# Group Classification and Representations
# of Invariant Solutions
# of the full Boltzmann Equation

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**Abstract.** Group analysis developed especially for differential equations allows systematic study of solutions of the full Boltzmann kinetic equation. The study is connected with the admitted Lie group of infinitesimal transformations. Group classification of admitted group gives all representations of essentially different invariant solutions. Ususally the group classification is quite difficult to do: it requires the application of special methods developed in group analysis. The lucky fact for the Boltzmann equation is that this equation and the system of gas dynamics equations admit isomorphic Lie groups. It allows using results of the group classification obtained for the gas dynamics equations.

In this report the representations of all invariant solutions of the full Boltzmann equation and its Fourier representation when they are reduced to the equations with one or two independent variables were constructed.

## INTRODUCTION

The construction of exact solutions of the Boltzmann equation still attracts attention of scientists. This equation is difficult for studying because of its nonlinearity and presence of many independent variables. Almost all exact solutions of differential and integro–differential equations were found by reducing the number of independent variables: the reduced systems were obtained by assuming a representation of the solution. There are some approaches for constructing a representation of solution: most of them are *ad hoc* methods. One of the regular methods which allows forming the representation is a group analysis. An application of it consists of some steps. Shortly they can be described as: finding an admitted Lie group $G$, classifying of all subgroups (construction of optimal system of subgroups), seeking of invariant or partially invariant solutions.

We note that the group analysis method was developed especially for differential equations. An application of it to integro–differential equations presents some difficulties. The main one arises from the integral (nonlocal) terms presented in these equations. There are several ways by which one can overcome these difficulties: short review of them can be found in [1].

As it was mentioned the first step of applying the group analysis is a finding of admitted Lie group. Earlier symmetry and invariant H-solutions of the full Boltzmann equation (BE) of the kinetic gas theory were studied by *ad hoc* methods. In most of these studies a form of admitted transformation was postulated a priori (see [2,3] for a review). By authors of [4–6] it was proved that the full BE admits an 11-parameter Lie group $G^{11}$ of point transformations. Some extensions of the $G^{11}$ group for special intermolecular potentials are also known. We should note here that the question about completeness of the found admitted groups is open up to now. In spite of this fact it is very useful to carry out a classification of the set of H-solutions for a constructive description of invariance with respect to these Lie group solutions of the BE. This classification allows separating the set of H-solutions into non-intersecting essentially different classes, obtaining representations of the H-solutions for different classes and reducing the full BE to factor–equations. The classification demands a construction of an optimal system of subgroups (subalgebras) of an admitted Lie group (algebra) [7]. The general algorithms for

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constructing such systems are known in group theory. However, their realization for large dimension groups such as $G^{11}$ requires extremely long and tedious calculations. For example, one can point to the Euler gas dynamics (EGD) system of equations that admits Lie groups of similar dimensions for different state equations. Despite the fact that these admitted groups were obtained in the seventies [7] their optimal systems were only calculated in the last 5 years [8–10].

The lucky fact for the full BE is that there is the isomorphism of the Lie algebras $L^{11}$ admitted by the full Boltzmann kinetic equation with arbitrary differential cross section and by the EGD–system of equations with general state equation [11]. Also there are isomorphisms of the extensions of $L^{11}$ up to the $L^{12}$ and $L^{13}$ for specified intermolecular potentials and studied in gas dynamics admitted Lie algebras for polytropic gas. The isomorphisms allows solving a problem of classification of invariant H–solutions of the full Boltzmann equation by using optimal systems of subalgebras known for the Euler system.

In this report the isomorphisms of Lie groups (algebras) admitted by the full BE and EGD–system is set up. The proved isomorphism allows using the optimal systems of subalgebras obtained for the EGD–system in the papers cited above for classifications of invariant solutions of the full BE and obtaining on this basis representations of essentially different H–solutions of the spatially inhomogeneous Boltzmann equation with one and two independent invariant variables in explicit form.

