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Relaxation of nonlinear impulsive controlled systems on Banach spaces[☆]

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Abstract

Relaxation control for a class of semilinear impulsive controlled systems is investigated. Existence of mild solutions for semilinear impulsive controlled systems is proved. By introducing a regular countably additive measure, we convexify the original control systems and obtain the corresponding relaxed control systems. The existence of optimal relaxed controls and relaxation results is also proved.

Keywords: Impulsive systems; Banach spaces; Semilinear evolution equations; Relaxation

1. Introduction

Let $I \equiv [0, T]$ be a closed and bounded interval of the real line. Let $D \equiv \{t_1, t_2, \dots, t_n\}$ be a partition on $(0, T)$ such that $0 < t_1 < t_2 < \dots < t_n < T$. A semilinear impulsive controlled system can be described by the following evolution equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)) & t \in (0, T) \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), & i = 1, 2, \dots, n, \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in a Banach space X, the functions f, J_i, $i = 1, 2, \dots$, are continuous nonlinear operators from X to X, and $\Delta x(t_i) \equiv x(t_i + 0) - x(t_i - 0) = x(t_i + 0) - x(t_i)$. This system contains the jump in the state x at time t_i with J_i determining the size of the jump at t_i. In this paper, we aim to prove the existence of state–control pairs of the system (1.1). Moreover, by defining the objective functional $J(x, u) = \int_0^T L(t, x(t), u(t))\mathrm{d}t$, we shall find sufficient conditions to guarantee the existence of optimal state–control pairs when convexity conditions on a certain orientor field are not assumed. This is the relaxation

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problem. By introducing regular countable additive measures, we convexify the original control systems and obtain the corresponding relaxed control systems. Under some reasonable assumptions, we prove that the set of original trajectories is dense in the set of relaxed trajectories in an appropriate space. The existence of optimal relaxed controls is obtained under some regularity hypotheses concerning the cost functional.

In recent years, relaxed systems have attracted much attention since some orientor fields do not satisfy the convexity condition. See, for instance, [1,6,7]. Ahmed [1] dealt with this problem and introduced measure-valued controls in which the control space is compact and values of relaxed control are countable additive measures, while Papageorgious [6] and other authors including us continue to discuss this problem in another direction. However, to our knowledge, there are few authors who have studied the problem of relaxed controls of systems governed by impulsive evolution equations, particularly, relaxation on semilinear impulsive evolution equations. We organize the paper as follows. In Section 2, we describe the original control systems and the corresponding relaxed control systems. The properties of relaxed trajectories are given in Section 3. Section 4 is devoted to the existence of relaxed optimal controls and relaxation theorems.

2. Original and relaxed controlled systems

In what follows, let the Banach space $(X, \|\cdot\|_X)$ be the state space, $I \equiv [0, T]$ be a closed and bounded interval of the real line, $C(I, X)$ denote the space of continuous functions, and $C^1(I, X)$ denote the space of first-order continuous differentiable functions. Let $L(X, Y)$ denote the space of bounded linear operators from X to Y and $L(X)$ denote the space of bounded linear operators from X to X.

We denote the ball $\{x \in X : \|x\| \leq r\}$ by B_r. Define $PC(I, X) \equiv \{x : I \to X : x(t)$ is continuous at $t \neq t_i$, left continuous at $t = t_i$, and the right hand limit $x(t_i^+)$ exists$\}$. Equipped with the supremum norm topology, it is a Banach space.

We introduce the following assumptions.

[A]: The operator A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on X.

[F]: $f : I \times X \to X$ is an operator such that

(1) $t \to f(t, \xi)$ is measurable and locally Lipschitz continuous with respect to the last variable, i.e., for any finite number $\rho > 0$, there exists a constant $L_1(\rho) > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\|_X \leq L_1(\rho)\|x_1 - x_2\|_X,$$

$\forall x_1, x_2 \in B_\rho$.

(2) There exists a constant $k > 0$ such that $\|f(t, x)\|_X \leq k(1 + \|x\|_X)$.

[J]: $J_i : X \to X$ is an operator such that

(1) J_i maps a bounded set to a bounded set.

(2) There exist constants $h_i > 0, i = 1, 2, \ldots, n$, such that

$$\|J_i(x) - J_i(y)\| \leq h_i\|x - y\|, \quad x, y \in X.$$

Consider the following impulsive systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) & t \in [0, T] \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), & i = 1, 2, \ldots, n. \end{cases} \tag{2.1}$$

By a mild solution of (2.1), we shall mean that a function $x \in PC(I, X)$ satisfies the following integral equation:

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))\mathrm{d}\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)).$$

Theorem 1. *Suppose the assumptions* [A], [F], *and* [J] *hold; then for every $x_0 \in X$ the system* (2.1) *has a unique mild solution $x \in PC(I, X)$ and the mild solution depends continuously on the initial conditions—that is, if $x_0, y_0 \in X$ and if $x(t), y(t)$ are mild solutions of Eq.* (2.1) *which satisfy $x(0) = x_0$ and $y(0) = y_0$. Then there exists a constant $C > 0$ s.t.*

$$\sup_{t\in[0,T]} \|x(t) - y(t)\| \leq C\|x_0 - y_0\|_X.$$

Proof. Firstly, we consider the following general differential equation without impulse:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) & t > 0, \\ x(0) = x_0. \end{cases} \tag{2.1.1}$$

