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Group classification of two-dimensional stable viscous gas equations

S.V. Meleshko

School of Mathematics, Surenaree University of Technology, Nakhon Ratchasima, 30000, Thailand[1]

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Abstract

Group classification of viscous gas equations in two-dimensional case is made. Exact solutions of simplified equations and complete equations of viscous gas are compared on the one model problem. This comparison shows that simplified equations have the same order as boundary layer equations. © 1998 Published by Elsevier Science Ltd. All rights reserved.

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1. Introduction

One of the important and difficult problem of computational aerodynamics is investigation of the flows near bodies. The viscous gas dynamics equations allow us to obtain full information about the structures of flows with the usual temperature and pressure. Yet despite progress in numerical methods and techniques, the calculation of complex flows remains a difficult problem.

Numerical methods, powerful and irreplaceable tools in solving new problems, are frequently difficult to justify and are labor intensive. The necessary reasonable interpretation and representation of results of mass calculations on the computer, and explanation of various mathematical difficulties, encountered during computations, is closely tied with deep analytical theory of mathematical nature. Research of specific problems involve the overcoming of significant mathematical difficulties mainly due to either nonlinearity, or the presence of a large number of variables in the initial equations. Therefore, the analytic study of properties of partial differential equations (PDE) play an important role in applied mathematics and mathematical physics. Among these methods, analytical research based on knowledge of separate classes of the partial solutions have received widespread attention. Each exact solution has value, first, as the exact description of the real process in frameworks of the given model; secondly, as a model to compare various numerical methods; thirdly, as theory to improve the used models.

[1] Institute of Theoretical and Applied Mechanics, Russian Federation.

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Besides the complete viscous gas equations, simplified equations are used as well. There are simplified equations from the complete set through eliminating some elements using various assumptions concerning the character of flows [1–4]. One of the most simple of the simplified equations is the boundary layer model. Another approach is based on the consideration of parabolic Navier–Stokes equations, considered accurate in a wider range of parameters of the flow. The precise range of applicability of these simplified equations remains an unresolved theoretical problem. The numerical research of various models is frequently hampered by the difficulty involved in distinguishing errors due to the numerical method involved from the description of the model itself.

The first part of our article is devoted to the group classification of two-dimensional stable viscous gas equations. In the second part we consider the one invariant exact solution of the complete equations of viscous gas and some of their simplifications. We then compare these solutions with respect to various entrance parameters.

2. General statement of a problem

We consider the two-dimensional stable viscous gas equations

$$u\rho_x + v\rho_y + \rho u_x + \rho v_y + \nu \frac{\rho u}{x} = 0,$$

$$\rho(uu_x + vu_y) = -p_x - \frac{2}{3}\left(\mu\left(u_x + v_y + \nu \frac{u}{x}\right)\right)_x + 2(\mu u_x)_x + (\mu(u_y + v_x))_y + 2\nu\mu\left(\frac{u}{x}\right)_x,$$

$$\rho(uv_x + vv_y) = -p_y - \frac{2}{3}\left(\mu\left(u_x + v_y + \nu \frac{u}{x}\right)\right)_y + 2(\mu v_y)_y + (\mu(u_y + v_x))_x + \nu \frac{\mu}{x}\left(u_y + v_x\right),$$

$$up_x + vp_y + A(p,\rho)\left(u_x + v_y + \nu \frac{u}{x}\right) = B(p,\rho)\left(\left(\kappa\left(\frac{p}{\rho}\right)_x\right)_x + \left(\kappa\left(\frac{p}{\rho}\right)_y\right)_y + \nu \frac{\kappa}{x}\left(\frac{p}{\rho}\right)_x + 2\mu\left(\frac{2}{3}\left(u_x^2 - u_x v_y + v_y^2 + \nu \frac{u}{x}\left(\frac{u}{x} - u_x - v_y\right)\right) + \frac{1}{2}(v_x + u_y)^2\right)\right). \tag{2.1}$$

Here ρ is the density; p is the pressure; μ is the coefficient of viscosity; κ is the coefficient of heat conductivity; ε is the internal energy of viscous gas; T is the temperature; $\nu = 0$ for the plane flows and $\nu = 1$ for the axi-symmetrical flows;

$$A = \frac{p - \rho^2 \varepsilon_\rho}{\rho \varepsilon_p}, \quad B = \frac{1}{\rho \varepsilon_p}. \tag{2.2}$$

