Algorithms for Shape Preserving Local Approximation with Automatic Selection of Tension Parameters

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Abstract

This paper describes the problem of shape preserving approximation for data with specified tolerances. Using the tool of generalized B-splines (GB-splines for short), simple one- and three-point algorithms of shape preserving local approximation with automatic choice of the tension parameters are developed. In the two-dimensional case, tensor products of one-dimensional splines are employed. The results of numerical calculations are given.

Keywords: Interval data; GB-splines; Shape preserving local approximation; Automatic selection of tension parameters; Tensor product surfaces.

1. Introduction

The tool of generalized splines is widely used to solve shape preserving interpolation problems (e.g., see Boor, 1978; Gregory, 1986; Sakai and Silanes, 1986; Beatson and Wolkowitz, 1989; Schaback, 1990). By introducing tension parameters into the spline structure, one can preserve various characteristics of the initial data including positivity, convexity, linear and planar sections. Here the main challenge is to develop algorithms that choose these parameters automatically. The currently available algorithms (Miroshnichenko, 1984; Sapidis et al., 1988; McCartin, 1990) mainly make use of the piecewise representation of splines. On the same basis, the problem of shape preserving approximation (not interpolation) was treated in the work of Pruess (1978) and Schmidt and Scholz (1990) as spline smoothing.

The method of local approximation (Lyche and Schumaker, 1975), combined with recurrence algorithms for computing polynomial B-splines (Boor, 1978), was found to be efficient in practical applications. Such approximation providing a variation diminishing property has many useful data shape preserving properties (Schumaker, 1981). However it gives a curve which only

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approximates the data, but changes the data shape substantially. For solving the shape preserving approximation problem, this approach was used in (Grebennikov, 1983) with rather theoretical algorithms for data refinement.

The method of local approximation can be based on GB-splines. Until recently, local support bases for computations with generalized splines have been available only for some special types of splines (Boor, 1972; Lyche and Winther, 1979; Schumaker, 1982, 1983; Lyche, 1985; Dyn and Ron, 1988). This limited the choice of methods when using generalized splines. In (Koch and Lyche, 1989, 1991, 1993) exponential GB-splines were introduced and their application to interpolation problems was considered. Hyperbolic GB-splines with tension were obtained in (Marušić, 1996). In (Kvasov, 1996a) the author constructed GB-splines allowing the tension parameters to vary from interval to interval.

In practical calculations we usually treat data with specified tolerances. Therefore we should develop methods for constructing fair-shape-preserving approximations which satisfy these tolerances and inherit geometric properties of the data. In this paper such a setting of the problem is formalized by introducing the notion of a class of shape preserving functions. We develop one- and three-point algorithms of shape preserving local approximation based on GB-splines (Kvasov, 1996a) with automatic choice of the tension parameters. We choose the tension parameters to satisfy the given tolerances and the monotonicity and convexity conditions for the initial data. These algorithms generalize the preliminary results of (Kvasov 1996c, 1998) for shape preserving splines.

In the approximation of surfaces the initial data is assumed to be given as a set of pointwise-assigned non-intersecting curvilinear sections of a three-dimensional solid. First, using the shape preserving interpolation algorithm of (Kvasov, 1996b), we construct a system of curves along the initial sections. A two-dimensional surface spline is defined as the tensor product of one-dimensional splines, generating a family of generalized local approximation splines in the orthogonal direction. This yields a finite system of curvilinear coordinate lines on the surface which form a regular grid. Along those lines we can preserve various properties of the initial data including convexity, monotonicity, rectilinear and planar sections.

2. The Problem of Shape Preserving Approximation

Let the grid $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ be given on the interval $[a, b]$ together with a set of intervals $F = \{F_i \,|\, i = 0, \ldots, N\}$, $F_i \equiv [f_i - \varepsilon_i, f_i + \varepsilon_i]$, where $\varepsilon_i > 0$ are given small numbers. We want to construct a smooth approximating function $S \in C^2[a, b]$, such that $S(x_i) \in F_i$, $i = 0, \ldots, N$, and, in addition, S preserves the shape of the initial data.

To formalize the problem let us introduce the notation

$$\Delta_i S = (S(x_{i+1}) - S(x_i))/h_i, \quad h_i = x_{i+1} - x_i, \quad i = 0, \ldots, N-1,$$
$$\delta_i S = \Delta_i S - \Delta_{i-1} S, \quad i = 1, \ldots, N-1$$

and use interval differences (Moor, 1966; Shokin, 1981)

$$\Delta_i F = h_i^{-1}(F_{i+1} - F_i) = [\Delta_i f - e_i, \Delta_i f + e_i], \quad e_i = h_i^{-1}(\varepsilon_i + \varepsilon_{i+1}), \quad i = 0, \ldots, N-1,$$

$$\delta_i F = \Delta_i F - \Delta_{i-1} F = [\delta_i f - E_i, \delta_i f + E_i], \quad E_i = e_{i-1} + e_i, \quad i = 1, \ldots, N-1,$$

$$[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1] > 0, \quad \text{provided that} \quad a_1 > b_2.$$

The initial data is non-negative (non-positive) on the interval $[x_R, x_K]$, $K \geq R$, if $F_i \geq 0$ $(F_i \leq 0)$, $i = R, \ldots, K$. The initial data is said to be monotonically increasing (monotonically decreasing) on the interval $[x_R, x_K]$, $K > R$, if $\Delta_i F > 0$ $(\Delta_i F < 0)$, $i = R, \ldots, K-1$. The data is called convex downwards (upwards) in $[x_R, x_K]$, $K > R+1$, if $\delta_i F > 0$ $(\delta_i F < 0)$, $i = R+1, \ldots, K-1$.

The intervals $\Delta_i F$, $\delta_i F$ are assumed to contain no zeros for any i, that is, the initial data satisfies the constraints

$$(\Delta_i f)^2 > e_i^2, \quad i = 0, \ldots, N-1; \quad (\delta_i f)^2 > E_i^2, \quad i = 1, \ldots, N-1. \qquad (1)$$

Geometrically, this means that neighbouring intervals do not intersect ($F_i \cap F_{i+1} = \emptyset$ for all i) and we cannot draw a straight line through any three consecutive intervals F_j, $j = i-1, i, i+1$, $i = 1, \ldots, N-1$.

If the inequalities (1) are satisfied in the entire interval $[a, b]$, then the initial data uniquely defines the conditions of convexity and monotonicity of the approximating function S. If, for the values of a certain function S, we have $S(x_i) \in F_i$, $i = 0, \ldots, N$, then $\Delta_i S \in \Delta_i F$, $i = 0, \ldots, N-1$, $\delta_i S \in \delta_i F$, $i = 1, \ldots, N-1$. Bearing in mind the constraints on the initial data, we obtain

$$\Delta_i S \, \Delta_i f > 0, \quad i = 0, \ldots, N-1, \quad \delta_i S \, \delta_i f > 0, \quad i = 1, \ldots, N-1. \qquad (2)$$

Definition 1. *A set of functions $I(\Delta, F)$ is called a class of shape preserving functions if, for any function $S \in I(\Delta, F)$, the following conditions are satisfied:*

(i) *$S \in C^2[a, b]$;*

(ii) *$S(x_i) \in F_i$, $i = 0, \ldots, N$;*

(iii) *S is monotone on $[x_i, x_{i+1}]$, $i = 1, \ldots, N-2$, for $\Delta_{i-1} f \Delta_i f > 0$ and $\Delta_i f \Delta_{i+1} f > 0$, monotone on $[x_0, x_1]$ for $\Delta_0 f \Delta_1 f > 0$ and on $[x_{N-1}, x_N]$ for $\Delta_{N-2} f \Delta_{N-1} f > 0$;*
S' has a change of sign on $[x_{i-1}, x_{i+1}]$, $i = 1, \ldots, N-1$, for $\Delta_{i-1} f \Delta_i f < 0$; the number of sign changes of S' on $[a, b]$ is equal to the number of sign changes in the sequence $\Delta_0 f, \Delta_1 f, \ldots, \Delta_{N-1} f$;

(iv) $S''(x_i)\delta_i f \geq 0$, $i = 1, \ldots, N-1$; *the number of sign changes of the function* S'' *on* $[a, b]$ *is equal to the number of sign changes in the sequence* $\delta_1 f, \delta_2 f, \ldots, \delta_{N-1} f$.