Define a closed ball $\bar{B}(x_0, 1)$ as follows:

$$\bar{B}(x_0, 1) = \{x \in C([0, T_1], X), \|x(t) - x_0\| \le 1, 0 \le t \le T_1\},$$

where T_1 will be chosen later. Define a map P on $\bar{B}(x_0, 1)$ by

$$(Px)(t) = T(t)x_0 + \int_0^t T(t-\tau) f(\tau, x(\tau))\mathrm{d}\tau$$

and let $M \equiv \sup_{t\in[0,T]} \|T(t)\|$. Using assumption [F], one can verify that P maps $\bar{B}(x_0, 1)$ to $\bar{B}(x_0, 1)$. To prove this, we note that

$$\begin{aligned} \|(Px)(t) - x_0\| &\le \|T(t)x_0 - x_0\| + \int_0^t \|T(t-\tau)\| \|f(\tau, x(\tau))\|\mathrm{d}\tau \\ &\le Mk(1+\rho)t + \|T(t)x_0 - x_0\|. \end{aligned}$$

Since $T(t)$ is the strongly continuous C_0-semigroup, there exists $T_{11} > 0$ such that for all $t \in [0, T_{11}]$, $\|T(t)x_0 - x_0\| \le \frac{1}{2}$. Now, let $0 < T_{22} < \frac{1}{2Mk(1+\rho)}$. Set $T_1' = \min\{T_{11}, T_{22}\}$; hence for all $t \in [0, T_1']$ we have $\|(Px)(t) - x_0\| \le 1$. Hence $P : \bar{B}(x_0, 1) \to \bar{B}(x_0, 1)$.

Let $x_1, x_2 \in \bar{B}(x_0, 1)$. By assumption [F](1), we have

$$\begin{aligned} \|(Px_1)(t) - (Px_2)(t)\| &\le \int_0^t \|T(t-\tau)\| \|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\|\mathrm{d}\tau \\ &\le MtL_1(\rho)\|x_1 - x_2\|. \end{aligned}$$

Now, let $0 < T_1'' = \frac{1}{2ML_1(\rho)}$; then $\|(Px_1)(t) - (Px_2)(t)\| \le \frac{1}{2}\|x_1 - x_2\|$. Hence, we shall choose $T_1 = \min\{T_1', T_1''\}$ to guarantee that P is a contraction map on $\bar{B}(x_0, 1)$. This implies that (2.1.1) has a unique mild solution on $[0, T_1]$. Again, using the assumption [F], we can obtain the a priori estimate of mild solutions of Eq. (2.1.1). To see this, we note that

$$\begin{aligned} \|x(t)\| &\le \|T(t)x_0\| + \int_0^t \|T(t-\tau)\| \|f(\tau, x(\tau))\|\mathrm{d}\tau \\ &\le M\|x_0\| + MkT + Mk\int_0^t \|x(\tau)\|\mathrm{d}\tau. \end{aligned}$$

By the Gronwall inequality, we obtain

$$\begin{aligned} \|x(t)\| &\le (M\|x_0\| + MkT)\,\mathrm{e}^{Mk\int_0^t \mathrm{d}\tau} \\ &\le (M\|x_0\| + MkT)\,\mathrm{e}^{MkT} \equiv \overline{M}. \end{aligned}$$

That is, there exists a constant $\overline{M} = (M\|x_0\| + MkT)\,\mathrm{e}^{MkT} > 0$ such that for $t \in [0, T]$ we have $\|x(t)\| \le \overline{M}$. Then we can prove the global existence of the mild solution of system (2.1.1) on $[0, T]$.

Now, we are ready to construct a mild solution for the impulsive system (2.1). For $t \in [0, t_1)$, the above result implies that $x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))\mathrm{d}\tau$ is the unique mild solution of the system (2.1) on $[0, t_1]$. Clearly the solution is continuous on $[0, t_1)$ and since $T(t)$ is a continuous semigroup, then $x(t)$ can be extended continuously until the point of time t_1 which is denoted by $x(t_1)$. It is easy to see that $x(t_1) \in X$. Since J_1 maps bounded sets to bounded subsets of X, the jump is uniquely determined by the expression

$$x(t_1 + 0) = x(t_1 - 0) + J_1(x(t_1 - 0)) \equiv x(t_1) + J_1(x(t_1)) \equiv x_1.$$

Consider the time $t \in (t_1, t_2)$. We have

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))\mathrm{d}\tau + T(t-t_1)J_1(x(t_1)).$$

Again, $x \in C((t_1, t_2), X)$ and can be extended continuously until the point of time t_2 which is denoted by $x(t_2) \in X$. By the previous result, $x(\cdot)$ is a mild solution of Eq. (2.1) on $(t_1, t_2]$. Because J_2 maps bounded sets to bounded sets, the jump is uniquely determined by

$$x(t_2 + 0) = x(t_2 - 0) + J_2(x(t_2 - 0)) \equiv x(t_2) + J_2(x(t_2)) \equiv x_2.$$

This procedure can be repeated on $t \in (t_2, t_3], (t_3, t_4], \ldots, (t_n, T]$. Thus we obtain a unique mild solution of problem (2.1) on $[0, T]$ and it is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))\mathrm{d}\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)), \quad 0 \le t \le T.$$

For the proof of continuous dependence on the initial value, one can use the Gronwall inequality to find a constant C such that $\|x(t) - y(t)\| \le C\|x_0 - y_0\|_X$ for all $t \in [0, T]$. The proof is now complete. □

Now, we introduce the admissible controls space U_{ad}.

