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GROUP CLASSIFICATION OF THE BOLTZMANN EQUATION WITH A SOURCE FUNCTION

Amornrat Suriyawichitseranee

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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GROUP CLASSIFICATION OF THE BOLTZMANN EQUATION WITH A SOURCE FUNCTION

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

Thesis Examining Committee

____

(Asst. Prof. Dr. Eckart Schulz)

Chairperson

____

(Prof. Dr. Sergey Meleshko)

Member (Thesis Advisor)

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สมการโบลต์ซมันน์สามารถอธิบายพฤติกรรมเชิงสถิติของของไหลด้วยพจน์ของฟังก์ชันการแจกแจงระดับโมเลกุลได้ สำหรับความยากในการหาผลเฉลยของสมการนี้ เนื่องมาจากโครงสร้างที่ซับซ้อนในเชิงคณิตศาสตร์ของพจน์การชนของสมการดังกล่าวเป็นหลัก สำหรับการศึกษากรณีเฉพาะบางกรณีของสมการโบลต์ซมันน์ เอวี โบบีเลฝ สามารถลดความซับซ้อนของสมการโบลต์ซมันน์ชนิดไอโซทรอปิกและเป็นเอกพันธุ์เชิงปริภูมิ โดยการใช้การแปลงฟูเรียร์ และวิทยานิพนธ์ฉบับนี้ได้นำเสนอการศึกษาสมการที่ถูกทำให้ง่ายขึ้นนั้น ซึ่งอยู่ในรูปสมการอินทิโกร-ดิฟเฟอเรนเชียลที่มีฟังก์ชันแหล่งต้นทาง โดยวิธีการวิเคราะห์กรุป

ส่วนแรกของวิทยานิพนธ์นำเสนอการหาผลเฉลยของสมการโดยใช้แนวคิดของการแปลงฟังก์ชันก่อกำเนิดชนิดโมเมนต์ แม้ว่าสมการที่ได้ใหม่จากการแปลงดังกล่าวจะยังคงมีสมบัติไม่เฉพาะที่ แต่ก็เป็นสมการที่ง่ายกว่าในการหาผลเฉลย ขั้นตอนวิธีที่ใช้ในการหาผลเฉลยประกอบด้วยการจำแนกกรุปเทียบกับฟังก์ชันแหล่งต้นทาง ซึ่งผลที่ได้จะเป็นส่วนที่ช่วยเสริมงานวิจัยอื่นที่เคยมีการศึกษาสมการดังกล่าวก่อนหน้านี้แต่ยังไม่สมบูรณ์

ส่วนที่สองของวิทยานิพนธ์เป็นการประยุกต์ใช้การวิเคราะห์กรุปของสมการอินทิโกร-ดิฟเฟอเรนเชียลที่ได้จากการแปลงฟูเรียร์ของสมการโบลต์ซมันน์ชนิดไอโซทรอปิกและเป็นเอกพันธุ์เชิงปริภูมิที่มีฟังก์ชันแหล่งต้นทาง ในการศึกษานี้จะแปลงสัมประสิทธิ์ที่ปรากฏในสมการกำหนดให้อยู่ในรูปอนุกรมเทย์เลอร์ซึ่งทำให้สามารถหาผลเฉลยของสมการดังกล่าวได้ทั้งหมด วิทยานิพนธ์นี้ได้นำเสนอการจำแนกกรุปบริบูรณ์และผลเฉลยยืนยงทั้งหมดด้วย

จากการศึกษาพบว่าทั้งสองกลวิธีสามารถใช้ได้ดีในการแก้สมการและหาผลเฉลยยืนยงของสมการที่ได้จากการแปลงฟูเรียร์ของสมการโบลต์ซมันน์ชนิดไอโซทรอปิกและเป็นเอกพันธุ์เชิงปริภูมิ

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา____________________

ปีการศึกษา 2556 ลายมือชื่ออาจารย์ที่ปรึกษา____________________

AMORNRAT SURIYAWICHITSERANEE : GROUP CLASSIFICATION OF THE BOLTZMANN EQUATION WITH A SOURCE FUNCTION. THESIS ADVISOR : PROF. SERGEY V. MELESHKO, Ph.D. 71 PP.

BOLTZMANN EQUATION/ ADMITTED LIE GROUP/ GROUP CLASSIFICATION/ INVARIANT SOLUTION

The Boltzmann equation describes the statistical behavior of fluids in terms of a molecular distribution function. The difficulties for solving the equation are mainly due to the complex mathematical structure of its collision term. In the particular case of the spatially homogeneous and isotropic Boltzmann equation, A.V. Bobylev succeeded in reducing the equation to a simpler one by applying the Fourier transform. The purpose of this thesis is to study this simpler integro-differential equation by the group analysis method, even under the presence of a source function.

The first part of the thesis deals with solving the equation by the approach of a moment generating function. Although the equation governing the moment generating function is still nonlocal, it is simpler than the original equation. The algorithm applied in this thesis yields a complete group classification of the equation with respect to the source function, thus correcting the deficiencies of earlier studies.

The second part of the thesis is devoted to the group analysis of the integro-differential equation arising as the Fourier image of the studied equation with a source function. The coefficients in the determining equation are represented by the Taylor series and the determining equation is successfully solved. The complete group classification and all invariant solutions of the equations are presented in the

thesis.

Both techniques perform well for solving and finding invariant solutions for the Fourier image of the spatially homogeneous and isotropic Boltzmann equation with a source function.

School of Mathematics Student's Signature___________

Academic Year 2013 Advisor's Signature___________

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Amornrat Suriyawichitseranee

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CHAPTER I

INTRODUCTION

1.1 Background and History

Integro-differential equations, together with delay differential equations and stochastic differential equations, are equations with nonlocal operators. They have been studied for a long time in numerous scientific and engineering applications and in mathematics. Many well-known integro-differential equations are kinetic equations which form the bases in the kinetic theories of rarefied gases, plasma and radiation transfer. One of them is the Boltzmann kinetic equation in rarefied gas dynamics.

More than a hundred years ago, Ludwig Boltzmann proposed a nonlinear integro-differential equation which has come to be the fundamental equation in the kinetic theory of rarefied gases. The Boltzmann equation describes the statistical behaviour of fluid in terms of a molecular distribution function, and is of great interest in connection with initiation of high threshold processes by hot particles, rarefied gas flows over catalytic surfaces, disturbances of upper earth atmosphere by solar flares, among others.

The main difficulties to solving this equation are largely due to the complex mathematical structure of the collision term. The detailed form of this term depends on the precise nature of the intermolecular forces. However, the exact solution for general intermolecular forces and arbitrary initial conditions is not known. The solutions are known under only restrictive conditions, therefore it is of great importance to study simplified models, for which one can obtain special solutions or preferably the general solution for arbitrary initial conditions. In constructing model-Boltzmann equations, one looks for special intermolecular forces between the particles in the gas, such that the differential scattering cross-section or collision rate has a simple dependence on the energies of the colliding particles or on the scattering angle. The best known examples in this category are the so called Maxwell molecules. After the studies of the class of the Maxwell molecules, new classes of invariant solutions were constructed in the 1960s (Ernst, 1981).

The most important motivation came from finding an exact solution of the nonlinear Boltzmann equation for Maxwell molecules, found by Bobylev (1975) and independently by Krook and Wu (1976). This exact solution is referred to as the BKW-solution. This solution has been generalized in many directions. Another development was reviewed in Ernst (1981).

One of powerful methods for finding analytical solutions of differential equations is group analysis (Ovsiannikov, 1978). Group analysis was introduced by Sophus Lie in 1870. It has been applied for finding solutions of many types of ordinary differential equations and partially differential equations and it continues to be developed for nonlocal equations, e.g. integro-differential equations (Grigoriev and Meleshko, 1986).

Group analysis involves the study of symmetries of differential equations, with emphasis on finding solutions by using their symmetries. In case of ordinary differential equations, the existence of a symmetry can be used to reduce the order of an equation, and then the solutions are found from solving the reduced equation. For a given system of partial differential equations, group analysis may lead to an easier form of the system, or to special types of the system. Group analysis has been applied to delay differential equations (Tanthanuch and Meleshko, 2002) and integro-differential equations (Grigoriev and Meleshko, 1987).

1.2 Statement of the Problem

This thesis is devoted to group classification of invariant solutions of the Boltzmann equation in the spatially homogeneous and isotropic case with an arbitrary source term. Several approaches have been worked out during the study of invariant solutions of integro-differential equations, and an introduction to these approaches can be seen in Chapter II. One of these approaches is the transition to an equation for a moment generating function, which was first considered by Krook and Wu (1976), where the BKW solution was also obtained. Therefore, this thesis separats the group classification of the spatially homogeneous and isotropic Boltzmann equation with sources into two parts: firstly, the equation for a moment generating function is studied; and secondly, the spatially homogeneous and isotropic Boltzmann equation with an integral term is investigated. While, the spatially homogeneous and isotropic Boltzmann equation with sources was first studied by Nonnenmacher (1984), it was not taken into account that this equation is, in fact, a nonlocal partial differential equation (Grigoriev and Meleshko, 2012); this problem is corected in the thesis. The two studied equations are introduced in the next two subsections.

1.2.1 The Fourier Image of the Spatially Homogeneous and Isotropic Boltzmann Equation

The Fourier image of the spatially homogeneous and isotropic Boltzmann equation with sources has the form (Bobylev, 1975)

$$\varphi_t(y,t) + \varphi(y,t)\varphi(0,t) = \int_0^1 \varphi(ys,t)\varphi(y(1-s),t)\ ds + \hat{q}(y,t). \tag{1.1}$$

Here, the function $\varphi(y,t)$ is related with the Fourier transform $\tilde{\varphi}(k,t)$ of the isotropic distribution function $f(x,t)$ by the formula

$$\varphi(k^2/2,t) = \tilde{\varphi}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv)f(v,t)\,dv.$$

Similarly, the transform of the source function $q(v,t)$ is

$$Q(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv)q(v,t)\,dv,$$

and

$$Q(k,t) = \hat{q}(k^2/2,t).$$

The inverse Fourier transform of $\tilde{\varphi}(k,t)$ gives the distribution function

$$f(v,t) = \frac{4\pi}{v}\int_0^\infty k\sin(kv)\tilde{\varphi}(k,t)\,dk.$$

1.2.2 The Moment Generating Function of the Spatially Homogeneous and Isotropic Boltzmann Equation

Normalized moments of the distribution function f are introduced by the formula (Grigoriev and Meleshko, 2012)

$$M_n(t) = \frac{4\pi}{(2n+1)!!}\int_0^\infty f(v,t)v^{2n+2}dv, \qquad n = 0,1,2\dots. \tag{1.2}$$

Following (Bobylev, 1984), one can obtain a system of equations for the moments (1.2) from (1.1). It is sufficient to substitute the power series expansions

$$\varphi(y,t) = \sum_{n=0}^{\infty}(-1)^n M_n(t)\frac{y^n}{n!}, \qquad \hat{q}(y,t) = \sum_{n=0}^{\infty}(-1)^n q_n(t)\frac{y^n}{n!},$$

into (1.1), where

$$q_n(t) = \frac{4\pi}{(2n+1)!!}\int_0^\infty q(v,t)v^{2n+2}dv, \qquad n = 0,1,2\dots,$$

are the normalized moments of the source function. As a result, one derives the moment system

$$\frac{dM_n(t)}{dt} + M_n(t)M_0(t) = \frac{1}{n+1}\sum_{k=0}^{n} M_k(t)M_{n-k}(t) + q_n(t). \tag{1.3}$$

Let us define moment generating functions for the distribution function $f(v,t)$ and for the source function $q(v,t)$:

$$G(\omega,t) = \sum_{n=0}^{\infty} \omega^n M_n(t), \qquad S(\omega,t) = \sum_{n=0}^{\infty} \omega^n q_n(t).$$

Multiplying Equations (1.3) by ω^n, and summing over all n, one obtains the equation for $G(\omega,t)$

$$\frac{\partial^2(\omega G)}{\partial t \partial \omega} + M_0(t)\frac{\partial(\omega G)}{\partial \omega} = G^2 + \frac{\partial(\omega S)}{\partial \omega}. \tag{1.4}$$

Here, the obvious relations are used

$$\sum_{n=0}^{\infty}(n+1)\omega^n M_n(t) = \frac{\partial(\omega G)}{\partial \omega}, \quad \sum_{n=0}^{\infty}(n+1)\omega^n q_n(t) = \frac{\partial(\omega S)}{\partial \omega},$$

$$\sum_{n=0}^{\infty}\omega^n \sum_{k=0}^{n} M_k(t)M_{n-k}(t) = G^2.$$

In contrast to the case of homogeneous relaxation with $q(v,t) = 0$, the gas density $M_0(t) \equiv \varphi(0,t)$ is not constant. From Equation (1.3) for $n = 0$ one can obtain

$$M_0(t) = \int_0^t q_0(t')dt' + M_0(0).$$

Notice also that

$$M_0(t) = G(0,t). \tag{1.5}$$

This is the reason why Equation (1.4) has a nonlocal term. Nonenmacher (1984) has not taken this fact into account in the process of finding an admitted Lie group. Neglecting this condition can lead to incorrect admitted Lie groups.

Equation (1.4) is conveniently rewritten in the form

$$(xu_t)_x - u^2 + u(0)(xu)_x = g, \tag{1.6}$$

where $u(0) = u(0,t)$. Here, $\omega = x$, $G = u$ and $(\omega S)_\omega = g$.

In this thesis, the approach for integro-differential equation and algebraic method are used. This approach contains the following steps:

(1) Find the admitted Lie groups of Equation (1.1) and Equation (1.6). Notice that we use a different method to finding the admitted Lie groups of these equations, as compared with partial differential equation;

(2) Classify the admitted Lie algebras of Equation (1.1) and Equation (1.6);

This thesis is organized as follows. Chapter II introduces some background knowledge of Lie group analysis and application of group analysis for integro-differential equations. Chapter III presents the algorithm in finding an admitted Lie group of Equation (1.6), followed by its classification. The method for finding the admitted Lie group, the classification, and the invariant solutions of Equation (1.1) are given in Chapter IV. Lastly, the conclusion is presented in Chapter V.

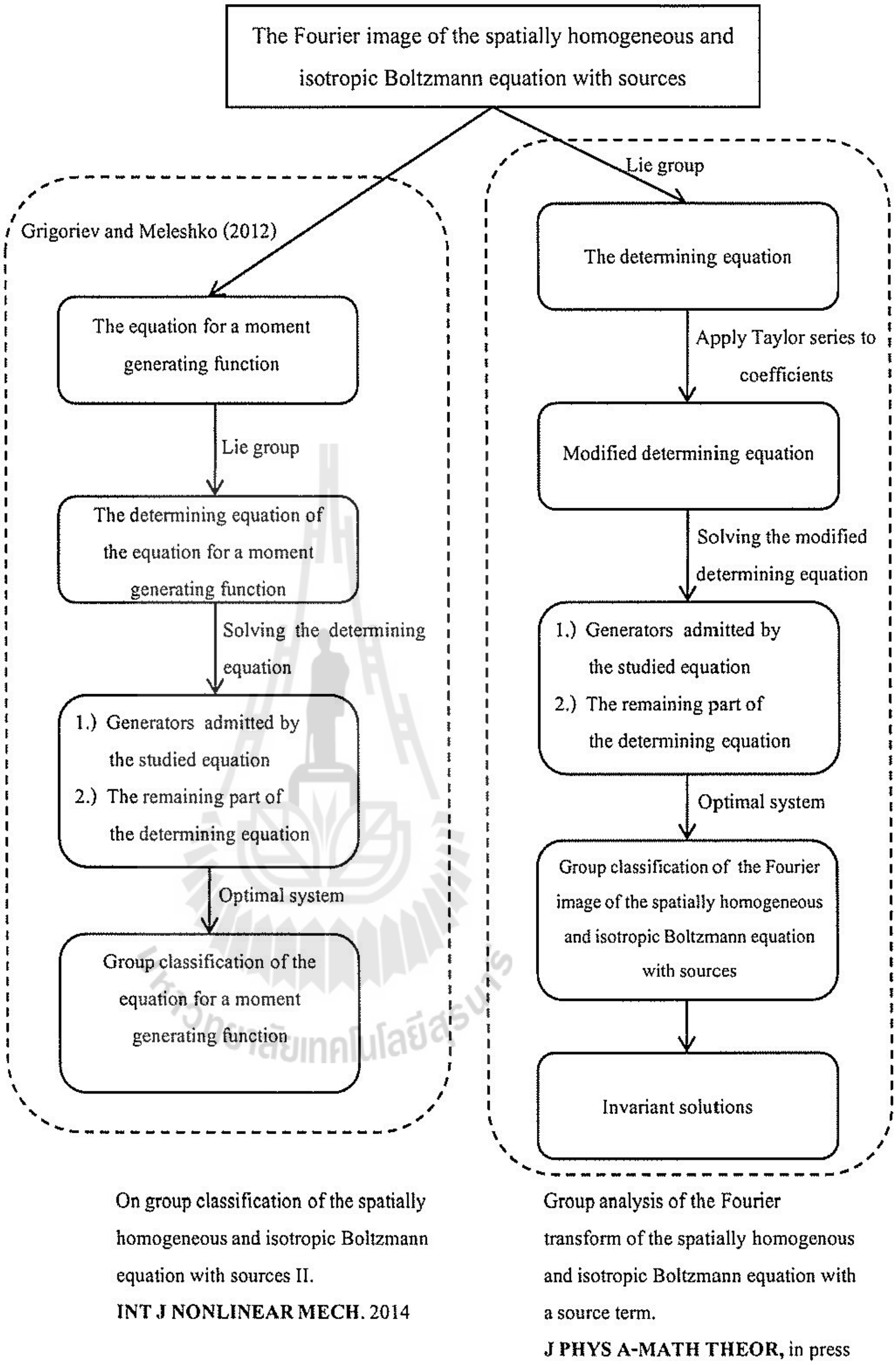

Figure 1.1 Work flow of this thesis.