## AN ADMITTED LIE ALGEBRA

The full BE that describes evolution of the distribution function $f(t, \mathbf{x}, \mathbf{v})$ in the product space $R_+^1 \times R_{\mathbf{x}}^3 \times R_{\mathbf{v}}^3$ is as follows [12]

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} = I(f, f), \tag{1}$$

$$J(f, f) = \int d\mathbf{w} \, d\mathbf{n} g \sigma(g, \frac{\mathbf{gn}}{g})[f(\mathbf{v}^*)f(\mathbf{w}^*) - f(\mathbf{v})f(\mathbf{w})], \tag{2}$$

$$\mathbf{v}^* = \frac{1}{2}(\mathbf{v} + \mathbf{w} + g\mathbf{n}), \ \mathbf{w}^* = \frac{1}{2}(\mathbf{v} + \mathbf{w} - g\mathbf{n}), \mathbf{u} = \mathbf{v} - \mathbf{w}, g = |\mathbf{v} - \mathbf{w}| = |\mathbf{v}^* - \mathbf{w}*|, |\mathbf{n}| = 1.$$

Here $t \in R_+^1$ is time, $\mathbf{x} = (x, y, z) \in R_{\mathbf{x}}^3$ is space variable and $\mathbf{v} = (u, v, w) \in R_{\mathbf{v}}^3$ is molecular velocity; $\sigma(g, \frac{\mathbf{gn}}{g})$ is a differential scattering cross section, $d\mathbf{w}$ denotes an volume element of $R_{\mathbf{v}}^3$, $d\mathbf{n}$ is a surface element of unit sphere in $R^3$.

For power intermolecular potentials $U(r) \propto r^{-(\nu-1)} (\nu > 2)$ there is $\sigma = g^\gamma I(\frac{\mathbf{gn}}{g})$, $\gamma = (\nu - 5)/(\nu - 1)$. A value $\nu = 5$ ($\gamma = 0$) corresponds to Maxwellian molecules and a limit $\nu \to \infty$ corresponds to hard sphere molecules.

In [5] an admitted Lie group $G(T_a)$ of point transformations $T_a$ of the BE (1) were being looked for in the form

$$f = \varphi(t, \mathbf{x}, \mathbf{v}; a)f', \ t' = \tau(t, \mathbf{x}; a), \ \mathbf{x}' = \mathbf{h}(t, \mathbf{x}; a), \quad \mathbf{v}' = B(t, \mathbf{x}; a)\mathbf{v} + \mathbf{b}(t, \mathbf{x}; a), \tag{3}$$

where $a$ is a group parameter, $B$–some $3 \times 3$ matrix. A feature of this group is that the nonlinear integral collision operator has the generalized "scaling" property

$$J(f', f') = \psi(t', \mathbf{x}', \mathbf{v}')J(f, f) \tag{4}$$

Undefined functions in (3) were found from the main property of an admitted Lie group: the BE (1) admits a Lie group $G(T_a)$ if for each $a$ a transformation (3) converts any solution of the BE into some solution of the same equation. Because there is a one–to–one correspondence between a Lie group and a Lie algebra further we deal with Lie algebras. The result of the searching of the admitted Lie group (3) is the following.

For arbitrary cross section $\sigma$ there is the Lie algebra $L^{11}(X)$ with the basis of infinitesimal generators:

$$X_1 = \partial_x, \ X_2 = \partial_y, \ X_3 = \partial_z, \ X_4 = t\partial_x + \partial_u, \ X_5 = t\partial_y + \partial_v, \ X_6 = t\partial_z + \partial_w,$$

$$X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v, \ X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,$$

$$X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u, \quad X_{10} = \partial_t, \quad X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z - f\partial_f.$$

The generators $X_1, X_2, X_3, X_{10}$ correspond to simple shifts with respect to the space variables and time; the generators $X_4, X_5, X_6$ correspond to the Galilean transformations; the generators $X_7, X_8, X_9$ correspond to rotations; the generator $X_{11}$ corresponds to a scale transformation.

