ON THE COMPLETE GROUP CLASSIFICATION OF THE NONLINEAR KLEIN-GORDON EQUATION WITH A DELAY

Fengshan Long

A Thesis Submitted in Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy in Applied Mathematics

Suranaree University of Technology

Academic Year 2015

การจำแนกกลุ่มแบบบริบูรณ์ของสมการไคลน์-กอร์ดอนไม่เชิงเส้น
ที่มีการประวิง

นายฟังซาน หลง

มหาวิทยาลัยเทคโนโลยีสุรนารี

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์
มหาวิทยาลัยเทคโนโลยีสุรนารี
ปีการศึกษา 2558

ON THE COMPLETE GROUP CLASSIFICATION OF THE NONLINEAR KLEIN-GORDON EQUATION WITH A DELAY

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

Thesis Examining Committee

(Asst. Prof. Dr. Eckart Schulz)

Chairperson

(Prof. Dr. Sergey V. Meleshko)

Member (Thesis Advisor)

(Prof. Dr. Adrian E. Flood)

Member

(Asst. Prof. Dr. Prapart Pue-on)

Member

(Asst. Prof. Dr. Jessada Tanthanuch)

Member

(Prof. Dr. Sukit Limpijumnong)

Vice Rector for Academic Affairs

and Innovation

(Prof. Dr. Santi Maensiri)

Dean of Institute of Science

มหาวิทยาลัยเทคโนโลยีสุรนารี

ฟังซาน หลง : การจำแนกกลุ่มแบบบริบูรณ์ของสมการไคลน์-กอร์ดอนไม่เชิงเส้นที่มีการประวิง (ON THE COMPLETE GROUP CLASSIFICATION OF THE NONLINEAR KLEIN-GORDON EQUATION WITH A DELAY) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร. เซอร์เก เมเลชโก, 123 หน้า.

สมการไคลน์-กอร์ดอนที่ไม่มีการประวิงได้ถูกนำไปใช้เพื่อสร้างตัวแบบของปรากฏการณ์แบบไม่เชิงเส้นต่าง ๆ อย่างมากมาย ในกระบวนการทางวิทยาศาสตร์สมัยใหม่นั้น ได้เพิ่มการประวิงเข้ามาในการสร้างตัวแบบทางคณิตศาสตร์ด้วย สำหรับความมุ่งหมายของวิทยานิพนธ์นี้ คือการวิเคราะห์กลุ่มของสมการไคลน์-กอร์ดอนที่มีการประวิง

ส่วนแรกของวิทยานิพนธ์นี้ เกี่ยวข้องกับการสร้างสมการกำหนดของสมการเชิงอนุพันธ์ประวิง โดยการประวิงนั้นแปรตามเวลา ถึงแม้ว่าการสร้างสมการกำหนดสำหรับกรณีที่มีการประวิงคงตัวนั้น จะเป็นที่ทราบกันอยู่แล้ว แต่ในปัจจุบันยังไม่มีการนำการวิเคราะห์กลุ่ม มาประยุกต์ใช้กับสมการเชิงอนุพันธ์ประวิงที่มีการประวิงแบบแปรผันตามเวลาได้ ดังนั้นการวิเคราะห์สำหรับกรณีดังกล่าวจึงได้ถูกพัฒนา ดังที่ปรากฏในวิทยานิพนธ์นี้

ผลการศึกษาส่วนที่สองของวิทยานิพนธ์จะเป็นการศึกษาการจำแนกกลุ่มของสมการไคลน์-กอร์ดอนที่มีการประวิงทั้งแบบหนึ่งมิติ และสองมิติ สำหรับสมการไคลน์-กอร์ดอนหนึ่งมิติที่พิจารณาจะมีการประวิงคงตัว แต่สมการไคลน์-กอร์ดอนสองมิติ จะเป็นชนิดที่มีการประวิงแบบแปรผันตามเวลา ผลการศึกษาได้แสดงถึงสมการกำหนดที่เกี่ยวข้องและผลเฉลยทั่วไปของสมการกำหนด นอกจากนี้ได้แสดงถึงการจำแนกกลุ่มลีที่ถูกแอดมิทโดยสมการดังกล่าว และการวิเคราะห์ของผลเฉลยยืนยงทั้งหมด

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา Fengshan Long

ปีการศึกษา 2558 ลายมือชื่ออาจารย์ที่ปรึกษา ______

FENGSHAN LONG : ON THE COMPLETE GROUP CLASSIFICATION OF THE NONLINEAR KLEIN-GORDON EQUATION WITH A DELAY. THESIS ADVISOR : PROF. SERGEY V. MELESHKO, Ph.D. 123 PP.

KLEIN-GORDON EQUATION/TIME DELAY DIFFERENTIAL EQUATION/LIE GROUP/INVARIANT SOLUTION

The Klein-Gordon equation without delay is used to model many different nonlinear phenomena. In modern science many mathematical models also include delay in consideration. The purpose of this thesis is group analysis of the Klein-Gordon equation with a delay.

The first part of the thesis results is related with constructing determining equations of delay differential equations, where the delay depends on time. In spite of the fact that the method of constructing determining equations with constant delay is known, at present there are no applications of group analysis to time-varying delay differential equations. In the thesis, this analysis is developed.

The second part of the thesis is devoted to group classification of the one- and two-dimensional Klein-Gordon equation with a delay. The one-dimensional Klein-Gordon equation is considered with a constant delay, whereas for the two-dimensional Klein-Gordon equation delay is time-varying. Corresponding determining equations and their general solutions are obtained. Classifications of the admitted Lie groups are given. Analysis of all invariant solutions is presented.

School of Mathematics

Academic Year 2015

Student's Signature Fengshan Long

Advisor's Signature ____________

ACKNOWLEDGEMENTS

I am deeply grateful to my advisor, Prof. Dr. Sergey V. Meleshko for his support, encouragement, patient help, suggestions and teaching me English; this thesis would not have been possible without him. I have learned from him how to do research, behave and work, which are the most valuable assets in my life forever.

I would like to express my appreciation to Asst. Prof. Dr. Eckart Schulz for carefully reading and checking my manuscript including my thesis proposal and papers, and for teaching and helping me during the course of my study here in SUT.

I am particularly grateful to Assoc. Prof. Dr. Prapasri Asawakun, Prof. Dr. Santi Maensiri, and Prof. Dr. Yupeng Yan, who have created a joint educational program on behalf of SUT with my university, and for giving me the opportunity to learn in SUT. I also acknowledge the financial support of Guizhou University of Finance and Economics, Guiyang, China.

In addition, I would like to express my special thanks to my friends who always have given their friendly help including living and studying in SUT, and especially to Chompit Kaewmanee, Giovanna Fae Oguis and Wiparat Worapitpong for helping me a lot.

Lastly, I am forever grateful to my family, especially to my wife, for constantly giving me support, understanding, encouragement and love.

Fengshan Long

CONTENTS

		Page
ABSTRACT IN THAI		I
ABSTRACT IN ENGLISH		II
ACKNOWLEDGEMENTS		III
CONTENTS		IV
LIST OF TABLES		VII
CHAPTER		
I	**INTRODUCTION**	1
	1.1 Delay Differential Equation	1
	1.2 How DDEs Arise	2
	1.3 Solving DDEs	4
	1.4 Klein-Gordon Equation With Delay	7
II	**GROUP ANALYSIS**	9
	2.1 One-parameter Lie Group of Transformations	9
	2.2 Canonical Lie-Bäcklund Operators	11
	2.3 Determining Equations	13
	2.4 Lie Algebra of Generators	14
	2.5 Classification of Subalgebras	16
	2.6 Invariant Solutions	19
III	**ADMITTED LIE GROUP DELAY DIFFERENTIAL EQUATIONS**	21
	3.1 Time-Varying Delay	21

CONTENTS (Continued)

Page

3.2 Equivalence Lie Group of Transformations for Constant Delay . . . 26

3.3 Determining Equation of The Vary-time Delay 29

IV APPLYING GROUP ANALYSIS OF THE ONE-DIMENSIONAL NONLINEAR KLEIN-GORDON EQUATION WITH CONSTANT DELAY 31

4.1 Equivalence Lie Group of Equation (4.1) 31

4.2 Admitted Lie Group of Equation (4.1) 34

4.3 Extensions of the Kernel 37

4.3.1 Case $\Delta \neq 0$ 38

4.3.2 Case $\Delta = 0$ 39

4.4 Summary of the Group Classification 47

4.5 Invariant Solutions 48

4.5.1 Invariant Solutions of (4.1) with $g(u, \bar{u}) = e^u H(\bar{u} - u)$ 49

4.5.2 Invariant Solutions of (4.1) with $g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$ 49

4.5.3 Invariant Solutions of (4.1) with the Function $g(u, \bar{u})$ of Forms $No.3$-$No.6$ in Table 4.1 50

4.5.4 Invariant Solutions of (4.1) with $g(u, \bar{u}) = k_1\bar{u} + k_2u$ 61

V APPLYING GROUP ANALYSIS OF THE TWO-DIMENSIONAL NONLINEAR KLEIN-GORDON EQUATION WITH TIME-VARYING DELAY 66

5.1 Admitted Lie Group of Equation (5.1) 66

5.2 Extensions of the Kernel 70

5.2.1 Case $\Delta \neq 0$ 70

CONTENTS (Continued)

Page

5.2.2 Case $\Delta = 0$ 72

5.3 Summary of the Group Classification 78

5.4 Invariant Solutions 81

5.4.1 Invariant Solutions of (5.1) with $g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$ 81

5.4.2 Invariant Solutions of (5.1) with $g(u, \bar{u}) = e^u H(\bar{u} - u)$ 87

5.4.3 Invariant Solutions of (5.1) with $g(u, \bar{u}) = u^k H(\frac{\bar{u}}{u})$ 92

5.4.4 Invariant Solutions of (5.1) with $g(u, \bar{u}) = k_0 u + H(\bar{u} + k_1 u)$ 94

5.4.5 Invariant Solutions of (5.1) with $g(u, \bar{u}) = H(\bar{u} + k_0 u)$ 99

5.4.6 Invariant Solutions of (5.1) with $g(u, \bar{u}) = k_2 e^{\bar{u} + k_0 u}$ 102

5.4.7 Invariant Solutions of (5.1) with $g(u, \bar{u}) = k_2(\bar{u} + k_0 u + \frac{\alpha_0}{\alpha_1})^{1+\frac{1}{\alpha_1}}$ 104

5.4.8 Invariant Solutions of (5.1) with $g(u, \bar{u}) = k_1 \bar{u} + k_2 u$ 106

VI CONCLUSIONS 114

REFERENCES 115

CURRICULUM VITAE 123

มหาวิทยาลัยเทคโนโลยีสุรนารี

LIST OF TABLES

Table **Page**

4.1 Group classification of the equation $u_{tt} = u_{xx} + g(u, \bar{u})$ 48

4.2 Invariant solutions for $g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$ 50

4.3 Invariant solutions for $g(u, \bar{u}) = k_0 u + H(\bar{u} + k_1 u)$ 54

4.4 Invariant solutions for $g(u, \bar{u}) = H(\bar{u} - u)$ 56

4.5 Invariant solutions for $g(u, \bar{u}) = k_2 \ln |\bar{u} - u - \beta| + k_3$ 56

4.6 Invariant solutions for $g(u, \bar{u}) = k_2(\bar{u} - u + \beta)^k + k_3$ 56

4.7 Invariant solutions for $g(u, \bar{u}) = H(\bar{u} - ue^{-\alpha_1 \tau})$ 60

4.8 Invariant solutions for $g(u, \bar{u}) = k_2 \ln |\bar{u} - ue^{-\alpha_1 \tau} - \beta| + k_3$ 60

4.9 Invariant solutions for $g(u, \bar{u}) = k_2(\bar{u} - ue^{-\alpha_1 \tau} + \beta)^k + k_3$ 61

4.10 Invariant solutions for $g(u, \bar{u}) = k_1 \bar{u} + k_2 u$ 65

5.1 Group classification the equation $u_{tt} = u_{xx} + u_{yy} + g(u, \bar{u})$ 80

5.2 Invariant solutions for $g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$ 86

5.3 Invariant solutions for $g(u, \bar{u}) = e^u H(\bar{u} - u)$ 91

5.4 Invariant solutions for$g(u, \bar{u}) = u^k H(\frac{\bar{u}}{u})$, $k = 1 - (\alpha_1)^{-1}$ 93

5.5 Invariant solutions for $g(u, \bar{u}) = k_0 u + H(\bar{u} + k_1 u)$ 98

5.6 Invariant solutions for $g(u, \bar{u}) = H(\bar{u} - u)$ 101

5.7 Invariant solutions for $g(u, \bar{u}) = k_2 e^{\bar{u} + k_0 u}$, $k_0 = -1 - 2(\alpha_2)^{-1}$ 103

5.8 Invariant solutions for $g(u, \bar{u}) = k_2(\bar{u} + k_0 u + \frac{\alpha_0}{\alpha_1})^k$, $k = 1 + (\alpha_1)^{-1}$ . . 105

5.9 Invariant solutions for $g(u, \bar{u}) = k_1 \bar{u} + k_2 u$ 112

CHAPTER I

INTRODUCTION

1.1 Delay Differential Equation

In many physical, engineering and biological phenomena the rate of variation in the system state depends on past states, a characteristic which is called delay. Time-delay systems (shortly, TDS) are also called systems with aftereffect or dead-time, hereditary systems, equations with deviating argument or differential-difference equations. J. P. Richard (Richard, 2003) has given a four point explanation for interest and development of TDS. In particular, this framework is very general and allows both simple (constant) and complex (variable or distributed) forms of delays to be modeled. Mathematical modelling of biological and physical systems with delays is mainly based on delay differential equations (DDEs), which were first discovered in biological systems and later found in many engineering systems, such as mechanical transmissions, fluid transmissions, metallurgical processes and networked control systems. In modern science, one encounters many mathematical models described by delay differential equations (Myshkis, 1972; Bellman and Cooke, 1963; Hale, 1977; Driver, 1977; Kolmanovskii and Myshkis, 1992; Wu, 1996; Smith, 2010), for example in population dynamics, bioscience problems, control problems, and electrical networks containing lossless transmission lines. Delay differential equations, which describe the rate of change of the unknown quantity in a system depend not only on the current state of the system, but also on its entire previous evolution, that is, on values of the unknown at certain times in the past. In particular, the complex form of delays systems, that is, having time-varying delay, which varies in an interval with a nonzero lower bound is characterized as interval time-varying delay and is encountered in the wide range of engineering applications spread from chemical reactors and combustion engines to the networked control systems and recurrent neural networks (Li et al., 2011; Farnam and Esfanjani, 2014). During the recent years, stability analysis and stabilization of time-invariant systems with time-varying delay has been an active research field (Phat et al., 2012; Zhang et al., 2010; Peng and Tian, 2008; Kwon et al., 2011; Shao, 2009; Park, 1994; Louisell, 2001; Niculescu et al., 1998; Sun et al., 1997; Verriest, 1994).

1.2 How DDEs Arise

Interest in using delay differential equations often arises when traditional pointwise modeling assumptions are replaced by more realistic distributed assumptions. As an example, let us demonstrate one of the well-known models describing the dynamics of a population (see, for example, (Kolmanovskii, 1996)). In 1838 P.F.Verhulst applied the equation

$$\dot{N}(t) = \lambda N(t)(1 - N(t))$$

for describing the dynamics of a population, where $N(t)$ is the population density. The general solution of this equation is $N(t) = \left(1 + Ce^{-\lambda(t-t_o)}\right)$. This model, however, has some shortcomings. For example, it implies that the population growth is monotone, whereas in reality the growth oscillates. Another weakness is that according to this model, the population reacts immediately to a change of population, whereas in reality it is not so: the rate of change $\dot{N}(t)$ of the population density does not immediately react to the change of population $N(t)$.

Thus, a more realistic model was proposed by G.E.Hutchinson (1948)

$$\dot{N}(t) = \lambda N(t)(1 - N(t-\tau)),$$

where τ is a delay time. This model is more realistic, because it takes into account inertia of the reaction to a change of population, and the oscillating tendency of the population approaching a stationary state.

Most equations applied in mathematical modelling include time and space variables. As a rule, physical phenomena have diffusion with respect to the space variables. This leads to systems of partial differential equations with delay. For example, equations of the forms

$$\mathbf{u}_t = \mathbf{u}_{xx} + \mathbf{F}(\mathbf{u}, \mathbf{w})$$

or

$$\mathbf{u}_{tt} = \mathbf{u}_{xx} + \mathbf{F}(\mathbf{u}, \mathbf{u}_t, \mathbf{w})$$

are generalizations of the models $\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{w})$ or $\mathbf{u}_{tt} = \mathbf{F}(\mathbf{u}, \mathbf{u}_t, \mathbf{w})$. Here $\mathbf{u} = \mathbf{u}(t)$ is the vector of the dependent variables, and $\mathbf{w} = \mathbf{u}(t-\tau)$. There is also another way for deriving partial differential equations with delay. Let us give a demonstration starting with the reaction-diffusion equation

$$u_t = u_{xx} + F(u).$$

This equation arises in biology, biophysics, biochemistry, chemistry, medicine, control, climate model theory, ecology, economics and many other areas. Due to a number of factors depending on the area of applications there is a delay in the studied processes. From the physical viewpoint the delay is responsible for inertia in mass/heat transfer processes: the system does not respond to an action immediately at the time t when the action is applied as in the classical local-equilibrium case, but at a relaxation time τ later. This brings us to a partial differential equation with delay,

$$u_t = u_{xx} + F(u, w),\ w = u(t - \tau, x), \tag{1.1}$$

where τ is constant.

1.3 Solving DDEs

Despite of the importance of exact solutions, there is a lack of methods for constructing exact solutions of delay differential equations. From among the known approaches for constructing exact solutions of delay differential equations one mentions the following:

(i) travelling wave type solutions;

(ii) reduction to differential equations;

(iii) use of representation of a solution on the base of a priori simplified assumptions.

มหาวิทยาลัยเทคโนโลยีสุรนารี

Travelling waves are applied in many areas of science and engineering. Solutions $u(t, x)$ of travelling wave type have the representation $u = U(x - Dt)$, where x is a space variable, t is time, and D is constant. A wavefront propagates along x with constant phase velocity D.

Reductions of equations with nonlocal terms are applied in viscoelastic materials, nonlinear optics and other areas of mathematical physics. For example, consider one of the most general evolution equations used in nonlinear wave physics (Rudenko and Soluyan, 1977; Rudenko et al., 1974):

$$(u_x - uu_t - w_{tt})_t = u_{yy} + u_{zz}, \tag{1.2}$$

where $w = \int_0^\infty K(s)\, u(t-s)\, ds$. Special cases of equation (1.2) which take the form of a partial differential equation are well-known (Rudenko, 2010): the Khokhlov-Zabolotskaya equation (Zabolotskaya and Khokhlov, 1969; Rudenko, 2010),

$$(u_x - uu_t)_t = u_{yy} + u_{zz},$$

the Kadomtsev-Petviashvili equation

$$(u_x - uu_t - u_{ttt})_t = u_{yy} + u_{zz},$$

and the delay differential equation (Ibragimov et al., 2011)

$$(u_x - uu_t - u_t + w_t)_t = u_{yy} + u_{zz},\ w(t,x) = u(t-1,x),$$

which is simpler than the original equation (1.2) and other models (Rudenko, 2010).

As an example of the third approach one can consider the approach (Polyanin and Zhurov, 2014e) applied to nonlinear delay reaction-diffusion equation (1.1). In (Polyanin and Zhurov, 2014e) solutions of equation (1.1) were sought in the form

$$u = \sum_{n=1}^{N} \phi_n(x)\psi_n(t),$$

where the functions $\phi_n(x)$ and $\psi_n(t)$ are to be determined subsequently, and are sought in a form so that it is possible to apply the method of invariant subspaces (Galaktionov and Svirshchevskii, 2007) which for delay differential equations requires additional restrictions. This approach was also applied by the authors of (Polyanin and Zhurov, 2014e) to the delay reaction-diffusion equation (and pair of equations) (Polyanin and Zhurov, 2013; Polyanin and Zhurov, 2014a; Polyanin and Zhurov, 2014b; Polyanin and Zhurov, 2014c; Polyanin and Zhurov, 2014d; Polyanin and Zhurov, 2014f; Polyanin and Zhurov, 2014g; Polyanin and Zhurov, 2015).

Throughout the years, many methods for obtaining exact solutions of differential equations instead of approximating solutions have been developed. One of them is the group analysis method. Group analysis was initially introduced in the 1870s by a Norwegian mathematician, Sophus Lie. He found a new method for integrating differential equations. This method is universal and effective for solving nonlinear differential equations analytically. It involves the study of symmetries of differential equations, with the emphasis on using the symmetries to find solutions. The theory of group analysis has been applied to both ordinary and partial differential equations in (Ovsiannikov, 1978; Olver, 1986; Ibragimov, 1999) and more mathematical models (Ibragimov, 1996). One of its applications to differential equations is the problem of group classification of differential equations. Group classification means to classify given differential equations with respect to arbitrary elements. The group classification problem of differential equations was first formulated by Lie (Lie, 1883). He has given a classification of ordinary differential equations in terms of their symmetry groups, thereby identifying the full set of equations which could be solved or reduced to lower-order equations by this method.

มหาวิทยาลัยเทคโนโลยีสุรนารี

In this thesis, the group analysis method is used, which is used not only for solving differential equations, but also for developing new models. The approach for applying group analysis to partial differential equations with simple (constant) forms of delays was introduced in (Tanthanuch and Meleshko, 2004; Meleshko, 2005; Grigoriev et al., 2010), and the method for constructing and solving determining equations was presented in (Tanthanuch and Meleshko, 2004). They show that the presence of a symmetry in a delay differential equation allows finding an invariant solution and reducing the number of independent variables of the equation, which is similar to the theory for partial differential equations. However, at present, there are no results related with applications of group analysis to the time-varying delay differential equations. It should be mentioned here that the study of time-varying delay differential equations is a new area in the theory of the group analysis method.

1.4 Klein-Gordon Equation With Delay

In this thesis the nonlinear delay Klein-Gordon equation of the form

$$u_{tt} = \Delta u + g(u, \bar{u}) \tag{1.3}$$

is studied by the group analysis method, where $\bar{u}(t, x_1, x_2, ..., x_n) = u(t - \tau(t), x_1, x_2, ..., x_n)$, $\tau(t)$ depends on t and $\tau(t) > 0$.

The Klein-Gordon equation plays an important role in mathematical physics. This equation was deduced in the 1920s and 1930s independently by Oskar Klein (Swedish physicist) and Walter Gordon (German physicist). The Klein-Gordon equation is a fundamental equation in relativistic quantum mechanics and quantum field theory. The nonlinear Klein-Gordon equation without delay has the form

$$u_{tt}(x, t) = \Delta u(x, t) + g(u(x, t)). \tag{1.4}$$

It is well known that the Klein-Gordon equation (1.4) is used to model many different nonlinear phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles and of Josephson junctions (see (Drazin and Johnson, 1989) Chap. 8.2 for details). It has also been the subject of detailed investigation in studies of solitons and nonlinear science in general. It is probably best known as the sine-Gordon equation

$$u_{tt} - \Delta u + \sin u = 0,$$

although it also appears with $F(u) = \sinh u$, polynomial $F(u)$, and other nonlinear functions. Details of existence, uniqueness, and other analytic properties of solutions of equation (1.4) can be found in (Polyanin and Zaitsev, 2012; Bullough and Caudrey, 1980; Novikov, 1984; Grundland and Infeld, 1992; Zhdanov, 1994; Andreev et al., 1998), and a more general discussion, including applications and numerical approximations, can be found in (Drazin and Johnson, 1989) and (Dodd et al., 1982). According to the fact that the system does not respond to an action immediately at the time t when the action is applied, as in the classical local-equilibrium case, but at a relaxation time $\tau(t)$ later, one therefore considers it necessary to study equation (1.3) with time-varying delay. This leads to the one-dimensional nonlinear constant delay Klein-Gordon equation of the form

$$u_{tt} = u_{xx} + g(u, \bar{u}), \tag{1.5}$$

and the two-dimensional nonlinear Klein-Gordon equation with time-varying delay of the form

$$u_{tt} = u_{xx} + u_{yy} + g(u, \bar{u}), \tag{1.6}$$

where $\bar{u}(t, x) = u(t - \tau, x)$, $\tau(t)$ is constant in equation (1.5), in equation (1.6) $\bar{u}(t, x, y) = u(t - \tau(t), x, y)$, $\tau(t)$ depends on t and $\tau(t) > 0$, which denotes the time-varying delay.

CHAPTER II

GROUP ANALYSIS

Before discussion of the main research in the Chapter IV, some background and basic concepts of group analysis are presented here. In 1870, a Norwegian mathematician, Sophus Lie, introduced the theory of continuous transformation groups that are now known as Lie groups. The main concept of the group analysis method for constructing exact solutions of differential equations is the concept of admitted Lie group. This method is a successful method for integration of linear and nonlinear differential equations. Many results obtained by this method are collected in the Handbook of Lie Group Analysis (1994), (1995), (1996). Group analysis was also applied to integro-differential, stochastic and delay differential equations in (Grigoriev et al., 2010).

In this chapter, some background of Lie group analysis is reviewed such as a one-parameter Lie group of transformations, canonical Lie-Bäcklund operators, determining equations, Lie algebra of generators, classification of subalgebras. In the last section, notions of invariant solutions are presented.

2.1 One-parameter Lie Group of Transformations

Let $g : V \times \triangle \to V$ be an invertible transformation, where V is the set of variables $z = (x, u)$. Here $x = (x_1, \ldots, x_n)$ is the set of independent variables, and $u = (u_1, \ldots, u_m)$ is the set of dependent variables. Furthermore, $\triangle \subset \mathbb{R}$ is a symmetric interval with respect to zero. The variable a is considered as a parameter of the transformation g, which transforms the variable $z = (x, u)$ into

$\bar{z} = (\bar{x}, \bar{u})$ of the same space.

Let $g(z; a)$ be denoted by $g_a(z)$. The set of functions g_a forms a one-parameter Lie group of transformations of the space V if the following properties hold:

1. $g_0(z) = z$ for any $z \in V$;
2. $g_a(g_b(z)) = g_{a+b}(z)$ for any $a, b, a + b \in \triangle$ and $z \in V$;
3. if $g_a(z) = z$ for any $z \in V$, then $a = 0$; and
4. $g \in C^k(V \times \triangle)$.

For a Lie group define a set of functions

$$\zeta(z) = (\zeta^1(z), \zeta^2(z), \ldots, \zeta^N(z)) = \frac{dg}{da}(z, 0).$$

The operator

$$X = \zeta^i(z)\partial_{z_i} \tag{2.1}$$

is called an infinitesimal generator of the Lie group.

A Lie group of transformations (2.1) is completely defined by the solution of the Cauchy problem:

$$\frac{d\bar{z}}{da} = \zeta(\bar{z}) \tag{2.2}$$

$$\bar{z}|_{a=0} = z. \tag{2.3}$$

Here the initial data (2.3) are taken at the point $a = 0$. Equations (2.2) are called Lie equations. The Lie Theorem establishes a one-to-one correspondence between the Lie group of transformations and the infinitesimal generator (2.1).

The space $Z = R^n \times R^m$ is prolonged by introducing the additional variables $p = (p^j_\alpha)$. Here, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index. For a multi-index the notations $|\alpha| \equiv \alpha_1 + \alpha_2 + ... + \alpha_n)$ and $\alpha, i \equiv (\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, ..., \alpha_n)$ are used. The variable (p^j_α) plays the role of a derivative

$$p^j_\alpha = \frac{\partial^{|\alpha|}u^j}{\partial x^\alpha} = \frac{\partial^{|\alpha|}u^j}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}.$$

The space J^l of the variables

$$x = (x_i),\ u = (u^j),\ p = (p^j_\alpha)$$

$$(i = 1, 2, ..., n; j = 1, 2, ..., m; |\alpha| \leq l)$$

is called the $l - th$ prolongation of the space Z.

Definition 2.1. The generator

$$\underset{l}{X} = X + \sum_{j,\alpha} \eta^j_\alpha \partial_{p^j_\alpha},\ \ (j = 1, 2, ..., m; |\alpha| \leq l),$$

with the coefficients:

$$\eta^j_{\tilde{\alpha},k} = D_k \eta^j_{\tilde{\alpha}} - \sum_i p^j_{\tilde{\alpha},i} D_k \xi^i,\ \ (|\tilde{\alpha}| \leq l - 1) \tag{2.4}$$

is called the $l - th$ prolongation of the generator X, where $X = \xi^i \partial_{x_i} + \eta^j \partial_{u^j}$, and D_k are the total derivative operators with respect to x_k:

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j,\alpha} p^j_{\alpha,k} \frac{\partial}{\partial_{p^j_\alpha}},\ \ (k = 1, 2, ..., n),$$

and $\eta^j_0 = \eta^j$.

2.2 Canonical Lie-Bäcklund Operators

Consider operators of the form

$$\xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{2.5}$$

where ξ^i and η^α depend on the independent variables x, the dependent variables u, and finite set of their derivatives. The prolongation to all derivatives is

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + ..., \tag{2.6}$$

where

$$\begin{aligned} u_{i_1 \ldots i_k}^\alpha &= \frac{\partial^k u^\alpha}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}}, \\ \zeta_i^\alpha &= D_i\big(\eta^\alpha - \xi^j u_j^\alpha\big) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}\big(\eta^\alpha - \xi^j u_j^\alpha\big) + \xi^j u_{j i_1 i_2}^\alpha, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{2.7}$$

Definition 2.2. An operator given by formula (2.6) and (2.7) is called a Lie-Bäcklund operator.

The Lie-Bäcklund operator (2.6) is often written in the abbreviated form

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + ... \tag{2.8}$$

Definition 2.3. Two Lie-Bäcklund operations X_1 and X_2 are said to be equivalent if $X_1 - X_2 = \xi^i D_i$.