If gas is perfect (i.e. it obeys to the Clapeyron equation $T = p/(R\rho)$), then $\varepsilon = \varepsilon(T)$ and $A = p(1 + \varepsilon')/\varepsilon'$, $B = A/p - 1$. For our considerations, the coefficients of viscosity and heat conductivity depend only on p/ρ

$$\mu = \mu(p/\rho), \quad \kappa = \kappa(p/\rho), \tag{2.3}$$

the function $A = A(p, \rho)$ is arbitary function and

$$B = \frac{A - p}{p}.$$

We wish to obtain a group classification with respect to arbitrary elements μ, κ, A[2] for the Eqs. (2.1). The gas is considered essentially viscous and conductivity: $\mu \neq 0$, $\kappa \neq 0$.

We will use a classical group analysis for constructing solutions of the system of Eqs. (2.1). The advanced approach to this kind of problems is given in [5].

The application of group analysis implies some steps. The first step is group classification. An admissible group is found at this step. The next step is a construction of an optimal system of subalgebras. Then one can attempt to find an invariant solution for each subalgebra of the optimal system.

The group classification problem consists in searching for admissible group of transformations,

[2] When the article was written author noticed that if $B = A/p - 1$ then from Eq. (2.2) the $\varepsilon = \varepsilon(p/\rho)$ is obtained.

which is admitted for all arbitrary elements of the system (in our case it is an arbitrary state equation $A(p, \rho)$) and all specifications of arbitrary elements. Specialization of arbitrary elements can extend the admissible group.

For each subalgebra of admissible algebra we can try to find an invariant solution. There are an infinite number of subalgebras. But if two subalgebras are similar, i.e., they are connected to each other by a transformation from the symmetry group, then their corresponding invariant solutions are connected with each other by the same transformation. Since the set of subalgebras can be divided into classes of similar subalgebras, therefore, it is sufficient to find only one representative solution from each similar class of subalgebras. A set of representatives of equivalent subalgebra classes is called the optimal system of subalgebras.

Here we will give some comments on the application of group analysis to the two-dimensional steady gas flow equations (2.1).

3. Equivalence transformations

The first stage of group classification requires determining the groups of equivalence transformations of Eqs. (2.1). An equivalence transformation is a nondegenerate change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to the system of equations of the same class. It allows us to use the simplest representation of given equations. We will follow the new approach to the calculation of equivalence transformations [6].

Since arbitrary elements satisfy restrictions (2.3) and $A = A(p, \rho)$, we will have to adjoin the equations

$$A_x = 0, \quad A_y = 0, \quad A_u = 0, \quad A_v = 0,$$

$$\mu_x = 0, \quad \mu_y = 0, \quad \mu_u = 0, \quad \mu_v = 0,$$

$$\kappa_x = 0, \quad \kappa_y = 0, \quad \kappa_u = 0, \quad \kappa_v = 0,$$

$$\rho\mu_\rho + p\mu_p = 0, \quad \rho\kappa_\rho + p\kappa_p = 0$$

to the Eqs. (2.1). Here we give a construct to the group of equivalence transformations without restrictions on the representation of equivalence transformations [7], so that all coefficients of infinitesimal generator

$$X^{\mathrm{e}} = \zeta^x\partial_x + \zeta^y\partial_y + \zeta^u\partial_u + \zeta^v\partial_v + \zeta^\rho\partial_\rho + \zeta^p\partial_p + \zeta^A\partial_A + \zeta^\mu\partial_\mu + \zeta^\kappa\partial_\kappa$$

are dependent on all independent, dependent variables and arbitrary elements [6]

$$x, y, u, v, \rho, p, A, \mu, \kappa.$$

With the following notation:

$$u^1 = u, \quad u^2 = v, \quad u^3 = \rho, \quad u^4 = p,$$

$$a^1 = A, \quad a^2 = \mu, \quad a^3 = \kappa$$

and

$$z^1 = x,\ z^2 = y,\ z^3 = u,\ z^4 = v,\ z^5 = \rho,\ z^6 = p,$$

the coefficients of the prolonged operator

$$\bar{X}^{\mathrm{e}} = X^{\mathrm{e}} + \sum_i (\zeta^{u^i_x}\partial_{u^i_x} + \zeta^{u^i_y}\partial_{u^i_y}) + \sum_{k,j}\zeta^{a^k_{z^j}}\partial_{a^k_{z^j}} + \cdots$$

can be constructed with the prolongation formulas:

$$\zeta^{u^i_x} = D_x\zeta^{u^i} - u^i_x D_x\xi^x - u^i_y D_x\xi^y,$$

$$\zeta^{u^i_y} = D_y\zeta^{u^i} - u^i_x D_y\xi^x - u^i_y D_y\xi^y,$$

$$\zeta^{u^i_{xx}} = D_x\zeta^{u^i_x} - u^i_{xx} D_x\xi^x - u^i_{xy} D_x\xi^y,$$

$$\zeta^{u^i_{xy}} = D_y\zeta^{u^i_x} - u^i_{xx} D_y\xi^x - u^i_{xy} D_y\xi^y,$$

$$\zeta^{u^i_{yy}} = D_y\zeta^{u^i_y} - u^i_{xy} D_y\xi^x - u^i_{yy} D_y\xi^y,$$

$$\zeta^{a^k_{z^j}} = D^{\mathrm{e}}_{z^j}\zeta^{a^k} - \sum_\alpha a^k_{z^\alpha} D^{\mathrm{e}}_{z^j}\zeta^{z^\alpha}.$$

Here operators D_x, D_y denote the total derivative operator with respect to x and y, respectively. For example,

$$D_x = \partial_x + \sum_\alpha u^\alpha_x\partial_{u^\alpha} + \sum_i (a^i_x + \sum_j a^i_{u^j}u^j_x)\partial_{a^i} + \cdots.$$

When we use the operator $D^{\mathrm{e}}_{z^j}$ we consider $z^1, \ldots, z^6$ as independent variables and a^1, a^2, a^3 as dependent variables, we obtain:

$$D^{\mathrm{e}}_{z^j} = \partial_{z^j} + \sum_i a^i_{z^j}\partial_{a^i} + \cdots.$$

All necessary calculations were carried on the computer using the symbolic manipulation program REDUCE [8].

The calculation showed that the group of equivalence transformations of Eqs. (2.1) corresponds to Lie algebra with generators

$\partial_y, x\partial_x + y\partial_y + \mu\partial_\mu + \kappa\partial_\kappa,$

$x\partial_x + y\partial_y + u\partial_u + v\partial_v - 2\rho\partial_\rho,$

$\rho\partial_\rho + p\partial_p + A\partial_A + \mu\partial_\mu + \kappa\partial_\kappa,$

If $v = 0$ (in the case of plane flows), two more generators

$\partial_x, \ - y\partial_x + x\partial_y - v\partial_u + u\partial_v$

are adjoined to the above list.

4. The admissible group

The finding of an admissible group consists in seeking solutions of determining equations which can be constructed in a standard way [7] by the generator

$$X = \zeta^x\partial_x + \zeta^y\partial_y + \zeta^u\partial_u + \zeta^v\partial_v + \zeta^\rho\partial_\rho + \zeta^p\partial_p$$

acting on the given system of equations (2.1).

Integrating the determining equations we obtain the following classification relations:

$$(c_1 + 2\omega c_3)\,(\rho A_\rho + pA_p - A) \ - 2c_3\rho A_\rho = 0, \quad (4.1)$$

$$c_3(p\mu' - \omega\rho\mu) = 0, \quad c_3(p\kappa' - \omega\rho\kappa) = 0, \quad (4.2)$$

$$vc_2 = 0, vc_4 = 0.$$

Here the representation of the solution of determining equations has the form

$$\zeta^x = (c_3 - c_1)\; x + c_2 y + c_4,$$

$$\zeta^y = (c_3 - c_1) \quad y - c_2 x + c_6,$$

$$\zeta^u = c_3 u + c_2 v, \quad \zeta^v = c_3 v - c_2 u,$$

$$\zeta^\rho \ = \ (c_1 + 2(\omega - 1)\ c_3)\rho, \quad \zeta^p \ = \ (c_1 + 2\omega c_3)\ p,$$

$c_1, c_2, \ldots, c_6$ are arbitrary constants.

