# วิธีเลอช็องดร์-กาเลอร์คินแบบเร็วสำหรับการไหลสโตกส์เกรเดียนต์รอบทรงกลมสองลูก

นางสาวพิกุล ภูผาสุข

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

สาขาวิชาคณิตศาสตร์ประยุกต์

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ปีการศึกษา 2551

# FAST LEGENDRE-GALERKIN METHOD FOR THE GRADIENT STOKES FLOW AROUND TWO RIGID SPHERES

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**A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics**

**Suranaree University of Technology**

**Academic Year 2008**

**FAST LEGENDRE-GALERKIN METHOD FOR THE GRADIENT STOKES FLOW AROUND TWO RIGID SPHERES**

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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พิกุล ภูผาสุข : วิธีเลอช็องดร์-กาเลอร์คินแบบเร็วสำหรับการไหลสโตกส์เกรเดียนต์รอบทรงกลมคงรูปสองลูก (FAST LEGENDRE-GALERKIN METHOD FOR THE GRADIENT STOKES FLOW AROUND TWO RIGID SPHERES)
อาจารย์ที่ปรึกษา : รองศาสตราจารย์ ดร. นิโคไลน์ มอสกิน, อาจารย์ที่ปรึกษาร่วม : ศาสตราจารย์ ดร. คริสโต คริสตอพ, 74 หน้า.


ในการศึกษาความหนืดยังผลของการแขวนลอยนั้น จะพิจารณาการไหลเกรเดียนต์ซึ่งไหลผ่านอนุภาคทรงกลม เมื่อสมการของการไหลคือ $u|_{\infty} \simeq U + Gx$ โดยที่ U คือกระแสเอกรูป G คือเกรเดียนต์ของความเร็วที่อนันต์ซึ่งเป็นค่าคงตัวและ x คือเวกเตอร์ตำแหน่ง ความหนืดยังผลอันดับหนึ่ง ที่ขึ้นอยู่กับสัดส่วนเชิงปริมาตรของเฟสของอนุภาค สามารถหาได้จากการพิจารณาการไหลรอบๆ ทรงกลมเดี่ยว ในการหาความหนืดยังผลอันดับสองนั้น จะต้องแก้ปัญหาของการไหลรอบๆ ทรงกลมที่ไม่เท่ากันสองลูก ภายใต้เกรเดียนต์ของความเร็วที่อนันต์ซึ่งเป็นปัญหาในสามมิติ

ในการศึกษาครั้งนี้ ได้มีการใช้การสมมาตรของการไหล และได้ทำการลดทอนปัญหาในสามมิติให้เป็นระบบของปัญหาในสองมิติได้ห้าระบบ โดยระบบที่ง่ายที่สุดถูกปรับให้เป็นระบบของฟังก์ชันเส้นกระแส ซึ่งฟังก์ชันนี้หาค่าได้จากการแก้สมการด้วยตัวดำเนินการไบสโตกเซียน พิกัดไบสเฟียริเคิลถูกใช้สำหรับขอบของทรงกลมที่เป็นพื้นผิววิเคราะห์ วิธีสเปกตรัมแบบเร็วซึ่งใช้พหุนามเลอช็องดร์ถูกนำเสนอในการแก้สมการไบสโตกเซียนด้วยการลู่เข้าแบบเลขชี้กำลัง อีกทั้งวิธีฟังก์ชันก่อกำเนิดถูกใช้สำหรับพหุนามเชบบีเชพและพหุนามเลอช็องดร์สมทบ และนอกจากนั้นยังได้ระบบพีชคณิตแบบปิดสำหรับระบบที่พิจารณาข้างต้น


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PIKUL PUPHASUK : FAST LEGENDRE-GALERKIN METHOD FOR THE GRADIENT STOKES FLOW AROUND TWO RIGID SPHERES. THESIS ADVISOR : ASSOC. PROF. NIKOLAY MOSHKIN, Ph.D. THESIS CO-ADVISOR : PROF. CHRISTO CHRISTOV, Ph.D. 74 PP.

STOKES FLOW / TWO SPHERES / CONSTANT VELOCITY GRADIENT AT INFINITY / BI-STOKESIAN EQUATION / ASSOCIATED LEGENDRE POLYNOMIALS / SPECTRAL METHOD / GENERATING FUNCTION METHOD

When the effective viscosity of suspensions is modeled, the main gradient flow

$$\mathbf{u}|_{\infty} \simeq \mathbf{U} + \mathbb{G}\mathbf{x},$$

is perturbed by the presence of spherical inclusions. Here $\mathbf{U}$ is the uniform stream, $\mathbb{G}$ is the constant velocity gradient at infinity and $\mathbf{x}$ is a position vector. The flow around a single sphere allows one to find the average contribution to the effective viscosity within the first order with respect to the volume fractions of the particulate phase. In order to obtain the second asymptotic order, one needs to solve the problem of the flow around two non-equal spheres under constant velocity gradient at infinity, which is a 3D problem.

In this study, the underlying symmetries of the flow are used, and the full 3D problem is reduced to five conjugated 2D problems. The simplest 2D problem is formulated in terms of the stream function, which requires solving equation with bi-Stokesian operator. Bi-spherical coordinates are used for which the boundaries of the spheres are also coordinate surfaces. To solve the bi-Stokesian equations, a fast spectral method based on Legendre polynomials is proposed with exponential convergence. The method of generating function is used for both Chebyshev and associated Legendre polynomials and closed algebraic systems are obtained for the systems under considerations.

School of Mathematics

Academic Year 2008

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# ACKNOWLEDGMENTS

First of all, I would like to express my gratitude to my thesis advisor, Assoc. Prof. Dr. Nikolay P. Moshkin, who provided me with valuable advice. He took care, helped and encouraged me during doing research with patience and kindness. I also thank my thesis co-advisor, Prof. Dr. Christo I. Christov from University of Louisiana at Lafayette, who posed the problem and suggested the method used in this dissertation. He guided, encouraged and inspired me to do research, and I appreciated working with him. I would like Dr. Moshkin and Dr. Christov to know that they are not just my advisors, but also my fathers.

I am grateful to my thesis committee: Assoc. Prof. Dr. Prapasri Assawakun, Prof. Dr. Sergey V. Meleshko, Asst. Prof. Dr. Eckart Schulz, Assoc. Prof. Dr. Adrian Flood and Asst. Prof. Dr. Julaporn Benjapiyaporn for their valuable comments and useful suggestions. Special thanks are due to my instructors at Suranaree University of Technology and staffs from both SUT and ULL.

I appreciate also the help of Abhinandan Chowdhury from UL Lafayette who shared with me his Mathematica code for a problem involving Legendre series. I have a big "thank you" for my friends at SUT, Neung, Krong, Jit, Kam, Tom, Bee, Nan, Boy, Run, and O for the wonderful time during my study here, and for my friends at ULL and UTA for everything that they did for me when I visited there.

The Ministry of University Affairs of Thailand (MUA) and Khon Kaen University are acknowledged for the financial support of this research possible.

The last, but not the least: I express my deep appreciation of the unlimited love, trust and encouragement, which I received from my lovely family and my soul mate. I do love them so much.

Finally, I want to say that I'm not on the top of my life. I've just started my research journey, so let's walk together.

Pikul Puphasuk

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# CHAPTER I

# INTRODUCTION

The hydrodynamic behavior of solid particles or fluid drops moving in a continuous medium at very small Reynolds numbers has a great importance for investigations in the fields of chemical, biochemical, and environmental engineering and science.

## 1.1 Previous research

The theoretical study of Stokes of the flow created by a translating rigid sphere in a viscous fluid has been extended by Hadamard and Rybczynski to the translation of a fluid sphere (droplet). In most practical applications, particles or drops are not isolated. Rather they interact through the disturbances that they introduce in the surrounding liquid. Hence, it is important to determine how the presence of neighboring particles affects the motion of the fluid inside and outside of droplets/particles, and determine their interaction. Estimating the effective transport coefficients of heterogeneous media is of great importance for many technological processes. The most typical examples of such media are suspensions, in which the second (particulate) phase is comprised by spherical particles (the filler) that are randomly dispersed throughout the continuous phase (the matrix). The different transport problems that can be considered for a suspension are the effective electric or heat conductivity, effective viscosity, effective elasticity.

The first successful attempt to estimate the effective electric conductivity is due to Maxwell (1873) who compared the potential created by $n$ spheres of radius $a$ each to the potential of a sphere that encompasses the swarm of small spheres and has an equivalent electric conductivity in the sense of yielding the same potential at a large distance. Using this approach, Maxwell obtained the contribution to the effective conductivity of first order with respect to the volume fraction of the particulate phase. The same idea was applied by Einstein (1906) for computing the first-order in volume fraction contribution to the effective viscosity of a suspension. For the elastic moduli of a suspension, the same approach was applied (Walpole, 1972).

Jeffrey (1973) argued that the method proposed by Maxwell can give correctly only the first order in the volume fraction and went on to discuss the statistical properties of the centers of spheres. He extended the arguments of Batchelor and Green (1972) and justified the conclusion that the second order approximation in the volume fraction can be obtained only if the solution around two spheres is obtained. A comprehensive review of the works on viscosity of suspensions can be found by Herczynski and Pienkovska (1980).

The method of functional expansions (Volterra–Wiener series) with random point basis function (RPF) for rigorous treatment of the statistical properties of materials with random structure has originated in (Christov, 1981). The application to estimating the effective heat conduction modulus of monodisperse suspension was presented in (Christov and Markov, 1985a), while the elastic moduli were treated in (Christov and Markov, 1985b). After the generalization of the RPF expansion to marked random point functions was outlined (Christov, 1985a), the most general case of a polydisperse suspension of perfect disorder type became amenable to the Volterra–Wiener method (Christov and Markov, 1985c). Nowadays, it can be considered as proven that the two-sphere solution does rigorously lead to the second-order approximation with respect to the volume fraction as it gives precisely the second-order kernel in the formal Volterra-Wiener expansion. Since the effective transport properties are aimed at, one needs to solve the two-sphere problem under constant gradient of the main field at infinity. This defines the main goal of the present thesis: to develop an efficient numerical tool for solving the two-sphere problem.

A method to solve the Laplace equation, called currently "twin-pole expansion" (Jeffrey, 1973) was proposed by Hicks (1879). The method consists in expanding the solution in spherical harmonics around two poles. This method was used on numerous occasions.Its main advantage lies in the fact that the integrals needed to compute the overall transport coefficients are easy to evaluate. For this reason, Jeffrey (1973) went on to suggest that, in the context of the statistical theory of suspension, the twin-pole expansion is superior to the method involving bi-spherical coordinates. This claim is not immediately verifyable because the twin-pole expansion actually involves two levels of approximation: the first level is the truncation of the Legendre series. The second level of approximation stems from the fact that the functional coefficients of the series which depend on the radial coordinate, cannot be found in closed form. Rather, the solution is sought in asymptotic series with respect to the small parameter $r/D$, where $r$ is the radius of the bigger of the spheres, and $D$ is the distance between their centers.