Remark 1. We do not give special consideration to the non-negative (non-positive) approximation because we always obtain such data approximation automatically just as a consequence of the approximation monotonicity.
Remark 2. One needs to choose the values of the parameters ε_i so as to obtain a balance between the exactness of approximation and the smoothness of the curve.

The search for a function $S \in I(\Delta, F)$ is referred to as the *problem of shape preserving approximation.* A solution to this problem will be sought in the form of a tension generalized spline.

3. Tension Generalized Splines

Let a partition $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a, b]$ be given to which we associate a space of functions S_4^G whose restriction to the subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$ is spanned by the system of four linearly independent functions $\{1, x, \Phi_i, \Psi_i\}$ and where every function in S_4^G has two continuous derivatives.

Definition 2. *A tension generalized spline is a function* $S \in S_4^G$ *such that*
(i) *for any* $x \in [x_i, x_{i+1}]$, $i = 0, \ldots, N-1$

$$\begin{aligned} S(x) =& [S(x_i) - \Phi_i(x_i)S''(x_i)](1-t) + [S(x_{i+1}) - \Psi_i(x_{i+1})S''(x_{i+1})]t \\ &+ \Phi_i(x)S''(x_i) + \Psi_i(x)S''(x_{i+1}), \end{aligned} \tag{3}$$

where $t = (x - x_i)/h_i$, *and the functions* Φ_i *and* Ψ_i *are subject to the constraints*

$$\Phi_i^{(r)}(x_{i+1}) = \Psi_i^{(r)}(x_i) = 0, \quad r = 0, 1, 2; \quad \Phi_i''(x_i) = \Psi_i''(x_{i+1}) = 1;$$

(ii) $S \in C^2[a, b]$.

The functions Φ_i and Ψ_i depend on the tension parameters which influence the behaviour of S fundamentally. We call them the *defining functions.* In practice, one takes

$$\begin{aligned} \Phi_i(x) &= \varphi_i(t)h_i^2 = \psi(p_i, 1-t)h_i^2, \\ \Psi_i(x) &= \psi_i(t)h_i^2 = \psi(q_i, t)h_i^2, \quad 0 \leq p_i, q_i < \infty. \end{aligned} \tag{4}$$

In the limiting case when $p_i, q_i \to \infty$ we require that $\lim_{p_i \to \infty} \Phi_i(p_i, x) = 0$, $x \in (x_i, x_{i+1}]$, and $\lim_{q_i \to \infty} \Psi_i(q_i, x) = 0$, $x \in [x_i, x_{i+1})$, so that the function S in formula (3) is a linear function. Additionally, we require that if $p_i = q_i = 0$ for all i we get a conventional cubic spline with $\varphi_i(t) = (1-t)^3/6$ and $\psi_i(t) = t^3/6$.

Let us consider a basis for the space S_4^G consisting of functions with local supports of minimum length B_i, $i = -1, \ldots, N+1$, having the following properties

$$B_i(x) > 0, \quad x \in (x_{i-2}, x_{i+2}),$$
$$B_i(x) \equiv 0, \quad x \notin (x_{i-2}, x_{i+2}),$$
$$\sum_{j=-1}^{N+1} B_j(x) \equiv 1, \quad x \in [a, b].$$

It was shown in (Kvasov, 1996a) that such splines, called GB-splines, have the form

$$B_i(x) = \begin{cases} \Psi_{i-2}(x) B_i''(x_{i-1}), \quad x \in [x_{i-2}, x_{i-1}), \\ \dfrac{x - y_{i-1}}{y_i - y_{i-1}} + \Phi_{i-1}(x) B_i''(x_{i-1}) + \Psi_{i-1}(x) B_i''(x_i), \\ \qquad\qquad x \in [x_{i-1}, x_i), \\ \dfrac{y_{i+1} - x}{y_{i+1} - y_i} + \Phi_i(x) B_i''(x_i) + \Psi_i(x) B_i''(x_{i+1}), \\ \qquad\qquad x \in [x_i, x_{i+1}), \\ \Phi_{i+1}(x) B_i''(x_{i+1}), \quad x \in [x_{i+1}, x_{i+2}), \\ 0, \quad \text{otherwise}, \end{cases} \tag{5}$$

where

$$y_j = x_j - \frac{z_j}{z_j'}, \quad z_j^r \equiv z_j^{(r)}(x_j) = \Psi_{j-1}^{(r)}(x_j) - \Phi_j^{(r)}(x_j), \quad r = 0, 1$$

and

$$B_i''(x_j) = \frac{y_{i+1} - y_{i-1}}{z_j' \omega_{i-1}'(y_j)}, \quad j = i-1, i, i+1$$

with $\omega_{i-1}(x) = (x - y_{i-1})(x - y_i)(x - y_{i+1})$.

Using the results of (Kvasov, 1996a) we can write down any spline $S \in S_4^G$ as a linear combination of the GB-splines

$$S(x) = \sum_{j=-1}^{N+1} b_j B_j(x) \quad \text{for} \quad x \in [a, b] \tag{6}$$

with certain constant coefficients b_j.

In what follows we will only consider the case where the “averaged knots” of GB-splines $y_i = x_i - z_i/z_i'$, $i = 0, \ldots, N$, coincide with the knots of the basic grid Δ, that is, $z_i = \Psi_{i-1}(x_i) - \Phi_i(x_i) = 0$, $i = 0, \ldots, N$, and $x_{-i} = x_0 - ih_0$, $x_{N+i} = x_N + ih_{N-1}$, $i = 1, 2, 3$. Then according to (5), expression (6) for the spline S on the interval $[x_i, x_{i+1}]$ can be put in the form

$$S(x) = b_i + \Delta_i b(x - x_i) + (z_i')^{-1} \delta_i b \, \Phi_i(x) + (z_{i+1}')^{-1} \delta_{i+1} b \, \Psi_i(x), \tag{7}$$

where $\delta_j b = \Delta_j b - \Delta_{j-1} b$, $j = i, i+1$, $\Delta_j b = (b_{j+1} - b_j)/h_j$.

Whence we have the formulae

$$S(x_i) = b_i + \delta_i b H_i^{-1}, \tag{8a}$$

$$S'(x_i) = (z_i')^{-1}[\Delta_i b\, \Psi_{i-1}'(x_i) - \Delta_{i-1} b\, \Phi_i'(x_i)], \tag{8b}$$

$$S''(x_i) = (z_i')^{-1}\delta_i b, \tag{8c}$$

where

$$H_i = \frac{\Psi_{i-1}'(x_i)}{\Psi_{i-1}(x_i)} - \frac{\Phi_i'(x_i)}{\Phi_i(x_i)}.$$

Conversely

$$\begin{aligned} b_{i-1} &= S(x_i) - h_{i-1}S'(x_i) + \bar{b}_{i-1}S''(x_i), \\ b_i \;\; &= S(x_i) - \Phi_i(x_i)S''(x_i), \\ b_{i+1} &= S(x_i) + h_iS'(x_i) + \bar{a}_iS''(x_i), \quad i = 0, \ldots, N \end{aligned} \tag{9}$$

with the notation

$$\bar{a}_i = -\Phi_i(x_i) - h_i\Phi_i'(x_i), \quad \bar{b}_{i-1} = -\Psi_{i-1}(x_i) + h_{i-1}\Psi_{i-1}'(x_i).$$

The choice of the defining functions Φ_i and Ψ_i will be subject to the conditions (4). In addition, we will assume that $d^r\psi(q,t)/dt^r$, $r = 0, 1, 2$ are non-negative monotone functions of their arguments $q \geq 0$, $0 \leq t \leq 1$, and also $\psi(q,1) < \psi'(q,1)$ and $\psi(q,1)$, $\psi'(q,1)/\psi(q,1)$ are strictly monotonically increasing functions of q.