Let Γ be a compact Polish space (i.e., a separable complete metric space).

We define

$$U_{\mathrm{ad}} = \{u : [0, T] \to \Gamma | u \text{ is strongly measurable}\}.$$

By the measurable selection theorem, $U_{\mathrm{ad}} \neq \phi$ (see [3]). We make the following assumptions for our control systems.

Assumptions

[F1] $f : I \times X \times \Gamma \to X$ is an operator such that

(1) $t \mapsto f(t, \xi, \eta)$ is measurable, and $(\xi, \eta) \mapsto f(t, \xi, \eta)$ is continuous on $X \times \Gamma$.

(2) For any finite number $\rho > 0$, there exists a constant $L(\rho) > 0$ such that

$$\|f(t, x_1, \sigma) - f(t, x_2, \sigma)\|_X \le L(\rho)\|x_1 - x_2\|_X,$$

for all $\|x_1\| < \rho$, $\|x_2\| < \rho$, and $t \in I, \sigma \in \Gamma$.

(3) There exists a constant $k_F > 0$ such that

$$\|f(t, x, \sigma)\|_X \le k_F(1 + \|x\|_X) \quad (t \in I, \sigma \in \Gamma).$$

Consider the following original control system:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad u(\cdot) \in U_{\mathrm{ad}}. \end{cases} \tag{2.2}$$

Theorem 2. *Suppose the assumptions* [A], [J], *and* [F1] *hold. Then for every $x_0 \in X$ and $u \in U_{\mathrm{ad}}$, the system* (2.2) *has a unique mild solution $x \in PC(I, X)$ which satisfies*

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau), u(\tau))\mathrm{d}\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)).$$

Proof. Let $u \in U_{\mathrm{ad}}$ and define $g_u(t, x) = f(t, x, u)$. Since f is measurable, then $g_u : I \times X \to X$ is measurable on $[0, T]$ for each fixed $x \in X$. Hence g_u satisfies the assumption [F]. By Theorem 1, the system (2.2) has a unique mild solution $x \in PC(I, X)$.

In order to introduce the relaxed control system corresponding to (2.2), we need some preparations which are drawn from ([4], p. 618–650). Let Γ be a compact Polish space, and $C(\Gamma)$ consist of all continuous real-valued functions. Endowed with the supremum norm, $C(\Gamma)$ is a Banach space. Let $\Phi(\mathbf{C})$ be a σ-field generated by the collection $\mathbf{C}$ of all closed sets of Γ and let $\Sigma_{\mathrm{rca}}(\Gamma)$ be the space of all regular countably additive measures on the measurable space $(\Gamma, \Phi(\mathbf{C}))$. For $\mu \in \Sigma_{\mathrm{rca}}(\Gamma)$, $|\mu|$ denotes the total variation of μ. □

Lemma 3. *The dual space $C(\Gamma)^*$ can be identified algebraically and metrically with $\Sigma_{\text{rca}}(\Gamma)$ with the norm*

$$\|\mu\|_{\Sigma_{\text{rca}}(\Gamma)} = |\mu|(\Gamma).$$

The duality pairing of $C(\Gamma)$ and $\Sigma_{\text{rca}}(\Gamma)$ is given by

$$\langle f, \mu\rangle = \int_\Gamma f(\sigma)\mu(\mathrm{d}\sigma)$$

for $f \in C(\Gamma)$, $\mu \in \Sigma_{\text{rca}}(\Gamma)$.

Let $L^1(I, C(\Gamma))$ be the space of all (the equivalence class of) strongly measurable $C(\Gamma)$-valued functions $u(\cdot)$ defined on I such that

$$\|u\| = \int_I \|u(t)\|\mathrm{d}t < +\infty.$$

$L^1(I, C(\Gamma))$ is a Banach space. $L^\infty_w(I, C(\Gamma)^*)$ is the space of all $C(\Gamma)^*$-valued $C(\Gamma)$-weakly measurable functions $g(\cdot)$ such that there exists $C > 0$ with

$$|\langle g(t), y\rangle| \le C\|y\|_{C(\Gamma)} \quad \text{a.e. in } 0 \le t \le T, \tag{2.2.1}$$

for each $y \in C(\Gamma)$ (the null set where (2.2.1) fails to hold may depend on y). Two functions $g(\cdot), h(\cdot)$ are said to be equivalent in $L^\infty_w(I, C(\Gamma)^*)$ (in symbols, $g \approx h$) if $\langle g(t), y\rangle = \langle f(t), y\rangle$ a.e. in $0 \le t \le T$ for each $y \in C(\Gamma)$.

