant consists of the invariants

$$\bar{J} = \left(uy^{-1/z}, vy^{-1/z}, \tau y^{-2/z}, p\right) \qquad \bar{\bar{J}} = x/y.$$

If $z = \alpha + 2\beta = 0$, then there is an invariant solution only if $h \neq 0$ and the universal invariant is

$$\bar{J} = \left(ue^{-y/h}, ve^{-y/h}, \tau e^{-2y/h}, p\right) \qquad \bar{\bar{J}} = x.$$

5. Invariant solutions

In this section we demonstrate the construction of reduced systems for invariant solutions. As an example the subalgebra $\{Y + hX_2\}$ for an ideal gas $(T = R^{-1}p\tau)$ of type (c) is taken. Note that in the case of an ideal gas $B = \gamma - 1$, $A = \gamma p$, where γ is a constant. The obtained reduced systems are systems of ordinary differential equations. For solving the ordinary differential equations, one can use well-developed numerical methods.

5.1. The case $2\beta + 1 \neq 0$

At first, let us consider $z = 2\beta + 1 \neq 0$. In this case the representation of the invariant solution is

$$u = U(\xi)y^q \qquad v = V(\xi)y^q \qquad \tau = G(\xi)y^{2q} \qquad p = P(\xi) \qquad \xi = \frac{x}{y}$$

where $q = z^{-1}$. The next step in obtaining the reduced system is the substitution of the representation of the invariant solution into the initial system of viscous gas dynamics equations. For example, the equation of mass conservation becomes

$$\frac{\mathrm{d}G}{\mathrm{d}\xi}(U - \xi V) - G\frac{\mathrm{d}U}{\mathrm{d}\xi}(U - \xi V) + (q-1)GV = 0.$$

If $q = 1$ (or $\beta = 0$), then the last equation can be integrated:

$$U = \xi V + c_1 G$$

where c_1 is an arbitrary constant. For the sake of simplicity, we present the reduced system for this case ($\beta = 0$) and also assume that the functions $\Lambda(p)$, $M(p)$, $K(p)$ are constants. The remaining equations in this case are

$$\begin{aligned}
&\frac{\mathrm{d}G}{\mathrm{d}\xi}c_1^2 + c_1\left(2V - (\xi^2+1)\,\mathrm{Re}^{-1}(\tilde\lambda+2)\frac{\mathrm{d}^2G}{\mathrm{d}\xi^2}\right) + (\xi^2+1)\left(\frac{\mathrm{d}P}{\mathrm{d}\xi} - 2(\tilde\lambda+2)\,\mathrm{Re}^{-1}\frac{\mathrm{d}V}{\mathrm{d}\xi}\right) = 0\\
&c_1\left(\mathrm{Re}^{-1}\frac{\mathrm{d}^2G}{\mathrm{d}\xi^2}\xi(\tilde\lambda+1) + \frac{\mathrm{d}V}{\mathrm{d}\xi}\right) + 2\,\mathrm{Re}^{-1}\frac{\mathrm{d}V}{\mathrm{d}\xi}\xi(\tilde\lambda+1) - \frac{\mathrm{d}P}{\mathrm{d}\xi}\xi\\
&\qquad - \mathrm{Re}^{-1}\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}(\xi^2+1) + G^{-1}V^2 = 0\\
&2(\gamma-1)^{-1}\,\mathrm{Re}\,PV + c_1^2\left(2\frac{\mathrm{d}G}{\mathrm{d}\xi}G\xi - \left(\frac{\mathrm{d}G}{\mathrm{d}\xi}\right)^2(\tilde\lambda+2+\xi^2) - G^2\right)\\
&\qquad + c_1\left(\frac{\mathrm{d}G}{\mathrm{d}\xi}(\mathrm{Re}\,P - 4V(\tilde\lambda+1)) + 2\frac{\mathrm{d}V}{\mathrm{d}\xi}G(\xi^2-1) - 2\frac{\mathrm{d}G}{\mathrm{d}\xi}\frac{\mathrm{d}V}{\mathrm{d}\xi}\xi(\xi^2+1)\right)\\
&\qquad + c_1(\gamma-1)^{-1}\,\mathrm{Re}\left(\frac{\mathrm{d}G}{\mathrm{d}\xi}P + \frac{\mathrm{d}P}{\mathrm{d}\xi}G\right) + 2V(\mathrm{Re}\,P - 2V(\tilde\lambda+1))\\
&\qquad - \left(\frac{\mathrm{d}V}{\mathrm{d}\xi}\right)^2(\xi^2+1)^2 + \frac{\gamma\,\mathrm{Pr}^{-1}}{(\gamma-1)}\left(2\left(\frac{\mathrm{d}G}{\mathrm{d}\xi}P\xi + \frac{\mathrm{d}P}{\mathrm{d}\xi}G\xi - GP\right)\right.\\
&\qquad \left.- (\xi^2+1)\left(2\frac{\mathrm{d}G}{\mathrm{d}\xi}\frac{\mathrm{d}P}{\mathrm{d}\xi} + \frac{\mathrm{d}^2G}{\mathrm{d}\xi^2}P + \frac{\mathrm{d}^2P}{\mathrm{d}\xi^2}G\right)\right) = 0.
\end{aligned} \tag{27}$$

Here Re is the Reynolds number and Pr is the Prandtl number. The nondimensional dependent variables $\tilde G$, $\tilde V$, $\tilde P$, $\tilde c_1$ are related to the dimensional variables G, V, P, c_1 by the formulae

$$G = L^{-2}\tau_0\tilde G \qquad V = Lv_0\tilde V \qquad P = p_0\tilde P \qquad \lambda = \mu\tilde\lambda \qquad c_1 = v_0L\tau_0^{-1}\tilde c_1$$

$$p_0 = v_0^2\tau_0^{-1} \qquad \mathrm{Pr} = \frac{\gamma R}{\kappa(\gamma-1)}\mu \qquad \mathrm{Re} = Lv_0\mu^{-1}\tau_0^{-1}$$

where L is the reference length, v_0 is the reference velocity and τ_0 is the reference specific volume. In system (27) the wave '∼' is dropped. The system of equations (27) is invariant with respect to the transformation

$$\xi' = -\xi \qquad c_1' = -c_1.$$

Therefore, it is enough to study this system for $c_1 \geqslant 0$. If $c_1 \neq 0$, then the system can be solved with respect to the second derivatives of the functions G, V and P. If $c_1 = 0$, then from

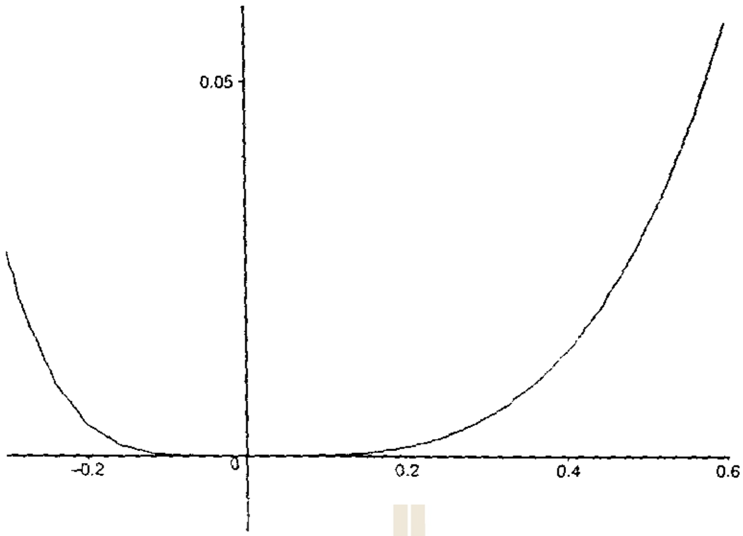

Figure 1. The functions $V(\xi)$ for $c_1 = 0$ and $c_1 = 0.5$.