The procedure of asymptotic solution can be interpreted physically as adding to the solution created by the boundary condition on one of the boundaries, a solution that is reflected from the other boundary. The procedure is also known as the "method of reflections" (Happel and Brenner, 1983). It has been successfully applied in various problems with two boundaries (e.g., for Stokes flow around a sphere in a cylindrical pipe (Zimmerman, 2004)).

For the case of closely situated spheres when one of the radii is much greater than the other, the said parameter can actually tend to unity, which can make the respective series very slowly convergent. The bi-spherical coordinates offer an approach that is free of this limitation.

Without belittling the importance of the twin-pole expansion, a numerical solution with controlled convergence is still in demand, if for no other reason, but at least for estimating the region of convergence of the twin-pole expansion. The approach based on bi-spherical coordinates gives the solution in closed form, albeit in an infinite series with respect to the Legendre polynomials.

A successful numerical (e.g., spectral) solution is contingent on finding the appropriate curvilinear coordinates in which the boundaries of the domain of the solution are coordinate lines. The fact that the bi-spherical coordinates are the best suited tool for solving a transport problem in a medium containing two spherical solutions was first emphasized by Lord Kelvin. He was apparently the first to introduce the bi-spherical coordinates in 1846 in a letter to Liouville (Thomson, 1884). The first detailed application of the bi-spherical coordinates for solving the Laplace equation was given by G. B. Jeffery (1912) for the potential flow around two spheres. The important difference here is that the flow stream has a constant gradient at infinity. The situation with the two-particle problem is much more complicated because of technical difficulties connected with the solution. Legendre *et al.* (2003), numerically solved the three dimensional flow past two identical spherical bubbles moving side by side in a viscous fluid for Reynolds number $0.02 \leq Re \leq 500$ and calculated the drag and the lift forces. Similar results were given by Kim *et al.* (1993), where the drag and the lift forces were discussed for two rigid spheres placed in a uniform stream perpendicular to their line of centers. Ardekani and Rangel (2006), studied the unsteady motion of two solid spherical particles in an unbounded incompressible Newtonian flow, where the background flow can be time dependent. The application of the bi-spherical coordinates to the heat conduction problem around two spheres with constant gradient at infinity was studied by Christov (1985b), where the Legendre-series method is worked out analytically but no numerical results were presented. An important advance in that paper was the proposed effective way to reduce an essentially 3D problem to a set of three 2D problems. The numerical results for this case have recently been obtained by Chowdhury and Christov (2009). They present the solutions for the temperature distribution with both longitudinal and transverse gradients at infinity, and demonstrated the very fast convergence of the Legendre-series method for problems of the type they considered.

## 1.2 Objectives and overview of the thesis

Due to the specific dimension of the particles and the intraparticle distances (characteristic lengths) in suspensions, the particles can be considered as being very small and their movements can be considered to be very slow. As a result of these assumptions, the problems under consideration are, in fact, quasisteady, and the explicit dependence on time can be neglected in the equations. In this study we shall be interested in the problem of the gradient creeping (Stokes) flow around two spheres. Following the gist of the works of Christov (1985b), and Chowdhury and Christov (2009), we generalized the idea there, and succeeded in reducing the original 3D problem to five 2D problems. This radically reduces the complexity of the problem when treated numerically. Note that the flow of viscous liquid is mathematically speaking significantly more complicated than the problem of heat conduction, where the 3D problem was equivalent to just three 2D problems. The five systems are one of the important contributions of the present work, and are shown in Chapter III. The decision is made there to focus only on the first of the system derived there, and it is recast in bi-spherical coordinates in terms of stream function.

In Chapter IV, the objective is to find a semi-analytical solution in Legendre series for the stream function from the first of these systems. To make the exposition self-contained and to facilitate the reader, the pertinent properties of the Legendre polynomials are compiled in Chapter II. The important contribution of Chapter IV is that for the first time in the literature, a Legendre series technique is applied for solving equation with bi-Stokesian operator. We show how to obtain a closed system for the the unknown coefficients of the Legendre series for the stream function by satisfying the boundary conditions which are expanded into series with respect to associated Legendre polynomials. The chapter shows how the algebraic systems for these boundary conditions are derived.

The method created in Chapter IV is implemented numerically in Chapter V, and it is demonstrated that the theoretically expected exponential convergence is splendidly confirmed by our numerical experiments. After the validations performed in Chapter V, the method is judged to be reliable and very efficient, and some specific preliminary result about the flow around two spheres are presented in Chapter VI for the stream lines of the flow. Finally, conclusions and recommendation for the future research are provided in chapter VII.

# CHAPTER II

# LEGENDRE POLYNOMIALS

In this chapter we compile the necessary information on the Legendre polynomials in order to make the presentation self-contained (Andrews, 1985).

We recall the binomial series

$$(1-u)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n u^n, \quad |u| < 1. \tag{2.1}$$

Upon setting $u = t(2x-t)$, we find that

$$w(x,t) = (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n t^n (2x-t)^n, \tag{2.2}$$

which is valid for $|2xt - t^2| < 1$. For $|t| < 1$, it follows that $|x| \leq 1$. The factor $(2x-t)^n$ is simply a finite binomial series, and thus (2.2) can further be expressed as

$$w(x,t) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n t^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k (2x)^{n-k} t^k \qquad \text{or} \tag{2.3}$$

$$w(x,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{n} \binom{n}{k} (-1)^{n+k} (2x)^{n-k} t^{n+k}. \tag{2.4}$$

Since our goal is to obtain a power series involving powers of $t$ by a single index, the change of indice $n \to n-k$ is suggested. Thus, Eq. (2.4) can be written in the equivalent form

$$w(x,t) = \sum_{n=0}^{\infty} \Big\{ \sum_{k=0}^{[n/2]} \binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} (-1)^n (2x)^{n-2k} \Big\} t^n. \tag{2.5}$$

The innermost summation in (2.5) is of finite length and therefore represents a polynomial in $x$, which happens to be of degree $n$. If we denote this polynomial by

$$P_n(x) = \sum_{k=0}^{[n/2]} \binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} (-1)^n (2x)^{n-2k}, \tag{2.6}$$

then Eq. (2.5) leads to the intended result

$$w(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n, \quad |x| \le 1, |t| < 1, \tag{2.7}$$

where $w(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}}$. The polynomial $P_n(x)$ are called the *Legendre polynomials* in honor of their discoverer. Since

$$\binom{-\frac{1}{2}}{n} = (-1)^n \binom{n-\frac{1}{2}}{n} = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}, \tag{2.8}$$

it follows that the product of binomial coefficients is

$$\binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} = \frac{(-1)^{n-k}(2n-2k)!}{2^{2n-2k}(n-k)!k!(n-2k)!}, \tag{2.9}$$

and hence

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)! x^{n-2k}}{2^n (n-k)!k!(n-2k)!} \tag{2.10}$$

We note here that when $n$ is an even number, the polynomial $P_n(x)$ is an even function, and when $n$ is odd the polynomial is an odd function. Therefore,

$$P_n(-x) = (-1)^n P_n(x), \quad n = 0, 1, 2, ... \tag{2.11}$$

## 2.1 Basic properties of $P_n(x)$

The Legendre polynomials are rich in recurrence relations and identities. Central to the development of many of these is the *generating function relation*

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n, \quad |x| \le 1, \ |t| < 1. \tag{2.12}$$

In order to obtain the desired recurrence relations, we observe that the function $w(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}}$ satisfies the differential equation

$$(1 - 2xt + t^2) \frac{\partial w}{\partial t} + (t - x) w = 0. \tag{2.13}$$

Direct substitution of the series (2.7) for $w(x,t)$ into (2.13) yields

$$(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x)t^n = 0. \tag{2.14}$$

After algebraic manipulations, one obtains

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \tag{2.15}$$

where $n = 1, 2, 3, ...$

A relation involving derivatives of the Legendre polynomials can be derived in the same fashion by first making the observation that $w(x,t)$ satisfies

$$(1-2xt+t^2)\frac{\partial w}{\partial t} - tw = 0, \tag{2.16}$$

where this time the differentiation is with respect to $x$. Substituting the series for $w(x,t)$ directly into (2.16) leads to

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} = 0. \tag{2.17}$$

After rearrangement, we get

$$P_{n+1}'(x) - 2xP_n'(x) + P_{n-1}'(x) - P_n(x) = 0, \tag{2.18a}$$

$$P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x), \tag{2.18b}$$

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x), \tag{2.18c}$$

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x), \tag{2.18d}$$

where $n = 1, 2, 3, ...$

All the recurrence relations that have been derived thus far involve successive Legendre polynomials. We may well wonder if any relation exists between derivatives of the Legendre polynomials and Legendre polynomials of the same index. The answer is in the affirmative, but to derive this relation we must consider second derivatives of the polynomials.

By taking the derivative of both sides of (2.18d), we get

$$\frac{d}{dx}\Big[(1-x^2)P_n'(x)\Big] = nP_{n-1}'(x) - nP_n(x) - nxP_n'(x), \tag{2.19}$$

and then, using (2.18c) to eliminate $P_{n-1}'(x)$, we arrive at the derivative relation

$$\frac{d}{dx}\Big[(1-x^2)P_n'(x)\Big] + n(n+1)P_n(x) = 0, \tag{2.20}$$

which holds for $n = 0, 1, 2, ....$

Expanding the product term in (2.20) yields

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0, \tag{2.21}$$

and thus we deduce that the Legedre polynomial $y = P_n(x)$ $(n = 0, 1, 2, ...)$ is a solution of the linear second order DE

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \tag{2.22}$$

which is called *Legendre's differential equation.*

Perhaps the most natural way in which Legendre polynomials arise in applications is in their virtue of being the solutions of Legendre's equation. In such problems the basic model is generally a partial differential equation. Solving the partial DE by the separation of variables leads to a system of ODEs, and sometimes one of these is Legendre's ODE, Eq. (2.22).