The kernel of the fundamental Lie algebra is made up of the following generators: for the axisymmetric flows ($v = 1$):

∂_y

and for the plane flows ($v = 0$)

$$X_1 = \partial_x, X_2 = \partial_y, X_3 = - y\partial_x + x\partial_y - v\partial_u + u\partial_v.$$

Table 1

N	A	Extending operators
1	γp	X_4, X_5
2	$pf(\rho)$	$X_5 - (2\omega - 1)\ X_4$
3	$\rho f(p/\rho)$	X_4
4	$f(p)$	$X_5 - (2\omega + 1)\ X_4$

Extension of the kernel of the main Lie algebra occurs by specializing the functions $\mu(p/\rho)$, $\kappa(p/\rho)$ and $A(p, \rho)$.

If $\mu(p/\rho)$ or $\kappa(p/\rho)$ are not exponential functions of (p/ρ), then $c_3 = 0$ and group classification is reduced to the considering the equation

$$c_1(\rho A_\rho + pA_p - A) = 0.$$

In this case extension of the kernel occurs only on generator

$$X_4 = - x\partial_x - y\partial_y + \rho\partial_\rho + p\partial_p.$$

when

$$A = \rho\varphi(p/\rho)$$

with arbitrary function $\varphi(p/\rho)$.

If $\mu = \mu_0(p/\rho)^\omega$, $\kappa = \kappa_0(p/\rho)^\omega$, ($\mu_0, \kappa_0$ are constants), then group classification is broader. The study of Eq. (4.1) is completed in the same way as for ideal inviscous gas [9]. The results of group classification are represented in the Table 1.

Here

$$X_5 = u\partial_u + v\partial_v + (2\omega - 1)\,\rho\partial_\rho \ + \ (2w + 1)\,p\partial_p.$$

In practice, usually model of politropic gas is used. We further suppose

$$A = \gamma p, \quad \mu = \mu_0(p/\rho)^\omega, \quad \kappa = \kappa_0(p/\rho)^\omega.$$

5. The optimal system of subalgebras

We construct the optimal system of subalgebras of algebra $L_5 = \{X_1, X_2, X_3, X_4, X_5\}$. The table of commutators (Table 2).

From the table of commutators we see that X_5 is the center of L_5, composition series of ideals

$$O \subset \{X_1, X_2, X_3\} \ \subset \ \{X_1, X_2, X_3, X_4\}$$
$$\subset \{X_1, X_2, X_3, X_4, X_5\} = L_5$$

and a group of inner automorphisms consists of the form as shown in Table 3.

We also have one involution E_1, that corresponds to changing dependent and independent variables:

$$x' = -x, \quad y' = -y, \quad u' = -u, \quad v' = -v.$$

Lie algebra L_5 has following decomposition $L_5 = \{\{\{X_1, X_2, X_3\} \oplus X_4\} \oplus X_5\}$.

Normal optimal system of subalgebras of algebra $L_4 = \{X_1, X_2, X_3, X_4\}$ is listed in Table 4.

Normal optimal system of subalgebras of algebra $L_5 = \{X_1, X_2, X_3, X_4, X_5\}$ is listed in Table 5.

Here r denotes the subalgebra dimension. The latter column presents the identificator of normalizer shown in a given line of subalgebra. The first number in the normalizer indentificator denotes its dimension, the second number is its number among the subalgebras of a given dimension. Self-normalized subalgebras are marked by the sign of equality.

