Generalized Tension B-splines

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Abstract. Explicit formulae and recurrence relations for calculation of generalized tension B-splines of arbitrary degree are given. We derive main properties of GB-splines and their series, i.e. partition of unity, shape preserving properties, invariance with respect to linear transformations, etc. It is shown that such splines, providing the variation diminishing property, are Chebyshev splines.

§1. Introduction

Fitting curves and surfaces to functions and data requires the availability of methods which preserve the shape of the data. In practical calculations we usually deal with data given with prescribed accuracy. Therefore we should develop methods for constructing fair-shape-preserving approximations that satisfy given tolerances and inherit geometric properties of the data such as positivity, monotonicity, convexity, presence of linear sections, etc.

Such approximation can be based on generalized B-splines. Until recently, local support bases for computations with generalized splines have been available for only some special types of splines [1,8,11]. This limits the choice of methods when using generalized splines in tension. In [4,5,9] local support basis functions for exponential splines were introduced and their application to interpolation problems was considered. A recurrence relation for rational B-splines with prescribed poles was recently obtained in [2]. In this paper we expand the main results of [6,7] on generalized tension B-splines of arbitrary degree allowing the tension parameters to vary from interval to interval.

§2. Generalized B-splines of Arbitrary Degree

Let a partition $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ be given on the segment $[a, b]$ to which we associate a space of splines S_n^G whose restriction to a subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$ is spanned by the system of linearly independent functions $\{1, x, \ldots, x^{n-2}, \Phi_{i,n}(x), \Psi_{i,n}(x)\}$, $n \geq 1$, and where any function in S_n^G has $n-1$ continuous derivatives.

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Definition 2.1. *The generalized spline of degree n is a function $S(x) \in S_n^G$ such that*

(1) for any $x \in [x_i, x_{i+1}]$, $i = 0, \dots, N-1$

$$S(x) = P_{i,n-2}(x) + S^{(n-1)}(x_i)\Phi_{i,n}(x) + S^{(n-1)}(x_{i+1})\Psi_{i,n}(x),$$

where $P_{i,n-2}(x)$ is a polynomial of degree $n-2$, and

$$\begin{aligned} \Phi_{i,n}^{(r)}(x_{i+1}) &= \Psi_{i,n}^{(r)}(x_i) = 0, \quad r = 0, \dots, n-1 \\ \Phi_{i,n}^{(n-1)}(x_i) &= \Psi_{i,n}^{(n-1)}(x_{i+1}) = 1; \end{aligned} \tag{2.1}$$

(2) $S(x) \in C^{n-1}[a, b]$.

Consider the problem of constructing a basis in the space S_n^G consisting of functions with local support of minimal length. For this, it is convenient to extend the mesh Δ by adding points $x_{-n} < \cdots < x_{-1} < a$, $b < x_{N+1} < \cdots < x_{N+n}$. As $dim(S_n^G) = (n+1)N - n(N-1) = N + n$, it is sufficient to construct a system of linearly independent splines $B_{j,n}(x)$, $j = -n, \dots, N-1$ in S_n^G such that $B_{j,n}(x) > 0$ if $x \in (x_j, x_{j+n+1})$ and $B_{j,n}(x) \equiv 0$ outside (x_j, x_{j+n+1}).

For $n > 1$ we require the fulfillment of the normalization condition

$$\sum_{j=-n}^{N-1} B_{j,n}(x) \equiv 1 \quad \text{for} \quad x \in [a, b]. \tag{2.2}$$

