# Shape Preserving $C^2$ Spline Interpolation

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**Abstract.** In this paper we summarize the main results of [2] where an algorithm of shape preserving $C^2$ spline interpolation for arbitrary $1-D$ discrete data is developed. We consider a classification of such data to separate the sections of linearity, the angles and the breaks. For remaining data we give a local algorithm of $C^2$ interpolation by generalized splines with automatic choice of the parameters to retain the monotonicity and convexity properties of the data.

## §1. Introduction

It is well known that polynomial splines generally do not retain the geometric properties of the given data. To obtain the necessary solution many authors [1,3,4,5] introduce some parameters in the structure of the spline. Then they choose these parameters in such a way to satisfy the geometric constraints. The key idea here is to develop algorithms for automatic selection of these parameters.

This paper defines a class of functions $I(V)$ having shape properties determined by a given set of points $V = \{P_i = (x_i, f_i) \in \mathbb{R}^2 : x_0 < x_1 < \cdots < x_N\}$. Based on the definition, necessary and sufficient inequality conditions on $V$ are given in order that $I(V)$ be non-empty. A local algorithm for covex and monotone interpolation by $C^2$ generalized splines with automatic choice of the parameters is obtained. Its application enables us to give a complete solution to the shape preserving interpolation problem for $1-D$ data of arbitrary form, and to isolate the sections of linearity, the angles and the breaks.

## §2. The Class of Shape Preserving Interpolants

Let the sequence of points $V = \{P_i \,|\, i = 0, 1, \ldots, N\}$, $P_i = (x_i, f_i)$, on the plane $\mathbb{R}^2$ be fixed, where $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ forms a partition of the interval $[a, b]$. We introduce the notation for the first two devided differences $\Delta_i f = (f_{i+1} - f_i)/h_i$, $h_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, N-1$; $\delta_i f = \Delta_i f - \Delta_{i-1} f$, $i = 1, 2, \ldots, N-1$. As usual, we shall say that the initial data increases monotonically

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(decreases monotonically) on the subinterval $[x_n, x_k]$, $n > k$, if $\Delta_i f > 0$ ($\Delta_i f < 0$), $i = n, \ldots, k-1$. We say it is convex down (up) on $[x_n, x_k]$, $k > n+1$ if $\delta_i f > 0$ ($\delta_i f < 0$), $i = n, \ldots, k-2$.

We call the problem of searching for a sufficiently smooth function $S(x)$ such that $S(x_i) = f_i$, $i = 0, 1, \ldots, N$, and $S(x)$ preserves the form of the initial data, a *shape preserving interpolation problem*. It means that $S(x)$ should monotonically increase or decrease if the data has the same behaviour. Analogously, $S(x)$ should also be convex (concave) in data convexity (concavity) intervals.

Evidently the solution of the shape preserving interpolation problem is not unique. We formalize the class of functions in which we search for the solution.

**Definition 2.1.** *The set of functions $I(V)$ is called the class of shape preserving functions if for any $S(x) \in I(V)$ the following conditions are met:*
*(1)* $S(x) \in C^2[a, b]$;
*(2)* $S(x_i) = f_i$, $i = 0, 1, \ldots, N$;
*(3)* $S'(x)\Delta_i f \geq 0$ *if* $\Delta_i f \neq 0$ *and* $S'(x) = 0$ *if* $\Delta_i f = 0$ *for all* $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$; *and*
*(4)* $S''(x_i)\delta_i f \geq 0$, $i = 1, 2, \ldots, N-1$; $S''(x)\delta_j f \geq 0$, $x \in [x_i, x_{i+1}]$, $j = i, i+1$ *if* $\delta_i f \delta_{i+1} f \geq 0$; *$S(x)$ has no more than one inflection point* $\overline{x}$ *in the interval* $(x_i, x_{i+1})$ *if* $\delta_i f \delta_{i+1} f < 0$ *and also* $S''(x)\delta_i f \geq 0$ *for* $x \in [x_i, \overline{x}]$ *and the number of inflection points in the interval* $(x_{i-1}, x_{i+1})$ *does not exceed the number of sign changes in the sequence* $\delta_{i-1} f, \delta_i f, \delta_{i+1} f$.

**Remark.** When counting the number of sign changes in the sequence $\delta_{i-1} f, \delta_i f, \delta_{i+1} f$, the zeros are omitted.

The following propositions, characterizing the properties of shape preserving interpolants, are proved by using simple geometric considerations.