For the power intermolecular potentials there is an extension of the Lie algebra $L^{11}(X)$ to the algebra $L^{12}(X)$ by the generator

$$X_{12} = t\partial_t - u\partial_u - v\partial_v - w\partial_w + (\gamma + 2)f\partial_f$$

And for a special case $\gamma = -1$ there is one more generator:

$$X_{13} = t^2\partial_t + t\mathbf{x}\partial_{\mathbf{x}} + (\mathbf{x} - t\mathbf{v})\partial_{\mathbf{v}},$$

which corresponds to a projective transformation.

Here an action of the derivative $\partial_f$ in the generators $X_{11}$ and $X_{12}$ onto integral operator (2) has to be considered as the Freschet derivative.

**Remark 1**. The representation of the admitted group (3) was indirectly confirmed by applying the direct method [5,13] for finding an admitted group, which was applied to the Fourier–representation of the spatially homogeneous and isotropic Boltzmann equation

$$\Phi \equiv \frac{\partial \varphi(x,t)}{\partial t} + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t) \ ds = 0. \tag{5}$$

An another application of the direct method was for the similar system of equations, which describes multi-component gas mixture [14].

**Remark 2**. In [4] the Lie subalgebra with the generators $X_1, X_2, X_3, X_4, X_5, X_6, X_{10}$ was originally calculated for the Bhatnagar–Gross–Krook kinetic equation [12]. Then it was directly verified that these generators are admitted by the BE (1). There the generator $X_{12}$ was presented with reference on A.A. Nikol'skii (as a private communication). A mutual relation between $X_{13}$ and a point transformation of the BE for $\gamma = -1$ found by Nikol'skii [15] was pointed out in [16].

**Remark 3**. In [6,17] a Lie group with generators presented here was announced as a full (complete) Lie group admitted by the full BE (1). But calculations were practically carried out in the similar *ad hoc* approach as outlined above. We must emphasize that a rigorous proof of completeness of an admitted group can only be given by deriving a general solution of the determining equations for coefficients of the generators. For some kinetic equations including the BE with some additional symmetry properties such proofs were presented in [13,14,18]. The same proof for the multi-dimensional $G^{11}$ group is ahead.

## CLASSIFICATION OF SUBALGEBRAS

The Lie algebra $L^{11}(X)$ and its extensions by generators $X_{12}, X_{13}$ defines a particular *group classification* of the BE (1) with respect to specifications of the collision cross section $\sigma$ as *a parameter* (in terminology of [7]).

Here we present a classification of all H–solutions invariant with respect to the Lie group $G = G^{11}$ of the BE (1). The classification separates a set of H–solutions into equivalent (similar) classes. Any two H–solutions $f_1$ and $f_2$ are elements of the same equivalent class if there exists a transformation $T_a \in G$ that $f_2 = T_a f_1$. Otherwise $f_1$, $f_2$ belong to different classes and they are called essentially different H–solutions. A list of all essentially different H–solutions (one representative from each class) composes an optimal system of invariant solutions that defines the searched classification. An optimal system of invariant solutions is connected with an optimal system of subalgebras $\Theta_L$ of the Lie algebra $L = L^{11}(X)$ [7]. For low dimension of an algebra $L$ calculations of $\Theta_L$ are sufficiently simple. Optimal systems for some kinetic equations with high symmetry were obtained in [4,13,18]. The higher the dimension of algebra $L$, the greater the difficulty in the construction of an optimal system.

However, for the Lie algebra $L^{11}(X)$ admitted by the BE (1) there is a remarkable circumstance that allows avoiding tedious calculations.

**Theorem 1**. The Lie algebra $L^{11}(X)$ admitted by the full BE is isomorphic to the Lie algebra $L^{11}(Y)$ admitted by the EGD–system.