The first few Legendre polynomials are listed in Table 2.1. The graphs of $P_n(x), n = 0, 1, 2, 3, 4, 5$ are sketched in Fig. 2.1

A representation of the Legendre polynomials involving differentiation is given by the *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\Big[(x^2-1)^n\Big], \quad n = 0, 1, 2, \ldots. \tag{2.23}$$

In order to verify (2.23), we start with the binomial series

$$(x^2-1)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} x^{2n-2k}, \tag{2.24}$$

| |
|---|
| $P_0(x) = 1$ |
| $P_1(x) = x$ |
| $P_2(x) = \frac{1}{2}(-1 + 3x^2)$ |
| $P_3(x) = \frac{1}{2}(-3x + 5x^3)$ |
| $P_4(x) = \frac{1}{8}(3 - 30x^2 + 35x^4)$ |
| $P_5(x) = \frac{1}{8}(15x - 70x^3 + 63x^5)$ |
| $P_6(x) = \frac{1}{16}(-5 + 105x^2 - 315x^4 + 231x^6)$ |
| $P_7(x) = \frac{1}{16}(-35x + 315x^3 - 693x^5 + 429x^7)$ |
| $P_8(x) = \frac{1}{128}(35 - 1260x^2 + 6930x^4 - 12012x^6 + 6435x^8)$ |

**Table 2.1** Legendre polynomials $P_n(x)$

and differentiate $n$ times. Noting that

$$\frac{d^n}{dx^n} x^m = \begin{cases} \frac{m!}{(m-n)!} x^{m-n}, & n \leq m \\ 0, & n > m \end{cases} \tag{2.25}$$

we infer that

$$\frac{d^n}{dx^n}\left[(x^2-1)^n\right] = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} = 2^n n! P_n(x) \tag{2.26}$$

from which Eq.(2.23) now follows.

The orthogonality property for Legendre polynomials is

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1}, & n = m \end{cases} \tag{2.27}$$

The theorem for convergence of Legendre series is given by (see Andrews (1985))

**Theorem 2.1.** *If the function $f$ is piecewise smooth in the closed interval $-1 \leq x \leq 1$, then the Legendre series*

$$f(x) = \sum_{k=0}^{\infty} c_k P_k(x)\,, \quad c_k = (k + \frac{1}{2}) \int_{-1}^{1} f(x) P_k(x) dx, \tag{2.28}$$

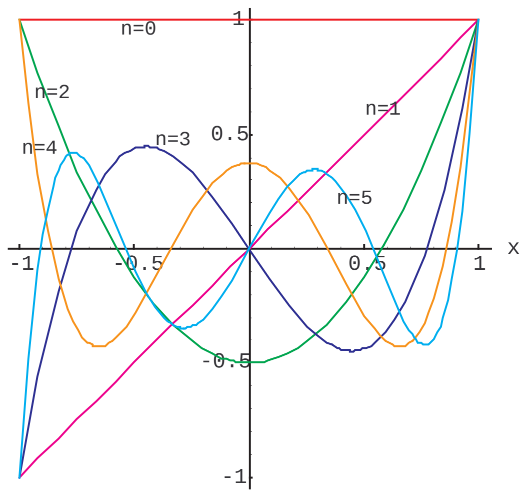



**Figure 2.1** Graph of $P_n(x), n = 0, 1, 2, 3, 4, 5$

*converges pointwise to $f(x)$ at every continuity point of the function $f$ in the interval $-1 < x < 1$. At points of discontinuity of $f$ in the interval $-1 < x < 1$, the series converges to the average values $\frac{1}{2}[f(x^+) + f(x^-)]$. Finally, at $x = -1$ the series converges to $f(-1^+)$, and at $x = 1$ it converges to $f(1^-)$*

## 2.2 Associated Legendre polynomials

In applications involving either the Laplace or the Helmholtz equation in spherical, oblate spheroidal, or prolate spheroidal coordinates, it is not Legendre's equation that ordinarily arises but rather the *associated Legendre equation*

$$(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]y = 0. \tag{2.29}$$

Observe that for $m = 0$, (2.29) reduces to Legendre's equation. The DE (2.29) and its solutions, called *associated Legendre functions*, can be developed directly from Legendre's equation and its solutions.

If $z$ is a solution of Legendre's equation, i.e. if

$$(1 - x^2)z'' - 2xz' + n(n+1)z = 0, \tag{2.30}$$

then

$$y = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} z, \tag{2.31}$$

is a solution of (2.29).

We define the *associated Legendre functions* by

$$P_n^{(m)}(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x)\,, \quad m = 0, 1, 2, ..., n. \tag{2.32}$$

The first few associated Legendre polynomials $P_n^{(1)}(x)$ are listed in Table 2.2 The graphs of $P_n^{(1)}(x), n = 1, 2, 3, 4, 5$ are sketched in Fig. 2.2

| |
|---|
| $P_1^{(1)}(x) = -(1-x^2)^{\frac{1}{2}}$ |
| $P_2^{(1)}(x) = -3x(1-x^2)^{\frac{1}{2}}$ |
| $P_3^{(1)}(x) = -\frac{3}{2}(1-x^2)^{\frac{1}{2}}(-1+5x^2)$ |
| $P_4^{(1)}(x) = -\frac{5}{2}(1-x^2)^{\frac{1}{2}}(-3x+7x^3)$ |
| $P_5^{(1)}(x) = -\frac{15}{8}(1-x^2)^{\frac{1}{2}}(1-14x^2+21x^4)$ |
| $P_6^{(1)}(x) = -\frac{21}{8}(1-x^2)^{\frac{1}{2}}(5x-30x^3+33x^5)$ |
| $P_7^{(1)}(x) = -\frac{7}{16}(1-x^2)^{\frac{1}{2}}(-5+135x^2-495x^4+429x^6)$ |
| $P_8^{(1)}(x) = -\frac{9}{16}(1-x^2)^{\frac{1}{2}}(-35x+285x^3-1001x^5+715x^7)$ |

**Table 2.2** Associated Legendre polynomials $P_n^{(1)}(x)$

## 2.3 Basic properties of $P_n^{(m)}(x)$

Using the Rodrigues formula (2.23), it is possible to write (2.32) in the form

$$P_n^{(m)}(x) = \frac{(-1)^m}{2^n n!}(1-x^2)^{\frac{m}{2}} \frac{d^{n+m}}{dx^{n+m}}\Big[(x^2-1)^n\Big]. \tag{2.33}$$

Here we make the interesting observation that the right hand side of (2.33) is well defined for all values of $m$ such that $n+m \geq 0$, i.e., for $m \geq -n$, whereas (2.32)

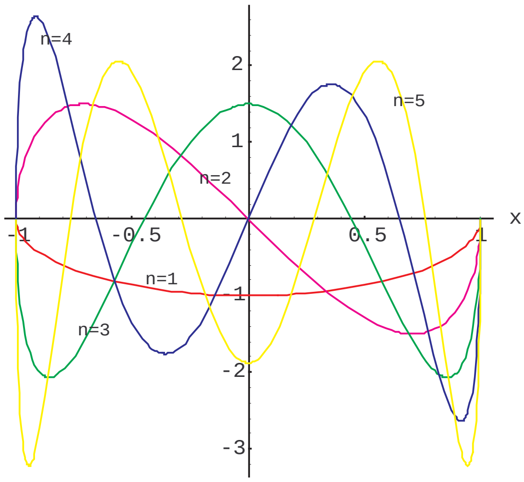



**Figure 2.2** Graph of $P_n^{(1)}(x), n = 1, 2, 3, 4, 5$

is valid only for $m \geq 0$. Thus, (2.33) may be used to extend the definition of $P_n^{(m)}(x)$ to include all integer values of $m$ such that $-n \leq m \leq n$ (If $m > n$, then necessarily $P_n^{(m)}(x) \equiv 0$, which we leave to the reader to prove). The functions for negative $n$ are defined by $P_{-n}^{(m)} = P_{n-1}^{(m)}$. Moreover, using the Leibniz formula

$$\frac{d^m}{dx^m}(fg) = \sum_{k=0}^{m} \left( \begin{array}{c} m \\ k \end{array} \right) \frac{d^{m-k} f}{dx^{m-k}} \frac{d^k g}{dx^k}, \quad m = 1, 2, 3, \ldots, \tag{2.34}$$

it can be shown that

$$P_n^{(-m)}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^{(m)}(x). \tag{2.35}$$

Lastly, we note that for $m = 0$ we get the special case

$$P_n^{(0)}(x) = P_n(x). \tag{2.36}$$

The associated Legendre functions $P_n^{(m)}(x)$ satisfy many recurrence relations, several of which are generalizations of the recurrence formulas for $P_n(x)$. But because $P_n^{(m)}(x)$ has two indices instead of just one, there exists a wider variety of possible relations than for $P_n(x)$. To derive the *three-term recurrence* *formula* for $P_n^{(m)}(x)$, we start with the known relation

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0. \tag{2.37}$$

Upon differentiating the last expression $m$ times, and using the recurrence relations from the previous section one gets

$$(n-m+1)P_{n+1}^{(m)}(x) - (2n+1)xP_n^{(m)}(x) + (n+m)P_{n-1}^{(m)}(x) = 0. \tag{2.38}$$

Additional recurrence relations include the following

$$(1-x^2)P_n'^{(m)}(x) = (n+m)P_{n-1}^{(m)}(x) - nxP_n^{(m)}(x), \tag{2.39}$$

$$(1-x^2)P_n'^{(m)}(x) = (n+1)xP_n^{(m)}(x) - (n-m+1)xP_{n+1}^{(m)}(x), \tag{2.40}$$

$$(1-x^2)^{\frac{1}{2}}P_n^{(m)}(x) = \frac{1}{2n+1}\Big[P_{n+1}^{(m+1)}(x) - P_{n-1}^{(m+1)}(x)\Big], \tag{2.41}$$

$$\begin{aligned}(1-x^2)^{\frac{1}{2}}P_n^{(m)}(x) &= \frac{1}{2n+1}\Big[(n+m)(n+m-1)P_{n-1}^{(m-1)}(x), \\ &\quad - (n-m+1)(n-m+2)P_{n+1}^{(m-1)}(x)\Big],\end{aligned} \tag{2.42}$$

$$\begin{aligned}P_n^{(m+1)}(x) &= 2mx(1-x^2)^{-\frac{1}{2}}P_n^{(m)}(x) \\ &\quad - [n(n+1) - m(m-1)]P_n^{(m-1)}(x),\end{aligned} \tag{2.43}$$

$$(1-x^2)^{1/2}P_l^{(m+1)}(x) = (l-m)xP_n^{(m)}(x) - (n+m)P_{n-1}^{(m)}(x), \tag{2.44}$$

$$(x^2-1)P_n^{(m)\prime}(x) = lxP_n^{(m)}(x) - (n+m)P_{n-1}^{(m)}(x), \tag{2.45}$$

$$\begin{aligned}(x^2-1)P_n^{(m)\prime}(x) &= -(n+m)(n-m+1)(1-x^2)^{1/2}P_n^{(m-1)}(x) \\ &\quad - mxP_n^{(m)}(x).\end{aligned} \tag{2.46}$$

The generating function for $P_n^{(m)}(x)$ is

$$\frac{(1-x^2)^{\frac{m}{2}}}{(1-2tx+t^2)^{m+1/2}} = \frac{(-1)^m 2^m m!}{(2m)!}\sum_{n=0}^{\infty} t^n P_{n+m}^{(m)}(x)t^n\,, \quad |x|<1, |t|<1. \tag{2.47}$$