By definition 2.1, we will seek basis splines in the form

$$B_{j,n}(x) = \begin{cases} B_{j,n}^{(n-1)}(x_{j+1})\Psi_{j,n}(x), \quad x_j \le x \le x_{j+1} \\ P_{j,l,n-2}(x) + B_{j,n}^{(n-1)}(x_{j+l})\Phi_{j+l,n}(x) \\ \qquad + B_{j,n}^{(n-1)}(x_{j+l+1})\Psi_{j+l,n}(x) \\ \qquad x_{j+l} \le x \le x_{j+l+1}, \quad l = 1, \dots, n-1 \\ B_{j,n}^{(n-1)}(x_{j+n})\Phi_{j+n,n}(x), \quad x_{j+n} \le x \le x_{j+n+1} \\ 0, \quad x \notin (x_j, x_{j+n+1}). \end{cases} \tag{2.3}$$

The form of $B_{j,n}(x)$ in (2.3) for $x \in [x_{j+k}, x_{j+1+k}]$, $k = 0, n$ has been simplified in virtue of the conditions $B_{j,n}^{(r)}(x_j) = B_{j,n}^{(r)}(x_{j+n+1}) = 0$, $r = 0, \dots, n-1$ and the properties (2.1) of functions $\Phi_{j,n}(x)$, $\Psi_{j,n}(x)$.

Taking into account the continuity conditions for polynomials $P_{j,l,n-2}(x)$, $l = 1, \dots,\ n-1$ in (2.3) we have the relations

$$P_{j,l,n-2}(x) = P_{j,l-1,n-2}(x) + B_{j,n}^{(n-1)}(x_{j+l}) \sum_{r=0}^{n-2} z_{j+l,n}^{(r)} (x - x_{j+l})^r / r! \qquad l = 1, \dots, n \tag{2.4}$$

with $z^{(r)}_{j+l,n} = \Psi^{(r)}_{j+l-1,n}(x_{j+l}) - \Phi^{(r)}_{j+l,n}(x_{j+l})$, $r = 0, \ldots, n-2$.

As in (2.3), polynomials $P_{j,l,n-2}(x) \equiv 0$ when $l = 0$ and $l = n$. Then by repeated application of the formula (2.4) we have

$$P_{j,l,n-2}(x) = \sum_{l'=1}^{l} B^{(n-1)}_{j,n}(x_{j+l'}) \sum_{r=0}^{n-2} z^{(r)}_{j+l',n}(x - x_{j+l'})^r/r!$$

$$= -\sum_{l'=l+1}^{n} B^{(n-1)}_{j,n}(x_{j+l'}) \sum_{r=0}^{n-2} z^{(r)}_{j+l',n}(x - x_{j+l'})^r/r!, \quad l = 1, \ldots, n-1.$$

In particular, the following identity is valid

$$\sum_{l=1}^{n} B^{(n-1)}_{j,n}(x_{j+l}) \sum_{r=0}^{n-2} z^{(r)}_{j+l,n}(x - x_{j+l})^r/r! \equiv 0. \tag{2.5}$$

Using the expansion of polynomials (2.5) by powers of x we arrive at a system of $n-1$ linear algebraic equations which defines the unknown quantities $B^{(n-1)}_{j,n}(x_{j+l})$, $l = 1, \ldots, n$. To obtain the unique solution of this system we can use the normalization condition (2.2). We can eliminate the unknowns analogously as has been done in [6,7].

§3. Recurrence Algorithm for Calculation of GB-splines

Let us define the function

$$B_{j,1}(x) = \begin{cases} \Psi^{(n-1)}_{j,n}(x), & x_j \le x \le x_{j+1} \\ \Phi^{(n-1)}_{j+1,n}(x), & x_{j+1} \le x \le x_{j+2} \\ 0, & x \notin (x_j, x_{j+2}) \end{cases} \tag{3.1}$$

where the functions $\Psi^{(n-1)}_{j,n}(x)$, $\Phi^{(n-1)}_{j+1,n}(x)$ are assumed to be positive and monotone on (x_j, x_{j+1}) and (x_{j+1}, x_{j+2}) respectively.

