รหัสโครงการ SUT1-103-42-12-15

รายงานการวิจัย

การวิเคราะห์มัลติรีโซลูชันบนทรงกลม

Multiresolution Analysis on the Sphere

คณะผู้วิจัย

หัวหน้าโครงการ

Asst. Prof Dr. Eckart Schulz

สาขาวิชาคณิตศาสตร์

สำนักวิชาวิทยาศาสตร์

มหาวิทยาลัยเทคโนโลยีสุรนารี

ได้รับทุนอุดหนุนการวิจัยจากมหาวิทยาลัยเทคโนโลยีสุรนารี ปีงบประมาณ 2542

ผลงานวิจัยเป็นความรับผิดชอบของหัวหน้าโครงการวิจัยแต่เพียงผู้เดียว

ตุลาคม 2545

Acknowledgements

The author wishes to acknowledge with gratitude the financial support received from Suranaree University of Technology in the form of a research grant.

บทคัดย่อ

งานวิจัยฉบับนี้ได้นำเสนอวิธีการสร้างกรอบเวฟเล็ตบนทรงกลม วิธีการนี้สามารถกระทำได้โดยการส่งทรงกลมไปทั่วถึงระนาบขยายในลักษณะที่นำการหมุนไปสู่การเลื่อนขนาน ดังนั้นเราสามารถนำประโยชน์ของการสร้างเวฟเล็ตที่หลากหลายในระนิาบเพื่อให้ได้กรอบตรึงแน่นแบบต่อเนื่อง ภินทนะ และ มัลติรีโซลูชัน (multiresolution) บนทรงกลม นอกจากนี้ยังได้แสดงถึงการมีอยู่ของกรอบเรียบลายชนิดบนทรงกลม

Abstract

We present a method for constructing wavelet frames on the sphere. This is achieved by mapping the sphere onto the extended plane in a way which takes rotations to translations. We thus can make use of the various wavelet constructions in the plane to obtain continuous frames, discrete tight frames and multiresolutions on the sphere. In addition, we show that there exists a great variety of smooth frames on the sphere.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background and Rationale

Continuous wavelet transforms and discrete wavelet series are now widely used to analyze functions and signals in Euclidean space. Often termed "Windowed Fourier Transform", the continuous wavelet transform permits analysis of signals in both spatial and time domains. Discrete wavelet series on the other hand are often used in image or data compression where the numerical reconstruction of a signal from its wavelet coefficients is desired.

In the general mathematical setting one starts with a closed group H of invertible $n \times n$ matrices acting in the usual way on Euclidean space, and considers the natural representation π of the semi-direct product G of the two groups on the space of square integrable functions $L^2(\mathbf{R}^n)$ which is given by $\pi(a, z) = D_a T_z$, where D_a denotes the dilation operator associated with a matrix a of H, and T_z the translation operator determined by a vector z in $\mathbf{R}^n$. For a given square integrable function w, the wavelet transform associated with w maps an element f in $L^2(\mathbf{R}^n)$ to a function Wf defined on G by means of the inner product, $Wf(a,z) = \langle f, \pi(a, z)w \rangle$.

It is now natural to ask under what conditions the original function can be reconstructed from its wavelet transform. A frequently used sufficient condition is that W be an isometry with regards to the L^2 norms, in which case we call w a *tight frame generator* although this condition can be relaxed somehow. While it was first thought that square integrability of the representation □□was necessary [2, 4, 6, 7] it later turned out that this condition is not required, and that loosely speaking, the group H must not be too large and must act on $\mathbf{R}^n$ in some regular fashion [10, 12].

In practical reconstruction, one wishes both the dilation and the translation parameters to be discrete. That is, one chooses a to lie in some discrete subgroup of H and z in some sublattice of

R^n. Appropriate choices for these groups and lattices are discussed in [2] and [9]. Multiresolution decomposition [14, 15] is a method of constructing such wavelet series in a way so that they can be computed with particularly efficient algorithms. Only in the one-dimensional setting or in the case of some special one-parameter groups of diagonal matrices, however, is it known how to generate a multiresolution analysis in a systematic way.

Since the earth and many objects of interest are of spherical shape, one is also interested in doing wavelet analysis on the sphere. The problem one faces here is that there does not exist a natural concept of dilation because the sphere is compact. In addition, rotations on the sphere do not form an abelian group and thus the usual Fourier transform techniques used in wavelet analysis can not be used. To overcome this problem, many authors have introduced mappings from the sphere or subsets of the sphere into the plane.

Torresani [13] maps a hemisphere into the tangent plane, thus introducing some notion of dual group. However, there is still no notion of dilation, and thus only local Gabor type analysis is achieved. Holschneider [8] makes use of spherical harmonics, but introduces dilations in an ad-hoc and rather unsystematic way. Antoine and Vandergheynst [1] discuss continuous wavelet transforms using an elegant group-theoretical approach. They are able to specify necessary conditions for a function to be a wavelet frame, but can not obtain tight frames. In addition, they only use the simplest type of dilation which stretches in all directions at the same rate, and their approach does not lend itself easily to obtaining discrete frames. A number of authors have addressed this question by constructing multiresolution analysis on the sphere, either using the tensor product of multiresolutions on the unit interval [3], or by devising subdivision schemes on the sphere itself [5, 11]. None of these multiresolutions are derived from group representations.

Thus, it is still of interest to investigate other methods for obtaining discrete frames on the sphere, and in particular, for discretizing continuous wavelet frames.

1.2 Research Objectives

The objective of this project was to

1. find new ways to introduce continuous wavelet frames on the sphere,
2. investigate ways for disretizing these continuous frames,
3. investigate methods for obtaining multiresolutions on the sphere.

1.3 Scope and Limitations

A variety of schemes for introducing frames are known today, each having its own advantages and weak points. This project did not intend to obtain a "best" or most "elegant" method, but to shed some insight into the relationship between continuous and discrete frames. Nor did it intend to compare these schemes, or consider the construction of algorithms for the practical computation of wavelet coefficients or series.

1.4 Benefits from Research

This project adds to the variety of methods for continuous and discrete wavelet analysis on the sphere. Its results may be of use to engineers and scientists requiring data analysis and compression tools.

CHAPTER 2

METHODOLOGY

2.1 Constructions

Most of the authors discussing wavelet analysis on the sphere do so by either mapping points of the sphere into the plane [3, 13], or trying to adapt the standard wavelet analysis techniques to the sphere [1, 8].

We choose the first method, and introduce two maps not yet described in the literature. The first map takes the sphere minus its poles onto the infinite cylinder. We then need to define and discuss the construction of frames on the cylinder which are generated by translations in all directions and dilation along its axis. The second map takes the sphere minus a median onto the plane, thus making the well-established machinery of wavelet analysis in the plane available. In particular, we can introduce dilations which scale differently in each direction.

CHAPTER 3

RESULTS

3.1 Main Results

We have been able to establish the following results:

1. Using a smooth map from a dense subset of the sphere onto the infinite cylinder, we have introduced a type of wavelet analysis on the sphere which consists of rotations in longitudional and latitudional directions, and dilations in a single direction. However, frame generators are never smooth, so that this case is only sketched in the attached preprint.
2. Using a smooth map from a dense subset of the sphere onto the plane, we have introduced a type of wavelet analysis on the sphere which consists of rotations around the polar and equatorial axes, and dilations in both directions at different scaling factors and generated by a single matrix. We have shown that there exist many smooth tight frame generators provided that the dilation matrix is a proper contraction. Finally, we have been able to describe a method to show how these smooth frame generators can generate discrete wavelet frames.
3. Using the map discussed in 2., any multiresolution on the plane can easily be transferred onto the sphere.

Further details can be seen from the preprint which is included in the appendix.

3.2 Discussion

We have introduced two relatively simple schemes for obtaining continuous and discrete wavelet frames on the sphere. The particular new feature is that one can choose from a wide range of dilations, either of unidirectional type or of omnidiretional type and where the dilation scale varies with direction or where there is even a rotational dilation component present which happens when the dilation matrix has complex eigenvalues.