Definition 2.4. The operators of the form

$$Y = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + ...$$

are called canonical Lie-Bäcklund operators.

Any Lie-Bäcklund operator X is equivalent to a canonical Lie-Bäcklund operator with the coefficients (Ibragimov, 1999).

$$\begin{aligned} \zeta^\alpha &= \eta^\alpha - \xi^j u_j^\alpha \\ \zeta_i^\alpha &= D_i\big(\zeta^\alpha\big) \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}\big(\zeta^\alpha\big) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{2.9}$$

2.3 Determining Equations

Relations between differential equations and Lie groups are presented in this section.

Consider a manifold

$$S = \{(x, u, p) | \quad F^k(x, u, p) = 0, \quad (k = 1, 2, ..., s)\},$$

which is defined by a system of partial differential equations:

$$F^k(x, u, p) = 0, \qquad (k = 1, 2, ..., s), \tag{2.10}$$

where p are partial derivatives of u with respect to x. The manifold S is assumed to be regular, i.e.,

$$rank\left(\frac{\partial(F)}{\partial(u, p)}\right) = s.$$

A manifold S is said to be invariant with respect to the group of transformations (2.1), if every point of the manifold S is moved by transformations (2.1) into this manifold S. Accordingly, if the Lie group of transformations (2.1) is admitted by system (2.10), then system (2.10) is not changed under the Lie group of transformations.

Theorem 2.1. *(Ovsiannikov, 1978) Differential equations (2.10) admit a Lie group with generator X if and only if the following equations hold*

$$\underset{l}{X} F^k(x, u, p)\Big|_{(S)} = 0, \quad (k = 1, 2, ..., s), \tag{2.11}$$

where $\underset{l}{X}$ is the prolonged operator of the generator X; the notation $|_{(S)}$ means that the relations $\underset{l}{X} F^k$ are evaluated on the manifold S.

Equations (2.11) are called determining equations.

The algorithm for finding a generator X of a Lie group admitted by differential equations (2.10) consists of the following steps:

1. Form the admitted generator

$$X = \xi^i(x,u)\partial_{x_i} + \eta^j(x,u)\partial_{u^j}$$

   with unknown coefficients $\xi^i(x,u)$, $\eta^j(x,u)$.

2. Construct the prolonged operator $\underset{l}{X}$. The coefficients of the operator $\underset{l}{X}$ are defined by formula (2.4).

3. Apply the prolonged operator $\underset{l}{X}$ to each equation of the system (2.10).

4. Split the determining equations with respect to the parametric derivatives.

5. Solve the over-determined system of equations. The solution of the determining equations give us the coefficients of an admitted generator.

2.4 Lie Algebra of Generators

Let

$$X_i = \zeta_i^\alpha(z)\partial_{z_\alpha}, \quad (i = 1, 2) \tag{2.12}$$

be two infinitesimal generators. The generator

$$X_3 = \zeta_3^\alpha(z)\partial_{z_\alpha}$$

with the coefficients

$$\zeta_3^\alpha = X_1(\zeta_2^\alpha) - X_2(\zeta_1^\alpha),$$

is called a commutator of the generators X_1 and X_2. It is denoted by

$$X_3 = [X_1, X_2].$$

The operation of commutation satisfies the properties:

(1) bilinearity:

$$[\alpha X_1 + \beta X_2, X_3] = \alpha[X_1, X_3] + \beta[X_2, X_3],$$

$$[X_1, \alpha X_2 + \beta X_3] = \alpha[X_1, X_2] + \beta[X_1, X_3],$$

where α and β are arbitrary constant,

(2) antisymmetry: $[X_1, X_2] = -[X_2, X_1]$,

(3) the Jacobi identity:

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0.$$

Definition 2.5. A vector space L with a commutator operation satisfying these properties is called a Lie algebra.

Definition 2.6. A vector space of generators L is a Lie algebra if the commutator $[X_\mu, X_\nu]$ of any two generators in L belongs to L.

Theorem 2.2. *(Ovsiannikov, 1978) A commutator is invariant with respect to any change of variables.*

Theorem 2.3. *(Ovsiannikov, 1978) The operator of prolongation commutes with the operation of taking a commutator.*

Theorem 2.4. *(Ovsiannikov, 1978) If a system* (S) *admits generators* X *and* Y*, then it admits their commutator* $[X, Y]$.

The latter theorem means that the vector space LS of all admitted generators is a Lie algebra (admitted by the system (S)). This Lie algebra is called a principal algebra. To construct exact solutions one uses subalgebras of the admitted algebra.

Let L be a Lie algebra of generators.

Definition 2.7. A vector subspace $L' \subset L$ of Lie algebra L is called a subalgebra if it is a Lie algebra.

In other words, for arbitrary vectors X_μ and X_ν from L', their commutator $[X_\mu, X_\nu]$ belongs to L'.

Definition 2.8. Let $I \subset L$ be a subspace of Lie algebra L with the property, $[X, Y] \in I, \quad \forall X \in I$ and $\forall Y \in L$. The subspace I is called an ideal.

2.5 Classification of Subalgebras

Since any solution of a system of differential equations is mapped by a transformation from admitted Lie group into a solution of the same system, the problem of separating solutions into classes of essentially different solutions appears.

A linear one-to-one map f of a Lie algebra L onto a Lie algebra K is called an isomorphism if

$$f([X_\mu, X_\nu]_L) = [f(X_\mu), f(X_\nu)]_K,$$

where the indices L and K are used to denote the commutator in the corresponding Lie algebra. An isomorphism of L onto itself is called an automorphism. The set of all subalgebras can be classified with respect to automorphism.

Let L_r be an r-dimensional Lie algebra of generators with a basis $X_1, X_2, ..., X_r$, then

$$[X_\mu, X_\nu] = c^\lambda_{\mu\nu} X_\lambda$$

for any two generators X_μ and X_ν. The constants $c^\lambda_{\mu\nu}$ are called structure constants.

Notice that two Lie algebras are isomorphic if and only if they have the same structure constants in an appropriately chosen basis.

For any $X \in L_r$, then

$$X = x_\mu X_\mu.$$

Hence, elements of L_r are represented by vector $x = (x_1, x_2, ..., x_r)$. Let L_r^A be the Lie algebra spanned by the following operators:

$$E_\mu = c_{\mu\nu}^\lambda x_\nu \frac{\partial}{\partial x_\lambda}, \;\; \mu = 1, 2, ..., r.$$

The Lie algebra L_r^A generates a group G^A of linear transformations of x_μ. These transformations determine automorphisms of the Lie algebra L_r known as inner automorphisms. This set is denoted by $Int(L_r)$. Accordingly, G^A is called the group of inner automorphisms of L_r. Two subalgebras L_p and L_q of L_r are called similar, if one can be transformed to another by an element of $Int(L_r)$. Similar subalgebras of the same dimension compose an equivalence class.

Definition 2.9. The set of all classes (one representative from each class) is called an optimal system of subalgebras.

Thus, the optimal system of subalgebras of a Lie algebra L with inner automorphisms $A = Int(L)$ is a set of subalgebras $\Theta_A(L)$ such that:

1. there are no two elements of this set which can be transformed into each other by an inner automorphism of the Lie algebra L.
2. any subalgebra of the Lie algebra L can be transformed into one of subalgebras of the set $\Theta_A(L)$.

It is known (Ovsiannikov, 1978) that the problem of finding all automorphisms is reduced to the problem for finding automorphisms A_k for the basis vectors $y = e^k$, $(k = 1, 2, ..., s)$:

$$\frac{d}{dt}\hat{x}^\lambda = \hat{x}^\alpha c_{\alpha k}^\lambda, \;\; \hat{x}_{|t=0}^\lambda = x^\lambda, \;\; (\lambda = 1, 2, ..., r).$$

Here $\{e^k\}_{k=1}^r$ is the canonical basis in R^r. The automorphism A_k corresponds to the Lie group of transformations with the generator

$$x^\alpha c_{\alpha k}^\lambda \partial_{x^\lambda}.$$

Calculations of an optimal system of subalgebras is easy enough for low-dimensional Lie algebras (Patera and Winternitz, 1977). For high-dimensional Lie algebras one can use a two-step algorithm (Ovsiannikov, 1993). This algorithm reduces the problem of constructing an optimal system of subalgebras with high dimensions to a problem with low dimensions.

Assume that the Lie algebra L is decomposed into $L = I \oplus F$, where I is a proper ideal of the Lie algebra L and F is a subalgebra. The set of inner automorphisms $A = Int(L)$ of the Lie algebra L is also decomposed $A = A_I A_F$, where A_I and A_F are subsets of A which correspond to the elements of I and F, respectively, as follows.

Let $x \in L$ be decomposed as $x = x_I + x_F$, where $x_I \in I$ and $x_F \in F$. Any automorphism $C \in A$ can be written as $C = C_I C_F$, where $C_I \in A_I$ and $C_F \in A_F$. The automorphisms C_I and C_F have the following properties:

$$C_I x_F = x_F, \ \forall x_F \in F, \ \forall C_I \in A_I,$$

$$C_F x_I \in I, \ C_F x_F \in F, \ \forall x_I \in I, \ \forall x_F \in F, \ \forall C_F \in A_F.$$

At the first step, an optimal system of subalgebras $\Theta_{A_F}(F) = (F_1, F_2, ..., F_p, F_{p+1})$ of the algebra F is formed, here $F_{p+1} = \{0\}$ and the optimal system of subalgebras $\Theta_{A_F}(F)$ is constructed with respect to the automorphisms A_F. For each subalgebra F_j of F, $(j = 1, 2, ..., p+1)$, one has to find its stabiliser $St(F_j) \subset A$ as

$$St(F_j) = \{C \in A | C(F_j) = F_j\}.$$

Note that $St(F_{p+1}) = A$.

The second step consists of forming optimal system $\Theta_{St_{F_j}}(I \oplus F_j)$, $(j = 1, 2, ..., p+1)$. The optimal system of subalgebras $\Theta_A(L)$ of the Lie algebra L is a collection of $\Theta_{St_{F_j}}(I \oplus F_j)$, $(j = 1, 2, ..., p+1)$. If the subalgebra F can also be decomposed, the two-step algorithm can be used for constructing $\Theta_{A_F}(F)$.

2.6 Invariant Solutions

For a system S of partial differential equations (2.10), the coefficients of an admitted generator are found by solving the determining equations. Then one may obtain the transformation group G admitted by solving the Lie equations.

Let H be a subgroup of G.

Definition 2.10. A solution $u = U(x)$ of equations (2.10) is called an $H-$ *invariant* solution of the system S if the manifold $u = U(x)$ is an invariant manifold with respect to any transformation by elements of the group H.

Let H be the $r-$ *parameter* subgroup generated by the generators

$$X_\nu = \xi^i_\nu \frac{\partial}{\partial x_i} + \eta^j_\nu \frac{\partial}{\partial u^j}, \tag{2.13}$$

where $i = 1, 2, ...n$, $\nu = 1, 2, ...r$ and $j = 1, 2, ..., m$. Let $k = rank(\xi^i_\nu,\ \eta^j_\nu)$, then H has $n + m - k$ functionally independent invariants

$$J_1(x, u),\ J_2(x, u), ..., J_{n+m-k}(x, u).$$

Suppose that $rank(\frac{\partial J_\beta(x,u)}{\partial u^\alpha}) = m$, where $\beta,\ \alpha = 1, 2, ...m$. Then, setting

$$v^\beta = J_\beta(x, u),\ \ \lambda^l = J_{m+l}(x, u),$$

where $\beta = 1, 2, ...m$ and $l = 1, 2, ...n - k$, one can write the representation of the invariant solutions of the system S in the form

$$v^\beta = \phi^\beta(\lambda^1,\ \lambda^2 ..., \lambda^{n-k}),\ \ (\beta = 1, 2, ...m),$$

where ϕ^β is some function of its variables.

มหาวิทยาลัยเทคโนโลยีสุรนารี

CHAPTER III

ADMITTED LIE GROUP OF DELAY DIFFERENTIAL EQUATIONS

For group classification of delay differential equations, the method of constructing and solving determining equations for constant delay was presented by Tanthanuch and Meleshko (Tanthanuch and Meleshko, 2002). In Meleshko and Moyo (Meleshko and Moyo, 2008), this method was applied to the reaction-diffusion delay partial differential equation, where a complete group classification of the equation was obtained. Further applications of group analysis to delay differential equations can be found in Pue-on and Meleshko (2010) and Tanthanuch (2012). However, at present, there are no results related with the application of group analysis to time-varying delay differential equations.

3.1 Time-Varying Delay

This section is devoted to studying a change of a delay differential equation with a time-varying $\tau = \tau(t)$ under a Lie group of transformations. For the sake of simplicity, one considers here that $u = u(t, x)$. For the more general case $u = (u_1, u_2, ..., u_r)$ and $u_i = u_i(t, x_1, x_2, ..., x_p)$, where $(i = 1, 2, ..., r)$, a similar approach can be applied.

One starts the study from a Lie group of transformations:

$$\hat{t} = \varphi(t, x, u; a), \ \ \hat{x} = \chi(t, x, u; a), \ \ \hat{u} = \psi(t, x, u; a),$$

with the generator

$$X = \eta(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \zeta(t,x,u)\frac{\partial}{\partial u}.$$

If $u_0(t,x)$ is some function, then the transformed function is

$$\widetilde{u}(\widehat{t},\widehat{x}) = \psi\left(T(\widehat{t},\hat{x},a), X(\widehat{t},\hat{x},a), u_0\left(T(\widehat{t},\hat{x},a), X(\widehat{t},\hat{x},a)\right); a\right),$$

$$t = T(\widehat{t},\hat{x},a), \quad x = X(\widehat{t},\hat{x},a),$$

where

$$\begin{aligned} \hat{t} &= \varphi\left(T(\widehat{t},\hat{x},a), X(\widehat{t},\hat{x},a), u_0\left(T(\widehat{t},\hat{x},a), X(\widehat{t},\hat{x},a)\right); a\right), \\ \hat{x} &= \chi\left(T(\widehat{t},\hat{x},a), X(\widehat{t},\hat{x},a), u_0\left(T(\widehat{t},\hat{x},a), X(\widehat{t},\hat{x},a)\right); a\right), \end{aligned} \tag{3.1}$$

and

$$t = T\left(\varphi(t,x,u;a), \chi(t,x,u;a), a\right), \quad x = X\left(\varphi(t,x,u;a), \chi(t,x,u;a), a\right). \tag{3.2}$$

If $\tau(t)$ is some delay function, then

$$\tilde{\tau} = \varphi(t,x,u(t,x);a) - \varphi(t_-,x,u(t_-,x);a),$$

where $t_- = t - \tau(t)$. Thus, the prolongation of the generator of the Lie group for the delay parameter τ is

$$X = \eta\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial \tau}, \tag{3.3}$$

where $\mu = \mu(t,x,u,t_-,u_-) = \eta(t,x,u) - \eta(t_-,x,u_-)$.

Here it is assumed that all functions are defined. Later, when the determining equations are constructed, this requirement will be omitted.

Differentiating identities (3.1) with respect to a and setting $a = 0$, one has

$$\begin{aligned} &\frac{\partial}{\partial a}(\varphi(T(\widehat{t},\hat{x},a), X(\widehat{t},\hat{x},a), u_0(T(\widehat{t},\hat{x},a), X(\widehat{t},\hat{x},a)); a))_{|a=0} \\ &= \frac{\partial}{\partial a}(T(\widehat{t},\hat{x},a))_{|a=0} + \eta(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x})) = 0, \end{aligned}$$

$$\frac{\partial}{\partial a}(\chi(T(\widehat{t},\hat{x},a),X(\widehat{t},\hat{x},a),u_0(T(\widehat{t},\hat{x},a),X(\widehat{t},\hat{x},a));a))_{|a=0}$$
$$=\frac{\partial}{\partial a}(X(\widehat{t},\hat{x},a))_{|a=0}+\xi(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x}))=0.$$

Hence,

$$\begin{aligned}\frac{\partial}{\partial a}\left(T(\widehat{t},\hat{x},a)\right)_{|a=0}&=-\eta(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x})),\\ \frac{\partial}{\partial a}\left(X(\widehat{t},\hat{x},a)\right)_{|a=0}&=-\xi(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x})).\end{aligned}\tag{3.4}$$

Since the transformed functions are defined as follows

$$\widetilde{u}(\widehat{t},\hat{x})=\psi\left(T(\widehat{t},\hat{x},a),X(\widehat{t},\hat{x},a),u_0\left(T(\widehat{t},\hat{x},a),X(\widehat{t},\hat{x},a)\right);a\right),$$

$$\tilde{\tau}(\widehat{t})=\varphi(t,x,u_0(t,x);a)-\varphi(t-\tau(t),x,u_0(t-\tau(t),x);a),$$

where $t=T(\widehat{t},\hat{x},a)$ and $x=X(\widehat{t},\hat{x},a)$ are substituted, one derives

$$\tilde{\tau}(\widehat{t})=\varphi(T(\widehat{t},\hat{x},a),X(\widehat{t},\hat{x},a),u_0(T(\widehat{t},\hat{x},a),X(\widehat{t},\hat{x},a));a)$$

$$-\varphi(T(\widehat{t},\hat{x},a)-\tau(T(\widehat{t},\hat{x},a)),X(\widehat{t},\hat{x},a),u_0(T(\widehat{t},\hat{x},a)-\tau(T(\widehat{t},\hat{x},a)),X(\widehat{t},\hat{x},a));a).$$

Because of the requirement that a solution of equation (1.3) is mapped into a solution of the same equation, one has $\tilde{\tau}(\widehat{t})=\tau(\widehat{t})$. After differentiating this equation with respect to a, setting $a=0$, and using $T(\widehat{t},\hat{x},a)_{|a=0}=\widehat{t}$, $X(\widehat{t},\hat{x},a)_{|a=0}=\hat{x}$, one obtains

$$\frac{\partial}{\partial a}\left(\tilde{\tau}(\widehat{t})\right)_{|a=0}=(1-\tau'(\widehat{t}))\eta(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x}))-\eta(\widehat{t}-\tau(\widehat{t}),\hat{x},u_0(\widehat{t}-\tau(\widehat{t}),\hat{x}))=0.\tag{3.5}$$

Identities (3.1) imply that

$$\hat{t}-\widetilde{\tau}(\widehat{t})=$$
$$\varphi\left(T(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),X(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),u_0\left(T(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),X(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a)\right);a\right),$$
$$\hat{x}=$$
$$\chi\left(T(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),X(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),u_0\left(T(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),X(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a)\right);a\right).$$

Differentiating these equations with respect to a, setting $a=0$, and using the identity (3.5), one has

$$\frac{\partial}{\partial a}\left(T(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a)\right)_{|a=0}=-(1-\tau'(\widehat{t}))\eta(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x})),\tag{3.6}$$

and

$$\frac{\partial}{\partial a}\left(X(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a)\right)_{|a=0} = -\xi\left(\widehat{t}-\tau(\widehat{t}),\hat{x},u_0\left(\widehat{t}-\tau(\widehat{t}),\hat{x}\right)\right). \tag{3.7}$$

Because of

$$\widetilde{u}(\widehat{t},\hat{x}) = \psi\left(T(\widehat{t},\hat{x},a),X(\widehat{t},\hat{x},a),u_0\left(T(\widehat{t},\hat{x},a),X(\widehat{t},\hat{x},a)\right);a\right),$$

one gets

$$\widetilde{u}(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x}) =$$
$$\psi(T(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),X(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),u_0(T(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a),X(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x},a));a).$$

Hence

$$\begin{aligned}\frac{\partial}{\partial a}\left(\widetilde{u}(\widehat{t},\hat{x})\right)_{|a=0} = \zeta(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x})) - u_{0t}(\widehat{t},\hat{x})\eta(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x}))\\ -u_{0x}(\widehat{t},\hat{x})\xi(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x}))\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\frac{\partial}{\partial a}\left(\widetilde{u}(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x})\right)_{|a=0} = \zeta(\widehat{t}-\tau(\widehat{t}),\hat{x},u_0(\widehat{t}-\tau(\widehat{t}),\hat{x}))\\ -u_{0t}(\widehat{t}-\tau(\widehat{t}),\hat{x})(1-\tau'(\widehat{t}))\eta(\widehat{t},\hat{x},u_0(\widehat{t},\hat{x}))\\ -u_{0x}(\widehat{t}-\tau(\widehat{t}),\hat{x})\xi(\widehat{t}-\tau(\widehat{t}),\hat{x},u_0(\widehat{t}-\tau(\widehat{t}),\hat{x})).\end{aligned} \tag{3.9}$$

Let $\widetilde{\widetilde{u}}(\widehat{t},\hat{x}) = \widetilde{u}(\widehat{t}-\tau(\widehat{t}),\hat{x})$ and $\bar{u}_0(t,x) = u_0(t-\tau(t),x)$, then

$$\bar{u}_{0t}(\widehat{t},\hat{x}) = u_{0t}(\widehat{t}-\tau(\widehat{t}),\hat{x})(1-\tau'(\widehat{t}))$$

and

$$\bar{u}_{0x}(\widehat{t},\hat{x}) = u_{0x}(\widehat{t}-\tau(\widehat{t}),\hat{x}).$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial a}\left(\widetilde{u}(\widehat{t}-\widetilde{\tau}(\widehat{t}),\hat{x})\right)_{|a=0} = \frac{\partial}{\partial a}\left(\widetilde{\widetilde{u}}(\widehat{t},\hat{x})\right)_{|a=0} = \zeta(\widehat{t}-\tau(\widehat{t}),\hat{x},\bar{u}_0(\widehat{t},\hat{x}))\\ -\bar{u}_{0t}(\widehat{t},\hat{x})\eta(\widehat{t},\hat{x},\bar{u}_0(\widehat{t}+\tau(\widehat{t}),\hat{x})) - \bar{u}_{0x}(\widehat{t},\hat{x})\xi(\widehat{t}-\tau(\widehat{t}),\hat{x},\bar{u}_0(\widehat{t},\hat{x})).\end{aligned} \tag{3.10}$$

Hence

$$\frac{\partial}{\partial a}\left(\widetilde{\widetilde{u}}_{\widehat{t}}(\widehat{t},\hat{x})\right)_{|a=0} = \frac{\partial}{\partial a}\left(\frac{\partial}{\partial\widehat{t}}\left(\widetilde{\widetilde{u}}(\widehat{t},\hat{x})\right)\right)_{|a=0} = \frac{\partial}{\partial\widehat{t}}\left(\frac{\partial}{\partial a}\left(\widetilde{\widetilde{u}}(\widehat{t},\hat{x})\right)_{|a=0}\right),$$

and

$$\frac{\partial}{\partial a}\left(\widetilde{\bar{u}}_{\hat{x}}(\widehat{t},\hat{x})\right)_{|a=0}=\frac{\partial}{\partial a}\left(\frac{\partial}{\partial \widehat{x}}\left(\widetilde{\bar{u}}(\widehat{t},\hat{x})\right)\right)_{|a=0}=\frac{\partial}{\partial \hat{x}}\left(\frac{\partial}{\partial a}\left(\widetilde{\bar{u}}(\widehat{t},\hat{x})\right)_{|a=0}\right).$$

In a similar way, one has

$$\frac{\partial}{\partial a}\left(\widetilde{u}_{\widehat{t}}(\widehat{t},\hat{x})\right)_{|a=0}=\frac{\partial}{\partial \widehat{t}}\left(\frac{\partial}{\partial a}\left(\widetilde{u}(\widehat{t},\hat{x})\right)_{|a=0}\right),$$
$$\frac{\partial}{\partial a}\left(\widetilde{u}_{\hat{x}}(\widehat{t},\hat{x})\right)_{|a=0}=\frac{\partial}{\partial \hat{x}}\left(\frac{\partial}{\partial a}\left(\widetilde{u}(\widehat{t},\hat{x})\right)_{|a=0}\right),$$

and

$$\frac{\partial}{\partial a}\left(\widetilde{\tau}'(\widehat{t})\right)_{|a=0}=\frac{\partial}{\partial \widehat{t}}\left(\frac{\partial}{\partial a}\left(\widetilde{\tau}(\widehat{t})\right)_{|a=0}\right)=0.$$

Therefore, one derives the prolongation of the canonical Lie-Bäcklund which is equivalent to the generator X:

$$\begin{gathered}\bar{X}=\zeta^{u}\partial_{u}+\zeta^{\bar{u}}\partial_{\bar{u}}+\zeta^{u_t}\partial_{u_t}+\zeta^{u_x}\partial_{u_x}\\ +\zeta^{\bar{u}_t}\partial_{\bar{u}_t}+\zeta^{\bar{u}_x}\partial_{\bar{u}_x}+\zeta^{u_{tt}}\partial_{u_{tt}}+\zeta^{u_{xx}}\partial_{u_{xx}}+....,\end{gathered}\tag{3.11}$$

where the coefficients are

$$\begin{gathered}\zeta^{u}=\zeta(t,x,u(t,x))-u_t(t,x)\eta(t,x,u(t,x))-u_x(t,x)\xi(t,x,u(t,x)),\\ \zeta^{\bar{u}}=\zeta(t-\tau(t),x,u(t-\tau(t),x))-\bar{u}_t(t,x)\eta(t,x,u(t,x))\\ -\bar{u}_x(t,x)\xi(t-\tau(t),x,u(t-\tau(t),x)),\\ \zeta^{u_t}=D_t\left(\zeta^{u}\right),\,\zeta^{u_x}=D_x\left(\zeta^{u}\right),\,\zeta^{\bar{u}_t}=D_t\left(\zeta^{\bar{u}}\right),\,\zeta^{\bar{u}_x}=D_x\left(\zeta^{\bar{u}}\right),\\ \zeta^{u_{tt}}=D_t\left(D_t\left(\zeta^{u}\right)\right),\;\;\zeta^{u_{xx}}=D_x\left(D_x\left(\zeta^{u}\right)\right)\\ ....\end{gathered}$$

with $\bar{u}(t,x)=u(t-\tau(t),x)$, D_t and D_t are the total derivatives with respect to variables t and x, respectively.

Notice that the coefficient $\eta(t,x,u)$ has to satisfy the condition (3.5):

$$(1-\tau'(t))\eta(t,x,u(t,x))=\eta(t-\tau(t),x,u(t-\tau(t),x)).\tag{3.12}$$

In the particular case where the delay term $\tau(t)$ is constant, equation (3.12) becomes $\eta(t, x, u(t, x)) = \eta(t-\tau, x, u(t-\tau, x))$, which implies that

$$\begin{aligned}\frac{\partial}{\partial a}\left(\widetilde{u}(\widehat{t}-\widetilde{\tau}, \hat{x})\right)_{|a=0} &= \zeta(\widehat{t}-\tau, \hat{x}, u_0(\widehat{t}-\tau, \hat{x})) \\ &-u_{0t}(\widehat{t}-\tau, \hat{x})\eta(\widehat{t}-\tau, \hat{x}, u_0(\widehat{t}-\tau, \hat{x})) \\ &-u_{0x}(\widehat{t}-\tau, \hat{x})\xi(\widehat{t}-\tau, \hat{x}, u_0(\widehat{t}-\tau, \hat{x}))\end{aligned} \tag{3.13}$$

by equation (3.9) and $\mu = 0$ in (3.3).

Hence, under this case, one gets the prolongation of the canonical Lie-Bäcklund which is equivalent to the generator X:

$$\begin{aligned}\bar{X} &= \zeta^u\partial_u + \zeta^{\bar{u}}\partial_{\bar{u}} + \zeta^{u_t}\partial_{u_t} + \zeta^{u_x}\partial_{u_x} \\ &+\zeta^{\bar{u}_t}\partial_{\bar{u}_t} + \zeta^{\bar{u}_x}\partial_{\bar{u}_x} + \zeta^{u_{tt}}\partial_{u_{tt}} + \zeta^{u_{xx}}\partial_{u_{xx}} + ...,\end{aligned} \tag{3.14}$$

where the coefficients are

$$\begin{gathered}\zeta^u = \zeta(t, x, u(t, x)) - u_t(t, x)\eta(t, x, u(t, x)) - u_x(t, x)\xi(t, x, u(t, x)), \\ \zeta^{\bar{u}} = \zeta(t-\tau, x, u(t-\tau, x)) - \bar{u}_t(t, x)\eta(t-\tau, x, u(t-\tau, x)) \\ -\bar{u}_x(t, x)\xi(t-\tau, x, u(t-\tau, x)), \\ \zeta^{u_t} = D_t(\zeta^u), \zeta^{u_x} = D_x(\zeta^u), \zeta^{\bar{u}_t} = D_t(\zeta^{\bar{u}}), \zeta^{\bar{u}_x} = D_x(\zeta^{\bar{u}}), \\ \zeta^{u_{tt}} = D_t(D_t(\zeta^u)), \zeta^{u_{xx}} = D_x(D_x(\zeta^u)) \\ ....\end{gathered}$$

with $\bar{u}(t, x) = u(t-\tau, x)$, τ is constant, D_t and D_t are the total derivatives with respect to variables t and x, respectively.

3.2 Equivalence Lie Group of Transformations for Constant Delay

Most differential equations include arbitrary elements: constants and functions of the independent and dependent variables.

Definition 3.1. A transformation of the independent and dependent variables as well as the arbitrary elements is called an equivalence transformation of a system of differential equations if it preserves the differential structure of the equations themselves.

Definition 3.2. If a set of equivalence transformations of partial differential equations composes a Lie group of transformations, then the Lie group is called an equivalence Lie group.

For finding an equivalence Lie group of partial differential equations one can apply the infinitesimal approach (Ovsiannikov, 1978)*.

For delay differential equations, the notion of an equivalence Lie group was introduced in (Meleshko and Moyo, 2008; Grigoriev et al., 2010): it is a Lie group corresponding to an infinitesimal generator which satisfies determining equations. This Lie group of transformations provides a set of potential equivalence transformations. Notice that for partial differential equations these transformations are simply equivalence transformations.