4. A One-Point Algorithm of Shape Preserving Approximation

Algorithm 1. Set $b_i = f_i$, $i = 0, \ldots, N$, in formula (6). The coefficients b_{-1}, b_{N+1} can be computed in various ways, depending on the particular problem to be solved. For instance, they can be found from the boundary conditions (Beatson and Chacko, 1989): $S'(x_i) = f_i'$, $i = 0, N$. To find b_{-1}, b_{N+1} one can also apply other types of standard boundary conditions (Zavyalov et al., 1980).

The derivative values in the boundary conditions must be adjusted to the behaviour of the data. Otherwise we can obtain an incompatibility with the shape preserving restrictions. By this reason we will assume that they are subject to the constraints

$$\begin{aligned} (\Delta_0 f - f_0')\delta_1 f > 0, &\quad f_0'\Delta_0 f \geq 0, \\ (f_N' - \Delta_{N-1} f)\delta_{N-1} f > 0, &\quad f_N'\Delta_{N-1} f \geq 0, \end{aligned} \tag{10}$$

(by hypothesis, $\delta_1 f \neq 0$, $\delta_{N-1} f \neq 0$)

By virtue of the condition $z_i = 0$ or $\psi(q_{i-1},1)h_{i-1}^2 = \psi(p_i,1)h_i^2$, $i = 0, N$, and the strict monotonicity of the function $\psi(q,1)$, it follows from the equations $h_{-1} = h_0$, $h_N = h_{N-1}$ that $q_{-1} = p_0$, $q_{N-1} = p_N$. Thus, adding the first and third equations of (9), for $i = 0, N$, and taking into account the boundary conditions we obtain

$$b_{-1} = f_1 - 2h_0 f_0', \quad b_{N+1} = f_{N-1} + 2h_{N-1} f_N'. \tag{11}$$

First we choose values for the parameters q_{i-1}, p_i, $i = 1, \ldots, N-1$, so that $|S(x_i) - f_i| \le \varepsilon_i$.

According to the formulae (8a) and (4)

$$S(x_i) - f_i = \delta_i f \left[\frac{\psi'(q_{i-1},1)}{\psi(q_{i-1},1)} \frac{1}{h_{i-1}} + \frac{\psi'(p_i,1)}{\psi(p_i,1)} \frac{1}{h_i} \right]^{-1}, \quad i = 1, \ldots, N-1. \tag{12}$$

Since, when $h_{i-1} \le h_i$ the equation $z_i = 0$ leads to the relation $q_{i-1} \le p_i$, and when $h_{i-1} > h_i$, we have $q_{i-1} > p_i$, equation (12) provides a simple method for choosing the parameters q_{i-1}, p_i. For $h_{i-1} \le h_i$ due to the monotonicity of the function $\psi'(q,1)/\psi(q,1)$ in the variable q we have

$$|S(x_i) - f_i| \le |\delta_i f| \left[\left(\frac{1}{h_{i-1}} + \frac{1}{h_i} \right) \frac{\psi'(q_{i-1},1)}{\psi(q_{i-1},1)} \right]^{-1} \le \varepsilon_i.$$

Since for tension generalized splines the inequality $\psi'(q_{i-1},1)/\psi(q_{i-1},1) \ge 3$ is valid, we can define q_{i-1} by putting

$$\frac{\psi'(q_{i-1},1)}{\psi(q_{i-1},1)} - 3 = \max \left(\frac{|\delta_i f| h_{i-1} h_i}{h_{i-1} + h_i} \frac{1}{\varepsilon_i} - 3, 0 \right).$$

One finds the value of p_i from the condition $z_i = 0$ or

$$\psi(q_{i-1},1) h_{i-1}^2 = \psi(p_i,1) h_i^2, \quad i = 1, \ldots, N-1.$$

The case $h_i < h_{i-1}$ is analysed in the same way when $p_i < q_{i-1}$.

Similarly applying formula (8a) for $i = 0, N$ and using the boundary conditions in the form of equations (11) for finding p_0, q_{N-1} we have

$$\frac{\psi'(p_0,1)}{\psi(p_0,1)} - 3 = \max \left(\frac{h_0}{\varepsilon_0} |\Delta_0 f - f_0'| - 3, 0 \right),$$

$$\frac{\psi'(q_{N-1},1)}{\psi(q_{N-1},1)} - 3 = \max \left(\frac{h_{N-1}}{\varepsilon_N} |f_N' - \Delta_{N-1} f| - 3, 0 \right).$$

Lemma 1. *If $S''(x_i)S''(x_{i+1}) < 0$ then the function S'' changes its sign exactly once on the interval $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$. Otherwise, S'' does not change sign on $[x_i, x_{i+1}]$ at all.*

Proof: According to formulae (7) and (8c) for $x \in [x_i, x_{i+1}]$

$$S''(x) = S''(x_i)\Phi_i''(x) + S''(x_{i+1})\Psi_i''(x). \tag{13}$$

By hypothesis, the function $\Phi_i''(x) \ge 0$ is monotonically decreasing ($\Psi_i''(x) \ge 0$ is monotonically increasing) for $x \in [x_i, x_{i+1}]$. Hence for $S''(x_i)S''(x_{i+1}) > 0$, according to (13), the sign of $S''(x)$ remains unchanged for $x \in [x_i, x_{i+1}]$. When $S''(x_i)S''(x_{i+1}) < 0$, since the derivative

$$\frac{d}{dx} S''(x) = S''(x_i)\Phi_i'''(x) + S''(x_{i+1})\Psi_i'''(x)$$

is of constant sign, the function S'' is monotone in $[x_i, x_{i+1}]$. Thus it changes sign just once there. This proves the lemma.

Theorem 1. *If the conditions (10) are satisfied then the tension generalized spline S constructed by the one-point local approximation algorithm 1 will be a shape preserving function.*

Proof: According to (8c), $S''(x_i) = (z_i')^{-1}\delta_i f$, $i = 1, \ldots, N-1$. Since $z_i' > 0$ (Kvasov, 1996b), taking the conditions on the initial data (1) into account, one has $S''(x_i)\delta_i f > 0$, $i = 1, \ldots, N-1$.

It follows from (8c) and (11) that $S''(x_0) = (z_0')^{-1}\delta_0 b = 2(z_0')^{-1}(\Delta_0 f - f_0')$. Thus by virtue of (11) we obtain $S''(x_0)S''(x_1) > 0$. Similarly we have $S''(x_{N-1})S''(x_N) > 0$. Hence, it can be concluded on the basis of Lemma 1 that the number of sign changes of the function S'' on $[a, b]$ is equal to that in the sequence $\delta_i f$, $i = 1, \ldots, N-1$. Thus, conditions (iv) of Definition 1 are satisfied.

We now consider a grid $\gamma : a = v_0 < v_1 < \cdots < v_{N+1} = b$. Here for $S''(x_i)S''(x_{i+1}) \geq 0$, $i = 0, \ldots, N-1$, we put $v_{i+1} = \xi_{i+1} \in (x_i, x_{i+1})$ according to the equation $S(x_{i+1}) - S(x_i) = S'(\xi_{i+1})(x_{i+1} - x_i)$. For $S''(x_i)S''(x_{i+1}) < 0$ we choose $v_{i+1} = x^*$ from the condition $S''(x^*) = 0$, $x^* \in (x_i, x_{i+1})$.