The orthogonality of associated Legendre polynomials can be shown that

$$\int_{-1}^{1} P_n^{(m)}(x)P_k^{(m)}(x)dx = 0, \quad k \neq n, \tag{2.48}$$

and

$$\int_{-1}^{1} [P_n^{(m)}(x)]^2 dx = \frac{2(n+m)!}{(2n+1)(n-m)!}. \tag{2.49}$$

Moreover, it can be shown that

$$\int_{-1}^{1} \frac{P_n^{(m)}(x) P_n^{(k)}(x)}{1-x^2} dx = \begin{cases} 0, & k \neq m, \\ \frac{(n+m)!}{m(n-m)!}, & k = m \neq 0, \\ \infty\,, & k = m = 0. \end{cases} \tag{2.50}$$

## 2.4 Chebyshev Polynomials of the Second Kind

Chebyshev polynomials of the second kind of degree $n$ are denoted by $U_n$. They are the solutions to the Chebyshev differential equation:

$$(1-x^2)y'' - 3xy' + n(n+2)y = 0. \tag{2.51}$$

In general, a series solution about the origin will only converge for $|x| < 1$, when $n$ is an integer. The first few Chebyshev polynomials of the second kind are listed in Table 2.3. The graphs of $U_n(x), n = 0, 1, 2, 3, 4, 5$ are sketched in Fig. 2.3

| |
|---|
| $U_0(x) = 1$ |
| $U_1(x) = 2x$ |
| $U_2(x) = -1 + 4x^2$ |
| $U_3(x) = -4x + 8x^3$ |
| $U_4(x) = 1 - 12x^2 + 16x^4$ |
| $U_5(x) = 6x - 32x^3 + 32x^5$ |
| $U_6(x) = -1 + 24x^2 - 80x^4 + 64x^6$ |
| $U_7(x) = -8x + 80x^3 - 192x^5 + 128x^7$ |

**Table 2.3** Chebyshev polynomials $U_n(x)$

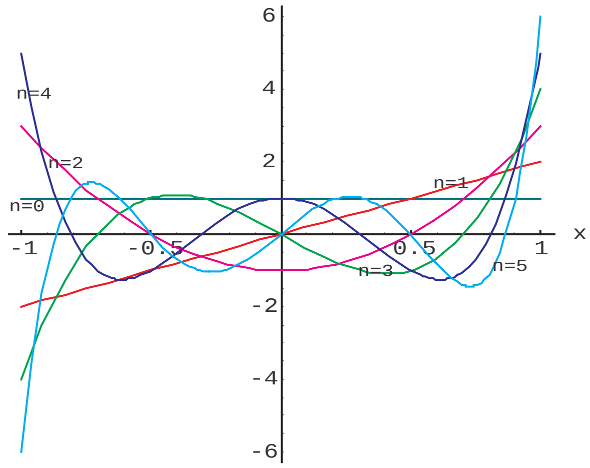



**Figure 2.3** Graph of $U_n(x), n = 0, 1, 2, 3, 4, 5$

## 2.5 Basic properties of $U_n(x)$

Rodrigues's Formula for $U_n(x)$ reads

$$U_n(x) = \frac{(-1)^n (n+1)\sqrt{\pi}}{2^{n+1}(n+\frac{1}{2})!(1-x^2)^{\frac{1}{2}}} \frac{d^n}{dx^n}(1-x^2)^{n+\frac{1}{2}}. \tag{2.52}$$

The polynomials can also be defined in term of the sum

$$U_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} (2x)^{n-2r}, \tag{2.53}$$

and they satisfy the orthogonality condition

$$\int_{-1}^{1} U_n(x) U_m(x) (1-x^2)^{\frac{1}{2}} dx = \begin{cases} 0, & n \neq m \\ \frac{\pi}{2}, & n = m. \end{cases} \tag{2.54}$$

The generating function for $U_n$ is

$$\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_n(x) t^n , \quad |x| < 1, |t| < 1. \tag{2.55}$$

The Chebyshev polynomials $U_n(x)$ of the second kind satisfy many recurrence relations. The list here those that are pertinents to the present work:

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \tag{2.56}$$

$$U'_{n+1}(x) + U'_{n-1}(x) = 2xU'_n(x) + 2U_n(x), \tag{2.57}$$

$$(1 - x^2)U'_n(x) = -nxU_n(x) + (n+1)U_{n-1}(x), \tag{2.58}$$

$$(1 - x^2)U''_n(x) = 3xU'_n(x) - n(n+2)U_n(x). \tag{2.59}$$

# CHAPTER III

# REDUCTION OF THE 3-DIMENSIONAL B.V.P. TO FIVE 2-DIMENSIONAL B.V.P.'S

One of the crucial elements of the present work is the reduction of the original 3D problem for the gradient flow around two spheres to five 2D problems. In order to elucidate the main idea, we will demonstrate the idea of reduction in terms of cylindrical coordinates, and only when the 2D problems are obtained we will move from cylindrical to bi-spherical coordinates.

## 3.1 Statement of the problem

Consider two rigid spheres of generally unequal radii $r_i, i = 1, 2$ suspended in an incompressible fluid of kinematic viscosity $\nu$. The centers of spheres are laid on the $z-$axis, and the distances of sphere centers from the origin are $d_1$ and $d_2$, respectively. Assuming creeping flow (very low Reynolds number), one can use the linear Stokes equations instead of the full Navier-Stokes equations. The boundary conditions include the non-slip condition on the sphere's boundaries, and the requirement for a constant velocity gradient at infinity. A sketch of the problem is shown in Figure 3.1.

The Stokes equations read

$$\nabla p = \nu \nabla^2 \mathbf{u}, \quad \mathbf{x} \in \Omega, \tag{3.1a}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{x} \in \Omega \tag{3.1b}$$

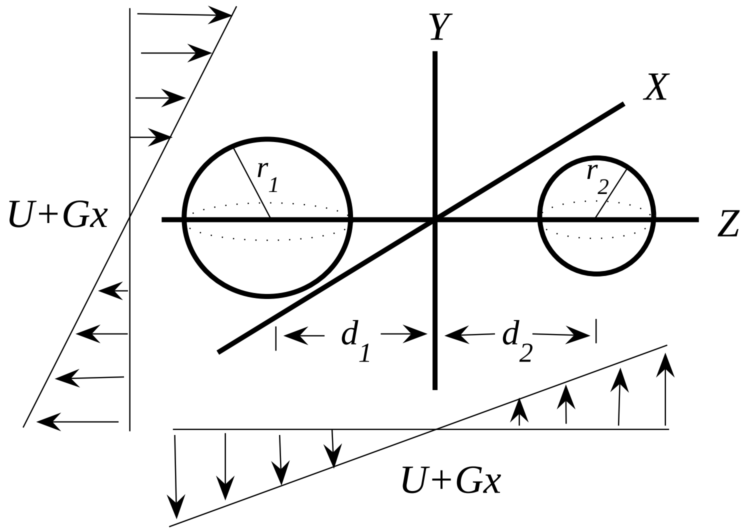



**Figure 3.1** Gradient flow around two spheres

The boundary conditions on the spheres are

$$\mathbf{u}|_{\Gamma} = 0, \tag{3.2a}$$

here $\Gamma$ is the composite boundary of the two inclusions (spheres), $\Omega$ is the region exterior to the spheres. The flow of interest here is the one with constant gradient at infinity, namely

$$\mathbf{u}|_{\mathbf{x}\to\infty} = \mathbf{U} + \mathbb{G}\mathbf{x}, \tag{3.2b}$$

where $\mathbf{U}$ is the constant velocity at infinity, while the tensor of the velocity gradient at infinity is denoted by $\mathbb{G}$. In the above formula we use the notations $\mathbf{x}$ for the position vector, and $\mathbf{u}$ for the velocity vector, namely

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} U^{(x)} \\ U^{(y)} \\ U^{(z)} \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}. \tag{3.3}$$

For the components of the velocity, the asymptotic boundary conditions read

$$u_x = U^{(x)} + G_{11}x + G_{12}y + G_{13}z, \tag{3.4a}$$

$$u_y = U^{(y)} + G_{21}x + G_{22}y + G_{23}z, \tag{3.4b}$$

$$u_z = U^{(z)} + G_{31}x + G_{32}y + G_{33}z. \tag{3.4c}$$

Eqs. (3.1) can be written for the perturbation

$$\mathbf{v} = \mathbf{u} - \widetilde{\mathbf{U}}, \tag{3.5}$$

where $\widetilde{\mathbf{U}} = \mathbf{U} + \mathbb{G}\mathbf{x}$, the velocity at infinity. Then

$$\nabla p = \nu \nabla^2 \mathbf{v}, \quad \mathbf{x} \in \Omega, \tag{3.6a}$$

$$\nabla \cdot \mathbf{v} = 0, \quad \mathbf{x} \in \Omega, \tag{3.6b}$$

with boundary conditions

$$\mathbf{v}|_\Gamma = -\widetilde{\mathbf{U}}|_\Gamma, \tag{3.7a}$$

$$\mathbf{v}|_{\mathbf{x}\to\infty} = 0. \tag{3.7b}$$

Note that the incompressibility condition imposes the following restriction on the components of the velocity gradient at infinity: $G_{11} + G_{22} + G_{33} = 0$.

## 3.2 Acknowledging the Symmetry of Boundary Conditions

The important difference between the considered problem and other similar studies is that the flow at infinity has a constant velocity gradient. This makes the problem three dimensional. Cases of two spheres in a uniform free stream are considered by many authors. An effective way to take advantage of the linearity of the problem when dealing with the boundary conditions was proposed in Christov

(1985) for the case of the temperature distribution around two spheres with a constant gradient at infinity. The gist of that method is to take advantage of the fact that there is no explicit dependence on the polar angle in the Laplace equation (what is called a 'cyclic variable'), and the authors presented the sought solution as a linear combination of Fourier functions of the polar angle as dictated only by the boundary conditions. We generalize this idea to our case. The essential difference in our case is that there are five functions of the polar coordinates that enter the boundary conditions.