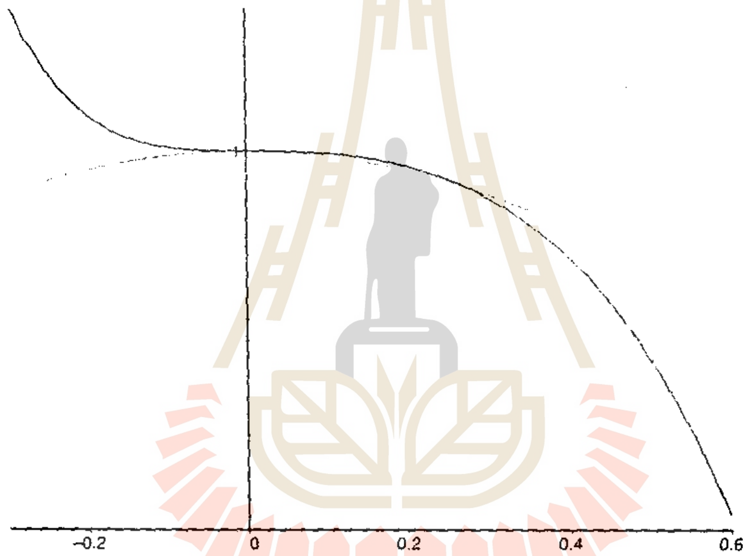

Figure 2. The functions $\tau(\xi)$ for $c_1 = 0$ and $c_1 = 0.5$.

the first equation one can find the first derivative of the function P and the remaining equations can be solved with respect to the second derivatives of the functions V and G. In this case if $V'(0) = G'(0) = P'(0) = 0$, then the functions $V(\xi)$, $G(\xi)$, $P(\xi)$ are symmetric. Note also that if $c_1 = 0$, then there is the particular solution

$$V = 0 \qquad P = C_2 \qquad G = C_3\xi + C_4(\xi^2 - 1).$$

In the figures two solutions with $c_1 = 0$ and $c_1 = 0.5$ are given. The functions for $V(\xi)$ are shown in figure 1. Note that for $c_1 = 0$ the function $V(\xi) = 0$. The functions for $\tau(\xi)$ are shown in figure 2. The function $\tau(\xi)$ for $c_1 = 0$ is symmetric. The functions for $P(\xi)$ are shown in figure 3. The initial values for these solutions are

$$V(0) = 0 \qquad V'(0) = 0 \qquad G(0) = 1.0$$
$$G'(0) = 0 \qquad P(0) = 1.0 \qquad P'(0) = 0$$

and $\mathrm{Pr} = 0.72$, $\mathrm{Re} = 10.0$, $\gamma = 1.4$. Note that increasing Re narrows the domain of validity of the solution.

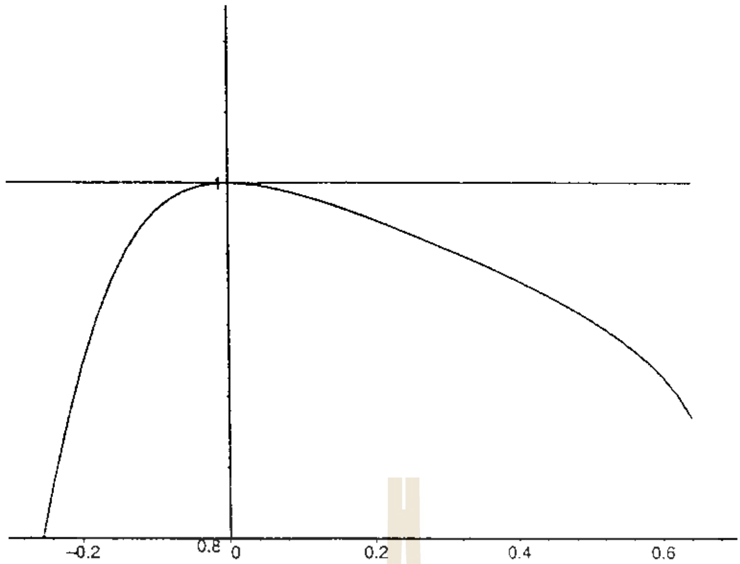

Figure 3. The functions $P(\xi)$ for $c_1 = 0$ and $c_1 = 0.5$.

5.2. The case $2\beta + 1 = 0$

If $2\beta + 1 = 0$, then the representation of the invariant solution is

$$u = U(x)\mathrm{e}^{qy} \qquad v = V(x)\mathrm{e}^{qy} \qquad \tau = G(x)\mathrm{e}^{2qy} \qquad p = P(x)$$

where $q = h^{-1}$. Substituting the representation of the invariant solution into the initial system of viscous gas dynamics equations gives the equations

$$G^{-1}\frac{\mathrm{d}G}{\mathrm{d}x}\left(2\frac{\mathrm{d}U}{\mathrm{d}x}M + \Lambda W\right) - 2\frac{\mathrm{d}P}{\mathrm{d}x}\left(2M'\frac{\mathrm{d}U}{\mathrm{d}x} + \Lambda' W\right) + 2\frac{\mathrm{d}P}{\mathrm{d}x}\sqrt{G}$$
$$- 4\frac{\mathrm{d}^2U}{\mathrm{d}x^2}M - 2\frac{\mathrm{d}W}{\mathrm{d}x}\Lambda + 2G^{-1/2}UW = 0$$

$$MG^{-1}\frac{\mathrm{d}G}{\mathrm{d}x}\left(\frac{\mathrm{d}V}{\mathrm{d}x} + qU\right) + 2G^{-1/2}\left(\frac{\mathrm{d}V}{\mathrm{d}x}U + qV^2\right)$$
$$- 2\frac{\mathrm{d}P}{\mathrm{d}x}M'\left(\frac{\mathrm{d}V}{\mathrm{d}x} + qU\right) - 2M\left(q\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right) = 0$$

$$\frac{\mathrm{d}G}{\mathrm{d}x}U - G\frac{\mathrm{d}U}{\mathrm{d}x} + qGV = 0$$

$$B^{-1}\sqrt{G}\left(PW + \frac{\mathrm{d}P}{\mathrm{d}x}U\right) - \left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2(\Lambda + 2M) + \frac{\mathrm{d}U}{\mathrm{d}x}(\sqrt{G}P - 2\Lambda qV) - \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 M$$
$$- 2\frac{\mathrm{d}V}{\mathrm{d}x}MqU + KR^{-1}\left(\frac{1}{2}\left(\frac{\mathrm{d}G}{\mathrm{d}x}\right)^2 PG^{-1} - \frac{3}{2}\frac{\mathrm{d}G}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}x} - \frac{\mathrm{d}^2G}{\mathrm{d}x^2}P\right.$$
$$\left. - \frac{\mathrm{d}^2P}{\mathrm{d}x^2}G - 2Pq^2G\right) - \frac{\mathrm{d}P}{\mathrm{d}x}R^{-1}K'\left(\frac{\mathrm{d}G}{\mathrm{d}x}P - \frac{\mathrm{d}P}{\mathrm{d}x}G\right)$$
$$+ q(\sqrt{G}PV - qV^2(\Lambda + 2M) - MqU^2) = 0$$

where $W = \frac{\mathrm{d}U}{\mathrm{d}x} + qV$. Note that from the equation of mass conservation, one can find the derivative $\frac{\mathrm{d}U}{\mathrm{d}x}$. The remaining equations are second-order ordinary differential equations with respect to G, V, P.

6. Conclusion

Thermodynamic state equations supplement the basic equations of fluid dynamics and thermodynamics by characterizing the specific fluid of interest. Many special real gas equations exist for specific fluids. The most commonly used thermal equation of state is the thermally perfect gas equation[5], where $p = R\rho T$. The thermally and calorically perfect gas ($\varepsilon = c_v T$) is a polytropic gas.

The general form of the thermal equation of state for real gases is [23]

$$p\tau = RTf(\tau, T)$$

where $f(\tau, T)$ is the gas compressibility factor. The equations of state $(f(\tau, T), \varepsilon(\tau, T))$, coefficients of viscosity and heat conductivity can be obtained from experimental data, derived from the kinetic theory or from an appropriate real gas equation of state. The latter approach is usually used in fluid dynamics. In our study the equations of state are obtained from the requirement of additional symmetry properties. Additional symmetries allow the construction of more exact solutions.