By construction, $S''(v_j)S''(x_i) \geq 0$, $j = i, i+1$. From the conditions on the initial data (1) and (8c) we have $S''(x_i) \neq 0$, $i = 0, \ldots, N$. Thus, by Lemma 1, S'' does not change sign, and S', accordingly, is monotone in $[v_i, v_{i+1}]$, $i = 1, \ldots, N-1$. In $[v_0, v_1]$ and $[v_N, v_{N+1}]$, the monotonicity of S' follows from the inequalities $S''(x_i)S''(x_{i+1}) > 0$, $i = 0, N-1$ and from Lemma 1.

We will now show that the inequality $S'(x^*)\Delta_i f > 0$ holds at any inflection point $x^* \in [x_i, x_{i+1}]$, $i = 1, \ldots, N-2$. By hypothesis, $\delta_i f \delta_{i+1} f < 0$ and there are two possibilities: either $\delta_i f \Delta_i f < 0$ or $\delta_i f \Delta_i f > 0$.

By (7), for $x \in [x_i, x_{i+1}]$

$$S'(x) = \Delta_i f + (z_i')^{-1}\delta_i f \Phi_i'(x) + (z_{i+1}')^{-1}\delta_{i+1} f \Psi_i'(x),$$

where $\Phi_i'(x) \leq 0$ and $\Psi_i'(x) \geq 0$.

Hence for $\delta_i f \Delta_i f < 0$, allowing for the signs of the functions Φ_i' and Ψ_i', we have $S'(x)\Delta_i f > 0$ and, therefore, $S'(v_{i+1})\Delta_i f = S'(x^*)\Delta_i f > 0$.

Now let $\delta_i f \Delta_i f > 0$. Consider the case $\delta_i f > 0$. Since the derivative has an extremum at a point of inflection, we have $\Delta_i S < S'(x^*)$. From the relation $\Delta_i S \Delta_i f > 0$ of (2), we again arrive at the inequality $S'(x^*)\Delta_i f > 0$. The case $\delta_i f < 0$ is analysed in a similar way.

Obviously by construction $S'(v_{i+1})\Delta_i f > 0$ when $S''(x_i)S''(x_{i+1}) \geq 0$, $i = 0, \ldots, N-1$. Thus, at the nodes of γ we have $S'(v_{i+1})\Delta_i f > 0$, $i = 0, \ldots, N-1$.

We have proved that S' is monotone on $[v_j, v_{j+1}]$, $j = i, i+1$, and $S'(v_{i+1})\Delta_i f > 0$. Thus S' is monotone on $[v_i, v_{i+2}]$. Now if $\Delta_j f \Delta_{j+1} f > 0$, $j = i-1, i$, then $S'(v_i)S'(v_{i+2}) > 0$. Thus, the sign of S' remains unchanged on $[v_i, v_{i+1}]$ and, in particular, on $[x_i, x_{i+1}]$. Hence, under this assumption S is monotone on $[x_i, x_{i+1}]$, $i = 1, \ldots, N-1$.

If $\Delta_0 f \Delta_1 f > 0$ then a function S' which is monotone on $[v_1, v_2]$ will be of constant sign there. Thus $S'(x_1)\Delta_0 f > 0$. We have already shown that $S'(x)$ is monotone on $[x_0, x_1]$. According to (11), $f_0'\Delta_0 f \geq 0$ and since $S'(x_1)\Delta_0 f > 0$, S is monotone on $[x_0, x_1]$. The case of the interval $[x_{N-1}, x_N]$ is considered in a similar manner.

Since $S'(x_0)\Delta_0 f > 0$, $S'(v_{i+1})\Delta_i f > 0$, $i = 0, \ldots, N-1$, $S'(x_N)\Delta_{N-1} f > 0$ and S' is monotone on $[v_i, v_{i+1}]$, $i = 0, \ldots, N$, the function S' changes sign on $[v_i, v_{i+1}]$ and, therefore, on $[x_{i-1}, x_{i+1}]$, $i = 1, \ldots, N-1$, if $\Delta_{i-1} f \times \Delta_i f < 0$. The number of sign changes of the function S' on $[a, b]$ is equal to that in the sequence $\Delta_0 f, \Delta_1 f, \ldots, \Delta_{N-1} f$. This proves the theorem.

5. A Three-Point Algorithm of Shape Preserving Approximation

Algorithm 2. The coefficients in (6) are computed using formulae (9) where $S''(x_i)$ is approximated by the second divided difference:

$$b_i = f_i - 2(h_{i-1} + h_i)^{-1}\Phi_i(x_i)\delta_i f, \quad i = 1, \ldots, N-1. \tag{14}$$

The coefficients b_i, $i = -1, 0, N, N+1$ are found by using boundary conditions $S^{(k)}(x_i) = f_i^{(k)}$, $i = 0, N$, $k = 0, 1$. One can also use other types of expanded standard boundary conditions (Zavyalov et al., 1980).

To adjust the values of the derivative in the boundary conditions with the behaviour of the data, that is, with shape preserving restrictions, we subject them to the constraints

$$\begin{aligned} \delta_1 f(\Delta_0 f - f_0') &> |\delta_1 f|\varepsilon_1 h_0^{-1}, & f_0'\Delta_0 f &\geq 0, \\ \delta_{N-1} f(f_N' - \Delta_{N-1} f) &> |\delta_{N-1} f|\varepsilon_{N-1} h_{N-1}^{-1}, & f_N'\Delta_{N-1} f &\geq 0. \end{aligned} \tag{15}$$

These conditions are more severe in comparison to the restrictions (10).

Using the boundary conditions, we can write out the explicit expressions for the coefficients b_i, $i = -1, 0, N, N+1$. According to the formulae (8) and (9) we obtain

$$\begin{aligned} b_{-1} &= b_1 - 2h_0 f_0', \\ b_0 &= f_0 - \frac{f_0 + h_0 f_0' - b_1}{1 - \psi'(p_0, 1)/\psi(p_0, 1)}, \\ b_N &= f_N - \frac{f_N - h_{N-1} f_N' - b_{N-1}}{1 - \psi'(q_{N-1}, 1)/\psi(q_{N-1}, 1)}, \\ b_{N+1} &= b_{N-1} + 2h_{N-1} f_N'. \end{aligned} \tag{16}$$

The parameters p_i, q_i, $i = 0, \ldots, N-1$ are determined from the shape preserving conditions of Definition 1 in two steps. First, according to (14) from the constraints

$$|b_i - f_i| = 2h_i^2(h_{i-1} + h_i)^{-1}\psi(p_i, 1)|\delta_i f| \leq \varepsilon_i, \quad i = 1, \ldots, N-1 \tag{17}$$

we find p_i and compute q_{i-1} from the relation $z_i = 0$.

The quantities p_0, q_{N-1} are chosen from the inequalities

$$\begin{aligned}|b_0 - f_0| &= \frac{|f_0 + h_0 f_0' - b_1|}{|1 - \psi'(p_0, 1)/\psi(p_0, 1)|} \le \varepsilon_0, \\ |b_N - f_N| &= \frac{|f_N - h_{N-1} f_N' - b_{N-1}|}{|1 - \psi'(q_{N-1}, 1)/\psi(q_{N-1}, 1)|} \le \varepsilon_N\end{aligned} \tag{18}$$

using (16).