Lemma 4. *The dual $L^1(I, C(\Gamma))^*$ is isometrically isomorphic to $L^\infty_w(I, C(\Gamma)^*)$. The duality pairing between the two spaces is given by*

$$\langle\langle g, f\rangle\rangle = \int_0^T \langle g(t), f(t)\rangle \mathrm{d}t,$$

where $g \in L^\infty_w(I, C(\Gamma)^*)$ and $f \in L^1(I, C(\Gamma))$.

Since Γ is a compact metric space, $C(\Gamma)^*$ is a separable Banach space (see [8], p. 265) and hence has the Radon–Nikodym property which tells us that $L^1(I, C(\Gamma))^* = L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$.

Definition 1. The space $R(I, \Gamma)$ of relaxed controls consists of all $\mu(\cdot)$ in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma)) = L^1(I, C(\Gamma))^*$ that satisfy

(i) if $f(\cdot,\cdot) \in L^1(I, C(\Gamma))$ is such that $f(t, \sigma) \ge 0$ for $\sigma \in \Gamma$ a.e. in $0 \le t \le T$ then

$$\int_0^T \int_\Gamma f(t, \sigma)\mu(t, \mathrm{d}\sigma)\mathrm{d}t \ge 0,$$

(ii) if $\chi(t)$ is the characteristic function of a measurable set $e \subseteq [0, T]$, and $\mathbf{1} \in C(\Gamma)$ is the function $\mathbf{1}(\sigma) = 1$, then

$$\int_0^T \int_\Gamma (\chi(t) \otimes \mathbf{1}(\sigma))\mu(t, \mathrm{d}\sigma)\mathrm{d}t = |e|.$$

Note that $\chi(\cdot) \otimes \mathbf{1}(\cdot) \in L^1(I, C(\Gamma))$.

We note that (ii) can be generalized to

$$\int_0^T \int_\Gamma (\phi(t) \otimes \mathbf{1}(\sigma))\mu(t, \mathrm{d}\sigma)\mathrm{d}t = \int_0^T \phi(t)\mathrm{d}t$$

for any $\phi(\cdot) \in L^1(I)$.

In fact, for $\mu(\cdot) \in R(I, \Gamma)$, we have

$$\|\mu\|_{L^\infty(I, \Sigma_{\text{rca}}(\Gamma))} \le 1, \quad \mu(t) \ge 0, \text{ and } \mu(t, \Gamma) = 1 \quad \text{a.e. in } 0 \le t \le T.$$

In particular,

$$\|\mu(t)\|_{\Sigma_{\text{rca}}(\Gamma)} = 1 \quad \text{a.e. in } 0 \le t \le T.$$

Lemma 5. *Let $\{\mu_n(\cdot)\}$ be a sequence in $R(I, \Gamma)$. Then there exists a subsequence which is $L^1(I, C(\Gamma))$-weakly convergent in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$ to $\mu(\cdot) \in R(I, \Gamma)$.*

Sometimes, using another equivalent definition of $R(I, \Gamma)$ is more convenient. We denote by $\Pi_{\text{rca}}(\Gamma)$ the set of all probability measures μ in $\Sigma_{\text{rca}}(\Gamma)$. We denote the Dirac measure with mass at u using the functional notation $\delta(\cdot - u)$ or δ_u. The set $D = \{\delta_u : u \in \Gamma\}$ of all Dirac measures is a subset of $\Pi_{\text{rca}}(\Gamma)$.

Lemma 6. *$\Pi_{\text{rca}}(\Gamma)$ is $C(\Gamma)$-weakly compact, also $C(\Gamma)$-weakly closed in $\Sigma_{\text{rac}}(\Gamma)$.*

Let $\overline{conv}$ denote the closed convex hull (closure taken in the weak $C(\Gamma)$-topology). Then

$$\Pi_{\text{rca}}(\Gamma) = \overline{conv}(D).$$

Since $C(\Gamma)$ is separable, the equivalent relation in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$ is equality almost everywhere. Let us define the set

$$R(I, \Pi_{\text{rca}}(\Gamma)) = \{u \in L^\infty(I, \Sigma_{\text{rca}}), \exists v \text{ s.t. } v \approx u \text{ and } v(t) \in \Pi_{\text{rca}}(\Gamma) \text{ a.e. in } 0 \le t \le T\}.$$

If $u(\cdot) \in U_{\text{ad}}$ then one can check that the Dirac delta with mass at $u(t)$ (written as $\delta(\cdot - u(t))$ is an element of $R(I, \Pi_{\text{rca}}(\Gamma))$. Hence we can identify U_{ad} as a subset of $R(I, \Pi_{\text{rca}}(\Gamma))$. We note further that $R(I, \Pi_{\text{rca}}(\Gamma)) = R(I, \Gamma)$ (see [4], Theorem 12.6.7).

Now, let us consider the new larger system known as the "relaxed impulsive system"*:*

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t))\mu(t), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad \mu(\cdot) \in U_r. \end{cases} \tag{2.3}$$

The admissible control space is $U_r = R(I, \Pi_{\text{rca}}(\Gamma))$. The function $F : I \times X \times \Sigma_{\text{rca}}(\Gamma) \to X$ is defined by

$$F(t, x)\mu = \int_\Gamma f(t, x, \sigma)\mu(\mathrm{d}\sigma).$$

The following theorem is an immediate consequence of Theorem 2.