The results obtained in this paper show that the classification of the function $A(p, \tau)$ is similar to the inviscid gas dynamics equations ([22], table 1). There is only one difference: the model 7 ([22], table 1) with the projective generator is absent in our study. The latter is because of (i) the presence of viscosity and (ii) steadiness of studied flows. Classifications of the first $\lambda(p, \tau)$ and second $\mu(p, \tau)$ coefficients of viscosity and the coefficient of heat conductivity $\kappa(p, \tau)$ are related to the classification of the function $A(p, \tau)$. If one uses an additional symmetry for constructing an invariant or a partially invariant solution, then these coefficients must have special representations.

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APPENDIX C

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On the Compatibility of Overdetermined Systems of Double Waves

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Abstract. Obtaining equations for double waves in the case of a general quasilinear system of partial differential equations poses some difficulties. They are connected with the complexity and awkwardness of the study of overdetermined systems, describing solutions of this class. However, there are general statements about double waves of autonomous quasilinear systems of equations. This article is devoted to the classification of irreducible double waves of autonomous nonhomogeneous systems.

Keywords: Partially invariant solutions, degenerate hodograph, multiple waves, double waves.

1. Introduction

A solution $u_i = u_i(x_1, x_2, \dots, x_n)$ $(i = 1, 2, \dots, m)$, of the autonomous quasilinear system of equations

$$\sum_{\alpha=1}^{n} A_\alpha(u) \frac{\partial u}{\partial x_\alpha} = f(u) \qquad (1)$$

is called a multiple wave of rank r if a rank of the Jacobi matrix $\partial(u_1, u_2, \dots, u_m)/\partial(x_1, x_2, \dots, x_n)$ is equal to r in a domain G of the independent variables $x_1, x_2, \dots, x_n$. Here A_α are rectangular $N \times m$ matrices with elements $a_{ij}^{\alpha}(u)$ and $f = (f_1(u), \dots, f_N(u))$.

Depending on the value of r, a multiple wave is called a simple $(r = 1)$, double $(r = 2)$ or triple $(r = 3)$ wave. The value $r = 0$ corresponds to uniform flow with constant u_i, $(i = 1, 2, \dots, m)$, and $r = n$ corresponds to the general case of nondegenerate solutions. Multiple waves of all ranks compose a class of degenerate hodograph solutions.

The singularity of the Jacobi matrix means that the functions $u_i(x)$ $(i = 1, 2, \dots, m)$ are functionally dependent (hodograph is degenerate), with $m - r$ number of functional constraints

$$u_i = \Phi_i(\lambda^1, \lambda^2, \dots, \lambda^r), \quad (i = 1, 2, \dots, m). \qquad (2)$$

The variables $\lambda^1(x), \lambda^2(x), \dots, \lambda^r(x)$ are called parameters of the wave. The solutions with a degenerate hodograph are a generalization of travelling waves: the wave parameters of the travelling waves are linear forms of independent variables. To find the r-multiple wave, it is necessary to substitute the representation (2) into system (1). We get an overdetermined

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system of differential equations for the wave parameters $\lambda^i(x)$ $(i = 1, 2, \ldots, r)$, which should be studied for compatibility. A review of applications of multiple waves in gas dynamics can be found in [1].

The main problem of the theory of solutions with a degenerate hodograph is getting a closed system of equations in the space of dependent variables (hodograph), establishing the arbitrariness of the general solution and determining flow in the physical space.

An arbitrary nonhomogeneous system (1) does not change under the transformations

$$x_i' = x_i + b_i, \quad (i = 1, 2, \ldots, n),$$

that compose a group G^n. For homogeneous systems (1) $(f = 0)$, there is one more scale transformation[1] $x_i' = ax_i$ $(i = 1, 2, \ldots, n)$. From the group analysis point of view, an r-multiple wave is a partially invariant solution with respect to G^n (or G^{n+1}) [2]. A class of partially invariant solutions of some group H is characterized by rank σ and defect δ: class $H(\sigma, \delta)$-solutions. If some class $H(\sigma, \delta)$-solutions are class $H_1(\sigma, \delta_1)$-solutions with fewer defects $\delta_1 < \delta$, then it is said that the class $H(\sigma, \delta)$-solutions are reduced to having fewer defects. For example, if $\delta_1 = 0$, then such a solution is reducible to an invariant solution with respect to the subgroup H_1.

A study of partially invariant solutions shows that classes of solutions of a given rank with fewer defects are easier to obtain. This is connected with the idea that the analysis of compatibility for the solutions with greater defects is more difficult. Therefore, it is useful to a priori clarify the structure properties of the overdetermined system.

There are only a few sufficient conditions of the reducibility [2] that allow us to predict a reduction on the basis of the structure properties of an overdetermined system. One of these conditions is a restriction on the ability to define all first derivatives of a solution (otherwise the solution is reduced to an invariant solution). Others are concerned with double waves. If in the process of obtaining compatibility conditions for the wave parameters of a double wave, we obtain $N = 2n - 1$ homogeneous equations of type (1), then this double wave is an invariant solution. In particular, plane nonisobaric double waves with the general state equation which has a defect of invariance $\delta = 2$ are isoentropic [2]. Another application of these conditions to double waves of gas dynamics equations leads to the result [3] that the class of irreducible to invariant solutions of plane isoentropic irrotational double waves is described by the flows obtained in [4]. For homogeneous systems of type (1) with $N = 2n - 2$ and $n = 3$, a full classification of double waves with the additional assumption about having functional arbitrariness of the solution was carried out in [5].

This article is devoted to the study of nonhomogeneous systems of type (1) with $N = 2n - 1$ equations, the solutions of which are not reducible to invariant.

2. Nonhomogeneous Systems ($N = 2n - 1$)

Let a system of $N = 2n - 1$ independent autonomous quasilinear equations on the wave parameters λ and μ of a double wave be of type (1). It can be obtained as a result of substitution of the representation of a double wave:

$$u_i = u_i(\lambda, \mu), \quad (i = 1, 2 \ldots, m)$$

[1] The full Lie group admissible by system (1) can be wider than G^n (or G^{n+1}).

into the initial system and some analysis of compatibility.[2] Without loss of generality equations, for the wave parameters can be rewritten as

$$\begin{aligned} \lambda_i &= p_i(\lambda, \mu)\lambda_1 + f_i(\lambda, \mu), \\ \mu_j &= q_j(\lambda, \mu)\lambda_1 + g_j(\lambda, \mu), \quad (i = 1, \dots, n;\ j = 1, \dots, n). \end{aligned} \tag{3}$$

Here $\lambda_i = \partial\lambda/\partial x_i$, $\mu_j = \partial\mu/\partial x_j$ and, for the sake of simplicity, we set $p_1 \equiv 1$, $f_1 \equiv 0$.

The problem is to classify systems of type (3), the solutions of which are irreducible to invariant solutions.

A classification is derived with respect to equivalence transformations, admitted by system (3):

(a) linear nondegenerate replacement of independent variables;
(b) replacement of wave parameters: $\lambda' = L(\lambda, \mu)$, $\mu' = M(\lambda, \mu)$.

In the last case, the coefficients p_i, q_i and the functions f_i, g_i are transformed by formulae:

$$p_1' = 1, \quad p_i' = \frac{p_i L_\lambda + q_i L_\mu}{L_\lambda + q_1 L_\mu}, \quad q_j' = \frac{p_j M_\lambda + q_j M_\mu}{L_\lambda + q_1 L_\mu},$$

$$f_1' = 0, \quad f_i' = f_i L_\lambda + g_i L_\mu - g_1 L_\mu p_i', \quad g_j' = f_j M_\lambda + g_j M_\mu - g_1 L_\mu q_j'.$$

$$(i = 2, \dots, n;\ j = 1, \dots, n).$$

As a result of such transformations (as in the homogeneous case [2]), it is possible to let $q_1 = 0$. For this purpose, it is enough to choose a function $L(\lambda, \mu)$, which satisfies the equation $L_\lambda + q_1 L_\mu = 0$.

If $\sum_i q_i^2 \neq 0$, then the coefficients of system (3) can be transformed to

$$q_1 = 0, \quad q_2 = 1. \tag{4}$$

Simultaneous to the equalities $q_1 = 0$, $q_2 = 1$ under replacement of the wave parameters, iff

$$M_\lambda = 0, \quad L_\lambda = M_\mu,$$

results in

$$L = \lambda M'(\mu) + \omega(\mu), \quad M = M(\mu). \tag{5}$$

Another case corresponds to system (3) with

$$q_i = 0 \quad (i = 1, 2, \dots, n). \tag{6}$$

There is no case (6) for homogeneous systems, because conditions (6) contradict the definition of a double wave for such a kind of systems: rank of the Jacobi matrix is less than two.