CHAPTER II

GROUP ANALYSIS

Before the discussion of the main research in the next chapter, it is useful to review some basic concepts of group analysis. In 1870, a Norwegian mathematician, Sophus Lie, introduced the theory of continuous transformation groups which are now known as Lie groups. Lie group analysis is a successful method for integration of linear and nonlinear differential equations by using their symmetries. Later, these groups were applied to many type of differential equations. An introduction to this method can be found in textbooks (cf. Ovsiannikov (1978), Ibragimov (1999)). The collection of results by using this method is in the Handbooks of Lie Group analysis (1994), (1995), (1996). The application of group analysis for integro-differential equations was developed in Grigoriev and Meleshko (1986), (1987), Meleshko (2005) and Grigoriev, Ibragimov, Kovalev and Meleshko (2010).

In this chapter, we review some background of Lie group analysis such as a one-parameter Lie group, The Lie algebra of generator, the Lie-Bäcklund operator, an invariant solution and an application of group analysis to integro-differential equations. In the last section is group classification.

2.1 One-parameter Lie Group of Transformations

Let us consider invertible transformations

$$\bar{z}^i = g^i(z; a), \tag{2.1}$$

where $i = 1, 2, .., N$, $z \in V \subset \mathbb{R}^N$ and a is a parameter, $a \in \triangle$. The set V is an open set in $\mathbb{R}^N$ and $\triangle$ is a symmetric interval in $\mathbb{R}$ with respect to zero.

Definition 1. A set G of transformations (2.1) is called a local one-parameter Lie group if it has the following properties:

(1) $g^i(z; 0) = z$ for all $z \in V$.

(2) $g^i(g^i(z; a), b) = g^i(z; a + b)$ for all $a, b, a + b \in \Delta, z \in V$.

(3) If, for $a \in \Delta$, we have $g^i(z; a) = z$ for all $z \in V$, then $a = 0$.

(4) $g^i \in C^\infty(V, \Delta)$.

For differential equations, the variable z is separated into two parts, $z = (x, u) \in V \subset \mathbb{R}^n \times \mathbb{R}^m, N = n + m$. Here, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the independent variable, $u = (u^1, u^2, ..., u^m) \in \mathbb{R}^m$ is the dependent variable. The transformations (2.1) can be decomposed as

$$\begin{aligned} \bar{x}_i &= \varphi^i(x, u; a), \qquad i = 1, ..., n, \\ \bar{u}^j &= \psi^j(x, u; a), \qquad j = 1, ..., m. \end{aligned} \tag{2.2}$$

Here, these transformations are assumed to be sufficiently differentiable with respect to the variables x_i and u^j. The expansion of the functions φ^i, ψ^j into their Taylor series in a near $a = 0$ yields the infinitesimal transformation of group G (2.1):

$$\bar{x}_i \approx x_i + \xi^i(x, u)a, \qquad \bar{u}^j \approx u^j + \eta^j(x, u)a, \tag{2.3}$$

where

$$\xi^i(x, u) = \left.\frac{\partial \varphi^i(x, u; a)}{\partial a}\right|_{a=0}, \quad \eta^j(x, u) = \left.\frac{\partial \psi^j(x, u; a)}{\partial a}\right|_{a=0} \tag{2.4}$$

Let us consider the relation between equations and Lie groups.

Definition 2. A group of transformations, which transforms a solution $u_0(x)$ to the new solution $u_a(x)$ of the same equation is called an admitted Lie group of transformations.

Suppose the equation

$$F(x,u) = 0 \tag{2.5}$$

and a solution $u_0(x)$ are given. The transformed function $\bar{u}_a(\bar{x})$ is obtained in the following way. Substituting $u_0(x)$ into (2.2), the transformations become

$$\bar{x}_i = \varphi^i(x, u_0(x); a), \qquad \bar{u}^j = \psi^j(x, u_0(x); a). \tag{2.6}$$

Using the inverse function theorem for the first equation of (2.6), we can find $x = \theta(\bar{x}; a)$. The transformed function is

$$\bar{u}_a(\bar{x}) = \psi(\theta(\bar{x}; a), u_0(\theta(\bar{x}; a)); a). \tag{2.7}$$

After applying an admitted Lie group of transformations, we obtain

$$F(\bar{x}, \bar{u}_a(\bar{x})) = 0. \tag{2.8}$$

Differentiating (2.8) with respect to the group parameter a and substituting $a = 0$, we get

$$\left.\frac{\partial F}{\partial x_i}\right|_{a=0} \left(\left.\frac{\partial \varphi^i}{\partial a}\right|_{a=0} \right) + \left.\frac{\partial F}{\partial u^j}\right|_{a=0} \left(\left.\frac{\partial \psi^j}{\partial a}\right|_{a=0} \right) = 0. \tag{2.9}$$

Applying formula (2.4), Equation (2.9) becomes

$$\xi^i(x,u)\frac{\partial F}{\partial x_i}(x,u) + \eta^j(x,u)\frac{\partial F}{\partial u^j}(x,u) = 0. \tag{2.10}$$

The last equation can be expressed as an action of the infinitesimal generator

$$X = \xi^i(x,u)\partial_{x_i} + \eta^j(x,u)\partial_{u^j}. \tag{2.11}$$

X is called the generator of the Lie group of transformations or infinitesimal generator, and the functions ξ^i and η^j are called the coefficients of the generator. Here an index is repeated once, it is a dummy index indicating a summation with the index running though the integers 1, 2, ... and this notation is used in the next of the thesis.

2.1.1 Prolongation of Group Transformations

In order to apply the Lie group of transformations (2.2) to the study of a differential equation, one needs to know how this group acts on derivatives. In the following the action of this group on $u(x)$ is the same as seen in the the previous subsection. For the sake of simplicity, let us first consider the case of one independent variable and one dependent variable. The transformation of the first derivative can be found as follows. Let us differentiate (2.7) with respect to $\bar{x}$

$$\bar{u}_{\bar{x}} = \frac{\partial \bar{u}_a(\bar{x})}{\partial \bar{x}} = \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial \psi}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \theta}{\partial \bar{x}} = \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}\frac{\partial u_0}{\partial x}\right)\frac{\partial \theta}{\partial \bar{x}}. \tag{2.12}$$

Substituting $x = \theta(\bar{x}; a)$ into the first transformation of (2.6) implies the identity

$$\bar{x} = \varphi(\theta(\bar{x}; a), u_0(\theta(\bar{x}; a)); a). \tag{2.13}$$

Differentiating this identity with respect to $\bar{x}$, we obtain

$$\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x}\right)\frac{\partial \theta}{\partial \bar{x}} = 1.$$

Since

$$\frac{\partial \varphi}{\partial x}(\theta(\bar{x}; 0), u_0(\theta(\bar{x}; 0)); 0) = 1, \quad \frac{\partial \varphi}{\partial u}(\theta(\bar{x}; 0), u_0(\theta(\bar{x}; 0)); 0) = 0,$$

then $\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x} \neq 0$ in the neighborhood of $a = 0$. Hence,

$$\bar{u}_{\bar{x}} = \frac{\frac{\partial \psi}{\partial x}(x, u_0; a) + \frac{\partial \psi}{\partial u}(x, u_0; a)\frac{\partial u_0}{\partial x}(x)}{\frac{\partial \varphi}{\partial x}(x, u_0; a) + \frac{\partial \varphi}{\partial u}(x, u_0; a)\frac{\partial u_0}{\partial x}(x)} \equiv \omega(x, u_0(x), \frac{\partial u_0}{\partial x}(x); a). \tag{2.14}$$

This is the first prolongation of the transformations (2.2). Similarly, the function ω can be written by Taylor series expansion with respect to the parameter a in the neighborhood of the point $a = 0$:

$$\omega(x, u, u_x; a) \approx u_x + a\zeta,$$

where

$$\zeta = \left.\frac{\partial \omega}{\partial a}\right|_{a=0} = D_x\eta - u_x D_x\xi, \quad \xi = \left.\frac{\partial \varphi}{\partial a}\right|_{a=0}, \quad \eta = \left.\frac{\partial \psi}{\partial a}\right|_{a=0}$$

Here, the operator

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + ...$$

and $\frac{\partial u}{\partial x} = u_x$ are used. It is an operator of the total derivatives with respect to x. The first prolongation of the generator (2.11) is given by

$$\underset{1}{X} = X + \zeta\frac{\partial}{\partial u_x}$$

The second and higher prolongations can be found in the same way.

Now consider the case where the number of independent and dependent variables is greater than one. Given $Z = \mathbb{R}^n \times \mathbb{R}^m$, the space Z is prolonged by introducing the additional variables $p = (p^j_\alpha)$. Here, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index. For a multi-index the notations $|\alpha| \equiv \alpha_1 + \alpha_2 + ... + \alpha_n$ and $\alpha, i \equiv (\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, ..., \alpha_n)$ are used. The variable p^j_α plays the role of a derivative,

$$p^j_\alpha = \frac{\partial^{|\alpha|}u^j}{\partial x^\alpha} = \frac{\partial^{|\alpha|}u^j}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}} .$$

The space J^l of the variables:

$$x = (x_i), \quad u = (u^j), \quad p = (p^j_\alpha),$$

$$(i = 1, 2, ..., n;\ j = 1, 2, ..., m;\ |\alpha| \le l)$$

is called the l-th prolongation of the space Z. For convenience the notation that $J^0 \equiv Z$ is used.

Definition 3. The generator

$$\underset{l}{X} = X + \sum_{j,\alpha} \eta^j_\alpha \partial_{p^j_\alpha}, \quad (j = 1, ..., m,\ |\alpha| \le l),$$

with the coefficients:

$$\eta^j_{\tilde{\alpha},k} = D_k \eta^j_{\tilde{\alpha}} - \sum_i p^j_{\tilde{\alpha},i} D_k \xi^i, \quad (|\tilde{\alpha}| \le l-1), \tag{2.15}$$

is called the l-th prolongation of the generator X.

Here, the total derivative operators with respect to x_k has the form

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j,\alpha} p^j_{\alpha,k} \frac{\partial}{\partial p^j_\alpha}, \quad (k = 1, 2, ..., n),$$

and $\eta^j_0 = \eta^j$, where ξ^i, η^j are defined as in formulae (2.4).

The correspondence between a one-parameter group G and its infinitesimal generater is explained by the following theorem.

Theorem 1. Let a function $g(z; a)$ satisfy the group properties and have expansion

$$\bar{z}^i = g^i(z; a) \approx z^i + \xi^i(z)a, \quad i = 1, 2, ..., N, \tag{2.16}$$

where

$$\xi^i(z) = \left. \frac{\partial g^i(z; a)}{\partial a} \right|_{a=0}$$

Then it solves the Cauchy problem

$$\frac{d\bar{z}^i}{da} = \xi^i(\bar{z}^i), \qquad \bar{z}^i\big|_{a=0} = z^i, \quad i = 1, 2, ..., N. \tag{2.17}$$

Conversely, given $\xi^i(z)$, the solution of the Cauchy problem (2.17) satisfies the group properties.

Equation (2.17) are called Lie equations. Precisely, this theorem establishes a one-to-one correspondence between Lie groups of transformations and infinitesimal generators.

2.1.2 Invariant Manifolds

The relation between differential equations and Lie groups is presented in this subsection by following the idea of an admitted Lie group of transformations.

Consider a manifold or surface M which is defined by a system of partial differential equations:

$$F^k(x, u, p) = 0, \qquad (k = 1, 2, ..., s). \tag{2.18}$$

Hence,

$$M = \{(x, u, p)| \quad F^k(x, u, p) = 0, \quad (k = 1, 2, ..., s)\},$$

where x is the independent variable, u is the dependent variable, and p are partial derivatives of u with respect to x. The manifold M is assumed to be regular, i.e.

$$rank\left(\frac{\partial(F)}{\partial(u, p)}\right) = s.$$

Then the system (2.18) determines a manifold M. A manifold M is said to be invariant with respect to the group of transformations G as shown in (2.2), if every point of the manifold M is moved by G along this manifold M, i.e.

$$F^k(\bar{x}, \bar{u}, \bar{p}) = 0, \qquad (k = 1, 2, ..., s).$$

Accordingly, the Lie group of transformations as shown in (2.2) is admitted by system (2.18), in other words, system (2.18) is not changed under the Lie group of transformations.

In order to find the infinitesimal generator of the group admitted by a system (2.18) one can use the following theorem.

Theorem 2. The differential equations (2.18) admit the group G with the generator X in the form as in Definition 3 if and only if the following equations hold

$$\underset{l}{X} F^k(x, u, p)\Big|_M = 0, \quad (k = 1, 2, ..., s), \tag{2.19}$$

where the notation $|_M$ means evaluated on manifold M. Equations (2.19) are called determining equations.

The algorithm for finding a generator of a Lie group admitted by differential equations as shown in (2.18), consists of the following steps:

(1) Forming the admitted generator

$$X = \xi^i(x,u)\partial_{x_i} + \eta^j(x,u)\partial_{u^j}$$

with the unknown coefficient $\xi^i(x,u), \eta^j(x,u)$ are given.

(2) Constructing the prolonged operator $\underset{l}{X}$. The coefficients of the operator $\underset{l}{X}$ are defined by formula (2.15).

(3) Applying the prolonged operator $\underset{l}{X}$ to each equation of the system (2.18).

(4) Considering on the manifold M as in Theorem 2.

(5) Solving the determining equations. The determining equations are split with respect to the parametric derivatives into over-determined systems. The solutions of the determining equations give us the coefficients of a generator.

2.2 Lie Algebra of Generators

Let

$$X_i = \zeta_i^\alpha(z)\partial_{z_\alpha}, \quad (i = 1, 2) \tag{2.20}$$

be two infinitesimal generators.

Definition 4. The generator

$$X_3 = \zeta_3^\alpha(z)\partial_{z_\alpha}$$

with the coefficients

$$\zeta_3^\alpha = X_1(\zeta_2^\alpha) - X_2(\zeta_1^\alpha)$$

is called a commutator of the generators X_1 and X_2.

The commutator is denoted by

$$X_3 = [X_1, X_2].$$

A commutator satisfies the axioms:

(1) Bilinearity: $[\alpha X_1 + \beta X_2, X_3] = \alpha[X_1, X_3] + \beta[X_2, X_3]$,
$[X_1, \alpha X_2 + \beta X_3] = \alpha[X_1, X_2] + \beta[X_1, X_3]$,
where α and β are arbitrary constant,

(2) Antisymmetry: $[X_1, X_2] = -[X_2, X_1]$,

(3) Jacobi identity: $[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0$.

Definition 5. A Lie algebra is a vector space L with a commutator operation, which satisfies the properties of bilinearity, antisymmetry, Jacobi identity and acts on this space.

Take for example the space of generators: A vector space of generators L is a Lie algebra if the commutator $[X_\mu, X_\nu]$ of any two generators in L belongs to L.

Lemma 3. A commutator is invariant with respect to any change of variables.

Theorem 4. If the system admits generators X and Y, then it admits their commutator $[X, Y]$.

Definition 6. A vector subspace $L' \subset L$ of Lie algebra L is called a subalgebra if it is a Lie algebra.

In other words, for arbitrary vectors X_μ and X_ν from L', their commutator $[X_\mu, X_\nu]$ belongs to L'.

Definition 7. Let $I \subset L$ be a subspace of Lie algebra L with the property, $[X,Y] \in I, \quad \forall X \in I$ and $\forall Y \in L$. The subspace I is called an ideal.