Theorem 7. *Assume that assumptions* [A], [J] *and* [F1] *hold. For every $\mu(\cdot) \in U_r$, the relaxed control system (2.3) has a unique solution.*

3. Properties of relaxed trajectories

In this section, we will denote the set of original trajectories and relaxed trajectories of the system (2.2) by X_0 and the system (2.3) by X_r, i.e.,

$$X_0 = \{x \in PC([0, T]; X) \mid x \text{ is a solution of (2.2) corresponding to } u(\cdot) \in U_{\text{ad}}\}$$

and

$$X_r = \{x \in PC([0, T]; X) \mid x \text{ is a solution of (2.3) corresponding to } \mu(\cdot) \in U_r\}.$$

Theorems 2 and 7 show that $X_0 \neq \phi$ implies $X_r \neq \phi$. Moreover, since $U_{\text{ad}} \subseteq U_r$ we have $X_0 \subseteq X_r$.

Next, we introduce one more hypothesis concerning the operator A.

[A1] An operator A is the infinitesimal generator of a compact C_0-semigroup $\{T(t), t \ge 0\}$.

Lemma 8. *Let A satisfy assumption* [A1] *on Banach space X. Let $1 < p$ and define*

$$S(g(\cdot)) = \int_0^\cdot T(\cdot - s)g(s)\mathrm{d}s \quad \forall g(\cdot) \in L^p(I, X).$$

Then $S : L^p(I, X) \to C(I, X)$ is compact.

Proof. See Lemma 3.2 in [5]. □

Lemma 9. *Let X be reflexive and separable. Suppose the assumptions* [A1] *and* [F1] *hold. If $\{\mu^n(\cdot)\}$ is a sequence in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$ with $\mu^n(\cdot) \to \mu(\cdot)$ $L^1(I, C(\Gamma))$-weakly as $n \to \infty$ then*

$$\rho_n(\cdot) = \int_0^{\cdot} T(\cdot - \tau) \int_\Gamma f(\tau, x(\tau), \sigma)(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma)\mathrm{d}\tau \to 0 \quad \text{in } C(I, X) \text{ as } n \to \infty,$$

where $x \in C([0, T], X)$.

Proof. Due to reflexivity of X, $\{T^*(t), t \geq 0\}$ is a C_0-semigroup in Banach space X^* (see [2], p. 47). Define $g_n(\tau) = \int_\Gamma f(\tau, x(\tau), \sigma)(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma)$; then

$$\begin{aligned}\|g_n(\tau)\| &\leq \int_\Gamma \|f(\tau, x(\tau), \sigma)\|(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma)\\ &\leq k_F(1 + \|x(\tau)\|)\|\mu^n(\tau) - \mu(\tau)\|_{\Sigma_{\text{rca}}(\Gamma)}\\ &\leq 2k_F(1 + \|x(\tau)\|).\end{aligned}$$

Since $x(t)$ is the solution of (2.3), then it is bounded by $\bar{M}$. This implies that $\{g_n(\cdot)\}$ is bounded in $L^p(I, X)$, $1 < p < +\infty$. Hence there exists a subsequence (denoted with the same symbol) with $g_n(\cdot) \xrightarrow{w} g(\cdot)$ in $L^p(I, X)$.

By Lemma 8, we have

$$\rho_n(\cdot) = \int_0^{\cdot} T(\cdot - \tau)g_n(\tau)\mathrm{d}\tau \xrightarrow{s} \int_0^{\cdot} T(\cdot - \tau)g(\tau)\mathrm{d}\tau \equiv \rho(\cdot) \quad \text{in } C(I, X).$$

For fixed $0 \leq t \leq T$, $h^* \in X^*$, we have

$$\begin{aligned}\langle \rho_n(t), h^*\rangle &= \int_0^t \langle T(t-\tau)g_n(\tau), h^*\rangle \mathrm{d}\tau\\ &= \int_0^t \langle g_n(\tau), T^*(t-\tau)h^*\rangle \mathrm{d}\tau\\ &= \int_0^t \int_\Gamma \langle f(\tau, x(\tau), \sigma), T^*(t-\tau)h^*\rangle(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma)\mathrm{d}\tau\\ &= \int_0^t \int_\Gamma \xi(\tau, \sigma)(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma)\mathrm{d}\tau\end{aligned}$$

where $\xi(\tau, \sigma) = \langle f(\tau, x(\tau), \sigma), T^*(t-\tau)h^*\rangle$.

By assumption [F1], for τ fixed, the map $\sigma \mapsto \xi(\tau, \sigma)$ is continuous. This implies that $\xi(\tau, \sigma) \in C(\Gamma)$ and

$$|\xi(\tau, \sigma)| \leq k(1 + \|x(\tau)\|).$$

Hence $\xi(\cdot, \cdot) \in L^1(I, C(\Gamma))$.