Let us first render Eqs. (3.6) into cylindrical coordinates $(r, \phi, z)^*$

$$x = r\cos\phi,\; y = r\sin\phi,\; z = z, \tag{3.8}$$

where $r \geq 0\,, \phi \in [0, 2\pi]\,, z \in (-\infty, \infty)$. Then Eqs. (3.6) adopt the form

$$\nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r v_r + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial^2 v_r}{\partial z^2}\Big] = \frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{3.9a}$$

$$\nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r v_\phi + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} + \frac{\partial^2 v_\phi}{\partial z^2}\Big] = \frac{1}{\rho r}\frac{\partial p}{\partial \phi}, \tag{3.9b}$$

$$\nu\Big[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2} + \frac{\partial^2 v_z}{\partial z^2}\Big] = \frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{3.9c}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r v_r + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0, \tag{3.9d}$$

where $v_r, v_\phi, v_z$ are the velocity components in terms of the cylindrical coordinates

$$v_r = v_x\cos\phi + v_y\sin\phi, \quad v_\phi = -v_x r\sin\phi + v_y r\cos\phi, \quad v_z = v_z. \tag{3.10}$$

In terms of the cylindrical coordinates, the boundary conditions on the sphere surfaces can be recast as follows

$$\begin{aligned} v_r|_\Gamma = &-(G_{11} + G_{22})\frac{r}{2} - (U^{(x)} + G_{13}z)\cos\phi - (U^{(y)} + G_{23}z)\sin\phi \\ &- (G_{11} - G_{22})\frac{r}{2}\cos 2\phi + (G_{12} + G_{21})\frac{r}{2}\sin 2\phi, \end{aligned} \tag{3.11a}$$

---

$^*$For details, see appendix A

$$v_\phi|_\Gamma = -(G_{21} - G_{12})\frac{r}{2} - (U^{(y)} + G_{23}z)\cos\phi + (U^{(x)} + G_{13}z)\sin\phi$$
$$- (G_{21} + G_{12})\frac{r}{2}\cos 2\phi - (G_{22} - G_{11})\frac{r}{2}\sin 2\phi, \qquad (3.11b)$$

$$v_z|_\Gamma = -(U^{(z)} + G_{33}z) - G_{31}r\cos\phi - G_{32}r\sin\phi. \qquad (3.11c)$$

Recall that at infinity we have $v_r = v_\phi = v_z = 0$ and that the variable $\phi$ is a cyclic variable, i.e., it does not enter the coefficients of equations (3.9). This means that the symmetry of the boundary conditions is entirely defined by Eqs. (3.11), which hints at the idea that one can seek the solutions of the 3D problems in the form of the following linear combinations

$$v_r(r,\phi,z) = v_r^{(0)}(r,z) + v_r^{(1)}(r,z)\cos\phi + v_r^{(2)}(r,z)\sin\phi + v_r^{(3)}(r,z)\cos 2\phi$$
$$+ v_r^{(4)}(r,z)\sin 2\phi, \qquad (3.12a)$$

$$v_z(r,\phi,z) = v_z^{(0)}(r,z) + v_z^{(1)}(r,z)\cos\phi + v_z^{(2)}\sin\phi + v_z^{(3)}(r,z)\cos 2\phi$$
$$+ v_z^{(4)}(r,z)\sin 2\phi, \qquad (3.12b)$$

$$v_\phi(r,\phi,z) = v_\phi^{(0)}(r,z) + v_\phi^{(1)}(r,z)\cos\phi + v_\phi^{(2)}(r,z)\sin\phi + v_\phi^{(3)}(r,z)\cos 2\phi$$
$$+ v_\phi^{(4)}(r,z)\sin 2\phi, \qquad (3.12c)$$

$$p(r,\phi,z) = p^{(0)}(r,z) + p^{(1)}(r,z)\cos\phi + p^{(2)}(r,z)\sin\phi + p^{(3)}(r,z)\cos 2\phi$$
$$+ p^{(4)}(r,z)\sin 2\phi. \qquad (3.12d)$$

Since the Stokes equations are linear and the functions $1, \cos\phi, \sin\phi, \cos 2\phi$ and $\sin 2\phi$ are linearly independent, the 3D governing equations (3.9) naturally split into the following five conjugated 2D problems.

**System 1.**

$$\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r v_r^{(0)} + \frac{\partial^2 v_r^{(0)}}{\partial z^2}\Big], \tag{3.13a}$$

$$\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial z} = \nu\Big[\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial v_z^{(0)}}{\partial r} + \frac{\partial^2 v_z^{(0)}}{\partial z^2}\Big], \tag{3.13b}$$

$$0 = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r v_\phi^{(0)} + \frac{\partial^2 v_\phi^{(0)}}{\partial z^2}\Big], \tag{3.13c}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r} r v_r^{(0)} + \frac{\partial v_z^{(0)}}{\partial z}. \tag{3.13d}$$

**System 2.**

$$\frac{1}{\rho}\frac{\partial p^{(1)}}{\partial r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r v_r^{(1)} - \frac{1}{r^2} v_r^{(1)} - \frac{2}{r^2} v_\phi^{(2)} + \frac{\partial^2 v_r^{(1)}}{\partial z^2}\Big], \tag{3.14a}$$

$$\frac{1}{\rho}\frac{\partial p^{(1)}}{\partial z} = \nu\Big[\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial v_z^{(1)}}{\partial r} - \frac{1}{r^2} v_z^{(1)} + \frac{\partial^2 v_z^{(1)}}{\partial z^2}\Big], \tag{3.14b}$$

$$\frac{p^{(2)}}{\rho r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r v_\phi^{(1)} - \frac{1}{r^2} v_\phi^{(1)} + \frac{2}{r^2} v_r^{(2)} + \frac{\partial^2 v_\phi^{(1)}}{\partial z^2}\Big], \tag{3.14c}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r} r v_r^{(1)} + \frac{\partial v_z^{(1)}}{\partial z} + \frac{1}{r} v_\phi^{(2)}. \tag{3.14d}$$

**System 3.**

$$\frac{1}{\rho}\frac{\partial p^{(2)}}{\partial r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r v_r^{(2)} - \frac{1}{r^2} v_r^{(2)} + \frac{2}{r^2} v_\phi^{(1)} + \frac{\partial^2 v_r^{(2)}}{\partial z^2}\Big], \tag{3.15a}$$

$$\frac{1}{\rho}\frac{\partial p^{(2)}}{\partial z} = \nu\Big[\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial v_z^{(2)}}{\partial r} - \frac{1}{r^2} v_z^{(2)} + \frac{\partial^2 v_z^{(2)}}{\partial z^2}\Big], \tag{3.15b}$$

$$-\frac{p^{(1)}}{\rho r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r v_\phi^{(2)} - \frac{1}{r^2} v_\phi^{(2)} - \frac{2}{r^2} v_r^{(1)} + \frac{\partial^2 v_\phi^{(2)}}{\partial z^2}\Big], \tag{3.15c}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r} r v_r^{(2)} + \frac{\partial v_z^{(2)}}{\partial z} - \frac{1}{r} v_\phi^{(1)}. \tag{3.15d}$$

**System 4.**

$$\frac{1}{\rho}\frac{\partial p^{(3)}}{\partial r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rv_r^{(3)} - \frac{4}{r^2}v_r^{(3)} - \frac{4}{r^2}v_\phi^{(4)} + \frac{\partial^2 v_r^{(3)}}{\partial z^2}\Big], \tag{3.16a}$$

$$\frac{1}{\rho}\frac{\partial p^{(3)}}{\partial z} = \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v_z^{(3)}}{\partial r} - \frac{4}{r^2}v_z^{(3)} + \frac{\partial^2 v_z^{(3)}}{\partial z^2}\Big], \tag{3.16b}$$

$$\frac{2p^{(4)}}{\rho r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rv_\phi^{(3)} - \frac{4}{r^2}v_\phi^{(3)} + \frac{4}{r^2}v_r^{(4)} + \frac{\partial^2 v_\phi^{(3)}}{\partial z^2}\Big], \tag{3.16c}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}rv_r^{(3)} + \frac{\partial v_z^{(3)}}{\partial z} + \frac{2}{r}v_\phi^{(4)}. \tag{3.16d}$$

**System 5.**

$$\frac{1}{\rho}\frac{\partial p^{(4)}}{\partial r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rv_r^{(4)} - \frac{4}{r^2}v_r^{(4)} + \frac{4}{r^2}v_\phi^{(3)} + \frac{\partial^2 v_r^{(4)}}{\partial z^2}\Big], \tag{3.17a}$$

$$\frac{1}{\rho}\frac{\partial v^{(4)}}{\partial z} = \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v_z^{(4)}}{\partial r} - \frac{4}{r^2}v_z^{(4)} + \frac{\partial^2 v_z^{(4)}}{\partial z^2}\Big], \tag{3.17b}$$

$$-\frac{2p^{(3)}}{\rho r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rv_\phi^{(4)} - \frac{4}{r^2}v_\phi^{(4)} - \frac{4}{r^2}v_r^{(3)} + \frac{\partial^2 v_\phi^{(4)}}{\partial z^2}\Big], \tag{3.17c}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}rv_r^{(4)} + \frac{\partial v_z^{(4)}}{\partial z} - \frac{2}{r}v_\phi^{(3)}. \tag{3.17d}$$

The above systems can be written in compact form as

$$\frac{1}{\rho}\frac{\partial p^{(j)}}{\partial r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rv_r^{(j)} + \frac{\partial^2 v_r^{(j)}}{\partial z^2} - \frac{\beta_j}{r^2}v_r^{(j)} - \frac{\delta_j}{r^2}v_\phi^{(j-(-1)^j)}\Big], \tag{3.18a}$$

$$\frac{1}{\rho}\frac{\partial p^{(j)}}{\partial z} = \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v_z^{(j)}}{\partial r} + \frac{\partial^2 v_z^{(j)}}{\partial z^2} - \frac{\beta_j}{r^2}v_z^{(j)}\Big], \tag{3.18b}$$

$$\frac{\delta_j}{2}\frac{p^{(j-(-1)^j)}}{\rho r} = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rv_\phi^{(j)} + \frac{\partial^2 v_\phi^{(j)}}{\partial z^2} - \frac{\beta_j}{r^2}v_\phi^{(j)} + \frac{\delta_j}{r^2}v_r^{(j-(-1)^j)}\Big], \tag{3.18c}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}rv_r^{(j)} + \frac{\partial v_z^{(j)}}{\partial z} + \frac{\delta_j}{2r}v_\phi^{(j-(-1)^j)}, \tag{3.18d}$$

where $j = 0, 1, 2, 3, 4$ and

$$\beta_j = \begin{cases} 0 & j = 0 \\ 1 & j = 1,2 \\ 4 & j = 3,4 \end{cases}, \qquad \delta_j = \begin{cases} 0 & j = 0 \\ (-1)^{j+1}2 & j = 1,2 \\ (-1)^{j+1}4 & j = 3,4 \end{cases}.$$

The boundary conditions for $v_r^{(j)}, v_\phi^{(j)}$ and $v_z^{(j)}$ are easily derived from (3.11) and (3.12). Thus, we have reduced the original 3D problem to five 2D problems, which significantly reduces the complexity of the problem.

We point out here that System 1 is uncoupled from the other four systems. Systems 2 and 3 are coupled with each other, but uncoupled from the other systems. Thus we can solve these systems together by suitable numerical method. Systems 4 and 5 can be treated as Systems 2 and 3 provided the solution of the latter is already know.