We will consider the sequence of B-splines defined by the recurrence formula

$$B_{j,k}(x) = \int_{x_j}^{x} \frac{B_{j,k-1}(\tau)}{c_{j,k-1}} d\tau - \int_{x_{j+1}}^{x} \frac{B_{j+1,k-1}(\tau)}{c_{j+1,k-1}} d\tau, \quad k = 2, \ldots, n \tag{3.2}$$

where

$$c_{j,k-1} = \int_{x_j}^{x_{j+k}} B_{j,k-1}(\tau) d\tau.$$

Differentiating formula (3.2) we obtain

$$B'_{j,k}(x) = B_{j,k-1}(x)/c_{j,k-1} - B_{j+1,k-1}(x)/c_{j+1,k-1}, \quad k = 2, \ldots, n. \tag{3.3}$$

Theorem 3.1. *The recurrence formulae (3.1) and (3.2) define the sequence of B-splines of the form*

$$B_{j,k}(x) = \begin{cases} B_{j,k}^{(k-1)}(x_{j+1})\Psi_{j,n}^{(n-k)}(x), \quad x_j \le x \le x_{j+1} \\ P_{j,l,k-2}(x) + B_{j,k}^{(k-1)}(x_{j+l})\Phi_{j+l,n}^{(n-k)}(x) \\ \qquad + B_{j,k}^{(k-1)}(x_{j+l+1})\Psi_{j+l,n}^{(n-k)}(x) \\ \qquad x_{j+l} \le x \le x_{j+l+1}, \quad l = 1, \ldots, k-1 \\ B_{j,k}^{(k-1)}(x_{j+k})\Phi_{j+k,n}^{(n-k)}(x), \quad x_{j+k} \le x \le x_{j+k+1} \\ 0, \quad x \notin (x_j, x_{j+k+1}) \end{cases} \tag{3.4}$$

$k = 1, \ldots, n$, *where*

$$\begin{aligned} P_{j,l,k-2}(x) &= \sum_{l'=1}^{l} B_{j,k}^{(k-1)}(x_{j+l'}) \sum_{r=n-k}^{n-2} z_{j+l',n}^{(r)}(x - x_{j+l'})^{r-n+k}/(r-n+k)! \\ &= -\sum_{l'=l+1}^{k} B_{j,k}^{(k-1)}(x_{j+l'}) \sum_{r=n-k}^{n-2} z_{j+l',n}^{(r)}(x - x_{j+l'})^{r-n+k}/(r-n+k)! \end{aligned} \tag{3.5}$$

and

$$\sum_{l=1}^{k} B_{j,k}^{(k-1)}(x_{j+l}) \sum_{r=n-k}^{n-2} z_{j+l,n}^{(r)}(x - x_{j+l})^{r-n+k}/(r-n+k)! \equiv 0$$
$$k = 2, \ldots, n.$$

This can be shown by induction using the differentiation formula (3.3).

To use the formulae (3.4) and (3.5) for calculations we first need to find the quantities $B_{j,k}^{(k-1)}(x_{j+l})$, $l = 1, \ldots, k$; $k = 2, \ldots, n$. According to (3.3),

$$B_{j,k}^{(k-1)}(x_{j+l}) = B_{j,k-1}^{(k-2)}(x_{j+l})/c_{j,k-1} - B_{j+1,k-1}^{(k-2)}(x_{j+l})/c_{j+1,k-1} \qquad l = 1, \ldots, k; \quad k = 2, \ldots, n. \tag{3.6}$$

In particular, it follows from here with $B_{j,1}(x_{j+1}) = 1$ that

$$\begin{aligned} B'_{j,2}(x_{j+1}) &= \frac{1}{c_{j,1}}, \quad & B''_{j,3}(x_{j+1}) &= \frac{1}{c_{j,1}c_{j,2}} \\ B'_{j,2}(x_{j+2}) &= -\frac{1}{c_{j+1,1}}, \quad & B''_{j,3}(x_{j+2}) &= -\frac{1}{c_{j+1,1}}\left(\frac{1}{c_{j,2}} + \frac{1}{c_{j+1,2}}\right) \\ & & B''_{j,3}(x_{j+3}) &= \frac{1}{c_{j+2,1}c_{j+1,2}} \end{aligned}$$

etc. Therefore to find the necessary values of the derivatives of the basis splines in interior nodes of their interval supports, it is necessary to know the quantities $c_{j,k}$, i. e. the integrals of the B-splines $B_{j,k}(x)$, $k = 1, \ldots, n-1$.