The disadvantage of this scheme is that by identifying translations in the plane with rotations on the compact sphere, the rate of spherical rotations decreases with increasing translation parameter. Thus, this method is well suited only for analyzing small-scale features on a well-localized domain. In order to analyze small-scale features over the whole sphere one may need to operate with a family of such frames, obtained by placing median where the sphere is cut at various locations across the sphere.

CHAPTER 4

CONCLUSION

4.1 Summary

We have described a new way of doing wavelet analysis on the sphere by introducing to new maps from the sphere onto locally Euclidean abelian groups. The map which takes the sphere onto the plane makes available a great variety of smooth continuous and discrete tight frame generators for the sphere, provided that one chooses a dilation generated by a proper contraction matrix.

4.2 Recommendations

The results of this project could be expanded on in several directions. First, it should be easy to adapt the maps used here to obtain continuous and discrete wavelet frames on the n-sphere. Secondly, we have only used one-parameter groups of dilations, and one naturally can introduce and investigate two-parameter groups of dilations on the sphere. Finally, several schemes for obtaining frames and multiresolutions on the sphere available now, and one could do a detailed comparison of all theses techniques by means of some practical examples.

REFERENCES

[1] J.-P. Antoine and P. Vandergheynst. 1999. *Wavelets on the 2-sphere: a group-theoretical approach*, Appl. Comp. Harm. Anal. **7**, 262-291.

[2] D. Bernier and K. F. Taylor. 1996. *Wavelets from square integrable representations*, SIAM J. Math. Anal. **27**, 594-608.

[3] S. Dahlke, W. Dahmen and I. Weinreich, *Multiresolution analysis and wavelets on S^2 and S^3*, Numer. Funct. Anal. Optim. **16**, 19-41.

[4] H. Fuehr. 1996. *Wavelet frames and admissibility in higher dimensions*, J. Math. Phys. **37**, 6353-6366.

[5] J. Goettelmann. 1999. *Locally supported wavelets on manifolds, with applications to the 2D sphere*, Appl. Comp. Harm. Anal. **7**, 1-33.

[6] A. Grossmann, J. Morlet and T. Paul. 1985. *Transforms associated to square integrable group representations I, general results*, J. Math. Phys **26**, 2473-2479.

[7] C. E. Heil and D. F. Walnut. 1989. *Continuous and discrete wavelet transforms*, SIAM Review **4**, 628-666.

[8] M. Holschneider. 1996. *Continuous wavelet transforms on the sphere*, J. Math. Phys. **37**, 4156-4165.

[9] R. Laugesen. 2002. *Translational averaging for completeness, characterization and oversampling of wavelets*, Collectanea Mathematica **53**, 211-249.

[10] R. Laugesen, N. Weaver, G. Weiss and E. Wilson. 2002. *A characterization of the higher dimensional groups associated with continuous wavelets*, J. Geom. Anal. **12**, 89-102.

[11] P. Schroeder and W. Sweldens. 1995. *Spherical wavelets: texture processing*, in: Rendering Techniques '95, P. Hanrahan and W. Purghathofer eds., Springer Verlag, 252-263.

[12] E. Schulz and K. F. Taylor. 2002. *Continuous and semi-discrete tight frames on R^n*, preprint.

[13] B. Torresani. 1995. *Phase space decompositions: Local Fourier analysis on spheres*, Signal Process. **43**, 341-346.

[14] G. Weiss and E. N. Wilson. 2001. *The mathematical theory of wavelets*, in: Proceedings of the NATO-ASI Meeting "Harmonic Analysis 2000 -- A Celebration", Kluwer.

[15] P. Wojtasczyk. 1997. *A Mathematical Introduction to Wavelets*, Cambridge University Press.

APPENDIX

PREPRINT OF RESEARCH RESULTS

CONTINUOUS AND DISCRETE FRAMES ON THE SPHERE

ECKART SCHULZ

ABSTRACT. In this note we present a method for constructing wavelet frames on the sphere. This is achieved by mapping the sphere onto the extended plane in a way which takes rotations to translations. We thus can make use of the various wavelet constructions in the plane to obtain continuous and discrete tight frames on the sphere. In addition, we show that there exists a great variety of smooth wavelets frames on the sphere.

1. INTRODUCTION

Continuous wavelet transforms and discrete wavelet series are now widely used to analyze functions and signals in Euclidean space. In the most general setting one starts with a semi-direct product of groups $H \ltimes \mathbb{R}^n$ [10], where H is a closed subgroup of $GL_n(\mathbb{R})$ acting on $\mathbb{R}^n$ by multiplication. There is then a natural representation π of $H \ltimes \mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ given by

$$\pi(a, \vec{z})w = D_a T_{\vec{z}} w \qquad (a \in H, \vec{z} \in \mathbb{R}^n)$$

where D_a denotes the dilation operator on $L^2(\mathbb{R}^n)$ associated with matrix multiplication, and $T_{\vec{z}}$ the translation operator determined by the vector $\vec{z}$. For a fixed square integrable function $w \in L^2(\mathbb{R}^n)$, the wavelet transform associated with w maps a function $f \in L^2(\mathbb{R}^n)$ to a function Wf on $H \ltimes \mathbb{R}^n$ through the inner product

$$Wf(a, \vec{z}) = < f, \pi(a, \vec{z})w > .$$

Numerous authors have investigated sufficient and necessary conditions on H and w which allow the reconstruction of the function f from its wavelet transform. While it was first thought that square integrability of π was necessary ([2], [4], [6], [7]) it later turned out that this condition is not required in many cases ([10], [12]) .

In practical computations, one wishes both parameters a and $\vec{z}$ to be discrete. That is, one chooses a to lie in some discrete subgroup of H, and $\vec{z}$ in some sublattice of $\mathbb{R}^n$. Appropriate choices for these groups and lattices are discussed in [2] and [9]. Multiresolution decomposition ([14], [15]) is a method of constructing such wavelet series in a way so that they can be computed with particularly efficient algorithms. Only in the one-dimensional setting or in the case of some special one-parameter groups of

Supported by a research grant from Suranaree University of Technology.

diagonal matrices, however, is it known how to generate a multiresolution analysis in a systematic way.

When considering wavelets on the sphere one encounters the problem that although the sphere is of simple topological structure, there is by its compactness no natural concept of dilation. Furthermore, rotations on the sphere do not form an abelian group and thus the usual Fourier transform techniques fail. To overcome this problem, many authors have introduced mappings from the sphere or subsets of the sphere into the plane.

Torresani [13] maps a hemisphere into the tangent plane, thus introducing some notion of dual group. However, there is still no notion of dilation, and thus only local Gabor-type analysis is achieved. Holschneider [8] makes use of spherical harmonics, but introduces dilations in an ad-hoc and rather unsystematic way. Antoine and Vandergheynst [1] discuss continuous wavelet transforms using an elegant group-theoretical approach. While they are able to specify necessary conditions for a function to be a wavelet frame, they can not obtain tight frames, and their approach does not lend itself easily to obtaining discrete frames.

Various authors have addressed this question by constructing multiresolution analysis on the sphere, either using the tensor product of multiresolutions in $L^2[0,1]$ ([3]), or by devising subdivision schemes on the sphere ([5],[11]). None of these methods is directly related with group representations.

In this note, we suggest a way to apply the wavelet theory of the plane to the sphere. We do so by mapping a dense subset of the sphere diffeomorphically onto the plane, in a way which maps rotations on the sphere to translations in the plane. We show that there exist a variety of smooth wavelets, and apply the discretization techniques in the plane to obtain discrete wavelet frames on the sphere. In addition, this map makes it possible to transfer a multiresolution analysis from the plane to the sphere.

This paper is organized as follows. In section 2 we review some of the standard techniques of wavelet analysis in the plane and adapt some results to our situation. In section 3 we introduce a map from a dense subset of the sphere onto the cylinder, thus obtaining a one-dimensional wavelet analysis on the sphere. In section 4 we then use a map from a dense subset of the sphere onto the plane and discuss how tight and smooth tight frames can be introduced onto the sphere by this map. Finally, the discretization of these frames is discussed in section 5.