Definition 8. Two Lie algebra of generators L' and L'' are similar if there exists a change of variable that transforms one into the other.

Hence, if Lie algebras L' and L'' are similar, then the generators $X = \zeta^\beta(z)\partial_{z_\beta} \in L'$ and $\hat{X} = \hat{\zeta}^\beta(\hat{z})\partial_{\hat{z}_\beta} \in L''$ of these algebras are related by the formula

$$\hat{\zeta}^\beta(\hat{z}) = X(q^\beta(z))\Big|_{z=q^{-1}(\hat{z})}.$$

A linear one-to-one map f of a Lie algebra L onto a Lie algebra K is called an isomorphism (algebra L and K are said to be isomorphic) if

$$f([X_\mu, X_\nu])_L = [f(X_\mu), f(X_\nu)]_K,$$

where the indice L and K are used to denote the commutator in the corresponding algebra. An isomorphism of L onto itself is termed an automorphism. Therefore the set of all subalgebras can be classified with respect to automorphisms.

Let a vector space L_r be given, and let $\{X_1, X_2, ..., X_r\}$ be its basis. Then L_r is a Lie algebra relative to a given commutator, if

$$[X_\mu, X_\nu] = c^\lambda_{\mu\nu} X_\lambda$$

with constant coefficients $c^\lambda_{\mu\nu}$ known as the structure constants.

Notice that two Lie algebras are isomorphic if they have the same structure constants in an appropriately chosen basis.

For a given Lie algebra L_r with basis $\{X_1, X_2, ..., X_r\}$, any $X \in L$ is written as

$$X = x_\mu X_\mu.$$

Hence, elements of L_r are represented by vectors $x = (x_1, ..., x_r)$. Let L_r^A be the Lie algebra spanned by the following operators,

$$E_\mu = c_{\mu\nu}^{\lambda} x_\nu \frac{\partial}{\partial x_\lambda}, \quad \mu = 1, ..., r,$$

with the commutator defined as in Definition 4. The algebra L_r^A generates the group G^A of linear transformations of $\{x_\mu\}$. These transformations determine automorphisms of the Lie algebra L_r known as inner automorphisms. This set is denoted by $\text{Int}(L_r)$. Accordingly, G^A is called the group of inner automorphisms of L_r, or the adjoint group of G. Two subalgebras L_s and L_q of L_r are called similar, if one can be transformed to another by an element of $\text{Int}(L_r)$. Similar subalgebras of the same dimension composes a class.

Definition 9. A set of representatives from all classes is called an optimal system of subalgebras.

Thus, an optimal system of subalgebras of a Lie algebra L with inner automorphisms $A = \text{Int}(L)$ is a set of subalgebras $\Theta_A(L)$ such that

(1) No two elements of this set can be transformed into each other by inner automorphisms of the Lie algebra L.

(2) Any subalgebra of the Lie algebra L can be transformed into one of the subalgebras of the set $\Theta_A(L)$

2.3 Lie-Bäcklund Operators

Consider operators of the form

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{2.21}$$

where ξ^i and η^α belong to the space of all infinitely differentiable functions, denoted by $\mathcal{A}$. Their prolongation to all derivatives is

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + ..., \tag{2.22}$$

where

$$\begin{aligned} \zeta_i^\alpha &= D_i\big(\eta^\alpha - \xi^j u_i^\alpha\big) + \xi^j u_{ji}^\alpha, \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}\big(\eta^\alpha - \xi^j u_i^\alpha\big) + \xi^j u_{j i_1 i_2}^\alpha, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{2.23}$$

Definition 10. An operator given by formula (2.22) and (2.23), where $\xi^i, \eta^\alpha \in \mathcal{A}$, is called a Lie-Bäcklund operator. The abbreviated operator (2.21) is also referred to as a Lie-Bäcklund operator.

In fact, the operator (2.22) is an infinite-order prolongation of operator (2.21).

It is an immediate consequence of the determining equation that every Lie-Bäcklund operator of the form

$$X_* = \xi^i D_i, \quad \xi^i \in \mathcal{A}, \tag{2.24}$$

is admitted by some differential equation, here D_i is the total derivative operator with respect to x_i. Moreover the set of all operators (2.24) is an ideal in the Lie algebra of all Lie-Bäcklund operators. Therefore, it is sufficient to consider Lie-Bäcklund operators with $\xi^i = 0$,

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \eta^\alpha \in \mathcal{A}, \tag{2.25}$$

which are called canonical Lie-Bäcklund operators. Notice that every operator (2.21) is transformed to the canonical form by the prescription

$$\bar{X} = X - \xi^i D_i.$$

2.4 Invariant Solutions

Theorem 2 is useful for finding the group admitted by a system of equations. When the functions $F^k(x,u,p)$ are known, the coefficients of a generator are determined from the determining equations. Then one may obtain the transformation group admitted by solving the Lie equations.

Conversely, in order to find an invariant system of equations for a given group G, it is convenient to use the following theorem on the representation of invariant equations via group invariants. Each one-parameter group G of transformations (2.16) has exactly $N-1$ functionally independent invariants. Any set of $N-1$ functionally independent invariants is called a basis of invariants for G. A basis of invariants for a group G with the generator

$$X = \xi^i(z)\frac{\partial}{\partial z^i}, \quad \xi^i(z) = \left.\frac{\partial g^i(z;a)}{\partial a}\right|_{a=0}$$

can be constructed by solving the characteristic system

$$\frac{dz^1}{\xi^1} = \frac{dz^2}{\xi^2} = \ldots = \frac{dz^N}{\xi^N}$$

Theorem 5. Let the system of equations,

$$F^k(z) = 0, \quad k = 1, \ldots, s, s < N, \tag{2.26}$$

admit the group G, and assume its tangent vector $\xi(z) = (\xi^1, \ldots, \xi^N)$ is not equal to zero on the manifold M determined by Equation (2.26). Then there exist functions Φ making it possible to rewrite Equation (2.26) in equivalent form as follows

$$\Phi_k(J_1(z), \ldots, J_{N-1}(z)) = 0, \quad k = 1, \ldots, s, \tag{2.27}$$

where $J_1(z), \ldots, J_{N-1}(z)$ is a basis of invariant of the group G. Equations (2.26) and (2.27) are equivalent in the sense that they determine the same manifold M.

If a solution is invariant, in the sense of every point of the manifold M being moved by G along this manifold under the action of some Lie group G, it is called a group invariant solution or an invariant solution.

2.5 Application of Group Analysis to Integro-Differential Equations

Group analysis is a powerful method for determining symmetries, and it has been applied with great success to ordinary differential equations and partial differential equations with local terms. In this thesis, the group analysis is method applied for studying equations with nonlocal terms. Although there exists an algorithm to find an admitted Lie group of differential equations as presented in section 2.1, a direct transference of the known scheme of the group analysis method to integro-differential equations is impossible. The main obstacle resides in the presence of nonlocal integral operators. Application of group analysis to integro-differential equations is presented in this section. Grigoriev and others (2010) presented several approaches that were discovered during a long history of studying invariant solutions of integro-differential equations. The main approaches can be classified as follows,

(1) Use of a presentation of a solution or an admitted Lie group of transformations on the basis of a priori simplified assumptions.

(2) Investigation of infinite systems of differential equations for power moments.

(3) Transformation of an original integro-differential equation into a differential equation.

(4) Direct derivation of a Lie group of transformations through corresponding

determining equations and construction of representations of invariant solutions of integro-differential equation.

This thesis uses approach (4) for studying the classification of the spatially homogeneous and isotropic Boltzmann equation with sources.

In applications of group analysis to equations with nonlocal operators, it is necessary to use successive steps as done in ordinary and partially differential equations. The first step involves constructing an admitted Lie group. The definition of an admitted Lie group given for differential equations cannot be applied to nonlocal equations. However, one can still define an admitted Lie group as a group solving a set of determining equations as outlined below. The main difficulty consists of solving the determining equations.

Let us consider a system of integro-differential equations:

$$\Phi(x, u) = 0. \tag{2.28}$$

Here, u is the vector of the dependent variables, and x is the vector of the independent variables. Assume that a one-parameter Lie group $G^1(x)$ of transformations

$$\bar{x} = \varphi(x, u; a), \quad \bar{u} = \psi(x, u; a) \tag{2.29}$$

with the generator

$$X = \xi^i(x, u)\partial_{x_i} + \eta^j(x, u)\partial_{u_j},$$

transforms a solution $u_0(x)$ of Equation (2.28) into the solution $u_a(x)$ of the same equation. The transformed function $u_a(x)$ is

$$u_a(\bar{x}) = \psi(x, u_0(x); a),$$

where $x = \theta(\bar{x}; a)$ is substituted into this expression. The function $\theta(\bar{x}; a)$ is found from the relation $\bar{x} = \varphi(x, u(x); a)$ using the inverse function theorem. Differentiating the equations $\Phi(x, u_a(x))$ with respect to the group parameter a and evaluating the result for the value $a = 0$, one obtains the equations

$$(\frac{\partial}{\partial a}\Phi(x, u_a(x)))_{|a=0} = 0. \tag{2.30}$$

These equations coincide with the equations

$$(\bar{X}\Phi)(x, u_0(x)) = 0 \tag{2.31}$$

obtained by the action of the canonical Lie-Bäcklund operator $\bar{X}$, which is equivalent to the generator X:

$$\bar{X} = \bar{\eta}^j \partial_{u^j},$$

where $\bar{\eta}^j = \eta^j(x, u) - \xi^i(x, u)p_i^j$. Equation (2.31) can be constructed without requiring the property that the Lie group should transform a solution into a solution. This allows the following definition.

Definition 11. A one-parameter Lie group G^1 of transformations (2.29) is a Lie group admitted by (2.28), if G^1 satisfies (2.31) for any solution $u_0(x)$ of (2.28). Equations (2.31) are called the determining equations.

Notice also that the determining equations of an integro-differential equation are integro-differential.

Grigoriev and others (2010) explain that the advantage of the above definition of an admitted Lie group provides a constructive method for obtaining the admitted Lie group. Another advantage of this definition is the possibility to apply it for seeking Lie-Bäcklund transformations, conditional symmetries and other types of symmetries for integro-differential equations. The main difficulty in obtaining an admitted Lie group consists of solving the determining equations. There are some methods for simplifying determining equations and as for partial differential equation, the main method is splitting. It should be noted that the splitting of integro-differential equations depends on the studied equations. Since the determining equations (2.31) have to be satisfied for any solution of the original equations, the arbitrariness of the solution of a Cauchy problem plays a key role in the process of solving the determining equations.

2.6 Group Classification

An equation which includes arbitrary elements, constants and functions of independent and dependent variables, they specify a process. The group classification is based on the enumeration of possible nonequivalent Lie algebras of operators admitted by the chosen type of equation. The first problem of group classification is constructing transformations which change arbitrary elements, while preserving the differential structure of the equations themselves. These transformations are called equivalence transformations. The group classification is regarded with respect to such transformations. In this thesis, the source function in the spatially homogeneous and isotropic Boltzmann equation is the chosen function for classification. There are considerable works on group classification of differential equations in Meleshko (2005), Grigoriev and others (2010) and the literature referenced therein.

In this thesis, the two-step algorithm of Ovsiannikov (1993) is used for classification of an admitted Lie group of the studied equations. This algorithm is useful for high-dimensional Lie algebras and it simplifies the problem of constructing an optimal system of subalgebras.

Let L be a Lie algebra with the basis $\{X_1, X_2, ..., X_r\}$. Assume that the Lie algebra L is decomposed as $L = I \bigoplus F$, where I is an ideal of the algebra L and F is a subalgebra. The set of inner automorphisms $A = \text{Int}(L)$ of the Lie algebra L is also decomposed as $A = A_I A_F$ where A_I and A_F are subsets of A which correspond to the elements of I and F respectively as follows:

Let $x \in L$ be decomposed as $x = x_I + x_F$, where $x_I \in I$ and $x_F \in F$. Any automorphism $C \in A$ can be written as $C = C_I C_F$, where $C_I \in A_I$ and $C_F \in A_F$. The automorphisms C_I and C_F have the properties,

$$C_I x_F = x_F, \quad \forall x_F \in F, \quad \forall C_I \in A_I,$$

$$C_F x_I \in I, \quad C_F x_F \in F, \quad \forall x_I \in I, \quad \forall x_F \in F, \quad \forall C_F \in A_F.$$

At the first step, an optimal system of subalgebras $\Theta_{A_F}(F) = \{F_1, F_2, ..., F_p, F_{p+1}\}$ of the algebra F is formed, here $F_{p+1} = \{0\}$ and the optimal system of subalgebras $\Theta_{A_F}(F)$ is constructed with respect to the automorphisms A_F. For each subalgebra F_j of F, $(j = 1, 2, .., p+1)$, one has to find its stabilizator $St(F_j) \subset A$ as

$$St(F_j) = \{C \in A | C(F_j) = F_j\}.$$

Note that $St(F_{p+1}) = A$.

The second step consists of forming optimal systems $\Theta_{St(F_j)}(I \bigoplus F_j), (j = 1, 2, .., p+1)$. The optimal system of subalgebras $\Theta_A(L)$ of the algebra L is a collection of $\Theta_{St(F_j)}(I \bigoplus F_j), (j = 1, 2, .., p+1)$. If the subalgebra F can also be decomposed, the two-step algorithm can be used when constructing $\Theta_{A_F}(F)$.

CHAPTER III

APPLYING GROUP ANALYSIS TO THE EQUATION FOR THE MOMENT GENERATING FUNCTION

In this chapter, the equation for the moment generating function of Equation (1.1), is studied. The equation is

$$(xu_t)_x - u^2 + u(0)(xu)_x = g, \tag{3.1}$$

where $u(0) = u(t,0)$. Note that Equation (3.1) is Equation (1.6) in Chapter I.

Because of the presence of the term $u(0)$, Equation (3.1) is not a partial differential equation. Therefore, the classical group analysis method cannot be applied to this equation. Instead the method developed for equations with nonlocal terms referred to in Chapter II can be used. In the next section, the latter method is applied for finding an admitted Lie group of Equation (3.1).

3.1 Admitted Lie Algebra of the Equation for the Moment Generating Function

Admitted generators are sought in the form

$$X = \tau(x,t,u)\partial_t + \xi(x,t,u)\partial_x + \zeta(x,t,u)\partial_u.$$

According to Definition 11, the determining equation for Equation (3.1) is

$$x\psi_{tx} + \psi_t + u(0)(x\psi_x + \psi) - 2\psi u + \psi(0)(xu)_x = 0, \tag{3.2}$$

where

$$\psi(x,t) = \zeta(x,t,u(x,t)) - u_t(x,t)\tau(x,t,u(t,x)) - u_x(x,t)\xi(x,t,u(x,t)),$$
$$\psi(0) = \psi(0,t).$$

After substituting the derivatives u_{tx}, u_{txx} and u_{ttx} found from Equation (3.1) and its derivatives with respect to x and t into (3.1), the determining equation becomes

$$\begin{aligned}
&\zeta_{tx}x^2 + \zeta_t x + \zeta_u g x + \zeta_u u^2 x + g\xi + u^2\xi - 2ux\zeta + ux\zeta(0) \\
&- x\left(g_t\tau + g_x\xi + g(\tau_t + \xi_x)\right) - \tau_t u^2 x - \xi_x u^2 x - xu_x(0)(u_x x + u)\xi(0) \\
&+ u(0)(\zeta_x x^2 - \zeta_u ux - u\xi + x\zeta + x\xi_x u + x\tau_t u) - \tau_x u_{tt}x^2 - x^2 u_x u_{tt}\tau_u \\
&- u_t u_{xx}\xi_u x^2 - u_{xx}\xi_t x^2 + u_t u_x x\big(\zeta_{uu}x - \tau_{tu}x + \tau_u u(0)x - \xi_{xu}x + \xi_u\big) \\
&+ u_t\big(\xi_x x + \zeta_{xu}x^2 - \xi - \tau_{tx}x^2 - 2\tau_u x(g + u^2) + u(0)x(2\tau_u u - x\tau_x)\big) \\
&+ u_t^2 x(\tau_u - x\tau_{xu}) + xu_t(0)(\tau - \tau(0))(u_x x + u) + u_x^2 x^2(\xi_u u(0) - \xi_{tu}) \\
&+ xu_x\big(x(\tau_t u(0) + \zeta(0) + \zeta_{tu}) - \xi_{tx}x - \xi_t - 2\xi_u g - 2\xi_u u^2 + 2\xi_u uu(0)\big) \\
&- u_t^2 u_x \tau_{uu} x^2 - u_t u_x^2 \xi_{uu} x^2 = 0.
\end{aligned} \tag{3.3}$$