Table 2

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_2	$-X_1$	0
X_2	0	0	$-X_1$	$-X_2$	0
X_3	$-X_2$	X_1	0	0	0
X_4	X_1	X_2	0	0	0
X_5	0	0	0	0	0

Table 3

	x_1	x_2
A_1	$x_1 - a_1x_4$	$x_2 + a_1x_3$
A_2	$x_1 - a_2x_3$	$x_2 - a_2x_4$
A_3	$x_1\cos(a_3) + x_2\sin(a_3)$	$-x_1\sin(a_3) + x_2\cos(a_3)$
A_4	x_1a_4	x_2a_4

Table 4

N	Subalgebra	Nor
$r = 4$		
1	X_1, X_2, X_3, X_4	= 4.1
$r = 3$		
1	$X_1, X_2, X_3 + \alpha X_4$	4.1
2	X_1, X_2, X_4	4.1
$r = 2$		
1	X_1, X_2	4.1
2	X_1, X_4	= 2.2
3	X_3, X_4	= 2.3
$r = 1$		
1	X_1	3.2
2	$X_3 + X\alpha_4$	2.3
3	X_4	= 2.3

Table 5

N	Subalgebra	Nor	N	Subalgebra	Nor
	$r = 5$			$r = 2$	
1	X_1, X_2, X_3, X_4, X_5	= 5.1	1	X_1, X_2	5.1
	$r = 4$		2	X_1, X_4	3.4
1	X_1, X_2, X_3, X_4	5.1	3	X_3, X_4	3.5
2	$X_1, X_2, X_3 + \alpha X_4, X_5 + \beta X_4$	5.1	4	$X_1, X_5 + \alpha X_4 + \beta X_2, \alpha \neq 0$	3.4
3	$X_1, X_2, X_4, X_5 + \alpha X_3$	5.1	5	$X_1, X_5 + \beta X_2$	4.3^0
	$r = 3$		6	$X_3 + \alpha X_4, X_5 + \beta X_4$	3.5
1	$X_1, X_2, X_3 + \alpha X_4$	5.1	7	$X_4, X_5 + \alpha X_3$	3.5
2	X_1, X_2, X_4	5.1		$r = 1$	
3	$X_1, X_2, X_5 + \alpha X_4 + \beta X_3$	5.1	1	X_1	4.3^0
4	X_1, X_4, X_5	= 3.4	2	$X_3 + \alpha X_4$	3.5
5	X_3, X_4, X_5	= 3.5	3	X_4	3.5
			4	$X_5 + \alpha X_4 + \beta X_3, \alpha^2 + \beta^2 \neq 0$	3.5
			5	$X_5 + \alpha X_1, \alpha \neq 0$	3.3^{00}
			6	X_5	5.1

6. Some invariant solutions

Now, we consider one plane ($v = 0$) invariant exact solution of model problem of viscous gas and some simplifications. In dimensionless form, they appear as

$$
\begin{aligned}
&u\rho_x + v\rho_y + \rho u_x + \rho v_y = 0,\\
&\rho(uu_x + vu_y) + p_x = \frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}(\mu u_y),\\
&\rho(uv_x + vv_y) + p_y = \frac{4\alpha}{3\mathrm{Re}}\frac{\partial}{\partial y}(\mu v_y),\\
&up_x + vp_y + \gamma p(u_x + v_y) = \frac{\gamma}{\mathrm{Pr}\,\mathrm{Re}}\frac{\partial}{\partial y}\left(\mu\left(\frac{p}{\rho}\right)_y\right) + \frac{\gamma-1}{\mathrm{Re}}\mu(u_y)^2,\\
&T = \gamma \mathrm{Ma}^2\frac{p}{\rho}.
\end{aligned}
\tag{6.1}
$$

Here Ma is Mach number, Re is Reynolds number, Pr is Prandtl number, α is equal to 1 or 0. If the coefficient of dynamic viscosity is constant, then we may assume that it is equal to 1. These equations are obtained from the complete equations of viscous compressible gas with the same assumptions about character of flows, as those of the equations of a boundary layer. Thus, in the equation of conservation of a pulse in cross direction (on an axis y) we retain only the terms of order $1/\sqrt{\mathrm{Re}}$. They include Euler equations and boundary layer equations.