Finally, we find p_i, q_i from the constraints

$$|S(x_i) - f_i| \le \varepsilon_i, \quad i = 0, \ldots, N.$$

From (8a) and (14) we have

$$\begin{aligned}S(x_i) = f_i + H_i^{-1}\Big\{ &- \frac{2\Phi_{i-1}(x_{i-1})}{(h_{i-2} + h_{i-1})h_{i-1}} \delta_{i-1} f - \frac{2\Phi_{i+1}(x_{i+1})}{h_i(h_i + h_{i+1})} \delta_{i+1} f \\ &+ \left[1 + 2\frac{\Phi_i(x_i)}{h_{i-1}h_i} - 2\frac{\Psi_{i-1}'(x_i) - \Phi_i'(x_i)}{h_{i-1} + h_i}\right] \delta_i f \Big\}.\end{aligned} \tag{19}$$

For tension generalized splines we have by virtue of the condition $z_i = 0$

$$\begin{aligned}\Psi_{i-1}(x_i) = \Phi_i(x_i) &\le \frac{1}{6} h_{i-1} h_i, \\ 0 \le \Psi_{i-1}'(x_i) - \Phi_i'(x_i) &\le \frac{1}{2}(h_{i-1} + h_i)\end{aligned}$$

independently of the relations between h_{i-1} and h_i. Thus, using the estimate (17), we obtain from (19)

$$|S(x_i) - f_i| \le H_i^{-1} \theta_i \le \varepsilon_i,$$

where $\theta_i = \varepsilon_{i-1} h_{i-1}^{-1} + \frac{4}{3}|\delta_i f| + \varepsilon_{i+1} h_i^{-1}$.

For $h_{i-1} \le h_i$ therefore, as in the algorithm 1, we find q_{i-1} from the relation

$$\frac{\psi'(q_{i-1}, 1)}{\psi(q_{i-1}, 1)} - 3 = \max\left(\frac{h_{i-1} h_i}{h_{i-1} + h_i} \frac{\theta_i}{\varepsilon_i} - 3, 0\right), \quad i = 2, \ldots, N-2. \tag{20}$$

We compute p_i from the condition $z_i = 0$.

For $i = 1, N-1$, bearing (16) and (18) in mind, we define the parameters q_0 and p_{N-1} similarly. In particular, for $h_0 \le h_1$ and $h_{N-1} \le h_{N-2}$ we have

$$\begin{aligned}\frac{\psi'(q_0, 1)}{\psi(q_0, 1)} - 3 &= \max\left(\frac{h_0 h_1}{h_0 + h_1} \frac{\tilde{\theta}_1}{\varepsilon_1} - 3, 0\right), \\ \frac{\psi'(p_{N-1}, 1)}{\psi(p_{N-1}, 1)} - 3 &= \max\left(\frac{h_{N-2} h_{N-1}}{h_{N-2} + h_{N-1}} \frac{\tilde{\theta}_{N-1}}{\varepsilon_{N-1}} - 3, 0\right),\end{aligned}$$

where

$$\tilde{\theta}_1 = \varepsilon_2 h_1^{-1} + \frac{4}{3}|\delta_1 f| + \frac{|\Delta_0 f - f_0'| + \varepsilon_1 h_0^{-1}}{|1 - \psi'(p_0, 1)/\psi(p_0, 1)|},$$

$$\tilde{\theta}_{N-1} = \varepsilon_{N-2} h_{N-1}^{-1} + \frac{4}{3}|\delta_{N-1} f| + \frac{|\Delta_{N-1} f - f_N'| + \varepsilon_{N-1} h_{N-1}^{-1}}{|1 - \psi'(q_{N-1}, 1)/\psi(q_{N-1}, 1)|}.$$

Theorem 2. *If the conditions (15) are satisfied then the tension generalized spline S constructed by the three-point local approximation algorithm 2 will be a shape preserving function.*

Proof: By virtue of the conditions on the initial data (1) and (2), the estimates (17) and (18) imply the relations

$$\begin{aligned} \Delta_i b\, \Delta_i f &> 0, \quad i = 0, \ldots, N-1, \\ (\Delta_i b - \Delta_{i-1} b)\delta_i f &> 0, \quad i = 1, \ldots, N-1. \end{aligned} \tag{21}$$

Whence, according to (8c), $S''(x_i)\delta_i f > 0$, $i = 1, \ldots, N-1$. From (8c) and (16) we also have

$$S''(x_0) = (z_0')^{-1}(\Delta_0 b - \Delta_{-1} b) = -h_0^{-2} \frac{b_1 - f_0 - h_0 f_0')}{\psi(p_0, 1) - \psi'(p_0, 1)}.$$

If $\delta_1 f > 0$, then according to (15) $f_0' < \Delta_0 f - \varepsilon_1 h_0^{-1}$. Thus, taking into account (17) we obtain

$$b_1 - f_0 - h_0 f_0' > b_1 - f_0 - h_0 \Delta_0 f + \varepsilon_1 = b_1 - f_1 + \varepsilon_1.$$

As for tension generalized splines the estimate $\psi(q, 1) < \psi'(q, 1)$ is valid for all $q \geq 0$ then $S''(x_0) \geq 0$, that is, $S''(x_0)S''(x_1) \geq 0$. The same inequality applies in the case $\delta_1 f < 0$. The estimate $S''(x_{N-1})S''(x_N) \geq 0$ is established in the same way. Now applying Lemma 1, we find that conditions (iv) of Definition 1 for shape preserving functions are satisfied.

Since inequalities (21) are satisfied, the conditions of part (iii) of Definition 1 can be verified as in the corresponding proof of Theorem 1. This proves the theorem.

Remark 3. For $f(x) = 1$, $f(x) = x$, we find in both the one-point algorithm 1 and the three-point algorithms 2 by direct verification that, respectively, $b_i = 1$, $b_i = x_i$, $i = -1, \ldots, N+1$ and, therefore, according to (7), a shape preserving spline S recovers straight lines.

Remark 4. For $p_i = q_i = 0$ for all i the equations $z_i = 0$, $i = 1, \ldots, N-1$ are only possible for a uniform grid Δ. In that case, by (14) we obtain the well-known three-point scheme for local approximation by cubic splines (Zavyalov et al., 1980).

6. Shape Preserving Surface Approximation

Let the domain $G : [c, d] \times [0, 1]$ in the WU plane be divided into N rectangular subdomains by the straight lines $w = w_i$, $i = 0, \ldots, N$, of the grid $\Delta_w : c = w_0 < w_1 < \cdots < w_N = d$. Suppose that on each of the lines $w = w_i$ the grid $\Delta_u^i : 0 = u_0^i < u_1^i < \cdots < u_{M_i}^i = 1$, $i = 0, \ldots, N$, is given. The number of nodes and their position for the grids Δ_u^i, $i = 0, \ldots, N$, are independent of one another. At nodes u_j^i, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$, the values f_{ij} of a certain function f are given with allowable deviations ε_{ij}.

The algorithms of local spline approximation of Sections 3 and 4 can be generalized so that a surface of class $C^{2,2}(G)$ passing through the points $P_{ij} = (w_i, u_j^i, \tilde{f}_{ij})$, where $\tilde{f}_{ij} \in [f_{ij} - \varepsilon_{ij}, f_{ij} + \varepsilon_{ij}]$, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$, can be constructed. As well as being efficient, these algorithms preserve the shape of the data.

The surface is sought in the form of the function

$$S(w, u) = \sum_{i=-1}^{N+1} b_i(u) B_i(w),$$

where the GB-splines B_i are the same as in (6). The functions b_i, $i = -1, \ldots, N+1$, generalize the formulae of local approximation of Sections 3 and 4 (Algorithms 1 and 2), being linear combinations of one-dimensional shape preserving interpolating splines S_i, $i = 0, \ldots, N$ (Kvasov, 1996b) which fix the curves along sections $w = w_i$, $i = 0, \ldots, N$, and pass through the points (u_j^i, f_{ij}), $j = 0, \ldots, M_i$.

Formally, the required formulae (Algorithms 3 and 4) are obtained by replacing the quantities $f_j^{(k)}$ in Algorithms 1 and 2, respectively, by the functions $S_j^{(k)}$, $k = 0, 1, 2$. The boundary conditions are changed similarly. In the case of the "one-point" scheme we use the boundary conditions: $\frac{\partial}{\partial w} S(w_i, u) = \frac{\partial}{\partial w} f(w_i, u)$, $i = 0, N$. For the "three-point" scheme these boundary conditions must be supplemented by the conditions $S(w_i, u) = S_i(u)$, $i = 0, N$. Since the formulae for the functions b_i, $i = -1, \ldots, N+1$, are a direct generalization of the local approximation formulae of Sections 3 and 4, we will confine our analysis to a short description of the algorithms. We use the notation $g_i(u) = \frac{\partial}{\partial w} f(w_i, u)$, $i = 0, N$.