A study of the compatibility of system (3) consists of the following. As a result of a reduction of the overdetermined system (3) to an involutive system, we get equations with a structure of nonhomogeneous quadratic forms with respect to the derivative λ_1. If at least

[2] A case of homogeneous $N = 2n - 1$ equations was studied by Ovsiannikov [2].

one of the coefficients of these forms is not equal to zero, then it means that a solution of the system satisfies the overdetermined system of equations from which all first derivatives can be found. By virtue of the reduction theorem [2], it gives the reduction of this solution to an invariant solution. Therefore, these forms are decomposed on subsystems on functions p_i, q_j, f_i, g_j: quadratic, linear and 'zero' terms with respect to power of the derivative λ_1. Further simplifications are connected with more the detailed study of the compatibility conditions of systems of types (4) and (6).

3. Systems of Type (4)

The value of $\lambda_{11} = a\lambda_1 + b$ can be defined from the expression $D_1(\mu_2 - \lambda_1 - g_2) - D_2(\mu_1 - g_1) = 0$, where D_i is a total derivative with respect to x_i, $a = p_2 g_{1\lambda} + g_{1\mu} - g_{2\lambda}$, $b = f_2 g_{1\lambda} + g_2 g_{1\mu} - g_1 g_{2\mu}$. It can be noted that all second derivatives λ_{ij} and μ_{ij} can be found. Therefore arbitrariness of the general solution of system of type (4) is only constant. For example, the derivatives

$$\lambda_{i1} = p_{i\lambda}\lambda_1^2 + \lambda_1(ap_i + f_{i\lambda} + g_1 p_{i\mu}) + bp_i + g_1 f_{i\mu}, \quad (i = 2, 3, \dots, n)$$

can be found from the expressions $D_1(\lambda_i - p_i\lambda_1 - f_i) = 0$. After substituting them into $F_i \equiv D_1\mu_i - D_i\mu_1 = 0$, $(i = 2, 3, \dots, n)$, we obtain nonhomogeneous quadratic forms with respect to the derivative λ_1. By virtue of the prohibition of reduction of the solution of system (3) to an invariant, the coefficients of these quadratic forms F_i have to be equal to zero:

$$q_{i\lambda} = 0, \tag{7}$$

$$q_i(p_2 g_{1\lambda} - g_{2\lambda}) + g_1 g_{i\mu} + g_{i\lambda} - p_i g_{1\lambda} = 0, \tag{8}$$

$$q_i b + g_1 g_{i\mu} - f_i g_{1\lambda} - g_i g_{1\mu} = 0, \quad (i = 2, 3, \dots, n). \tag{9}$$

In the same way from the quadratic forms $D_i\lambda_j - D_j\lambda_i = 0$, we get

$$q_j p_{i\mu} = q_i p_{j\mu}, \tag{10}$$

$$\begin{aligned} &f_j p_{i\lambda} + g_j p_{i\mu} + q_j f_{i\mu} + p_i g_1 p_{j\mu} = f_i p_{j\lambda} + g_i p_{j\mu} + q_i f_{j\mu} + p_j g_1 p_{i\mu},\\ &f_j f_{i\lambda} + g_j f_{i\mu} + p_i g_1 f_{j\mu} = f_i f_{j\lambda} + g_i f_{j\mu} + p_j g_1 f_{i\mu}, \quad (i, j = 2, 3, \dots, n;\ i \neq j). \end{aligned} \tag{11}$$

And from the equalities $D_i\mu_j - D_j\mu_i = 0$, we find

$$q_j(p_{i\lambda} - q_{j\mu}) = q_i(p_{j\lambda} - q_{j\mu}), \tag{12}$$

$$\begin{aligned} &g_j q_{i\mu} + q_i(p_j a + f_{j\lambda} + g_1 p_{j\mu}) + p_j g_{i\lambda} + q_j g_{i\mu}\\ &\quad = g_i q_{j\mu} + q_j(p_i a + f_{i\lambda} + g_1 p_{i\mu}) + p_i g_{j\lambda} + q_i g_{j\mu}, \end{aligned} \tag{13}$$

$$\begin{aligned} &q_i(p_j b + g_1 f_{j\mu}) + f_j g_{i\lambda} + g_j g_{i\mu}\\ &\quad = q_j(p_i b + g_1 f_{i\mu}) + f_i g_{j\lambda} + g_i g_{j\mu}, \quad (i, j = 2, 3, \dots, n;\ i \neq j). \end{aligned} \tag{14}$$

We note that the expressions $D_1\lambda_{i1} - D_i\lambda_{11} = 0$ are cubic polynomials with respect to the derivative λ_1: $p_{i\lambda\lambda}\lambda_1^3 + \cdots = 0$. Therefore,

$$p_{i\lambda\lambda} = 0, \quad (i = 2, 3, \ldots, n).$$

With the help of equivalence transformations (5) that leave the conditions $q_1 = 0$, $q_2 = 1$ unchanged, because of the choice of functions $\omega(\mu)$ and $\psi(\mu)$, we can assume that $p_2 = 0$. Then from (6), (10), (12), we get

$$q_{i\lambda} = 0, \quad p_{i\mu} = 0, \quad p_{i\lambda} = q_{i\mu}, \quad (i = 2, 3, \ldots, n). \tag{15}$$

By using (15) in the expressions $D_1\lambda_{i1} - D_i\lambda_{11} = 0$ $(i = 2, 3, \ldots, n)$, we find

$$q_i a_\mu = 2ap_{i\lambda} + f_{i\lambda\lambda}, \tag{16}$$

$$f_i a_\lambda + g_i a_\mu + q_i b_\mu = 3bp_{i\lambda} + g_1(p_i a_\mu + 2f_{i\lambda\mu}) + g_{1\lambda} f_{i\mu}, \tag{17}$$

$$ag_1 f_{i\mu} + b_\lambda f_i + g_i b_\mu = bf_{i\lambda} + g_1(p_i b_\mu + g_1 f_{i\mu\mu} + g_{1\mu} f_{i\mu}). \tag{18}$$

The functions p_i, q_j, f_i, g_j must satisfy (8), (9), (11), (14), (13), (15–18) for the irreducibility of solutions of system (3) to invariant solutions.

We note that

$$p_i = \lambda A_i + B_i, \quad q_j = \mu A_i + C_i, \quad (i = 2, 3, \ldots, n),$$

are the general solutions of Equations (15), where

$$A_1 = 0, \quad B_1 = 1, \quad C_1 = 0, \quad A_2 = 0, \quad B_2 = 0, \quad C_2 = 1,$$

and A_i, B_i, C_i $(i = 3, \ldots, n)$ are arbitrary constants. Further simplifications of equations of system (3) are connected with an application of equivalence transformations, which correspond to a replacement of the independent variables. By means of the replacement

$$x_1' = B_\alpha x_\alpha, \quad x_2' = C_\alpha x_\alpha, \quad x_i' = x_i, \quad (i = 3, 4, \ldots, n)$$

we can obtain $B_i = 0$, $C_i = 0$, $(i = 3, 4, \ldots, n)$.

Further, we have to consider two cases: (a) all $A_i = 0$ $(i = 3, 4, \ldots, n)$ and (b) $\sum_i A_i^2 \neq 0$.

In the first case (a), system (3) has the form

$$\begin{aligned} \lambda_2 &= f_2, \quad \lambda_i = f_i, \\ \mu_1 &= g_1, \quad \mu_2 = \lambda_1 + g_2, \quad \mu_i = g_i, \quad i \geq 3. \end{aligned} \tag{19}$$

In the second case (b), without loss of generality, we can regard $A_3 \neq 0$. Then as a result of one more linear transformation of the independent variables

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = A_\alpha x_\alpha, \quad x_i' = x_i, \quad (i = 4, 5, \ldots, n),$$

system (3) becomes

$$\begin{aligned} \lambda_2 &= f_2, \quad \lambda_3 = \lambda\lambda_1 + f_3, \quad \lambda_i = f_i, \\ \mu_1 &= g_1, \quad \mu_2 = \lambda_1 + g_2, \quad \mu_3 = \mu\mu_2 + g_3, \quad \mu_i = g_i, \quad i \geq 4. \end{aligned} \tag{20}$$

Further successive simplifications of systems (19) and (20) are connected with the analysis of the constants C_i.