In this dissertation we focus on creating an efficient spectral technique for solving the boundary value problem for System 1.

## 3.3 Stream Function Formulation for System 1

The component $v_\phi^{(0)}$ satisfies an elliptic equation and $v_\phi^{(0)} \equiv 0$ in the case $G_{12} = G_{21}$. The three equations (3.13a), (3.13b) and (3.13d) for $v_r^{(0)}, v_z^{(0)}$ and $p^{(0)}$ are the 2D incompressible Stokes equations with radial symmetry ( no dependence on the variable $\phi$). Then it is possible to introduce stream function. We rewrite the momentum equations of System 1 as follows

$$\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial r} = \nu\Big[\frac{\partial^2 v_r^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial v_r^{(0)}}{\partial r} - \frac{v_r^{(0)}}{r^2} + \frac{\partial^2 v_r^{(0)}}{\partial z^2}\Big], \tag{3.19a}$$

$$\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial z} = \nu\Big[\frac{\partial^2 v_z^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial v_z^{(0)}}{\partial r} + \frac{\partial^2 v_z^{(0)}}{\partial z^2}\Big]. \tag{3.19b}$$

The stream function $\psi$ is introduced the standard way:

$$v_r^{(0)} = \frac{\partial \psi}{\partial z}, \quad v_z^{(0)} = -\frac{1}{r}\frac{\partial r\psi}{\partial r}. \tag{3.20}$$

Upon substituting the above expression for $v_r^{(0)}$ and $v_z^{(0)}$ in (3.19a), we obtain

$$\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial r} = \nu\Big[\frac{\partial^3 \psi}{\partial r^2 \partial z} + \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial z} - \frac{1}{r}\frac{\partial \psi}{\partial z} + \frac{\partial^3 \psi}{\partial z^3}\Big]. \tag{3.21}$$

Differentiating with respect to $z$ , gives us

$$\frac{1}{\rho}\frac{\partial^2 p^{(0)}}{\partial r\partial z} = \nu\Big[\frac{\partial^4\psi}{\partial r^2\partial z^2} + \frac{1}{r}\frac{\partial^3\psi}{\partial r\partial z^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^4\psi}{\partial z^4}\Big]. \tag{3.22}$$

Consider (3.19b) in the same manner:

$$\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial z} = -\nu\Big[\frac{\partial^2}{\partial r^2}(\frac{1}{r}\frac{\partial r\psi}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial r\psi}{\partial r}) + \frac{\partial^2}{\partial z^2}(\frac{1}{r}\frac{\partial r\psi}{\partial r})\Big]. \tag{3.23}$$

Differentiating the latter with respect to $r$ gives

$$\frac{1}{\rho}\frac{\partial^2 p^{(0)}}{\partial r\partial z} = -\nu\frac{\partial}{\partial r}\Big[\frac{\partial^2}{\partial r^2}(\frac{1}{r}\frac{\partial r\psi}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial r\psi}{\partial r}) + \frac{\partial^2}{\partial z^2}(\frac{1}{r}\frac{\partial r\psi}{\partial r})\Big]. \tag{3.24}$$

Now, subtracting (3.24) from (3.22), we get

$$0 = \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r\psi + 2(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r)\frac{\partial^2}{\partial z^2}\psi + \frac{\partial^4\psi}{\partial z^4}\Big]. \tag{3.25}$$

We introduce the notations

$$F^2 = \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r \quad \text{and} \quad D_z^2 = \frac{\partial^2}{\partial z^2},$$

and define the *Stokesian operator* as

$$E^2 = F^2 + D_z^2 \overset{\text{def}}{=} \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2}. \tag{3.26}$$

Then the equation for the stream function Eq. (3.25) adopts the form

$$(E^2)^2\psi = 0, \tag{3.27}$$

where (3.27) can be called the *bi-Stokesian equation* in an analogy of biharmonic equation, in which the Laplace operator is applied twice.

## 3.4 Bi-Stokesian Equation in Bi-Spherical Coordinates

At this junction we will introduce the bi-spherical coordinates, because as already mentioned they give a decisive advantage in the problem when the boundary involves two spheres. In the bi-spherical coordinate system, both boundaries of the rigid spheres can be represented as coordinate surfaces.

We introduce the bi-spherical coordinates $(\xi, \phi', \eta)$ via their connection to cylindrical coordinates $(r, \phi, z)$ namely

$$r = \frac{a \sin \xi}{\cosh \eta - \cos \xi}, \quad \phi' = \phi, \quad z = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}, \tag{3.28}$$

where the constant $a$ is called focal distance. Substituting into (3.8), gives us

$$x = \frac{a \sin \xi \cos \phi}{\cosh \eta - \cos \xi}\ , \ y = \frac{a \sin \xi \sin \phi}{\cosh \eta - \cos \xi}\ , \ z = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}. \tag{3.29}$$

The coordinates $(\xi, \phi, \eta)$ vary in the interval $[0, \pi], [0, 2\pi]$ and $[\eta_1, \eta_2]$, respectively. Surfaces of constant $\eta$ are given by the spheres

$$x^2 + y^2 + (z - a\ \coth \eta)^2 = a^2 \operatorname{csch}^2 \eta. \tag{3.30}$$

Surfaces of constant $\xi$ by the apples $(\xi < \frac{\pi}{2})$ or lemons $(\xi > \frac{\pi}{2})$

$$z^2 + (\sqrt{x^2 + y^2} - a\ \cot \xi)^2 = a^2\ \csc^2 \xi. \tag{3.31}$$

Finally, surface of constant $\phi$ by the half-planes

$$\tan \phi = \frac{y}{x}. \tag{3.32}$$

A sketch of the bi-spherical coordinate system is shown in Figure 3.2. The spheres' radii $r_1$ and $r_2$, and the distance of their centers $d_1$ and $d_2$ from the origin are computed by using the following relations

$$r_i = a \operatorname{csch}|\eta_i|\ , \quad d_i = a\ \coth |\eta_i|. \tag{3.33}$$

The center to center distance between the spheres is $d = d_1 + d_2$. If $r_1, r_2$ and $d$ are given, we can find $a, \eta_1$ and $\eta_2$ as follows

$$a = \frac{\sqrt{d^4 - 2d^2(r_1^2 + r_2^2) + (r_1^2 - r_2^2)^2}}{2\,d}, \tag{3.34}$$

$$\eta_1 = -\ln\left(\frac{a}{r_1} + \sqrt{\frac{a^2}{r_1^2} + 1}\right) = -\operatorname{arcsinh}\frac{a}{r_1}, \tag{3.35}$$

$$\eta_2 = \ \ln\left(\frac{a}{r_2} + \sqrt{\frac{a^2}{r_2^2} + 1}\right) = \ \operatorname{arcsinh}\frac{a}{r_2}. \tag{3.36}$$

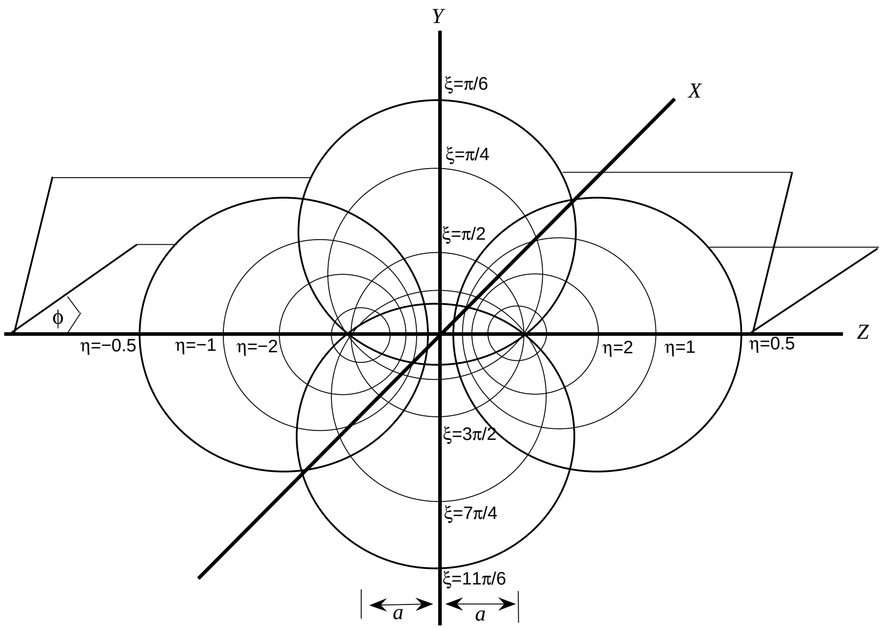



**Figure 3.2** Sketch of bi-spherical coordinates $(\xi, \eta, \phi)$.

Now, for the Stokesian differential operator we get the obvious rendition

$$E^2 = \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}. \tag{3.37}$$

The connection Eq. (3.28) between the polar and the bi-spherical coordinates can be differentiate to obtain

$$\frac{\partial}{\partial r} = \frac{\partial \xi}{\partial r}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial r}\frac{\partial}{\partial \eta}\ ,\ \frac{\partial}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}, \tag{3.38}$$

and

$$\frac{\partial}{\partial r} = \frac{\cos\xi\cosh\eta - 1}{a}\frac{\partial}{\partial \xi} - \frac{\sin\xi\sinh\eta}{a}\frac{\partial}{\partial \eta}, \tag{3.39a}$$

$$\frac{\partial}{\partial z} = -\frac{\sin\xi\sinh\eta}{a}\frac{\partial}{\partial \xi} - \frac{\cos\xi\cosh\eta - 1}{a}\frac{\partial}{\partial \eta}. \tag{3.39b}$$

Respectively, for the second derivatives we get

$$\frac{\partial^2}{\partial r^2} = \Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)^2\frac{\partial^2}{\partial\xi^2} + \Big(\frac{\sin\xi\sinh\eta}{a}\Big)^2\frac{\partial^2}{\partial\eta^2} \tag{3.40a}$$
$$- 2\Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)\Big(\frac{\sin\xi\sinh\eta}{a}\Big)\frac{\partial^2}{\partial\xi\partial\eta}$$
$$- \Big[\Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)\Big(\frac{\sin\xi\cosh\eta}{a}\Big) + \Big(\frac{\sin\xi\sinh\eta}{a}\Big)\Big(\frac{\cos\xi\sinh\eta}{a}\Big)\Big]\frac{\partial}{\partial\xi}$$
$$- \Big[\Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)\Big(\frac{\cos\xi\sinh\eta}{a}\Big) - \Big(\frac{\sin\xi\sinh\eta}{a}\Big)\Big(\frac{\sin\xi\cosh\eta}{a}\Big)\Big]\frac{\partial}{\partial\eta},$$

$$\frac{\partial^2}{\partial z^2} = \Big(\frac{\sin\xi\sinh\eta}{a}\Big)^2\frac{\partial^2}{\partial\xi^2} + \Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)^2\frac{\partial^2}{\partial\eta^2} \tag{3.40b}$$
$$+ 2\Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)\Big(\frac{\sin\xi\sinh\eta}{a}\Big)\frac{\partial^2}{\partial\xi\partial\eta}$$
$$+ \Big[\Big(\frac{\sin\xi\sinh\eta}{a}\Big)\Big(\frac{\cos\xi\sinh\eta}{a}\Big) + \Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)\Big(\frac{\sin\xi\cosh\eta}{a}\Big)\Big]\frac{\partial}{\partial\xi}$$
$$- \Big[\Big(\frac{\sin\xi\sinh\eta}{a}\Big)\Big(\frac{\sin\xi\cosh\eta}{a}\Big) - \Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)\Big(\frac{\cos\xi\sinh\eta}{a}\Big)\Big]\frac{\partial}{\partial\eta}.$$

Thus, for the Stokesian operator in bi-spherical coordinates we obtain

$$E^2 = \Big(\frac{\cosh\eta - \cos\xi}{a}\Big)^2\Big(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\Big) - \Big(\frac{\cosh\eta - \cos\xi}{a\sin\xi}\Big)^2$$
$$+ \Big(\frac{\cosh\eta - \cos\xi}{a}\Big)\Big[\frac{\cos\xi\cosh\eta - 1}{a\sin\xi}\frac{\partial}{\partial\xi} - \frac{\sinh\eta}{a}\frac{\partial}{\partial\eta}\Big]. \tag{3.41}$$

This operator will be used in next section for solving the bi-Stokesian equation.