**Lemma 2.1.** *If* $\Delta_{i-1} f \Delta_i f \leq 0$, *then for the function* $S(x)$ *to be shape preserving, it is necessary that* $S'(x_i) = 0$.

**Lemma 2.2.** *If* $\delta_i f = 0$ *and* $\delta_{i-1} f \delta_{i+1} f \geq 0$, *then the unique shape preserving function on the interval* $[x_{i-1}, x_{i+1}]$ *is the straight line passing through the points* $P_{i-1}, P_i, P_{i+1}$.

**Corollary 2.1.** *If* $\delta_i f = \delta_{i+1} f = 0$, *then the unique shape preserving function in the interval* $[x_{i-1}, x_{i+1}]$ *is the straight line passing through the points* $P_j$, $j = i-1, i, i+1, i+2$.

**Lemma 2.3.** *If* $\delta_i f = 0$ *and* $\delta_{i-1} f \delta_{i+1} f < 0$, *then for* $S(x) \in I(V)$ *it is necessary that one of the following conditions be met:*
*(1)* $S'(x_i)\delta_{i-1} f > \Delta_i f \delta_{i-1} f$, $S''(x_i) = 0$;
*(2)* $S'(x) = \Delta_i f$, $S''(x) = 0$ *for all* $x \in [x_{i-1}, x_{i+1}]$.

**Lemma 2.4.** *Let* $\delta_i f \neq 0$ *and* $S''(x_i)S''(x) \geq 0$ *for all* $x \in [z_1, z_2]$, $z_1, z_2 \in [x_i, x_{i+1}]$. *Then for* $S(x) \in I(V)$ *it is necessary that one of the following conditions be met:*

*(1)* $S'(z_1) < \Delta_z S < S'(z_2)$ *for* $\delta_i f > 0$,
*(2)* $S'(z_1) > \Delta_z S > S'(z_2)$ *for* $\delta_i f < 0$,
*(3)* $S'(x) = \Delta_z S$, $S''(x) = 0$ *for all* $x \in [z_1, z_2]$,
*where* $\Delta_z S = (S(z_2) - S(z_1))/(z_2 - z_1)$.

Lemma 2.4 immediately implies

**Corollary 2.2.** *If* $\delta_i f \delta_{i+1} f > 0$ *and* $S'(x_j) \neq \Delta_i f$, $j = i, i+1$, *then for* $S(x) \in I(V)$ *it is necessary that the condition*

$$S'(x_i)\delta_i f < \Delta_i f \delta_i f < S'(x_{i+1})\delta_i f$$

*holds.*

**Corollary 2.3.** *If* $\delta_{i-1} f \delta_i f > 0$ *and* $\delta_i f \delta_{i+1} f > 0$, *then for* $S(x)$ *to be shape preserving it is necessary that the inequalities hold:*

$$\min(\Delta_{i-1} f, \Delta_i f) \leq S'(x_i) \leq \max(\Delta_{i-1} f, \Delta_i f).$$

**Lemma 2.5.** *If* $S'(x_i) = 0$, *then for* $S(x)$ *to be shape preserving it is necessary that* $S''(x_i)\Delta_i f \geq 0$, $S''(x_i)\Delta_{i-1} f \leq 0$.

**Theorem 2.1.** *For the existence of a shape preserving function it is necessary and sufficient that none of the following conditions hold:*
*(1)* $\Delta_{i-1} f \Delta_i f \leq 0$, $\Delta_{i-1} f \neq 0$, $\delta_{i-2} f \delta_i f \geq 0$, $\delta_{i-1} f = 0$, $i = 3, \ldots, N-1$,
*(2)* $\Delta_{i-1} f \Delta_i f \leq 0$, $\Delta_i f \neq 0$, $\delta_i f \delta_{i+2} f \geq 0$, $\delta_{i+1} f = 0$, $i = 1, \ldots, N-3$,
*(3)* $\delta_i f \neq 0$, $\delta_{i-1} f = \delta_{i+1} f = 0$, $\delta_i f \delta_k f \geq 0$, $k = i-2, i+2$, $i = 3, \ldots, N-3$.

Necessity of this assertion is proved directly by using Lemmas 2.1–2.5. The proof of the sufficiency consists in local constructing the shape preserving function $S(x)$ which interpolates arbitrary data and for which the conditions (1)–(3) of the Theorem 2.1 are not satisfied.