For the sake of simplicity, one considers a one-dimensional dependent variable $u \in R^1$.

The class of considered differential equations with time delay is of the form

$$F^k(t, x, u, \bar{u}, p, \phi) = 0; \ \ (k = 1, 2, \dots, s), \tag{3.15}$$

where t and x are both the independent variables, $\bar{u} = u(x, t - \tau)$, τ is constant, p are derivatives of u with respect to x and $\phi = \phi(t, x, u, \bar{u})$ is arbitrary function.

Denoting $v(t, x) = \bar{u}(t, x)$, one gets a class of differential equations

$$G^k(t, x, u, v, p, \phi) = 0; \ \ (k = 1, 2, \dots, s + 1), \tag{3.16}$$

*Generalization of this approach is given in (Meleshko, 1996) where $G^k = F^k$ for $k = 1, 2, ..., s$, $G^{s+1} = \bar{u} - v$.

Here $(t, x, u, v) \in V \subset R^{1+n+1+1}$, and $\phi : V \to R$. The problem is to construct transformations of the space $R^{1+n+1+1+1}(t, x, u, v, \phi)$ which preserve the equations by changing only the representatives $\phi = \phi(t, x, u, v)$. For this purpose, a one-parameter Lie group of transformations of the space $R^{1+n+1+1+1}(t, x, u, v, \phi)$ with the group parameter a is applied

$$\widetilde{t} = f^t(t, x, u, v, \phi; a), \qquad \widetilde{x} = f^x(t, x, u, v, \phi; a),$$
$$\widetilde{u} = f^u(t, x, u, v, \phi; a), \qquad \widetilde{\phi} = f^\phi(t, x, u, v, \phi; a).$$

The generator of this group has the form

$$X^e = \gamma\partial_t + \xi^i\partial_{x_i} + \eta^u\partial_u + \eta^v\partial_v + \eta^\phi\partial_\phi, \qquad (3.17)$$

where the coordinates of the generator X^e are

$$\gamma = \gamma(t, x, u, v, \phi), \quad \xi^i = \xi^i(t, x, u, v, \phi),$$
$$\eta^u = \eta^u(t, x, u, v, \phi), \quad \eta^v = \eta^v(t, x, u, v, \phi),$$
$$\eta^\phi = \eta^\phi(t, x, u, v, \phi),$$
$$(i = 1, \dots, n).$$

Notice that earlier (Ovsiannikov, 1978), it was assumed that

$$\frac{\partial\xi^i}{\partial\phi} = 0, \ \frac{\partial\gamma}{\partial\phi} = 0, \ \frac{\partial\eta^u}{\partial\phi} = 0, \ \frac{\partial\eta^v}{\partial\phi} = 0,$$
$$(i = 1, \dots, n).$$

The canonical Lie-Bäcklund operator equivalent to the generator X^e is

$$\hat{X}^e = \zeta^u\partial_u + \zeta^v\partial_v + \zeta^\phi\partial_\phi. \qquad (3.18)$$

Here the coordinates are

$$\zeta^u = \eta^u - u_{x_i}\xi^i - u_t\gamma, \quad \zeta^v = \eta^v - v_{x_i}\xi^i - v_t\gamma,$$
$$\zeta^\phi = \eta^\phi - \xi^i D^e_{x_i}\phi - \gamma D^e_t\phi,$$

where

$$D^e_\lambda = \partial_\lambda + u_\lambda \partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi + ...$$
$$(\lambda = t, x_i).$$

The determining equations are

$$\widetilde{X}^e G^k(t, x, u, v, p, \phi)|_{[S]} = 0, \qquad (k = 1, 2, \ldots, s+1), \tag{3.19}$$

where the sign $|[S]$ means that the equations $\widetilde{X}^e G^k(x, u, p, \phi)$ are considered on any solution of equations (3.16). Here, $\widetilde{X}^e$ is the prolonged generator of the equivalence Lie group. Because equation (3.16) does not depend on differentiating of ϕ with respect to variables t, x_i, u and v, then $\widetilde{X}^e$ has

$$\widetilde{X}^e = \hat{X}^e + \zeta^{u_t}\partial_{u_t} + \zeta^{v_t}\partial_{v_t} + \zeta^{u_{x_i}}\partial_{u_{x_i}}$$
$$+\zeta^{v_{x_i}}\partial_{v_{x_i}} + \zeta^{u_{tt}}\partial_{u_{tt}} + \zeta^{u_{x_i x_j}}\partial_{u_{x_i x_j}} + ...,$$

where

$$\zeta^{u_{x_i}} = D^e_{x_i}\zeta^u,\ \zeta^{u_{x_i x_j}} = D^e_{x_j}\zeta^{u_{x_i}},\ \zeta^{v_t} = D^e_t\zeta^v,\ \ \zeta^{u_t} = D^e_t\zeta^u,\ \zeta^{u_{tt}} = D^e_t\zeta^{u_t}... \tag{3.20}$$

Noting that γ, ξ^i, η^u and η^v do not depend on ϕ, one derives that the coefficients of the prolonged generator of the equivalence Lie group become

$$\zeta^{u_{x_i}} = D_{x_i}\zeta^u,\ \zeta^{u_{x_i x_j}} = D_{x_j}\zeta^{u_{x_i}},\ \zeta^{v_t} = D_t\zeta^v,\ \ \zeta^{u_t} = D_t\zeta^u,\ \zeta^{u_{tt}} = D_t\zeta^{u_t}... \tag{3.21}$$

where D_{x_i} and D_t are operators of the total derivatives with respect to x_i and t, respectively.

3.3 Determining Equation of The Time-Varying Delay

Let the system of delay differential equations be of the form

$$F^k(t, x, u, \bar{u}, p) = 0;\ \ (k = 1, 2, \ldots, s). \tag{3.22}$$

Consider a one-parameter Lie group of transformations of the space $R^{1+n+1}(t,x,u)$ with the group parameter a is given by

$$\widetilde{t} = f^t(t,x,u;a), \qquad \widetilde{x} = f^x(t,x,u;a), \qquad \widetilde{u} = f^u(t,x,u;a).$$

The generator of this group has the form

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial t} + \zeta \frac{\partial}{\partial u}, \tag{3.23}$$

where the coordinates are

$$\xi^i = \xi^i(t,x,u), \qquad \eta = \eta(t,x,u), \qquad \zeta = \zeta(t,x,u),$$
$$(i = 1, \ldots, n).$$

The determining equations are defined as follows

$$\bar{X}F^k(t,x,u,\bar{u},p)|_{[S]} = 0, \ (k = 1,2,\ldots,s), \ (1-\tau'(t))\eta = \bar{\eta}. \tag{3.24}$$

Here, $\bar{X}$ is the prolongation of the canonical Lie-Bäcklund operator equivalent to the generator X, namely,

$$\bar{X} = \zeta^u \partial_u + \zeta^{\bar{u}} \partial_{\bar{u}} + \zeta^{u_t} \partial_{u_t} + \zeta^{u_{x_i}} \partial_{u_{x_i}}$$
$$+ \zeta^{u_{tt}} \partial_{u_{tt}} + \zeta^{u_{x_i x_j}} \partial_{u_{x_i x_j}} + \ldots,$$

where the coefficients

$$\zeta^u = \zeta - u_{x_i}\xi^i - u_t\eta, \ \zeta^{\bar{u}} = \bar{\zeta} - \bar{u}_{x_i}\bar{\xi}^i - \bar{u}_t\eta,$$
$$\zeta^{u_{x_i}} = D_{x_i}\zeta^u, \ \zeta^{u_{x_i x_j}} = D_{x_i}\zeta^{u_{x_j}},$$
$$\zeta^{u_t} = D_t\zeta^u, \ \zeta^{u_{tt}} = D_t\zeta^{u_t},$$

....

The sign bar over a function $f(x,t,u)$ means $\bar{f}(x,t) = f(x, t-\tau(t), u(x, t-\tau(t)))$. In particular, for a function $\bar{\eta}(x,t,u)$ it is defined as $\eta(x, t-\tau(t), u(x, t-\tau(t)))$.

CHAPTER IV

GROUP ANALYSIS OF THE ONE-DIMENSIONAL NONLINEAR KLEIN-GORDON EQUATION WITH CONSTANT DELAY

In this chapter, a complete Lie group classification of the one-dimensional nonlinear delay Klein-Gordon equation

$$u_{tt} = u_{xx} + g(u, \bar{u}), \;\; g_{\bar{u}}(u, \bar{u}) \neq 0, \tag{4.1}$$

where $\bar{u} = u(t-\tau, x)$, τ is constant, is presented. For group classification it is also necessary to use the equivalence Lie group of equation (4.1) which is presented in the next section. In further sections the determining equation is derived and its general solution is found. Then the complete group classification and representations of all invariant solutions are obtained.

4.1 Equivalence Lie Group of Equation (4.1)

To simplify the study, introduce the new dependent variable v, which is related with u by the formula

$$v(x, t) = u(x, t - \tau). \tag{4.2}$$

Equation (4.1) becomes the partial differential equation with two dependent variables

$$S \equiv u_{tt} - (u_{xx} + g) = 0, \tag{4.3}$$

where the arbitrary element is $g = g(u, v)$. The generator of the Lie group of equivalence transformation takes the form

$$X^e = \xi\partial_x + \eta\partial_t + \zeta\partial_u + \zeta^v\partial_v + \zeta^g\partial_g,$$

where $\xi,\ \eta,\ \zeta,\ \zeta^v$ and ζ^g are functions of the variables $t,\ x,\ u,\ v$ and g.

The Lie-Bäcklund form of this generator is

$$\hat{X}^e = \zeta^u\partial_u + \zeta^{v_1}\partial_v + \zeta^{g_1}\partial_g$$

where

$$\zeta^u = \zeta - \xi u_x - \eta u_t$$

$$\zeta^{g_1} = \zeta^g - \xi D_x^e g - \eta D_t^e g$$

$$\zeta^{v_1} = \zeta^v - \xi v_x - \eta v_t.$$

The prolonged operator for the equivalence Lie group is

$$\tilde{X}^e = \hat{X}^e + \zeta^{u_x}\partial_{u_x} + \zeta^{u_t}\partial_{u_t} + \zeta^{u_{xx}}\partial_{u_{xx}} + \zeta^{u_{tt}}\partial_{u_{tt}}.$$

Applying the algorithm described earlier to equation (4.3), one obtains the determining equation

$$\left(\tilde{X}^e(u_{tt} - (u_{xx} + g))\right)_{|(4.3)} = 0$$

or

$$(\zeta^{u_{tt}} - \zeta^{u_{xx}} - \zeta^g + \xi D_x g + \eta D_t g)_{|(S)} = 0, \tag{4.4}$$

where

$$\zeta^{u_t} = D_t^2(\zeta - \xi u_x - \eta u_t),\ \zeta^{u_{xx}} = D_x^2(\zeta - \xi u_x - \eta u_t), \tag{4.5}$$

D_x and D_t denote the total derivatives with respect to x and t respectively.

The determining equation related with the equation (4.2) is

$$\begin{gathered}\{\zeta^v(z(t,x)) - \zeta(z(t-\tau,x)) - v_t(t,x)\,(\xi(z(t,x)) - \xi(z(t-\tau,x))) \\ -v_x(t,x)\,(\eta(z(t,x)) - \eta(z(t-\tau,x)))\}_{|(S)} = 0,\end{gathered} \tag{4.6}$$

where

$$z(t,x) = (t, x, u(t,x), v(t,x), g\,(u(t,x), v(t,x)))\,.$$

Substituting the coefficients (4.5) into (4.4) and replacing the derivatives

$$u_{tt} = u_{xx} + g, \quad v_{tt} = v_{xx} + \bar{g}, \quad u_{ttx} = v_x g_v + g_u u_x + u_{xxx},$$

$$u_{ttt} = v_t g_v + g_u u_t + u_{xxt},$$

found from (4.3), the determining equation (4.4) becomes

$$\begin{aligned}
&-2\eta_{ut}u_t^2 - 2\eta_{vt}u_t v_t - \eta_{tt}u_t - 2\eta_t u_{xx} - 2\eta_t g - 2\eta_{uv}u_t^2 v_t + 2\eta_{uv}u_t u_x v_x\\
&+2\eta_{ux}u_t u_x - \eta_{uu}u_t^3 + \eta_{uu}u_t u_x^2 - 3\eta_u g u_t - 2\eta_u u_t u_{xx} + 2\eta_u u_{tx}u_x + 2\eta_{vx}u_t v_x\\
&-\eta_{vv}u_t v_t^2 + \eta_{vv}u_t v_x^2 - 2\eta_v g v_t - \eta_v \bar{g} u_t + 2\eta_v u_{tx}v_x - 2\eta_v u_{xx}v_t + \eta_{xx}u_t + 2\eta_x u_{tx}\\
&-2\xi_{ut}u_t u_x - 2\xi_{vt}u_x v_t - \xi_{tt}u_x - 2\xi_t u_{tx} - 2\xi_{uv}u_t u_x v_x + 2\xi_{uv}u_x^2 v_x + 2\xi_{ux}u_x^2\\
&-\xi_{uu}u_t^2 u_x + \xi_{uu}u_x^3 - \xi_u g u_x - 2\xi_u u_t u_{tx} + 2\xi_u u_{xx}u_x + 2\xi_{vx}u_x v_x - \xi_{vv}u_x v_t^2\\
&+\xi_{vv}u_x v_x^2 - 2\xi_v \bar{g} u_x - 2\xi_v u_{tx}v_t - 2\xi_v u_{tx}v_t + 2\xi_v u_{xx}v_x + \xi_{xx}u_x\\
&+2\xi_x u_{xx} + 2\zeta_{ut}u_t + 2\zeta_{vt}v_t + \zeta_{tt} + 2\zeta_{uv}u_t v_t - 2\zeta_{uv}u_x v_x - 2\zeta_{ux}u_x\\
&+\zeta_{uu}u_t^2 - \zeta_{uu}u_x^2 + \zeta_u g - 2\zeta_{vx}v_x + \zeta_{vv}v_t^2 - \zeta_{vv}v_x^2 + \zeta_v \bar{g} - \zeta_{xx} - \zeta^g = 0.
\end{aligned}$$

Splitting this equation with respect to u_x, u_t, v_x, v_t, u_{tx}, u_{xx} and $\bar{g}$, one can obtain

$$-2\eta_t g + \zeta_{tt} + \zeta_u g - \zeta_{xx} - \zeta^g = 0, \tag{4.7}$$

$$\eta_x = \xi_t,\ \eta_u = 0,\ \eta_v = 0,\ \eta_t = \xi_x,\ \xi_v = 0,\ \xi_u = 0, \tag{4.8}$$

$$\zeta_{uu} = 0,\ \zeta_{ux} = 0,\ \zeta_{ut} = 0,\ \zeta_v = 0. \tag{4.9}$$

From (4.8), one has

$$\eta_{xx} = \eta_{tt},\ \xi_{xx} = \xi_{tt}. \tag{4.10}$$

The general solution of (4.9) is $\zeta = k_1 u + \zeta_0$, where k_1 is constant, and $\zeta_0 = \zeta_0(t,x)$.

Splitting the determining equation (4.6) with respect to v_x, v_t, one gets

$$\zeta^v = \bar{\zeta},\ \eta = \bar{\eta},\ \xi = \bar{\xi}, \tag{4.11}$$

where

$$\bar{\zeta} = \zeta(z(t-\tau,x)), \bar{\xi} = \xi(z(t-\tau,x)), \bar{\eta} = \eta(z(t-\tau,x)).$$

The assumption that the function g does not depend on t and x leads to the equations $\zeta_t = 0,\ \zeta_x = 0,\ \zeta_t^g = 0, \zeta_x^g = 0$, which implies that $\zeta = k_1u + k_4$. From equation (4.7) one obtains $\zeta^g = -2\eta_t + k_1 g$, differentiating it with respect to x and t respectively, one obtains $\eta_{tt} = 0$ and $\eta_{tx} = 0$. Equations (4.8) , (4.10) and (4.11) give $\eta = k_2,\ \ \xi = k_3$, which implies that $\zeta^g = k_1 g$.

Therefore

$$\xi = k_3,\ \eta = k_2,\ \zeta = k_1 u + k_4,\ \zeta^g = k_1 g.$$

Equation (4.6) becomes $\zeta^v(z(t,x)) = k_1 v + k_4$. Hence the generators of the equivalence Lie group are

$$X^e = k_1 X_1^e + k_2 X_2^e + k_3 X_3^e + k_4 X_4^e + \zeta^v \partial_v,$$

where

$$X_1^e = g\partial_g + u\partial_u,\ X_2^e = \partial_t,\ X_3^e = \partial_x,\ X_4^e = \partial_u.$$

มหาวิทยาลัยเทคโนโลยีสุรนารี

4.2 Admitted Lie Group of Equation (4.1)

This section is devoted to the study of admitted Lie groups of the one-dimensional nonlinear Klein-Gordon equation with constant delay. For finding an admitted Lie group, the algorithm for constructing determining equation of delay differential equation is used in previous chapter. Let the generator of a Lie group admitted by equation (4.1) be

$$X = \xi\partial_x + \eta\partial_t + \zeta\partial_u,$$

where ξ, η and ζ are functions of x, t and u.

The prolongation of the canonical Lie-Bäcklund operator equivalent to the generator X is

$$\bar{X} = \zeta^u \partial_u + \zeta^{u_t} \partial_{u_t} + \zeta^{u_x} \partial_{u_x} + \zeta^{u_{tt}} \partial_{u_{tt}} + \zeta^{u_{xx}} \partial_{u_{xx}} + \zeta^{\bar{u}} \partial_{\bar{u}}$$

where the coefficients are

$$\zeta^u = \zeta - u_x \xi - u_t \eta,\ \zeta^{\bar{u}} = \bar{\zeta} - \bar{u}_x \bar{\xi} - \bar{u}_t \bar{\eta},\ \zeta^{u_x} = D_x \zeta^u,$$
$$\zeta^{u_{xx}} = D_x \zeta^{u_x},\ \zeta^{u_t} = D_t \zeta^u,\ \zeta^{u_{tt}} = D_t \zeta^{u_t}.$$

Here D_x and D_t are operators of the total derivatives with respect to x and t, respectively, the bar over a function $f(x,t,u)$ means $\bar{f} = f(x, t-\tau, u(x, t-\tau))$.

According to the algorithm for constructing the determining equations of an admitted Lie group (Grigoriev et al., 2010), one obtains

$$\left(\bar{X}(u_{tt} - (u_{xx} + g))\right)_{|(4.1)} = 0$$

or

$$(-\zeta^{u_{tt}} + \zeta^{u_{xx}} + g_u \zeta^u + g_{\bar{u}} \zeta^{\bar{u}})_{|(4.1)} = 0. \tag{4.12}$$

It is assumed that equation (4.12) is satisfied for any solution $u(x,\, t)$ of equation (4.1).

Substituting the coefficients of the prolonged generator into the determining equation (4.12), and replacing the derivatives found from equation (4.1) and its prolongations

$$u_{tt} = u_{xx} + g,\ u_{ttx} = \bar{u}_x g_{\bar{u}} + g_u u_x + u_{xxx},\ u_{ttt} = \bar{u}_t g_{\bar{u}} + g_u u_t + u_{xxt},$$

the determining equation (4.12) becomes

$$
\begin{gathered}
+2\eta_{ut}u_t^2+\eta_{tt}u_t+2\eta_t u_{xx}+2\eta_t g-2\eta_{ux}u_t u_x+\eta_{uu}u_t^3-\eta_{uu}u_t u_x^2\\
+3\eta_u g u_t+2\eta_u u_t u_{xx}-2\eta_u u_{tx}u_x-\eta_{xx}u_t-2\eta_x u_{tx}\\
+2\xi_{ut}u_t u_x+\xi_{tt}u_x+2\xi_t u_{tx}-2\xi_{ux}u_x^2-\xi_{uu}u_t^2 u_x-\xi_{uu}u_x^3\\
+\xi_u g u_x+2\xi_u u_t u_{tx}-2\xi_u u_{xx}u_x-\xi_{xx}u_x-2\xi_x u_{xx}\\
-2\zeta_{ut}u_t-\zeta_{tt}+2\zeta_{ux}u_x-\zeta_{uu}u_t^2+\zeta_{uu}u_x^2-\zeta_u g+\zeta_{xx}\\
+g_u\zeta+g_{\bar{u}}\eta\bar{u}_t-g_{\bar{u}}\bar{\eta}\bar{u}_t+g_{\bar{u}}\bar{u}_x\xi-g_{\bar{u}}\bar{u}_x\bar{\xi}+g_{\bar{u}}\bar{\zeta}=0.
\end{gathered}
$$

Splitting this equation with respect to u_x, u_t, $\bar{u}_x$, $\bar{u}_t$, u_{tx}, u_{xx}, and using the condition that $g_{\bar{u}} \neq 0$, one obtains

$$2\eta_t g+g_u\zeta+g_{\bar{u}}\bar{\zeta}-\zeta_{tt}-\zeta_u g+\zeta_{xx}=0, \tag{4.13}$$

$$\eta=\bar{\eta},\ \ \xi=\bar{\xi}, \tag{4.14}$$

$$\eta_u=0,\ \ \xi_u=0, \tag{4.15}$$

$$\eta_t=\xi_x,\ \ \eta_x=\xi_t, \tag{4.16}$$

$$\zeta_{uu}=0,\ \ \zeta_{ux}=0,\ \ \zeta_{ut}=0. \tag{4.17}$$

The general solution of equations (4.15) and (4.16) is

$$\xi(x,t,u)=\varphi(x+t)+\psi(x-t),\ \ \eta(x,t,u)=\varphi(x+t)-\psi(x-t),$$

where the functions φ and ψ are arbitrary functions of a single independent variable. Conditions (4.14) give that these functions have to be periodic with the period τ:

$$\varphi(z)=\varphi(z-\tau),\ \psi(z)=\psi(z-\tau).$$

Remark. One can derive that if $\eta_t = 0$, then η and ξ are constant.

From equation (4.17), one gets

$$\zeta=c_1u+\zeta_0,$$

where c_1 is constant and $\zeta_0 = \zeta_0(x,t)$. Equation (4.13) becomes

$$\zeta_{0xx} - \zeta_{0tt} + c_1(ug_u + \bar{u}g_{\bar{u}} - g) + \zeta_0 g_u + \bar{\zeta}_0 g_{\bar{u}} + 2\eta_t g = 0. \qquad (4.18)$$

Equation (4.18) is a classifying equation. For the kernel of admitted Lie groups one has to assume that equation (4.18) is satisfied for any function $g(u,\bar{u})$. This gives that $c_1 = 0$, $\zeta_0 = 0$ and $\eta_t = 0$. Hence

$$\eta = c_2 x + c_3, \xi = c_2 t + c_4,$$

where c_i, $(i = 2,3,4)$ are constant. Substituting into equation (4.14), one has that $c_2 = 0$. Hence, the generators

$$X_1 = \partial_t,\ X_2 = \partial_x$$

compose a basis of the kernel of admitted Lie algebras of equation (4.1).

4.3 Extensions of the Kernel

Extensions of the kernel of admitted Lie algebras are additional symmetries to the kernel which are admitted by equations for a particular function $g(u,\bar{u})$. In this section the extensions are found.

Differentiating equation (4.18) with respect to u and $\bar{u}$ respectively, one obtains

$$\zeta_0 g_{uu} + \bar{\zeta}_0 g_{u\bar{u}} = -c_1(ug_{uu} + \bar{u}g_{\bar{u}u}) - 2\eta_t g_u. \qquad (4.19)$$

$$\zeta_0 g_{\bar{u}u} + \bar{\zeta}_0 g_{\bar{u}\bar{u}} = -c_1(ug_{\bar{u}u} + \bar{u}g_{\bar{u}\bar{u}}) - 2\eta_t g_{\bar{u}}. \qquad (4.20)$$

Equations (4.19) and (4.20) are linear algebraic equations with respect to ζ_0 and $\bar{\zeta}_0$. The determinant of the matrix of this linear system of equations is equal to

$$\Delta = g_{\bar{u}u}^2 - g_{uu}g_{\bar{u}\bar{u}}.$$

4.3.1 Case $\Delta \neq 0$

Since $\Delta \neq 0$, one can find ζ_0 and $\bar{\zeta_0}$ from equations (4.19) and (4.20)

$$\zeta_0 = -c_1 u + 2\Delta^{-1}\eta_t(g_u g_{\bar{u}\bar{u}} - g_{\bar{u}} g_{\bar{u}u}), \tag{4.21}$$

$$\bar{\zeta_0} = -c_1 \bar{u} + 2\Delta^{-1}\eta_t(g_{\bar{u}} g_{uu} - g_u g_{\bar{u}u}). \tag{4.22}$$

Differentiating equation (4.21) with respect to u and $\bar{u}$ respectively, one gets

$$2\eta_t \frac{\partial}{\partial u}(\Delta^{-1}(g_u g_{\bar{u}\bar{u}} - g_{\bar{u}} g_{\bar{u}u})) = c_1, \ \ \eta_t \frac{\partial}{\partial \bar{u}}(\Delta^{-1}(g_u g_{\bar{u}\bar{u}} - g_{\bar{u}} g_{\bar{u}u})) = 0. \tag{4.23}$$

Notice that by virtue of the Remark and the first equation of (4.23), the extension of the kernel of admitted Lie algebras only occurs if $\eta_t \neq 0$. Hence, for the existence of the extension the second equation of (4.23) implies that

$$\Delta^{-1}(g_{\bar{u}} g_{\bar{u}u} - g_u g_{\bar{u}\bar{u}}) = h_1,$$

where $h_1 = h_1(u)$ is some function. Substituting the last expression into the first equation in (4.23), one finds that

$$h_1(u) = k_1 u + k_{10}, \ \ c_1 = 2\eta_t k_1,$$

where k_1 and k_{10} are constant. By virtue of the periodicity of η, the last equation gives that if $k_1 \neq 0$, then $\eta_t = 0$. Thus, for the existence of the extension of the kernel of admitted Lie algebras it is necessary to require that $k_1 = 0$, which leads to $c_1 = 0$ and

$$(g_{\bar{u}} g_{\bar{u}u} - g_u g_{\bar{u}\bar{u}}) = k_{10}\Delta. \tag{4.24}$$

Applying a similar study to the equation (4.22), one derives that

$$(g_u g_{\bar{u}u} - g_{\bar{u}} g_{uu}) = k_{20}\Delta, \tag{4.25}$$

มหาวิทยาลัยเทคโนโลยีสุรนารี

where k_{20} is constant. Equations (4.21) and (4.22) become

$$\zeta_0 = 2\eta_t k_{10},\ \bar{\zeta}_0 = 2\eta_t k_{20}.$$

Because of the periodicity of η and the condition $\eta_t \neq 0$, one gets that $k_{10} = k_{20}$. Notice that for $k_{10} = 0$ equations (4.24) and (4.25) lead to contradiction of the assumption $\Delta \neq 0$. For $k_{10} \neq 0$ the general solution of equations (4.24) and (4.25) is

$$g = e^{\alpha u} H(\bar{u} - u),$$

where $\alpha = k_{10}^{-1}$ and H is an arbitrary function of a single variable. Notice that

$$\Delta = -\alpha^2 e^{2\alpha u}(HH'' - (H')^2) \neq 0.$$

Without loss of generality one can assume that $\alpha = 1$, which implies $k_{10} = 1$.

Thus the set of admitted generators is

$$X = (\varphi(x+t) + \psi(x-t))\partial_x + (\varphi(x+t) - \psi(x-t))\partial_t + 2(\varphi'(x+t) + \psi'(x-t))\partial_u,$$

where the functions φ and ψ are arbitrary functions of a single independent variable and satisfy the condition $\varphi'(x+t) + \psi'(x-t) \neq 0$.

4.3.2 Case $\Delta = 0$

Case 1: $g_{\bar{u}\bar{u}} \neq 0$

The general solution of the equation $\Delta = 0$ is

$$g_u = \phi(g_{\bar{u}}), \tag{4.26}$$

where ϕ is an arbitrary function of the integration. Equations (4.19) and (4.20) become

$$(\zeta_0\phi' + \bar{\zeta}_0 + c_1(u\phi' + \bar{u}))\phi' g_{\bar{u}\bar{u}} = -2\eta_t\phi, \ \ (\zeta_0\phi' + \bar{\zeta}_0 + c_1(u\phi' + \bar{u}))g_{\bar{u}\bar{u}} = -2\eta_t g_{\bar{u}}. \tag{4.27}$$

Excluding $\bar{\zeta}_0$ from the latter equations, one finds that

$$\eta_t(g_{\bar{u}}\phi' - \phi) = 0.$$

Consider the case where $g_{\bar{u}}\phi' - \phi \neq 0$. This assumption leads to the condition $\eta_t = 0$. According to the remark, it implies that

$$\eta = c_2,\ \xi = c_3$$

where c_2 and c_3 are constants. Equations (4.27) reduce to

$$(\zeta_0 + c_1 u)\phi' = -(\bar{\zeta}_0 + c_1\bar{u}). \tag{4.28}$$

Notice that if $\phi' = 0$, then the latter equation implies $c_1 = 0$ and $\zeta_0 = 0$, which means that there is no extension of the kernel of admitted Lie algebra. Hence, one needs to study the case where $\phi' \neq 0$.