# CHAPTER IV

# LEGENDRE SPECTRAL METHOD FOR BI-STOKESIAN EQUATION

In this chapter, we create a spectral method based on associated Legendre polynomials for solving the bi-Stokesian equation $E^4\psi = 0$. The unknown coefficients of the series with respect to the associated Legendre polynomials are obtained by satisfying the boundary conditions which are expanded in series with respect to the associated Legendre polynomials. Consequently, the closed algebraic system for the unknown coefficients are represented.

## 4.1 General solution of bi-Stokesian equation

The stream function can be defined to satisfy the continuity equation (3.13d) for $v_r^{(0)}$ and $v_z^{(0)}$

$$v_r^{(0)} = \frac{\partial \psi}{\partial z}\,,\ v_z^{(0)} = -\frac{1}{r}\frac{\partial r\psi}{\partial r}. \tag{4.1}$$

The introduced stream function satisfies the so called bi-Stokesian equation

$$E^2(E^2\psi) = 0. \tag{4.2}$$

To find the general solution of (4.2), let us consider a related coupled system i.e.,

$$E^2\psi = \chi, \tag{4.3a}$$

$$E^2\chi = 0. \tag{4.3b}$$

Consider first (4.3b),

$$\Big(\frac{\cosh\eta-\cos\xi}{a}\Big)^2\Big(\frac{\partial^2\chi}{\partial\xi^2}+\frac{\partial^2\chi}{\partial\eta^2}\Big)-\Big(\frac{\cosh\eta-\cos\xi}{a\sin\xi}\Big)^2\chi$$
$$+\Big(\frac{\cosh\eta-\cos\xi}{a}\Big)\Big[\frac{\cos\xi\cosh\eta-1}{a\sin\xi}\frac{\partial\chi}{\partial\xi}-\frac{\sinh\eta}{a}\frac{\partial\chi}{\partial\eta}\Big]=0. \quad (4.4)$$

It is well known that by means of the substitution (see, e.g., Tikhonov and Samaraskii, 1990)

$$\chi=\sqrt{\cosh\eta-\cos\xi}\;\Phi, \quad (4.5)$$

the above equation can be made separable, namely

$$\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}+\cot\xi\frac{\partial\Phi}{\partial\xi}-\Big(\frac{1}{4}+\frac{1}{\sin^2\xi}\Big)\Phi=0. \quad (4.6)$$

The solution of Eq. (4.6) can be determined by separation of variables though stipulating

$$\Phi(\xi,\eta)=B(\xi)C(\eta).$$

Then, the following two independent ordinary differential equations are obtained

$$\frac{d^2C}{d\,\eta^2}=\lambda^2C, \quad (4.7)$$

$$\frac{d^2B}{d\,\xi^2}+\cot\xi\frac{dB}{d\xi}+\Big(\lambda^2-\frac{1}{4}-\frac{1}{\sin^2\xi}\Big)B=0. \quad (4.8)$$

Let now $B=D(\mu)$ where $\mu=\cos\xi$. Then Eq. (4.8) transforms to

$$\frac{d}{d\mu}\Big[(1-\mu^2)\frac{dD}{d\mu}\Big]+\Big[(\lambda-\frac{1}{2})(\lambda+\frac{1}{2})-\frac{1}{1-\mu^2}\Big]D=0, \quad (4.9)$$

which is the associated Legendre equation. Hence, the solution of Eq. (4.9) is in the form of the associated Legendre polynomials

$$D=P^{(1)}_{\lambda-1/2}(\mu) \quad (4.10)$$

which are defined for positive integer $m=\lambda-\frac{1}{2}>0$. Then, the general solution for $\Phi$ is given by

$$\Phi(\eta,\mu)=\sum_{m=1}^{\infty}\Big[L^{(m)}e^{(m+1/2)\eta}+N^{(m)}e^{-(m+1/2)\eta}\Big]P^{(1)}_m(\mu). \quad (4.11)$$

Then for the original function $\chi$, it follows that

$$\chi = (\cosh\eta - \cos\xi)^{\frac{1}{2}} \sum_{m=1}^{\infty} \Big[ L^{(m)} e^{(m+1/2)\eta} + N^{(m)} e^{-(m+1/2)\eta} \Big] P_m^{(1)}(\mu). \tag{4.12}$$

Consider now Eq. (4.3a) in bi-spherical coordinates, namely

$$\Big(\frac{\cosh\eta - \cos\xi}{a}\Big)^2 \Big(\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2}\Big) - \Big(\frac{\cosh\eta - \cos\xi}{a\sin\xi}\Big)^2 \psi$$
$$+ \Big(\frac{\cosh\eta - \cos\xi}{a}\Big) \Big[\frac{\cos\xi\cosh\eta - 1}{a\sin\xi}\frac{\partial\psi}{\partial\xi} - \frac{\sinh\eta}{a}\frac{\partial\psi}{\partial\eta}\Big] = \chi. \tag{4.13}$$

In the same manner, we can change the dependent variable

$$\psi = \sqrt{\cosh\eta - \cos\xi}\ \varphi, \tag{4.14}$$

and to obtain equation with a separable left-hand side, namely

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \cot\xi\frac{\partial\varphi}{\partial\xi} - \Big(\frac{1}{4} + \frac{1}{\sin^2\xi}\Big)\varphi = \frac{a^2}{(\cosh\eta - \cos\xi)^{5/2}}\,\chi. \tag{4.15}$$

At this junction we encounter the most important difficulty in applying the Legendre series, namely the fact that the r.h.s, of Eq. (4.15) is not immediately expandable in the same Legendre series (with the same coefficients) as the function $\chi$, because of the presence of the term $a^2(\cosh\eta - \cos\xi)^{-5/2}$. In order to identify the actual form of the series for the r.h.s. of Eq. (4.15), we substitute Eq. (4.12) into it to obtain

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \cot\xi\frac{\partial\varphi}{\partial\xi} - \Big(\frac{1}{4} + \frac{1}{\sin^2\xi}\Big)\varphi$$
$$= \Big(\frac{a}{\cosh\eta - \mu}\Big)^2 \sum_{m=1}^{\infty} \Big[ L^{(m)} e^{(m+1/2)\eta} + N^{(m)} e^{-(m+1/2)\eta} \Big] P_m^{(1)}(\mu). \tag{4.16}$$

The key idea here is to make use of the generating function for Chebyshev polynomials of the second kind, $U_m(\mu)$, and express the right-hand side of (4.16) into series with respect to the associated Legendre polynomials. This is one of the main contributions of the present dissertation.

The generating function for Chebyshev polynomials of the second kind is defined as follow

$$G(t,\mu) = \frac{1}{1-2\mu t+t^2} = \sum_{m=0}^{\infty} t^m U_m(\mu) \quad \text{for} \quad |t|<1, |\mu|<1.$$

For the derivative of the generating function, we get

$$\frac{\partial G}{\partial \mu} = \frac{2t}{(1-2\mu t+t^2)^2} = \sum_{m=1}^{\infty} t^m U_m^{'}(\mu);\; U_0^{'}(\mu)=0,$$

and since

$$\frac{a^2}{(\cosh\eta-\mu)^2} = \frac{a^2}{(\frac{e^{\eta}+e^{-\eta}}{2}-\mu)^2} = \frac{a^2(2t)^2}{(1-2\mu t+t^2)^2} \qquad ;\; t=e^{\eta},$$

then we find that

$$\frac{a^2}{(\cosh\eta-\mu)^2} = 2a^2 \begin{cases} \sum_{m=1}^{\infty} e^{(m+1)\eta} U_m^{'}(\mu) & \eta \le 0, \\ \sum_{m=1}^{\infty} e^{-(m+1)\eta} U_m^{'}(\mu) & \eta > 0. \end{cases} \tag{4.17}$$

Another convenient way to write the last equation is

$$\Big(\frac{a}{\cosh\eta-\mu}\Big)^2 = 2a^2 \sum_{m=1}^{\infty} e^{-(m+1)|\eta|} U_m^{'}(\mu). \tag{4.18}$$

Thus, we get

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \cot\xi\frac{\partial\varphi}{\partial\xi} - \Big(\frac{1}{4}+\frac{1}{\sin^2\xi}\Big)\varphi$$

$$= 2a^2\Big[\sum_{l=1}^{\infty} e^{-(l+1)|\eta|} U_l^{'}(\mu)\Big] \sum_{m=1}^{\infty} \Big[L^{(m)} e^{(m+1/2)\eta} + N^{(m)} e^{-(m+1/2)\eta}\Big] P_m^{(1)}(\mu). \tag{4.19}$$

At this stage, we need to derive a representation of the products of Chebyshev polynomials of the second kind with associated Legendre polynomials into series with respect to the associated Legendre polynomials. Formally, such a series can be written as

$$P_m^{(1)}(\mu) U_l^{'}(\mu) = \sum_{k=1}^{\infty} p_k^{ml} P_k^{(1)}(\mu), \qquad \text{where} \quad p_k^{ml}=0 \quad \text{for} \quad k \ge m+l. \tag{4.20}$$