We define now the admissible values $S_i^{(r)} = S^{(r)}(x_i)$, $r = 1, 2$, in the knots of the mesh $\Delta$. The choice of these values should be subjected to the following constraints:

$$\begin{gathered}\min(\Delta_{i-1} f, \Delta_i f) < S'_i < \max(\Delta_{i-1} f, \Delta_i f) \quad \text{and} \quad \delta_i f S''_i \geq 0 \\ \text{if} \quad \delta_i f \neq 0, \quad 1 \leq i \leq N-1;\end{gathered} \tag{2.1}$$

$$\begin{gathered}(S'_i - \Delta_i f)\delta_{i-1} f > 0, \; S'_i \Delta_i f \geq 0, \; S''_i = 0 \\ \text{if} \quad \delta_i f = 0, \; \delta_{i-1} f \delta_{i+1} f < 0, \quad 2 \leq i \leq N-2;\end{gathered} \tag{2.2}$$

$$\begin{gathered}(S'_1 - \Delta_1 f)\delta_2 f < 0, \; S'_1 \Delta_1 f \geq 0, \; S''_1 = 0 \quad \text{if} \quad \delta_1 f = 0, \\ (S'_{N-1} - \Delta_{N-1} f)\delta_{N-2} f > 0, \; S'_{N-1}\Delta_{N-1} f \geq 0, \; S''_{N-1} = 0 \\ \text{if} \; \delta_{N-1} f = 0;\end{gathered} \tag{2.3}$$

$$
\begin{aligned}
&(\Delta_0 f - S_0')\delta_1 f > 0,\ S_0'\Delta_0 f \geq 0\,(\Delta_0 f \neq 0),\ S_0''\delta_1 f \geq 0 \ \text{ if } \ \delta_1 f \neq 0,\\
&(\Delta_0 f - S_0')\delta_2 f < 0,\ S_0'\Delta_0 f \geq 0\,(\Delta_0 f \neq 0),\ S_0''\delta_2 f \leq 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad \delta_1 f = 0,\ \delta_2 f \neq 0;
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
&(S_N' - \Delta_{N-1} f)\delta_{N-1} f > 0,\ S_N'\Delta_{N-1} f \geq 0\,(\Delta_{N-1} f \neq 0),\ S_N''\delta_{N-1} f \geq 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad \delta_{N-1} f \neq 0,\\
&(S_N' - \Delta_{N-1} f)\delta_{N-2} f < 0,\ S_N'\Delta_{N-1} f \geq 0\,(\Delta_{N-1} f \neq 0),\ S_N''\delta_{N-2} f \leq 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad \delta_{N-1} f = 0, \delta_{N-2} f \neq 0.
\end{aligned}
\tag{2.5}
$$

For the constructing of the shape preserving function $S(x)$ it is sufficient to eliminate from the consideration the intervals of the $S(x)$ linearity and to define $S(x)$ in arbitrary subinterval $[x_i, x_{i+1}]$ for the following possible configurations of the data:

(A) $\delta_i f \delta_{i+1} f > 0,\ \ 0 \leq i \leq N-1;$

(B) $\delta_i f = 0,\ \ \delta_{i-1} f \delta_{i+1} f < 0,\ 1 \leq i \leq N-1;$

(C) $\delta_i f \delta_{i+1} f < 0,\ \ 1 \leq i \leq N-2$

(if $i = 0, N$, then we formally set $\delta_i f = S_i''$).

By introducing on the straight line, joining the points $P_i$, $P_{i+1}$, an additional inflection point extending the mesh $\Delta$ the case (C) is reduced to the case (B). In cases (A) and (B) the problem of the shape preserving function construction can be reduced [2] to the solution in $[x_i, x_{i+1}]$ of the Hermite interpolation problem by the given values $S_j^{(r)}$, $r = 0, 1, 2;\ j = i, i+1$ with the function monotonicity and convexity requirement in this interval and additional restrictions

$$\min(S_i', S_{i+1}') < \Delta_i f < \max(S_i', S_{i+1}'). \tag{2.6}$$

$$\Delta_i f S_j' \geq 0, \quad j = i, i+1. \tag{2.7}$$

$$S_j''/(S_{i+1}' - S_i') \geq 0, \quad j = i, i+1. \tag{2.8}$$

According to the Definition 2.1 the following relations should be satisfied too:

$$S''(x)S''(x_j) \geq 0, \quad j = i, i+1; \quad x \in [x_i, x_{i+1}]. \tag{2.9}$$

## §3. The Solution of the Hermite Interpolation Problem with Constraints

The question of local construction of the shape preserving function $S(x)$ can be solved by using generalized cubic splines [4,5].