**Proof**. The EGD–system is written as

$$\frac{d\rho}{dt} + \rho\nabla\mathbf{u} = 0, \qquad \rho\frac{d\mathbf{u}}{dt} + \nabla p = 0, \qquad \frac{dp}{dt} + A(\rho,p)\nabla\mathbf{u} = 0, \tag{6}$$

where $\rho$, $p$ are density and pressure of a gas, $\nabla$ is a nabla operator, $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \mathbf{v}\nabla$. As above $t \in R_+^1$, $\quad \mathbf{x} = (x,y,z) \in R_{\mathbf{x}}^3$, $\quad \mathbf{v} = (u,v,w) \in R_{\mathbf{v}}^3$, but now $\mathbf{v}$ is a vector of gas macroscopic velocity.

For an arbitrary state equation system (6) admits the 11-parameter Lie group of transformations [7] with the generators:

$$Y_1 = \partial_x,\ Y_2 = \partial_y,\ Y_3 = \partial_z,\ Y_4 = t\partial_x + \partial_u,\ Y_5 = t\partial_y + \partial_v,\ Y_6 = t\partial_z + \partial_w,$$

$$Y_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v,\ Y_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,$$

$$Y_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u, \quad Y_{10} = \partial_t, \quad Y_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z.$$

Let $Q(X) = Y$ be a linear transformation of $L^{11}(X)$ onto $L^{11}(Y)$, defined by $Q(X_k) = Y_k, k = 1,\ldots,11$. It is directly verified that $Q$ saves the commutators

$$Q([Y_k, Y_j]) = [Q(Y_k), Q(Y_j)], \quad j,k = 1,2,\ldots,11, \tag{7}$$

where $[A,B] = AB - BA$. It means that the Lie algebras $L^{11}(X)$ and $L^{11}(Y)$ are isomorphic and $Q$ is an isomorphism.

**Remark 4**. If the function $A(p,\rho) = \kappa\rho$ (it corresponds to polytropic gas), then there is an extension of the Lie algebra $L^{11}(Y)$ by two additional generators:

$$Y_{12} = t\partial_t - u\partial_u - v\partial_v - w\partial_w + 2\rho\partial_\rho,\ Y_{13} = \rho\partial_\rho + p\partial_p$$

up to the Lie algebra $L^{13}(Y) = L^{12}(Y) \oplus \{Y_{13}\}$. The operator $Y_{13}$ composes a center of the algebra $L^{13}$. It is analogously verified that the Lie algebra $L^{12}(X)$ is isomorphic to the subalgebra $L^{12}(Y)$.

**Remark 5**. In the case of a monatomic gas $\kappa = (n+2)/n$ ($n$ is a dimension of flow) the EGD–system (6) admits one more generator:

$$Y_{14} = t^2\partial_t + t\mathbf{x}\partial_{\mathbf{x}} + (\mathbf{x} - t\mathbf{v})\partial_{\mathbf{v}} - nt\rho\partial_\rho - (n+2)p\partial_p.$$

A connection between generators $X_{13}$ and $Y_{14}$ was noted in [19].

By virtue of the proven isomorphism of the Lie algebras $L^{11}(X)$ and $L^{11}(Y)$ their optimal systems of subalgebras are also isomorphic.

**Consequence**. For classifying and constructing essentially different H–solutions of the BE (1) one can use the optimal system of subalgebras constructed for the EGD–system (6).

Indeed, it is known [7] that a construction of the optimal system of subalgebras a given Lie algebra is completely defined by a table of commutators of basic generators. From (7) it follows that the tables of commutators of both algebras $L^{11}(X)$ and $L^{11}(Y)$ coincide. It proves the consequence.

**Remark 6**. As was noted in **Remark 4** the generator $Y_{13}$ is a center of the Lie algebra $L^{13}(Y)$: $[Y_i, Y_{13}] = 0, i = 1,\ldots,12$. This means that for a classification of subalgebras of $L^{12}(X)$ we can use the optimal system of subalgebras of $L^{12}(Y) \subset L^{13}(Y)$ without $Y_{13}$. The optimal system of subalgebras for $L^{13}$ was constructed in [9].