Since $\mu^n(\cdot) \to \mu(\cdot)$ $L^1(I, C(\Gamma))$-weakly in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$, then

$$\int_0^t \int_\Gamma \xi(\tau, \sigma)(\mu^n(\tau) - \mu(\tau))(\mathrm{d}\sigma)\mathrm{d}t \longrightarrow 0 \quad \text{as } n \to \infty.$$

This implies that, for fixed $t \in I$,

$$\langle \rho_n(t), h^*\rangle \longrightarrow 0 \quad \forall h^* \in X^*.$$

Hence $\rho_n(t) \xrightarrow{w} 0$ as $n \to \infty$. Thus $\rho(t) \equiv 0$. This means that $\rho_n(\cdot) \longrightarrow 0$ as $n \to \infty$ in $C(I, X)$. □

Remark. Using the same proof, one can see that the result of Lemma 9 is also true when $x \in PC([0, T], X)$.

Theorem 10. *Let X be reflexive and separable. Suppose the assumptions* [A1], [J], *and* [F1] *hold. If $x(\cdot, \mu)$ is the solution of* (2.3) *corresponding to μ then, for every $\varepsilon > 0$, there exists $u(\cdot) \in U_{\text{ad}}$ such that $x(\cdot, u)$ is a solution of* (2.2) *corresponding to u and satisfying*

$$\|x(\cdot, \mu) - x(\cdot, u)\|_{PC(I,X)} < \varepsilon, \quad t \in I.$$

Proof. Let $\mu(\cdot) \in U_r$; since $U_{\text{ad}} \subseteq U_r$ and U_{ad} is dense in U_r, there thus exists a sequence $\{u_n\} \subseteq U_{\text{ad}}$ such that $u_n \xrightarrow{w^*} \mu$. Let $x_n(\cdot) = x(\cdot, u_n)$ be the solution of (2.2) corresponding to u_n and $x(\cdot) = x(\cdot, \mu)$ be the solution of (2.3) corresponding to μ. Since

$$\begin{aligned} x_n(t) &= T(t)x_0 + \int_0^t T(t-\tau) f(\tau, x_n(\tau), u_n(\tau))\mathrm{d}\tau + \sum_{0<t_i<t} T(t-t_i) J_i(x_n(t_i)) \\ &= T(t)x_o + \int_0^t T(t-\tau) \left[\int_\Gamma f(\tau, x_n(\tau), \sigma)\delta_{u_n}(\tau)(\mathrm{d}\sigma) \right] \mathrm{d}\tau + \sum_{0<t_i<t} T(t-t_i) J_i(x_n(t_i)) \end{aligned}$$

and

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau) \left[\int_\Gamma f(\tau, x(\tau), \sigma)\mu(\tau)(\mathrm{d}\sigma) \right] \mathrm{d}\tau + \sum_{0<t_i<t} T(t-t_i) J_i(x(t_i)),$$

we have

$$\begin{aligned} x_n(t) - x(t) &= \int_0^t T(t-\tau) \left[\int_\Gamma (f(\tau, x_n(\tau), \sigma)\delta_{u_n}(\tau) - f(\tau, x(\tau), \sigma)\delta_{u_n}(\tau))(\mathrm{d}\sigma) \right] \mathrm{d}\tau \\ &\quad + \int_0^t T(t-\tau) \left[\int_\Gamma f(\tau, x(\tau), \sigma)(\delta_{u_n}(\tau) - \mu(\tau))(\mathrm{d}\sigma) \right] \mathrm{d}\tau \\ &\quad + \sum_{0<t_i<t} T(t-t_i)[J_i(x_n(t_i)) - J_i(x(t_i))] \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

By the Lipschitz condition [F1], we get

$$|I_1| \le M \int_0^t L(\rho)\|x_n(\tau) - x(\tau)\|,$$

where $I_1 \equiv \int_0^t T(t-\tau)[\int_\Gamma (f(\tau, x_n(\tau), \sigma)\delta_{u_n}(\tau) - f(\tau, x(\tau), \sigma)\delta_{u_n}(\tau))(\mathrm{d}\sigma)]\mathrm{d}\tau$, and M is a bound for $\|T(t)\|$ in $0 \le t \le T$.

Using assumption [J](2), we have

$$|I_3| \le \sum_{0<t_i<t} Mh_i \|x_n(t_i) - x(t_i)\|,$$

where $I_3 \equiv \sum_{0<t_i<t} T(t-t_i)[J_i(x_n(t_i)) - J_i(x(t_i))]$.

We denote the second integral I_2 by $\rho_n(t)$, i.e.,

$$\rho_n(t) \equiv I_2 \equiv \int_0^t T(t-\tau) \left[\int_\Gamma f(\tau, x(\tau), \sigma)(\delta_{u_n}(\tau) - \mu(\tau))(\mathrm{d}\sigma) \right] \mathrm{d}\tau.$$

Thus

$$\|x_n(t) - x(t)\| \le M \int_0^t L(\rho)\|x_n(\tau) - x(\tau)\|\mathrm{d}\tau + \|\rho_n(t)\| + \sum_{0<t_i<t} Mh_i\|x_n(t_i) - x(t_i)\|.$$

By the impulsive Gronwall inequality, we get

$$\|x_n(t) - x(t)\| \le C\|\rho_n(t)\|,$$

where $C \equiv \prod_{0<t_i<t}(1 + Mh_i)\exp(ML(\rho)t)$.