Here,

$$\tau(0) = \tau(0,t,u(0,t)), \quad \xi(0) = \xi(0,t,u(0,t)), \quad \zeta(0) = \zeta(0,t,u(0,t)),$$
$$u_t(0) = u_t(0,t), \quad u_x(0) = u_x(0,t).$$

Differentiating the determining equation (3.3) with respect to u_{tt}, u_{xx}, and then with respect to u_t and u_x, $\tau_u = 0$, $\tau_x = 0$, $\xi_u = 0$, $\xi_t = 0$ are found. Therefore,

$$\tau = \tau(t), \quad \xi = \xi(x),$$

and hence $\tau(0) = \tau(t)$. Differentiating the determining equation with respect to u_t, and then u_x, we find $\zeta_{uu} = 0$, i.e.,

$$\zeta(x,t,u) = u\zeta_1(x,t) + \zeta_0(x,t).$$

The coefficient with $u_x u_x(0)$ in the determining equation (3.3) gives $\xi(0) = 0$. Continuing with splitting the determining equation (3.3) with respect to u_t and then with respect to u_x, one finds

$$\zeta_1(x,t) = -x^{-1}\xi(x) + \zeta_{10}(t).$$

Hence, $\zeta(0) = \zeta(0,t) = u(0)(\zeta_{10}(t) - \xi'(0)) + \zeta_0(0,t)$. The coefficient with $u_x u(0)$ leads to the condition

$$\zeta_{10} = -\tau' + \xi'(0).$$

Differentiating the determining equation with respect to u twice, we has

$$\xi_x = 2\frac{\xi}{x} - \xi'(0).$$

The general solution of this equation is

$$\xi = x(c_1 x + c_0).$$

Equating the coefficient with u_x to zero, one derives $\tau_{tt}(t) = \zeta_0(0,t)$. The coefficient with $u(0)$ in the determining equation (3.3) gives $x\zeta_{0x} + \zeta_0 = 0$. This equation has a unique solution which is nonsingular at $x = 0$,

$$\zeta_0(x,t) = 0.$$

Therefore,

$$\tau = c_2 t + c_3$$

and

$$\zeta = u(-c_1 x - c_2).$$

The remaining part of the determining equation (3.3) becomes

$$g_t(c_2 t + c_3) + x g_x(c_1 x + c_0) = -2g(c_1 x + c_2). \qquad (3.4)$$

Thus, each admitted generator has the form

$$X = c_0X_0 + c_1X_1 + c_2X_2 + c_3X_3,$$

where

$$X_0 = x\partial_x, \quad X_1 = x(x\partial_x - u\partial_u), \quad X_2 = t\partial_t - u\partial_u, \quad X_3 = \partial_t. \tag{3.5}$$

The values of the constants c_0, c_1, c_2 and c_3 and relations between them depend on the function $g(t, x)$.

The trivial case of the function

$$g = 0$$

satisfies Equation (3.4), and corresponds to the case of the spatially homogeneous and isotropic Boltzmann equation without a source term. In this case, the complete group classification of the Boltzmann equation was carried out in Grigoriev and Meleshko (1986), using its Fourier image (1.1) with $\hat{q}(y, t) = 0$. The four–dimensional Lie algebra $L^4 = \{Y_1,\ Y_2,\ Y_3,\ Y_4\}$ spanned by the generators

$$Y_0 = y\partial_y, \quad Y_1 = y\varphi\partial_\varphi, \quad Y_2 = t\partial_t - \varphi\partial_\varphi, \quad Y_3 = \partial_t \tag{3.6}$$

defines the complete admitted Lie group G^4 of (1.1). There are direct relations between the generators (3.5) and (3.6).

Indeed, since the functions $\varphi(y, t)$ and $u(x, t)$ are related through the moments $M_n(t)$, $n = 0, 1, 2, \ldots$, it is sufficient to check that the transformations of moments defined through these functions coincide.

Consider the transformations corresponding to the generators Y_0 and X_0,

$$Y_0 = y\partial_y: \quad \bar{t} = t, \quad \bar{y} = ye^a, \quad \bar{\varphi} = \varphi;$$

$$X_0 = x\partial_x: \quad \bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u.$$

The transformed functions are

$$\bar{\varphi}(\bar{y},\bar{t}) = \varphi(\bar{y}e^{-a},\bar{t}), \quad \bar{u}(\bar{x},\bar{t}) = u(\bar{x}e^{-a},\bar{t}).$$

The transformations of moments are, respectively:

$$\begin{aligned}\bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y},\bar{t})}{\partial \bar{y}^n}_{|\bar{y}=0} = (-1)^n \frac{\partial^n \varphi(\bar{y}e^{-a},\bar{t})}{\partial \bar{y}^n}_{|\bar{y}=0} \\ &= (-1)^n e^{-na}\frac{\partial^n \varphi}{\partial y^n}(0,\bar{t}) = e^{-na}M_n(\bar{t}); \\ \bar{M}_n(\bar{t}) &= \frac{1}{n!}\frac{\partial^n u(\bar{x}e^{-a},\bar{t})}{\partial \bar{x}^n}_{|\bar{x}=0} = e^{-na}\frac{1}{n!}\frac{\partial^n u}{\partial x^n}(0,\bar{t}) = e^{-na}M_n(\bar{t}).\end{aligned}$$

Hence, one can see that the transformations of moments defined through the functions $\varphi(y,t)$ and $u(x,t)$ coincide.

The transformations corresponding to the generators Y_1 and X_1,

$$\begin{aligned}&Y_1 = y\varphi\partial_\varphi: \quad && \bar{t} = t, \quad \bar{y} = y, \quad \bar{\varphi} = \varphi e^{ya}; \\ &X_1 = x(x\partial_x - u\partial_u): \quad && \bar{t} = t, \quad \bar{x} = \frac{x}{1-ax}, \quad \bar{u} = (1-ax)u,\end{aligned}$$

act on the functions $\varphi(y,t)$ and $u(x,t)$ and their moments in the following way:

$$\begin{aligned}&\bar{\varphi}(\bar{y},\bar{t}) = e^{\bar{y}a}\varphi(\bar{y},\bar{t}), \quad \bar{u}(\bar{x},\bar{t}) = \frac{1}{1+a\bar{x}}u\left(\bar{t},\frac{\bar{x}}{1+a\bar{x}}\right), \\ &\bar{M}_n(\bar{t}) = (-1)^n\frac{\partial^n\bar{\varphi}(\bar{y},\bar{t})}{\partial\bar{y}^n}_{|\bar{y}=0} = (-1)^n\frac{\partial^n(e^{\bar{y}a}\varphi(\bar{y},\bar{t}))}{\partial\bar{y}^n}_{|\bar{y}=0} \\ &\qquad = (-1)^n\left(\left(\frac{\partial}{\partial y}+a\right)^n\varphi\right)(0,\bar{t}); \\ &\bar{M}_n(\bar{t}) = \frac{1}{n!}\frac{\partial^n\bar{u}(\bar{x},\bar{t})}{\partial\bar{x}^n}_{|\bar{x}=0} = \frac{1}{n!}\frac{\partial^n}{\partial\bar{x}^n}\left(\frac{1}{1+a\bar{x}}u\left(\bar{t},\frac{\bar{x}}{1+a\bar{x}}\right)\right)_{|\bar{x}=0},\end{aligned}$$

respectively. Using computer symbolic calculations with REDUCE one can check that these transformations of moments also coincide.

The vector fields Y_2 and X_2 generate the following transformations:

$$\begin{aligned}&Y_2 = t\partial_t - \varphi\partial_\varphi: \quad \bar{t} = te^a, \quad \bar{y} = y, \quad \bar{\varphi} = \varphi e^{-a}; \\ &X_2 = t\partial_t - u\partial_u: \quad \bar{t} = te^a, \quad \bar{x} = x, \quad \bar{u} = ue^{-a},\end{aligned}$$

which map the functions $\varphi(y,t)$ and $u(x,t)$ to the functions $\bar{\varphi}(\bar{y},\bar{t}) = e^{-a}\varphi(\bar{y},\bar{t}e^{-a})$ and $\bar{u}(\bar{x},\bar{t}) = e^{-a}u(\bar{x},\bar{t}e^{-a})$, respectively. The transformations of moments are

$$\begin{aligned}\bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y},\bar{t})}{\partial \bar{y}^n}_{|\bar{y}=0} = (-1)^n e^{-a} \frac{\partial^n \varphi(\bar{y},\bar{t}e^{-a})}{\partial \bar{y}^n}_{|\bar{y}=0} \\ &= (-1)^n e^{-a} \frac{\partial^n \varphi}{\partial y^n}(0,\bar{t}e^{-a}) = M_n(\bar{t}e^{-a})e^{-a}; \\ \bar{M}_n(\bar{t}) &= \frac{1}{n!} \frac{\partial^n \bar{u}(\bar{x},\bar{t})}{\partial \bar{x}^n}_{|\bar{x}=0} = \frac{1}{n!} e^{-a} \frac{\partial^n u(\bar{x},\bar{t}e^{-a})}{\partial \bar{x}^n}_{|\bar{x}=0} \\ &= \frac{1}{n!} e^{-a} \frac{\partial^n u}{\partial x^n}(0,\bar{t}e^{-a}) = M_n(\bar{t}e^{-a})e^{-a}.\end{aligned}$$

The case where the transformations of moments corresponding to the generators $Y_3 = \partial_t$ and $X_3 = \partial_t$ coincide is trivial. These direct relations between the Lie algebras confirm correctness of our calculations.

3.2 Equivalence Transformations of the Equation for the Moment Generating Function

For the group classification, one needs to know equivalence transformations. Let us find some of them using the generators (3.5) and considering their transformations of the left hand side of Equation (3.1)

$$Lu = xu_{tx} + u_t - u^2 + u(0)(xu_x + u). \tag{3.7}$$

The transformations corresponding to the generator $X_0 = x\partial_x$ map a function $u(x,t)$ into the function $\bar{u}(\bar{x},\bar{t}) = u(\bar{x}e^{-a},\bar{t})$, where a is the group parameter.

The transformed expression of (3.7) becomes

$$\begin{aligned}
\bar{L}\bar{u} &= \bar{x}\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{t}}(\bar{x},\bar{t}) + \frac{\partial\bar{u}}{\partial\bar{t}}(\bar{x},\bar{t}) - \bar{u}^2(\bar{x},\bar{t}) + \bar{u}(0,\bar{t})\big(\bar{x}\frac{\partial\bar{u}}{\partial\bar{x}}(\bar{x},\bar{t}) + \bar{u}(\bar{x},\bar{t})\big) \\
&= \bar{x}\frac{\partial^2 u}{\partial\bar{x}\partial\bar{t}}(\bar{x}e^{-a},\bar{t}) + \frac{\partial u}{\partial\bar{t}}(\bar{x}e^{-a},\bar{t}) - u^2(\bar{x}e^{-a},\bar{t}) \\
&\quad + u(0,\bar{t})\big(\bar{x}\frac{\partial u}{\partial\bar{x}}(\bar{x}e^{-a},\bar{t}) + u(\bar{x}e^{-a},\bar{t})\big) \\
&= \bar{x}e^{-a}\frac{\partial^2 u}{\partial x\partial t}(x,t) + \frac{\partial u}{\partial t}(x,t) - u^2(x,t) + u(0,t)\big(\bar{x}e^{-a}\frac{\partial u}{\partial x}(x,t) + u(x,t)\big) \\
&= x\frac{\partial^2 u}{\partial x\partial t}(x,t) + \frac{\partial u}{\partial t}(x,t) - u^2(x,t) + u(0,t)\big(x\frac{\partial u}{\partial x}(x,t) + u(x,t)\big) \\
&= Lu.
\end{aligned}$$

This defines the Lie group of equivalence transformations of Equation (3.1)

$$\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u, \quad \bar{g} = g. \tag{3.8}$$

Similarly, one derives that the transformations corresponding to the generator $X_3 = \partial_t$ define the equivalence Lie group:

$$\bar{t} = t + a, \quad \bar{x} = x, \quad \bar{u} = u, \quad \bar{g} = g. \tag{3.9}$$

The transformations corresponding to the generator $X_2 = t\partial_t - u\partial_u$ map a function $u(x,t)$ into the function $\bar{u}(\bar{x},\bar{t}) = e^{-a}u(\bar{x},\bar{t}e^{-a})$, which gives

$$\begin{aligned}
\bar{L}\bar{u} &= \bar{x}\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{t}}(\bar{x},\bar{t}) + \frac{\partial\bar{u}}{\partial\bar{t}}(\bar{x},\bar{t}) - \bar{u}^2(\bar{x},\bar{t}) + \bar{u}(0,\bar{t})\big(\bar{x}\frac{\partial\bar{u}}{\partial\bar{x}}(\bar{x},\bar{t}) + \bar{u}(\bar{x},\bar{t})\big) \\
&= \bar{x}e^{-a}\frac{\partial^2 u}{\partial\bar{x}\partial\bar{t}}(\bar{x},\bar{t}e^{-a}) + e^{-a}\frac{\partial u}{\partial\bar{t}}(\bar{x},\bar{t}e^{-a}) - e^{-2a}u^2(\bar{x},\bar{t}e^{-a}) \\
&\quad + e^{-a}u(0,\bar{t})\big(\bar{x}e^{-a}\frac{\partial u}{\partial\bar{x}}(\bar{x},\bar{t}e^{-a}) + e^{-a}u(\bar{x},\bar{t}e^{-a})\big) \\
&= \bar{x}e^{-2a}\frac{\partial^2 u}{\partial x\partial t}(x,t) + e^{-2a}\frac{\partial u}{\partial t}(x,t) - e^{-2a}u^2(x,t) \\
&\quad + e^{-2a}u(0,t)\big(\bar{x}\frac{\partial u}{\partial x}(x,t) + u(x,t)\big) \\
&= e^{-2a}\left(x\frac{\partial^2 u}{\partial x\partial t}(x,t) + \frac{\partial u}{\partial t}(x,t) - u^2(x,t) + u(0,t)\big(x\frac{\partial u}{\partial x}(x,t) + u(x,t)\big)\right) \\
&= e^{-2a}Lu.
\end{aligned}$$

Hence, we can conclude that the transformations

$$\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u, \quad \bar{g} = ge^{-2a} \tag{3.10}$$

compose an equivalence Lie group of Equation (3.1).

The transformations corresponding to the generator $X_1 = x(x\partial_x - u\partial_u)$ map a function $u(x,t)$ into the function $\bar{u}(\bar{x},\bar{t}) = \frac{1}{1+a\bar{x}}u\left(\frac{\bar{x}}{1+a\bar{x}},\bar{t}\right)$. Similarly, we obtain $\bar{L}\bar{u} = (1-ax)^2 Lu$ and the transformations

$$\bar{t} = t, \quad \bar{x} = \frac{x}{1-ax}, \quad \bar{u} = (1-ax)u, \quad \bar{g} = (1-ax)^2 g \tag{3.11}$$

compose an equivalence Lie group of Equation (3.1).

Thus, the Lie groups shown as Equations (3.8) - (3.11) corresponding to the generators

$$X_0^{\mathrm{e}} = x\partial_x, \quad X_1^{\mathrm{e}} = x(x\partial_x - u\partial_u - 2g\partial_g), \quad X_2^{\mathrm{e}} = t\partial_t - u\partial_u - 2g\partial_g, \quad X_3^{\mathrm{e}} = \partial_t$$

are equivalence Lie groups of Equation (3.1).

There are also two involutions corresponding to the changes

$$E_1: \ \bar{x} = -x; \qquad E_2: \ \bar{t} = -t, \quad \bar{u} = -u.$$

An involution is a transformation that is its own inverse.

3.3 Group Classification

Group classification of Equation (3.1) is carried out up to the equivalence transformations considered above.

Equation (3.4) can be rewritten in the form

$$c_0h_0 + c_1h_1 + c_2h_2 + c_3h_3 = 0, \tag{3.12}$$

where

$$h_0 = xg_x, \quad h_1 = x(xg_x + 2g), \quad h_2 = tg_t + 2g, \quad h_3 = g_t. \tag{3.13}$$

One of the methods for analyzing relations between the constants c_0, c_1, c_2 and c_3 is employing the algorithm developed for the gas dynamics equations

(Ovsiannikov, 1978) by analyzing the vector space Span(V), where the set V consists of the vectors

$$v = (h_0, h_1, h_2, h_3)$$

with t and x are varied. This algorithm allows one to study all possible admitted Lie algebras of Equation (3.1) without omission. Unfortunately, this is difficult to achieve. After several observations and studies of Equation (3.1), an algebraic algorithm was found which essentially reduces this study to a simpler problem. Observe here that because of the nonlinearity of the equivalence transformations corresponding to the generator X_1, it is difficult to select out equivalent cases with respect to these transformations, whereas the algebraic algorithm does not have such a complication. The following is an algebraic algorithm.