Here we restrict our consideration to the Lie algebra $L^{11}(X)$ admitted by the BE (1) with arbitrary cross section $\sigma$. An application of the optimal system of subalgebras of $L^{11}(X)$ for constructing invariant solutions of the full BE and EGD–system is different. It is connected with different numbers of the independent variables and unknown functions. Thus H–solutions of the BE (1) with one and two independent variables can only be obtained for subalgebras with dimension more than six. In our case the study of the optimal system [7] gives 11 different classes of invariant solutions with one independent variable and 38 with two independent variables. Their functional expressions are presented in **Table 1** and **Table 2**.

**TABLE 1.** Representations of H–solutions with one independent invariant variable.

| No. | f | | | No. | f | |
|---|---|---|---|---|---|---|
| 1 | $e^{\varepsilon\theta}\varphi(q)$ | 6.3 | | 7 | $\varphi(u)$ | 6.11 |
| 2 | $t^{-1}\varphi(W^2 + (V - rt^{-1})^2)$ | 6.4 | | 8 | $\varphi(u - t)$ | 6.23 |
| 3 | $t^{-1}\varphi(q)$ | 6.5 | | 9 | $e^{\varepsilon\theta}\varphi(w),\ \varepsilon \neq 0$ | 6.18 |
| 4 | $t^{-1}\varphi(u - xt^{-1})$ | 6.7 | | 10 | $t^{-1}\varphi(q/t)$ | 7.2 |
| 5 | $t^{-1}\varphi(u - \varepsilon\ ln\ t)$ | 6.20 | | 11 | $x^{-1}\varphi(u)$ | 7.3 |
| 6 | $\varphi(t)$ | 6.14 | | | | |

In the **Tables 1, 2** $\alpha, \beta, \varepsilon$ are arbitrary constants. In the second column representations of H–solutions are given. In the last column pair indices $m.i$ means a representative of the optimal system of subalgebras: $m$ is a dimension of corresponding subalgebra and $i$ is its number in the **Table 6** in [8]. In addition a capital S means that a given representation should be considered in a spherical coordinate system $(x, \varphi, \theta, u, V, W)$, where:

$$x = r\ \sin\ \theta\ \cos\ \varphi, \quad y = r\ \sin\ \theta\ \sin\ \varphi, \quad z = r\ \cos\ \theta,$$

$$u = U\ \sin\ \theta\ \cos\ \varphi + V\ \cos\ \theta\ \cos\ \varphi - W\ \sin\ \varphi, \qquad v = U\ \sin\ \theta\ \sin\ \varphi + V\ \cos\ \theta\ \sin\ \varphi + W\ \cos\ \varphi,$$

$$w = U\ \cos\ \theta - V\ \sin\ \theta, \quad Q = \sqrt{u^2 + v^2 + w^2} = \sqrt{U^2 + V^2 + W^2}$$

A capital C corresponds to a cylindrical coordinate system $(x, r, \theta, u, V, W)$, where:

$$y = r\ \cos\ \theta,\ z = r\ \sin\ \theta,\ v = V\ \cos\ \theta - W\ \sin\ \theta,\ w = V\ \sin\ \theta + W\ \cos\ \theta,$$

$$q = \sqrt{v^2 + w^2} = \sqrt{V^2 + W^2}.$$

Others representations are considered in a Cartesian coordinate system.