3.1. System (19)

In this case, Equations (8), (9), (11), (14) are reduced to

$$g_i = C_i\mu + K_i, \quad f_i = C_i\lambda + R_i,$$
$$C_i(\lambda g_{1\lambda} + \mu g_{1\mu} - g_1) + R_i g_{1\lambda} + K_i g_{1\mu} = 0,$$
$$C_i(\lambda g_{2\lambda} + \mu g_{2\mu} - g_2) + R_i g_{2\lambda} + K_i g_{2\mu} = 0,$$
$$C_i(\lambda f_{2\lambda} + \mu f_{2\mu} - f_2) + R_i f_{2\lambda} + K_i f_{2\mu} = 0,$$
$$C_i R_j = C_j R_i, \quad C_i K_j = C_j K_i, \quad (i, j = 3, 4, \dots, n), \tag{21}$$

where C_i, R_i, K_i are arbitrary constants.

3.1.1. *Case* $C_3 \neq 0$

If at least one of the constants C_i is not equal to zero (without loss of generality, we can take $C_3 \neq 0$), then with the help of transformations

$$\lambda' = \lambda + \frac{R_3}{C_3}, \quad \mu' = \mu + \frac{K_3}{C_3},$$

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = \sum_{\alpha=3}^{n} C_\alpha x_\alpha, \quad x_i' = x_i, \quad (i = 4, \dots, n),$$

system (19) becomes

$$\lambda_3 = \lambda, \quad \mu_3 = \mu, \quad \lambda_i = 0, \quad \mu_i = 0, \quad (i = 4, 5, \dots, n),$$
$$\lambda_2 = \lambda F(\mu/\lambda), \quad \mu_1 = \lambda\Psi_1(\mu/\lambda), \quad \mu_2 = \lambda_1 + \lambda\Psi_2(\mu/\lambda). \tag{22}$$

The functions F, Ψ_1, Ψ_2 must satisfy a system of three ordinary differential equations of the second order. This system is obtained after substitution of

$$f_2 = \lambda F(\mu/\lambda), \quad g_1 = \lambda\Psi_1(\mu/\lambda), \quad g_2 = \lambda\Psi_2(\mu/\lambda),$$

into Equations (16–18):

$$\Psi_1'' + y\Psi_2'' - y^2 F'' = 0,$$
$$(y^2 F - y\Psi_2 - \Psi_1)F'' = 0, \quad (y^2 F - y\Psi_2 - \Psi_1)\Psi_2'' = 0,$$

where $y \equiv \mu/\lambda$.

It can be noted that system (22) is invariant with respect to the transformation: $\lambda' = -\lambda$, $\mu' = -\mu$. Therefore, we can consider that $\lambda > 0$. It allows one more simplification by transformation:

$$\lambda' = \frac{\mu}{\lambda}, \quad \mu' = \ln(\lambda), \quad x_1' = x_2, \quad x_2' = x_1, \quad x_i = x_i, \quad (i = 3, 4, \dots, n).$$

System (22) is reduced to

$$\lambda_2 + \lambda\lambda_1 = \hat{\Psi}_1(\lambda), \quad \lambda_i = 0, \quad (i = 3, 4, \ldots, n).$$
$$\mu_1 = F(\lambda), \quad \mu_2 = \lambda_1 + \hat{\Psi}_2(\lambda), \quad \mu_3 = 1, \quad \mu_i = 0, \quad (i = 4, \ldots, n). \tag{23}$$

Here $\hat{\Psi}_1(\lambda) = \Psi_1(\lambda) + \lambda\Psi_2(\lambda) - \lambda^2 F(\lambda)$, $\hat{\Psi}_2(\lambda) = -\Psi_2(\lambda) + \lambda F(\lambda)$.

Let us make some remarks about solutions of system (23). A solution of (23) has the form

$$\lambda = \Lambda(x_1, x_2), \quad \mu = x_3 + G(x_1, x_2),$$

where the function $G(x_1, x_2)$ can be found from the totally integrable compatible system of differential equations. These solutions are invariant solutions of Equations (23) with respect to algebra with generators:

$$\partial_{x_3} + \partial_\mu, \quad \partial_{x_i}, \quad (i = 4, \ldots, n). \tag{24}$$

Assume that the functions $\Lambda(x_1, x_2)$ and $G((x_1, x_2)$ are functionally dependent, then the Jacobian

$$W(x_1, x_2) = \frac{\partial(\lambda, \mu)}{\partial(x_1, x_2)} = \lambda_1^2 + \lambda_1(\hat{\Psi}_2 + \lambda F) - F\hat{\Psi}_1 = 0.$$

This equation supplies the sufficient conditions for the reducibility of the solution of system (23) to an invariant solution with respect to $H \subset G^n$. Therefore, for irreducible solutions, the functions $\Lambda(x_1, x_2)$ and $G((x_1, x_2)$ are functionally independent or $W(x_1, x_2) \neq 0$.

We note that if $\hat{\Psi}_1 \neq 0$, then functions F, Ψ_1, Ψ_2 are linear: $F = k_1\lambda + k_2$, $\Psi_2 = k_3\lambda + k_4$, $\Psi_2 = k_5\lambda + k_6$ with arbitrary constants k_i $(i = 1, 2, \ldots, 6)$. If $\hat{\Psi}_1 = 0$, then $\hat{\Psi}_2'(\lambda) + \lambda F'(\lambda) = 0$ and $\Lambda = x_1/x_2$ up to shifts of the independent variables and because of $W = x_2^{-2}(1 + x_2\hat{\Psi}_2 + x_1 F) \neq 0$, then the solution is not reducible to an invariant solution of $H \subset G^n$.

3.1.2. *Case* $C_i = 0$ $(i = 3, 4, \ldots, n)$

Let us consider the case with all constants zero, $C_i = 0$.

Firstly, assume that at least one of the constants K_i is not equal to zero (without loss of generality, we can consider that $K_3 \neq 0$). Then from (21) we get

$$g_1 = g_1(\lambda - R\mu), \quad g_2 = g_2(\lambda - R\mu), \quad f_2 = f_2(\lambda - R\mu),$$

where $R = R_3/K_3$. If $g_1' = g_2' = f_2' = 0$, then the solution of system (23) is linear with respect to the independent variables, i.e. it is invariant with respect to some subgroup $H \subset G^n$. Therefore a prohibition of reducibility to an invariant solution leads to conditions $(g_1')^2 + (g_2')^2 + (f_2')^2 \neq 0$ or from (21) we have $R_i = RK_i$. After the transformation

$$x_3' = \sum_{i=3}^{n} K_i x_i, \quad x_i' = x_i, \quad i \neq 3$$

we obtain $f_3 = R$, $g_3 = 1$, $g_i = 0$, $f_i = 0$, $(i = 4, 5, \ldots, n)$. In addition we can reckon that $R = 0$. Really, if it is not so, then after one more transformation

$$\lambda' = \lambda - R\mu, \quad \mu' = R\mu,$$
$$x_1' = R^{-1}x_1 - x_2, \quad x_2' = x_2, \quad x_3' = Rx_3,$$

the same system can be obtained, but with $R = 0$. Irreducibility conditions (16–18) in this case become

$$f_2 = k_1\lambda + k_2, \quad g_1'' f_2 = 0, \quad g_2'' f_2 = 0$$

with arbitrary constants k_1, k_2. We note that if $f_2 = 0$ ($k_1 = 0, k_2 = 0$), then a solution of (19) is $\lambda = \varphi(x_1)$, $\mu = x_3 + cx_2 + \psi(x_1)$, which is invariant with respect to some subalgebra $H \subset G''$. Here c is a constant. Therefore, for systems irreducible to invariant solutions, we have to consider only the case when $f_2 \neq 0$. In this case, functions g_1 and g_2 are linear $g_1 = k_3\lambda + k_4$, $g_2 = k_5\lambda + k_6$ and system (19) is

$$\begin{aligned} \lambda_2 &= k_1\lambda + k_2, \quad \lambda_i = 0, \quad (i = 3, 4, \ldots, n), \\ \mu_1 &= k_3\lambda + k_4, \quad \mu_2 = \lambda_1 + k_5\lambda + k_6, \quad \mu_3 = 1, \quad \mu_j = 0, \quad (j = 4, 5, \ldots, n). \end{aligned} \tag{25}$$