First we study the Lie algebra L_4 composed by the generators X_0, X_1, X_2 and X_3. The commutator table is

	X_0	X_1	X_2	X_3
X_0	0	X_1	0	0
X_1	$-X_1$	0	0	0
X_2	0	0	0	$-X_3$
X_3	0	0	X_3	0

Using algorithm as in section 2.2, the inner automorphisms are

$$A_0: \quad \hat{x}_1 = x_1 e^a,$$

$$A_1: \quad \hat{x}_1 = x_1 + a x_0,$$

$$A_2: \quad \hat{x}_3 = x_3 e^a,$$

$$A_3: \quad \hat{x}_3 = x_3 + a x_2,$$

where only the changed coordinates are presented.

Second, one can notice that the results of using the equivalence transformations corresponding to the generators $X_0^e, X_1^e, X_2^e, X_3^e$ are similar to changing coordinates of a generator X with regards to the basis change. These changes are similar to the inner automorphisms. Indeed, the coefficients of the generator X are changed according to the relation,

$$X = (X\bar{t})\partial_{\bar{t}} + (X\bar{x})\partial_{\bar{x}} + (X\bar{u})\partial_{\bar{u}}.$$

Any generator X can be expressed as a linear combination of the basis generators:

$$\hat{x}_0\hat{X}_0 + \hat{x}_1\hat{X}_1 + \hat{x}_2\hat{X}_2 + \hat{x}_3\hat{X}_3 = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3, \qquad (3.14)$$

where

$$\hat{X}_0 = \bar{x}\partial_{\bar{x}}, \quad \hat{X}_1 = \bar{x}(\bar{x}\partial_{\bar{x}} - \bar{u}\partial_{\bar{u}}), \quad \hat{X}_2 = \bar{t}\partial_{\bar{t}} - \bar{u}\partial_{\bar{u}}, \quad \hat{X}_3 = \partial_{\bar{t}}.$$

Using the invariance of a generator with respect to a change of the variables, the basis generators X_i $(i = 0, 1, 2, 3)$ and $\hat{X}_j$ $(j = 0, 1, 2, 3)$ in corresponding equivalence transformations are related as follows:

$$X_0^e: \quad X_0 = \hat{X}_0, \quad X_1 = e^{-a}\hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = \hat{X}_3;$$

$$X_1^e: \quad X_0 = \hat{X}_0 + a\hat{X}_1, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = \hat{X}_3;$$

$$X_2^e: \quad X_0 = \hat{X}_0, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = e^{a}\hat{X}_3;$$

$$X_3^e: \quad X_0 = \hat{X}_0, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2 - a\hat{X}_3, \quad X_3 = \hat{X}_3.$$

Substituting these relations into the identity (3.14), one obtains that the coordinates of the generator X in the basis X_0, X_1, X_2, X_3 and in the basis $\hat{X}_0, \hat{X}_1, \hat{X}_2, \hat{X}_3$

are changed as follows:

$$X_0^e: \quad \hat{x}_1 = x_1 e^a,$$

$$X_1^e: \quad \hat{x}_1 = x_1 + a x_0,$$

$$X_2^e: \quad \hat{x}_3 = x_3 e^a,$$

$$X_3^e: \quad \hat{x}_3 = x_3 + a x_2,$$

They are similar to the changes defined by the inner automorphisms.

This observation allows us to use an optimal system of subalgebras of the Lie algebra L_4 for studying Equation (3.1). Construction of such an optimal system is not difficult. Moreover, a two-step algorithm as in Chapter II is applied. Let us decompose Lie algebra L_4 into the form $L_4 = I \bigoplus F$, where $I = \{X_0, X_1\}$ is an ideal and $F = \{X_2, X_3\}$ is subalgebra of the Lie algebra L_4. The result of construction of an optimal system of subalgebras is presented in Table 3.1.

Notice also that in constructing the optimal system of subalgebras we also used transformations corresponding to the involutions E_1 and E_2:

$$E_1: \ \hat{x}_1 = -x_1; \qquad E_2: \ \hat{x}_3 = -x_3.$$

The functions $g(x, t)$ are obtained by using the optimal system of subalgebras. We need to substitute the constants c_i corresponding to the basis generators of a subalgebra into Equation (3.4), and solve the system of equations thus obtained. The result of group classification is presented in Table 3.2, where α, $\beta \neq 1, \gamma \neq -2$ and κ are constant, and the function Φ is an arbitrary function of its argument.

Table 3.1 Optimal system of subalgebras of L_4 for the equation for the moment generating function.

No.	Basis	No.	Basis
1	$X_0,\ X_1,\ X_2,\ X_3$	11	$\alpha X_0 + X_2,\ X_1$
2	$\alpha X_0 + X_2,\ X_1,\ X_3$	12	$X_0 + X_3,\ X_1$
3	$X_0,\ X_1,\ X_3$	13	$X_1,\ X_3$
4	$X_0,\ X_1,\ X_2$	14	$X_0,\ X_1$
5	$X_0,\ X_2,\ X_3$	15	$\alpha X_0 + X_2$
6	$X_2,\ X_3$	16	$X_1 + X_2$
7	$X_2 - X_0,\ X_1 + X_3$	17	$X_0 + X_3$
8	$\gamma X_2 - 2X_0,\ X_3$	18	$X_1 + X_3$
9	$X_1 + X_2,\ X_3$	19	X_0
10	$X_0,\ X_2$	20	X_1
		21	X_3

3.3.1 Illustrative Examples of Obtaining the Function $g(x,t)$

For the algebra $\{\alpha X_0 + X_2,\ X_1,\ X_3\}$, we have three sets of coefficients c_i $(i = 0, 1, 2, 3)$:

$$\begin{aligned} \alpha X_0 + X_2 &: \ c_0 = \alpha,\ c_1 = 0,\ c_2 = 1,\ c_3 = 0; \\ X_1 &: \ c_0 = 0,\ c_1 = 1,\ c_2 = 0,\ c_3 = 0; \\ X_3 &: \ c_0 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 1. \end{aligned}$$

Table 3.2 Group classification of the equation for the moment generating function.

No.	$g(x,t)$	Generators
1.	0	$X_0,\ X_1,\ X_2,\ X_3$
2.	κx^{-2}	$X_0+X_2,\ X_1,\ X_3$
3.	$\kappa x^2(xt+1)^{-4}$	$X_2-X_0,\ X_1+X_3$
4.	κx^{γ}	$\gamma X_2-2X_0,\ X_3$
5.	$\kappa x^{-2}e^{2x^{-1}}$	$X_1+X_2,\ X_3$
6.	κt^{-2}	$X_0,\ X_2$
7.	$\kappa x^{-2}t^{2(\beta-1)}$	$\beta X_0+X_2,\ X_1$
8.	$\kappa x^{-2}e^{2t}$	$X_0+X_3,\ X_1$
9.	κx^{-2}	$X_1,\ X_3$
10.	$t^{-2}\Phi(xt^{-\alpha})$	αX_0+X_2
11.	$x^{-2}e^{2x^{-1}}\Phi(te^{x^{-1}})$	X_1+X_2
12.	$\Phi(xe^{-t})$	X_0+X_3
13.	$x^{-2}\Phi(t+x^{-1})$	X_1+X_3
14.	$\Phi(t)$	X_0
15.	$x^{-2}\Phi(t)$	X_1
16.	$\Phi(x)$	X_3

Note $\beta \neq 1, \gamma \neq -2$

These sets of constants define the following overdetermined system of partial differential equations for the function $g(x,t)$:

$$tg_t+\alpha xg_x=-2g,\quad xg_x=-2g,\quad g_t=0.$$

The general solution of these equations is $\alpha = 1$ and

$$g(x,t) = \kappa x^{-2},$$

where κ is constant. This case corresponds to No.2 in Table 3.2.

For the algebra $\{X_3\}$, there is only one equation for the function $g(x,t)$:

$$g_t = 0.$$

Hence, the function sought is $g(x,t) = \Phi(x)$, where $\Phi(x)$ is an arbitrary function. This case corresponds to No.16 in Table 3.2.

3.3.2 Illustrative Examples of Representations of Invariant Solutions

For the function $g(x,t) = \Phi(x)$, the admitted Lie algebra consists of the generator X_3 only, its invariant solution has the representation

$$u = r(x).$$

Substituting this representation of an invariant solution into Equation (3.1), one obtains the reduced equation

$$r(0)(xr_x + r) - r^2 = \Phi(x).$$

Since $r(0)$ is constant, the reduced equation is similar to the Ricatti equation. This allows us to state that in the case of a general functions $\Phi(x)$, the solution of this equation cannot be presented in terms of elementary functions. However, the presence of an admitted Lie group allows reducing the number of independent variables, thus assisting in obtaining solutions of Equation (3.1).

As another example, consider the function $g(x,t) = t^{-2}\Phi(xt^{-\alpha})$. There is also only one admitted generator $X = \alpha X_0 + X_2 = t\partial_t - u\partial_u + \alpha x\partial_x$. After finding invariants of the generator X, one defines the representation of an invariant solution

$$u(x,t) = t^{-1}\phi(xt^{-\alpha})$$

Substituting $u(t,x)$ into Equation (3.1), one also obtains a reduced equation of one independent variable:

$$\alpha z^2\phi''(z) + 2\alpha z\phi'(z) + \phi^2(z) + \phi(0)(z - \phi(z)) = \Phi(z),$$

where $z = xt^{-\alpha}$

CHAPTER IV

APPLYING GROUP ANALYSIS TO THE SPATIALLY HOMOGENEOUS AND ISOTROPIC BOLTZMANN EQUATION

4.1 The Equation under Study

The Fourier image of the spatially homogeneous and isotropic Boltzmann equation with sources has the form

$$\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) = \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\, ds + \hat{q}(x,t). \tag{4.1}$$

Here, t is time, $\varphi(x,t) = \tilde{\varphi}(\frac{k^2}{2},t)$ and $\hat{q}(x,t) = \tilde{\hat{q}}(\frac{k^2}{2},t)$ where $\tilde{\varphi}(k,t)$ and $\tilde{\hat{q}}(k,t)$ are the Fourier transforms of the distribution function and the source function, respectively.

Remark 1) Equation (4.1) is Equation (1.1) in Chapter I with variable y replaced by variable x. 2) The variables x in Equations (4.1) and (3.1) are different.

This section presents the complete solution of the determining equation. The solution was found by constructing necessary conditions for the coefficients of the admitted generator. These conditions are obtained by using a particular class of solutions of Equation (4.1). It is worth to note that the particular class of solutions allows us to find the general solution of the determining equation.

Differentiating (4.1) and substituting φ_t found from (4.1), we obtain

$$\varphi_{xt}(x,t) = -\varphi_x(x,t)\varphi(0,t) + 2\int_0^1 s\varphi_x(xs,t)\varphi(x(1-s),t)\,ds + \hat{q}_x(x,t),$$

$$\begin{aligned}\varphi_{tt}(x,t) = \varphi(x,t)\varphi^2(0,t) - 3\varphi(0,t)\int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,ds \\ + 2\int_0^1\int_0^1 \varphi(x(1-s),t)\varphi(xss',t)\varphi(xs(1-s'),t)ds'ds \\ - \hat{q}(0,t)\varphi(x,t) - \hat{q}(x,t)\varphi(0,t) + 2\int_0^1 \varphi(x(1-s),t)\hat{q}(xs,t)ds + \hat{q}_t(x,t).\end{aligned}$$

The generator of the admitted Lie group is sought in the form

$$X = \xi(x,t,\varphi)\partial_x + \eta(x,t,\varphi)\partial_t + \zeta(x,t,\varphi)\partial_\varphi.$$

The determining equation for Equation (4.1) is

$$D_t\psi(x,t) + \psi(0,t)\varphi(x,t) + \psi(x,t)\varphi(0,t) - 2\int_0^1 \varphi(x(1-s),t)\psi(xs,t)ds = 0, \quad (4.2)$$

when D_t is total derivative with respect to t and the function $\psi(x,t)$ is

$$\psi(x,t) = \zeta(x,t,\varphi(x,t)) - \xi(x,t,\varphi(x,t))\varphi_x(x,t) - \eta(x,t,\varphi(x,t))\varphi_t(x,t).$$

The determining equation (4.2) has to be satisfied for any solution of Equation (4.1). Assume that the coefficient of the infinitesimal generator X are represented by the formal Taylor series with respect to φ:

$$\begin{aligned}\xi(x,t,\varphi) &= \sum_{l\geq 0} q_l(x,t)\varphi^l(x,t), \\ \eta(x,t,\varphi) &= \sum_{l\geq 0} r_l(x,t)\varphi^l(x,t), \\ \zeta(x,t,\varphi) &= \sum_{l\geq 0} p_l(x,t)\varphi^l(x,t).\end{aligned} \quad (4.3)$$

A particular class of solutions is considered. This class is defined by the initial conditions

$$\varphi(x,t_0) = \varphi_0(x) = bx^n \quad (4.4)$$

at a given (arbitrary) time $t = t_0$. Here, n is a positive integer. For solving the determining equation, we apply the solutions corresponding to the initial data (4.4) by varying the parameter b and the degree n.

In the case of $n = 0$, the derivatives of the function $\varphi(x,t)$ at the time $t = t_0$ become

$$\varphi_t(x,t) = \hat{q}(x,t), \qquad \varphi_{xt}(x,t) = \hat{q}_x(x,t),$$
$$\varphi_{tt}(x,t) = -\hat{q}(0,t)b - \hat{q}(x,t)b + 2\int_0^1 b\hat{q}(xs,t)ds + \hat{q}_t(x,t).$$

The determining equation becomes

$$\begin{aligned}\zeta_t(x,t,b) + \hat{q}(x,t)\zeta_\varphi(x,t,b) - \hat{q}_x(x,t)\xi(x,t,b) - \eta_t(x,t,b)\hat{q}(x,t) - \hat{q}^2(x,t)\eta_\varphi(x,t,b)\\ -\eta(x,t,b)(-\hat{q}(0,t)b - \hat{q}(x,t)b + 2\int_0^1 b\hat{q}(xs,t)ds + \hat{q}_t(x,t))\\ +(\zeta(0,t,b) - \eta(0,t,b)\hat{q}(0,t))b + (\zeta(x,t,b) - \eta(x,t,b)\hat{q}(x,t))b\\ -2\int_0^1 b(\zeta(xs,t,b) - \eta(xs,t,b)\hat{q}(xs,t))ds = 0.\end{aligned} \tag{4.5}$$

Using the decompositions (4.3), from this equation one obtains

$$\begin{aligned}\frac{\partial p_0(x,t)}{\partial t} + \hat{q}(x,t)p_1(x,t) - \hat{q}_x(x,t)q_0(x,t) - \hat{q}(x,t)\frac{\partial r_0(x,t)}{\partial t}\\ -\hat{q}^2(x,t)r_1(x,t) - \hat{q}_t(x,t)r_0(x,t) = 0\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\frac{\partial p_{l+1}(x,t)}{\partial t} + (l+2)\hat{q}(x,t)p_{l+2}(x,t) - \hat{q}_x(x,t)q_{l+1}(x,t) - \hat{q}(x,t)\frac{\partial r_{l+1}(x,t)}{\partial t}\\ -(l+2)\hat{q}^2(x,t)r_{l+2}(x,t) + \hat{q}(0,t)r_l(x,t) - 2r_l(x,t)\int_0^1 \hat{q}(xs,t)ds\\ -\hat{q}_t(x,t)r_{l+1}(x,t) + p_l(0,t) - \hat{q}(0,t)r_l(0,t) + p_l(x,t) - 2\int_0^1 p_l(xs,t)ds\\ +2\int_0^1 r_l(xs,t)\hat{q}(xs,t)ds = 0\end{aligned} \tag{4.7}$$

where $l = 0, 1, 2, 3, \ldots$

If $n \geq 1$ the derivatives of the function $\varphi(x,t)$ at the time $t = t_0$ are

$$\varphi_t(x,t) = b^2x^{2n}P_n + \hat{q}(x,t), \quad \varphi_{xt}(x,t) = 2nb^2x^{2n-1}P_n + \hat{q}_x(x,t),$$
$$\varphi_{tt}(x,t) = b^3x^{3n}Q_n - \hat{q}(0,t)bx^n + 2bx^n\int_0^1 (1-s)^n\hat{q}(xs,t)ds + \hat{q}_t(x,t).$$

Here, the notations

$$P_n = \frac{(n!)^2}{(2n+1)!}, \quad Q_n = 2P_n\frac{(2n)!n!}{(3n+1)!}$$

are used. For further calculations it is also worth to notice that

$$B(m+1,n+1) = \int_0^1 s^m(1-s)^n ds = \frac{m!n!}{(m+n+1)!}.$$

In this case, the determining equation (4.2) becomes

$$\begin{aligned}
&\zeta_t + \hat{q}\zeta_\varphi - \hat{q}_x\xi - \hat{q}\eta_t - \hat{q}^2\eta_\varphi - \hat{q}_t\eta \\
&+b\Big(- nx^{n-1}\xi_t - nx^{n-1}\hat{q}\xi_\varphi + x^n\hat{q}(0,t)\eta - 2x^n\eta\int_0^1(1-s)^n\hat{q}(xs,t)ds \\
&\quad + x^n\zeta(0,t,0) - 2x^n\int_0^1(1-s)^n\zeta(xs,t,b(xs)^n)ds \\
&\quad - x^n\hat{q}(0,t)\eta(0,t,0) + 2x^n\int_0^1(1-s)^n\eta(xs,t,b(xs)^n)\hat{q}(xs,t)ds\Big) \\
&+b^2\Big(x^{2n}P_n\zeta_\varphi - 2nx^{2n-1}P_n\xi - x^{2n}P_n\eta_t - 2x^{2n}P_n\hat{q}\eta_\varphi - \delta_{n1}x^{2n-1}\xi(0,t,0) \\
&\quad + 2nx^{2n-1}\int_0^1(1-s)^n s^{n-1}\xi(xs,t,b(xs)^n)ds\Big) \\
&+b^3\Big(- nP_nx^{3n-1}\xi_\varphi - Q_nx^{3n}\eta + 2P_nx^{3n}\int_0^1(1-s)^n s^{2n}\eta(xs,t,b(xs)^n)ds\Big) \\
&+b^4\Big(- P_n^2x^{4n}\eta_\varphi\Big) = 0.
\end{aligned} \tag{4.8}$$

Using the arbitrariness of the value b and equating to zero the coefficients with respect to b^α ($\alpha = 0, 1, 2, ...$), the determining equation can be split into a series of equations.