Theorem 3.2. *The integrals of the generalized basis splines* $B_{j,k}(x)$, $k = 1,\dots,n-1$ *are given by the formula*

$$c_{j,k} = \int_{x_j}^{x_{j+k+1}} B_{j,k}(\tau)d\tau = \sum_{l=1}^{k} B_{j,k}^{(k-1)}(x_{j+l}) \sum_{r=n-k-1}^{n-2} z_{j+l,n}^{(r)} \frac{(x_{j+\alpha} - x_{j+l})^{r-n+k+1}}{(r-n+k+1)!}$$
$$\alpha = 1,\dots,k; \quad k = 1,\dots,n-1 \qquad (3.7)$$

This can be proven by induction using formulae for B-splines (3.4) and (3.5).

To construct the basis spline $B_{j,k}(x)$, $k = 2,\dots,n$, we apply formulae (3.6) and (3.7), and consecutively calculate the quantities $B_{j,k}^{(k-1)}(x_{j+\alpha})$, $\alpha = 1,\dots,k$, $k = 1,\dots,n$, and $c_{j+\beta}$, $\beta = 0,\dots,n-k$, $k = 1,\dots,n-1$.

§4. Properties of Generalized B-splines and Their Series

Let us formulate some properties of GB-splines which are mainly analogous to the properties of polynomial B-splines [10].

Theorem 4.1. *The functions* $B_{j,k}(x)$, $k = 1,\dots,n$ *have the following properties:*

1. $B_{j,k}(x) > 0$ *if* $x \in (x_j, x_{j+k+1})$ *and* $B_{j,k}(x) \equiv 0$ *if* $x \notin (x_j, x_{j+k+1})$;
2. *The splines* $B_{j,k}(x)$ *have* $k-1$ *continuous derivatives;*
3. *for* $k \geq 2$ $\sum_{j=-k}^{N-1} B_{j,k}(x) = 1$ *if* $x \in [a,b]$;

$$\Psi_{j,n}^{(r)}(x) = (\prod_{k=1}^{n-r-1} c_{j,k})B_{j,n-r}(x), \quad \Phi_{j,n}^{(r)}(x) = \prod_{k=1}^{n-r-1}(-c_{j-k,k})B_{j-n+r,n-r}(x)$$

if $x \in [x_j, x_{j+1}]$, $j = 0,\dots,N-1$, $r = 0,\dots,n-1$, $c_{j,k} = \int_{x_j}^{x_{j+k+1}} B_{j,k}(\tau)d\tau$.

We denote by S_k^G the set of splines $S(x) \in C^{k-1}[a,b]$ which are spanned by linear combinations of the functions $\{1,\dots,x^{k-2}, \Phi_i^{(n-k)}(x), \Psi_i^{(n-k)}(x)\}$, $k = 1,\dots,n$, in any subinterval $[x_i, x_{i+1}]$, $i = 0,\dots,N-1$. Using the methods [12], it is easy to show that the splines $B_{j,k}(x)$, $j = -k,\dots,N-1$, $k = 1,\dots,n$ have minimum-length supports, are linearly independent and form a basis in S_k^G, i.e. any generalized spline $S(x) \in S_k^G$, $k = 1,\dots,n$ can be uniquely represented in the form

$$S(x) = \sum_{j=-k}^{N-1} b_{j,k}B_{j,k}(x) \quad \text{for} \quad x \in [a,b] \qquad (4.1)$$

for some constant coefficients $b_{j,k}$.