If $k_1 \neq 0$, then by equivalence transformations we can consider that $k_1 = 1$, $k_2 = 0$. In this case

$$\lambda = \varphi(x_1)e^{x_2}, \quad \mu = (\varphi' + k_5\varphi)e^{x_2} - k_6x_2 + x_3,$$

where the function $\varphi = \varphi(x_1)$ satisfies the homogeneous linear ordinary differential equation

$$\varphi'' - k_3\varphi' + k_5\varphi = 0.$$

If $k_1 = 0$, but $k_2 \neq 0$, then, as in previous case, via equivalence transformations we can put $k_1 = 0$, $k_2 = 1$. And then

$$\lambda = x_2 + \varphi(x_1), \quad \mu = x_3 + x_2\left(\varphi' + \frac{k_5}{2}x_2 + k_5\varphi + k_6\right) + \psi,$$

where the functions $\varphi = \varphi(x_1)$ and $\psi = \psi(x_1)$ satisfy the ordinary differential equations

$$\varphi'' + k_5\varphi' - k_3 = 0, \quad \psi' = k_3\varphi + k_4.$$

Now let all constants $K_i = 0$. If at least one of the constants R_i is not equal to zero (without loss of generality, we can account that $R_3 \neq 0$), then by transformation

$$\lambda' = \mu, \quad \mu' = \lambda, \quad x_1' = x_2, \quad x_2' = x_1, \quad x_i = x_i, \quad (i = 3, 4, \ldots, n),$$

the same system is obtained as was considered in the previous case. If all $R_i = 0$, then for such a solution

$$\lambda = \Lambda(x_1, x_2), \quad \mu = G(x_1, x_2)$$

and it is invariant with respect to the subalgebra $H \subset G''$, which corresponds to the subalgebra $\{\partial_{x_3}, \partial_{x_4}, \ldots, \partial_{x_n}\}$.

3.2. SYSTEM (20)

A study of compatibility of system (20) is more cumbersome. In this case, Equations (8), (9), (11), (14), (16–18) can be reduced to

$$
\begin{aligned}
&g_{3\lambda} = \lambda g_{1\lambda} + \mu g_{2\lambda} - g_1,\\
&s_2 \equiv \mu b + g_1 g_{3\mu} - f_3 g_{1\lambda} - g_3 g_{1\mu} = 0,\\
&f_{3\mu} = \mu f_{2\mu} - f_2,\\
&f_2 f_{3\lambda} + g_2 f_{3\mu} + \lambda g_1 f_{2\mu} = f_3 f_{2\lambda} + g_3 f_{2\mu},\\
&g_2 + \mu f_{2\lambda} + g_{3\mu} = \lambda g_{2\lambda} + \mu g_{2\mu} + f_{3\lambda},\\
&s_6 \equiv \mu g_1 f_{2\mu} + f_2 g_{3\lambda} + g_2 g_{3\mu} - (f_3 g_{2\lambda} + g_3 g_{2\mu} + \lambda b + g_1 f_{3\mu}),\\
&f_i = 0, \quad g_i = 0, \quad (i = 4, 5, \ldots, n),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&a_\mu = f_{2\lambda\lambda},\\
&\mu a_\mu = 2a + f_{3\lambda\lambda},\\
&f_2 a_\lambda + g_2 a_\mu + b_\mu = g_1(2 f_{2\lambda\mu}) + g_{1\lambda} f_{2\mu},\\
&f_3 a_\lambda + g_3 a_\mu + \mu b_\mu = 3b + g_1(\lambda a_\mu + 2 f_{3\lambda\mu}) + g_{1\lambda} f_{3\mu},\\
&a g_1 f_{2\mu} + b_\lambda f_2 + g_2 b_\mu = b f_{2\lambda} + g_1(g_1 f_{2\mu\mu} + g_{1\mu} f_{2\mu}),\\
&a g_1 f_{3\mu} + b_\lambda f_3 + g_3 b_\mu = b f_{3\lambda} + g_1(\lambda b_\mu + g_1 f_{3\mu\mu} + g_{1\mu} f_{3\mu}).
\end{aligned}
\tag{27}
$$

The problem is to find a general solution (up to equivalence transformation) of system (26), (27). Because Equations (26) and (27) are not sufficient for irreducibility of a solution of system (20) to invariant solution, then the next problem is to try to analyze a solution of (20) with the found functions f_i, g_j and coefficients p_i, q_j.

All further intermediate calculations in the study of the compatibility of system (26) were made on a computer using the system REDUCE [6]. Here we give the method of computations and final results.

Let us input the new function $G_3 = g_3 - \mu g_2$ instead of g_3. From $(26)_1$ and $(26)_5$, we find $G_{3\lambda}$, $G_{3\mu}$ and from $(27)_1$: $f_{2\lambda\lambda}$ and $f_{3\lambda\lambda}$. After substitution of the found expressions into $\partial G_{3\lambda}/\partial\mu - \partial G_{3\mu}/\partial\lambda = 0$, we get the equation $(\lambda(g_{1\mu} - g_{2\lambda}))_\lambda = 0$. Without loss of generality, the last equation can be integrated:

$$
g_1 = \varphi_\lambda, \quad g_2 = \varphi_\mu + \psi_1 \log \lambda,
\tag{28}
$$

where $\varphi = \varphi(\lambda, \mu)$ and $\psi_1 = \psi_1(\mu)$ are arbitrary functions. After substitution of (28) into expressions for $f_{2\lambda\lambda}$ and $f_{3\lambda\lambda}$, we get

$$
f_{2\lambda\lambda} = -\frac{\psi_1'}{\lambda}, \quad f_{3\lambda\lambda} = \frac{2\psi_1 - \mu\psi_1'}{\lambda}.
$$

Integration of the last expressions allows us to find the functions

$$
f_2 = \lambda\psi_1'(1 - \log\lambda) + \lambda\psi_2 + \psi_3, \quad f_3 = \lambda(\mu\psi_1' - 2\psi_1)(1 - \log\lambda) + \lambda\psi_4 + \psi_5
$$

with arbitrary functions $\psi_i = \psi_i(\mu)$ $(i = 2, 3, 4, 5)$. From $(26)_3$, we have

$$\lambda(\psi_2 + \psi_4' - \mu\psi_2') + \psi_3 + \psi_5' - \mu\psi_3' = 0.$$

After splitting with respect to λ, we get

$$\psi_4' = \mu\psi_2' - \psi_2, \quad \psi_5' = \mu\psi_3' - \psi_3$$

or, if we input a new function $\psi_6 = \psi_6(\mu)$ by $\psi_4 = \psi_6' + \mu\psi_2 - \psi_1$, then $\psi_2 = (\psi_1' - \psi_6'')/2$. In this case,

$$\frac{\partial G_3}{\partial \lambda} = -\varphi_\lambda + \lambda\varphi_{\lambda\lambda}, \quad \frac{\partial G_3}{\partial \mu} = -2\varphi_\lambda + \lambda\varphi_{\lambda\mu} + \psi_6'$$

which can be integrated as $G_3 = -2\varphi + \lambda\varphi_\lambda + \psi_6$.

A composition of differentiating $(26)_6$ with respect to λ and subtracting it by differentiating $(26)_2$ with respect to μ and adding it to $(27)_3$ is

$$\psi_1\varphi_{\lambda\mu} - \psi_1'\varphi_\lambda + \frac{\psi_1}{\lambda} = 0.$$

If $\psi_1 \neq 0$, then we can get a contradiction. Really, let $\psi_1 \neq 0$, then the last equation can be integrated

$$\varphi = \psi_1(G - \mu \log \lambda) + \psi_7,$$

where $G = G(\lambda)$ and $\psi_7 = \psi_7(\mu)$ are arbitrary functions. In this case, Equation $(26)_4$ has the form

$$G(a_1\lambda \log \lambda + a_2\lambda + a_3) + a_4\lambda \log^2 \lambda + a_5\lambda \log \lambda + a_6\lambda + a_7 \log \lambda + a_8 = 0. \qquad (29)$$

where a_i, $(i = 1, 2, \ldots, 8)$ are polynomials of functions $\psi_1, \psi_3, \psi_5, \psi_6, \psi_7$ and their derivatives. It can be shown that (29) is possible only if $\psi_1 = 0$. But it contradicts the original assumption about ψ_1. Therefore, we have to consider $\psi_1 = 0$.