2. Frames on Locally Euclidean Abelian Groups

Let us first review the concept of frames on Euclidean Spaces generated by one-parameter groups, as discussed in [10] or [12]. Given a fixed matrix $A \in GL_n(\mathbb{R})$, consider the families of translation operators $T_{\vec{z}}$ with

$$(T_{\vec{z}}w)(\vec{x}) = w(\vec{x} - \vec{z})$$

and dilation operators D_k with

$$(D_k w)(\vec{x}) = \delta^{-k/2} w(A^{-k}\vec{x})$$

for $w \in L^2(\mathbb{R}^n)$, $\vec{x}, \vec{z} \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, and where $\delta = |\det(A)|$. Here we have written elements of $\mathbb{R}^n$ as column vectors. Since the Lebesgue measure on $\mathbb{R}^n$ is translation invariant and

$$|\det(A)| \int_{\mathbb{R}^n} f(A\vec{x})\, d\vec{x} = \int_{\mathbb{R}^n} f(\vec{x})\, d\vec{x}$$

for $f \in L^1(\mathbb{R}^n)$, both families constitute groups of unitary operators on $L^2(\mathbb{R}^n)$. For a fixed $w \in L^2(\mathbb{R}^n)$, the *wavelet transform associated with* w is the map $f \to Wf$ with

$$Wf(k, \vec{z}) = < f, D_k T_{\vec{z}} w >$$

for $f \in L^2(\mathbb{R}^n)$. A function $w \in L^2(\mathbb{R}^n)$ is called a *tight frame generator* (with frame bound one) if the wavelet transform constitutes a partial isometry from $L^2(\mathbb{R}^n)$ onto a subspace of $L^2(\mathbb{Z} \times \mathbb{R}^n)$, that is if

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} | < f, D_k T_{\vec{z}} w > |^2\, d\vec{z} \tag{2.1}$$

for all $f \in L^2(\mathbb{R}^n)$. One can then reconstruct the function f from its wavelet transform. In fact, from the polarization identity it follows that for all $f, g \in L^2(\mathbb{R}^n)$,

$$< f, g > = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} << f, D_k T_{\vec{z}} w > D_k T_{\vec{z}} w, g > d\vec{z}$$

that is,

$$f = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} Wf(k, \vec{z})\, D_k T_{\vec{z}} w\, d\vec{z} \tag{2.2}$$

weakly in $L^2(\mathbb{R}^n)$.

In order to discuss tight frame generators, one needs to work with Fourier transforms. As usual, $\widehat{\mathbb{R}^n}$ denotes the dual group of $\mathbb{R}^n$ which can be identified with $\mathbb{R}^n$ itself through the pairing $(\vec{\gamma}, \vec{x}) \to e^{-2i\pi\vec{\gamma}\cdot\vec{x}}$, where elements $\vec{\gamma}$ of $\widehat{\mathbb{R}^n}$ are now written as row vectors. In case $n = 1$ we will omit the vector notation and simply write $e^{-2i\pi\gamma x}$. The Fourier transform

$$f(\vec{x}) \to \hat{f}(\vec{\gamma}) = \int_{\mathbb{R}^n} f(\vec{x}) e^{-2i\pi\vec{\gamma}\cdot\vec{x}}\, d\vec{x}$$

maps $L^1(\mathbb{R}^n)$ into $C_o(\widehat{\mathbb{R}^n})$, the set of continuous functions vanishing at infinity, and its restriction to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ extends to a Hilbert space isomorphism between $L^2(\mathbb{R}^n)$ and $L^2(\widehat{\mathbb{R}^n})$, also denoted by $f \to \hat{f}$, which takes the translation operator $T_{\vec{z}}$ to the phase shift operator $E_{-\vec{z}}$ and the dilation operator D_k to the dilation operator D_{-k}. Here,

$$(E_{-\vec{z}}\hat{w})(\vec{\gamma}) = e^{2i\pi\vec{\gamma}\cdot(-\vec{z})}\hat{w}(\vec{\gamma}) \quad \text{and} \quad (D_{-k}\hat{w})(\vec{\gamma}) = \delta^{k/2}\hat{w}(\vec{\gamma}A^k)$$

for $\hat{w} \in L^2(\widehat{\mathbb{R}^n})$, $\vec{z} \in \mathbb{R}^n$, $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ and $k \in \mathbb{Z}$. Condition (2.1) then becomes

$$\|\hat{f}\|^2 = \sum_{k \in \mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} | < \hat{f}, D_{-k} E_{-\vec{z}} \hat{w} > |^2 \, d\vec{z}$$

for all $\hat{f} \in L^2(\widehat{\mathbb{R}^n})$.

In [10] and [12] it is shown that there exist tight frame generators if and only if $|\det A| \neq 1$. Furthermore, $w \in L^2(\mathbb{R}^n)$ is a tight frame generator if and only if the mean square value of $\hat{w}$ over orbits is essentially constant, that is,

$$\sum_{k \in \mathbb{Z}} |\hat{w}(\vec{\gamma} A^k)|^2 = 1$$

for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

As in the next section we will work over the space $\mathbb{R} \times \Pi$, we first generalize these results to spaces $\mathbb{R}^n \times \Pi^m$ ($n \geq 1, m \geq 0$) where Π denotes the unit circle. These spaces are direct products of two locally compact abelian groups, and thus their dual groups and the corresponding Fourier transforms are derived from those of the two component groups in an obvious way.

Recall that the dual group of the n-torus Π^n is the discrete group $\mathbb{Z}^n$. Parametrizing elements of Π^n by n-vectors $\Phi = (\varphi_1, \ldots, \varphi_n)^T$ with $0 \leq \varphi_i < 1$, and elements of $\mathbb{Z}^n$ as discrete row vectors $\vec{n}$, this duality is given through the pairing $(\vec{n}, \Phi) \to e^{-2i\pi\vec{n}\cdot\Phi}$. While there is no natural dilation on $L^2(\Pi^n)$, rotation by a multiangle $\Theta = (\theta_1, \ldots, \theta_n)^T$ gives rise to a natural translation operator T_Θ, given by

$$(T_\Theta w)(\Phi) = w(\Phi - \Theta)$$

where subtraction on the right denotes componentwise subtraction modulo one. The Fourier transform

$$f(\Phi) \to \hat{f}(\vec{n}) = \int_{\mathbb{R}^n} f(\Phi) e^{-2i\pi\vec{n}\cdot\Phi} \, d\Phi$$

now extends to an isomorphism between the Hilbert spaces $L^2(\Pi^n)$ and $L^2(\mathbb{Z}^n)$ mapping translation T_Θ to phase shift $E_{-\Theta}$.

As the Lebesgue measure on $\mathbb{R}^n \times \Pi^m$ is $d(\vec{z}, \Theta) = d\Theta \, d\vec{z}$, the translation and dilation operators on $L^2(\mathbb{R}^n \times \Pi^m)$ given by

$$(T_{(\vec{z},\Theta)} w)(\vec{x}, \Phi) = w(\vec{x} - \vec{z}, \Phi - \Theta)$$

and

$$(D_k w)(\vec{x}, \Phi) = \delta^{-k/2} w(A^{-k}\vec{x}, \Phi)$$

for $w \in L^2(\mathbb{R}^n \times \Pi^m)$, $k \in \mathbb{Z}$, $\vec{x}, \vec{z} \in \mathbb{R}^n$ and $\Phi, \Theta \in \Pi^m$ are again unitary. As before, the Fourier Transform takes these operators to phase shift and dilation operators, with

$$(E_{(-\vec{z},-\Theta)} \hat{w})(\vec{\gamma}, \vec{n}) = e^{2i\pi\vec{\gamma}\cdot(-\vec{z})} e^{2i\pi\vec{n}\cdot(-\Theta)} \hat{w}(\vec{\gamma}, \vec{n})$$

and

$$(D_{-k}\hat{w})(\vec{\gamma},\vec{n}) = \delta^{k/2}\hat{w}(\vec{\gamma}A^k,\vec{n})$$

for $\hat{w} \in L^2(\widehat{\mathbb{R}^n} \times \mathbb{Z}^m)$, $k \in \mathbb{Z}$, $\vec{z} \in \mathbb{R}^n$, $\vec{\gamma} \in \widehat{\mathbb{R}^n}$, $\Theta \in \Pi^n$ and $\vec{n} \in \mathbb{Z}^m$. In this setup, $w \in L^2(\mathbb{R}^n \times \Pi^m)$ is called a tight frame generator provided that

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} \int_{\Pi^m} | < f, D_k T_{(\vec{z},\Theta)} w > |^2 \, d\Theta \, d\vec{z}$$

for all $f \in L^2(\mathbb{R}^n \times \Pi^m)$.