As the tests for numerical methods [2, 10] following problem was considered:

On a plate ($y = 0$) constant flow is given

$$
u = u_w, \quad v = v_w \tag{6.2}
$$

and flux of a heat

$$
\frac{\partial T}{\partial y} = \frac{Q_w}{x}. \tag{6.3}
$$

On the straight line, leaving from a beginning of coordinates ($y = \xi_0 x$, $\xi_0 = \mathrm{const}$), constant flows are given longitudinal speed and temperature

$$
u = u_{00}, \quad T = T_{00}. \tag{6.4}
$$

Further we assume

$$
u_w = 0, \quad v_w = 0, \quad Q_w = 0, \quad u_{00} = 1, \quad T_{00} = 1.
$$

The solution of problem (6.1)–(6.3) studied using the area $L = \{(x, y)\,|\,0 \leqslant y \leqslant \xi_0 x\}$. The value $\delta = \xi_0 x$ is similar to the thickness of a boundary layer. Therefore, we consider ξ to be of order $1/\sqrt{\mathrm{Re}}$ just as in the case of the boundary layer.

The solution of problem (6.1)–(6.3) lies in a class of invariant solutions comprising

$$
u = U(\xi), \quad v = v(\xi), \quad \rho = \frac{R(\xi)}{x}, \quad p = \frac{P(\xi)}{x}, \quad \xi = \frac{y}{x}.
$$

For Eq. (6.1) ($\alpha = 0$) solution of problem (6.2), (6.3) coincides with the solution of the compressible boundary layer equations, when the dimensionless pressure in the external flow p can be expressed as

$$
p = \frac{k}{x}\,(P_0 = k = \mathrm{const}).
$$

Values of temperature and the speeds are expressed by the formulas

$$
\begin{aligned}
&T_0 = 1 - (\gamma - 1)\,\mathrm{Pr}(u^2 - 1)\mathrm{Ma}^2/2, \quad v = \xi u,\\
&\int_0^u \mu(T_0(u))\,\mathrm{d}u = -\,\mathrm{Re}(c_1\xi + k\xi^2/2),
\end{aligned}
\tag{6.5}
$$

where $k = \mathrm{const}$,

$$
P_0 = k, \quad c_1 = -\frac{1}{\mathrm{Re}\,\xi_0}\int_0^1 \mu(T_0(u))\,\mathrm{d}u - k\xi_0^2/2.
$$

In particular, if $\mu = 1$, then

$$
u = u_0(\xi) = -\,\mathrm{Re}\,\xi\left(k(\xi - \xi_0) - \frac{2}{\mathrm{Re}\,\xi_0}\right)\Big/2.
$$

For Eq. (6.1) ($\alpha = 1$) the pressure P explicitly expressed through $u = u_1(\xi)$, ξ is

$$
P = P_1(\xi) = \left[k + \frac{4}{3}\left(\frac{\mu}{\mathrm{Re}}u_1(\xi) - \xi c_2\right)\right]\Big/\left(1 + \frac{4\xi^2}{3}\right), \tag{6.6}
$$

and with speed and temperature included, we have a system of ordinary differential equations

$$\frac{\mu}{\mathrm{Re}}\,u_1' = -\left(c_2 + k\xi + \frac{4\mu}{3\mathrm{Re}}\,\xi u_1(\xi)\right)\Big/\left(1+\frac{4\xi^2}{3}\right),$$

$$v = \xi u_1(\xi), \tag{6.7}$$

$$\frac{\gamma}{\mathrm{Re}\,\mathrm{Pr}}\,(\mu(T_1'))' = (\gamma-1)\,(u_1P_1 + u_1'(P_1\xi + c_2)).$$

If $\mu = 1$, then we integrate the equation for longitudinal component of speed to obtain

$$u_1 = -\mathrm{Re}\left(\frac{c_2\frac{\sqrt{3}}{2}\ln(\frac{2}{\sqrt{3}}\,\xi + \sqrt{1+4\xi^2/3}) - \frac{3}{4}\,k}{\sqrt{1+4\xi^2/3}} + \frac{3}{4}k\right), \tag{6.8}$$

where

$$c_2 = -\tfrac{\sqrt{3}}{2}\,(k - \sqrt{1+4\xi_0^2/3}(k+4/(3\mathrm{Re})))/\ln(\tfrac{2}{\sqrt{3}}\,\xi_0 + \sqrt{1+4\xi_0^2/3}).$$