For $\alpha = 0$ the corresponding coefficient vanishes due to Equation (4.6). For

$\alpha = 1$ the corresponding coefficient is

$$
\begin{aligned}
x^n\bigg(&\frac{\partial p_1(x,t)}{\partial t} + 2p_2(x,t)\hat{q}(x,t) - \hat{q}_x(x,t)q_1(x,t) - \hat{q}(x,t)\frac{\partial r_1(x,t)}{\partial t} \\
&- 2\hat{q}^2(x,t)r_2(x,t) - \hat{q}_t(x,t)r_1(x,t) + \hat{q}(0,t)r_0(x,t) + p_0(0,t) \\
&- \hat{q}(0,t)r_0(0,t) - 2r_0(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds \\
&- 2\int_0^1 (1-s)^n p_0(xs,t)ds + 2\int_0^1 (1-s)^n\hat{q}(xs,t)r_0(xs,t)ds\bigg) \\
+x^{n-1}\bigg(&-n\frac{\partial q_0(x,t)}{\partial t} - n\hat{q}(x,t)q_1(x,t)\bigg) = 0.
\end{aligned}
$$

After substituting $\dfrac{\partial p_1}{\partial t}$ found from Equation (4.7) the latter equation is reduced to the equation

$$
\begin{aligned}
x\bigg(&2r_0(x,t)\int_0^1 \hat{q}(sx,t)ds - p_0(x,t) + 2\int_0^1 p_0(xs,t)ds \\
&- 2\int_0^1 r_0(xs,t)\hat{q}(xs,t)ds - 2r_0(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds \\
&+ 2\int_0^1 (1-s)^n\hat{q}(xs,t)r_0(xs,t)ds - 2\int_0^1 (1-s)^n p_0(xs,t)ds\bigg) \\
+\bigg(&-n\frac{\partial q_0(x,t)}{\partial t} - n\hat{q}(x,t)q_1(x,t)\bigg) = 0.
\end{aligned}
\tag{4.9}
$$

Dividing the latter equation by n and letting n approach infinity , we obtain

$$
\frac{\partial q_0}{\partial t} + \hat{q}q_1 = 0 \tag{4.10}
$$

and

$$
\begin{aligned}
x\bigg(&2r_0(x,t)\int_0^1 \hat{q}(sx,t)ds - p_0(x,t) + 2\int_0^1 p_0(xs,t)ds \\
&- 2\int_0^1 r_0(xs,t)\hat{q}(xs,t)ds - 2r_0(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds \\
&+ 2\int_0^1 (1-s)^n\hat{q}(xs,t)r_0(xs,t)ds - 2\int_0^1 (1-s)^n p_0(xs,t)ds\bigg) = 0
\end{aligned}
\tag{4.11}
$$

Substituting (4.11) into (4.9) and letting n approach infinity again, we get

$$
\begin{aligned}
&2r_0(x,t)\int_0^1 \hat{q}(sx,t)ds - p_0(x,t) + 2\int_0^1 p_0(xs,t)ds \\
&- 2\int_0^1 r_0(xs,t)\hat{q}(xs,t)ds = 0.
\end{aligned}
\tag{4.12}
$$

Thus, Equation (4.9) is reduced to the equation

$$r_0(x,t)\int_0^1 (1-s)^n \hat{q}(sx,t)ds - \int_0^1 (1-s)^n r_0(xs,t)\hat{q}(xs,t)ds \\ + \int_0^1 (1-s)^n p_0(xs,t)ds = 0. \tag{4.13}$$

For analyzing Equation (4.13), we use representation of the functions $\hat{q}, r_0$ and p_0 in the formal Taylor series:

$$\hat{q}(x,t) = \sum_{i=0}^{\infty} h_i(t)x^i, \qquad r_0(x,t) = \sum_{i=0}^{\infty} r_{0i}(t)x^i, \qquad p_0(x,t) = \sum_{i=0}^{\infty} p_{0i}(t)x^i. \tag{4.14}$$

All coefficients with respect to x^k in Equation (4.13) have to vanish

$$\sum_{i=0}^{k} \left[r_{0i}h_{k-i}\left(\frac{(k-i)!(n+k+1)!}{k!(n+k-i+1)!} - 1\right)\right] + p_{0k} = 0.$$

Due to arbitrariness of n, we find

$$r_{0i}h_{k-i} = 0, \qquad p_{0k} = 0, \tag{4.15}$$

for all $i \geq 1$ and for all $k \geq 0$. Since $\hat{q} \neq 0$, then there exists k_0 such that $h_{k_0} \neq 0$. Choosing $k = i + k_0$ for any $i \geq 1$, we get from the first equation of (4.15) and the second formula of (4.14) that

$$r_0(x,t) = r_{00}(t). \tag{4.16}$$

The second condition of (4.15) provides that

$$p_0(x,t) = 0. \tag{4.17}$$

For $\alpha = 2$ the corresponding equation is

$$x^{2n}\bigg(\frac{\partial p_2(x,t)}{\partial t} + 3p_3(x,t)\hat{q}(x,t) - \hat{q}_x(x,t)q_2(x,t) - \hat{q}(x,t)\frac{\partial r_2(x,t)}{\partial t} + \hat{q}(0,t)r_1(x,t)$$
$$- 3\hat{q}^2(x,t)r_3(x,t) - \hat{q}_t(x,t)r_2(x,t) - 2r_1(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds$$
$$- 2\int_0^1 (1-s)^n s^n p_1(xs,t)ds + 2\int_0^1 (1-s)^n s^n \hat{q}(xs,t)r_1(xs,t)ds$$
$$+ P_n p_1(x,t) - P_n\frac{\partial r_0(x,t)}{\partial t} - 2P_n\hat{q}(x,t)r_1(x,t)\bigg)$$
$$+x^{2n-1}\bigg(- n\frac{\partial q_1(x,t)}{\partial t} - 2n\hat{q}(x,t)q_2(x,t) - 2nP_n q_0(x,t) - \delta_{n1}q_0(0,t)$$
$$+ 2n\int_0^1 (1-s)^n s^{n-1} q_0(xs,t)ds\bigg) = 0.$$

Substituting $\dfrac{\partial p_2}{\partial t}$ found from Equation (4.7) into the latter equation, it becomes

$$x\bigg(- p_1(x,t) - p_1(0,t) + 2\int_0^1 p_1(xs,t)ds + 2r_1(x,t)\int_0^1 \hat{q}(sx,t)ds + \hat{q}(0,t)r_1(0,t)$$
$$- 2\int_0^1 r_1(xs,t)\hat{q}(xs,t)ds - 2r_1(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds$$
$$- 2\int_0^1 (1-s)^n s^n p_1(xs,t)ds + 2\int_0^1 (1-s)^n s^n\hat{q}(xs,t)r_1(xs,t)ds$$
$$+ P_n p_1(x,t) - P_n\frac{\partial r_0(x,t)}{\partial t} - 2P_n\hat{q}(x,t)r_1(x,t)\bigg)$$
$$+\bigg(- n\frac{\partial q_1(x,t)}{\partial t} - 2n\hat{q}(x,t)q_2(x,t) + 2n\int_0^1 (1-s)^n s^{n-1}q_0(xs,t)ds$$
$$- 2nP_n q_0(x,t) - \delta_{n1}q_0(0,t)\bigg) = 0. \tag{4.18}$$

Dividing by n and letting n approach infinity, we have

$$\frac{\partial q_1}{\partial t} + 2\hat{q}q_2 = 0. \tag{4.19}$$

Equation (4.18) becomes

$$x\Big(- p_1(x,t) - p_1(0,t) + 2\int_0^1 p_1(xs,t)ds + 2r_1(x,t)\int_0^1 \hat{q}(sx,t)ds + \hat{q}(0,t)r_1(0,t)$$
$$- 2\int_0^1 r_1(xs,t)\hat{q}(xs,t)ds - 2r_1(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds$$
$$- 2\int_0^1 (1-s)^n s^n p_1(xs,t)ds + 2\int_0^1 (1-s)^n s^n \hat{q}(xs,t)r_1(xs,t)ds$$
$$+ P_n p_1(x,t) - P_n\frac{\partial r_0(x,t)}{\partial t} - 2P_n\hat{q}(x,t)r_1(x,t)\Big)$$
$$+\Big(2n\int_0^1 (1-s)^n s^{n-1} q_0(xs,t)ds - 2nP_n q_0(x,t) - \delta_{n1}q_0(0,t)\Big) = 0. \tag{4.20}$$

Letting n approach infinity again, we obtain

$$\begin{aligned} x\Big(- p_1(x,t) - p_1(0,t) + 2\int_0^1 p_1(xs,t)ds + 2r_1(x,t)\int_0^1 \hat{q}(sx,t)ds \\ + \hat{q}(0,t)r_1(0,t) - 2\int_0^1 \hat{q}(xs,t)r_1(xs,t)ds\Big) - \delta_{n1}q_0(0,t) = 0. \end{aligned} \tag{4.21}$$

Subtracting (4.21) from (4.20), we get

$$x\Big(- 2r_1(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds + 2\int_0^1 (1-s)^n s^n \hat{q}(xs,t)r_1(xs,t)ds$$
$$- 2\int_0^1 (1-s)^n s^n p_1(xs,t)ds + P_n p_1(x,t) - P_n\frac{\partial r_0(x,t)}{\partial t} - 2P_n\hat{q}(x,t)r_1(x,t)\Big)$$
$$+\Big(2n\int_0^1 (1-s)^n s^{n-1} q_0(xs,t)ds - 2nP_n q_0(x,t)\Big) = 0. \tag{4.22}$$

Using representations (4.14) and

$$r_1(x,t) = \sum_{i=0}^{\infty} r_{1i}(t)x^i, \qquad p_1(x,t) = \sum_{i=0}^{\infty} p_{1i}(t)x^i,$$

Equation (4.22) becomes

$$\begin{aligned} \sum_{k=0}^{\infty} x^{k+1}\Big[\sum_{i=0}^{k} r_{1i}h_{k-i}\Big(\frac{n!(k-i)!}{(n+k-i+1)!} - \frac{n!(n+k)!}{(2n+k+1)!} + \frac{n!n!}{(2n+1)!}\Big) \\ + p_{1k}\Big(\frac{n!(n+k)!}{(2n+k+1)!} - \frac{n!n!}{2(2n+1)!}\Big)\Big] + x\frac{n!n!}{2(2n+1)!}\frac{\partial r_{00}}{\partial t} \\ + \sum_{k=0}^{\infty} x^k q_{0k}\Big(\frac{nn!n!}{(2n+1)!} - \frac{nn!(n+k-1)!}{(2n+k)!}\Big) = 0. \end{aligned} \tag{4.23}$$

Equating coefficients with respect to x^0 and x^1 to zero, we find

$$q_{00} = 0, \tag{4.24}$$

and

$$r_{10}h_0 + \frac{n!(n+1)!}{2(2n+1)!}(p_{01} + \frac{\partial r_{00}}{\partial t}) = 0.$$

Due to arbitrariness of n, these equations provide that

$$r_{10}h_0 = 0,$$

and

$$p_{10} + \frac{\partial r_{00}}{\partial t} = 0. \tag{4.25}$$

Equating the coefficients with respect to x^k in (4.23) for $k \geq 2$, we have

$$\sum_{i=0}^{k-1}\left[r_{1i}h_{k-1-i}\left(\frac{(k-1-i)!}{(n+k-i)!} - \frac{(n+k-1)!}{(2n+k)!} + \frac{n!}{(2n+1)!}\right)\right] + p_{1(k-1)}\left(\frac{(n+k-1)!}{(2n+k)!} - \frac{n!}{2(2n+1)!}\right) + q_{0k}\left(\frac{n(n!)}{(2n+1)!} - \frac{n((n+k-1)!)}{(2n+k)!}\right) = 0,$$

which provide that

$$r_{1i}h_{k-1-i} = 0, \qquad q_{0k} = 0, \qquad p_{1k} = 0,$$

for all $i = 0, 1, 2, ..., k-1$ and $k = 2, 3, 4, ...$ From the obtained conditions, we can conclude that

$$r_1(x,t)\hat{q}(x,t) = 0, \tag{4.26}$$

$$q_0(x,t) = q_{01}(t)x, \tag{4.27}$$

$$p_1(x,t) = p_{10}(t) + p_{11}(t)x. \tag{4.28}$$

Since $\hat{q} \neq 0$, Equation (4.26) gives that $r_1 = 0$.