It should be noted that for many representations from **Tables 1, 2** some H–solutions either do not exist or do not have a physical meaning. For some of these it can be easily seen. In general case it is necessary to substitute a representation of solution into the BE (1) and to study a corresponding factor equation. But in difference from the EGD–system (6) an obtaining a factor equation for the BE with complicated collision integral (2) is sufficiently difficult. As an example of the difficulties the factor equation of H–solution (number 38 in **Table 2**), derived in [20], can be presented:

$$\frac{\partial f(t, Q)}{\partial t} = 8\frac{\pi^2\sigma^2}{Q}\int\limits_0^Q \int\limits_{\sqrt{Q^2 - P^2}}^Q f(t, P)f(t, R)\sqrt{P^2 + R^2 - Q^2}\,PR\,dP\,dR + 8\pi^2\sigma^2\left(\int\limits_Q^\infty f(t, P)P\,dP\right)^2 +$$

$$16\frac{\pi^2\sigma^2}{Q}\int\limits_0^Q f(t, P)P^2\,dP\int\limits_Q^\infty f(t, P)P\,dP - \frac{2}{3}\frac{\pi^2\sigma^2}{Q}f(t, Q)\int\limits_0^\infty f(t, P)[(Q + P)^3 - |Q - P|^3]P\,dP$$

**TABLE 2.** Representations of H–solutions with two independent invariant variables.

| No. | f | | No. | f | |
|---|---|---|---|---|---|
| 1 | $r^{-1}\varphi(U,q)$ | S 5.1 | 17 | $\varphi(t,u)$ | 5.17 |
| 2 | $r^{-1}\varphi(V,W)$ | C 5.2 | 18 | $\varphi(x-t^2/2,\,u-t)$ | 5.18 |
| 3 | $x^{-1}\varphi(u,q)$ | 5.3 | 19 | $\varphi(x,u)$ | 5.19 |
| 4 | $e^{-\alpha\theta}\varphi(u-\beta\theta,q)$ | C 5.4 | 20 | $\varphi(q,\,arcsin(v/q)+t)$ | 5.20 |
| 5 | $t^{-1}\varphi(u-x/t,(v-y/t)^2+ +(w-z/t)^2)$ | 5.5 | 21 | $x^{-1}\varphi(u,w-\beta\,ln\,x)$ | 5.21 |
| | | | 22 | $e^{-\beta u}\varphi(v,w)$ | 5.22 |
| 6 | $t^{-1}\varphi(u-x/t,q)$ | 5.6 | 23 | $t^{-1}\varphi(v-y/t,w-z/t)$ | 5.24 |
| 7 | $t^{-1}\varphi(u-\beta\,ln\,t+ +\alpha\,arcsin((v-y/t)/q),\ q)$ $q=\sqrt{(v-y/t)^2+(w-z/t)^2}$ | 5.7 | 24 | $t^{-1}\varphi(u-x/t, v-\alpha^{-1}(x-\beta\,ln\,t))$ | 5.25 |
| 8 | $t^{-1}\varphi(u-\beta\,ln\,t+ +\alpha\,arcsin(v/q),\ q)$ | 5.8 | 25 | $t^{-1}\varphi(x/t-\beta\,ln\,t, u-\beta\,ln\,t)$ | 5.26 |
| | | | 26 | $t^{-1}\varphi(u-x/t,v-\beta\,ln\,t)$ | 5.27 |
| 9 | $\varphi(t,q),$ $q=\sqrt{(v-y/t)^2+(w-z/t)^2}$ | $(5.9)_{\alpha=0}$ | 27 | $t^{-1}\varphi(u-\beta\,ln\,t,v)$ | 5.28 |
| | | | 28 | $t^{-1}\varphi(v,w)$ | 5.29 |
| 10 | $t^{-1}\varphi(q,\,arcsin((v-y/t)/q)+ +\alpha^{-1}\,ln\,t),$ $q=\sqrt{(v-y/t)^2+(w-z/t)^2}$ | $5.9_{\alpha\neq0}$ | 29 | $\varphi(u-t,v-\alpha^{-1}(x-t^2/2))$ | 5.30 |
| | | | 30 | $\varphi(x-t^2/2,u-t)$ | 5.31 |
| | | | 31 | $\varphi(u,v-x)$ | 5.32 |
| 11 | $t^{-1}\varphi(x/t,u-\alpha^{-1}\beta\,ln\,t)$ | 5.10 | 32 | $\varphi(x,u)$ | 5.33 |
| 12 | $t^{-1}\varphi(arcsin(v/q)-\alpha^{-1}\,ln\,t,q)$ | 5.11 | 33 | $\varphi(u-t,v)$ | 5.34 |
| 13 | $\varphi(u-\alpha\,arcsin(v/q),q)$ | 5.12 | 34 | $\varphi(t,u-x/t)$ | 5.35 |
| 14 | $\varphi(t,u-x/t)$ | 5.13 | 35 | $\varphi(t,w+ut-x)$ | 5.36 |
| 15 | $\varphi(t,q)$ | 5.15 | 36 | $\varphi(t,u)$ | 5.37 |
| 16 | $\varphi(t,q),$ $q=(v-(yt+z)/(1+t^2))^2+ +(w+(y-zt)/(1+t^2))^2$ | 5.16 | 37 | $\varphi(t,(\mathbf{u}-\mathbf{x}/t)^2)$ | 6.9 |
| | | | 38 | $\varphi(t,Q)$ | 6.10 |