**Definition 3.1.** *Our generalized cubic spline on the mesh $\Delta$ will be a function $S(x) \in C^2[a,b]$ such that in any subinterval $[x_j, x_{j+1}]$ it has the form*

$$S(x) = [S_j - \varphi_j(0)h_j^2 S_j''](1-t) + [S_{j+1} - \psi_j(1)h_j^2 S_{j+1}'']t \\ +\varphi_j(t)h_j^2 S_j'' + \psi_j(t)h_j^2 S_{j+1}'', \tag{3.1}$$

*where $t = (x - x_j)/h_j$ and the functions $\varphi_j(t)$, $\psi_j(t)$ satisfy the conditions*

$$\varphi_j^{(r)}(1) = \psi_j^{(r)}(0) = 0, \quad r = 0,1,2; \quad \varphi_j''(0) = \psi_j''(1) = 1.$$

*We assume that $\varphi_j''(t)$, $\psi_j''(t)$ are continuous monotonic functions of the variable $t \in [0,1]$ values and*

$$\varphi_j(t) = \varphi(p_j, t), \quad \psi_j(t) = \varphi(q_j, 1-t), \quad p_j, q_j \geq 0. \tag{3.2}$$

To solve the Hermite interpolation problem with constraints on the interval $[x_i, x_{i+1}]$ let us define a function

$$S(x) = \begin{cases} S(x, x_i, x_{i1}) & \text{if} \quad x \in [x_i, x_{i1}]; \\ S(x, x_{i1}, x_{i+1}) & \text{if} \quad x \in [x_{i1}, x_{i+1}], \end{cases}$$

which has the form (3.1) on the intervals $[x_i, x_{i1}]$, $[x_{i1}, x_{i+1}]$ and satisfies the interpolation and smoothness conditions

$$S^{(r)}(x_j) = f_j^{(r)}, \quad S^{(r)}(x_{i1} - 0) = S^{(r)}(x_{i1} + 0), \quad r = 0,1,2; \quad j = i, i+1.$$

We assume that the inequalities (2.6)–(2.8) are fulfilled and according to (2.7) we have $S_i' S_{i+1}' \geq 0$.

Let us introduce the notations

$$h_{i1} = x_{i1} - x_i, \quad \mu_{i1} = 1 - \lambda_{i1} = h_{i1}/h_i, \quad \tau_i = \frac{S_{i+1}' - \Delta_i f}{S_{i+1}' - S_i'},$$

$$\alpha_i = \frac{S_{i1} - f_i}{h_{i1}}, \quad \beta_i = \frac{f_{i+1} - S_{i1}}{h_i - h_{i1}}, \quad \sigma_j = \frac{h_i S_j''}{S_{i+1}' - S_i'}, \quad j = i, i+1.$$

According to these notations and by the inequalities (2.6) we have

$$\Delta_i f = \tau_i S_i' + (1 - \tau_i) S_{i+1}', \quad 0 < \tau_i < 1. \tag{3.3}$$

Using the formula (3.1) we obtain

$$\alpha_i = \frac{1}{\psi_i'(1)} \left\{ h_{i1} T_i^{-1} S_i'' - [\psi_i(1) - \psi_i'(1)] S_i' + \psi_i(1) S_{i1}' \right\}, \\ \beta_i = \frac{1}{\varphi_{i1}'(0)} \left\{ (h_i - h_{i1}) T_{i1}^{-1} S_{i+1}'' - \varphi_{i1}(0) S_{i1}' + [\varphi_{i1}(0) + \varphi_{i1}'(0)] S_{i+1}' \right\}, \\ T_j^{-1} = [\varphi_j(0) + \varphi_j'(0)][\psi_j(1) - \psi_j'(1)] - \varphi_j(0)\psi_j(1), \quad j = i, i1. \tag{3.4}$$

By the continuity of the spline second derivative in the knot $x_{i1}$ we have the equation

$$S'_{i1} = \left[\frac{\lambda_{i1}}{\psi'_i(1)} - \frac{\mu_{i1}}{\varphi'_{i1}(0)}\right]^{-1}\left[\frac{\lambda_{i1}}{\psi'_i(1)}S'_i - \frac{\mu_{i1}}{\varphi'_{i1}(0)}S'_{i+1}\right.$$
$$\left. - \frac{\varphi'_i(0)}{\psi'_i(1)}\lambda_{i1}h_{i1}S''_i + \frac{\psi'_{i1}(1)}{\varphi'_{i1}(0)}\mu_{i1}(h_i - h_{i1})S''_{i+1}\right]. \tag{3.5}$$