Further consideration is based on the analysis of the compatibility of Equations $(26)_4$ and $\partial s_2/\partial\mu - \partial s_6/\partial\lambda = 0$, which have the forms:

$$\varphi_\mu h - 2\varphi h' + \psi_6 h' - \psi_3(\mu\psi_6'' - 2\psi_6') + \psi_5\psi_6'' = 0. \qquad (30)$$

$$-3\varphi_\lambda\varphi_{\mu\mu} + \varphi_\lambda\psi_6'' + 3\varphi_\mu\varphi_{\lambda\mu} - \varphi_{\lambda\lambda}h = 0. \qquad (31)$$

where $h = \lambda\psi_6'' - 2\psi_3$.

Assume that $h = 0$, so $\psi_3 = 0$, $\psi_6 = c_1\mu + c_2$, where c_1 and c_2 are constants. We note that in this case $\psi_5' = 0$. Analysis of (31) requires that we need to study two cases: (a) $\varphi_\mu = 0$ and (b) $\varphi_\mu \neq 0$.

Let $\varphi_\mu = 0$, then from (31) we get

$$(c_1\lambda + \psi_5)\varphi_{\lambda\lambda} - c_1\varphi_\lambda = 0.$$

If $c_1 \neq 0$, then without loss of generality, system (20) can be written as

$$\lambda_2 = 0, \quad \lambda_3 = \lambda\lambda_1 + \lambda, \quad \lambda_i = 0,$$
$$\mu_1 = 2c\lambda, \quad \mu_2 = \lambda_1, \quad \mu_3 = \mu\lambda_1 + \mu + c_2, \quad \mu_i = 0, \quad i \geq 4. \qquad (32)$$

A solution of this system is

$$\lambda = -x_1\phi(x_3), \quad \mu = (cx_1^2 + x_2 + c_2e^{x_3})\phi(x_3),$$

where $\phi(x_3) = e^{x_3}/(e^{x_3} - 1)$.

If $c_1 = 0$ and $\psi_5 \neq 0$, then without loss of generality, system (20) can be written as

$$\lambda_2 = 0, \quad \lambda_3 = \lambda\lambda_1 + 1, \quad \lambda_i = 0,$$
$$\mu_1 = c, \quad \mu_2 = \lambda_1, \quad \mu_3 = \mu\lambda_1 - c\lambda + c_2, \quad \mu_i = 0, \quad i \geq 4. \tag{33}$$

A solution of this system is

$$\lambda = -\frac{x_1}{x_3} + \frac{x_3}{2}, \quad \mu = c\left(x_1 - \frac{x_3^2}{6}\right) - \frac{x_2}{x_3},$$

where c is an arbitrary constant.

If $c_1 = 0$ and $\psi_5 = 0$, then without loss of generality, system (20) can be written as

$$\lambda_2 = 0, \quad \lambda_3 = \lambda\lambda_1, \quad \lambda_i = 0,$$
$$\mu_1 = \varphi', \quad \mu_2 = \lambda_1, \quad \mu_3 = \mu\lambda_1 + \lambda\varphi' - 2\varphi, \quad \mu_i = 0, \quad i \geq 4, \tag{34}$$

where $\varphi = \varphi(\lambda)$ is an arbitrary function of λ. A solution of this system is

$$\lambda = -\frac{x_1}{x_3}, \quad \mu = -\frac{x_2}{x_3} - x_3\varphi(\lambda).$$

Let $\varphi_\mu \neq 0$, then from (31) we get $\varphi = F(\xi)$, where $\xi = \mu + \psi(\lambda)$. The functions $\psi(\lambda)$ and $F(\xi)$ are functions of one argument ($F' \neq 0$), which have to satisfy the equations

$$\psi''(c_1\lambda + \psi_5) = 0, \quad F''(2F - c_1\xi - c_3) + c_1F' - (F')^2 = 0.$$

Here, by virtue of the first equation, $c_3 \equiv \psi'(c_1\lambda + \psi_5) - c_1\psi$ is a constant.

If $c_1 \neq 0$, then as a result of equivalence transformations, we can set $c_1 = 1$, $\psi_5 = 0$, $\psi = 0$, and system (20) can be written as

$$\lambda_2 = 0, \quad \lambda_3 = \lambda\lambda_1 + \lambda, \quad \lambda_i = 0,$$
$$\mu_1 = 0, \quad \mu_2 = \lambda_1 + F', \quad \mu_3 = \mu\lambda_1 + \mu + \mu F' - 2F, \quad \mu_i = 0, \quad i \geq 4, \tag{35}$$

where the function $F = F(\mu)$ satisfies

$$(\mu - 2F)F'' = F'(1 - F'), \quad (F' \neq 0).$$

A solution of this system is

$$\lambda = \frac{x_1 e^{x_3}}{1 - e^{x_3}}, \quad \mu = \mu(x_2, x_3),$$

where the function $\mu(x_2, x_3)$ satisfies a compatible overdetermined system of equations.

If $c_1 = 0$ and $\psi_5 \neq 0$, then without loss of generality and because of equivalence transformations, system (20) can be written as

$$\lambda_2 = 0, \quad \lambda_3 = \lambda\lambda_1 + 1, \quad \lambda_i = 0,$$
$$\mu_1 = 0, \quad \mu_2 = \lambda_1 + 2c\mu, \quad \mu_3 = \mu\lambda_1, \quad \mu_i = 0, \quad i \geq 4, \tag{36}$$

where $c \neq 0$ is a constant. The solution of this system (up to scaling x_1, x_2, x_3 and μ) is

$$\lambda = -\frac{x_1}{x_3} + x_3, \quad \mu = \frac{1}{x_3}(\gamma\, e^{x_2} + 1),$$

where $\gamma = 0$ or $\gamma = 1$. If $\gamma = 0$, then the solution is invariant with respect to the subalgebra $\partial_{x_2}, \partial_{x_i}, (i = 4, 5, \ldots, n)$.

If $c_1 = 0$ and $\psi_5 = 0$, then without loss of generality, system (20) can be written as

$$\begin{aligned}
\lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1, \quad \lambda_i = 0,\\
\mu_1 &= \psi' F', \quad \mu_2 = \lambda_1 + F',\\
\mu_3 &= \mu\lambda_1 + (\mu + \psi'\lambda)F' - 2F, \quad \mu_i = 0, \quad i \geq 4,
\end{aligned} \tag{37}$$

where $\psi = \psi(\lambda)$ is an arbitrary function, $F = c(\xi + c_3)^2$, $\xi = \mu + \psi(\lambda)$ and c, c_3 are constants ($c \neq 0$). With the help of equivalence transformation, this system can be simplified to

$$\begin{aligned}
\lambda_2 &= 0, \quad \lambda_3 = \lambda\lambda_1, \quad \lambda_i = 0,\\
\mu_1 &= \psi'(\mu + \lambda_1), \quad \mu_2 = \lambda_1 + \mu,\\
\mu_3 &= \mu\lambda_1 + (\lambda\psi' - \psi)(\mu + \lambda_1), \quad \mu_i = 0, \quad i \geq 4.
\end{aligned} \tag{38}$$

The general solution of this system is (up to equivalence transformation)

$$\lambda = -\frac{x_1}{x_3}, \quad \mu = \frac{1}{x_3}(\gamma\, e^{x_2 - x_3\psi} + 1),$$

where $\gamma = 0$ or $\gamma = 1$. If $\gamma = 0$, then the solution is invariant with respect to the subalgebra $\partial_{x_2}, \partial_{x_i}, (i = 4, 5, \ldots, n)$.

Now we consider the case $h \equiv \lambda\psi_6'' - 2\psi_3 \neq 0$.