Differentiating equation (4.28) with respect to $\bar{u}$, one gets

$$(\zeta_0 + c_1 u)\phi'' g_{\bar{u}\bar{u}} = -c_1. \tag{4.29}$$

Further differentiation of equation (4.29) with respect to t and x, respectively, give

$$\phi''\zeta_{0t} = 0,\ \phi''\zeta_{0x} = 0. \tag{4.30}$$

Assume that $\phi'' \neq 0$. This assumption leads to the condition that ζ_0 is constant, say $\zeta_0 = k_1$. By virtue of the inverse function theorem, from equation (4.28) one has

$$g_{\bar{u}} = h(\frac{\alpha + \beta\bar{u}}{\alpha + \beta u}), \tag{4.31}$$

where α and β are constant. Because of the condition $g_{\bar{u}\bar{u}} \neq 0$ then $\beta \neq 0$. Using the equivalence transformation corresponding to the generators X_4^e, one can account that $\alpha = 0$. Integrating equations (4.31) and using condition (4.26), one derives that

$$g(u,\bar{u}) = uH(z) + k_0,\ z = \frac{\bar{u}}{u},$$

where k_0 is the integrating constant, and $H'' \neq 0$. Equation (4.18) becomes

$$k_1(H + H'(1 - z)) = c_1 k_0.$$

Since $H'' \neq 0$, the latter equation gives that $k_1 = 0$, and for existence of an extension of the kernel of admitted Lie algebras one needs to assume that $k_0 = 0$. Thus

$$g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$$

and the extension of the kernel of admitted Lie algebras is given by the generator

$$X_3 = u\partial_u.$$

Assuming that $\phi'' = 0$, one obtains that there exist constants k_1 and k_0 such that

$$g_u = k_1 g_{\bar{u}} + k_0,\ k_0 \neq 0.$$

By virtue of the condition $\phi' \neq 0$ one has to assume that $k_1 \neq 0$. The general solution of the latter equation is

$$g(u, \bar{u}) = k_0 u + H(\bar{u} + k_1 u), \tag{4.32}$$

where H is arbitrary function such that $H' \neq 0$. Notice that by scaling the independent variables t and x as well the delay parameter τ one can assume that $k_0 = \pm 1$.

Equation (4.28) becomes

$$(\zeta_0 + c_1 u)k_1 + (\bar{\zeta}_0 + c_1 \bar{u}) = 0, \tag{4.33}$$

which implies that $c_1 = 0$ and $\bar{\zeta}_0 = -k_1 \zeta_0$. Equation (4.18) is reduced to the Klein-Gordon equation

$$\zeta_{0tt} = \zeta_{0xx} + k_0 \zeta_0.$$

Thus, for the function (4.32), one obtains that if there exists a nontrivial solution $q(t,x)$ of the linear Klein-Gordon equation

$$q_{tt} = q_{xx} + k_0 q, \tag{4.34}$$

satisfying the condition

$$q(t-\tau, x) = -k_1 q(t,x), \tag{4.35}$$

then the extension of the kernel is given by the generator

$$X_3 = q(t,x)\partial_u.$$

Notice that the set of functions $g(u,\bar{u})$ for which there exists a nontrivial solution of (4.34) and (4.35) is not empty*. For example, if $k_1 = -e^{\sqrt{k_0}\tau}$, then $q = e^{-\sqrt{k_0}t}$ is a nontrivial solution of (4.34) and (4.35).

Let us consider the case where $g_{\bar{u}}\phi' - \phi = 0$. This means that

$$g_u = k_1 g_{\bar{u}},$$

where k_1 is constant. The general solution of the latter equation is

$$g(u,\bar{u}) = H(z), \quad z = \bar{u} + k_1 u,$$

where ρ is a function of a single variable such that $H'' \neq 0$. Equation (4.20) becomes

$$k_1\zeta_0 + \bar{\zeta}_0 + c_1 z = -2\eta_t \frac{H'}{H''}. \tag{4.36}$$

Differentiating equation (4.36) two times with respect to z, one obtains

$$\eta_t \left(\frac{H'}{H''}\right)'' = 0. \tag{4.37}$$

---

*Some particular solutions of the linear Klein-Gordon equation can be found in Polyanin A.D, Handbook of Linear partial Differential Equation for Engineers and Scientists, Chapman and Hall/CRC, 2002

The assumption $\left(\frac{H'}{H''}\right)'' \neq 0$ leads to the condition that $\eta_t = 0$. From equation (4.36) one gets $c_1 = 0$ and $\bar{\zeta}_0 = -k_1\zeta_0$. Equation (4.18) is reduced to the wave equation $\zeta_{0tt} = \zeta_{0xx}$. Thus,

$$\zeta_0(t,x) = N(x-t) + G(x+t),$$

where $N(y)$ and $G(y)$ are arbitrary functions of a single variable satisfying the conditions

$$N(y+\tau) + k_1N(y) = c_0, \;\; G(y-\tau) + k_1G(y) = -c_0, \tag{4.38}$$

with constant c_0. The extension of the kernel of admitted Lie algebras is given by the generator

$$X_3 = \zeta_0(t,x)\partial_u.$$

Remark. The functions

$$N(y) = e^{-ay}, \;\; G(y) = e^{ay}$$

and constants

$$k_1 = -e^{a\tau}, \;\; c_0 = 0$$

provide an example of such functions.

The assumption $\left(\frac{H'}{H''}\right)'' = 0$ implies that

$$\frac{H'}{H''} = \alpha z + \beta, \tag{4.39}$$

where α and β are constant. Since $H'' \neq 0$, then $\alpha z + \beta \neq 0$. Equation (4.36) becomes

$$k_1\zeta_0 + \bar{\zeta}_0 + c_1 z = -2\eta_t(\alpha z + \beta). \tag{4.40}$$

Splitting the latter equation with respect to z, one obtains

$$c_1 + 2\alpha\eta_t = 0, \;\; k_1\zeta_0 + \bar{\zeta}_0 + 2\eta_t\beta = 0.$$

If $\alpha \neq 0$, solving equation (4.39), one obtains

$$g(z) = \begin{cases} k_2 \ln|z-\beta| + k_3 & , \\ k_2(z+\beta)^k + k_3 & , \quad k(k-1) \neq 0, \end{cases}$$

where $k = \alpha^{-1} + 1$, $k_2 \neq 0$ and k_3 are constant.

By virtue of periodicity of η one has that $\eta_t = 0$ and $c_1 = 0$. Equations (4.18) and (4.40) become

$$k_1\zeta_0 + \bar{\zeta}_0 = 0, \quad \zeta_{0tt} = \zeta_{0xx}.$$

Thus, the extension of the kernel of admitted Lie algebras is given by the generator

$$X_3 = (N(x-t) + G(x+t))\partial_u,$$

where N and G are arbitrary functions of a single variable satisfying the conditions

$$N(y+\tau) + k_1 N(y) = c_0, \quad G(y-\tau) + k_1 G(y) = -c_0,$$

for some constants c_0.

The assumption $\alpha = 0$ implies that $\beta \neq 0$ and $c_1 = 0$. The general solution of equation (4.39) is

$$g = k_2 e^{\gamma z} + k_3, \qquad (4.41)$$

where $\gamma = \beta^{-1}$, k_2 and k_3 are constant. Here without loss of generality one can assume that $\gamma = 1$.

Substituting the function g of (4.41) into equation (4.13), and using the relation that $\eta(x,t,u) = \varphi(x+t) - \psi(x-t)$, one gets

$$\zeta_{tt} = \zeta_{xx} + 2k_3(\varphi'(x+t) + \psi'(x-t)).$$

The general solution of the latter equation is

$$2\zeta(x,t) = -k_3\left((x-t)\varphi(x+t) + (x+t)\psi(x-t)\right) + \nu(x+t) + \mu(x-t),$$

where ν and μ are functions of a single variable. Using the conditions that the function φ and ψ are periodic with period τ, then

$$2\left(\bar{\zeta}_0 + k_1\zeta_0 + 2\beta\eta_t\right) = -k_3((k_1+1)w+\tau)\varphi(y) - k_3((k_1+1)y-\tau)\psi(w) + \nu(y-\tau) \\ +\mu(w+\tau) + k_1\left(\nu(y)+\mu(w)\right) + 4\beta\left(\varphi'(y)+\psi'(w)\right) = 0, \tag{4.42}$$

where $w = x - t$, $y = x + t$.

Differentiating equation (4.42) twice with respect to w, one gets

$$4\beta\psi'''(w) - k_3((k_1+1)y-\tau)\psi''(w) \\ +\mu''(w+\tau) + k_1\mu''(w) = 0.$$

Splitting the latter equation with respect to y, one obtains that

$$k_3(k_1+1)\psi''(w) = 0, \tag{4.43}$$

$$4\beta\psi'''(w) + k_3\tau\psi''(w) + \mu''(w+\tau) + k_1\mu''(w) = 0. \tag{4.44}$$

Integrating equations (4.43) and (4.44), one has

$$k_3(k_1+1)\psi(w) = b_1 w + b_0, \\ 4\beta\psi'(w) + k_3\tau\psi(w) + \mu(w+\tau) + k_1\mu(w) = b_3 w + b_2 \tag{4.45}$$

where b_i, $(i = 0, 1, 2, 3)$ are arbitrary constants.

In a similar way, one obtains that the functions φ and ν satisfy the following conditions

$$k_3(k_1+1)\varphi(y) = -b_1 y + b_3, \\ 4\beta\varphi'(y) - k_3\tau\varphi(y) + \nu(y-\tau) + k_1\nu(y) = b_0 y - b_2. \tag{4.46}$$

Thus, the set of admitted generators is

$$X = (\varphi(x+t) + \psi(x-t))\partial_x + (\varphi(x+t) - \psi(x-t))\partial_t + q(x,t)\partial_u,$$

where ψ, μ and φ, ν are solutions of equations (4.45) and (4.46), and

$$q(t,x) = \frac{1}{2}[-k_3((x-t)\varphi(x+t) + (x+t)\psi(x-t)) + \nu(x+t) + \mu(x-t)].$$

Notice that for $k_3 = 0$ and $k_1 = -1$ the function (4.41) is a particular case of the function $g(u, \bar{u}) = e^u H(\bar{u} - u)$, hence, one can assume that $k_3^2 + (k_1 + 1)^2 \neq 0$.

Case 2: $g_{\bar{u}\bar{u}} = 0,\ g_{uu} \neq 0$

In this case

$$g(u, \bar{u}) = k_1\bar{u} + h(u), \tag{4.47}$$

where $k_1 \neq 0$ is a constant and $h'' \neq 0$. By virtue of equation (4.20), one finds that $\eta_t = 0$. According to the Remark, one has that

$$\eta = c_2,\ \xi = c_3,$$

where c_2 and c_3 are constants.

Equation (4.19) gives that $\zeta = 0$. Thus, in this case there is no extension of the kernel of admitted Lie algebras.

Case 3: $g_{\bar{u}\bar{u}} = 0,\ \ g_{uu} = 0$

This case corresponds to a linear delay differential equation with

$$g(u, \bar{u}) = k_1\bar{u} + k_2 u + k, \tag{4.48}$$

where k, $k_1 \neq 0$ and k_2 are constant.

Notice that the constant k can be reduced to zero by the change

$$u = \tilde{u} - \frac{\tilde{k}_1}{2}x^2 + \tilde{k}_2.$$

Indeed, choosing the constants $\tilde{k}_1$ and $\tilde{k}_2$ such that

$$\tilde{k}_1(k_1 + k_2) = 0,\ \ \tilde{k}_1 - \tilde{k}_2(k_1 + k_2) = k$$

the function $\tilde{u}$ satisfies the equation

$$\tilde{u}_{tt} = \tilde{u}_{xx} + k_1\bar{\tilde{u}} + k_2\tilde{u}.$$

The determining equations reduce to the equations

$$\eta = c_2, \quad \xi = c_3, \;\; \zeta = \zeta_0 + c_1 u,$$

where c_2, c_3 are constant, and $\zeta_0(t,x)$ satisfies the equation

$$\zeta_{0tt} = \zeta_{0xx} + k_1\bar{\zeta_0} + k_2\zeta_0. \tag{4.49}$$

The extension of the kernel is given by the generators $X_3 = u\partial_u$ and $X_{\zeta_0} = \zeta_0(t,x)\partial_u$.

In particular, if $k_0 = k_1 e^{\sqrt{k_0}\tau} + k_2$, then $\zeta_0 = e^{-\sqrt{k_0}t}$ is a particular solution of equation (4.49).

4.4 Summary of the Group Classification

By the discussions of the previous section one obtains the following complete group classification of the delay partial differential equation

$$u_{tt} = u_{xx} + g(u, \bar{u}),$$

where $g_{\bar{u}} \neq 0$. The results of the group classification are presented in Table 4.1, where the function H is a function of a single argument, $\varphi = \varphi(x+t)$ and $\psi = \psi(x-t)$ are arbitrary periodic functions; the functions $\psi_0 = \psi_0(x-t)$ and $\varphi_0 = \varphi_0(x+t)$ are also periodic, and satisfy conditions (4.45) and (4.46) with the functions $\mu_0 = \mu_0(x-t)$, and $\nu_0 = \nu_0(x+t)$; the functions $G = G(x+t)$ and $N = N(x-t)$ satisfy the equations

$$N(y+\tau) + k_1 N(y) = c_0, \;\; G(y-\tau) + k_1 G(y) = -c_0, \tag{4.50}$$

for some constant c_0, while the coefficients $q_i(t,x)$, $(i = 1,2,3)$ satisfy the equations

$$q_{1tt}(t,x) = q_{1xx}(t,x) + k_0 q_1(t,x), \;\; q_1(t-\tau, x) = -k_1 q_1(t,x), \tag{4.51}$$

$$q_2(t,x) = -k_3((x-t)\varphi_0(x+t) + (x+t)\psi_0(x-t)) + \nu_0(x+t) + \mu_0(x-t), \quad (4.52)$$

$$q_{3tt}(t,x) = q_{3xx}(t,x) + k_2 q_3(t,x) + k_1 q_3(t-\tau, x). \quad (4.53)$$

Table 4.1 Group classification of the equation $u_{tt} = u_{xx} + g(u, \bar{u}), \quad (g_{\bar{u}} \neq 0)$.

No.	$g(u,\bar{u})$	Conditions	Extensions
1	$e^u H(\bar{u} - u)$	$\varphi' + \psi' \neq 0$	$(\varphi + \psi)\partial_x$ $+(\varphi - \psi)\partial_t$ $+2(\varphi' + \psi')\partial_u$
2	$uH(\frac{\bar{u}}{u})$	$H'' \neq 0$	$u\partial_u$
3	$k_0 u + H(\bar{u} + k_1 u)$	$k_0 k_1 H' \neq 0$	$q_1\partial_u$
4	$H(\bar{u} + k_1 u)$	$\left(\frac{H'}{H''}\right)'' \neq 0$	$(N + G)\,\partial_u$
5	$k_2 \ln\lvert\bar{u} + k_1 u - \beta\rvert + k_3$		$(N + G)\,\partial_u$
6	$k_2(\bar{u} + k_1 u + \beta)^k + k_3$	$k(k-1) \neq 0$	$(N + G)\,\partial_u$
7	$k_2 e^{\bar{u} + k_1 u} + k_3$	$k_3^2 + (k_1 + 1)^2 \neq 0$	$(\varphi_0 + \psi_0)\partial_x$ $+(\varphi_0 - \psi_0)\partial_t$ $+q_2\partial_u$
8	$k_1\bar{u} + k_2 u$	$k_1 \neq 0$	$u\partial_u, \ \ q_3\partial_u$

มหาวิทยาลัยเทคโนโลยีสุรนารี

4.5 Invariant Solutions

Invariant solutions are sought for subalgebras of the admitted Lie algebra. All different invariant solutions can be obtained on the base of an optimal system of subalgebras. Notice that for the cases 3 to 8 there is a generator of the form $q(t,x)\partial_u$. Usually such kind of generators are omitted in construction of invariant solutions. However the use of these generators also provides invariant solutions. This will be demonstrated further.

4.5.1 Invariant Solutions of (4.1) with $g(u, \bar{u}) = e^u H(\bar{u} - u)$

Consider the case where the function $\psi = 0$. A representation of an invariant solution is

$$u = \ln|\varphi(x+t)| + h(x-t) \tag{4.54}$$

where the function h is an arbitrary function. Substituting the representation of the invariant solution into (4.1), one has

$$H(h(x-t+\tau) - h(x-t)) = 0.$$

In particular, if the function h is periodic $h(y+\tau) = h(y)$, and $H(0) = 0$, then (4.54) provides a solution of equation (4.1). Notice that (4.54) is a d'Alambert solution of the wave equation with the additional property: the functions φ and h are periodic.

4.5.2 Invariant Solutions of (4.1) with $g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$

1. **Optimal System of Subalgebras**

As the admitted Lie algebra defined by the generators

$$X_1 = \partial_t, \;\; X_2 = \partial_x, \;\; X_3 = u\partial_u$$

is Abelian, an optimal system of one-dimensional admitted subalgebras consists of

$$\{H_i\}, \;\; (i = 1, 2, 3),$$

where

$$H_1 = X_1 + \alpha X_2 + \beta X_3, \;\; H_2 = X_2 + \alpha X_3, \;\; H_3 = X_3,$$

with arbitrary constants α and β.

2. **Invariant Solutions**

Notice that there are no invariant solutions corresponding to the subalgebra H_3.

Case 1**: Subalgebra** H_1**.** Solving the characteristic system related with the generator H_1, one gets that a representation of an invariant solution is $u = \phi(\theta)e^{\beta t}$, where ϕ is a function of a single variable $\theta = x - \alpha t$. For $\beta = 0$ this class of solutions is called traveling wave. Substituting this representation of a solution into equation (4.1) with the function $g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$, it becomes

$$(\alpha^2 - 1)\phi''(\theta) = 2\alpha\beta\phi'(\theta) + \phi(\theta)(H(\frac{\phi(\theta + \alpha\tau)}{\phi(\theta)}e^{-\beta\tau}) - \beta^2).$$

Case 2**: Subalgebra** H_2**.** A representation of an invariant solution is $u = \phi(t)e^{\alpha x}$, where ϕ is a function of a single variable. The reduced equation is

$$\phi''(t) = \phi(t)(\alpha^2 + H(\frac{\phi(t - \tau)}{\phi(t)})).$$

In summary, the representations of all invariant solutions and reduced equations are given in Table 4.2, where $\theta = x - \alpha t$.

Table 4.2 Invariant solutions for $g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \phi(\theta)e^{\beta t}$	$(\alpha^2 - 1)\phi''(\theta) = 2\alpha\beta\phi'(\theta)$
			$+\phi(\theta)(H(\frac{\phi(\theta+\alpha\tau)}{\phi(\theta)}e^{-\beta\tau}) - \beta^2)$,
2	H_2	$u = \phi(t)e^{\alpha x}$	$\phi''(t) = \phi(t)(\alpha^2 + H(\frac{\phi(t-\tau)}{\phi(t)}))$.

4.5.3 Invariant Solutions of (4.1) with the Function $g(u, \bar{u})$ of Forms $No.3$-$No.6$ in Table 4.1

For these functions the admitted Lie algebra L is spanned by X_1, X_2 and X_3, where

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = q(t, x)\partial_u. \qquad (4.55)$$

The commutator table is

$[,]$	X_1	X_2	X_3
X_1	0	0	$q_t\partial_u$
X_2	0	0	$q_x\partial_u$
X_3	$-q_t\partial_u$	$-q_x\partial_u$	0

The requirement that L is a Lie algebra implies existence of constants α_1 and α_2 such that

$$q_t(t,x) = \alpha_1 q(t,x),\ q_x(t,x) = \alpha_2 q(t,x). \tag{4.56}$$

The general solution of equations (4.56) is $q(t,x) = ce^{(\alpha_1 t+\alpha_2 x)}$, where c is constant and $c \neq 0$. Because X_1, X_2, X_3 is a basis of Lie algebra L, one can choose $c = 1$.

1. **For Function** $g(u,\bar{u}) = k_0 u + H(\bar{u} + k_1 u)$

As $q(t,x)$ satisfies relations (4.51), one derives that

$$k_0 = \alpha_1^2 - \alpha_2^2,\ k_1 = -e^{-\alpha_1\tau}, \tag{4.57}$$

where $\alpha_1 \neq \pm\alpha_2$. For obtaining automorphisms one has to solve the Lie equations. The automorphisms are:

$$\begin{aligned} A_1:&\quad \hat{x_3} = x_3 e^{\alpha_1 a};\\ A_2:&\quad \hat{x_3} = x_3 e^{\alpha_2 a};\\ A_3:&\quad \hat{x_3} = x_3 - a(\alpha_1 x_1 + \alpha_2 x_2). \end{aligned} \tag{4.58}$$

For obtaining an optional system of subalgebra one uses the two-step algorithm (Ovsiannikov, 1993). Before constructing an optimal system, one studies the algebraic structure of the Lie algebra L. Consider the vector space L^1 spanned by the operators $\{X_1, X_2\}$. One can verify that it is a subalgebra of the Lie algebra L, and the vector space I spanned by the operator $\{X_3\}$ is an ideal of the Lie algebra L. Hence, the Lie algebra L is decomposed as $I \oplus L^1$. Because the subalgebra L^1

is Abelian, an optimal system of one-dimensional subalgebras is

$$\{K_i\}, \ \ (i = 1, 2),$$

where

$$K_1 = X_2, \ \ K_2 = X_1 + p_1 X_2,$$

with an arbitrary constant p_1.

According to the two-step algorithm for classifying Lie algebra L, it is sufficient to consider the following forms of one-dimensional subalgebras:

$$W_1 = \{X_2 + a_{13}X_3\},\ W_2 = \{X_1 + p_1X_2 + a_{13}X_3\},\ W_3 = \{X_3\}, W_0 = \{0\}, \quad (4.59)$$

where a_{13} is constant. Here W_0 corresponds to the ideal I.

For further study one needs to simplify (4.59) by applying automorphisms (4.58).

First, consider the case W_1.

Case 1: $a_{13} = 0$**.** One gets the one-dimensional subalgebra $\{X_2\}$.

Case 2: $a_{13} \neq 0$**.** If $\alpha_2 \neq 0$, then, using the automorphism A_3, a_{13} can be changed to 0: one gets the one-dimensional subalgebra $\{X_2\}$. If $\alpha_2 = 0$, then, using the automorphism A_1, a_{13} can be changed to ϵ: one gets the one-dimensional subalgebra $\{X_2 + \epsilon X_3\}$.

Consider the case W_2.

Case 1: $\alpha_1 + p_1\alpha_2 \neq 0$**.** Using the automorphism A_3, a_{13} can be changed to 0, this gives the one-dimensional subalgebra $\{X_1 + p_1X_2\}$.

Case 2: $\alpha_1 + p_1\alpha_2 = 0$**.** One needs to consider $a_{13} = 0$ and $a_{13} \neq 0$. The first case corresponds to the one-dimensional subalgebra $\{X_1+p_1X_2\}$. The second case, using the automorphism A_2, one obtains the one-dimensional subalgebra $\{X_1 + p_1X_2 + \epsilon X_3\}$.

The obtained above results are summarized as follows.

Theorem 4.1. *An optimal system of one-dimensional subalgebras of the Lie algebra L with the basis generators (4.55) are*

$$H_1 = \{X_1 + p_1X_2 + \epsilon X_3\}_{|\alpha_1+p_1\alpha_2=0},\ H_2 = \{X_2 + \epsilon X_3\}_{|\alpha_2=0},$$

$$H_3 = \{X_1 + p_1X_2\},\ H_4 = \{X_2\},\ H_5 = \{X_3\},$$

where $\epsilon = \pm 1$, p_1 *is constant and the symbol* $|$ *means conditions.*

Using the obtained optimal system of subalgebras, all invariant solutions are analyzed below.

Notice that there are no invariant solutions corresponding to the subalgebra H_5.

Case 1**: Subalgebra** H_1**.** In this case, condition is $\alpha_1 + p_1\alpha_2 = 0$, and the characteristic system is

$$\frac{dt}{1} = \frac{dx}{p_1} = \frac{du}{\epsilon e^{(\alpha_1 t+\alpha_2 x)}}.$$

Solving the characteristic system, one gets that a representation of an invariant solution is $u = \epsilon e^{\alpha_2\theta} + \phi(\theta)$, where ϕ is a function of a single variable $\theta = x - p_1 t$. Substituting this representation of a solution into equation (4.1) with the function $g(u, \bar{u}) = k_0 u + H(\bar{u} + k_1 u)$, and using (4.57), one derives the reduced equation

$$(p_1^2 - 1)\phi''(\theta) = (\alpha_1^2 - \alpha_2^2)\phi(\theta) + H(\phi(p_1\tau + \theta) - e^{-\alpha_1\tau}\phi(\theta)).$$

Case 2**: Subalgebra** H_2**.** A representation of an invariant solution is $u = \epsilon x e^{\alpha_1 t} + \phi(t)$, where ϕ is a function of a single variable. Using (4.57), the reduced equation is

$$\phi''(t) = \alpha_1^2\phi(t) + H(\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t)).$$

Case 3**: Subalgebra** H_3**.** A representation of an invariant solution is $u = \phi(\theta)$, where ϕ is a function of a single variable $\theta = x - p_1 t$. Using condition (4.57), the reduced equation is

$$(p_1^2 - 1)\phi''(\theta) = (\alpha_1^2 - \alpha_2^2)\phi(\theta) + H(\phi(p_1\tau + \theta) - e^{-\alpha_1\tau}\phi(\theta)).$$

Case 4**: Subalgebra** H_4**.** A representation of an invariant solution is $u = \phi(t)$, where ϕ is a function of a single variable. Using condition (4.57) the reduced equation is

$$\phi''(t) = (\alpha_1^2 - \alpha_2^2)\phi(t) + H(\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t)).$$

In summary, the representations of invariant solutions and reduced equations are given in Table 4.3, where $\phi(\theta)$ is a function of the single variable $\theta = x - p_1 t$.

Table 4.3 Invariant solutions for $g(u, \bar{u}) = k_0 u + H(\bar{u} + k_1 u)$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \epsilon e^{\alpha_2\theta} + \phi(\theta)$	$(p_1^2 - 1)\phi''(\theta) = (\alpha_1^2 - \alpha_2^2)\phi(\theta)$ $+H(\phi(p_1\tau + \theta) - e^{-\alpha_1\tau}\phi(\theta))$
2	H_2	$u = \epsilon x e^{\alpha_1 t} + \phi(t)$	$\phi''(t) = \alpha_1^2\phi(t)$ $+H(\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t))$
3	H_3	$u = \phi(\theta)$	$(p_1^2 - 1)\phi''(\theta) = (\alpha_1^2 - \alpha_2^2)\phi(\theta)$ $+H(\phi(p_1\tau + \theta) - e^{-\alpha_1\tau}\phi(\theta))$
4	H_4	$u = \phi(t)$	$\phi''(t) = (\alpha_1^2 - \alpha_2^2)\phi(t)$ $+H(\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t))$

2. **For Functions** $g(u, \bar{u})$ **of Forms** $No.4$**-**$No.6$ **in Table 4.1**

For these cases, because $q(t, x) = N(x - t) + G(x + t)$ and the functions $N(x - t)$ and $G(x + t)$ satisfy the conditions (4.50), one derives that $q(t, x) = e^{\alpha_1(t+\epsilon x)}$, $c_0 = 0$ and

$$\alpha_2 = \epsilon\alpha_1,\ k_1 = -e^{-\alpha_1\tau}, \tag{4.60}$$

where $\epsilon = \pm 1$. One needs to consider $\alpha_1 = 0$ and $\alpha_1 \neq 0$.

Case 2.1**:** $\alpha_1 = 0$**.** One derives that $\alpha_2 = 0$, $k_1 = -1$, $q(t, x) = 1$ and the admitted Lie algebra is spanned by the generators $X_1 = \partial_t$, $X_2 = \partial_x$, and

$X_3 = \partial_u$. This Lie algebra is Abelian, and an optimal system of one-dimensional subalgebras is

$$H_1 = \{X_1 + \alpha X_2 + \gamma X_3\},\ \ H_2 = \{X_2 + \alpha X_3\},\ \ H_3 = \{X_3\},$$

with arbitrary constants α and γ.

Notice that there are no invariant solutions corresponding to the subalgebra H_3.

Case 2.1.1**: Subalgebra** H_1**.** A representation of an invariant solution is $u = \phi(\theta) + \gamma t$, where ϕ is a function of a single variable $\theta = x - \alpha t$. Substituting this representation of a solution into equation (4.1) with the functions $g(u, \bar{u}) = H(\bar{u} - u)$, $g(u, \bar{u}) = k_2 \ln|\bar{u} - u - \beta| + k_3$ and $g(u, \bar{u}) = k_2(\bar{u} - u + \beta)^k + k_3$, respectively, using (4.60), one derives the reduced equations

$$(\alpha^2 - 1)\phi''(\theta) = H(\phi(\alpha\tau + \theta) - \phi(\theta) - \gamma\tau),$$

$$(\alpha^2 - 1)\phi''(\theta) = k_2 \ln|\phi(\alpha\tau + \theta) - \phi(\theta) - \gamma\tau - \beta| + k_3,$$

and

$$(\alpha^2 - 1)\phi''(\theta) = k_2(\phi(\alpha\tau + \theta) - \phi(\theta) - \gamma\tau + \beta)^k + k_3.$$

Case 2.1.2**: Subalgebra** H_2**.** A representation of an invariant solution is $u = \phi(t) + \alpha x$, where ϕ is a function of a single variable. Substituting this representation of a solution into equation (4.1) with the functions $g(u, \bar{u}) = H(\bar{u} - u)$, $g(u, \bar{u}) = k_2 \ln|\bar{u} - u - \beta| + k_3$ and $g(u, \bar{u}) = k_2(\bar{u} - u + \beta)^k + k_3$, respectively, using (4.60), one derives the reduced equations

$$\phi''(t) = H(\phi(t - \tau) - \phi(t)),$$

$$\phi''(t) = k_2 \ln|\phi(t - \tau) - \phi(t) - \beta| + k_3,$$

and

$$\phi''(t) = k_2(\phi(t - \tau) - \phi(t) + \beta)^k + k_3.$$

In summary, the representations of all invariant solutions and reduced equations are given in Tables 4.4-4.6, where $\theta = x - \alpha t$.