Applying the differentiation formula (3.3), we obtain for $r \leq k-1$

$$S^{(r)}(x) = \sum_{j=-k+r}^{N-1} b_{j,k}^{(r)} B_{j,k-r}(x)$$

where

$$b_{j,k}^{(l)} = \begin{cases} b_{j,k}, & l = 0 \\ \dfrac{b_{j,k}^{(l-1)} - b_{j-1,k}^{(l-1)}}{c_{j,k-l}}, & l = 1, 2, \ldots, r. \end{cases}$$

If now $b_j^{(k)} > 0$, $k = 0, 1, 2$, $j = -3 + k, \ldots, N - 1$, then the spline $S(x)$ will be a positive monotonically increasing and convex function.

Let $Z_{[a,b]}(f(x))$ be the number of isolated zeros of a function $f(x)$ on the segment $[a, b]$.

Lemma 4.1. *If the spline* $S(x) = \sum_{j=-k}^{N-1} b_{j,k} B_{j,k}(x)$, $k = 1, \ldots, n$ *does not vanish on any subsegment of* $[a, b]$, *then* $Z_{[a,b]}(S(x)) \leq N + k - 1$.

Denote by $supp\ B_{j,k}(x) = \{x | B_{j,k}(x) \neq 0\}$, $k = 1, \ldots, n$, the support of the spline $B_{j,k}(x)$, i.e. the interval (x_j, x_{j+k+1}).

Theorem 4.2. *Assume that* $\tau_{-k} < \tau_{-k+1} < \cdots < \tau_{N-1}$, $k = 1, \ldots, n$. *Then*

$$D = det\,(B_{j,k}(\tau_i)) \neq 0, \quad i, j = -k, \ldots, N - 1$$

if and only if

$$\tau_j \in supp\ B_{j,k}(x), \quad j = -k, \ldots, N + 1. \tag{4.2}$$

If condition (4.2) is satisfied, then $D > 0$.

The following three statements follow immediately from the theorem 4.2.

Corollary 4.1. *The system of generalized B-splines* $\{B_{j,k}(x)\}$, $j = -k, \ldots, N - 1$, $k = 1, \ldots, n$, *is a weak Chebyshev system in the sense of [3], i.e. for any* $\tau_{-k} < \tau_{-k+1} < \cdots < \tau_{N-1}$ *we have* $D \geq 0$ *and* $D > 0$ *if and only if condition (4.2) is satisfied. If the latter is satisfied, then the generalized spline* $S(x) = \sum_{j=-k}^{N-1} b_{j,k} B_{j,k}(x)$, $k = 1, \ldots, n$ *has no more than* $N + k - 1$ *isolated zeros.*

Corollary 4.2. *If the conditions of Theorem 4.2 are satisfied, the solution of the interpolation problem* $S(\tau_i) = f_i$, $i = -k, \ldots, N - 1$, $f_i \in \mathbb{R}$ *exists and is unique.*

Let $A = \{a_{ij}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, be a rectangular $(m \times n)$ matrix with $m \leq n$. The matrix A is said to be totally nonnegative (totally positive) [3] if the minors of all orders of the matrix are all nonnegative (positive), i.e. for all $1 \leq l \leq m$ we have $det(a_{i_p j_q}) \geq 0$ (> 0) for all $1 \leq i_1 < \ldots < i_l \leq m$, $1 \leq j_1 < \ldots < j_l \leq n$.