Let $\psi_6'' \neq 0$, then system (30), (31) is compatible (up to equivalence transformations) only if system (20) has the form

$$\begin{aligned}
\lambda_2 &= (\lambda + \alpha)\mu, \quad \lambda_3 = \lambda\lambda_1, \quad \lambda_i = 0,\\
\mu_1 &= 0, \quad \mu_2 = \lambda_1 + \mu(\mu + \beta), \quad \mu_3 = \mu\lambda_1, \quad \mu_i = 0, \quad i \geq 4,
\end{aligned} \tag{39}$$

where α, β are constants. A solution of this system depends on β.

If $\beta \neq 0$, then the solution is (up to equivalence transformation)

$$\lambda = \frac{x_1 - \alpha\gamma\, e^{x_2}}{\gamma\, e^{x_2} - x_3}, \quad \mu = -\frac{1 + \beta^2\gamma\, e^{x_2}}{\gamma\, e^{x_2} - x_3},$$

where $\gamma = 0$ or $\gamma = 1$. If $\gamma = 0$, then the solution is invariant with respect to the subalgebra $\partial_{x_2}, \partial_{x_i}, (i = 4, 5, \ldots, n)$.

If $\beta = 0$, then the solution is (up to equivalence transformation)

$$\lambda = -\frac{x_1 + \alpha x_2^2}{x_3 + x_2^2}, \quad \mu = -\frac{x_2}{x_3 + x_2^2}.$$

Let $\psi_6'' = 0$ or $\psi_6 = c_1\mu + c_2$ and $\psi_3 \neq 0$. Changing the function φ to $Q(\lambda, \mu) = (\varphi - \psi_6/2)/h^2$ simplifies Equations (30) and $(27)_3$, further. Equation $(27)_3$ can be integrated:

$$\frac{\partial Q}{\partial \lambda} = 6Q^2 \frac{\psi_3 \psi_3'' - (\psi_3')^2}{\psi_3} - 3Q \frac{c_1 \psi_3'}{2\psi_3^2} + \psi_8,$$

where $\psi_8 = \psi_8(\mu)$. Then from these two equations by cross-differentiating, we get

$$AQ^2 + BQ + C = 0,$$

where $A = 6\psi_3^2(\psi_3^2\psi_3''' - 2\psi_3\psi_3'\psi_3'' + (\psi_3')^3)$, $B = 3c_1\psi_3(\psi_3')^2/2$, $C = \psi_8'\psi_3^4 - 3c_1^2\psi_3'/16$.

Further analysis depends on the value of Q_λ. There are only two possibilities: (a) $A = 0$, $B = 0$, $C = 0$ and (b) $Q_\lambda = 0$.

In case (a), because $B = 0$, we need to consider two cases. In the first case $\psi_3' = 0$, and then, without loss of generality, system (20) can be reduced to

$$\begin{aligned} \lambda_2 &= 1, \quad \lambda_3 = \lambda(\lambda_1 + c_1) - \mu + c_2, \\ \mu_1 &= k, \quad \mu_2 = \lambda_1 + c_1, \quad \mu_3 = \mu\lambda_1 - k\lambda + k_1, \end{aligned} \tag{40}$$

where k and k_1 are constants and c_1 attains two values: either $c_1 = 1$ or $c_1 = 0$. In the second case, $c_1 = 0$, and without loss of generality, the system (20) can be reduced to

$$\begin{aligned} \lambda_2 &= -\frac{1}{2}(\mu - k)^2, \quad \lambda_3 = \lambda\lambda_1 - \frac{1}{6}(\mu + 2k)(\mu - k)^2, \\ \mu_1 &= \frac{(\mu - k)^4}{6(\lambda - k_1)^2}, \quad \mu_2 = \lambda_1 - \frac{2(\mu - k)^3}{3(\lambda - k_1)}, \\ \mu_3 &= \mu\lambda_1 - \frac{(\mu - k)^2(\lambda\mu + 3k\lambda - 2k_1\mu - 2kk_1)}{6(\lambda - k_1)^2}, \end{aligned} \tag{41}$$

where k and k_1 are constants.

Let us now consider case (b) $Q_\lambda = 0$. From $s_6 = 0$ we get $Q\psi_3'' = 0$. If $c_1 = 0$, then system (20) can be reduced to

$$\begin{aligned} \lambda_2 &= \psi_3, \quad \lambda_3 = \lambda\lambda_1 + \psi_5, \\ \mu_1 &= 0, \quad \mu_2 = \lambda_1 + k\psi_3\psi_3', \quad \mu_3 = \mu\lambda_1 + k\psi_3\psi_5', \end{aligned} \tag{42}$$

where k is a constant and ψ_3 is an arbitrary function of one argument and the function ψ_5 is connected with ψ_3 by: $\psi_5' = \mu\psi_3' - \psi_3$. If $c_1 \neq 0$, then system (20) can be reduced to

$$\begin{aligned} \lambda_2 &= 1, \quad \lambda_3 = \lambda(\lambda_1 + 1) - \mu + k_1, \\ \mu_1 &= 0, \quad \mu_2 = \lambda_1 + 1, \quad \mu_3 = \mu\lambda_1 + k, \end{aligned} \tag{43}$$

where k and k_1 are constants.

We can thus formulate the following theorem:

THEOREM. *System (19) can have solutions irreducible to invariant solutions only if it is equivalent to one of the systems: (23), (25), (32–36), (37) (or (38)).*

4. Systems of Type (6)

Systems of the type (6) have the form

$$\lambda_i = p_i(\lambda, \mu)\lambda_1 + f_i(\lambda, \mu), \quad \mu_j = g_j(\lambda, \mu), \quad (i = 1, \ldots, n;\ j = 1, \ldots, n). \tag{44}$$

As with systems of type (4), we can obtain the necessary irreducibility conditions from expressions $D_i\mu_j - D_j\mu_i = 0$:

$$g_{i\lambda} = p_i g_{1\lambda}, \quad g_{i\mu}g_1 = f_i g_{1\lambda} + g_i g_{1\mu}, \quad (p_j f_i - p_i f_j)g_{1\lambda} + g_i g_{j\mu} - g_j g_{i\mu} = 0. \tag{45}$$

and

$$(p_i p_{j\mu} - p_j p_{i\mu})g_1 + p_{i\lambda} f_j + p_{i\mu} g_j - p_{j\lambda} f_i - p_{j\mu} g_i = 0,$$

$$(p_i f_{j\mu} - p_j f_{i\mu})g_1 + f_{i\lambda} f_j + f_{i\mu} g_j - f_{j\lambda} f_i - f_{j\mu} g_i = 0, \tag{46}$$

from expressions $D_i\lambda_j - D_j\lambda_i = 0$. Here $i, j = 2, 3, \ldots, n$.

Assume that $g_1 \neq 0$. If $g_{1\lambda} = 0$, then without loss of generality, we can consider $g_1 = 1$. In this case, from (45) we can conclude that g_i, $(i, j = 2, 3, \ldots, n)$ are constants, even up to equivalence transformations we can regard them as $g_i = 0$, $(i, j = 2, 3, \ldots, n)$. Solution of such a system is $\mu = x_1$, which is partially invariant with defect $\delta \leq 1$. It is possible to obtain a further simplification of system (44).

If $g_{1\lambda} \neq 0$, then without loss of generality we can consider $g_1 = \lambda$. Because in this case, from (45) we have

$$p_i = g_{i\lambda}, \quad f_i = \lambda g_{i\mu}, \quad (i = 2, 3, \ldots, n).$$

It gives that the first $n - 1$ equations $\lambda_i = p_i\lambda_1 + f_i$, 0, $(i, j = 2, 3, \ldots, n)$ are consequences of the other equations. But we have assumed that the equations of system (44) are not dependent.

If $g_1 = 0$, then without loss of generality we can consider that $g_2 = 1$. From (45) and changing the independent variables, we can obtain $g_j = 0$, $(j = 3, 4, \ldots, n)$. The solution of such a system is $\mu = x_2$, which is partially invariant with defect $\delta \leq 1$. As before, it is possible for a further simplification of system (44).

5. Conclusion

In this paper, the classification of systems of type (3) with $N = 2n - 1$ for double waves of nonhomogeneous quasilinear equations is performed.

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