Table 4.4 Invariant solutions for $g(u, \bar{u}) = H(\bar{u} - u)$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \phi(\theta) + \gamma t$	$(\alpha^2 - 1)\phi''(\theta) =$ $H(\phi(\alpha\tau + \theta) - \phi(\theta) - \gamma\tau)$
2	H_2	$u = \phi(t) + \alpha x$	$\phi''(t) = H(\phi(t - \tau) - \phi(t))$

Table 4.5 Invariant solutions for $g(u, \bar{u}) = k_2 \ln \lvert\bar{u} - u - \beta\rvert + k_3$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \phi(\theta) + \gamma t$	$(\alpha^2 - 1)\phi''(\theta) = k_3$ $+k_2 \ln \lvert\phi(\alpha\tau + \theta) - \phi(\theta) - \gamma\tau - \beta\rvert$
2	H_2	$u = \phi(t) + \alpha x$	$\phi''(t) = k_3$ $k_2 \ln \lvert\phi(t - \tau) - \phi(t) - \beta\rvert$

Table 4.6 Invariant solutions for $g(u, \bar{u}) = k_2(\bar{u} - u + \beta)^k + k_3$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \phi(\theta) + \gamma t$	$(\alpha^2 - 1)\phi''(\theta) = k_3$ $+k_2(\phi(\alpha\tau + \theta) - \phi(\theta) - \gamma\tau + \beta)^k$
2	H_2	$u = \phi(t) + \alpha x$	$\phi''(t) = k_3$ $k_2(\phi(t - \tau) - \phi(t) + \beta)^k$

Case 2.2**:** $\alpha_1 \neq 0$**.** Because of $\alpha_2 = \epsilon\alpha_1$, then $q(t, x) = e^{\alpha_1(t+\epsilon x)}$. For obtaining automorphisms one has to solve Lie equations. The automorphisms are:

$$A_1: \quad \hat{x_3} = x_3 e^{\alpha_1 a};$$

$$A_2: \quad \hat{x_3} = x_3 - a\alpha_1(x_1 + \epsilon x_2).$$

For obtaining an optional system of subalgebras one uses the two-step algorithm (Ovsiannikov, 1993). Before constructing an optimal system, one studies the algebraic structure of the Lie algebra L. Consider the vector space L^1 spanned by the operators $\{X_1, X_2\}$. One can verify that it is a subalgebra of the Lie algebra L, and the vector space I spanned by the operator $\{X_3\}$ is an ideal of the Lie algebra L. Hence, the Lie algebra L is decomposed as $I \oplus L^1$. The subalgebra L^1 is Abelian, then an optimal system of one-dimensional subalgebras is

$$K_1 = \{X_2\}, \;\; K_2 = \{X_1 + pX_2\},$$

with an arbitrary constant p.

According to the two-step algorithm for classifying the Lie algebra L, it is sufficient to consider the following forms of one-dimensional subalgebras:

$$W_1 = \{X_2 + a_{13}X_3\}, \; W_2 = \{X_1 + pX_2 + a_{13}X_3\}, \; W_3 = \{X_3\},$$

where a_{13} is constant.

For further study one needs to simplify the latter one-dimensional subalgebras W_i by using automorphisms.

First, consider the case W_1. Using the automorphism A_2, a_{13} can be changed to 0. This gives the one-dimensional subalgebra $\{X_2\}$.

Consider the case W_2.

Case 2.2.1**:** $\alpha_2 = \alpha_1$. If $p = -1$ and $\beta \neq 0$, then, the using automorphism A_1, one gets the one-dimensional subalgebra $\{X_1 - X_2 + \epsilon X_3\}$. If $p = -1$ and $\beta = 0$, then W_2 is reduced to the one-dimensional subalgebra $\{X_1 - X_2\}$. If $p \neq -1$, then, using the automorphism A_2, β can be changed to 0: one gets the one-dimensional subalgebra $\{X_1 + pX_2\}$. From the above discussion, one gets the subalgebras $\{X_1 - X_2 + \epsilon X_3\}$ and $\{X_1 + pX_2\}$, where p is arbitrary.

Case 2.2.2**:** $\alpha_2 = -\alpha_1$. If $p = 1$ and $\beta \neq 0$, then, using the automorphism A_1, one has the one-dimensional subalgebra $\{X_1 + X_2 + \epsilon X_3\}$. If $p = 1$ and $\beta = 0$, then one obtains the one-dimensional subalgebra $\{X_1 + X_2\}$. If $p \neq 1$, then, using the automorphism A_2, β can be changed to 0: one gets the one-dimensional subalgebra $\{X_1 + pX_2\}$. From the above discussion, one has the subalgebras $\{X_1 + X_2 + \epsilon X_3\}$ and $\{X_1 + pX_2\}$ with arbitrary constant p.

The results obtained above are summarized as follows.

Theorem 4.2. *An optimal system of one-dimensional subalgebras of the Lie algebra L consists of the subalgebras*

$$H_1 = \{X_1 - X_2 + \epsilon X_3\}_{|\alpha_2=\alpha_1},\ H_2 = \{X_1 + X_2 + \epsilon X_3\}_{|\alpha_2=-\alpha_1},$$
$$H_3 = \{X_1 + pX_2\},\ H_4 = \{X_2\},\ H_5 = \{X_3\},$$

where $\epsilon = \pm 1$, *p is an constant and and the symbol* $|$ *means conditions.*

Using the obtained optimal system of subalgebras, all invariant solutions are analyzed below.

Notice that there are no invariant solutions corresponding to the subalgebra H_5.

Subalgebra H_1**.** A representation of an invariant solution is $u = \phi(\theta) - \epsilon x e^{\alpha_1 \theta}$, where ϕ is a function of a single variable $\theta = x + t$. Substituting this representation of a solution into equation (4.1) with the functions $g(u, \bar{u}) = H(\bar{u} - u)$, $g(u, \bar{u}) = k_2 \ln|\bar{u} - u - \beta| + k_3$ and $g(u, \bar{u}) = k_2(\bar{u} - u + \beta)^k + k_3$, respectively, using (4.60), one derives the reduced equations

$$2\epsilon\alpha_1 e^{\alpha_1\theta} = H(\phi(\theta - \tau) - e^{\alpha_1\tau}\phi(\theta)),$$

$$2\epsilon\alpha_1 e^{\alpha_1\theta} = k_2 \ln|\phi(\theta - \tau) - e^{\alpha_1\tau}\phi(\theta) - \beta| + k_3.$$

and

$$2\epsilon\alpha_1 e^{\alpha_1\theta} = k_2(\phi(\theta - \tau) - e^{\alpha_1\tau}\phi(\theta) + \beta)^k + k_3.$$

Subalgebra H_2**.** A a representation of an invariant solution is $u = \phi(\delta) + \epsilon x e^{\alpha_1 \delta}$, where ϕ is a function of a single variable $\delta = t - x$. Substituting this representation of a solution into equation (4.1) with the functions $g(u, \bar{u}) = H(\bar{u} - u)$, $g(u, \bar{u}) = k_2 \ln |\bar{u} - u - \beta| + k_3$ and $g(u, \bar{u}) = k_2(\bar{u} - u + \beta)^k + k_3$, respectively, using (4.60), one derives the reduced equations

$$2\epsilon\alpha_1 e^{\alpha_1 \delta} = H(\phi(\delta - \tau) - e^{\alpha_1 \tau}\phi(\delta)),$$

$$2\epsilon\alpha_1 e^{\alpha_1 \delta} = k_2 \ln |\phi(\delta - \tau) - e^{\alpha_1 \tau}\phi(\delta) - \beta| + k_3.$$

and

$$2\epsilon\alpha_1 e^{\alpha_1 \delta} = k_2(\phi(\delta - \tau) - e^{\alpha_1 \tau}\phi(\delta) + \beta)^k + k_3.$$

Subalgebra H_3**.** A representation of an invariant solution is $u = \phi(\gamma)$, where ϕ is a function of a single variable $\gamma = x - pt$. Substituting this representation of a solution into equation (4.1) with the functions $g(u, \bar{u}) = H(\bar{u} - u)$, $g(u, \bar{u}) = k_2 \ln |\bar{u} - u - \beta| + k_3$ and $g(u, \bar{u}) = k_2(\bar{u} - u + \beta)^k + k_3$, respectively, using (4.60), one derives the reduced equations

$$(p^2 - 1)\phi''(\gamma) = H(\phi(p\tau + \gamma) - e^{\alpha_1 \tau}\phi(\theta)),$$

$$(p^2 - 1)\phi''(\gamma) = k_2 \ln |\phi(p\tau + \gamma) - e^{\alpha_1 \tau}\phi(\theta) - \beta| + k_3.$$

and

$$(p^2 - 1)\phi''(\gamma) = k_2(\phi(p\tau + \gamma) - e^{\alpha_1 \tau}\phi(\theta) + \beta)^k + k_3.$$

Subalgebra H_4**.** A representation of an invariant solution is $u = \phi(t)$, where ϕ is a function of a single variable. Substituting this representation of a solution into equation (4.1) with the functions $g(u, \bar{u}) = H(\bar{u} - u)$, $g(u, \bar{u}) = k_2 \ln |\bar{u} - u - \beta| + k_3$ and $g(u, \bar{u}) = k_2(\bar{u} - u + \beta)^k + k_3$, respectively, using (4.60), one derives the reduced equations

$$\phi''(t) = H(\phi(t - \tau) - e^{-\alpha_1 \tau}\phi(t)),$$

$$\phi''(t) = k_2 \ln|\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t) - \beta| + k_3.$$

and

$$\phi''(t) = k_2(\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t) + \beta)^k + k_3.$$

In summary, the representations of all invariant solutions and reduced equations are given in Tables 4.7-4.9, where $\theta = x + t$, $\delta = t - x$, $\gamma = x - pt$.

Table 4.7 Invariant solutions for $g(u, \bar{u}) = H(\bar{u} - ue^{-\alpha_1\tau})$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \phi(\theta) - \epsilon x e^{\alpha_1\theta}$	$2\epsilon\alpha_1 e^{\alpha_1\theta} =$ $H(\phi(\theta-\tau) - e^{\alpha_1\tau}\phi(\theta))$
2	H_2	$u = \phi(\delta) + \epsilon x e^{\alpha_1\delta}$	$2\epsilon\alpha_1 e^{\alpha_1\delta} =$ $H(\phi(\delta-\tau) - e^{\alpha_1\tau}\phi(\delta))$
3	H_3	$u = \phi(\gamma)$	$(p^2-1)\phi''(\gamma) =$ $H(\phi(p\tau+\gamma) - e^{\alpha_1\tau}\phi(\theta))$
4	H_4	$u = \phi(t)$	$\phi''(t) = H(\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t))$

Table 4.8 Invariant solutions for $g(u, \bar{u}) = k_2 \ln|\bar{u} - ue^{-\alpha_1\tau} - \beta| + k_3$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \phi(\theta) - \epsilon x e^{\alpha_1\theta}$	$2\epsilon\alpha_1 e^{\alpha_1\theta} = k_3$ $+k_2 \ln|\phi(\theta-\tau) - e^{\alpha_1\tau}\phi(\theta) - \beta|$
2	H_2	$u = \phi(\delta) + \epsilon x e^{\alpha_1\delta}$	$2\epsilon\alpha_1 e^{\alpha_1\delta} = k_3$ $+k_2 \ln|\phi(\delta-\tau) - e^{\alpha_1\tau}\phi(\delta) - \beta|$
3	H_3	$u = \phi(\gamma)$	$(p^2-1)\phi''(\gamma) = k_3$ $+k_2 \ln|\phi(p\tau+\gamma) - e^{\alpha_1\tau}\phi(\theta) - \beta|$
4	H_4	$u = \phi(t)$	$\phi''(t) = k_3$ $+k_2 \ln|\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t) - \beta|$

Table 4.9 Invariant solutions for $g(u,\bar{u}) = k_2(\bar{u} - ue^{-\alpha_1\tau} + \beta)^k + k_3$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \phi(\theta) - \epsilon x e^{\alpha_1\theta}$	$2\epsilon\alpha_1 e^{\alpha_1\theta} = k_3$
			$+k_2(\phi(\theta - \tau) - e^{\alpha_1\tau}\phi(\theta) + \beta)^k$
2	H_2	$u = \phi(\delta) + \epsilon x e^{\alpha_1\delta}$	$2\epsilon\alpha_1 e^{\alpha_1\delta} = k_3$
			$+k_2(\phi(\delta - \tau) - e^{\alpha_1\tau}\phi(\delta) + \beta)^k$
3	H_3	$u = \phi(\gamma)$	$(p^2 - 1)\phi''(\gamma) = k_3$
			$+k_2(\phi(p\tau + \gamma) - e^{\alpha_1\tau}\phi(\theta) + \beta)^k$
4	H_4	$u = \phi(t)$	$\phi''(t) = k_3$
			$+k_2(\phi(t - \tau) - e^{-\alpha_1\tau}\phi(t) + \beta)^k$

4.5.4 Invariant Solutions of (4.1) with $g(u,\bar{u}) = k_1\bar{u} + k_2 u$

1. **Optimal System**

Consider the Lie algebra $L_4 = \{X_1, X_2, X_3, X_4\}$, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = u\partial_u$ and $X_4 = q_3(t,x)\partial_u$. The commutation relations are

$[,]$	X_1	X_2	X_3	X_4
X_1	0	0	0	$q_{3t}\partial_u$
X_2	0	0	0	$q_{3x}\partial_u$
X_3	0	0	0	$-X_4$
X_4	$-q_{3t}\partial_u$	$-q_{3x}\partial_u$	X_4	0

The assumption that L_4 is a Lie algebra gives

$$q_{3t}(t,x) = \alpha_1 q_3(t,x),\ q_{3x}(t,x) = \alpha_2 q_3(t,x), \tag{4.61}$$

where α_1 and α_2 are constant. The general solution of equations (4.61) is $q_3(t,x) = ce^{(\alpha_1 t + \alpha_2 x)}$, where c is constant. Because $\{X_1, X_2, X_3, X_4\}$ is a basis of the Lie algebra L_4, one can choose that $X_4 = e^{(\alpha_1 t + \alpha_2 x)}\partial_u$, i.e. $q_3(t,x) = e^{(\alpha_1 t + \alpha_2 x)}$. As

มหาวิทยาลัยเทคโนโลยีสุรนารี

$q_3(t,x)$ satisfies equation (4.53), one has that

$$k_2 = -k_1 e^{-\alpha_1 \tau} + \alpha_1^2 - \alpha_2^2. \tag{4.62}$$

The automorphisms of the Lie algebra L_4 are

$$A_1: \quad \hat{x_4} = x_4 e^{\alpha_1 a};$$

$$A_2: \quad \hat{x_4} = x_4 e^{\alpha_2 a};$$

$$A_3: \quad \hat{x_4} = x_4 e^{-a};$$

$$A_4: \quad \hat{x_4} = x_4 - a(\alpha_1 x_1 + \alpha_2 x_2 - x_3).$$

From the commutator table, one can derive that the Lie algebra L_4 decomposes as $I \oplus L^3$, where $L^3 = \{X_1, X_2, X_3\}$ is a subalgebra and $I = \{X_4\}$ is an ideal of the Lie algebra L_4, respectively. As the subalgebra L^3 is Abelian, an optimal system of one-dimensional subalgebras is

$$K_1 = \{X_1 + \alpha X_2 + \beta X_3\}, \;\; K_2 = \{X_2 + \alpha X_3\}, \;\; K_3 = \{X_3\},$$

with arbitrary constants α and β.

According to the two-step algorithm (Ovsiannikov, 1993) for classifying the Lie algebra L, it is sufficient to consider the following forms of one-dimensional subalgebras:

$$W_1 = \{X_1 + \alpha X_2 + \beta X_3 + a_{14} X_4\}, \; W_2 = \{X_2 + \alpha X_3 + a_{14} X_4\},$$
$$W_3 = \{X_3 + a_{14} X_4\}, \; W_4 = \{X_4\},$$

where a_{14} is constant.

For further study one needs to simplify the latter one-dimensional subalgebras W_i by applying automorphisms.

Consider the subalgebra W_1.

Case 1: $\alpha_1 + \alpha_2 \alpha - \beta = 0$. If $a_{14} \neq 0$, then, using the automorphism A_3, a_{14} can be changed to ϵ: one gets the one-dimensional subalgebra $\{X_1 + \alpha X_2 + \beta X_3 + \epsilon X_4\}$. If $a_{14} = 0$, then one gets the one-dimensional subalgebra $\{X_1 + \alpha X_2 + \beta X_3\}$.

Case 2: $\alpha_1+\alpha_2\alpha-\beta \neq 0$**.** Using the automorphism A_4, a_{14} can be changed to 0: one gets the one-dimensional subalgebra $\{X_1 + \alpha X_2 + \beta X_3\}$.

For the subalgebra W_2.

Case 1: $\alpha_2 = \alpha$**.** If $a_{14} \neq 0$, then, using the automorphism A_3, a_{14} can be changed to ϵ: one gets the one-dimensional subalgebra $\{X_2 + \alpha X_3 + \epsilon X_4\}$. If $a_{14} = 0$, then one gets the one-dimensional subalgebra $\{X_2 + \alpha X_3\}$.

Case 2: $\alpha_2 \neq \alpha$**.** Using the automorphism A_4, a_{14} can be changed to 0: one gets the one-dimensional subalgebra $\{X_2 + \alpha X_3\}$.

For case W_3, using the automorphism A_4, a_{14} can be changed to 0, thus one gets the one-dimensional subalgebra $\{X_3\}$.

The results obtained above are summarized as follows.

Theorem 4.3. *An optimal system of one-dimensional subalgebras of the Lie algebra L_4 is defined by the subalgebras*

$$H_1 = \{X_1 + \alpha X_2 + \beta X_3\},\ H_2 = \{X_2 + \alpha X_3\},$$

$$H_3 = \{X_1 + \alpha X_2 + \beta X_3 + \epsilon X_4\}_{|\alpha_1+\alpha_2\alpha-\beta=0},$$

$$H_4 = \{X_2 + \alpha X_3 + \epsilon X_4\}_{|\alpha=\alpha_2},\ H_5 = \{X_3\},\ H_6 = \{X_4\},$$

where $\epsilon = \pm 1$, the symbol $|$ means conditions.

2. **Invariant Solutions**

In this subsection, the obtained optimal systems of subalgebras are used for deriving all invariant solutions.

Notice that $X_3 = u\partial_u$ and $X_4 = e^{(\alpha_1 t+\alpha_2 x)}\partial_u$, which implies that there are no solutions invariant with respect to the subalgebras H_5 and H_6.

Case 2.1: Subalgebra H_1**.** A representation of an invariant solution is $u = \phi(\theta)e^{\beta t}$, where ϕ is a function of a single variable, $\theta = x - \alpha t$. Substituting this representation of a solution into equation (4.1) with the function $g(u, \bar{u}) = k_1\bar{u} + k_2u$, using (4.62), the reduced equation is

$$(\alpha^2 - 1)\phi''(\theta) - 2\alpha\beta\phi'(\theta) + (\beta^2 - k_2)\phi(\theta) - k_1e^{-\beta\tau}\phi(\alpha\tau + \theta) = 0.$$

Case 2.2**: Subalgebra** H_2**.** A representation of an invariant solution is $u = \phi(t)e^{\alpha x}$, where ϕ is a function of a single variable. Substituting this representation of a solution into equation (4.1) with the function $g(u, \bar{u}) = k_1\bar{u}+k_2u$, using (4.62), the reduced equation is

$$\phi''(t) - \phi(t)(\alpha^2 + k_2) - k_1\phi(t - \tau) = 0.$$

Case 2.3**: Subalgebra** H_3**.** A representation of an invariant solution is $u = (\phi(\theta) + \epsilon te^{\alpha_2\theta})e^{\beta t}$, where ϕ is a function of a single variable, $\theta = x - \alpha t$. Substituting this representation of a solution into equation (4.1) with the function $g(u, \bar{u}) = k_1\bar{u} + k_2u$, using (4.62), the reduced equation is

$$(\alpha^2-1)\phi''(\theta)-2\alpha\beta\phi'+(\beta^2-k_2)\phi(\theta)-k_1e^{-\beta\tau}\phi(\alpha\tau+\theta)+\epsilon(k_1e^{-\beta\tau}\tau+2\alpha_1)e^{\alpha_2\theta} = 0.$$

Case 2.4**: Subalgebra** H_4**.** A representation of an invariant solution is $u = (\epsilon xe^{\alpha_1 t} + \phi(t))e^{\alpha x}$, where ϕ is a function of a single variable. Substituting this representation of a solution into equation (4.1) with the function $g(u, \bar{u}) = k_1\bar{u} + k_2u$, using (4.62), the reduced equation is

$$\phi''(t) - (\alpha^2 + k_2)\phi(t) - k_1\phi(t - \tau) - 2\epsilon\alpha e^{\alpha_1 t} = 0.$$

In summary, the representations of all invariant solutions and reduced equations are given in Table 4.10, where $\theta = x - \alpha t$.

Table 4.10 Invariant solutions for $g(u, \bar{u}) = k_1\bar{u} + k_2u$.

No.	Algebra	Inv. solutions	Reduced equation
1	H_1	$u = \phi(\theta)e^{\beta t}$	$(\alpha^2 - 1)\phi''(\theta) - 2\alpha\beta\phi'(\theta)$ $+(\beta^2 - k_2)\phi(\theta)$ $-k_1e^{-\beta\tau}\phi(\alpha\tau + \theta) = 0$
2	H_2	$u = \phi(t)e^{\alpha x}$	$\phi''(t) - \phi(t)(\alpha^2 + k_2)$ $-k_1\phi(t - \tau) = 0$
3	H_3	$u = (\phi(\theta) + \epsilon te^{\alpha_2\theta})e^{\beta t}$	$(\alpha^2 - 1)\phi''(\theta) - 2\alpha\beta\phi'$ $+(\beta^2 - k_2)\phi(\theta)$ $-k_1e^{-\beta\tau}\phi(\alpha\tau + \theta)$ $+\epsilon(k_1e^{-\beta\tau}\tau + 2\alpha_1)e^{\alpha_2\theta} = 0$
4	H_4	$u = (\epsilon xe^{\alpha_1 t} + \phi(t))e^{\alpha x}$	$\phi''(t) - (\alpha^2 + k_2)\phi(t)$ $-k_1\phi(t - \tau) - 2\epsilon\alpha e^{\alpha_1 t} = 0$

มหาวิทยาลัยเทคโนโลยีสุรนารี

CHAPTER V

GROUP ANALYSIS OF THE TWO-DIMENSIONAL NONLINEAR KLEIN-GORDON EQUATION WITH TIME-VARYING DELAY

The purpose of this chapter is to apply group analysis to the two-dimensional nonlinear Klein-Gordon equation with a time-varying delay

$$u_{tt} = u_{xx} + u_{yy} + g(u, \bar{u}),\ g_{\bar{u}}(u, \bar{u}) \neq 0, \tag{5.1}$$

where $\bar{u}(t, x, y) = u(t - \tau(t), x, y)$, $\tau(t)$ depends on t and $\tau(t) > 0$.

5.1 Admitted Lie Group of Equation (5.1)

This section is devoted to the study of admitted Lie groups of the two-dimensional nonlinear Klein-Gordon equation (5.1). The algorithm of constructing the determining equation is expressed in Chapter III.

Let the generator of a Lie group admitted by equation (5.1) be

$$X = \xi\partial_x + \gamma\partial_y + \eta\partial_t + \zeta\partial_u,$$

where ξ, η, γ and ζ are functions of variables x, y, t and u.

The prolongation of the canonical Lie-Bäcklund operator equivalent to the generator X is

$$\bar{X} = \zeta^u\partial_u + \zeta^{u_{tt}}\partial_{u_{tt}} + \zeta^{u_{xx}}\partial_{u_{xx}} + \zeta^{u_{yy}}\partial_{u_{yy}} + \zeta^{\bar{u}}\partial_{\bar{u}}$$

where the coefficients are

$$\zeta^u = \zeta - u_x\xi - u_y\gamma - u_t\eta,\ \zeta^{\bar{u}} = \bar{\zeta} - \bar{u}_x\bar{\xi} - \bar{u}_y\bar{\gamma} - \bar{u}_t\eta,\ \zeta^{u_t} = D_t\zeta^u,\ \zeta^{u_x} = D_x\zeta^u,$$

$$\zeta^{u_{xx}} = D_x\zeta^{u_x},\ \zeta^{u_y} = D_y\zeta^u,\ \zeta^{u_{yy}} = D_y\zeta^{u_y},\ \zeta^{u_t} = D_t\zeta^u,\ \ \zeta^{u_{tt}} = D_t\zeta^{u_t}.$$

Here D_x, D_y and D_t are operators of the total derivatives with respect to x, y and t, respectively, and the bar over a function $f(t,x,y,u)$ means $\bar{f} = f(t - \tau(t), x, y, u(t-\tau(t), x, y))$.

According to the algorithm for constructing the determining equations, one obtains

$$\left(\bar{X}(-u_{tt} + u_{xx} + u_{yy} + g)\right)_{|(5.1)} = 0$$

or

$$(-\zeta^{u_{tt}} + \zeta^{u_{xx}} + \zeta^{u_{yy}} + g_u\zeta^u + g_{\bar{u}}\zeta^{\bar{u}})_{|(5.1)} = 0. \tag{5.2}$$

It is also assumed that the determining equation is satisfied for any solution $u(t,x,y)$ of equation (5.1).

Substituting the coefficients of the prolonged generator into the determining equation (5.2), and replacing the derivatives found from equation (5.1) and its prolongations:

$$u_{tt} = u_{xx} + u_{yy} + g,\quad u_{ttx} = \bar{u}_x g_{\bar{u}} + g_u u_x + u_{xxx} + u_{xyy},$$

$$u_{ttt} = \bar{u}_t g_{\bar{u}} + g_u u_t + u_{txx} + u_{tyy}, u_{tty} = \bar{u}_y g_{\bar{u}} + g_u u_y + u_{yxx} + u_{yyy},$$

determining equation (5.2) becomes

$$
\begin{gathered}
2\eta_{tu}u_t^2 + \eta_{tt}u_t + 2\eta_t g + 2\eta_t u_{xx} + 2\eta_t u_{yy} - 2\eta_{ux}u_t u_x \\
-2\eta_{uy}u_t u_y + \eta_{uu}u_t^3 - \eta_{uu}u_t u_x^2 - \eta_{uu}u_t u_y^2 + 3\eta_u g u_t + 2\eta_u u_t u_{xx} \\
+2\eta_u u_t u_{yy} - 2\eta_u u_{tx}u_x - 2\eta_u u_{ty}u_y - \eta_{xx}u_t - 2\eta_x u_{tx} - \eta_{yy}u_t \\
-2\eta_y u_{ty} + g_u\zeta + g_{\bar{u}}\gamma\bar{u}_y - g_{\bar{u}}\bar{\gamma}\bar{u}_y + g_{\bar{u}}\bar{u}_x\xi - g_{\bar{u}}\bar{u}_x\bar{\xi} \\
+g_{\bar{u}}\bar{\zeta} + 2\gamma_{tu}u_t u_y + \gamma_{tt}u_y + 2\gamma_t u_{ty} - 2\gamma_{ux}u_x u_y - 2\gamma_{uy}u_y^2 \\
+\gamma_{uu}u_t^2 u_y - \gamma_{uu}u_x^2 u_y - \gamma_{uu}u_y^3 + \gamma_u g u_y + 2\gamma_u u_t u_{ty} - 2\gamma_u u_x u_{xy} \\
-2\gamma_u u_y u_{yy} - \gamma_{xx}u_y - 2\gamma_x u_{xy} - \gamma_{yy}u_y - 2\gamma_y u_{yy} + 2\xi_{tu}u_t u_x \\
+\xi_{tt}u_x + 2\xi_t u_{tx} - 2\xi_{ux}u_x^2 - 2\xi_{uy}u_x u_y + \xi_{uu}u_t^2 u_x - \xi_{uu}u_x^3 \\
-\xi_{uu}u_x u_y^2 + \xi_u g u_x + 2\xi_u u_t u_{tx} - 2\xi_u u_x u_{xx} - 2\xi_u u_{xy}u_y \\
-\xi_{xx}u_x - 2\xi_x u_{xx} - \xi_{yy}u_x - 2\xi_y u_{xy} - 2\zeta_{tu}u_t - \zeta_{tt} + 2\zeta_{ux}u_x \\
+2\zeta_{uy}u_y - \zeta_{uu}u_t^2 + \zeta_{uu}u_x^2 + \zeta_{uu}u_y^2 - \zeta_u g + \zeta_{xx} + \zeta_{yy} = 0.
\end{gathered}
$$

Splitting this determining equation with respect to $u_x, u_t, u_y, \bar{u}_x, \bar{u}_t, \bar{u}_y, u_{tx}, u_{xx}, u_{ty}, u_{xy}, u_{yy}$ and using the condition that $g_{\bar{u}} \neq 0$, one obtains

$$2\eta_t g + g_u\zeta + g_{\bar{u}}\bar{\zeta} - \zeta_{tt} - \zeta_u g + \zeta_{xx} + \zeta_{yy} = 0, \tag{5.3}$$

$$\gamma_u = 0,\ \eta_u = 0,\ \xi_u = 0, \tag{5.4}$$

$$\eta_t = \gamma_y = \xi_x,\ \gamma_x = -\xi_y,\ \eta_y = \gamma_t,\ \eta_x = \xi_t, \tag{5.5}$$

$$\zeta_{uu} = 0,\ 2\zeta_{uy} = \gamma_{xx} + \gamma_{yy} - \gamma_{tt},\ 2\zeta_{ut} = -\eta_{xx} - \eta_{yy} + \eta_{tt},\ 2\zeta_{ux} = \xi_{xx} + \xi_{yy} - \xi_{tt}, \tag{5.6}$$

$$\xi = \bar{\xi},\ \ \gamma = \bar{\gamma}. \tag{5.7}$$

From equation (5.5) one obtains

$$\eta_{xx} = \eta_{tt} = \eta_{yy},\ \xi_{xx} = \xi_{tt} = -\xi_{yy},\ -\gamma_{xx} = \gamma_{tt} = \gamma_{yy}. \tag{5.8}$$

From equation (5.4), setting $\eta_t = \gamma_y = \xi_x = \varphi(t, x, y)$, by equation (5.8) and (5.5), one gets

$$\eta_{ttt} = \varphi_{tt} = 0,\ \xi_{xxx} = \varphi_{xx} = 0,\ \gamma_{yyy} = \varphi_{yy} = 0,$$

that is,

$$\eta = \alpha_2 t^2 + \alpha_1 t + \alpha_0,\ \xi = \beta_2 x^2 + \beta_1 x + \beta_0,\ \gamma = \theta_2 y^2 + \theta_1 y + \theta_0,$$

where $\alpha_i = \alpha_i(x,y)$, $\beta_i = \beta_i(t,y)$, $\theta_i = \theta_i(t,x)$, $(i = 0,1,2)$. From equation (5.8), one derives

$$2\beta_2 = \beta_{2tt}x^2 + \beta_{1tt}x + \beta_{0tt} = -(\beta_{2yy}x^2 + \beta_{1yy}x + \beta_{0yy}),$$

which implies that $2\beta_2 = \beta_{0tt} = -\beta_{0yy}$, $\beta_{2tt} = \beta_{2yy} = 0$ and $\beta_{1tt} = \beta_{1yy} = 0$. Solving these equations, one can get $\beta_i = k_{i1}t + k_{i2}ty + k_{i3}y + k_{i4}$, where $k_{ij}\,(i = 1,2,\ j = 1,2,3,4)$ is constant. Since $\xi = \bar{\xi}$, one has $\beta_i = \bar{\beta}_i$, $i = 0,1,2$, which implies that $\beta_2 = 0$, $\beta_1 = k_{13}y + k_{14}$, $\beta_0 = k_{03}y + k_{04}$, and $\xi = (k_{13}y + k_{14})x + k_{03}y + k_{04}$, where $k_{i3}\,(i = 0,1)$ is constant. In a similar way, one can obtain $\gamma = (b_{13}x + b_{14})y + b_{03}x + b_{04}$, where $b_{i3}\,(i = 0,1)$ is constant.