Theorem 1. *Let $H = \mathbb{R}^n \times \Pi^m$ $(n \geq 1, m \geq 0)$ and let $A \in GL_n(\mathbb{R})$. Then $w \in L^2(\mathbb{R}^n \times \Pi^m)$ is a tight frame generator if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^k,\vec{n})|^2 = 1 \tag{2.3}$$

for all $\vec{n} \in \mathbb{Z}^m$ and almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

Proof. The proof is a simple modification of the standard result for $m = 0$ presented in [12]. For all $f \in L^2(\mathbb{R}^n \times \Pi^m)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\Pi^m} \left| < \hat{f}, D_{-k} E_{(-\vec{z},-\Theta)} \hat{w} > \right|^2 d\Theta \, d\vec{z} \\
&= \int_H \left| \sum_{\vec{n}\in\mathbb{Z}^m} \int_{\widehat{\mathbb{R}^n}} \delta^{k/2} \, \hat{f}(\vec{\gamma},\vec{n}) \, \overline{\hat{w}(\vec{\gamma}A^k,\vec{n})} \, e^{2i\pi\vec{\gamma}A^k\cdot\vec{z}} \, e^{2i\pi\vec{n}\cdot\Theta} \, d\vec{\gamma} \right|^2 d(\vec{z},\Theta) \\
&= \int_H \left| \sum_{\vec{n}\in\mathbb{Z}^m} \int_{\widehat{\mathbb{R}^n}} \delta^{-k/2} \, \hat{f}(\vec{\gamma}A^{-k},\vec{n}) \, \overline{\hat{w}(\vec{\gamma},\vec{n})} \, e^{2i\pi\vec{\gamma}\cdot\vec{z}} \, e^{2i\pi\vec{n}\cdot\Theta} d\vec{\gamma} \right|^2 d(\vec{z},\Theta) \\
&= \delta^{-k} \int_H \left| \int_{\widehat{H}} \varphi_k(\vec{\gamma},\vec{n}) \, e^{2i\pi(\vec{\gamma},\vec{n})\cdot(\vec{z},\Theta)} \, d(\vec{\gamma},\vec{n}) \right|^2 d(\vec{z},\Theta)
\end{aligned}$$

where we have set $\varphi_k(\vec{\gamma},\vec{n}) = \hat{f}(\vec{\gamma}A^{-k},\vec{n}) \, \overline{\hat{w}(\vec{\gamma},\vec{n})} \in L^1(\widehat{H})$. Now the inner integral is precisely the inverse Fourier transform $\check{\varphi}_k$ of φ_k, so that by

Plancherel's formula

$$\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\Pi^m}\left|< \hat{f}, D_{-k}E_{(-\vec{z},-\Theta)}\hat{w} >\right|^2 d\Theta\, d\vec{z}$$

$$= \sum_{k\in\mathbb{Z}} \delta^{-k}\int_H |\check{\varphi}_k(\vec{z},\Theta)|^2\, d(\vec{z},\Theta)$$

$$= \sum_{k\in\mathbb{Z}} \delta^{-k}\int_{\widehat{H}} |\varphi_k(\vec{\gamma},\vec{n})|^2\, d(\vec{\gamma},\vec{n})$$

$$= \sum_{k\in\mathbb{Z}} \delta^{-k}\sum_{\vec{n}\in\mathbb{Z}^m}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma}A^{-k},\vec{n})\,\overline{\hat{w}(\vec{\gamma},\vec{n})}|^2\, d\vec{\gamma}$$

$$= \sum_{k\in\mathbb{Z}}\sum_{\vec{n}\in\mathbb{Z}^m}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma},\vec{n})|^2\, |\hat{w}(\vec{\gamma}A^{k},\vec{n})|^2\, d\vec{\gamma}$$

$$= \sum_{\vec{n}\in\mathbb{Z}^m}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma},\vec{n})|^2 \left(\sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^{k},\vec{n})|^2\right) d\vec{\gamma}.$$

It follows that w is a semi-discrete tight frame generator if and only if

$$\|\hat{f}\|^2 = \sum_{\vec{n}\in\mathbb{Z}^m}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma},\vec{n})|^2 \left(\sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^{k},\vec{n})|^2\right) d\vec{\gamma}$$

for all $\hat{f} \in L^2(\widehat{\mathbb{R}^n})$. Now this identity holds if and only if $\sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^k,\vec{n})|^2 = 1$ almost everywhere, and thus the assertion follows. □

Theorem 2. *Let $H = \mathbb{R}^n \times \Pi^m$ $(n \geq 1, m \geq 0)$ and let $A \in GL_n(\mathbb{R})$. Then there exists a tight frame generator $w \in L^2(\mathbb{R}^n \times \Pi^m)$ if and only if $|\det(A)| \neq 1$.*

Proof. Suppose first that $|\det(A)| \neq 1$. Then by theorem 3 in [12], there exists a tight frame generator $v(\vec{x})$ for the wavelet transform in $L^2(\mathbb{R}^n)$ associated with A, and then by theorem 1,

$$\sum_{k\in\mathbb{Z}} |\hat{v}(\vec{\gamma}A^k)|^2 = 1 \qquad \text{a.e. } \vec{\gamma}.$$

Now list all the elements of $\mathbb{Z}^m$ in a sequence $\{\vec{n}_l\}_{n=1}^{\infty}$, pick integers p_l so that

$$M := \sum_{l=1}^{\infty} |\det(A)|^{p_l} < \infty$$

and set

$$\hat{w}(\vec{\gamma},\vec{n}_l) = \hat{v}(\vec{\gamma}A^{-p_l})$$

for $(\vec{\gamma}, \vec{n}_l) \in \widehat{\mathbb{R}^n} \times \mathbb{Z}^m$. Then $\hat{w}$ is measurable and

$$\begin{aligned}\sum_{\vec{n}\in\mathbb{Z}^m} \int_{\widehat{\mathbb{R}^n}} |\hat{w}(\vec{\gamma},\vec{n})|^2 \, d\vec{\gamma} &= \sum_{l=1}^{\infty} \int_{\widehat{\mathbb{R}^n}} |\hat{v}(\vec{\gamma}A^{-p_l})|^2 \, d\vec{\gamma} \\ &= \sum_{l=1}^{\infty} \int_{\widehat{\mathbb{R}^n}} \delta^{p_l} \, |\hat{v}(\vec{\gamma})|^2 \, d\vec{\gamma} \\ &= M \, \|\hat{v}\|^2 \; < \; \infty,\end{aligned}$$

that is, $\hat{w} \in L^2(\widehat{\mathbb{R}^n} \times \mathbb{Z}^m)$. Furthermore, for each l,

$$\sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^k, \vec{n}_l)|^2 \;=\; \sum_{k\in\mathbb{Z}} |\hat{v}(\vec{\gamma}A^{k-p_l})|^2 \;=\; \sum_{k\in\mathbb{Z}} |\hat{v}(\vec{\gamma}A^k)|^2 \;=\; 1$$

for almost all $\vec{\gamma}$, which shows that w is a tight frame generator.