Returning to Equation (4.8), for $\alpha = 3$, the corresponding equation is

$$x^{3n}\bigg(\frac{\partial p_3(x,t)}{\partial t} + 4p_4(x,t)\hat{q}(x,t) - \hat{q}_x(x,t)q_3(x,t) - \hat{q}(x,t)\frac{\partial r_3(x,t)}{\partial t}$$
$$+ \hat{q}(0,t)r_2(x,t) - 4\hat{q}^2(x,t)r_4(x,t) - \hat{q}_t(x,t)r_3(x,t)$$
$$- 2r_2(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds + 2\int_0^1 (1-s)^n s^{2n}\hat{q}(xs,t)r_2(xs,t)ds$$
$$- 2\int_0^1 (1-s)^n s^{2n} p_2(xs,t)ds + 2P_n p_2(x,t) - P_n\frac{\partial r_1(x,t)}{\partial t}$$
$$- 4P_n\hat{q}(x,t)r_2(x,t) - Q_n r_0(x,t) + 2P_n\int_0^1 (1-s)^n s^{2n} r_0(xs,t)ds\bigg)$$
$$+x^{3n-1}\bigg(-n\frac{\partial q_2(x,t)}{\partial t} - 3n\hat{q}(x,t)q_3(x,t) - 2nP_n q_1(x,t)$$
$$+ 2n\int_0^1 (1-s)^n s^{2n-1} q_1(xs,t)ds - nP_n q_1(x,t)\bigg) = 0,$$

which after substitution of $\dfrac{\partial p_3}{\partial t}$ found from Equation (4.7) becomes

$$x\bigg(-p_2(x,t) - p_2(0,t) + 2\int_0^1 p_2(xs,t)ds + 2r_2(x,t)\int_0^1 \hat{q}(sx,t)ds + \hat{q}(0,t)r_2(0,t)$$
$$- 2\int_0^1 r_2(xs,t)\hat{q}(xs,t)ds - 2r_2(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds$$
$$- 2\int_0^1 (1-s)^n s^{2n} p_2(xs,t)ds + 2\int_0^1 (1-s)^n s^{2n}\hat{q}(xs,t)r_2(xs,t)ds$$
$$+ 2P_n p_2(x,t) - P_n\frac{\partial r_1(x,t)}{\partial t} - 4P_n\hat{q}(x,t)r_2(x,t) - Q_n r_0(x,t)$$
$$+ 2P_n\int_0^1 (1-s)^n s^{2n} r_0(xs,t)ds\bigg)$$
$$+\bigg(-n\frac{\partial q_2(x,t)}{\partial t} - 3n\hat{q}(x,t)q_3(x,t) - 3nP_n q_1(x,t)$$
$$+ 2n\int_0^1 (1-s)^n s^{2n-1} q_1(xs,t)ds\bigg) = 0. \tag{4.29}$$

Similarly, dividing the latter equation by n, letting n approach infinity and using the representations of the functions $\hat{q}, r_0, r_1, r_2, q_1$ and p_2 in the formal Taylor series, we obtain

$$\frac{\partial q_2}{\partial t} + 3\hat{q}q_3 = 0, \tag{4.30}$$

$$q_1(x,t)=0,\quad \frac{\partial r_1(x,t)}{\partial t}=0,\quad p_2(x,t)=0,\quad r_2(x,t)\hat{q}(x,t)=0. \tag{4.31}$$

The last equation gives that $r_2 = 0$. From Equations (4.10), (4.27) and the first condition of (4.31) we have

$$q_0(x,t)=c_0x. \tag{4.32}$$

For $\alpha = 4 + l$ ($l \geq 0$), Equation (4.8) is reduced to the equation

$$\begin{aligned}
x^{(4+l)n}\Bigg(&\frac{\partial p_{4+l}(x,t)}{\partial t}+(5+l)p_{5+l}(x,t)\hat{q}(x,t)-\hat{q}_x(x,t)q_{4+l}(x,t)\\
&-\hat{q}(x,t)\frac{\partial r_{4+l}(x,t)}{\partial t}+\hat{q}(0,t)r_{3+l}(x,t)-(5+l)\hat{q}^2(x,t)r_{5+l}(x,t)\\
&-\hat{q}_t(x,t)r_{4+l}(x,t)-2r_{3+l}(x,t)\int_0^1(1-s)^n\hat{q}(xs,t)ds\\
&-2\int_0^1(1-s)^n s^{3+l}p_{3+l}(xs,t)ds+(3+l)P_np_{3+l}(x,t)\\
&+2\int_0^1(1-s)^n s^{(3+l)n}\hat{q}(xs,t)r_{3+l}(xs,t)ds-P_n\frac{\partial r_{2+l}(x,t)}{\partial t}\\
&-2(3+l)P_n\hat{q}(x,t)r_{3+l}(x,t)+2P_n\int_0^1(1-s)^n s^{(3+l)n}r_{1+l}(xs,t)ds\\
&-(1+l)P_n^2r_{1+l}(x,t)-Q_nr_{1+l}(x,t)\Bigg)\\
+x^{(4+l)n-1}\Bigg(&-n\frac{\partial q_{3+l}(x,t)}{\partial t}-(4+l)n\hat{q}(x,t)q_{4+l}(x,t)-2nP_nq_{2+l}(x,t)\\
&-(2+l)nP_nq_{2+l}(x,t)+2n\int_0^1(1-s)^n s^{(3+l)n-1}q_{2+l}(xs,t)ds\Bigg)=0,
\end{aligned}$$

which after substitution $\dfrac{\partial p_{4+l}}{\partial t}$ found from Equation (4.7), becomes

$$
\begin{aligned}
x\Big(& -p_{3+l}(x,t) - p_{3+l}(0,t) + 2\int_0^1 p_{3+l}(xs,t)ds + 2r_{3+l}(x,t)\int_0^1 \hat{q}(sx,t)ds \\
& + \hat{q}(0,t)r_{3+l}(0,t) - 2\int_0^1 r_{3+l}(xs,t)\hat{q}(xs,t)ds - 2r_{3+l}(x,t)\int_0^1 (1-s)^n\hat{q}(xs,t)ds \\
& - 2\int_0^1 (1-s)^n s^{3n} p_{3+l}(xs,t)ds + 2\int_0^1 (1-s)^n s^{3n}\hat{q}(xs,t)r_{3+l}(xs,t)ds \\
& + (3+l)P_n p_{3+l}(x,t) - P_n\frac{\partial r_{2+l}(x,t)}{\partial t} - 2(3+l)P_n\hat{q}(x,t)r_{3+l}(x,t) - Q_n r_{1+l}(x,t) \\
& + 2P_n\int_0^1 (1-s)^n s^{3n} r_{1+l}(xs,t)ds - (l+1)P_n^2 r_{1+l}(x,t)\Big) \\
+\Big(& -n\frac{\partial q_{3+l}(x,t)}{\partial t} - (4+l)n\hat{q}(x,t)q_{4+l}(x,t) - 2nP_n q_{2+l}(x,t)) - (2+l)nP_n q_{2+l}(x,t) \\
& + 2n\int_0^1 (1-s)^n s^{(3+l)n-1} q_{2+l}(xs,t)ds\Big) = 0.
\end{aligned}
\tag{4.33}
$$

Similar as in the previous case, dividing the latter equation by n and letting n approach infinity and using the representations of the functions, we obtain

$$
\frac{\partial q_{3+l}(x,t)}{\partial t} + (4+l)\hat{q}(x,t)q_{4+l}(x,t) = 0, \quad q_{2+l}(x,t) = 0, \frac{\partial r_{2+l}(x,t)}{\partial t} = 0,
$$
$$
p_{3+l}(x,t) = 0, \quad r_{3+l}(x,t)\hat{q} = 0, \quad r_{1+l}(x,t) = 0, \quad (l \geq 0).
$$

Thus,

$$
p_0(x,t) = 0, \quad p_1(x,t) = p_{10}(t) + p_{11}(t)x, \quad q_0(x,t) = c_0 x,
$$
$$
\frac{\partial r_k(x,t)}{\partial x} = 0, \quad p_{k+1} = 0, \quad q_k = 0, \quad r_k = 0, \quad k \geq 1.
$$

Substituting p_0, p_2, q_1, r_1, r_2 and $r_0(x,t) = r_{00}(t)$ into Equation (4.7) in case $l = 0$, we find

$$
\frac{\partial p_{10}}{\partial t} = 0, \quad \frac{\partial p_{11}}{\partial t} = 0, \tag{4.34}
$$

i.e. $p_{10} = c_2$, $p_{11}(t) = c_1$, where c_1 and c_2 are constant. Hence

$$
p_1(x,t) = c_2 + c_1 x, \tag{4.35}
$$

Equation (4.25) gives

$$r_{00}(t) = -c_2 t + c_3, \tag{4.36}$$

where c_3 is constant. Therefore, the coefficients of the generator X are

$$\xi(x,t,\varphi) = c_0 x, \quad \eta(x,t,\varphi) = -c_2 t + c_3, \quad \zeta(x,t,\varphi) = (c_2 + c_1 x)\varphi, \tag{4.37}$$

where $c_0, c_1, c_2$and c_3 are arbitrary constant.

The remaining part of the determining equation (4.2) becomes

$$(c_2 t - c_3)\hat{q}_t - c_0 x \hat{q}_x + (c_1 x + 2c_2)\hat{q} = 0. \tag{4.38}$$

Thus, each admitted generator has the form

$$X = c_0 X_0 + c_1 X_1 + c_2 X_2 + c_3 X_3,$$

where

$$X_0 = x\partial_x, \quad X_1 = x\varphi\partial_\varphi, \quad X_2 = \varphi\partial_\varphi - t\partial_t, \quad X_3 = \partial_t. \tag{4.39}$$

The values of the constants c_0, c_1, c_2, and c_3 and relations between them depend on the function $\hat{q}$. The trivial case where the function $\hat{q}$ is equal to zero satisfies Equation (4.38). This case corresponds to the spatially homogeneous and isotropic Boltzmann equation without a source term. Grigoriev and Meleshko (1987) studied the Boltzmann equation with $\hat{q} = 0$, They have shown that the admitted Lie algebra is four-dimensional and spanned by the generators

$$X_0 = x\partial_x, \quad X_1 = x\varphi\partial_\varphi, \quad X_2 = \varphi\partial_\varphi - t\partial_t, \quad X_3 = \partial_t.$$

4.2 On Equivalence Transformations

In this section we find some of the equivalence transformations. Let

$$L\varphi = \varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\, ds. \tag{4.40}$$

Considering the transformations of $L\varphi$ corresponding to Equation (4.39), the equivalence transformations are obtained.

The transformations corresponding to the generator $X_0 = x\partial_x$ map a function $\varphi(x,t)$ into the function $\bar{\varphi}(\bar{x},\bar{t}) = \varphi(\bar{x}e^{-a},\bar{t})$, where a is the group parameter. The transformed expression of (4.40) becomes

$$\begin{aligned}
\bar{L}\bar{\varphi} &= \bar{\varphi}_{\bar{t}}(\bar{x},\bar{t}) + \bar{\varphi}(\bar{x},\bar{t})\bar{\varphi}(0,\bar{t}) - \int_0^1 \bar{\varphi}(\bar{x}s,\bar{t})\bar{\varphi}(\bar{x}(1-s),\bar{t})\,ds \\
&= \varphi_{\bar{t}}(\bar{x}e^{-a},\bar{t}) + \varphi(\bar{x}e^{-a},\bar{t})\varphi(0,\bar{t}) - \int_0^1 \varphi(\bar{x}e^{-a}s,\bar{t})\varphi(\bar{x}e^{-a}(1-s),\bar{t})\,ds \\
&= \varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,ds \\
&= L\varphi.
\end{aligned}$$

This defines the Lie group of equivalence transformations of Equation (4.1)

$$\bar{x} = xe^a, \quad \bar{t} = t, \quad \bar{\varphi} = \varphi, \quad \bar{\hat{q}} = \hat{q}.$$

Similarly, the transformations corresponding to the generator $X_3 = \partial_t$ define the equivalence Lie group

$$\bar{x} = x, \quad \bar{t} = t + a, \quad \bar{\varphi} = \varphi, \quad \bar{\hat{q}} = \hat{q}.$$

The transformations corresponding to the generator $X_2 = \varphi\partial_\varphi - t\partial_t$ map a function $\varphi(x,t)$ into the function $\bar{\varphi}(\bar{x},\bar{t}) = e^a\varphi(\bar{x},\bar{t}e^a)$, which gives

$$\begin{aligned}
\bar{L}\bar{\varphi} &= \bar{\varphi}_{\bar{t}}(\bar{x},\bar{t}) + \bar{\varphi}(\bar{x},\bar{t})\bar{\varphi}(0,\bar{t}) - \int_0^1 \bar{\varphi}(\bar{x}s,\bar{t})\bar{\varphi}(\bar{x}(1-s),\bar{t})\,ds \\
&= e^a\varphi_{\bar{t}}(\bar{x},\bar{t}e^a) + e^a\varphi(\bar{x},\bar{t}e^a)e^a\varphi(0,\bar{t}e^a) - \int_0^1 e^{2a}\varphi(\bar{x}s,\bar{t}e^a)\varphi(\bar{x}(1-s),\bar{t}e^a)\,ds \\
&= e^{2a}\varphi_t(\bar{x},\bar{t}e^a) + e^{2a}\varphi(\bar{x},\bar{t}e^a)\varphi(0,\bar{t}e^a) - e^{2a}\int_0^1 \varphi(\bar{x}s,\bar{t}e^a)\varphi(\bar{x}(1-s),\bar{t}e^a)\,ds \\
&= e^{2a}\big(\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,ds\big) \\
&= e^{2a}L\varphi.
\end{aligned}$$

Hence, we can conclude that the transformations

$$\bar{x} = x, \quad \bar{t} = te^{-a}, \quad \bar{\varphi} = \varphi e^a, \quad \bar{\hat{q}} = \hat{q}e^{2a}$$

compose an equivalence Lie group of Equation (4.1).

The transformations corresponding to the generator $X_1 = x\varphi\partial_\varphi$ map a function $\varphi(x,t)$ into the function $\bar\varphi(\bar x,\bar t) = e^{\bar x a}\varphi(\bar x,\bar t)$, thus

$$\begin{aligned}
\bar L\bar\varphi &= \bar\varphi_{\bar t}(\bar x,\bar t) + \bar\varphi(\bar x,\bar t)\bar\varphi(0,\bar t) - \int_0^1 \bar\varphi(\bar x s,\bar t)\bar\varphi(\bar x(1-s),\bar t)\,ds \\
&= e^{\bar x a}\varphi_{\bar t}(\bar x,\bar t) + e^{\bar x a}\varphi(\bar x,\bar t)\varphi(0,\bar t) - \int_0^1 e^{\bar x a s}\varphi(\bar x s,\bar t)e^{\bar x(1-s)a}\varphi(\bar x(1-s),\bar t)\,ds \\
&= e^{\bar x a}\varphi_t(\bar x,\bar t) + e^{\bar x a}\varphi(\bar x,\bar t)\varphi(0,\bar t) - e^{\bar x a}\int_0^1 \varphi(\bar x s,\bar t)\varphi(\bar x(1-s),\bar t)\,ds \\
&= e^{xa}\big(\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,ds\big) \\
&= e^{xa}L\varphi,
\end{aligned}$$

which gives the equivalence Lie group of transformations

$$\bar x = x, \quad \bar t = t, \quad \bar\varphi = \varphi e^{xa}, \quad \bar{\hat q} = \hat q e^{xa}.$$

Thus, it has been shown that the generators

$$X_0^e = x\partial_x, \quad X_1^e = x\varphi\partial_\varphi + x\hat q\partial_{\hat q}, \quad X_2^e = \varphi\partial_\varphi - t\partial_t + x\hat q\partial_{\hat q}, \quad X_3^e = \partial_t \tag{4.41}$$

define an equivalence Lie group of Equation (4.1). The transformation

$$E: \quad \bar t = -t, \quad \bar\varphi = -\varphi \tag{4.42}$$

does not change Equation (4.1). This is an involution.

Let us study the change of the generators $X = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3$ under the transformations corresponding to these equivalence transformations. After the change defined by equivalence transformation one gets the generator

$$X = \hat x_0\hat X_0 + \hat x_1\hat X_1 + \hat x_2\hat X_2 + \hat x_3\hat X_3, \tag{4.43}$$

where

$$\hat X_0 = \bar x\partial_{\bar x}, \quad \hat X_1 = \bar x\bar\varphi\partial_{\bar\varphi}, \quad \hat X_2 = \bar\varphi\partial_{\bar\varphi} - \bar t\partial_{\bar t}, \quad \hat X_3 = \partial_{\bar t}.$$

The corresponding transformations of the basis generators are

$$X_0^{\mathrm{e}}:\ X_0 = \hat{X}_0, X_1 = e^{-a}\hat{X}_1, X_2 = \hat{X}_2, X_3 = \hat{X}_3;$$

$$X_1^{\mathrm{e}}:\ X_0 = \hat{X}_0 + a\hat{X}_1, X_1 = \hat{X}_1, X_2 = \hat{X}_2, X_3 = \hat{X}_3;$$

$$X_2^{\mathrm{e}}:\ X_0 = \hat{X}_0, X_1 = \hat{X}_1, X_2 = \hat{X}_2, X_3 = e^{-a}\hat{X}_3;$$

$$X_3^{\mathrm{e}}:\ X_0 = \hat{X}_0, X_1 = \hat{X}_1, X_2 = \hat{X}_2 + a\hat{X}_3, X_3 = \hat{X}_3,$$

or the coordinates of the generator X are changed as follows

$$X_0^{\mathrm{e}}:\quad \hat{x}_1 = x_1 e^{-a},$$

$$X_1^{\mathrm{e}}:\quad \hat{x}_1 = x_1 + a x_0,$$

$$X_2^{\mathrm{e}}:\quad \hat{x}_3 = x_3 e^{-a},$$

$$X_3^{\mathrm{e}}:\quad \hat{x}_3 = x_3 + a x_2.$$

4.3 Group Classification

In Equation (4.1), the source term is assumed to be arbitrary. Group classification of Equation (4.1) is carried out up to the equivalence transformations considered in the previous section. The method for classifying the source function $\hat{q}$ is similar to the method which was used for classifying the equation for the moment generating function in Chapter III. First of all, we note that actions of equivalence transformations corresponding to the equivalence Lie algebra spanned by the generators (4.41) are equivalent to the automorphisms of the Lie algebra L_4 spanned by the generators X_0, X_1, X_2, X_3.

In fact, the commutators of these generators are

	X_0	X_1	X_2	X_3
X_0	0	X_1	0	0
X_1	$-X_1$	0	0	0
X_2	0	0	0	$-X_3$
X_3	0	0	X_3	0

Using table of commutators, the inner automorphisms are obtained

$$A_0: \quad \hat{x}_1 = x_1 e^a,$$

$$A_1: \quad \hat{x}_1 = x_1 + a x_0,$$

$$A_2: \quad \hat{x}_3 = x_3 e^a,$$

$$A_3: \quad \hat{x}_3 = x_3 + a x_2,$$

where only changed coordinates are presented.