The values of pressure, temperature and speed, calculated from the complete equations of viscous gas, are expressed with the formulas ($u = u_2(\xi)$, $p = P_2(\xi)$, $T = T_2(\xi)$) as

$$P_2 = \frac{k - c_3\xi}{1+\xi^2} + \frac{4\mu}{3\mathrm{Re}}\,u, \quad v = \xi u, \tag{6.9}$$

$$T_2 = 1 - (\gamma-1)\,\mathrm{Pr}((1+\xi^2)\,u^2 - (1+\xi_0^2))\,\mathrm{Ma}^2/2,$$

$$\int_0^z \mu(T_2(z))\,\mathrm{d}z = \mathrm{Re}\left(\frac{k - c_3\xi}{\sqrt{1+\xi^2}} - k\right),$$

$$z = u\sqrt{1+\xi^2}, \tag{6.10}$$

where

$$c_3 = k\,\frac{1-\sqrt{1+\xi_0^2}}{\xi_0} - \frac{\sqrt{1+\xi_0^2}}{\xi_0\mathrm{Re}}\int_0^{\sqrt{1+\xi_0^2}} \mu(T_2(z))\,\mathrm{d}z.$$

Thus, if $\mu = 1$, then

$$u_2 = \mathrm{Re}\,\frac{k - c_3\xi - k\sqrt{1+\xi^2}}{1+\xi^2}, \tag{6.11}$$

and

$$c_3 = k\,\frac{1-\sqrt{1+\xi_0^2}}{\xi_0} - \frac{1+\xi_0^2}{\xi_0\,\mathrm{Re}}.$$

If ξ_0 has order about $1/\sqrt{\mathrm{Re}}$, k about 1, then it is possible to show that the constructed solutions satisfy those conditions, used to obtain the simplified Eqs. (6.1). Thus, we get the following estimations of differences of the solutions of the complete equations of viscous gas (2.1) from the solutions of the simplified Eqs. (6.1)

$$|u_2 - u_0| = \xi_0^2|\zeta - \zeta^3 + 9a\zeta(\zeta^3 - 2\zeta^2 - 1)| + \mathrm{O}\left(\frac{1}{\mathrm{Re}^2}\right),$$

$$|u_2 - u_1| = \xi_0^2|\zeta - \zeta^3 + a\zeta(3\zeta^3 - 2\zeta^2 - 12\zeta - 7)| + \mathrm{O}\left(\frac{1}{\mathrm{Re}^2}\right),$$

$$|P_2 - P_0| = \frac{7}{3\mathrm{Re}}\left|\zeta + 18a\left(\zeta - \frac{10}{7}\,\zeta^2\right)\right| + \mathrm{O}\left(\frac{1}{\mathrm{Re}^2}\right), \tag{6.12}$$

$$|P_2 - P_1| = \frac{1}{3\mathrm{Re}}\,|\zeta + 18a(\zeta - 2\zeta^2)| + \mathrm{O}\left(\frac{1}{\mathrm{Re}^2}\right), \tag{6.13}$$

where $\zeta = \xi/\xi_0$, $a = k\,\mathrm{Re}\,\xi_0^2/36$.

Moreover, if we make the limiting transition of $\xi_0 \to 0$ in Eqs. (6.7)–(6.11) then the corresponding limiting values of speed and temperature will coincide

$$\int_0^u \mu(T(u))\,\mathrm{d}u = \zeta\int_0^1 \mu(T(u))\,\mathrm{d}u, \quad v = 0.$$

$$T = 1 - (\gamma-1)\,\mathrm{Pr}(u^2-1)\,\mathrm{Ma}^2/2.$$

We have for pressure the following:
in the case of simplified Eqs. (6.1) with $\alpha = 0$

$$P_0 = 1,$$

for simplified Eqs. (6.1) with $\alpha = 1$

$$P_1 = 1 + \frac{4\mu}{3\mathrm{Re}}\,u,$$

for complete Eqs. (2.1)

$$P_2 = 1 + \frac{4\mu}{3\mathrm{Re}}\,u + b\zeta, \quad b = \frac{1}{\mathrm{Re}}\int_0^1 \mu(T(u))\,\mathrm{d}u.$$

These results imply that for small values ξ_0

$$|P_1 - P_2| \leqslant |P_2 - P_1|.$$

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