Now suppose to the contrary that $|\det(A)| = 1$ but there exists a tight frame generator w in $L^2(\mathbb{R}^n \times \Pi^m)$. Set $\hat{v}(\vec{\gamma}) = \hat{w}(\vec{\gamma}, 0)$. Then $\hat{v} \in L^2(\mathbb{R}^n)$ and

$$\sum_{k\in\mathbb{Z}} |\hat{v}(\vec{\gamma}A^k)|^2 \;=\; \sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^k, 0)|^2 \;=\; 1$$

that is, v is a tight frame generator for $L^2(\mathbb{R})$ contradicting theorem 3 in [12]. This proves the theorem. □

3. Frames on the sphere generated by unidirectional dilations

Let S^2 denote the 2-sphere. Using spherical coordinates, we can describe its points by the pair of Euler angles (θ, φ) with $0 \le \theta \le \pi$ and $0 \le \varphi < 2\pi$. In these coordinates, the Lebesgue measure is $d(\theta, \varphi) = \sin\theta \, d\theta \, d\varphi$. Since this measure is rotation invariant, rotations take the natural role of translations on the sphere. However, there is no notion of dilation because the sphere is compact.

Our idea is to map the sphere onto a space on which we can do wavelet analysis, in a way which maps rotations along the Euler angles to translations. Since this space must possess dilations and can thus not be compact, our map can obviously not preserve measures. In this section, we choose this space to be the infinite cylinder $\mathbb{R} \times \Pi$, while in the next section we will choose it to be the plane.

Let $p = (0, \varphi)$ and $q = (\pi, \varphi)$ denote the two poles on the sphere, respectively, and define a diffeomorphism $\Pi : S^2 \backslash \{p, q\} \to \mathbb{R} \times \Pi$ by

$$\Pi(\theta, \varphi) = (\cot\theta, \varphi).$$

Then Π induces an isomorphism $\widetilde{\Pi}$ between $L^2(S^2)$ and $L^2(\mathbb{R} \times \Pi)$ which is given by

$$(\widetilde{\Pi} f)(x, \varphi) = \frac{1}{(x^2+1)^{3/4}} \, f(\cot^{-1} x, \varphi)$$

for $f \in L^2(S^2)$. Conjugating twodirectional translations on the cylinder as well as unidirectional dilations in direction of its axis with the map $\widetilde{\Pi}$ we obtain corresponding translation operators $\widetilde{T}_{(z,\phi)} = \widetilde{\Pi}^{-1} T_{(z,\phi)} \widetilde{\Pi}$ and dilation operators in longitudinal direction, $\widetilde{D}_k = \widetilde{\Pi}^{-1} D_k \widetilde{\Pi}$ on $L^2(S^2)$. In fact, one easily computes that

$$(\widetilde{T}_{(z,\phi)} f)(\theta,\varphi) = \frac{\csc^{3/2}\theta}{[(\cot\theta - z)^2 + 1]^{3/4}}\, f(\cot^{-1}(\cot\theta - z), \varphi - \phi) \tag{3.1}$$

and

$$(\widetilde{D}_k f)(\theta,\varphi) = |a|^k \frac{\csc^{3/2}\theta}{[\cot^2\theta + a^{2k}]^{3/4}}\, f(\cot^{-1}(a^{-k}\cot\theta), \varphi) \tag{3.2}$$

for $f \in L^2(S^2)$, $k \in \mathbb{Z}$, $z \in \mathbb{R}$ and $0 \le \phi < 2\pi$, where $a \neq 0$ is a fixed dilation parameter.

From theorem 2 we see directly that tight frame generators exists, and we formulate this observation as a theorem:

Theorem 3. *Let a be a dilation parameter with $|a| \neq 0, 1$. Then there exists a tight frame generator w in $L^2(S^2)$ for the wavelet transform associated with the operators $\widetilde{T}_z$ and $\widetilde{D}_k$ defined in (3.1) and (3.2).*

4. Frames on the sphere generated by omnidirectional dilations

In order to map the sphere onto the Euclidean plane, we slice it along a meridian $M = \{(\theta, 0) \in S^2 : 0 \le \theta \le \pi\}$ and flatten and stretch it. That is, we use the diffeomorphism $\Gamma : S^2 \backslash M \to \mathbb{R}^2$ given by

$$\Gamma(\theta,\varphi) = \big(\cot\theta, 2\cot\frac{\varphi}{2}\big).$$

The corresponding isomorphism $\widetilde{\Gamma}$ from $L^2(S^2)$ onto $L^2(\mathbb{R}^2)$ is

$$(\widetilde{\Gamma} f)(x,y) = \frac{2}{(x^2+1)^{3/4}\,(y^2+4)^{1/2}}\, f\big(\cot^{-1} x, 2\cot^{-1}\frac{y}{2}\big)$$

and it gives rise to translation operators $\widetilde{T}_{\vec{z}} = \widetilde{\Gamma}^{-1} T_{\vec{z}} \widetilde{\Gamma}$ acting on $L^2(S^2)$ through

$$(\widetilde{T}_{\vec{z}} f)(\theta,\varphi) = \frac{\csc^{3/2}\theta\ \csc\left(\frac{\varphi}{2}\right)}{\left[(\cot\theta - z_1)^2 + 1\right]^{3/4}\ \left[\left(\cot\left(\frac{\varphi}{2}\right) - \frac{z_2}{2}\right)^2 + 1\right]^{1/2}}$$
$$f\left(\cot^{-1}(\cot\theta - z_1),\ 2\cot^{-1}\left(\cot\left(\frac{\varphi}{2}\right) - \frac{z_2}{2}\right)\right) \tag{4.1}$$

for $f \in L^2(S^2)$ and $\vec{z} = (z_1, z_2) \in \mathbb{R}^2$. Dilations are now determined by an invertible 2×2 matrix A giving rise to omnidirectional dilations

$$\widetilde{D}_k = \widetilde{\Gamma}^{-1} D_k \widetilde{\Gamma} \tag{4.2}$$

in $L^2(S^2)$. If we choose the matrix A to be diagonal, $A = \operatorname{diag}(a_1, a_2)$, then the dilations (4.2) simply become

$$(\widetilde{D}_k f)(\theta,\varphi) = |a_1|^k\, |a_2|^{k/2}\, \frac{\csc^{3/2}\theta\ \csc\left(\frac{\varphi}{2}\right)}{\left[\cot^2\theta + a_1^{2k}\right]^{3/4}\ \left[\cot^2\left(\frac{\varphi}{2}\right) + a_2^{2k}\right]^{1/2}}\, f\left(\ \cot^{-1}\left(a_1^{-k}\cot\theta\right),\ 2\cot^{-1}\left(a_2^{-k}\cot\left(\frac{\varphi}{2}\right)\right)\ \right)$$

for $f \in L^2(S^2)$ and $k \in \mathbb{Z}$.

From theorem 2 we obtain again:

Theorem 4. *Let $A \in GL_2(\mathbb{R})$ be a dilation matrix with $|\det(A)| \neq 1$. Then there exists a tight frame generator w in $L^2(S^2)$ for the wavelet transform associated with the operators $\widetilde{T}_{\bar{z}}$ and $\widetilde{D}_k$ defined in (4.1) and (4.2).*

5. Smooth frame generators

Ideally, one wishes frame generators to possess nice properties. They should be smooth and compactly supported, or at least vanish rapidly at infinity. We now show that given a proper contraction matrix, there exist tight frames generators on the plane which are in the class of Schwartz functions. Then we show that the corresponding frame generators on the sphere are smooth.