Algorithm 3. The one-point scheme:

$$\begin{aligned} b_{-1}(u) &= S_1(u) - 2h_0 g_0(u), \\ b_i(u) &= S_i(u), \quad i = 0, \ldots, N, \\ b_{N+1}(u) &= S_{N-1}(u) + 2h_{N-1} g_N(u). \end{aligned} \tag{22}$$

Algorithm 4. The three-point scheme:

$$\begin{aligned} b_{-1}(u) &= b_1(u) - 2h_0 g_0(u), \\ b_0(u) &= S_0(u) - \frac{S_0(u) + h_0 g_0(u) - b_1(u)}{1 - \psi'(p_0, 1)/\psi(p_0, 1)}, \end{aligned}$$

$$
\begin{aligned}
b_i(u) &= S_i(u) - 2(h_{i-1}+h_i)^{-1}\Phi_i(w_i)\delta_i S(u), \quad i = 1,\dots,N-1, \\
b_N(u) &= S_N(u) - \frac{S_N(u) - h_{N-1}g_N(u) - b_{N-1}(u)}{1-\psi'(q_{N-1},1)/\psi(q_{N-1},1)}, \qquad (23) \\
b_{N+1}(u) &= b_{N-1}(u) + 2h_{N-1}g_N(u),
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_i S(u) &= \Delta_i S(u) - \Delta_{i-1}S(u), \\
\Delta_j S(u) &= [S_{j+1}(u) - S_j(u)]/h_j, \quad j = i-1, i.
\end{aligned}
$$

The boundary conditions can be computed using second- and third-degree one-parameter Lagrange interpolating polynomials. Corresponding to the shape preserving constraints (10), (15) (Kvasov, 1996b), we set

$$
g_0(u) = \begin{cases} \frac{\partial}{\partial w}L_{2,0}(w_0,u) & \text{if } \frac{\partial}{\partial w}L_{2,0}(w_0,u)\Delta_0 S(u) \ge 0,\ \delta_1 S(u) \ne 0, \\ \frac{\partial}{\partial w}L_{3,0}(w_0,u) & \text{if } \frac{\partial}{\partial w}L_{3,0}(w_0,u)\Delta_0 S(u) \ge 0,\ \delta_1 S(u) = 0, \\ 0, & \text{otherwise,} \end{cases}
$$

$$
g_N(u) = \begin{cases} \frac{\partial}{\partial w}L_{2,N-2}(w_N,u) & \text{if } \frac{\partial}{\partial w}L_{2,N-2}(w_N,u)\Delta_{N-1} S(u) \ge 0, \\ & \qquad \delta_{N-1} S(u) \ne 0, \\ \frac{\partial}{\partial w}L_{3,N-3}(w_N,u) & \text{if } \frac{\partial}{\partial w}L_{3,N-3}(w_N,u)\Delta_{N-1} S(u) \ge 0, \\ & \qquad \delta_{N-1} S(u) = 0, \\ 0, & \text{otherwise,} \end{cases} \qquad (24)
$$

where

$$
\begin{aligned}
L_{2,i}(w,u) &= S_i(u) + (w-w_i)[\Delta_i S(u) + (w-w_{i+1})\delta_{i+1}S(u)/(w_{i+2}-w_i)], \\
L_{3,i}(w,u) &= [L_{2,i}(w,u)(w_{i+3}-w) + L_{2,i+1}(w,u)(w-w_i)]/(w_{i+3}-w_i).
\end{aligned}
$$

Instead of g_i, $i = 0, N$, we could consider the interpolating shape preserving splines (Kvasov, 1996b) constructed from given values of $\partial f(w_j, u_j^i)/\partial w$, $j = 0,\dots,M_i$, $i = 0, N$.

In practice, it is often necessary to adjust the assigned values of f_{ij} on an initial irregular grid to the nodes of a regular grid in domain G, that is, to points $(\tilde{w}_n, \tilde{u}_m)$, $m = 0,\dots,\tilde{M}$, $n = 0,\dots,\tilde{N}$. In that case it is sufficient to know the quantities $g_j(\tilde{u}_m)$, $m = 0,\dots,\tilde{M}$, $j = 0, N$, which can be found from formulae (24).

The shape preserving spline S possesses the following data shape preserving properties.

Property 1. Let the functions S_j, $j = i-1,\dots,i+2$, $1 \le i \le N-2$, be monotone and/or convex in the interval $[\tilde{u}_m, \tilde{u}_{m+1}]$. Then the generalized spline S constructed by Algorithm 3 for any fixed $\tilde{w} \in [w_i, w_{i+1}]$ will be monotone and/or convex on $[\tilde{u}_m, \tilde{u}_{m+1}]$.

Property 2. Let the functions S_j, $j = i-1,\dots,i+2$, $1 \le i \le N-1$, be monotone and/or convex in the interval $[\tilde{u}_m, \tilde{u}_{m+1}]$ and satisfy the conditions

$$
S_j^{(k)}\delta_j^{(k)} f < 0, \quad j \ne 0, N, \quad S_j^{(k)} g_j^{(k)} < 0, \quad j = 0, N,
$$

where, respectively, $k = 1$ and/or $k = 2$. Then the generalized spline S constructed by Algorithm 4 for any fixed $\tilde{w} \in [w_i, w_{i+1}]$, $2 \le i \le N-3$, will be monotone and/or convex on $[\tilde{u}_m, \tilde{u}_{m+1}]$.

These assertions can be proved by using the formulae

$$\frac{\partial^k}{\partial u^k} S(w,u) = \sum_{i=-1}^{N+1} b_i^{(k)}(u) B_i(w), \quad k = 1, 2,$$

employing expressions (22) and (23) for the coefficients b_i and taking into account the fact that GB-splines are finite: $B_i(w) > 0$ for $w \in (w_{i-2}, w_{i+2})$ and $B_i(w) = 0$ for $w \notin (w_{i-2}, w_{i+2})$.

Property 3. Suppose that for any $\tilde{S}_j(u)$ for which $\Delta_i \tilde{S}(u)$, $\delta_i \tilde{S}(u)$ do not change sign for any $u \in [0,1]$, the choice of parameters p_i, q_i, $i = -2, \ldots, N+2$, of the generalized spline S, gives

$$|\tilde{S}_j(u) - S_j(u)| \le E_j(u), \quad j = 0, \ldots, N,$$

where E_j are given functions. Then for any fixed u the spline $S_u(w) = S(w,u)$ is a shape preserving function.

The proof follows from the arguments for one-dimensional local approximation splines given above.

The values of the spline S are computed most efficiently, in the sense that the minimum number of arithmetic operations are performed, when the regular resultant grid mentioned above is used. In that case we first find the coefficients $b_i(\tilde{u}_m)$, $i = -1, \ldots, N+1$, and then the values $S(\tilde{w}_n, \tilde{u}_m)$, $n = 0, \ldots, \tilde{N}$, $m = 0, \ldots, \tilde{M}$ using the formulae for GB-splines.

A non-single-valued shape preserving surface, assigned pointwise in the form of a family of, generally speaking, curvilinear non-intersecting sections can be constructed by introducing the standard parametrization:

$$x = S^x(w,u), \quad y = S^y(w,u), \quad z = S^z(w,u). \tag{25}$$

In this case the initial points $T_{ij} = (x_{ij}, y_{ij}, z_{ij})$, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$, are assumed to belong to the parallelepiped $\prod_{ij} = \{\tilde{\chi}_{ij} | |\tilde{\chi}_{ij} - \chi_{ij}| \le \varepsilon_{ij}^{\chi}\}$, where for each of the coordinate functions (25) we have put $\chi_{ij} = \chi(w_i, u_j)$, and ε_{ij}^{χ} is the allowable error with respect to the corresponding variable. The resultant surface will be obtained as a triple of shape preserving splines constructed using the above algorithm.