# FOURIER REPRESENTATION OF THE FULL BOLTZMANN EQUATION

For molecules with exponential potential, the Fourier representation is [21]

$$\frac{\partial\phi}{\partial t}+i\frac{\partial^2\phi}{\partial\mathbf{x}\partial\mathbf{w}}=$$

$$=C(\gamma)\int d\mathbf{w}_1 d\mathbf{n}\ g_\gamma\left(\frac{\mathbf{un}}{|\mathbf{u}|}\right)\phi\left(\frac{\mathbf{w}}{2}+\mathbf{w}_1\right)\phi\left(\frac{\mathbf{w}}{2}-\mathbf{w}_1\right)[|\mathbf{w}_1-\frac{|\mathbf{w}|}{2}\mathbf{n}|^{-(\gamma+3)}-|\mathbf{w}_1-\frac{\mathbf{w}}{2}|^{-(\gamma+3)}],$$

where $\mathbf{u}=\mathbf{w}-\mathbf{w}_1$, $C(\gamma)=\dfrac{2^\gamma\pi^{-3/2}\Gamma((\gamma+3)/2)}{\Gamma(-\gamma/2)}$. Note that the Fourier transformation for the BE allowed to Bobylev [22] to construct the exact solution for (pseudo) Maxwellian molecules, which is called now as BKW–solution. By application of the same method (as it was presented in the previous section) to the Fourier representation of the BE we obtained the following generators of the admitted Lie algebra

$$Y_1=\partial_x,\ Y_2=\partial_y,\ Y_3=\partial_z,$$

$$Y_4=y\partial_z-z\partial_y+v\partial_w-w\partial_v,\ Y_5=z\partial_x-x\partial_z+w\partial_u-u\partial_w,$$

$$Y_6=x\partial_y-y\partial_x+u\partial_v-v\partial_u,\ Y_7=\partial_t,\ Y_8=t\partial_t+x\partial_x+y\partial_y+z\partial_z-\phi\partial_\phi,$$

$$Y_9=t\partial_t+u\partial_u+v\partial_v+w\partial_w+(\gamma-1)\phi\partial_\phi.$$

A self-normalized optimal system of subalgebras of the algebra $L_9=\{Y_1,Y_2,Y_3,Y_4,Y_5,\ Y_6,Y_7,Y_8,Y_9\}$ was constructed by using the two–steps algorithm developed in [8]. This algorithm allows reducing the problem of construction the optimal system of subalgebras of $L_9$ to a classification of subalgebras with less dimensions.