From equation (5.5), one derives

$$b_{13} = k_{13} = 0,\ b_{14} = k_{14},\ b_{03} = -k_{03}$$

and

$$\eta = b_{14}t + \alpha_0,$$

where α_0 is an arbitrary constant. By equation (5.6), one gets $\zeta_{ut} = \zeta_{ux} = \zeta_{uy} = 0$, say $\zeta = c_6 u + \zeta_0$, where ζ_0 is a function of variables t, x and y.

Therefore, the obtained above results are summarized as follows:

$$\eta = c_5 t + c_1,\ \xi = c_5 x + c_4 y + c_2,\ \gamma = c_5 y - c_4 x + c_3,\ \zeta = c_6 u + \zeta_0,$$

where constant c_i is arbitrary , $(i = 1,2,3,4,5,6)$, ζ_0 depends on variables t, x and y.

Therefore, determining equation (5.3) becomes

$$g_u\zeta_0 + g_{\bar{u}}\bar{\zeta}_0 + c_6 u g_u + c_6 \bar{u} g_{\bar{u}} + (2c_5 - c_6)g - \zeta_{0tt} + \zeta_{0xx} + \zeta_{0yy} = 0. \qquad (5.9)$$

For finding the kernel of admitted Lie groups one has to assume that equation (5.3)-(5.7) are satisfied for any function $g(u,\bar{u})$. One derives $\zeta = 0$ and $c_5 = 0$ by equation (5.3). Hence, the generators

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = y\partial_x - x\partial_y$$

compose a basis of the kernel of admitted Lie algebras of equation (5.1).

5.2 Extension of Kernel

Extensions of the kernel of admitted Lie algebras are additional symmetries to the kernel which are admitted by equations for a particular function $g(u,\bar{u})$. In this section the extensions are found.

Differentiating equation (5.9) with respect to u and $\bar{u}$, one obtains

$$g_{uu}\zeta_0 + g_{u\bar{u}}\bar{\zeta}_0 = -c_6(ug_{uu} + \bar{u}g_{u\bar{u}}) - 2c_5 g_u, \tag{5.10}$$

$$g_{u\bar{u}}\zeta_0 + g_{\bar{u}\bar{u}}\bar{\zeta}_0 = -c_6(ug_{u\bar{u}} + \bar{u}g_{\bar{u}\bar{u}}) - 2c_5 g_{\bar{u}}. \tag{5.11}$$

Equations (5.10) and (5.11) are linear algebraic equations with respect to ζ_0 and $\bar{\zeta}_0$. The determinant of the matrix of this linear system of equations is equal to

$$\Delta = g_{u\bar{u}}^2 - g_{uu}g_{\bar{u}\bar{u}}.$$

5.2.1 Case $\Delta \neq 0$

If $\Delta \neq 0$, then, solving this linear system, one can find ζ_0 as follow:

$$\zeta_0 = -c_6 u + 2c_5 h(u,\bar{u}), \tag{5.12}$$

where $h(u,\bar{u}) = \Delta^{-1}(g_u g_{\bar{u}\bar{u}} - g_{\bar{u}} g_{u\bar{u}})$. Differentiating equation (5.12) with respect to u and $\bar{u}$, respectively, one has

$$2c_5 h(u,\bar{u})_u = c_6,\ 2c_5 h(u,\bar{u})_{\bar{u}} = 0,$$

which implies that

$$2c_5h(u,\bar{u})_{uu} = 0,\ 2c_5h(u,\bar{u})_{\bar{u}} = 0.$$

If $c_5 = 0$, then one derives $\zeta = 0$, thus there is no extension of the kernel of admitted Lie algebras. Therefore, for existence of an extension of the kernel one needs $c_5 \neq 0$, which implies that $h(u,\bar{u})_{uu} = 0$, $h(u,\bar{u})_{\bar{u}} = 0$. Setting $h(u,\bar{u}) = \alpha_1 u + \alpha_0$, one has

$$\zeta_0 = 2c_5\alpha_0,\ \ c_6 = 2c_5\alpha_1,$$

where α_0 and α_1 are arbitrary constants. Substituting $\zeta = 2c_5\alpha_1 u + 2c_5\alpha_0$ into equation (5.9), one gets

$$(\alpha_1 u + \alpha_0)g_u + (\alpha_1\bar{u} + \alpha_0)g_{\bar{u}} = (\alpha_1 - 1)g. \tag{5.13}$$

Solving equation (5.13), one needs to consider two cases: $\alpha_1 = 0$ and $\alpha_1 \neq 0$.

If $\alpha_1 = 0$, then equation (5.13) becomes $\alpha_0 g_u + \alpha_0 g_{\bar{u}} = -g$, since $g \neq 0$, one derives $\alpha_0 \neq 0$. Solving this equation, one gets

$$g(u,\bar{u}) = e^{\alpha u}H(\bar{u} - u),$$

where $\alpha = -\alpha_0^{-1}$ and H is an arbitrary function of a single variable. Notice that

$$\Delta = -\alpha^2 e^{2\alpha u}(HH'' - (H')^2) \neq 0.$$

Without loss of generality one can assume that $\alpha = 1$, which implies that $\alpha_0 = -1$. Thus, the extension of the kernel of admitted Lie algebras is given by the generator

$$\tilde{X}_5 = t\partial_t + x\partial_x + y\partial_y - 2\partial_u.$$

If $\alpha_1 \neq 0$, then, solving equation (5.13), one can get

$$g(u,\bar{u}) = (u + \frac{\alpha_0}{\alpha_1})^k H(\frac{\bar{u} + \frac{\alpha_0}{\alpha_1}}{u + \frac{\alpha_0}{\alpha_1}}),$$

where $k = 1 - \frac{1}{\alpha_1}$. Without loss of generality one can assume that $\alpha_0 = 0$, one derives

$$\Delta = -\frac{1}{\alpha_1} u^{2k-4}((k-1)HH'' - k(H')^2) \neq 0.$$

Hence, one gets

$$g(u, \bar{u}) = u^k H(\frac{\bar{u}}{u}),$$

where H is an arbitrary function of a single variable with satisfying $(k-1)HH'' - k(H')^2 \neq 0$, and the extension of the kernel of admitted Lie algebras is given by the generator

$$\hat{X}_5 = t\partial_t + x\partial_x + y\partial_y + 2\alpha_1 u\partial_u.$$

where $k = 1 - \frac{1}{\alpha_1}$ and $\alpha_1 \neq 0$ are both constants.

5.2.2 Case $\Delta = 0$

1. **Case $g_{\bar{u}\bar{u}} \neq 0$.**

If $g_{\bar{u}\bar{u}} \neq 0$, then the general solution of the equation $\Delta = 0$ is

$$g_u = \phi(g_{\bar{u}}), \tag{5.14}$$

where ϕ is an arbitrary function of the integration.

Equations (5.10) and (5.11) are reduced to

$$(\phi')^2 g_{\bar{u}\bar{u}}\zeta_0 + \phi' g_{\bar{u}\bar{u}}\bar{\zeta}_0 = -c_6(u(\phi')^2 g_{\bar{u}\bar{u}} + \bar{u}\phi' g_{\bar{u}\bar{u}}) - 2c_5\phi, \tag{5.15}$$

$$\phi' g_{\bar{u}\bar{u}}\zeta_0 + g_{\bar{u}\bar{u}}\bar{\zeta}_0 = -c_6(u\phi' g_{\bar{u}\bar{u}} + \bar{u} g_{\bar{u}\bar{u}}) - 2c_5 g_{\bar{u}}, \tag{5.16}$$

respectively, which imply that $2c_5(g_{\bar{u}}\phi' - \phi) = 0$.

Case 1.1: Assumption that $g_{\bar{u}}\phi' - \phi \neq 0$, which implies that $c_5 = 0$. One has

$$(\zeta_0 + c_6 u)\phi' = -(\bar{\zeta}_0 + c_6\bar{u}). \tag{5.17}$$

If $\phi' = 0$, then $\bar{\zeta_0} = -c_6\bar{u}$, which implies that $\zeta = 0$, thus there is no any extension of the kernel of admitted Lie algebras. Hence, for extension of the kernel one needs to study $\phi' \neq 0$.

Differentiating equation (5.17) with respect to $\bar{u}$, one gets

$$(\zeta_0 + c_6 u)\phi'' g_{\bar{u}\bar{u}} = -c_6. \quad (5.18)$$

Differentiating equation (5.18) with respect to t, x and y, respectively, one gets

$$\phi''\zeta_{0t} = 0,\ \phi''\zeta_{0x} = 0,\ \phi''\zeta_{0y} = 0. \quad (5.19)$$

Assume that $\phi'' \neq 0$, which implies that ζ_0 is constant, say $\zeta_0 = k_1$. By virtue of the inverse function theorem, from equation (5.17) one has

$$g_{\bar{u}} = h(\frac{\alpha + \beta\bar{u}}{\alpha + \beta u}), \quad (5.20)$$

where α and β are constants. Because of the condition $g_{\bar{u}\bar{u}} \neq 0$, one has $\beta \neq 0$. The transformation

$$\tilde{t} = t,\ \tilde{x} = x,\ \tilde{y} = y,\ \tilde{u} = u + a \quad (5.21)$$

is an equivalence transformation of equation (5.1) for any constant a. Integrating equations (5.20), using the condition (5.14), one derives that

$$g(u, \bar{u}) = uH(z) + k_0,\ z = \frac{\bar{u}}{u},$$

where k_0 is a integrating constant, $H'' \neq 0$.

Determining equation (5.9) becomes

$$k_1(H - zH' + H') = c_6 k_0.$$

Since $H'' \neq 0$, the latter equation gives that $k_1 = 0$. For existence of an extension of the kernel of admitted Lie algebras one needs to assume that $k_0 = 0$. Hence,

$$g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$$

and the extension of the kernel of admitted Lie algebras is given by the generator

$$X_5 = u\partial_u.$$

Assume that $\phi'' = 0$, which implies that there exist constants k_1 and k_0 such that

$$g_u = k_1 g_{\bar{u}} + k_0,\ k_0 \neq 0.$$

By virtue of the condition $\phi' \neq 0$ one has to assume that $k_1 \neq 0$. The general solution of the latter equation is

$$g(u, \bar{u}) = k_0 u + H(\bar{u} + k_1 u), \tag{5.22}$$

where H is a function such that $H' \neq 0$. Equation (5.17) becomes

$$(\zeta_0 + c_6 u)k_1 + (\bar{\zeta}_0 + c_6 \bar{u}) = 0, \tag{5.23}$$

and one derives $c_6 = 0$ and $\bar{\zeta}_0 = -k_1\zeta_0$. Equation (5.9) is reduced to the two-dimensional Klein-Gordon equation

$$\zeta_{0tt} = \zeta_{0xx} + \zeta_{0yy} + k_0\zeta_0.$$

Thus, for the function (5.22) one obtains that if there exists a nontrivial solution $q(t, x, y)$ of the linear Klein-Gordon equation

$$q_{tt} = q_{xx} + q_{yy} + k_0 q, \tag{5.24}$$

satisfying the condition

$$q(t - \tau(t), x, y) = -k_1 q(t, x, y), \tag{5.25}$$

then the extension of the kernel is given by the generator

$$X_q = q(t, x, y)\partial_u.$$

Notice that the set of functions $g(u,\bar{u})$ for which there exists a nontrivial solution of (5.24) and (5.25) is not empty. For example, when τ is constant, and $k_1 = -e^{\sqrt{k_0}\tau}$, then $q = e^{-\sqrt{k_0}t}$ is a nontrivial solution of (5.24) and (5.25).

Case 1.2: Assume that $g_{\bar{u}}\phi' - \phi = 0$. Solving this equation, one can get that the general solution of the latter equation is

$$g(u,\bar{u}) = H(z),\ z = \bar{u} + k_0 u, \tag{5.26}$$

where k_0 is an arbitrary constant, H is function of a single variable such that $H'' \neq 0$. Substituting equation (5.26) into equation (5.16), one gets

$$-2c_5 h(z) = c_6 z + k_0\zeta_0 + \bar{\zeta}_0, \tag{5.27}$$

where $h(z) = \frac{H'(z)}{H''(z)}$. Differentiating equation (5.27) with respect to z by once and twice, one can get

$$-2c_5 h'(z) = c_6,\ -2c_5 h''(z) = 0.$$

Case 1.2.1: Assume that $h''(z) \neq 0$, that is, $(\frac{H'(z)}{H''(z)})'' \neq 0$, which implies that $c_5 = 0$ and $c_6 = 0$. Substituting these equations into equation (5.27), one has $k_0\zeta_0 + \bar{\zeta}_0 = 0$, one derives $\zeta_{0tt} = \zeta_{0xx} + \zeta_{0yy}$ by equation (5.9). Thus, if there exists a nontrivial solution $q(t,x,y)$ of the linear Klein-Gordon equation

$$q_{tt} = q_{xx} + q_{yy}, \tag{5.28}$$

satisfying the condition

$$q(t-\tau(t),x,y) + k_0 q(t,x,y) = 0, \tag{5.29}$$

then the extension of the kernel is given by the generator

$$X_q = q(t,x,y)\partial_u,$$

where k_0 is an arbitrary constant.

Case 1.2.2: Assumption that $h''(z) = 0$, one obtains $h(z) = \alpha_1 z + \alpha_0$, that is,

$$\frac{H'(z)}{H''(z)} = \alpha_1 z + \alpha_0, \tag{5.30}$$

where α_0 and α_1 are arbitrary constants. By equation (5.27), one gets

$$k_0\zeta_0 + \bar{\zeta}_0 = -2c_5\alpha_0,\ c_6 = -2c_5\alpha_1, \tag{5.31}$$

which implies that ζ_0 depends on c_5, say $\zeta_0(t, x, y) = c_5 q(t, x, y)$.

Notice that if $c_5 = 0$, then $\zeta = 0$, which implies that there is no extension of the kernel of admitted Lie algebras. For existence of an extension of the kernel of admitted Lie algebras one needs to assume that $c_5 \neq 0$. Here $q(t, x, y)$ is an arbitrary function satisfying

$$k_0 q + \bar{q} = -2\alpha_0. \tag{5.32}$$

Thus, the extension of the kernel is given by the generator

$$X_5 = t\partial_t + x\partial_x + y\partial_y + (q(t, x, y) - 2\alpha_1 u)\partial_u.$$

Because $L = \{X_1,\ X_2,\ X_3,\ X_4,\ X_5\}$ is an admitted Lie algebra, by definition of a Lie algebra, one can get $q_t(t, x, y) = 0$, $q_x(t, x, y) = 0$ and $q_y(t, x, y) = 0$, one derives $q(t, x, y) = \alpha_2$, where α_2 is an arbitrary constant satisfying $\alpha_2(k_0 + 1) = -2\alpha_0$.

Solving equation (5.30), one derives the following three cases: $\alpha_1 = 0$, $\alpha_1 = -1$ and $\alpha_1(\alpha_1 + 1) \neq 0$.

If $\alpha_1 = 0$, then $c_6 = 0$, which implies that $\alpha_0 \neq 0$ by $H''(z) \neq 0$, thus $\alpha_2 \neq 0$. Solving equation (5.30), one has

$$g(u, \bar{u}) = k_2 e^{\frac{1}{\alpha_0}(\bar{u} + k_0 u)} + k_1. \tag{5.33}$$

Without loss of generality one can assume that $\alpha_0 = 1$; then $k_0 = -1 - \frac{2}{\alpha_2}$. Substituting equation (5.33) into equation (5.9), one gets $k_1 = 0$. Hence, the extension of the kernel is given by the generator

$$X_5 = t\partial_t + x\partial_x + y\partial_y + \alpha_2\partial_u.$$

If $\alpha_1 = -1$, then $c_6 = 2c_5$, solving equation (5.30), one has

$$g(u,\bar{u}) = k_2 \ln|\bar{u} + k_0 u - \alpha_0| + k_1,\ k_2 \neq 0. \tag{5.34}$$

Substituting equation (5.34) into equation (5.9), one gets $k_2 = 0$, which is contradiction with the condition $k_2 \neq 0$. Thus, there is no extension of the kernel of admitted Lie algebras under this case.

If $\alpha_1(\alpha_1+1) \neq 0$, solving equation (5.30), one has

$$g(u,\bar{u}) = k_2(\bar{u} + k_0 u + \frac{\alpha_0}{\alpha_1})^{1+\frac{1}{\alpha_1}} + k_1,\ k_2 \neq 0. \tag{5.35}$$

Substituting equation (5.35) into equation (5.9), one gets $k_1 = 0$. Thus,

$$g(u,\bar{u}) = k_2(\bar{u} + k_0 u + \frac{\alpha_0}{\alpha_1})^{1+\frac{1}{\alpha_1}},\ k_2 \neq 0$$

and the extension of the kernel is given by the generator

$$X_5 = t\partial_t + x\partial_x + y\partial_y + (\alpha_2 - 2\alpha_1 u)\partial_u,$$

where $\alpha_2(k_0+1) = -2\alpha_0$.

2. **Case $g_{\bar{u}\bar{u}} = 0$, $g_{uu} \neq 0$.** This assumption implies that $g_{\bar{u}u} = 0$. Because of condition $g_{\bar{u}} \neq 0$, one has

$$g(u,\bar{u}) = k_1\bar{u} + h(u), \tag{5.36}$$

where $k_1 \neq 0$ is a constant and $h'' \neq 0$. By equations (5.10) and (5.11), one has

$$\zeta = 0,\ c_5 = 0,$$

which means that there is no extension of the kernel of the admitted Lie algebra.

3. **Case** $g_{\bar{u}\bar{u}} = 0$, $g_{uu} = 0$.

This case corresponds to a linear delay differential equation with

$$g(u, \bar{u}) = k_1 \bar{u} + k_2 u + k, \tag{5.37}$$

where k, $k_1 \neq 0$ and k_2 are constant.

Notice that the constant k can be reduced to zero by the change

$$u = \tilde{u} - \frac{\tilde{k}_1}{4} x^2 - \frac{\tilde{k}_1}{4} y^2 + \tilde{k}_2.$$

Indeed, choosing the constants $\tilde{k}_1$ and $\tilde{k}_2$ such that

$$\tilde{k}_1 (k_1 + k_2) = 0, \;\; \tilde{k}_1 - \tilde{k}_2 (k_1 + k_2) = k$$

the function $\tilde{u}$ satisfies the equation

$$\tilde{u}_{tt} = \tilde{u}_{xx} + \tilde{u}_{yy} + k_1 \bar{\tilde{u}} + k_2 \tilde{u}.$$

Substituting equation (5.37) into equation (5.11), one gets $c_5 = 0$. Determining equation (5.9) becomes

$$\zeta_{0tt} = \zeta_{0xx} + \zeta_{0yy} + k_1 \bar{\zeta}_0 + k_2 \zeta_0. \tag{5.38}$$

Hence, the extension of the kernel is given by the generators $X_5 = u\partial_u$ and $X_{\zeta_0} = \zeta_0(t, x, y)\partial_u$. In particular, if τ is constant and $k_0 = k_1 e^{\sqrt{k_0}\tau} + k_2$, then $\zeta_0 = e^{-\sqrt{k_0}t}$ is a particular solution of equation (5.38).

5.3 Summary of the Group Classification

By the discussions of the previous section one obtains the following complete group classification of the time-varying delay partial differential equation

$$u_{tt} = u_{xx} + u_{yy} + g(u, \bar{u}), \; \bar{u}(t, x, y) = u(t - \tau(t), x, y),$$

where $g_{\bar{u}} \neq 0$. The results of the group classification are presented in Table 5.1, where the function H is a function of a single argument, and the coefficients $q_i(t,x)$, $(i = 1, 2, 3)$ satisfy the equations:

$$\begin{gathered} q_{1tt}(t,x,y) = q_{1xx}(t,x,y) + q_{1yy}(t,x,y) + k_0 q_1(t,x,y), \\ q_1(t-\tau,x,y) = -k_1 q_1(t,x,y),\ k_0 q_1(t,x,y) \neq 0 \end{gathered} \tag{5.39}$$

$$\begin{gathered} q_{2tt}(t,x,y) = q_{2xx}(t,x,y) + q_{2yy}(t,x,y), \\ q_2(t-\tau,x,y) = -k_0 q_2(t,x,y),\ q_2(t,x,y) \neq 0 \end{gathered} \tag{5.40}$$

$$q_{3tt}(t,x,y) = q_{3xx}(t,x,y) + q_{3yy}(t,x,y) + k_2 q_3(t,x,y) + k_1 q_3(t-\tau,x,y). \tag{5.41}$$

Table 5.1 Group classification of the equation $u_{tt} = u_{xx} + u_{yy} + g(u, \bar{u}), \quad (g_{\bar{u}} \neq 0)$.

No.	$g(u, \bar{u})$	Conditions	Extensions
1	$uH(\frac{\bar{u}}{u})$	$H'' \neq 0$	$X_6 = u\partial_u$
2	$e^u H(\bar{u} - u)$	$(H')^2 - HH'' \neq 0$	$\tilde{X}_5 = t\partial_t + x\partial_x + y\partial_y - 2\partial_u$
3	$u^k H(\frac{\bar{u}}{u})$	$k = 1 - \frac{1}{\alpha_1}$, $(k-1)HH'' - k(H')^2 \neq 0$	$\hat{X}_5 = t\partial_t + x\partial_x + y\partial_y + 2\alpha_1 u\partial_u$
4	$k_0 u + H(\bar{u} + k_1 u)$	$k_0 k_1 H' \neq 0$	$q_1\partial_u$
5	$H(\bar{u} + k_0 u)$	$H''(\frac{H'}{H''})'' \neq 0$	$q_2\partial_u$
6	$k_2 e^{\bar{u} + k_0 u}$	$k_0 = -1 - \frac{2}{\alpha_2}, k_2 \neq 0$	$Z_5 = t\partial_t + x\partial_x + y\partial_y + \alpha_2\partial_u$
7	$k_2(\bar{u} + k_0 u + \frac{\alpha_0}{\alpha_1})^{1+\frac{1}{\alpha_1}} + k_1$	$\alpha_2(k_0 + 1) = -2\alpha_0$, $k_2(\alpha_1 + 1) \neq 0$	$Y_5 = t\partial_t + x\partial_x + y\partial_y + (\alpha_2 - 2\alpha_1 u)\partial_u$
8	$k_1\bar{u} + k_2 u$	$k_1 \neq 0$	X_6, $q_3\partial_u$

5.4 Invariant Solutions

The purpose of the group analysis method is to construct exact solutions of partial differential equations; here finding invariant solutions is an additional purpose. This section is devoted to constructing invariant solutions of equation (5.1) for each of the functions $g(u,\bar{u})$ in Table 5.1. Using an optimal system of two-dimensional subalgebra of the admitted Lie algebra, one derives invariant solutions. Notice that for the cases $4, 5$ and 8 in Table 5.1, there is a generator of the form $q(t,x,y)\partial_u$. Usually such kind of generators are omitted in construction of invariant solutions. However the use of these generators also provides invariant solutions. This will be demonstrated further.

5.4.1 Invariant solutions of (5.1) with $g(u,\bar{u}) = uH(\frac{\bar{u}}{u})$

For this function the admitted Lie algebra L_1 is spanned by $\{X_1, X_2, X_3, X_4, X_5\}$, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = u\partial_u$. The commutator table is

[,]	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	0
X_2	0	0	0	$-X_3$	0
X_3	0	0	0	X_2	0
X_4	0	X_3	$-X_2$	0	0
X_5	0	0	0	0	0

Solving the corresponding to Lie equations, the automorphisms are

$$A_1: \quad \hat{x_3} = x_3 + a_1 x_4;$$

$$A_2: \quad \hat{x_2} = x_2 - a_2 x_4;$$

$$A_3: \quad \hat{x_2} = x_2\cos(a_3) + x_3\sin(a_3), \;\; \hat{x_3} = -x_2\sin(a_3) + x_3\cos(a_3).$$

Before constructing an optimal system, one studies the algebraic structure of the Lie algebra L_1. Consider the vector space L^3 spanned by the operators $\{X_1,\ X_4,\ X_5\}$, One can verify that it is a subalgebra of the Lie algebra L_1, and the vector space I^2 spanned by the operators $\{X_2, X_3\}$ is an ideal of algebra L_1. Hence, the Lie algebra L_1 is decomposed as $I^2 \oplus L^3$. Because the subalgebra L^3 is Abelian, an optimal system of one-dimensional admitted subalgebras consists of

$$\{H_i\},\ \ (i = 1, 2, 3),$$

where

$$H_1 = X_4 + \alpha X_1 + \beta X_5,\ \ H_2 = X_1 + \alpha X_5,\ \ H_3 = X_5,$$

and an optimal system of two-dimensional admitted subalgebras consists of D_1, D_2, D_3, where

$$D_1 = \{X_4 + \beta X_5, X_1 + \alpha X_5\},\ D_2 = \{X_4 + \alpha X_1, X_5\},\ D_3 = \{X_1, X_5\}$$

with arbitrary constants α and β.

Let $Y_i = a_{i1}X_1 + a_{i2}X_2 + a_{i3}X_3 + a_{i4}X_4 + a_{i5}X_5\ (i = 1, 2)$ which constitutes a two-dimensional subalgebras of the Lie algebra L_1 and is denoted by matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{pmatrix}, \tag{5.42}$$

with the requirement that rank of the matrix (5.42) is equal to 2. According to the two-step algorithm (Ovsiannikov, 1993) for classifying the Lie algebra L_1, it is sufficient to consider the following forms of two-dimensional subalgebras:

$$\begin{pmatrix} \alpha & a_{12} & a_{13} & 1 & \beta \\ 0 & a_{22} & a_{23} & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & \alpha \\ 0 & a_{22} & a_{23} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 1 \\ 0 & a_{22} & a_{23} & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & a_{12} & a_{13} & 1 & 0 \\ 0 & a_{22} & a_{23} & 0 & 1 \end{pmatrix}, \tag{5.43}$$
$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & a_{12} & a_{13} & 1 & \alpha \\ 1 & a_{22} & a_{23} & 0 & \beta \end{pmatrix},$$

where $a_{ij}, (i = 1, 2; j = 2, 3)$ and α are arbitrary constants.

For further study one needs to simplify (5.43) by applying automorphisms $A_i \, (i = 1, 2, 3)$ and transformation of matrix; the results are summarized as follows.

Theorem 5.1. *An optimal system of two-dimensional subalgebras of the Lie algebra L_1 consists of the subalgebras*

$$M_1 = \{X_2, X_3\}, \; M_2 = \{X_2, X_5 + \alpha X_3\},$$
$$M_3 = \{X_2, X_1 + \gamma X_3 + \beta X_5\}, \; M_4 = \{X_5, X_4 + \alpha X_1\},$$
$$M_5 = \{X_5 + \alpha X_2, X_1 + \beta X_2 + \gamma X_3\}, \; M_6 = \{X_4 + \alpha X_5, X_1 + \beta X_5\}$$

with arbitrary constants α, β and γ.

Proof : Here we only repsent the processes of calculating for the case

$$\begin{pmatrix} 0 & a_{12} & a_{13} & 1 & \alpha \\ 1 & a_{22} & a_{23} & 0 & \beta \end{pmatrix}, \tag{5.44}$$

which denotes subalgebra $\{Y_1, Y_2\}$, where $Y_1 = 0 \cdot X_1 + a_{12}X_2 + a_{13}X_3 + X_4 + \alpha X_5$, $Y_2 = X_1 + a_{22}X_2 + a_{23}X_3 + 0 \cdot X_4 + \beta X_5$. Calculation of other the optimal systems of two-dimensional subalgebra of the Lie algebra L_1 are similar.