The algorithms given here can be classed as Gordon type algorithms (Faux and Pratt, 1979; Gordon, 1969), with the essential difference, however, that instead of the functions b_i, $i = 0, \ldots, N$, being "blended" there with the help of fundamental splines, a local approximation of those functions is used and the surface is constructed in the space of shape preserving splines.

7. Numerical Examples

The approximating generalized splines were proved to be shape preserving under constraints (1) on the initial data which are required in order to have unique monotonicity and convexity conditions. In fact, the algorithms work well with more general data, as the examples below show. The reason is as follows. Using the algorithms we always satisfy the given tolerances. If the algorithms fail in the data monotonicity and/or convexity on some intervals then we have to increase the values of the corresponding tension parameters or to vary the tolerance parameters ε_i. This provides the properties of monotonicity and convexity for any data. However in the first case the resulting curve can be rather "angular". To avoid this situation it is often better to increase the values of the parameters ε_i which control the shape of the resulting curve and apply the algorithms repeatedly. This permits us to influence the "smoothness" of the resulting curve and yet remain within the given tolerances.

The use of tension generalized splines in the approximation of pointwise given curves and surfaces is illustrated in the figures. The defining functions were taken in the form (4) with

$$\psi(q_i, t) = Q_i t^3/[1 + q_i t(1-t)], \quad Q_i^{-1} = 2(1+q_i)(3+q_i)$$

which corresponds to rational splines with quadratic denominator. Other examples of defining functions for rational, exponential, hyperbolic splines and splines with additional knots can be found in (Kvasov, 1996a).

The more exact three-point formulae of local approximation (Algorithms 2 and 4) were used in each case. To find q_{i-1} we apply the formula (20) which takes the form

$$q_{i-1} = \max\Big(\frac{h_{i-1}h_i}{h_{i-1}+h_i}\frac{\theta_i}{\varepsilon_i} - 3, 0\Big).$$

Since we suppose $h_{i-1} \le h_i$, then the relation $z_i = 0$ gives us

$$p_i = -2 + \Big[1 + (1+q_{i-1})(3+q_{i-1})\Big(\frac{h_i}{h_{i-1}}\Big)\Big]^{1/2} \ge 0.$$

For comparison, the standard cubic spline interpolation of (Zavyalov et al., 1980) was used on the same data (in our case with $p_i = q_i = 0$ for all i). The derivatives at the endpoints were computed using second-degree Lagrange polynomials: $S'(x_0) = \mathbb{L}'_{2,0}(x_0)$, $S'(x_N) = \mathbb{L}'_{2,N-2}(x_N)$, and then corrected in accordance with the shape preserving conditions (15) (for surfaces, (23)). The tolerance from the initial data was 10%. The solid and dashed curves in the figures 1–8 denote the graphs of a rational spline S and a cubic spline S_3. The crosses indicate the initial points.

Figure 1a illustrates the approximation of the single pulse function $f(x) = \max(0, 1-4|x-1.75|)$ from points $x_i = 1+0.25i$, $i = 0,\ldots,6$. The cubic spline here is typified by the presence of oscillations. At the same time, the shape

(a) (b)

Fig. 1. Profiles of interpolation and shape preserving splines. (a) Approximation of a unit-pulse function; (b) data obtained by Späth (1969).

preserving spline is insensitive to these bursts. The "radius of curvature" of the corners can be influenced here by changing ε_i.

The data for Figs. 1b and 2 (Tables 1 and 2) are taken from (Späth, 1969, 1974). The cubic spline in Fig. 1b has superfluous points of inflection in the first, third, fourth and eighth intervals. The shape preserving spline does not have these oscillations. Figures 2 reflect the same general tendencies in the behaviour of the splines S_3 and S. By reducing ε_i where necessary the curve can be further "fitted" to the data (Fig. 2b), but becomes more "angular".

Table 1. Data for Figure 1b:

x_i	0.0	1.0	1.5	2.5	4.0	4.5	5.5	6.0	8.0	10.0
f_i	10.0	8.0	5.0	4.0	3.5	3.4	6.0	7.1	8.0	8.5

Table 2. Data for Figure 2:

x_i	0	2.0	2.5	3.5	5.5	6.0	7	8.5	10
f_i	2	2.5	4.5	5.0	4.5	1.5	1	0.5	0

The next test used a function with a discontinuous derivative obtained by joining intervals of a straight line and a semicircle: $f(x) = 1 + [1 - (x-4)^2]^{1/2}$ for $|x - 4| \leq 1$ and $f(x) = 1$ otherwise. From a geometrical point of view the curve of the interpolating cubic spline is invalid, whereas here the shape preserving spline gives a perfect profile (Fig. 3a).

The case of a quarter circle combined with a straight-line segment is considered in Fig. 3b. Here the curvature at the join is discontinuous. The vertical tangent at the left-hand boundary was approximated by the value $S'(a) = 50$. From the geometric point of view again the cubic interpolant is far from satisfactory, whereas even here the shape preserving spline gives no oscillations, automatically correcting the boundary conditions.

In many studies of shape preserving interpolation, tests are made using the data of Akima (1970):

(a) (b)

Fig. 2. Data obtained by Späth (1974). Variation of the shape preserving curve with decreasing tolerance parameters ε_i.

(a) (b)

Fig. 3. Joining of a part of the circle with line segments. (a) Semicircle; (b) one quadrant of a circle.

Table 3. Data for Figure 4a:

x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	56	60	85

The profiles of the splines S_3 and S obtained for this data are shown in Fig. 4a. On the "high gradient" interval, the tolerance from the initial data was increased to the maximum: $\varepsilon_7 = 10$, $\varepsilon_6 = \varepsilon_8 = \varepsilon_9 = 5$ with $\varepsilon_i = 1$ at all other points.

Figure 4b shows the results for the data taken from (Fritsch and Carlson, 1980): $\{x_i\} = \{7.99, 8.09, 8.19, 8.7, 9.2, 10, 12, 15, 20\}$, $\{f_i\} = \{0, 2.76429E{-}5$, $4.37498E - 2$, 0.169183, 0.469428, 0.943740, 0.998636, 0.999916, 0.999994$\}$. Here $\varepsilon_i = 0.1$ for all i.

(a) (b)

Fig. 4. Typical behaviour of interpolation and shape preserving splines, given fast- and slow-change sections of data. (a) Data obtained by Akima (1970); (b) data obtained by Fritsch and Carlson (1980).

As a numerical test of the two-dimensional algorithm 4 of shape preserving approximation, we tried to reconstruct the surface of a "Viking boat". The initial data, which the author obtained from Professor Tom Lyche of Oslo University, was defined pointwise in the form of the envelopes of the sides and the keel of the boat, as well as six ribs. Three-dimensional view of the data is given in Figure 5. After partial selection of the data, a system of non-intersecting, generally speaking curvilinear, pointwise assigned loft sections was constructed from this data. Each section, except the sections for ribs, contained 4 points.

Fig. 5. Three-dimensional view of the data.