Theorem 5. *Let $A \in GL_2(\mathbb{R})$ be a dilation matrix whose eigenvalues all lie in the open unit disk. Then there exists a tight frame generator $w \in L^2(\mathbb{R}^2)$ with $\hat{w} \in C_c^\infty(\mathbb{R}^2)$. In particular, w is in the class of Schwartz functions.*

Proof. Using the real Jordan form of A, one sees that A^k is equivalent to exactly one of

$$\begin{bmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{bmatrix}, \qquad \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \lambda\cos k\beta & \lambda\sin k\beta \\ -\lambda\sin k\beta & \lambda\cos k\beta \end{bmatrix}$$

with $0 < |\lambda|, |\mu| < 1$. Fix a number α such that $|\lambda|, |\mu| < \alpha < 1$. It is easy to see that there exists an integer k_o such that

$$\|A^k\| \leq \alpha^k \qquad \forall k \geq k_o \tag{5.1}$$

Now since A is invertible, there exists a constant c_1, $0 < c_1 < 1$ such that

$$c_1\|\vec{\gamma}\| \leq \|\vec{\gamma}A\| \tag{5.2}$$

for all $\vec{\gamma}$. Let us set

$$S = \{\vec{\gamma} \in \mathbf{R}^2 : c_1 \leq \|\vec{\gamma}\| \leq 1\}$$

and

$$S_\epsilon = \{\vec{\gamma} \in \mathbf{R}^2 : c_1 - \epsilon \leq \|\vec{\gamma}\| \leq 1 + \epsilon\}$$

for some fixed ϵ, $0 < \epsilon < c_1$.

Note that the orbit of each $\vec{\gamma} \neq 0$ intersects the annulus S. In fact, since

$$\lim_{k\to\infty} \vec{\gamma}A^k = 0 \qquad \text{and} \qquad \lim_{k\to-\infty} \|\vec{\gamma}A^k\| = \infty \tag{5.3}$$

there exists a largest $k = k_1$ such that $\|\vec{\gamma}A^{k_1}\| > 1$. Then by (5.2),

$$c_1 < c_1\|\vec{\gamma}A^{k_1}\| \leq \|\vec{\gamma}A^{k_1+1}\| \leq 1.$$

Now pick a function $f \in C_c^\infty(\mathbf{R}^2)$ whose support is contained in the larger annulus S_ϵ and such that $|f(\vec{\gamma})| \geq m > 0$ for $\vec{\gamma} \in S$. Set

$$\sigma(\vec{\gamma}) = \sum_{k\in\mathbf{Z}} |f(\vec{\gamma}A^k)|^2 \qquad (\vec{\gamma} \neq 0)$$

By (5.1) and (5.3) we see that for each $\vec{\gamma}$, there exists a neighborhood on which this sum is finite and that

$$m \leq \sigma(\vec{\gamma}) \leq M$$

for some $M > 0$ and all $\vec{\gamma} \neq 0$. In particular, $\sigma \in C^\infty(\mathbf{R}^2)$. It follows that the inverse Fourier transform of

$$\hat{w}(\vec{\gamma}) = \begin{cases} \dfrac{f(\vec{\gamma})}{\sqrt{\sigma(\vec{\gamma})}} & \vec{\gamma} \neq 0 \\ 0 & \vec{\gamma} = 0 \end{cases}$$

is the desired tight frame generator. □

Theorem 6. *Let $A \in GL_2(\mathbb{R})$ be a dilation matrix whose eigenvalues all lie in the open unit disk. Then there exists a tight frame generator w in $L^2(S^2)$ which is infinitely differentiable.*

Proof. Let w be as constructed in theorem 5. Since $\widetilde{\Gamma}$ is a diffeomorphism, the function $\tilde{w} := \widetilde{\Gamma}^{-1}w$ is a tight frame generator for $L^2(S^2)$ which actually lies in $C^\infty(S^2\backslash M)$.

By induction, one easily shows that any partial derivative $D\tilde{w}$ of $\tilde{w}$ on $S^2\backslash M$ is a finite linear combination of functions

$$v(\theta,\varphi) = (\csc\theta)^{p/2}(\cot\theta)^{q/2}\left(\csc\left(\tfrac{\varphi}{2}\right)\right)^r\left(\cot\left(\tfrac{\varphi}{2}\right)\right)^s \frac{\partial^{i+j}}{\partial^i x\,\partial^j y} w\left(\cot\theta, 2\cot\left(\tfrac{\varphi}{2}\right)\right)$$

for some non-negative integers p, q, r, s. Thus if $\Phi \in M$ is arbitrary, we have for each such v,

$$\lim_{(\theta,\varphi)\to\Phi} v(\theta,\varphi)$$
$$= \lim_{x^2+y^2\to\infty} (x^2+1)^{p/4} x^{q/2} \left(\tfrac{y^2+4}{4}\right)^{r/2} y^s \,\frac{\partial^{i+j}}{\partial^i x\,\partial^j y}\, w(x,y) = 0$$

because w is a Schwartz function. Thus

$$\lim_{(\theta,\varphi)\to\Phi} D\tilde{w}(\theta,\varphi) = 0$$

and we can extend all derivatives $D\tilde{w}$ to the whole of the sphere by assigning a value of zero on M. Again, since w is a Schwartz function we have for each θ with $0 < \theta < \pi$ and $\varphi \neq 0$,

$$\lim_{\varphi \to 0^+} \frac{v(\theta, \varphi)}{\varphi} = \lim_{y \to \infty} \frac{(x^2+1)^{p/4} x^{q/2} \left(\frac{y^2+4}{4}\right)^{r/2} y^s \frac{\partial^{i+j}}{\partial^i x \partial^j y} w(x,y)}{\cot^{-1} y} = 0$$

and similarly,

$$\lim_{\varphi \to 2\pi^-} \frac{v(\theta, 2\pi - \varphi)}{2\pi - \varphi} = 0, \quad \lim_{\theta \to 0^+} \frac{v(\theta, \varphi)}{\theta} = 0, \quad \text{and} \quad \lim_{\theta \to \pi^-} \frac{v(\theta, \varphi)}{\pi - \theta} = 0$$

so that we can in fact consider $\tilde{w}$ an element of $C^\infty(S^2)$, all of whose derivatives vanish on M. □

6. Discrete Frames

In practical applications it is easier to work with infinite sums than with integrals. We therefore would like to discretize the continuous frames constructed above. We begin by answering this question in $L^2(\mathbb{R}^n)$.

A function $w \in L^2(\mathbb{R}^n)$ is called *discrete tight frame generator* if there exists a lattice $\Delta \in \mathbb{R}^n$ so that

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} \sum_{\vec{l} \in \Delta} \left| < f, D_k T_{\vec{l}} w > \right|^2 \tag{6.1}$$

for all $f \in L^2(\mathbb{R}^n)$. The reconstruction formula (2.2) then becomes a strongly convergent infinite sum [7]

$$f = \sum_{k \in \mathbb{Z}} \sum_{\vec{l} \in \Delta} Wf(k, \vec{l})\, D_k T_{\vec{l}} w.$$

Applying the Fourier transform, (6.1) is equivalent to

$$\|\hat{f}\|^2 = \sum_{k \in \mathbb{Z}} \sum_{\vec{l} \in \Delta} \left| < \hat{f}, D_{-k} E_{-\vec{l}} \hat{w} > \right|^2$$

for all $\hat{f} \in L^2(\widehat{\mathbb{R}^n})$.

The next theorem is a generalization of the one dimensional situation discussed in [7].

Theorem 7. *Let $A \in GL_n(\mathbb{R})$ be a dilation matrix, and $w \in L^2(\mathbb{R}^n)$ a tight frame generator. Set*

$$S = \{\vec{\gamma} \in \widehat{\mathbb{R}^n} : \hat{w}(\vec{\gamma}) \neq 0\}.$$

If there exists $r > 0$ so that $S \cap S + r\vec{m}$ is a set of measure zero for all $\vec{m} \in \mathbb{Z}^n \backslash \{0\}$ then

$$\sum_{k \in \mathbb{Z}} \sum_{\vec{m} \in \mathbb{Z}^n} \left| < \hat{f}, D_{-k} E_{-b\vec{m}} \hat{w} > \right|^2 = b^{-n} \|\hat{f}\|^2$$

for all $\hat{f} \in L^2(\widehat{\mathbb{R}^n})$, *where* $b = \frac{1}{r}$. *That is,* $\{D_k T_{b\vec{m}}\, b^{\frac{n}{2}} w\}$ *is a discrete frame for* $L^2(\mathbb{R}^n)$.