Now taking into account the identity $\mu_{i1}\alpha_i + \lambda_{i1}\beta_i = \Delta_i f$ and substituting here the expressions for $\alpha_i$, $\beta_i$, $S'_{i1}$ from (3.4) and (3.5) we arrive at the equation with respect to $\mu_{i1}$

$$\Phi_i(\mu_{i1}) = A_i\mu_{i1}^3 + B_i\mu_{i1}^2 + C_i\mu_{i1} + D_i = 0. \tag{3.6}$$

If now to set $p_{i1} = q_i$ then according to (3.2) we have in (3.6) the coefficient $A_i = 0$ and to define $\mu_{i1}$ we obtain the quadratic equation

$$\Phi_i(\mu_{i1}) = \frac{1}{\psi'_i(1)}\left[\hat{B}_i\mu_{i1}^2 + \hat{C}_i\mu_{i1} + \hat{D}_i\right] = 0, \tag{3.7}$$

where

$$\begin{aligned}
\hat{B}_i =& [\psi_i(1)\varphi'_i(0) + T_i^{-1}]\sigma_i + [\psi_i(1)\psi'_{i1}(1) - T_{i1}^{-1}]\sigma_{i+1},\\
\hat{C}_i =& -\psi_i(1)\varphi'_i(0)\sigma_i + [-\psi_i(1)\psi_{i1}(1) + 2T_{i1}^{-1}]\sigma_{i+1} + 2\psi_i(1) - \psi'_i(1),\\
\hat{D}_i =& -T_{i1}^{-1}\sigma_{i+1} - \psi_i(1) + \tau_i\psi'_i(1).
\end{aligned}$$

Since

$$\begin{aligned}
\Phi_i(0) =& \tau_i - \frac{1}{\psi'_i(1)}\left[T_{i1}^{-1}\sigma_{i+1} + \psi_i(1)\right],\\
\Phi_i(1) =& -(1-\tau_i) + \frac{1}{\psi'_i(1)}\left[T_i^{-1}\sigma_i + \psi_i(1)\right],
\end{aligned}$$

we can find such $\overline{p}_i$, $\overline{q}_i$, $\overline{q}_{i1}$, that for all $p_i \geq \overline{p}_i$, $q_i \geq \overline{q}_i$, $q_{i1} \geq \overline{q}_{i1}$ according to (3.3) we have $\Phi_i(0) > 0$, $\Phi_i(1) < 0$. Thus the equation (3.7) has a unique root $\mu_{i1} \in (0,1)$.

As $p_{i1} = q_i$ we can rewrite the equation (3.5) in the form

$$S'_{i1} = S'_i + \mu_{i1}(S'_{i+1} - S'_i) - \lambda_{i1}\mu_{i1}h_i[\varphi'_i(0)S''_i + \psi'_{i1}(1)S''_{i+1}]. \tag{3.8}$$

Considering $f(x)$ as a sufficiently smooth function we assume that $S_j^{(r)} - f_j^{(r)} = O(h_i^{k+1-r})$, $r = 1,2$; $j = i, i+1$; $k = 2$ or $k = 3$. Then using (3.8) we obtain

$$\begin{aligned}
S'_{i1} - f'(x_{i1}) =& S'_i - f'_i - [\varphi'_i(0) + \psi'_{i1}(1)]\lambda_{i1}\mu_{i1}h_i f''_i\\
&+ h_{i1}(h_i - h_{i1})[1/2 - \psi'_{i1}(1)]f'''_i + O(h_i^k).
\end{aligned}$$

It implies that the approximation error order will increase for the derivative of the spline in the point $x_{i1}$ if according to (3.2) we set $q_{i1} = p_i$.

We consider now the question of the shape preserving properties for the generalized spline $S(x)$ in the interval $[x_i, x_{i+1}]$. The following criterion is valid.