By using Lemma 9, we show that $\rho_n(\cdot) \to 0$ as $n \to \infty$ in $PC([0, T], X)$. Hence $x_n(\cdot) \to x(\cdot)$ as $n \to \infty$ in $PC([0, T], X)$. The proof is complete. □

4. Relaxed optimal controls and relaxation theorems

Consider the following Lagrange optimal control (P$_r$): Find a control policy $\mu_0 \in U_r$ such that it imparts a minimum to the cost functional J given by

$$J(\mu) \equiv J(x_\mu, \mu) \equiv \int_I \int_\Gamma l(t, x_\mu(t), \sigma)\mu(t)(\mathrm{d}\sigma)\mathrm{d}t, \tag{P$_r$}$$

where x_μ is a solution of the system (2.3) corresponding to the control $\mu \in U_r$.

We make the following hypotheses concerning the integrand $l(\cdot, \cdot, \cdot)$.

[L] $l : I \times X \times \Gamma \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an operator such that
(1) $(t, \xi, \sigma) \mapsto l(t, \xi, \sigma)$ is measurable,
(2) $(\xi, \sigma) \mapsto l(t, \xi, \sigma)$ is lower semicontinuous,
(3) $|l(t, \xi, \sigma)| \le \theta_R(t)$ for almost all $t \in I$ provided that $\|\xi\|_X \le R, \sigma \in \Gamma$ and $\theta_R(t) \in L^1(I)$.

Before proving the existence of the relaxed control, we need a lemma.

Lemma 11. *Suppose $h : I \times X \times \Gamma \to \mathbb{R}$ satisfies*

(1) $t \mapsto h(t, \xi, \sigma)$ is measurable, $(\xi, \sigma) \mapsto h(t, \xi, \sigma)$ is continuous,
(2) $|h(t, \xi, \sigma)| \le \psi_R(t) \in L^1(I)$ provided that $\|\xi\|_X \le R$ and $\sigma \in \Gamma$.

If $x_n \to x$ in $C(I, X)$ then $h_n(\cdot, \cdot) \to h(\cdot, \cdot)$ in $L^1(I, C(\Gamma))$ as $n \to \infty$, where $h_n(t, \sigma) = h(t, x_n(t), \sigma)$ and $h(t, \sigma) = h(t, x(t), \sigma)$.

Proof. It follows immediately from the first hypothesis of this lemma that

$$h_n, h \in L^1(I, C(\Gamma)).$$

For each fixed $t \in I$, we shall show that $h_n(t, \cdot) \to h(t, \cdot)$ in $C(\Gamma)$ as $n \to \infty$.

By definition, we have

$$\sup_{\sigma \in \Gamma} |h_n(t, \sigma) - h(t, \sigma)| = \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)}.$$

Since Γ is compact, there exists $\sigma_n \in \Gamma$ such that

$$|h_n(t, \sigma_n) - h(t, \sigma_n)| = \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)}$$

and we can assume $\sigma_n \to \sigma^*$ as $n \to \infty$. We note that

$$\begin{aligned}\sup_{\sigma \in \Gamma} |h_n(t, \sigma) - h(t, \sigma)| &= |h_n(t, \sigma_n) - h(t, \sigma_n)| \\ &\le |h_n(t, \sigma_n) - h_n(t, \sigma^*)| + |h_n(t, \sigma^*) - h(t, \sigma^*)| + |h(t, \sigma^*) - h(t, \sigma_n)|.\end{aligned}$$

Then, by continuity of h, we have $|h_n(t, \sigma_n) - h(t, \sigma_n)| \to 0$ as $n \to \infty$.

This means

$$\|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)} \to 0 \quad \text{as } n \to \infty.$$

Assuming that $x_n \to x$ in $C(I, X)$ as $n \to \infty$ then there exists R such that $\|x_n(t)\|, \|x(t)\| \le R$.

Hence, by the second hypothesis of this lemma, we have

$$\|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)} \le \psi_R(t).$$

This implies that

$$\int_I \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)}\mathrm{d}t \to 0 \quad \text{as } n \to \infty.$$

We have

$$h_n(\cdot, \cdot) \to h(\cdot, \cdot) \quad \text{in } L^1(I, C(\Gamma)) \text{ as } n \to \infty.$$

This proves the lemma. □

Let $m_r = \inf\{J(\mu) : \mu \in U_r\}$. We have the following existence of relaxed optimal control.

Theorem 12. *Suppose assumptions* [A1], [F1], [J] *and* [L] *hold. Then there exists* $\mu^* \in U_r$ *such that* $J(\mu^*) = m_r$.

Proof. Let $\{\mu_n\}$ be a minimizing sequence so that $\lim_{n\to\infty} J(\mu_n) = m_r$. Recall that U_r is w^*-compact in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$; by passing to a subsequence if necessary, we may assume $\mu_n \xrightarrow{w^*} \mu^*$ in $L^\infty(I, \Sigma_{\text{rca}}(\Gamma))$ as $n \to \infty$. Next, we shall prove that (x, μ^*) is an optimal pair, where x is the solution of (2.3) corresponding to μ^*.