**TABLE 3.** Representations of H–solutions with one independent invariant variable of the Fourier–representation.

| No. | $\phi$ | No. | $\phi$ |
|---|---|---|---|
| 1 | $t^{\gamma\beta-1}\psi(Qt^{-\beta})$ | 6 | $u^{\gamma-1}x^{-1}\psi(q/u)$ |
| 2 | $Q^{\gamma}\psi(t)$ | 7 | $k^{\gamma-1}r^{-\gamma}\psi(rQ/k)$ |
| 3 | $Q^{\gamma}\psi(Qe^{-t})$ | 8 | $u^{\gamma}t^{-1}\psi(q/u)$ |
| 4 | $t^{-1}\psi(Q)$ | 9 | $u^{-\alpha}\psi(q/u)$ |
| 5 | $\psi(Q)$ | | |

**TABLE 4.** Representations of H–solutions with two independent invariant variables of the Fourier–representation.

| No. | $\phi$ | No. | $\phi$ |
|---|---|---|---|
| 1 | $\psi(t,Q)$ | 11 | $x^{-1}\psi(u,q)$ |
| 2 | $\frac{u^{\gamma-1}}{r}\psi(\frac{vy+wz}{ru},\frac{-vz+wy}{ru})$ | 12 | $(x-t)^{-1}\psi(u,q)$ |
| 3 | $u^{\gamma-1+\delta}e^{-\beta\theta}\psi(xu^{\delta}e^{-\beta\theta},q/u)$ | 13 | $t^{\gamma\beta-1}\psi(ut^{-\beta},qt^{-\beta})$ |
| 4 | $x^{-1}e^{(\gamma-\beta)\theta}\psi(qe^{-\beta\theta},ue^{-\beta\theta})$ | 14 | $u^{\gamma}\psi(t,q/u)$ |
| 5 | $x^{-1}u^{(\gamma-1)}\psi(\theta,q/u)$ | 15 | $e^{\gamma t}\psi(ue^{-t},qe^{-t})$ |
| 6 | $x^{-1}u^{(\gamma-1)}\psi(xu/t,q/u)$ | 16 | $t^{-1}\psi(u,q)$ |
| 7 | $x^{-1}u^{(\gamma-1)}\psi(q/u,ux^{\beta})$ | 17 | $\psi(u,q)$ |
| 8 | $e^{(\gamma-1)\beta(x-\theta)}\psi(ue^{\beta(\theta-x)},qe^{\beta(\theta-x)})$ | 18 | $t^{\gamma-1}r^{-\gamma}\psi(RU/t,Rq/t)$ |
| 9 | $e^{(\gamma-1)x}\psi(ue^{-x},qe^{-x})$ | 19 | $U^{\gamma-1-\alpha}\psi(RU^{-\alpha},q/U)$ |
| 10 | $u^{\gamma-1}\psi(x,q/u)$ | 20 | $R^{-1}\psi(U,q)$ |

The representations of invariant solutions are given in **Tables 3, 4**. There

$$Q=\sqrt{u^2+v^2+w^2},\quad \beta\neq 0, q=\sqrt{v^2+w^2},\quad r=\sqrt{y^2+z^2},\quad k=ux+vy+w$$

and in spherical coordinate system (S):

$$q=\sqrt{V^2+W^2},\quad R=\sqrt{x^2+y^2+z^2}$$

the constants $\alpha$ and $\beta$ are arbitrary.

**Remark 7** The well known BKW-solution for $\gamma=0$ is in the class 7.4.

**Remark 8** For a homogeneous case there is the additional generators

$$Y_{10}=u\phi\partial_{\phi},\quad Y_{11}=v\phi\partial_{\phi},\quad Y_{12}=w\phi\partial_{\phi}.$$

Moreover, for the (pseudo) Maxwellian molecules in the homogeneous case there is one more extension

$$Y_{13}=(u^2+v^2+w^2)\phi\partial_{\phi}.$$

## CONCLUSION

Isomorphism of Lie algebras admitted by the EGD–system and the BE allows construction of all invariant with respect to $L^{11}(X)$ solutions. Here there are all representations of invariant solutions with one and two independent variables. One applied consequence of the proved isomorphism was used. Further results for invariant solutions of the BE which can be obtained by use of optimal systems of subalgebras and a study of deeper connections between H–solutions of the BE and invariant solutions of the EGD–system are ahead.

## ACKNOWLEDGMENT

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