**Theorem 3.1.** *By the fulfillment the restrictions*

$$\frac{\psi_i(1)}{\psi_i'(1)} - \varphi_i'(0)\sigma_i < 1 - \tau_i, \quad \frac{\psi_i(1)}{\psi_i'(1)} - \varphi_i'(0)\sigma_{i+1} < \tau_i, \tag{3.9}$$

*the unique shape preserving generalized cubic spline $S(x)$ exists solving the Hermite interpolation problem with restrictions (2.6)–(2.9).*

**Proof:** The conditions (2.6) – (2.8) are fulfilled by the construction. The requirement (2.9) means the absence on $[x_i, x_{i+1}]$ of inflection points for the shape preserving function $S_f(x)$. Let us show that for the spline $S(x)$ this condition will be fulfilled if the inequalities are valid

$$\begin{gathered}\min(\alpha_i, \beta_i) < S_{i1}' < \max(\alpha_i, \beta_i), \\ \min(S_i', \Delta_i f) < \alpha_i < \max(S_i', \Delta_i f), \\ \min(S_{i+1}', \Delta_i f) < \beta_i < \max(S_{i+1}', \Delta_i f).\end{gathered}$$

It is convenient to rewrite these inequalities in the form

$$\begin{aligned}\alpha_i(S_{i+1}' - S_i')^{-1} &< S_{i1}'(S_{i+1}' - S_i')^{-1} < \beta_i(S_{i+1}' - S_i')^{-1}, \\ S_i'(S_{i+1}' - S_i')^{-1} &< \alpha_i(S_{i+1}' - S_i')^{-1} < \Delta_i f(S_{i+1}' - S_i')^{-1}, \\ \Delta_i f(S_{i+1}' - S_i')^{-1} &< \beta_i(S_{i+1}' - S_i')^{-1} < S_{i+1}'(S_{i+1}' - S_i')^{-1}.\end{aligned} \tag{3.10}$$

From (3.4) and (3.8) we find

$$\begin{aligned}\alpha_i &= S_i' + \mu_{i1}(S_{i+1}' - S_i')\frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_i + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right], \\ \beta_i &= S_{i+1}' - \lambda_{i1}(S_{i+1}' - S_i')\frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1} - \mu_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right].\end{aligned} \tag{3.11}$$

It enables us to write the conditions (3.10) in the form

$$\begin{gathered}\frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_i + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right] < 1 + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i), \\ \frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1} - \mu_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right] < 1 - \mu_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i), \\ 0 < \mu_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_i + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right] < 1 - \tau_i, \\ 0 < \lambda_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1} - \mu_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right] < \tau_i.\end{gathered}$$

To fulfill these inequalities and the conditions $\Phi_i(0) > 0$, $\Phi_i(1) < 0$ it is sufficient to choose the parameters $p_i$, $q_i$ in such a way that the restrictions (3.9) are satisfied.

According to (3.1)

$$S_i' = \alpha_i + [\varphi_i(0) + \varphi_i'(0)]h_{i1}S_i'' - \psi_i(1)h_{i1}S_{i1}''.$$

Then by substituting here the expression for $\alpha_i$ from (3.11) we have

$$S_{i1}'' = \frac{S_{i+1}' - S_i'}{h_i\psi_i'(1)}\big[1 + \varphi_i'(0)(\mu_{i1}\sigma_i + \lambda_{i1}\sigma_{i+1})\big]. \tag{3.12}$$

If the inequalities (3.9) are fulfilled, the expression in square parentheses in (3.12) is positive and $S_{i1}''(S_{i+1}' - S_i') \geq 0$. As $S_j''(S_{i+1}' - S_i') \geq 0$, $j = i, i+1$, we conclude from here that $S_{i1}''S_j'' \geq 0, \quad j = i, i+1$.

From (3.1) on the interval $[x_i, x_{i1}]$ we have

$$S''(x) = S_i''\varphi_i''(t) + S_{i1}''\psi_i''(t).$$

Since $\varphi_i''(t), \psi_i''(t) \geq 0$ for $t \in [0,1]$, then

$$S''(x)S_j'' \geq 0, \quad j = i, i1 \quad \text{for} \quad x \in [x_i, x_{i1}].$$

We arrive at an analogous conclusion by considering the subinterval $[x_{i1}, x_{i+1}]$. As a result the function $S''(x)$ is convex in the interval $[x_i, x_{i+1}]$ and $S'(x)$ is monotone. Because of the assumption $S_i'S_{i+1}' \geq 0$ the function $S(x)$ has the monotonicity property. The theorem is proved.

The given construction completes the proof of the sufficiency conditions of Theorem 2.1 from the previous section.

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