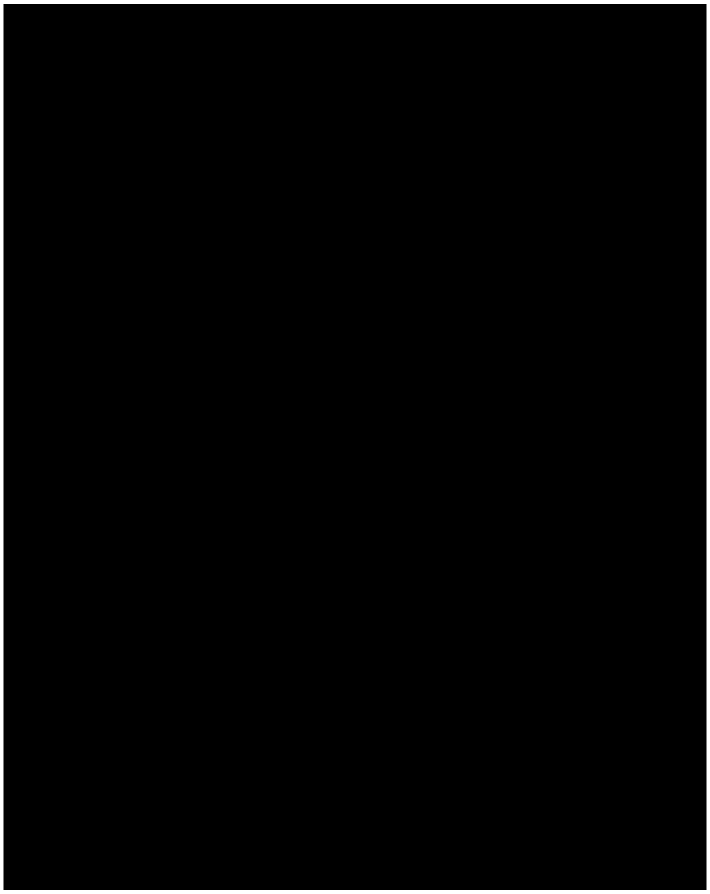

Fig. 6. Resulting shape preserving surface.

First, using the shape preserving interpolation algorithm of (Kvasov, 1996b) we construct a system of space curves along the selected sections. A two-dimensional spline is defined as the tensor product of one-dimensional splines, generating a family of generalized local approximation splines in the orthogonal direction by algorithm 4. This yields a finite system of curvilinear coordinate lines on the surface which form a regular grid. Properties of the initial data such as convexity, monotonicity, the presence of linear and plane segments, angles and non-smoothness are preserved along those lines.

The Euler coordinates of the multi-valued shape preserving surface were computed by the standard parametrization (25). In Figure 6 the resulting shape preserving surface is given with a mesh of lines 100×100.

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References

Akima, H. (1970), A new method of interpolation and smooth curve fitting based on local procedures, J. Assoc. Comput. Mech. 17, 589–602.

Beatson, R.K. and Chacko, E. (1989), A quantative comparison of end conditions for cubic spline interpolation, in: Chui, C.K., Schumaker, L.L. and Ward, J.D., eds., *Approximation Theory VI: Proceedings of the Sixth International Symposium on Approximation Theory*, Vol. I. Academic Press, Boston, 77–79.

Beatson, R.K. and Wolkowitz, H. (1989), Post-processing piecewise cubics for monotonicity, SIAM J. Numer. Anal. 26, 480–502.

de Boor, C. (1972), On calculating with B-splines, J. Approx. Th. 6, 50–62.

de Boor, C. (1978), *A Practical Guide to Splines*, Springer Verlag, New York.

Dyn, N. and Ron, A. (1988), Recurrence relations for Tchebicheffian B-splines, J. Analyse Math. 51, 118–138.

Faux, I.D. and Pratt, M.J. (1979), *Computational Geometry for Design and Manufacture*, Ellis Horwood, Chichester, UK.

Fritsch, F.N. and Carlson, R.E. (1980), Monotone piecewise cubic interpolation, SIAM J. Numer. Anal. 17, 238–246.

Gordon, W.J. (1969), Spline-blended surface interpolation through curve networks, J. of Math. and Mech. 18, 951–952.

Grebennikov, A.I. (1983), *Method of Splines and Solution of Ill-posed Problems of Approximation Theory*, Moscow State University Press, Moscow (in Russian).

Gregory, J.A. (1986), Shape preserving spline interpolation, Computer Aided Design 18, 53–57.

Koch, P.E. and Lyche, T. (1989), Exponential B-splines in tension, in: Chui C. K., Schumaker L.L., and Ward J.D., eds., *Approximation Theory VI: Proceedings of the Sixth International Symposium on Approximation Theory*, Vol. II. Academic Press, Boston, 361–364.

Koch, P.E. and Lyche, T. (1991), Construction of exponential tension B-splines of arbitrary order, In: Laurent, P.J., Le Méhaute, A., and Schumaker, L.L., eds., *Curves and Surfaces*, Academic Press, New York, 255–258.

Koch, P.E. and Lyche, T. (1993), Interpolation with Exponential B-splines in Tension, in: Farin G. et al., eds., *Geometric Modelling, Computing/Supplementum 8*, Springer-Verlag, Wien, 173–190.

Kvasov, B.I. (1996a), GB-splines and their properties, Annals of Numerical Mathematics 3, 139–149.

Kvasov, B.I. (1996b), Isogeometric interpolation by generalized splines, Rus. J. Numer. Anal. Math. Modelling 10, 223–246.

Kvasov, B.I. (1996c), Shape preserving spline approximation via local algorithms, in: Fontanella, F., Jetter, K., and Laurent, P. J., eds., *Advanced Topics in Multivariate Approximation*, World Scientific Publishing Co. Inc., Singapore, 181–196.

Kvasov, B.I. (1998), Shape Preserving C^2 Spline Interpolation, in: Tangmanee, S. and Schulz, E., eds., *Proceedings of The Second Asian Mathematical Conference 1995*, World Scientific Publishing Co. Pte. Ltd., Singapore, ISBN 981-02-3225-X, 430–438.

Lyche, T. (1985), A recurrence relation for Chebyshevian B-splines, Constr. Approx. 1, 155–173.

Lyche, T. and Schumaker, L.L. (1975), Local spline approximation methods, J. Aprox. Th. 15, 294–325.

Lyche, T. and Winther, R. (1979), A Stable Recurrence Relation for Trigonometric B-splines, J. Approx. Th. 25, 266–279.

Marušić, M. (1996), Stable calculation by splines in tension, submitted to Grazer Mathematische Berichte.

McCartin, B.J. (1990), Computation of exponential splines, SIAM J. Sci. Stat. Comput. 11, 242–262.

Miroshnichenko, V.L. (1984), Convex and monotone spline interpolation, in: *Constructive Theory of Functions'84*, Sofia, 610–620.

Moor, R.E. (1966), *Interval Analysis*, Prentice-Hall, Englewood Cliffs, New York.

Pruess, S. (1978), An algorithm for Computing Smoothing Splines in Tension, Computing 19, 365–373.

Sakai, M. and Lopes de Silanes, M.C. (1986), A simple rational spline and its application to monotone interpolation to monotonic data, Numer. Math. 50, 171–182.

Sapidis, N.S., Kaklis, P.D., and Loukakis, T.A. (1988), A Method for computing the tension parameters in convexity-preserving spline-in-tension interpolation, Numer. Math. 54, 179–192.

Schaback, R. (1990), Adaptive rational splines, Constructive Approximation 6, 167–179.

Schmidt, W.J. and Scholz, I. (1990), A dual algorithm for convex-concave data smoothing by cubic C^2-splines, Numer. Math. 57, 330–350.

Schumaker, L.L. (1981), *Spline functions: Basic theory*, John Wiley & Sons, New York.

Schumaker, L.L. (1982), On Recurrences for Generalized B-splines, J. Approx. Th. 36, 16–31.

Schumaker, L.L. (1983), On hyperbolic splines, J. Approx. Th. 38, 144–166.

Shokin, Yu.I. (1981), *Interval Analysis*, Nauka, Novosibirsk (in Russian).

Späth, H. (1969), Exponential spline interpolation, Computing 4, 225–233.

Späth, H. (1974), *Spline Algorithms for Curves and Surfaces*, Utilitas Mathematica Publishing, Inc., Winnipeg.

Zavyalov, Yu.S., Kvasov, B.I., and Miroshnichenko, V.L. (1980), *Methods of spline functions*. Nauka, Moscow (in Russian).