Corollary 4.3. *For arbitrary integers* $-k \leq \nu_{-k} < \ldots < \nu_{l-k-1} \leq N - 1$ *and* $\tau_{-k} < \tau_{-k+1} < \cdots < \tau_{l-k-1}$, $k = 1, \ldots, n$, *we have*

$$D_l = det\{B_{\nu_j,k}(\tau_i)\} \geq 0, \quad i, j = -k, \ldots, l - k - 1,$$

and $D_l > 0$ *if and only if* $\tau_j \in supp\ B_{\nu_j,k}(x)$, $j = -k, \ldots, l - k - 1$, *i.e. the matrix* $\{B_{j,k}(\tau_i)\}$, $i, j = -k, \ldots, N - 1$ *is totally nonnegative.*

Denote by $S^-(\mathbf{v})$ the number of sign changes (variations) in the sequence of components of the vector $\mathbf{v} = (v_1, \ldots, v_n)$, with zero being neglected. For a bounded real function $f(x)$, let $S^-(f) \equiv S^-(f(x))$ be the number of sign changes of the function $f(x)$ on the real axis $\mathbb{R}$ without taking into account the zeros

$$S^-(f(x)) = \sup_p S^-[f(\tau_1), \ldots, f(\tau_p)], \quad \tau_1 < \tau_2 < \ldots < \tau_p.$$

Theorem 4.3. *The spline $S(x) = \sum_{j=-k}^{N-1} b_{j,k} B_{j,k}(x)$, $k = 1, \ldots, n$ is a variation diminishing function, i.e. the number of sign changes $S(x)$ does not exceed the one in the sequence of its coefficients*

$$S^-_{\mathbb{R}}\left(\sum_{j=-k}^{N-1} b_{j,k} B_{j,k}(x)\right) \le S^-(\mathbf{b}), \quad \mathbf{b} = (b_{-k,k}, \ldots, b_{N-1,k}).$$

Let $\hat{S}_n^G$ be a set of generalized splines on the mesh $\hat{\Delta} = \{\hat{x}_i \,|\, \hat{x}_i = px_i + q, i = 0, \ldots, N\}$ which is obtained from the linear space S_n^G by linear transformation of the variable $\hat{x} = px + q$ where $p \neq 0$ and q are constant.

Theorem 4.4. *An approximating generalized spline $S(x) \in S_n^G$ is invariant with respect to a linear transformation of the real axis $\mathbb{R} = (-\infty, \infty)$.*

The proofs of the statements above are based on the methods of [10] for polynomial B-splines.

§5. Local Approximation by Generalized Splines

Using the locality of B-splines one can reduce the representation of a spline $S(x)$ as a linear combination of B-splines (4.1) for $k = n$ to the form

$$S(x) = \sum_{j=i-n}^{i} b_{j,n} B_{j,n}(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, \ldots, N-1. \tag{5.1}$$

Theorem 5.1. *The restriction (5.1) of the spline $S(x)$ to the interval $[x_i, x_{i+1}]$ can be written in the form*

$$S(x) = P_{i,n-2}(x) + b_{i-1,n}^{(n-1)} \Phi_{i,n}(x) + b_{i,n}^{(n-1)} \Psi_{i,n}(x),$$

where

$$P_{i,n-2}(x) = \sum_{k=0}^{n-2} b_{i-n+1+k,n}^{(k)} Q_{i,n-2}^{(n-2-k)}(x)$$

$$Q_{i,n-2}^{(k)}(x) = \begin{cases} Q_{i,n-2}(x)/c_{i-1,1}, & k = 0 \\ \dfrac{Q_{i-1,n-2}^{(k-1)}(x) - Q_{i,n-2}^{(k-1)}(x)}{c_{i-k-1,k+1}}, & k = 1, 2, \ldots, n-2 \end{cases}$$

$$Q_{j,n-2}(x) = \sum_{l=0}^{n-2} z_{j,n}^{(r)} \frac{(x - x_j)^r}{r!}, \quad j = i-n+2, \ldots, i, \quad Q_{i,n-2}^{(n-2)}(x) \equiv 1$$

$$b_{j,n}^{(k)} = \frac{b_{j,n}^{(k-1)} - b_{j-1,n}^{(k-1)}}{c_{j,n-k}}, \quad k = 1, \ldots, n-1; \quad b_{j,n}^{(0)} = b_{j,n}, \quad j = i-n, \ldots, i.$$

This assertion is new even for polynomial splines, and can be proven by induction.

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