Thus we can conclude that the changes corresponding to the equivalence transformations are similar to actions of the inner automorphisms. Because of this property for classifying Equation (4.1), we can use an optimal system of subalgebras.

4.3.1 Optimal System of Subalgebras

The commutator table of Lie algebra L_4 coincides with the commutator table considered in Chapter III, where group classification of the equation for a moment generating function was studied. The difference in constructing an optimal system here consists of the set of involutions, in the present case the involution corresponding to $\hat{x}_1 = -x_1$ is absent. The optimal system of subalgebras is presented in Table 4.1.

For the group classification of Equation (4.1) using the optimal system of subalgebras, the functions $\hat{q}(x,t)$ are obtained by substituting into Equation (4.38)

Table 4.1 Optimal system of subalgebras of L_4 for the Fourier image of the spatially homogeneous and isotropic Boltzmann equation.

No.	Basis	No.	Basis
1.	$X_0,\ X_1,\ X_2,\ X_3$	13.	$X_0+X_3,\ X_1$
2.	$\alpha X_0+X_2,\ X_1,\ X_3$	14.	$X_1,\ X_3$
3.	$X_0,\ X_1,\ X_3$	15.	$X_0,\ X_3$
4.	$X_0,\ X_1,\ X_2$	16.	$X_0,\ X_1$
5.	$X_0,\ X_2,\ X_3$	17.	αX_0+X_2
6.	$X_2,\ X_3$	18.	X_1+X_2
7.	$X_0+X_2,\ X_1+X_3$	19.	X_1-X_2
8.	$\alpha X_0+X_2,\ X_3$	20.	X_0+X_3
9.	$X_1+X_2,\ X_3$	21.	X_1+X_3
10.	$X_1-X_2,\ X_3$	22.	X_0
11.	$X_0,\ X_2$	23.	X_1
12.	$\alpha X_0+X_2,\ X_1$	24.	X_3

the constants c_i corresponding to the basis generators of a subalgebra of the optimal system of subalgebras and solving the obtained system of equations. The result of the group classification is presented in Table 4.2, where α and κ are constant, and the function Φ is an arbitrary function of its argument.

Table 4.2 Group classification of the Fourier image of the spatially homogeneous and isotropic Boltzmann equation.

No.	$\hat{q}(t,x)$	Generators
1.	0	$X_0,\ X_1,\ X_2\ X_3$
2.	$\kappa x^2 e^{tx}$	$X_0+X_2,\ X_1+X_3$
3.	κx^α	$2X_0+\alpha X_2,\ X_3$
4.	κt^{-2}	$X_0,\ X_2$
5.	$t^{-2}\Phi(xt^\alpha)$	αX_0+X_2
6.	$t^{-(x+2)}\Phi(x)$	X_1+X_2
7.	$t^{x-2}\Phi(x)$	X_1-X_2
8.	$\Phi(xe^{-t})$	X_0+X_3
9.	$e^{xt}\Phi(x)$	X_1+X_3
10.	$\Phi(t)$	X_0
11.	$\Phi(x)$	X_3

4.3.2 Illustrative Examples of Obtaining the Function $\hat{q}$

Let us consider the subalgebra $\{\alpha X_0+X_2,\ X_3\}$. For this Lie algebra, there are two sets of coefficients c_i, $(i=0,1,2,3)$:

$$\alpha X_0+X_2: \quad c_0=\alpha \quad c_1=0 \quad c_2=1 \quad c_3=0;$$

$$X_3: \quad c_0=0 \quad c_1=0 \quad c_2=0 \quad c_3=1.$$

These sets define the system of equations by substituting the coefficient c_i into Equation (4.38):

$$t\hat{q}_t+2\hat{q}-\alpha x\hat{q}_x=0, \qquad \hat{q}_t=0.$$

The general solution of these equations is $\hat{q} = \kappa x^{\alpha}$, where α and κ are a constant. Hence the Equation (4.1) with the source function $\hat{q} = \kappa x^{\alpha}$ admits the generator $\alpha X_2 + 2X_0,\ X_3$. This case corresponds to the No.3 in Table 4.2.

For the subalgebra $\{X_1 - X_2\}$, there is only one equation

$$t\hat{q}_t + 2\hat{q} - x\hat{q}_x = 0.$$

The general solution of the equation is $\hat{q} = t^{x-2}\Phi(x)$, where $\Phi(x)$ is an arbitrary function. This case corresponds to No.7 in Table 4.2.

4.4 Invariant Solutions

The study presented in this section is devoted to constructing invariant solutions of Equation (4.1). For each obtained function $\hat{q}$, we study the admitted Lie algebras. Using an optimal system of subalgebras of these Lie algebras, we derive invariant solutions. The set of all these solutions defines the set of all possible invariant solutions of Equation (4.1). It is shown here that similar to differential equations, equations for finding invariant solutions are reduced to equations with fewer the independent variables.

4.4.1 Invariant Solutions with $\hat{q} = \kappa x^2 e^{xt}$

For the source function $\hat{q} = \kappa x^2 e^{tx}$ the admitted Lie algebra of Equation (4.1) is $\{X_0 + X_2,\ X_1 + X_3\}$. An optimal system of subalgebras of this Lie algebra consists of subalgebras: $\{X_0 + X_2\}$, $\{X_1 + X_3\}$ and $\{X_0 + X_2,\ X_1 + X_3\}$.

A representation of an invariant solution corresponding to the subalgebra $\{X_0 + X_2\}$ is $\varphi = t^{-1}r(z)$, where $z = xt$. Substituting this representation of a solution into Equation (4.1), it becomes

$$zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\,ds = \kappa z^2 e^z. \tag{4.44}$$

The latter equation (4.44) is an equation with the single independent variable z

A representation of an invariant solution corresponding to the subalgebra $\{X_1 + X_3\}$ is $\varphi = e^{xt}r(x)$. Substituting this representation of a solution into Equation (4.1), we get the reduced equation,

$$xr(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))\, ds = \kappa x^2.$$

The subalgebra $\{X_0 + X_2,\ X_1 + X_3\}$ gives an invariant solution in the form $\varphi = Kxe^{xt}$, where K is constant. After substituting this representation into Equation (4.1), we get the equation for the constant K,

$$K^2 - 6K + 6\kappa = 0.$$

If $\kappa \leq \frac{3}{2}$, then $C = 3 \pm \sqrt{9-6k}$.

4.4.2 Invariant Solutions with $\hat{q} = \kappa x^\alpha$

An optimal system of subalgebras consists of the list: $\{2X_0 + \alpha X_2,\ X_3\}$, $\{X_3\}$ and either $\{2X_0 + \alpha X_2\}$ for $\alpha \neq 0$ or $\{\gamma X_0 + X_3\}$ for $\alpha = 0$.

A representation of an invariant solution corresponding to the subalgebra $\{2X_0 + \alpha X_2,\ X_3\}$ is $\varphi = Kx^{\frac{\alpha}{2}}$, where K is constant. After substituting this representation into Equation (4.1), we get the equation for the constant K,

$$K^2B + \kappa = 0,$$

where B is the beta function, $B = \int_0^1 (s(1-s))^{\frac{\alpha}{2}} ds$.

A representation of an invariant solution corresponding to the subalgebra $\{X_3\}$ is $\varphi = r(x)$. Substituting this representation into Equation (4.1), we get the reduced equation,

$$r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \kappa x^\alpha.$$

A representation of an invariant solution corresponding to the subalgebra $\{2X_0 + \alpha X_2\}$ where $\alpha \neq 0$ is $\varphi = t^{-1}r(z)$, where $z = t^2x^\alpha$. Substituting this representation into Equation (4.1), we get the reduced equation,

$$2zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \kappa z.$$

A representation of an invariant solution corresponding to the subalgebra $\{\gamma X_0 + X_3\}$ is $\varphi = r(z)$, where $z = xe^{-\gamma t}$. Substituting this representation into Equation (4.1), we get the reduced equation,

$$-\gamma zr'(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \kappa.$$

4.4.3 Invariant Solutions with $\hat{q} = \kappa t^{-2}$

For the source function $\hat{q} = \kappa t^{-2}$ the admitted Lie algebra of Equation (4.1) is $\{X_0,\ X_2\}$. An optimal system of subalgebras of this Lie algebra consists of subalgebras: $\{X_0,\ X_2\}$, $\{\alpha X_0 + X_2\}$ and $\{X_0\}$.

The invariant solution corresponding to the subalgebra $\{X_0,\ X_2\}$ is $\varphi = -\kappa t^{-1}$.

A representation of an invariant solution corresponding to the subalgebra $\{\alpha X_0 + X_2\}$ is $\varphi = t^{-1}r(z)$, where $z = xt^\alpha$. Substituting this representation into Equation (4.1), we get the reduced equation,

$$\alpha zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \kappa.$$

A representation of an invariant solution corresponding to the subalgebra $\{X_0\}$ is $\varphi = r(t)$. Substituting this representation into Equation (4.1), we get the invariant solution $\varphi = -\kappa/t + K$, where K is constant.

4.4.4 Invariant Solutions with $\hat{q} = t^{-2}\Phi(xt^{\alpha})$

In this case the admitted Lie algebra is $\{\alpha X_0 + X_2\}$. An invariant solution has the representation $\varphi = t^{-1}r(z)$, where $z = xt^{\alpha}$. Substituting this representation into Equation (4.1), the reduced equation is

$$\alpha z r'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \Phi(z).$$

4.4.5 Invariant Solutions with $\hat{q} = t^{-(x+2)}\Phi(x)$

The admitted Lie algebra is $\{X_1 + X_2\}$. An invariant solution has the representation $\varphi = t^{-(x+1)}r(x)$. Substituting this representation into Equation (4.1), the reduced equation is

$$-(x+1)r(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \Phi(x).$$

4.4.6 Invariant Solutions with $\hat{q} = t^{x-2}\Phi(x)$

The admitted Lie algebra is $\{X_1 - X_2\}$. An invariant solution has the representation $\varphi = t^{x-1}r(x)$. Substituting this representation into Equation (4.1), the reduced equation is

$$(x-1)r(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \Phi(x).$$

4.4.7 Invariant Solutions with $\hat{q} = \Phi(xe^{-t})$

For the function $\hat{q} = \Phi(xe^{-t})$, the admitted Lie algebra is $\{X_0 + X_3\}$. Its invariant solution has the representation $\varphi = r(z)$, where $z = xe^{-t}$. Substituting this representation into Equation (4.1), the reduced equation is

$$-zr'(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\ ds = \Phi(z).$$

In particular, for BKW-solution $r = 6e^{z}(1-z)$ which gives that $\Phi = 0$.

4.4.8 Invariant Solutions with $\hat{q} = e^{xt}\Phi(x)$

The admitted Lie algebra is $\{X_1 + X_3\}$. An invariant solution has the representation $\varphi = e^{xt}r(x)$. Substituting this representation into Equation (4.1), the reduced equation is

$$xr(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \Phi(x).$$

4.4.9 Invariant Solutions with $\hat{q} = \Phi(t)$

The admitted Lie algebra is $\{X_0\}$. Its invariant solution is

$$\varphi = \int \Phi(t)\, dt.$$

4.4.10 Invariant Solutions with $\hat{q} = \Phi(x)$

The admitted Lie algebra is $\{X_3\}$. An invariant solution has the representation $\varphi = r(x)$. Substituting this representation into Equation (4.1), the reduced equation is

$$r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \Phi(x).$$

Table 4.3 Invariant solutions.

$\hat{q}(x,t)$	Generators	Representation of solutions	Reduced equations
$\kappa x^2 e^{xt}$	X_0+X_2, X_1+X_3		$\varphi = Kxe^{xt}$, $K^2 - 6K + 6\kappa = 0$
	X_0+X_2	$\varphi = t^{-1}r(z), z = xt$	$zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \kappa z^2 e^z$
	X_1+X_3	$\varphi = e^{xt}r(x)$	$xr(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \kappa x^2$
κx^α	$2X_0+\alpha X_2, X_3$		$\varphi = Kx^{\frac{\alpha}{2}}, K^2B + \kappa = 0$ and $B = \int_0^1 (s(1-s))^{\frac{\alpha}{2}} ds$
	X_3	$\varphi = r(x)$	$r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \kappa x^\alpha$
	$2X_0+\alpha X_2$	$\varphi = t^{-1}r(z)$, $z = t^2 x^\alpha$	$2zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \kappa z$
	γX_0+X_3	$\varphi = r(z)$, $z = xe^{-\gamma t}$	$-\gamma zr'(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \kappa$
κt^{-2}	X_0, X_2		$\varphi = -\kappa t^{-1}$
	$\alpha X_0 + X_2$	$\varphi = t^{-1}r(z), z = xt^\alpha$	$\alpha zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \kappa$
	X_0	$\varphi = r(t)$	$\varphi = -\kappa/t + K$
$t^{-2}\Phi(xt^\alpha)$	$\alpha X_0 + X_2$	$\varphi = t^{-1}r(z)$, $z = xt^\alpha$	$\alpha zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \Phi(z)$
$t^{-(x+2)}\Phi(x)$	$X_1 + X_2$	$\varphi = t^{-(x+1)}r(x)$	$-(x+1)r(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \Phi(x)$
$t^{x-2}\Phi(x)$	$X_1 - X_2$	$\varphi = t^{x-1}r(x)$	$(x-1)r(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \Phi(x)$
$\Phi(xe^{-t})$	$X_0 + X_3$	$\varphi = r(z)$, $z = xe^{-t}$	$-zr'(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))ds = \Phi(z)$
$e^{xt}\Phi(x)$	$X_1 + X_3$	$\varphi = e^{xt}r(x)$	$xr(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \Phi(x)$
$\Phi(t)$	X_0		$\varphi = \int \Phi(t)\, dt$
$\Phi(x)$	X_3	$\varphi = r(x)$	$r(x)r(0) - \int_0^1 r(xs)r(x(1-s))ds = \Phi(x)$

CHAPTER V

CONCLUSION

The goal of this thesis was to study the integro-differential equation which arises as the Fourier image of the spatially homogeneous and isotropic Boltzmann equation with a source term,

$$\varphi_t + \varphi(0)\varphi = \int_0^1 \varphi(xs)\varphi(x(1-s))\,ds + q, \tag{5.1}$$

by the group analysis method. Since this equation is nonlocal, the classical group analysis method could not be applied. Instead, the recently developed algorithm applying group analysis to equations with nonlocal terms was made use of in this thesis.

This thesis is separated in two parts. The first part deals with the equation for a moment generating function obtained from equation (5.1),

$$(xu_t)_x - u^2 + u(0)(xu)_x = g. \tag{5.2}$$

Although this equation is still nonlocal, it is simpler than Equation (5.1). The algorithm applied in this thesis allowed us to present a complete group classification of Equation (5.2) with respect to the source term $q(x,t)$, thus correcting the deficiencies of earlier studies of this equation (Nonenmacher(1984)).

The results achieved from the study of Equation (5.2) made it possible to continue with the study of Equation (5.1) by methods of group analysis, as shown in the second part of the thesis. The main difficulty for Equation (5.1) consisted of solving the determining equation. This task has been achieved with success, and a complete group classification of Equation (5.1) could be made.

The general method of analyzing Equations (5.1) and (5.2) consists of the following steps:

(1) Construct the determining equation.

The determining equations were obtained using the Definition (11).

(2) Analysis of the determining equation.

The determining equation of Equation (5.2) was simplified by splitting it into several equations. For the analysis of the determining equation of Equation (5.1), a particular class of solutions of Equation (5.1) was used.

(3) Solve the reduced determining equations.

An algebraic approach was applied for solving these equations.

The group classification of the two equations thus obtained separates the source terms into several classes. For each class all the invariant solutions have been studied.

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CURRICULUM VITAE

NAME : Amornrat Suriyawichitseranee

EDUCATION BACKGROUND :

- B.Eng.(Civil Engineering), Suranaree University, Thailand, 1998.
- M.Eng.(Civil Engineering), Suranaree University, Thailand, 2005.

SCHOLARSHIPS :

- 2010-2012. The Center of Excellence in Mathematics (CEM), the Commission of Higher Education (CHE), Sriayudthaya Road, Bangkok 10140, Thailand.

CONFERENCES :

- Complete group classification of the equation for generating function of the Boltzmann equation with sources, **The 3rd Academic Conference on Natural Science for Master and PhD students from ASEAN countries**, November 11-15, 2013, Phnom Penh, Cambodia.

PUBLICATIONS :

- On the equation for the power moment generating function of Boltzmann equation. Group classification with respect to a source function (with Yu.N.Grigoriev and S.V.Meleshko), **to appear in Proceeding 6th Workshop "Group Analysis of Differential Equations & Integrable Systems" 2013**, University of Cyprus, Nicosia.
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