Since every lower semicontinuous measurable integrand is the limit of an increasing sequence of Caratheodory integrands, there exists an increasing sequence of Caratheodory integrands $\{l_k\}$ such that

$$l_k(t, \xi, \sigma) \uparrow l(t, \xi, \sigma) \quad \text{as } k \to \infty \text{ for all } t \in I, \sigma \in \Gamma.$$

Invoking the definition of weak topology and applying Lemma 11 on each subinterval of $[0, T]$, $l_k(t, x_n(t), \sigma) \to l_k(t, x(t), \sigma)$ as $n \to \infty$ for almost all $t \in I$ and all $\sigma \in \Gamma$, then

$$\begin{aligned} J(x, \mu^*) = J(\mu^*) &= \int_I \int_\Gamma l(t, x(t), \sigma)\mu^*(t)(\mathrm{d}\sigma)\mathrm{d}t \\ &= \lim_{k\to\infty} \int_I \int_\Gamma l_k(t, x(t), \sigma)\mu^*(t)(\mathrm{d}\sigma)\mathrm{d}t \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \int_I \int_\Gamma l_k(t, x_n(t), \sigma)\mu_n(t)(\mathrm{d}\sigma)\mathrm{d}t \\ &\le \lim_{n\to\infty} \int_I \int_\Gamma l(t, x_n(t), \sigma)\mu_n(t)(\mathrm{d}\sigma)\mathrm{d}t \\ &= m_r. \end{aligned}$$

However, by definition of m_r, it is obvious that $J(x, \mu^*) \ge m_r$. Hence $J(x, \mu^*) = m_r$.

This implies that (x, μ^*) is an optimal pair. □

If $J(u) = \int_I l(t, x(t), u(t))\mathrm{d}t$ is the cost function for the original problem, and $J(u_0) = \inf\{J(u), u \in U_{\text{ad}}\} = m_0$, in general, since $U_{\text{ad}} \subseteq U_r$, we have $m_r \le m_0$. It is desirable that $m_r = m_0$, i.e., our relaxation is reasonable. We have the following relaxation theorem. For this, we need hypotheses on l stronger than [L]:

[L1] $l : I \times X \times \Gamma \to \mathbb{R}$ is an operator such that
(1) $(t, \xi, \sigma) \to l(t, \xi, \sigma)$ is measurable,
(2) $(\xi, \sigma) \to l(t, \xi, \sigma)$ is continuous,
(3) $|l(t, \xi, \sigma)| \le \theta_R(t)$ for almost all $t \in I$, provided $\|\xi\|_X \le R, \sigma \in \Gamma$ and $\theta_R \in L^1(I)$.

Theorem 13. *If assumptions* [A1], [J], [F1], *and* [L1] *hold and* Γ *is compact then* $m_0 = m_r$.

Proof. Let (x, μ^*) be the optimal pair (the existence was guaranteed by the previous theorem); that is $m_r = J(x, \mu^*)$. By Theorem 10, there exists $\{u^n\} \subseteq U_{\text{ad}}$ and $\{x_n\} \subseteq PC(I, X)$ such that

$$\delta_{u_n}(\cdot) \to \mu^*(\cdot)\ L^1(I, C(\Gamma))\text{-weakly} \quad \text{in } L^\infty(I, \Sigma_{\text{rca}}(\Gamma)),$$

and $x_n \to x$ in $PC(I, X)$ as $n \to \infty$.

Applying Lemma 11 to each subinterval of $[0, T]$, one can verify that

$$l(\cdot, x_n(\cdot), \cdot) \to l(\cdot, x(\cdot), \cdot) \quad \text{in } L^1(I, C(\Gamma)).$$

By definition of the weak topology on U_r, we have

$$\begin{aligned} J(u_n) = J(\delta_{u_n}) &= \int_I \int_\Gamma l(t, x_n(t), \sigma)\delta_{u_n}(t)(\mathrm{d}\sigma)\mathrm{d}t \\ &\to \int_I \int_\Gamma l(t, x(t), \sigma)\mu^*(t)(\mathrm{d}\sigma)\mathrm{d}t = J(x, \mu^*) = m_r. \end{aligned}$$

But, by definition of m_0, $J(u_n) \ge m_0$. Hence $m_r = \lim_{n\to\infty} J(u_n) \geqslant m_0$. This implies $m_0 = m_r$. The proof is now complete. □

References

[1] N.U. Ahmed, Properties of relaxed trajectories for a class of nonlinear evolution equations on a Banach space, SIAM J. Control Optim. 21 (1983) 953–967.

[2] N.U. Ahmed, Semigroup Theory with Applications to System and Control, in: Pitman Research Notes in Maths series, vol. 246, Longman Scientific Technical, New York, 1991.

[3] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, 1990.

[4] H.O. Fattorini, Infinite Dimensional Optimization and Control Theory, Cambridge University Press, 1999.

[5] X. Li, J. Yong, Optimal Control Theory for Infinite Dimensional Systems, Birkhäuser, Boston, 1995.

[6] N.S. Papageorgious, Properties of the relaxed trajectories of evolutions and optimal control, SIAM J. Control Optim. 27 (2) (1989) 267–288.

[7] X. Xiang, P. Sattayatham, W. Wei, Relaxed controls for a class of strongly nonlinear delay evolution equations, Nonlinear Anal. 52 (2003) 703–723.

[8] J. Warga, Optimal Control of Differential and Functional Differential Equations, Springer, New York, 1996.