First, by the automorphisms A_1 and A_2, a_{12} and a_{13} can be changed to 0. Checking subalgebra conditions, one has

$$[X_4+\alpha X_5, X_1+a_{22}X_2+a_{23}X_3+\beta X_5] = a(X_4+\alpha X_5)+b(X_1+a_{22}X_2+a_{23}X_3+\beta X_5),$$

for some constants a and b. By calculating the left hand side and comparing the coefficients in the left hand with coefficients in the right hand side, one obtains

$$-a_{23}X_2 + a_{22}X_3 = bX_1 + ba_{22}X_2 + ba_{23}X_3 + aX_4 + (a\alpha + b\beta)X_5,$$

which implies that

$$a = 0,\ b = 0,\ a_{22} = ba_{23},\ -a_{23} = ba_{22},$$

and one derives $a_{23} = 0$ and $a_{22} = 0$. One obtains the two-dimensional subalgebra $\{X_4 + \alpha X_5, X_1 + \beta X_5\}$.

Thus, the proof is completed.

Using the optimal system of subalgebras obtained, the representations of all invariant solutions and reduced equations are given in Table 5.2, where $w_1 = y - \gamma t$, $w = x^2 + y^2$, $\psi(t)$ is an arbitrary function satisfying $\psi(t) > 0$.

Remark: Illustrating the representation of invariant solutions by case $N = 2$ in Table 5.1, others can be obtained using similar way. As the subalgebra is spanned by $\{X_2, X_5 + \alpha X_3\}$, then operators are $Y_1 = X_2$ and $Y_2 = X_5 + \alpha X_3$. If $\alpha = 0$, i.e. $Y_2 = X_5$, then one derives that there are no invariant solutions. Hence, for existence of an invariant solution, one has to assume $\alpha \neq 0$. For operator $Y_1 = X_2$, the invariant solution does not depend on variable x. For $X_5 + \alpha X_3$, solving the corresponding characteristic system, one gets that the representation of an invariant solution is $u = \phi(t)e^{\frac{1}{\alpha}y}$, where ϕ is a function of a single variable. Substituting this representation of a solution into equation (5.1) with the function $g(u, \bar{u}) = uH(\frac{\bar{u}}{u})$, the reduced equation is

$$\phi''(t) = \phi(t)(\frac{1}{\alpha^2} + H(\frac{\phi(t - \psi(t))}{\phi(t)})).$$

In addition, one needs to consider the condition that $(1-\tau'(t))\eta=\bar{\eta}$ for $\tau(t)$; by the generators $\{X_2\}$ and $\{X_5+\alpha X_3\}$, one derives that η are both equal to zero, which implies that it holds for arbitrary $\tau(t)$, by denoting $\psi(t)$.

มหาวิทยาลัยเทคโนโลยีสุรนารี

Table 5.2 Invariant solutions for $g(u,\bar{u}) = uH(\frac{\bar{u}}{u})$.

No.	Alg.	Repr. of inv. solutions	$Repr.of\tau$	Reduced equation
1	M_1	$u = \phi(t)$	$\psi(t)$	$\phi''(t) = \phi(t)H(\frac{\phi(t-\psi(t))}{\phi(t)})$
2	M_2	$u = \phi(t)e^{\frac{1}{\alpha}y}$	$\psi(t)$	$\phi''(t) = \phi(t)(\frac{1}{\alpha^2} + H(\frac{\phi(t-\psi(t))}{\phi(t)}))$
3	M_3	$u = e^{\beta t}\phi(w_1)$	$cons.$	$(1-\gamma^2)\phi''(w_1) = 2\gamma\beta\phi'(w_1)$
				$+(H(e^{-\beta\tau}\frac{\phi(w_1+\gamma\tau)}{\phi(w_1)}) - \beta^2)\phi(w_1)$
4	M_4	no		
5	M_5	$u = e^{\frac{1}{\alpha}(x-\beta t)}\phi(w_1)$	$cons.$	$(1-\gamma^2)\phi''(w_1) = -2\frac{\beta\gamma}{\alpha}\phi'(w_1)$
				$+(\frac{1}{\alpha^2}(1-\beta^2) + H(e^{\frac{\beta}{\alpha}\tau}\frac{\phi(w_1+\gamma\tau)}{\phi(w_1)}))\phi(w_1)$
6	M_6	$u = \phi(w)e^{(\alpha\arcsin\frac{x}{\sqrt{w}}+\beta t)}$	$cons.$	$4w\phi''(w) + 4\phi'(w) + (\alpha^2 w + H(e^{-\beta\tau}) - \beta^2)\phi(w) = 0.$

5.4.2 Invariant Solutions of (5.1) with $g(u,\bar{u}) = e^u H(\bar{u}-u)$

For this case, the admitted Lie algebra is $L_2 = \{X_1,\ X_2,\ X_3,\ X_4, X_5\}$, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = t\partial_t + x\partial_x + y\partial_y - 2\partial_u$. The commutator table is

$[,]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	X_1
X_2	0	0	0	$-X_3$	X_2
X_3	0	0	0	X_2	X_3
X_4	0	X_3	$-X_2$	0	0
X_5	$-X_1$	$-X_2$	$-X_3$	0	0

The corresponding to automorphisms are

$A_1:\quad \hat{x_1} = x_1 - a_1x_5;$

$A_2:\quad \hat{x_2} = x_2 - a_2x_5,\ \hat{x_3} = x_3 + a_2x_4;$

$A_3:\quad \hat{x_2} = x_2 - a_3x_4,\ \hat{x_3} = x_3 - a_3x_5;$

$A_4:\quad \hat{x_2} = x_2\cos(a_4) + x_3\sin(a_4),\ \hat{x_3} = -x_2\sin(a_4) + x_3\cos(a_4);$

$A_5:\quad \hat{x_1} = x_1e^{a_5},\ \hat{x_2} = x_2e^{a_5},\ \hat{x_3} = x_3e^{a_5}.$

From the commutator table, one can derive that the Lie algebra L_2 decomposes as $I^3 \oplus L^2$, where $I^3 = \{X_1, X_2, X_3\}$ is an ideal and $L^2 = \{X_4,\ X_5\}$ is a subalgebra of the Lie algebra L_2, respectively. Since the subalgebra L^2 is Abelian, an optimal system of one-dimensional admitted subalgebras consists of

$$\{H_i\},\ \ (i = 1, 2),$$

where

$$H_1 = X_4,\ \ H_2 = X_5 + \alpha X_4,$$

and an optimal system of two-dimensional admitted subalgebras consists of

$$D_1 = \{X_4, X_5\}$$

with arbitrary constants α.

Two-dimensional subalgebras of the Lie algebra L_2 are also denoted by matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{pmatrix}, \tag{5.45}$$

with the requirement that the rank of the matrix (5.45) is equal to 2. For obtaining an optimal system of subalgebras one uses the two-step algorithm (Ovsiannikov, 1993); it is sufficient to consider the following forms of two-dimensional subalgebras:

$$\begin{array}{cc} \begin{pmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \end{pmatrix}, & \begin{pmatrix} a_{11} & a_{12} & a_{13} & \alpha & 1 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \end{pmatrix}, & \begin{pmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 \end{pmatrix}, \end{array} \tag{5.46}$$

where $a_{ij}, (i = 1, 2; j = 1, 2, 3)$ and α are arbitrary constants.

For further study one needs to simplify (5.46) by applying automorphisms $A_i\,(i = 1, 2, 3, 4, 5)$ and transformation of matrix, the results are summarized as follows.

Theorem 5.2. *An optimal system of two-dimensional subalgebras of the Lie algebra L_2 is:*

$$M_1 = \{X_2, X_3\},\ M_2 = \{X_1, X_4\},\ M_3 = \{X_2, X_1 + \alpha X_3\},$$

$$M_4 = \{X_5, X_4\},\ M_5 = \{X_5, X_2 + \alpha X_1\},\ M_6 = \{X_1, X_5 + \alpha X_2 + \beta X_4\}$$

with arbitrary constants α and β.

Proof : As the algorithm of the optimal system of two-dimensional subalgebras of the Lie algebra L_2 are similar, hence, we only give the process of calculating for the case

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \alpha & 1 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \end{pmatrix}, \tag{5.47}$$

which denotes subalgebra $\{Y_1, Y_2\}$, where $Y_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \alpha X_4 + X_5$, $Y_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + 0 \cdot X_4 + 0 \cdot X_5$.

First, by successfully using the automorphisms A_1, A_3 and A_4, then a_{11}, a_{13} and a_{23} can be changed to 0 respectively. Thus the matrix 5.47 reduces to

$$\begin{pmatrix} 0 & a_{12} & 0 & \alpha & 1 \\ a_{21} & a_{22} & 0 & 0 & 0 \end{pmatrix}, \tag{5.48}$$

and $a_{21}^2 + a_{22}^2 \neq 0$. Checking subalgebra conditions, one has

$$[a_{12}X_2 + \alpha X_4 + X_5, a_{21}X_1 + a_{22}X_2] = a(a_{12}X_2 + \alpha X_4 + X_5) + b(a_{21}X_1 + a_{22}X_2),$$

for some constants a and b. By calculating the left hand side and comparing the coefficients in the left hand with coefficients in the right hand side, one gets

$$\alpha a_{22}X_3 - a_{21}X_1 - a_{22}X_2 = ba_{21}X_1 + (aa_{12} + ba_{22})X_2 + a\alpha X_4 + aX_5,$$

which implies that

$$a = 0,\ (b+1)a_{21} = 0,\ (b+1)a_{22} = 0,\ \alpha a_{22} = 0.$$

Since $a_{21}^2 + a_{22}^2 \neq 0$, one derives $b = -1$.

Case 1: $a_{22} = 0$. As $a_{21}^2 + a_{22}^2 \neq 0$, one gets $a_{21} \neq 0$. Dividing Y_2 by a_{21}, thus the operators $Y_2 = X_1$. One gets the two-dimensional subalgebra M_6 in theorem.

Case 2: $a_{22} \neq 0$. The assumption implies that $\alpha = 0$, dividing Y_2 by a_{22}, thus the operators $Y_2 = \beta X_1 + X_2$. Using transformation of matrix, a_{12} can be changed to 0. one gets

$$\begin{pmatrix} \gamma & 0 & 0 & 0 & 1 \\ \beta & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.49}$$

By the automorphism A_1, γ can be changed to 0: one gets the two-dimensional subalgebra M_5 in theorem.

Thus, the proof is completed.

Using the obtained optimal system of subalgebras, the representation of all invariant solutions and reduced equations are given in Table 5.3, where $w = x^2+y^2$, $w_1 = y-\alpha t$, $w_2 = tw^{-1/2}$, $w_3 = t/y$, $\psi(t)$ is a function satisfying $\psi(t) > 0$, and a is an arbitrary constant with $a > 0$.

Remark: For reducing equations and the existence of invariant solutions, one has to assume that some coefficients are equal to zero for some generator. For example, one needs to assume $\alpha = 0$ in the case 5 of Table 5.2

มหาวิทยาลัยเทคโนโลยีสุรนารี

Table 5.3 Invariant solutions for $g(u,\bar{u}) = e^{u}H(\bar{u}-u)$.

No.	Alg.	Repr. of inv. solutions	$Repr.of\tau$	Reduced equation
1	M_1	$u = \phi(t)$	$\psi(t)$	$\phi''(t) = e^{\phi(t)}H(\phi(t-\psi(t)) - \phi(t))$
2	M_2	*no*		
3	M_3	$u = \phi(w_1)$	*cons.*	$(1-\alpha^2)\phi''(w_1) + e^{\phi(w_1)}H(\phi(w_1+\alpha\tau) - \phi(w_1)) = 0$
4	M_4	$u = -\ln(w) + \phi(w_2)$	at	$(w_2^2-1)\phi''(w_2) + w_2\phi'(w_2)$ $+e^{\phi(w_2)}H(\phi((1-a)w_2) - \phi(w_2)) = 0$
5	M_5	$u = -2\ln(y) + \phi(w_3)$	at	$(w_3^2-1)\phi''(w_3) + 2w_3\phi'(w_3)$ $+e^{\phi(w_3)}H(\phi((1-a)w_3) - \phi(w_3)) + 2 = 0$
6	M_6	*no*		

5.4.3 Invariant Solutions of (5.1) with $g(u,\bar{u}) = u^k H(\frac{\bar{u}}{u}), (k \neq 1)$

For this function the admitted Lie algebra is $L_3 = \{X_1, X_2, X_3, X_4, X_5\}$, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = t\partial_t + x\partial_x + y\partial_y + 2\alpha_1 u\partial_u$. Comparing the commutator table in previous subsection, one derives all optimal systems of two-dimensional subalgebras of the algebra L_3 with the basis

$$M_1 = \{X_2, X_3\},\ M_2 = \{X_1, X_4\},\ M_3 = \{X_2, X_1 + \alpha X_3\},$$
$$M_4 = \{X_5, X_4\},\ M_5 = \{X_5, X_2 + \alpha X_1\},\ M_6 = \{X_1, X_5 + \alpha X_2 + \beta X_4\}$$

with arbitrary constants α and β.

Using the obtained optimal system of subalgebras, the representation of all invariant solutions and reduced equations are given in Table 5.4, where $w = x^2 + y^2$, $w_1 = y - \alpha t$, $w_2 = tw^{-1/2}$, $w_3 = t/y$, $\psi(t)$ is a function satisfying $\psi(t) > 0$, and a is an arbitrary constant with $a > 0$.

Table 5.4 Invariant solutions for $g(u,\bar{u}) = u^k H(\frac{\bar{u}}{u}), (k \neq 1,\ k = 1 - \frac{1}{\alpha_1})$.

No.	Alg.	Inv. solutions	*Repr.ofτ*	Reduced equation
1	M_1	$u = \phi(t)$	$\psi(t)$	$\phi''(t) = \phi^k(t) H(\frac{\phi(t-\psi(t))}{\phi(t)})$
2	M_2	*no*		
3	M_3	$u = \phi(w_1)$	*cons.*	$(1-\alpha^2)\phi''(w_1) + \phi^k(w_1) H(\frac{\phi(w_1+\alpha\tau)}{\phi(w_1)}) = 0$
4	M_4	$u = w^{\alpha_1}\phi(w_2)$	*at*	$(w_2^2-1)\phi''(w_2) - 2\alpha_1 w_2 \phi'(w_2) + 4\alpha_1^2 \phi(w_2)$
				$+\phi^k(w_2) H(\frac{\phi((1-a)w_2)}{\phi(w_2)}) = 0$
5	M_5	$u = y^{2\alpha_1}\phi(w_3)$	*at*	$(w_3^2-1)\phi''(w_3) - 2(\alpha_1-1) w_3 \phi'(w_3)$
				$+\phi^k(w_3) H(\frac{\phi((1-a)w_3)}{\phi(w_3)}) + 2\alpha_1(2\alpha_1-1)\phi(w_3) = 0$
6	M_6	*no*		

5.4.4 Invariant Solutions of (5.1) with $g(u,\bar{u}) = k_0u + H(\bar{u} + k_1u)$

For this case, the admitted algebra L_4 is spanned by X_1, X_2, X_3, X_4 and X_5, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = q(t,x,y)\partial_u$. The commutator table is

$[,]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	$q_t\partial_u$
X_2	0	0	0	$-X_3$	$q_x\partial_u$
X_3	0	0	0	X_2	$q_y\partial_u$
X_4	0	X_3	$-X_2$	0	$(yq_x - xq_y)\partial_u$
X_5	$-q_t\partial_u$	$-q_x\partial_u$	$-q_y\partial_u$	$-(yq_x - xq_y)\partial_u$	0

The requirement that L_4 is a Lie algebra gives existence of constants $\alpha_i\,(i = 1,2,3,4)$ such that

$$\begin{gathered} q_t(t,x,y) = \alpha_1 q(t,x,y),\ q_x(t,x,y) = \alpha_2 q(t,x,y), \\ q_y(t,x,y) = \alpha_3 q(t,x,y),\ yq_x(t,x,y) - xq_y(t,x,y) = \alpha_4 q(t,x,y). \end{gathered} \tag{5.50}$$

The general solution of equations (5.50) is $q(t,x,y) = ce^{\alpha_1 t}$, where c is constant and $c \neq 0$. Therefore, one can choose $q(t,x,y) = e^{\alpha_1 t}$. Since $q(t,x,y)$ satisfies relations (5.39), one derives

$$k_0 = \alpha_1^2,\ k_1 = -e^{-\alpha_1\tau}. \tag{5.51}$$

Notice that $k_0 \neq 0$, which implies that $\alpha_1 \neq 0$ and τ must be constant.

By the commutator table, the automorphisms are:

$A_1: \quad \hat{x_5} = x_5 e^{-a_1\alpha_1};$

$A_2: \quad \hat{x_3} = x_3 + a_2 x_4;$

$A_3: \quad \hat{x_2} = x_2 - a_3 x_4;$

$A_4: \quad \hat{x_2} = x_2\cos(a_4) + x_3\sin(a_4),\ \hat{x_3} = -x_2\sin(a_4) + x_3\cos(a_4);$

$A_5: \quad \hat{x_5} = x_5 + a_5\alpha_1 x_1.$

The Lie algebra L_4 decomposes as $I^3 \oplus L^2$, where $L^2 = \{X_1,\ X_4\}$ is a subalgebra and $I^3 = \{X_2, X_3, X_5\}$, is an ideal of the Lie algebra L_4, respectively. Since the subalgebra L^2 is Abelian, an optimal system of one-dimensional admitted subalgebras consists of

$$\{H_i\}, \ \ (i = 1, 2),$$

where

$$H_1 = X_1, \ \ H_2 = X_4 + \alpha X_1,$$

and an optimal system of two-dimensional admitted subalgebras

$$D_1 = \{X_1, X_4\}$$

with arbitrary constants α.

According to the two-step algorithm (Ovsiannikov, 1993) for classifying the Lie algebra L_4, it is sufficient to consider the following forms of two-dimensional subalgebras:

$$\begin{pmatrix} 1 & a_{12} & a_{13} & 0 & a_{15} \\ 0 & a_{22} & a_{23} & 0 & a_{25} \end{pmatrix}, \quad \begin{pmatrix} \alpha & a_{12} & a_{13} & 1 & a_{15} \\ 0 & a_{22} & a_{23} & 0 & a_{25} \end{pmatrix},$$
$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & a_{15} \\ 0 & a_{22} & a_{23} & 0 & a_{25} \end{pmatrix}, \quad \begin{pmatrix} 1 & a_{12} & a_{13} & 0 & a_{15} \\ 0 & a_{22} & a_{23} & 1 & a_{25} \end{pmatrix}, \tag{5.52}$$

where $a_{ij}, (i = 1, 2; j = 2, 3, 5)$ and α are constants.

For further study one needs to simplify (5.52) by applying automorphisms $A_i\,(i = 1, 2, 3, 4, 5)$ and matrix transformations; the results are summarized as follows.

Theorem 5.3. *An optimal system of two-dimensional subalgebras of the Lie algebra L_4 is:*

$$M_1 = \{X_1, X_4\},\ M_2 = \{X_4, X_5\},\ M_3 = \{X_5, X_4 + \gamma X_1\},$$
$$M_4 = \{X_2, X_3\},\ M_5 = \{X_2, X_5 + \alpha X_3\},\ M_6 = \{X_2, X_1 + \alpha X_3\},$$
$$M_7 = \{X_1 + \gamma X_2, X_5 + \beta X_3\}$$

with arbitrary constants α, β and $\gamma \neq 0$.

Proof : we show the processes of calculating for the case

$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & a_{15} \\ 0 & a_{22} & a_{23} & 0 & a_{25} \end{pmatrix}, \tag{5.53}$$

which denotes subalgebra $\{Y_1, Y_2\}$, where $Y_1 = 0\cdot X_1 + a_{12}X_2 + a_{13}X_3 + 0\cdot X_4 + a_{15}X_5$, $Y_2 = 0 \cdot X_1 + a_{22}X_2 + a_{23}X_3 + 0 \cdot X_4 + a_{25}X_5$.

Case 1: $a_{15} \neq 0$. Dividing Y_1 by a_{15}, a_{15} can be changed to 1. Using a matrix transformation, a_{25} can be changed to 0. Since the rank of matrix (5.53) is equal to 2, one has $a_{22}^2 + a_{23}^2 \neq 0$. Using the automorphism A_4, a_{23} can be changed to 0, hence, the matrix 5.53 reduces to

$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 1 \\ 0 & a_{22} & 0 & 0 & 0 \end{pmatrix}. \tag{5.54}$$

Since $a_{22} \neq 0$, dividing Y_2 by a_{22}, one has $Y_2 = X_2$. Using a matrix transformation, a_{12} can be changed to 0: one gets the two-dimensional subalgebra M_5 in the theorem.

Case 2: $a_{15} = 0$**.** The assumption implies that $a_{25} = 0$; the matrix 5.53 reduces to

$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 \end{pmatrix}. \tag{5.55}$$

Since the rank of matrix (5.55) is equal to 2, by a matrix transformation, matrix (5.55) can be changed to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}; \tag{5.56}$$

one derives the two-dimensional subalgebra M_4 in the theorem.

Other cases can be computed similarly.

Thus, the proof is completed.

Using the obtained optimal systems of two-dimensional subalgebras of the Lie algebra L_4 obtained in the previous theorem, all invariant solutions and reduced equations are presented in Table 5.5, where $w_1 = y - \alpha t$, $w_2 = x - \gamma t$.

มหาวิทยาลัยเทคโนโลยีสุรนารี

Table 5.5 Invariant solutions for $g(u,\bar{u}) = k_0u + H(\bar{u} + k_1u)$.

No.	Alg.	Inv. solutions	$Repr. of \tau$	Reduced equation
1	M_1	*no*		
2	M_2	*no*		
3	M_3	*no*		
4	M_4	$u = \phi(t)$	*cons.*	$\phi''(t) = \alpha_1^2\phi(t) + H(\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t))$
5	M_5	$u = \frac{y}{\alpha}e^{\alpha_1 t} + \phi(t)$	*cons.*	$\phi''(t) = \alpha_1^2\phi(t) + H(\phi(t-\tau) - e^{-\alpha_1\tau}\phi(t))$
6	M_6	$u = \phi(w_1)$	*cons.*	$(1-\alpha^2)\phi''(w_1) + \alpha_1^2\phi(w_1) + H(\phi(w_1 + \alpha\tau) - e^{-\alpha_1\tau}\phi(w_1)) = 0$
7	M_7	$u = \frac{1}{\beta}e^{\alpha_1 t}y + \phi(w_2)$	*cons.*	$(1-\gamma^2)\phi''(w_2) + \alpha_1^2 H(\phi(w_2 + \gamma\tau) - e^{-\alpha_1\tau}\phi(w_2)) = 0$

5.4.5 Invariant Solutions of (5.1) with $g(u,\bar{u}) = H(\bar{u} + k_0 u)$

For this case the admitted algebra L_5 is spanned by X_1, X_2, X_3, X_4 and X_5, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = q(t,x,y)\partial_u$. The commutator table is

[,]	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	$q_t\partial_u$
X_2	0	0	0	$-X_3$	$q_x\partial_u$
X_3	0	0	0	X_2	$q_y\partial_u$
X_4	0	X_3	$-X_2$	0	$(yq_x - xq_y)\partial_u$
X_5	$-q_t\partial_u$	$-q_x\partial_u$	$-q_y\partial_u$	$-(yq_x - xq_y)\partial_u$	0

The requirement that L_5 is a Lie algebra implies existence of constants $\alpha_i\,(i = 1,2,3,4)$ such that

$$\begin{aligned} q_t(t,x,y) = \alpha_1 q(t,x,y),\ q_x(t,x,y) = \alpha_2 q(t,x,y),\\ q_y(t,x,y) = \alpha_3 q(t,x,y),\ yq_x(t,x,y) - xq_y(t,x,y) = \alpha_4 q(t,x,y). \end{aligned} \tag{5.57}$$

The general solution of equations (5.57) is $q(t,x,y) = ce^{\alpha_1 t}$, where c is constant and $c \neq 0$. Therefore, one can choose $q(t,x,y) = e^{\alpha_1 t}$. Since $q(t,x,y)$ satisfies relations (5.40), one derives

$$\alpha_1 = 0,\ k_0 = -1. \tag{5.58}$$

The result leads to $q(t,x,y) = 1$; substituting it into the commutator table, which coincides with the commutator table of Lie algebra L_1, one thus derives an optimal system of two-dimensional subalgebras of the Lie algebra L_5:

$$M_1 = \{X_2, X_3\},\ M_2 = \{X_2, X_5 + \alpha X_3\},$$
$$M_3 = \{X_2, X_1 + \gamma X_3 + \beta X_5\},\ M_4 = \{X_5, X_4 + \alpha X_1\},$$
$$M_5 = \{X_5 + \alpha X_2, X_1 + \beta X_2 + \gamma X_3\},\ M_6 = \{X_4 + \alpha X_5, X_1 + \beta X_5\}$$

with arbitrary constants α, β and γ.

By the obtained optimal system, all invariant solutions and reduced equations are presented in Table 5.6, where $w_1 = y - \gamma t$, $w = x^2 + y^2$, $\psi(t)$ is a function satisfying $\psi(t) > 0$.

มหาวิทยาลัยเทคโนโลยีสุรนารี

Table 5.6 Invariant solutions for $g(u, \bar{u}) = H(\bar{u} - u)$.

No.	Alg.	Repr. of inv. solutions	$Repr. of \tau$	Reduced equation
1	M_1	$u = \phi(t)$	$\psi(t)$	$\phi''(t) = H(\phi(t - \psi(t)) - \phi(t))$
2	M_2	$u = \phi(t) + \frac{1}{\alpha} y$	$\psi(t)$	$\phi''(t) = H(\phi(t - \psi(t)) - \phi(t))$
3	M_3	$u = \beta t + \phi(w_1)$	$cons.$	$(1 - \gamma^2)\phi''(w_1) + H(\phi(w_1 + \gamma\tau) - \phi(w_1) - \beta\tau) = 0$
4	M_4	no		
5	M_5	$u = \frac{1}{\alpha}(x - \beta t) + \phi(w_1)$	$cons.$	$(1 - \gamma^2)\phi''(w_1) + H(\phi(w_1 + \gamma\tau) - \phi(w_1) - \frac{\beta}{\alpha}\tau) = 0$
6	M_6	$u = \phi(w) + \alpha \arcsin \frac{x}{\sqrt{w}} + \beta t$	$cons.$	$4w\phi''(w) + 4\phi'(w) + H(-\beta\tau) = 0$

มหาวิทยาลัยเทคโนโลยีสุรนารี

5.4.6 Invariant Solutions of (5.1) with $g(u, \bar{u}) = k_2 e^{\bar{u}+k_0 u}$

Consider the Lie algebra $L_6 = \{X_1, X_2, X_3, X_4, X_5\}$, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = t\partial_t + x\partial_x + y\partial_y + \alpha_2\partial_u$. Because the commutator table coincides with the commutator table of Lie algebra L_2, thus, one has an optimal system of two-dimensional subalgebras consisting

$$M_1 = \{X_2, X_3\},\ M_2 = \{X_1, X_4\},\ M_3 = \{X_2, X_1 + \alpha X_3\},$$
$$M_4 = \{X_5, X_4\},\ M_5 = \{X_5, X_2 + \alpha X_1\},\ M_6 = \{X_1, X_5 + \alpha X_2 + \beta X_4\}$$

with arbitrary constants α and β. The representations of all invariant solutions and reduced equations are given in Table 5.7, where $w = x^2 + y^2$, $w_1 = y - \alpha t$, $w_2 = tw^{-1/2}$, $w_3 = t/y$, $\psi(t)$ is a function satisfying $\psi(t) > 0$, and a is an arbitrary constant with $a > 0$.

Table 5.7 Invariant solutions for $g(u,\bar{u}) = k_2 e^{\bar{u}+k_0 u}$, $k_0 = -1 - \frac{2}{\alpha_2}$.

No.	Alg.	Repr. of inv. solutions	*Repr.of* τ	Reduced equation
1	M_1	$u = \phi(t)$	$\psi(t)$	$\phi''(t) = k_2 e^{\phi(t-\psi(t))+k_0\phi(t)}$
2	M_2	*no*		
3	M_3	$u = \phi(w_1)$	*cons.*	$(1-\alpha^2)\phi''(w_1) + k_2 e^{\phi(w_1+\alpha\tau)+k_0\phi(w_1)} = 0$
4	M_4	$u = \frac{\alpha_2}{2}\ln w + \phi(w_2)$	at	$(w_2^2-1)\phi''(w_2) + 2w_2\phi'(w_2) + k_2 e^{\phi((1-a)w_2)+k_0\phi(w_2)} = 0$
5	M_5	$u = \alpha_2 \ln y + \phi(w_3)$	at	$(w_3^2-1)\phi''(w_3) + 2w_3\phi'(w_3) + k_2 e^{\phi((1-a)w_3)+k_0\phi(w_3)} - \alpha_2 = 0$
6	M_6	*no*		

5.4.7 Invariant Solutions of (5.1) with $g(u,\bar{u}) = k_2(\bar{u} + k_0 u + \frac{\alpha_0}{\alpha_1})^{1+\frac{1}{\alpha_1}}$

Consider the Lie algebra $L_7 = \{X_1, X_2, X_3, X_4, X_5\}$, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = t\partial_t + x\partial_x + y\partial_y + (\alpha_2 - 2\alpha_1 u)\partial_u$. Since the commutator table coincides with the commutator table of Lie algebra L_2, thus, an optimal system of two-dimensional subalgebras consists of

$$M_1 = \{X_2, X_3\},\ M_2 = \{X_1, X_4\},\ M_3 = \{X_2, X_1 + \alpha X_3\},$$
$$M_4 = \{X_5, X_4\},\ M_5 = \{X_5, X_2 + \alpha X_1\},\ M_6 = \{X_1, X_5 + \alpha X_2 + \beta X_4\}$$

with arbitrary constants α and β. By the obtained optimal systems, the representations of all invariant solutions and reduced equations are presented in Table 5.8, where $w = x^2 + y^2$, $w_1 = y - \alpha t$, $w_2 = tw^{-1/2}$, $w_3 = t/y$, $\psi(t)$ is a function satisfying $\psi(t) > 0$, and a is an arbitrary constant with $a > 0$.