Proof. For each $k \in \mathbb{Z}$, set

$$F_k(\vec{\gamma}) \;=\; \sum_{\vec{m}\in\mathbb{Z}^n} \hat{f}(\,(\vec{\gamma} - r\vec{m})A^{-k}\,)\,\overline{\hat{w}(\vec{\gamma} - r\vec{m})}$$

Thus F_k is periodic; indeed for all $\vec{l} \in \mathbb{Z}^n$ we have

$$\begin{aligned} F_k(\vec{\gamma} - r\vec{l}) &= \sum_{\vec{m}\in\mathbb{Z}^n} \hat{f}(\,(\vec{\gamma} - r\vec{l} - r\vec{m})A^{-k}\,)\,\overline{\hat{w}(\vec{\gamma} - r\vec{l} - r\vec{m})} \\ &= \sum_{\vec{m}\in\mathbb{Z}^n} \hat{f}(\,(\vec{\gamma} - r(\vec{l} + \vec{m})\,)A^{-k}\,)\,\overline{\hat{w}(\,\vec{\gamma} - r(\vec{l} + \vec{m})\,)} \\ &= \sum_{\vec{m}\in\mathbb{Z}^n} \hat{f}(\,(\vec{\gamma} - r\vec{m})A^{-k}\,)\,\overline{\hat{w}(\vec{\gamma} - r\vec{m})} \\ &= F_k(\vec{\gamma}). \end{aligned}$$

Set $I_r = (-\frac{r}{2}, \frac{r}{2}]^n$. Then $F_k \in L^1(I_r)$. In fact, since $\widehat{\mathbb{R}^n} = \bigcup_{\vec{m}\in\mathbb{Z}^n}(I_r - r\vec{m})$ we have

$$\begin{aligned} \int_{I_r} |F_k(\vec{\gamma})|\, d\vec{\gamma} &= \int_{I_r} \left| \sum_{\vec{m}\in\mathbb{Z}^n} \hat{f}(\,(\vec{\gamma} - r\vec{m})A^{-k}\,)\,\overline{\hat{w}(\vec{\gamma} - r\vec{m})} \right| d\vec{\gamma} \\ &\leq \int_{I_r} \sum_{\vec{m}\in\mathbb{Z}^n} |\hat{f}(\,(\vec{\gamma} - r\vec{m})A^{-k}\,)|\; |\hat{w}(\vec{\gamma} - r\vec{m})|\; d\vec{\gamma} \\ &= \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma}A^{-k})|\, |\hat{w}(\vec{\gamma})|\, d\vec{\gamma} < \infty \end{aligned}$$

because $\hat{f}, \hat{w} \in L^2(\widehat{\mathbb{R}^n})$. In a similar way we have for $\vec{m} \in \mathbb{Z}^n$,

$$\begin{aligned} &\int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma}A^{-k})\,\overline{\hat{w}(\vec{\gamma})}\, e^{2i\pi\vec{\gamma}\cdot b\vec{m}}\, d\vec{\gamma} \\ &\qquad = \int_{I_r} \sum_{\vec{l}\in\mathbb{Z}^n} \hat{f}(\,(\vec{\gamma} - r\vec{l})A^{-k}\,)\; \overline{\hat{w}(\vec{\gamma} - r\vec{l})}\, e^{2i\pi(\vec{\gamma} - r\vec{l})\cdot b\vec{m}}\, d\vec{\gamma} \\ &\qquad = \int_{I_r} F_k(\vec{\gamma}) e^{2i\pi\vec{\gamma}\cdot b\vec{m}}\, d\vec{\gamma}. \end{aligned} \tag{6.2}$$

As the collection $\{\sqrt{b^n}e^{2i\pi\vec{\gamma}\cdot b\vec{m}}\}_{\vec{m}\in\mathbb{Z}^n}$ forms an orthonormal basis $\{e_{\vec{m}}\}_{\vec{m}\in\mathbb{Z}^n}$ for $L^2(I_r)$, we have

$$\begin{aligned}\sum_{\vec{m}\in\mathbb{Z}^n}\left|\int_{I_r} F_k(\vec{\gamma})e^{2i\pi\vec{\gamma}\cdot b\vec{m}}\, d\vec{\gamma}\right|^2 &= \frac{1}{b^n}\sum_{\vec{m}\in\mathbb{Z}^n}\left|\int_{I_r} F_k(\vec{\gamma})\sqrt{b^n}e^{2i\pi\vec{\gamma}\cdot b\vec{m}}\, d\vec{\gamma}\right|^2\\
&= \frac{1}{b^n}\sum_{\vec{m}\in\mathbb{Z}^n}|< F_k, e_{\vec{m}} >|^2\\
&= \frac{1}{b^n}\|F_k\|^2\\
&= \frac{1}{b^n}\int_{I_r}|F_k(\vec{\gamma})|^2\, d\vec{\gamma} \qquad (6.3)\end{aligned}$$

provided that $F_k \in L^2(I_r)$ as well. In this case, by (6.2) and (6.3),

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\sum_{\vec{m}\in\mathbb{Z}^n}\left|< \hat{f}, D_{-k}E_{-b\vec{m}}\hat{w} >\right|^2\\
&= \sum_{k\in\mathbb{Z}}\delta^{-k}\sum_{\vec{m}\in\mathbb{Z}^n}\left|\int_{\widehat{\mathbb{R}^n}}\hat{f}(\vec{\gamma}A^{-k})\overline{\hat{w}(\vec{\gamma})}e^{2i\pi\vec{\gamma}\cdot b\vec{m}}\, d\vec{\gamma}\right|^2\\
&= \sum_{k\in\mathbb{Z}}\delta^{-k}\sum_{\vec{m}\in\mathbb{Z}^n}\left|\int_{I_r}F_k(\vec{\gamma})e^{2i\pi b\vec{\gamma}\cdot\vec{m}}\, d\vec{\gamma}\right|^2\\
&= \sum_{k\in\mathbb{Z}}\delta^{-k}b^{-n}\int_{I_r}|F_k(\vec{\gamma})|^2\, d\vec{\gamma}\\
&= \sum_{k\in\mathbb{Z}}\delta^{-k}b^{-n}\int_{I_r}\left|\sum_{\vec{m}\in\mathbb{Z}^n}\hat{f}((\vec{\gamma}-r\vec{m})A^{-k})\overline{\hat{w}(\vec{\gamma}-r\vec{m})}\right|^2\, d\vec{\gamma}\\
&= \sum_{k\in\mathbb{Z}}\delta^{-k}b^{-n}\int_{I_r}\left(\sum_{\vec{l}\in\mathbb{Z}^n}\hat{f}((\vec{\gamma}-r\vec{l})A^{-k})\overline{\hat{w}(\vec{\gamma}-r\vec{l})}\right)\\
&\qquad\left(\sum_{\vec{m}\in\mathbb{Z}^n}\overline{\hat{f}((\vec{\gamma}-r\vec{m})A^{-k})}\hat{w}(\vec{\gamma}-r\vec{m})\right)\, d\vec{\gamma}\\
&= \sum_{k\in\mathbb{Z}}\delta^{-k}b^{-n}\sum_{\vec{l}\in\mathbb{Z}^n}\int_{I_r}\hat{f}((\vec{\gamma}-r\vec{l})A^{-k})\overline{\hat{w}(\vec{\gamma}-r\vec{l})}\\
&\qquad\left(\sum_{\vec{m}\in\mathbb{Z}^n}\overline{\hat{f}((\vec{\gamma}-r\vec{m})A^{-k})}\hat{w}(\vec{\gamma}-r\vec{m})\right)\, d\vec{\gamma}\\
&= \sum_{k\in\mathbb{Z}}\delta^{-k}b^{-n}\int_{\widehat{\mathbb{R}^n}}\hat{f}(\vec{\gamma}A^{-k})\overline{\hat{w}(\vec{\gamma})}\left(\sum_{\vec{m}\in\mathbb{Z}^n}\overline{\hat{f}((\vec{\gamma}-r\vec{m})A^{-k})}\hat{w}(\vec{\gamma}-r\vec{m})\right)\, d\vec{\gamma}\\
&= \sum_{k\in\mathbb{Z}}\delta^{-k}b^{-n}\int_{\widehat{\mathbb{R}^n}}\sum_{\vec{m}\in\mathbb{Z}^n}\hat{f}(\vec{\gamma}A^{-k})\overline{\hat{f}((\vec{\gamma}-r\vec{m})A^{-k})}\,\overline{\hat{w}(\vec{\gamma})}\,\hat{w}(\vec{\gamma}-r\vec{m})\, d\vec{\gamma}\end{aligned}$$