Table 5.8 Invariant solutions for $g(u,\bar{u}) = k_2(\bar{u} + k_0 u + \frac{\alpha_0}{\alpha_1})^k$, $k = 1 + \frac{1}{\alpha_1}$.

No.	Alg.	Inv. solutions	$Repr.of\tau$	Reduced equation
1	M_1	$u = \phi(t)$	$\psi(t)$	$\phi''(t) = k_2(\phi(t-\psi(t)) + k_0\phi(t) + \frac{\alpha_0}{\alpha_1})^k$
2	M_2	*no*		
3	M_3	$u = \phi(w_1)$	*cons.*	$(1-\alpha^2)\phi''(w_1) + k_2(\phi(w_1+\alpha\tau) + k_0\phi(w_1) + \frac{\alpha_0}{\alpha_1})^k = 0$
4	M_4	$u = w^{-\alpha_1}\phi(w_2) + \frac{\alpha_2}{2\alpha_1}$	at	$(w_2^2-1)\phi''(w_2) + (4\alpha_1+2)w_2\phi'(w_2)$
				$+4\alpha_1^2\phi(w_2) + k_2(\phi((1-a)w_2) + k_0\phi(w_2))^k = 0$
5	M_5	$u = y^{2\alpha_1}\phi(w_3)$	at	$(w_3^2-1)\phi''(w_3) + (4\alpha_1+2)w_3\phi'(w_3)$
				$+2\alpha_1(2\alpha_1+1)\phi(w_3) + k_2(\phi((1-a)w_3) + k_0\phi(w_3))^k = 0$
6	M_6	*no*		

5.4.8 Invariant Solutions of (5.1) with $g(u,\bar{u}) = k_1\bar{u} + k_2 u$

Consider the Lie algebra $L_8 = \{X_1, X_2, X_3, X_4, X_5, X_6\}$, where $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = u\partial_u$ and $X_6 = q(t,x,y)\partial_u$. The commutation relations are

$[,]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	0	$q_t\partial_u$
X_2	0	0	0	$-X_3$	0	$q_x\partial_u$
X_3	0	0	0	X_2	0	$q_y\partial_u$
X_4	0	X_3	$-X_2$	0	0	$Z\partial_u$
X_5	0	0	0	0	0	$-X_6$
X_6	$-q_t\partial_u$	$-q_x\partial_u$	$-q_y\partial_u$	$-Z\partial_u$	X_6	0

where $Z = yq_x - xq_y$.

Because L_8 is a Lie algebra, by definition of Lie algebra, one derives

$$\begin{gathered} q_t(t,x,y) = \alpha_1 q(t,x,y),\ q_x(t,x,y) = \alpha_2 q(t,x,y), \\ q_y(t,x,y) = \alpha_3 q(t,x,y),\ yq_x(t,x,y) - xq_y(t,x,y) = \alpha_4 q(t,x,y). \end{gathered} \tag{5.59}$$

The general solution of equations (5.59) is $q(t,x,y) = ce^{\alpha_1 t}$, where c is constant. Since $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ is a basis of algebra L_8, one can choose $X_6 = e^{\alpha_1 t}\partial_u$, i.e, $q(t,x,y) = e^{\alpha_1 t}$. Since $q(t,x,y)$ satisfies equation (5.41), one derives that

$$k_2 = -k_1 e^{-\alpha_1 \tau} + \alpha_1^2. \tag{5.60}$$

Notice that if $\alpha_1 = 0$, then $k_2 = -k_1$; if $\alpha_1 \neq 0$, then one derives that τ is constant.

By the commutator table, thus the automorphisms $A_i\,(i = 1,2,3,4,5,6)$ are as follows:

$$A_1: \quad \hat{x_6} = x_6 e^{-a_1\alpha_1};$$

$$A_2: \quad \hat{x_3} = x_3 + a_2 x_4;$$

$$A_3: \quad \hat{x_2} = x_2 - a_3 x_4;$$

$$A_4: \quad \hat{x_2} = x_2\cos(a_4) + x_3\sin(a_4),\ \hat{x_3} = -x_2\sin(a_4) + x_3\cos(a_4);$$

$$A_5: \quad \hat{x_6} = x_6 e^{a_5};$$

$$A_6: \quad \hat{x_6} = x_6 + a_6(\alpha_1 x_1 - x_5).$$

The Lie algebra L_8 decomposes as $I^3 \oplus L^3$, where $L^3 = \{X_1,\ X_4,\ X_4\}$ is a subalgebra and $I^3 = \{X_2, X_3, X_6\}$ is an ideal of the Lie algebra L_8, respectively. Since the subalgebra L^3 is Abelian, an optimal system of one-dimensional admitted subalgebras consists of

$$H_1 = \{X_5\},\ H_2 = \{X_4 + \alpha X_5\},\ H_3 = \{X_1 + \alpha X_4 + \beta X_5\}$$

and an optimal system of two-dimensional admitted subalgebras consists of

$$D_1 = \{X_4, X_5\},\ D_2 = \{X_1 + \alpha X_4, X_5\},\ D_3 = \{X_1 + \beta X_5, X_4 + \alpha X_5\}$$

with arbitrary constants α and β.

According to the two-step algorithm (Ovsiannikov, 1993) for classifying the Lie algebra L_8, it is sufficient to consider the following forms of two-dimensional subalgebras:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 & a_{16} \\ 0 & a_{22} & a_{23} & 0 & 0 & a_{26} \end{pmatrix},$$
$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 1 & a_{16} \\ 0 & a_{22} & a_{23} & 0 & 0 & a_{26} \end{pmatrix}, \quad \begin{pmatrix} 0 & a_{12} & a_{13} & 1 & \alpha & a_{16} \\ 0 & a_{22} & a_{23} & 0 & 0 & a_{26} \end{pmatrix},$$
$$\begin{pmatrix} 1 & a_{12} & a_{13} & \alpha & \beta & a_{16} \\ 0 & a_{22} & a_{23} & 0 & 0 & a_{26} \end{pmatrix}, \quad \begin{pmatrix} 0 & a_{12} & a_{13} & 1 & 0 & a_{16} \\ 0 & a_{22} & a_{23} & 0 & 1 & a_{26} \end{pmatrix}, \tag{5.61}$$
$$\begin{pmatrix} 1 & a_{12} & a_{13} & \alpha & 0 & a_{16} \\ 0 & a_{22} & a_{23} & 0 & 1 & a_{26} \end{pmatrix}, \quad \begin{pmatrix} 1 & a_{12} & a_{13} & 0 & \beta & a_{16} \\ 0 & a_{22} & a_{23} & 1 & \alpha & a_{26} \end{pmatrix},$$

where $a_{ij}, (i = 1, 2; j = 2, 3, 6)$ and α are arbitrary constants, the matrices denote two-dimensional subalgebras of the Lie algebra L_8.

For further study one needs to simplify (5.61) by using automorphisms $A_i\,(i = 1, 2, 3, 4, 5, 6)$ and matrix transformations, the results are summarized as follows.

Theorem 5.4. *An optimal system of two-dimensional subalgebras of the Lie algebra L_8 is defined by the subalgebras*

$$M_1 = \{X_2, X_3\},\ M_2 = \{X_2, X_5 + \alpha X_3\},$$
$$M_3 = \{X_2, X_1 + \gamma X_3 + \beta X_5\},\ M_4 = \{X_2, X_6 + \alpha X_3\},$$
$$M_5 = \{X_4, X_1 + \alpha_1 X_5 + \epsilon X_6\},\ M_6 = \{X_5, X_4 + \alpha X_1\},$$
$$M_7 = \{X_6, X_4 + \alpha X_5\},\ M_8 = \{X_6, X_4 + \alpha X_1 + \beta X_5\}_{|\alpha\neq 0},$$
$$M_9 = \{X_1 + \alpha X_2, X_5 + \beta X_2 + \gamma X_3\},\ M_{10} = \{X_3 + \epsilon X_6, X_1 + \alpha X_2 + \alpha_1 X_5\},$$
$$M_{11} = \{X_2, X_1 + \alpha X_3 + \alpha_1 X_5 + \epsilon X_6\},\ M_{12} = \{X_4 + \beta X_5, X_1 + \alpha X_5\},$$
$$M_{13} = \{X_2 + \epsilon X_6, X_1 + \beta X_2 + \alpha X_3 + \alpha_1 X_5\},\ M_{14} = \{X_6, X_1 + \alpha X_3 + \alpha_1 X_5\}$$

with arbitrary constants α, β, and γ, $\epsilon = \pm 1$, the symbol $|$ means conditions.

Proof : Consider the case

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \alpha & \beta & a_{16} \\ 0 & a_{22} & a_{23} & 0 & 0 & a_{26} \end{pmatrix}, \tag{5.62}$$

which denotes subalgebra $\{Y_1, Y_2\}$, where $Y_1 = X_1 + a_{12}X_2 + a_{13}X_3 + \alpha X_4 + \beta X_5 + a_{16}X_6$, $Y_2 = 0 \cdot X_1 + a_{22}X_2 + a_{23}X_3 + 0 \cdot X_4 + 0 \cdot X_5 + a_{26}X_6$.

Case 1**:** $\alpha = 0$**.** Checking subalgebra conditions, one has

$$\begin{aligned} &[X_1 + a_{12}X_2 + a_{13}X_3 + \beta X_5 + a_{16}X_6, a_{22}X_2 + a_{23}X_3 + a_{26}X_6] \\ &= a(X_1 + a_{12}X_2 + a_{13}X_3 + \beta X_5 + a_{16}X_6) + b(a_{22}X_2 + a_{23}X_3 + a_{26}X_6), \end{aligned}$$

for some constants a and b. By calculating the left hand side and comparing the coefficients in the left hand with coefficients in the right hand side, one gets

$$a_{26}(\alpha_1 - \beta)X_6 = aX_1 + (aa_{12} + ba_{22})X_2 + (aa_{13} + ba_{23})X_3 + a\eta X_5 + (aa_{16} + ba_{26})X_6,$$

one derives

$$a = 0,\ ba_{22} = 0,\ ba_{23} = 0,\ (\alpha_1 - \beta - b)a_{26} = 0.$$

Case 1.1**:** $b \neq 0$**.** This assumption implies that $a_{22} = a_{23} = 0$. Because rank of matrix (5.62) is equal to 2, then $a_{26} \neq 0$, one can choose $a_{26} = 1$. Using a matrix transformation, a_{16} can be changed to 0. Using the automorphism A_4, a_{13} can be changed to 0: one gets the two-dimensional subalgebra M_{14} in theorem.

Case 1.2**:** $b = 0$**.** This assumption implies that $(\alpha_1 - \beta)a_{26} = 0$.

Case 1.2.1**:** $a_{26} = 0$**.** The rank of matrix (5.62) is equal to 2, which implies that $a_{22}^2 + a_{23}^2 \neq 0$. Using automorphisms A_4, a_{23} can be changed to 0, and one can choose $a_{22} = 1$. Using a matrix transformation, a_{12} can be changed to 0: one gets the two-dimensional subalgebra $\{X_2, X_1 + a_{13}X_3 + \beta X_5 + a_{16}X_6\}$. If $a_{16} = 0$, then one has subalgebra M_3 in the theorem. If $a_{16} \neq 0$, then, using the automorphism A_5, a_{16} can be changed to ϵ: one obtains the subalgebra M_{11} in the theorem.

Case 1.2.2**:** $a_{26} \neq 0$**.** This assumption implies that $\beta = \alpha_1$ and one can choose $a_{26} = 1$. Using a matrix transformation, a_{16} can be changed to 0. Using the automorphism A_4, a_{23} can be changed to 0: one gets the two-dimensional subalgebra $\{X_6 + a_{22}X_2, X_1 + a_{12}X_2 + a_{13}X_3 + \alpha_1 X_5\}$. If $a_{22} = 0$, then, using the automorphism A_4, a_{13} can be changed to 0: one has the subalgebra M_{14} in the theorem. If $a_{12} \neq 0$, then, dividing Y_1 by a_{12}, and using the automorphism A_5, one gets the subalgebra M_{13} in the theorem.

Case 2**:** $\alpha \neq 0$**.** By successfully using the automorphisms A_2 and A_3, thus a_{12} and a_{13} can be changed to 0. Checking subalgebra conditions, one has

$$[X_1 + \alpha X_4 + \beta X_5 + a_{16}X_6, a_{22}X_2 + a_{23}X_3 + a_{26}X_6]$$
$$= a(X_1 + \alpha X_4 + \beta X_5 + a_{16}X_6) + b(a_{22}X_2 + a_{23}X_3 + a_{26}X_6),$$

for some constants a and b. By calculating the left hand side and comparing the coefficients in the left hand with coefficients in the right hand side, one gets

$$a_{26}(\alpha_1 - \beta)X_6 + \alpha a_{22}X_3 - \alpha a_{23}X_2$$
$$= aX_1 + ba_{22}X_2 + ba_{23}X_3 + a\alpha X_4 + a\eta X_5 + (aa_{16} + ba_{26})X_6,$$

which implies that

$$a = 0,\ ba_{22} = -\alpha a_{23},\ ba_{23} = \alpha a_{22},\ (\alpha_1 - \beta - b)a_{26} = 0,$$

one derives $a_{22} = a_{23} = 0$. Therefore, one has

$$\begin{pmatrix} 1 & 0 & 0 & \alpha & \beta & a_{16} \\ 0 & 0 & 0 & 0 & 0 & a_{26} \end{pmatrix}. \tag{5.63}$$

Since rank of matrix (5.63) is equal to 2, which implies that $a_{26} \neq 0$, one can choose $a_{26} = 1$. Using a matrix transformation, a_{16} can be changed to 0. Since $\alpha \neq 0$, dividing Y_1 by α, one thus gets the two-dimensional subalgebra M_8 in the theorem.

Other cases can be similarly discussed and computed.

Thus, the proof is completed.

By the obtained optimal system, representations of all invariant solutions and reduced equations are given in Table 5.9, where $w = x^2+y^2$, $w_1 = x-\alpha t$, $w_2 = y-\alpha t$, $w_3 = \gamma x - \alpha\gamma t - \beta y$, and $\psi(t)$ is a function satisfying $\psi(t) > 0$.

มหาวิทยาลัยเทคโนโลยีสุรนารี

Table 5.9 Invariant solutions for $g(u, \bar{u}) = k_1\bar{u} + k_2u$, $k_1 \neq 0$.

No.	Alg.	Inv. solutions	$Repr.of\tau$	Reduced equation
1	M_1	$u = \phi(t)$	$\psi(t)$	$\phi''(t) = k_1\phi(t - \psi(t)) + k_2\phi(t)$
2	M_2	$u = \phi(t)e^{\frac{1}{\alpha}y}$	$\psi(t)$	$\phi''(t) = \phi(t)(\frac{1}{\alpha^2} + k_2) + k_1\phi(t - \psi(t))$
3	M_3	$u = e^{\beta t}\phi(w_1)$	$cons.$	$(1 - \gamma^2)\phi''(w_1) = 2\gamma\beta\phi'(w_1)$
				$+(k_2 - \beta^2)\phi(w_1) + k_1\phi(w_1 + \gamma\tau)$
4	M_4	$u = \frac{y}{\alpha}e^{\alpha_1 t} + \phi(t)$	$\psi(t)$	$\phi''(t) = k_1\phi(t - \psi(t)) + k_2\phi(t)$
5	M_5	$u = (\epsilon t + \phi(w))e^{\alpha_1 t}$	$cons.$	$4w\phi''(w) + 4\phi'(w) - \epsilon k_1\tau e^{-\alpha_1\tau} - 2\epsilon\alpha_1 = 0$
6	M_6	no		
7	M_7	no		
8	M_8	no		
9	M_9	$u = e^{\frac{1}{\gamma}y}\phi(w_3)$	$cons.$	$(\gamma^2 + \beta^2 - (\alpha\gamma)^2)\phi''(w_3) - 2\frac{\beta}{\gamma}\phi'(w_3)$
				$+k_1\phi(w_3 + \alpha\gamma\tau) + (k_2 + \gamma^{-2})\phi(w_3) = 0$
10	M_{10}	$u = (\phi(w_1) + \epsilon y)e^{\alpha_1 t}$	$cons.$	$(1 - \alpha^2)\phi''(w_1) + 2\alpha\alpha_1\phi'(w_1) + (k_2 - \alpha_1^2)\phi(w_1)$
				$+k_1e^{-\alpha_1\tau}\phi(w_1 + \alpha\tau) = 0$
11	M_{11}	$u = (\phi(w_2) + \epsilon t)e^{\alpha_1 t}$	$cons.$	$(1 - \alpha^2)\phi''(w_2) + 2\alpha\alpha_1\phi'(w_2) - 2\epsilon\alpha_1$
				$+(k_2 - \alpha_1^2)\phi(w_2) + k_1e^{-\alpha_1\tau}(\phi(w_2 + \alpha\tau) - \epsilon\tau) = 0$

Table 5.9 (Continued) Invariant solutions for $g(u,\bar{u}) = k_1\bar{u} + k_2u,\ k_1 \neq 0$.

No.	Alg.	Inv. solutions	$Repr.of\tau$	Reduced equation
12	M_{12}	$u = \phi(w)e^{(\alpha \arcsin \frac{x}{\sqrt{w}} + \beta t)}$	*cons.*	$4w\phi''(w) + 4\phi'(w) + (\alpha^2 w + \alpha_1^2 - \beta^2)\phi(w) = 0$
13	M_{13}	$u = (\phi(w_2) + \epsilon x - \epsilon\beta t)e^{\alpha_1 t}$	*cons.*	$(1-\alpha^2)\phi''(w_2) + 2\alpha\alpha_1\phi'(w_2) + 2\epsilon\beta\alpha_1$
				$+k_1 e^{-\alpha_1\tau}(\phi(w_2+\alpha\tau) - \phi(w_2) + \beta\epsilon\tau) = 0$
14	M_{14}	*no*		

CHAPTER VI

CONCLUSIONS

The goal of this thesis is application of the group analysis method to the one-dimensional nonlinear Klein-Gordon equation with constant delay

$$u_{tt}(x,t) = u_{xx}(x,t) + g(u(x,t), u(x,t-\tau)), \quad g_{\bar{u}} \neq 0, \tag{6.1}$$

and the two-dimensional nonlinear Klein-Gordon equation with time-varying delay

$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) + g(u(x,y,t), u(x,y,t-\tau(t))), \; g_{\bar{u}} \neq 0. \tag{6.2}$$

Using the approach developed in (Tanthanuch and Meleshko, 2002), the complete group classification of equation (6.1) is obtained. Results of the group classification are presented in Table 4.1. Representations of all invariant solutions are given in Tables 4.2-4.10.

In addition, group analysis of a differential equation with time-varying delay is developed. This analysis is applied to equation (6.2). The complete group classification of this equation with respect to the arbitrary function g is obtained (Table 5.1). All admitted Lie algebras are classified. These classifications are used for deriving invariant solutions. Representations of all invariant solutions are presented in Tables 5.2-5.9.

REFERENCES

REFERENCES

Andreev, V. K., Kaptsov, O. V., Pukhnachov, V. V., and Rodionov, A. A. (1998). **Applications of Group-Theoretical Methods in Hydrodynamics**. Dordrecht: Kluwer.

Bellman, R. and Cooke, K. L. (1963). **Differential-Difference Equations**. New York: Academic Press.

Bullough, R. K. and Caudrey, P. J. (1980). **Solitons**. New York: Springer-Verlag.

Dodd, R. K., Eilbeck, I. C., Gibbon, J. D., and Morris, H. C. (1982). **Solitons and Nonlinear Wave Equations**. London: Academic Press.

Drazin, P. J. and Johnson, R. S. (1989). **Solitons: An Introduction**. Cambridge: Cambridge University Press.

Driver, R. D. (1977). **Ordinary and Delay Differential Equations**. New York: Springer-Verlag.

Farnam, A. and Esfanjani, R. M. (2014). Improved stabilization method for networked control systems with variable transmission delays and packet dropout. **ISA Trans** 53: 1746–1753.

Galaktionov, V. A. and Svirshchevskii, S. R. (2007). **Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics**. London: Chapman & Hall/CRC.

Grigoriev, Y. N., Ibragimov, N. H., Kovalev, V. F., and Meleshko, S. V.

มหาวิทยาลัยเทคโนโลยีสุรนารี

(2010). **Symmetries of Integro-Differential Equations**. New York: Springer.

Grundland, A. M. and Infeld, E. (1992). A family of non-linear Klein-Gordon equations and their solutions. **Math. Phys.** 33(2): 498–503.

Hale, J. (1977). **Functional Differential Equations**. New York: Springer-Verlag.

Ibragimov, N. H. (1996). **Handbook of Lie Group Analysis of Differential Equations**. Boca Raton: CRC Press.

Ibragimov, N. H. (1999). **Elementary Lie Group Analysis and Ordinary Differential Equations**. New York: John Wiley and Sons.

Ibragimov, N. H., Meleshko, S. V., and Rudenko, O. V. (2011). Group analysis of evolutionary integro-differential equations describing nonlinear waves: General model. **J. of Phys. A: Math. and Theor.** 44(31): 1751–8121.

Kolmanovskii, V. and Myshkis, A. (1992). **Applied Theory of Functional Differential Equations**. Dordrecht: Kluwer Academic Publishers.

Kolmanovskii, V. B. (1996). Equations with aftereffect and mathematical modeling. **Soros Educational J.** (4): 122–127.

Kwon, O., Lee, S., and Park, J. (2011). Linear matrix inequality approach to new delay-dependent stability criteria for uncertain dynamic systems with timevarying delays. **J. Optim. Theory Appl.** 149: 630–646.

Li, T., Song, A., Xue, M., and Zhang, H. (2011). Stability analysis on delayed neural networks based on an improved delay-partitioning approach. **J. Comput. Appl. Math.** 235: 3086–3095.

Lie S. (1883). Klassification und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten. **Arch. Math. Naturv.** 9: 371–393.

Louisell, J. (2001). Delay differential systems with time-varying delay: new directions for stability theory. **Kybernetika** 37:239–251.

Meleshko, S. V. (1996). Generalization of the equivalence transformations. **J. Non-Linear Math. Phys.** 3(1-2): 170–174.

Meleshko, S. V. (2005). **Methods for Constructing Exact Solutions of Partial Differential Equations**. New York: Springer.

Meleshko, S. V. and Moyo, S. (2008). On the complete group classification of the reaction-diffusion equation with a delay. **J. Math. Anal. Appl.** 338: 448–466.

Myshkis, A. D. (1972). **Linear Differential Equations With Retarded Argument**. Moscow: Nauka.

Niculescu, S. L., de. Souza, C. E., Dugard, L., and Dion, J. M. (1998). Robust exponential stability of uncertain systems with time-varying delays. **IEEE Trans. Automat. Control** 43: 743–748.

Novikov, S. (1984). **Theory of Solitons: The Inverse Scattering Method**. New York: Springer.

Olver, P. J. (1986). **Applications of Lie Groups to Differential Equations**. New York: Springer-Verlag.

Ovsiannikov, L. V. (1978). **Group Analysis of Differential Equations**. New York: Academic Press.

Ovsiannikov, L. V. (1993). On optimal system of subalgebras. **Docl. RAS** 333(6): 702–704.

Park, J. W. and Ko, C. J. (1994). Reciprocally convex approach to stability of systems with time-varying delays. **Automatica** 47: 235–238.

Patera, J. and Winternitz, P. (1977). Subalgebras of real three and four dimensional Lie algebras. **J. of Math. Phys.** 18: 1449–1455.

Peng, C. and Tian, Y. (2008). Delay-dependent robust stability criteria for uncertain systems with interval time-varying delay. **J. Comput. Appl. Math.** 214: 480–494.

Phat, V., Khongtham, Y., and Ratchagit, K. (2012). LMI approach to exponential stability of linear systems with interval time-varying delays. **Linear Alg. Appl.** 436: 243–251.

Polyanin, A. D. and Zaitsev, V. F. (2012). **Handbook of Nonlinear Partial Differential Equations**. Boca Raton: Hall/CRC Press.

Polyanin, A. D. and Zhurov, A. I. (2013). Exact solutions of linear and nonlinear differential-difference heat and diffusion equations with finite relaxation time. **Int. J. Non-Linear Mech.** 54: 115–126.

Polyanin, A. D. and Zhurov, A. I. (2014a). Exact separable solutions of delay reaction-diffusion equations and other nonlinear partial functional-differential equations. **Commun. Nonlinear Sci. Numer. Simul.** 19: 409–416.

Polyanin, A. D. and Zhurov, A. I. (2014b). The functional constraints method: application to non-linear delay reaction-diffusion equations with varying transfer coefficients. **Int. J. Non-Linear Mech.** 67: 267–277.

Polyanin, A. D. and Zhurov, A. I. (2014c). Functional constraints method for constructing exact solutions to delay reaction-diffusion equations and more complex nonlinear equations. **Commun. Nonlinear Sci. Numer. Simul.** 19: 417–430.

Polyanin, A. D. and Zhurov, A. I. (2014d). Generalized and functional separable solutions to nonlinear delay Klein-Gordon equations. **Commun. Nonlinear Sci. Numer. Simulat.** 19: 2676–2689.

Polyanin, A. D. and Zhurov, A. I. (2014e). New generalized and functional separable solutions to non-linear delay reaction-diffusion equations. **Int. J. Non-Linear Mech.** 59: 16–22.

Polyanin, A. D. and Zhurov, A. I. (2014f). Non-linear instability and exact solutions to some delay reaction-diffusion systems. **Int. J. Non-Linear Mech.** 62: 33–40.

Polyanin, A. D. and Zhurov, A. I. (2014g). Nonlinear delay reaction-diffusion equations with varying transfer coefficients: exact methods and new solutions. **Appl. Math. Lett.** 37: 43–48.

Polyanin, A. D. and Zhurov, A. I. (2015). The generating equations method: constructing exact solutions to delay reaction-diffusion system sand other non-linear coupled delay PDEs. **Int. J. Non-Linear Mech.** 71: 104–115.

Pue-on, P., and Meleshko, S. V. (2010). Group classification of second-order delay ordinary differential equations. **Commun. Nonlinear Sci. Numer. Simul.** 15: 1444–1453.

มหาวิทยาลัยเทคโนโลยีสุรนารี

Richard, J. P. (2003). Time-delay systems: an overview of some recent advances and open problems. **Automatica** 39: 1667–1694.

Rudenko, O. V. (2010). The 40th anniversary of the Khokhlov-Zabolotskaya equation. **Acoustical Phys.** 56(4): 457–466.

Rudenko, O. V. and Soluyan, S. I. (1977). **Theoretical Foundations of Nonlinear Acoustics**. New York: Plenum, Consultants Bureau.

Rudenko, O. V., Soluyan, S. I., and Khokhlov, R. V. (1974). Problems of the theory of nonlinear acoustics. **Sov. Phys. Acoustics** 20(3): 356–359.

Shao, H. (2009). New delay-dependent stability criteria for systems with interval delay. **Automatica** 45: 744–749.

Smith, H. L. (2010). **An Introduction to Delay Differential Equations with Applications to The Life Sciences**. New York: Springer.

Sun, Y. J., Hsieh, J. G., and Yang, H. C. (1997). On the stability of uncertain systems with multiple time-varying delays. **IEEE Trans. Automat. Control** 4: 101–104.

Tanthanuch, J. (2012). Symmetry analysis of the nonhomogeneous inviscid Burgers equation with delay. **Commun. Nonlinear Sci. Numer. Simul.** 17: 4978–4987.

Tanthanuch, J. and Meleshko, S. V. (2002). Application of group analysis to delay differential equations. **Nonlinear Acoustics at the Beginning of the 21st Century** 1: 607–610.

Tanthanuch, J. and Meleshko, S. V. (2004). On definition of an admitted Lie

มหาวิทยาลัยเทคโนโลยีสุรนารี

group for functional differential equations. **Commun. Nonlinear Sci. Numer. Simul.** 9(1): 117–125.

Verriest, E. I. (1994). Robust stability of time varying systems with unknown bounded delays. **Lake Buena Vista** 33: 417–422.

Wu, J. (1996). **Theory and Applications of Partial Functional Differential Equations**. New York: Springer-Verlag.

Zabolotskaya, E. A. and Khokhlov, R. V. (1969). Quasi-plane waves in the nonlinear acoustics of confined beams. **Sov. Phys. Acoustics** 15(1): 35–40.

Zhang, W., Cai, X., and Han, Z. (2010). Robust stability criteria for systems with interval time-varying delay and nonlinear perturbations. **J. Comput. Appl. Math.** 234: 174–180.

Zhdanov, R. Z. (1994). Separation of variables in the non-linear wave equation. **J. of Phys. A: Math. and General** 27(9): 91–97.

มหาวิทยาลัยเทคโนโลยีสุรนารี

CURRICULUM VITAE

NAME: Fengshan Long
GENDER: male
DATE OF BIRTH: August 10, 1979
NATIONALITY: Chinese
PLACE OF BIRTH: Guizhou, China
E-MAIL: lfsh88888@126.com

AFFILIATION AND ADDRESS:

- School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang, 550025, China

EDUCATION BACKGROUND:

- M. Sc. School of Mathematics and Computer Science, Guizhou Normal University, China, 2005.
- B. Sc. School of Mathematics and Computer Science, Guizhou Normal University, China, 2002.

มหาวิทยาลัยเทคโนโลยีสุรนารี

CONFERENCE:

- Symmetry analysis of the nonlinear Klein-Gordon equation with delay, **The 41st Congress on Science and Technology of Thailand (STT41)**. November 6-8, 2015, Suranaree University of Technology, Nakhonratchasima, Thailand.

PUBLICATION:

- F. S. Long and S. V. Meleshko. On the Complete Group Classification of the One-Dimensional Nonlinear Klein-Gordon Equation with a Delay. **Mathematical Methods in the Applied Sciences**. DOI: 10.1002/mma.3769.