Now by the assumption on the set S, the terms in the inner sum are zero a.e. for $\vec{m} \neq 0$, so that the above becomes

$$\sum_{k\in\mathbb{Z}} \sum_{\vec{m}\in\mathbb{Z}^n} \left| < \hat{f}, D_{-k}E_{-b\vec{m}}\hat{w} > \right|^2$$

$$= \sum_{k\in\mathbb{Z}} \delta^{-k} b^{-n} \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma}A^{-k})|^2 \, |\hat{w}(\vec{\gamma}|^2 \, d\vec{\gamma}$$

$$= b^{-n} \sum_{k\in\mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 \, |\hat{w}(\vec{\gamma}A^k)|^2 \, d\vec{\gamma}$$

$$= b^{-n} \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 \left(\sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^k)|^2 \right) d\vec{\gamma} \;=\; b^{-n} \|\hat{f}\|^2$$

by theorem 1. Going backwards, we conclude that the interchange of summation and integration in the above computations is justified and that indeed $F_k \in L^2(I_r)$. Thus, $b^{\frac{n}{2}} w$ is a discrete tight frame generator. □

Corollary 1. *Let $A \in GL_2(\mathbb{R})$ be a dilation matrix whose eigenvalues all lie in the open unit disk. Then there exists a discrete tight frame generator $w \in L^2(\mathbb{R}^2)$ which lies in the class of Schwartz functions.*

Proof. Let $w \in C_c^\infty$ denote the tight frame generator for $L^2(\mathbb{R}^2)$ constructed in the proof of theorem 5. Since $\text{supp}(\hat{w})$ lies inside a bounded set, the assumptions of theorem 7 are satisfied. □

Conjugating with the map $\widetilde{\Gamma}$ above we obtain:

Corollary 2. *Let $A \in GL_2(\mathbb{R})$ be a dilation matrix whose eigenvalues all lie in the open unit disk. Then there exists a discrete tight frame generator $w \in L^2(S^2)$ associated with the translations $\widetilde{T}_{\vec{z}}$ and $\widetilde{D}_k$ defined in (4.1) and (4.2) and which is infinitely differentiable.*

We note that we can show that in the case where A has one eigenvalue of absolute value one, or where A is a one-dimensional dilation as discussed in section 3, discrete tight frame generators w still exist. However, their Fourier transforms $\hat{w}$ do no longer vanish at infinity, so that these functions are not smooth, and we therefore do not provide details.

7. Conclusion

We have introduced two types of continuous tight frames on the sphere, one type associated with one-dimensional dilations, and a general type associated with two-dimensional dilations by making use of the tools of wavelet analysis on the real line and the plane, respectively. The latter admits a large number of smooth tight frame generators in $L^2(S^2)$, and using a periodization technique we were able to discretize these frames. A multiresolution analysis in the plane can easily be transfered onto the sphere by the map we have introduced. In addition, we can obtain frame generators which are well localized on the sphere.

The disadvantage of this scheme is that by mapping translations in the plane to rotations on the compact sphere, the rate of rotations decreases with increasing translation parameter. Thus, this method is well suited only for analyzing small-scale features on a well-localized domain. In order to analyze small-scale features over the whole sphere one may need to operate with a family of such frames, obtained by placing the cut M at various locations across the sphere.

REFERENCES

[1] J.-P. Antoine and P. Vandergheynst, *Wavelets on the 2-sphere: a group-theoretical approach*, Appl. Comp. Harm. Anal. **7**(1999), 262–291.

[2] D. Bernier and K. F. Taylor, *Wavelets from square integrable representations*, SIAM J. Math. Anal. **27**(1996), 594–608.

[3] S. Dahlke, W. Dahmen and I. Weinreich, *Multiresolution analysis and wavelets on S^2 and S^3*, Numer. Funct. Anal. Optim. **16**(1995), 19–41.

[4] H. Führ, *Wavelet frames and admissibility in higher dimensions*, J. Math. Phys. **37**(1996), 6353–6366.

[5] J. Göttelmann, *Locally supported wavelets on manifolds, with applications to the 2D sphere*, Appl. Comp. Harm. Anal. **7**(1999), 1–33.

[6] A. Grossmann, J. Morlet and T. Paul, *Transforms associated to square integrable group representations I, general results*, J. Math. Phys **26**(1985), 2473–2479.

[7] C. E. Heil and D. F. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review **4**(1989), 628–666.

[8] M. Holschneider, *Continuous wavelet transforms on the sphere*, J. Math. Phys. **37**(1996), 4156–4165.

[9] R. Laugesen, *Translational averaging for completeness, characterization and oversampling of wavelets*, Collectanea Mathematica **53**(2002), 211–249.

[10] R. Laugesen, N. Weaver, G. Weiss and E. Wilson, *A characterization of the higher dimensional groups associated with continuous wavelets*, J. Geom. Anal. **12**(2002), 89–102.

[11] P. Schröder and W. Sweldens, *Spherical wavelets: texture processing*, in: Rendering Techniques '95, P. Hanrahan and W. Purghathofer eds., Springer Verlag, 1995, 252–263.

[12] E. Schulz and K. F. Taylor, *Continuous and semi-discrete tight frames on $\mathbb{R}^n$*, preprint.

[13] B. Torrésani, *Phase space decompositions: Local Fourier analysis on spheres*, Signal Process. **43**(1995), 341–346.

[14] G. Weiss and E. N. Wilson, *The mathematical theory of wavelets*, in: Proceedings of the NATO-ASI Meeting "Harmonic Analysis 2000 – A Celebration", Kluwer, 2001.

[15] P. Wojtasczyk, *A Mathematical Introduction to Wavelets*, Cambridge University Press, 1997.

SCHOOL OF MATHEMATICS, SURANAREE UNIVERSITY OF TECHNOLOGY, 111 UNIVERSITY AVENUE, NAKHON RATCHASIMA, 30000, THAILAND

E-mail address: `eckart@math.sut.ac.th`

BIOGRAPHY OF RESEARCHER

Name: Eckart Schulz

Current Position: Assistant Professor

Suranaree University of Technology

Education:

1991 Ph.D. in Mathematics, University of Saskatchewan, Canada.

1985 M.Sc. in Mathematics, University of Saskatchewan, Canada.

Professional Experience:

1993 – present Suranaree University of Technology, Asst. Prof. in Mathematics.

1991-1993 Khon Kaen University, Special Lecturer in Mathematics.

1990-1991 University of Saskatchewan, Canada, Lecturer in Mathematics.

Professional Activities:

Editor, AMC95 Conference Proceedings.

Research Publications:

1. *On the Stable Rank of Crossed Products of Sectional C*-algebras by Compact Lie Groups,* Proc. Amer. Math. Soc., 112(1991), 733-744.
2. *Group C*-algebras as Fixed Point Algebras*, Proceedings of the First Annual Meeting in Mathematics, National Research Council of Thailand, 1993 (2536).
3. *Construction of Wavelets,* Proceedings of the Second Annual Meeting in Mathematics, National Research Council of Thailand, 1994 (2537).
4. *Extensions of the Heisenberg Group and Wavelet Analysis in the Plane* (with Keith F. Taylor), CRM Lecture Notes and Conference Proceedings, 18(2000).

Text Books:

Lecture Notes in Differential Equations, Suranaree University of Technology, 1995.