The coefficients $p_k^{ml}$ of the series are obtained by using the orthogonality of the associated Legendre polynomials, namely

$$\int_{-1}^{1} P_m^{(1)}(\mu)U_l'(\mu)P_s^{(1)}(\mu)d\mu = \int_{-1}^{1}\sum_{k=1}^{\infty} p_k^{ml} P_k^{(1)}(\mu)P_s^{(1)}(\mu)d\mu$$

$$= \sum_{k=1}^{\infty} p_k^{ml} \int_{-1}^{1} P_k^{(1)}(\mu)P_s^{(1)}(\mu)d\mu = \int_{-1}^{1}[P_k^{(1)}(\mu)]^2 d\mu \sum_{k=1}^{\infty} p_k^{ml}\delta_{ks}$$

$$= p_s^{ml}\int_{-1}^{1}[P_k^{(1)}(\mu)]^2 d\mu, \quad (4.21)$$

which gives that

$$p_k^{ml} = \frac{\int_{-1}^{1} P_m^{(1)}(\mu)U_l'(\mu)P_k^{(1)}(\mu)d\mu}{\int_{-1}^{1}[P_k^{(1)}(\mu)]^2 d\mu} = \frac{\int_{-1}^{1} P_m^{(1)}(\mu)U_l'(\mu)P_k^{(1)}(\mu)d\mu}{\frac{2k(k+1)}{2k+1}} \quad (4.22)$$

Now, Eq. (4.16) can be rewritten as follows

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \cot\xi\frac{\partial\varphi}{\partial\xi} - \Big(\frac{1}{4} + \frac{1}{\sin^2\xi}\Big)\varphi = \sum_{k=1}^{\infty} Q_k(\eta)P_k^{(1)}(\mu), \quad (4.23)$$

where the following notation is adopted

$$Q_k(\eta) \stackrel{\text{def}}{=} 2a^2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_k^{ml} e^{-(l+1)|\eta|}\Big(L^{(m)}e^{(m+1/2)\eta} + N^{(m)}e^{-(m+1/2)\eta}\Big). \quad (4.24)$$

Here, we can make use of the separability of the above equation and seek the solution in the form of series with respect to associated Legendre polynomials:

$$\varphi = \sum_{k=1}^{\infty} f_k(\eta)P_k^{(1)}(\mu). \quad (4.25)$$

After some tedious but straightforward computations, Eq. (4.23) adopts the following form

$$\sum_{k=1}^{\infty} f_k''(\eta)P_k^{(1)}(\mu) + (1-\mu^2)\sum_{k=1}^{\infty} f_k(\eta)P_k^{(1)''}(\mu) - 2\mu\sum_{k=1}^{\infty} f_k(\eta)P_k^{(1)'}(\mu)$$

$$- \Big(\frac{1}{4} + \frac{1}{1-\mu^2}\Big)\sum_{k=1}^{\infty} f_k(\eta)P_k^{(1)}(\mu) = \sum_{k=1}^{\infty} Q_k(\eta)P_k^{(1)}(\mu), \quad (4.26)$$

which breaks naturally into the two following independent equations

$$\sum_{k=1}^{\infty}\Big((1-\mu^2)P_k^{(1)''}(\mu)-2\mu P_k^{(1)'}(\mu)+\Big[k(k+1)-\frac{1}{1-\mu^2}\Big]P_k^{(1)}(\mu)\Big)=0, \quad (4.27a)$$

$$\sum_{k=1}^{\infty}\Big[f_k''(\eta)-\Big(k+\frac{1}{2}\Big)^2 f_k(\eta)\Big]P_k^{(1)}(\mu)=\sum_{k=1}^{\infty}Q_k(\eta)P_k^{(1)}(\mu). \quad (4.27b)$$

Therefore, we can find $f_k(\eta)$ by solving

$$f_k''(\eta)-\Big(k+\frac{1}{2}\Big)^2 f_k(\eta)=Q_k(\eta). \quad (4.28)$$

The general solution of the homogeneous equation is

$$f_{k_c}(\eta)=C_k^{(1)}e^{(k+\frac{1}{2})\eta}+C_k^{(2)}e^{-(k+\frac{1}{2})\eta}. \quad (4.29)$$

We will find a particular solution $f_{k_p}(\eta)$ of the non-homogeneous equation (4.28) by using the method of undetermined coefficients. Since, in the definition of the functions $Q_k(\eta)$ enters the exponent of $-|\eta|$, we have to consider two different cases.

**Case 1:** $\eta \leq 0$. In this case, Eq. (4.24) can be recast as follows

$$Q_k(\eta)=2a^2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}p_k^{ml}\Big[L^{(m)}e^{(l+m+\frac{3}{2})\eta}+N^{(m)}e^{(l-m+\frac{1}{2})\eta}\Big] \quad (4.30)$$

and the particular solution can be sought in the form

$$F_k(\eta)=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}e^{(l-m+\frac{1}{2})\eta}\Big] \quad (4.31)$$

provided that $k \neq l+m+1, k \neq m-l-1$ and $k \neq l-m$. Then

$$F_k'=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}(l+m+\frac{3}{2})e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}(l-m+\frac{1}{2})e^{(l-m+\frac{1}{2})\eta}\Big],$$

$$F_k''=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}(l+m+\frac{3}{2})^2e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}(l-m+\frac{1}{2})^2e^{(l-m+\frac{1}{2})\eta}\Big].$$

Substituting these expressions for $f_k^{''}, f_k$ in Eq. (4.28), we get

$$\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big\{S_k^{(ml)}\Big[(l+m+\frac{3}{2})^2-(k+\frac{1}{2})^2\Big]e^{(l+m+\frac{3}{2})\eta} + T_k^{(ml)}\Big[(l-m+\frac{1}{2})^2-(k+\frac{1}{2})^2\Big]e^{(l-m+\frac{1}{2})\eta}\Big\} = 2a^2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}p_k^{ml}\Big\{L^{(m)}e^{(l+m+\frac{3}{2})\eta}+N^{(m)}e^{(l-m+\frac{1}{2})\eta}\Big\}.$$

By comparing the coefficients, we get

$$S_k^{(ml)}=\frac{2a^2p_k^{ml}L^{(m)}}{(l+m+\frac{3}{2})^2-(k+\frac{1}{2})^2},\quad T_k^{(ml)}=\frac{2a^2p_k^{ml}N^{(m)}}{(l-m+\frac{1}{2})^2-(k+\frac{1}{2})^2}.$$

Note that the two ‘resonant’ cases have to be treated separately, because for them the right-hand side contains an exponential function with an exponent equal to one of the eigenvalues $\lambda$ of the operator of the homogeneous equation of Eq. (4.28), namely

$$f_k^{''}(\eta)-(k+\frac{1}{2})^2f_k(\eta)=e^{(l+m+\frac{3}{2})\eta}. \tag{4.32}$$

For this equation, the characteristic exponents of the homogeneous equation are $\lambda=\pm(k+\frac{1}{2})$. Then, if $l+m+\frac{3}{2}=\pm(k+\frac{1}{2})$ (or what is the same $k=l+m+1$), we have to seek for a particular solution in the form

$$F_k(\eta)=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}\eta e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}e^{(l-m+\frac{1}{2})\eta}\Big], \tag{4.33}$$

and therefore

$$F_k^{'}=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}\Big((l+m+\frac{3}{2})\eta+1\Big)e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}(l-m+\frac{1}{2})e^{(l-m+\frac{1}{2})\eta}\Big],$$

$$F_k^{''}=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}\Big((l+m+\frac{3}{2})^2\eta+2(l+m+\frac{3}{2})\Big)e^{(l+m+\frac{3}{2})\eta} + T_k^{(ml)}(l-m+\frac{1}{2})^2e^{(l-m+\frac{1}{2})\eta}\Big].$$

By substituting these expressions for $f_k^{''}, f_k$ in Eq. (4.28), one obtains

$$\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big\{S_k^{(ml)}\Big[2(l+m+\frac{3}{2})\Big]e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}\Big[(l-m+\frac{1}{2})^2-(k+\frac{1}{2})^2\Big]$$
$$\times\, e^{(l-m+\frac{1}{2})\eta}\Big\} = 2a^2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}p_k^{ml}\Big[L^{(m)}e^{(l+m+\frac{3}{2})\eta}+N^{(m)}e^{(l-m+\frac{1}{2})\eta}\Big].$$

Comparing the coefficients, we obtain

$$S_k^{(ml)} = \frac{a^2p_k^{ml}L^{(m)}}{l+m+\frac{3}{2}}, \qquad T_k^{(ml)} = \frac{2a^2p_k^{ml}N^{(m)}}{(l-m+\frac{1}{2})^2-(k+\frac{1}{2})^2}.$$

In the same manner we treat the other equation

$$f_k^{''}(\eta)-(k+\frac{1}{2})^2f_k(\eta) = e^{(l-m+\frac{1}{2})\eta}. \tag{4.34}$$

If $l-m+\frac{1}{2}=\pm(k+\frac{1}{2})$ (or what is the same $k=l-m$ or $k=m-l-1$), then we have to seek for the following particular solution

$$F_k(\eta) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}\eta e^{(l-m+\frac{1}{2})\eta}\Big]. \tag{4.35}$$

It follows that

$$F_k^{'} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}(l+m+\frac{3}{2})e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}\Big[(l-m+\frac{1}{2})\eta+1\Big]e^{(l-m+\frac{1}{2})\eta}\Big]$$

$$F_k^{''} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big[S_k^{(ml)}(l+m+\frac{3}{2})^2e^{(l+m+\frac{3}{2})\eta}$$
$$+\,T_k^{(ml)}\Big[(l-m+\frac{1}{2})^2\eta+2(l-m+\frac{1}{2})\Big]e^{(l-m+\frac{1}{2})\eta}\Big]$$

Replacing these expressions for $f_k^{''}, f_k$ in Eq. (4.28), we obtain

$$\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Big\{S_k^{(ml)}\Big[(l+m+\frac{3}{2})^2-(k+\frac{1}{2})^2\Big]e^{(l+m+\frac{3}{2})\eta}+T_k^{(ml)}\Big[2(l-m+\frac{1}{2})\Big]e^{(l-m+\frac{1}{2})\eta}\Big\}$$
$$= 2a^2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}p_k^{ml}\Big\{L^{(m)}e^{(l+m+\frac{3}{2})\eta}+N^{(m)}e^{(l-m+\frac{1}{2})\eta}\Big\}.$$

Comparing the coefficients, gives us

$$S_k^{(ml)} = \frac{2a^2p_k^{ml}L^{(m)}}{(l+m+\frac{3}{2})^2-(k+\frac{1}{2})^2}, \qquad T_k^{(ml)} = \frac{a^2p_k^{ml}N^{(m)}}{l-m+\frac{1}{2}}.$$

Hence the particular solution is represented by

$$f_{k_p}(\eta) = \sum_{m=1}^{\infty} L^{(m)} \varepsilon_m^k(\eta) + \sum_{m=1}^{\infty} N^{(m)} \lambda_m^k(\eta), \tag{4.36}$$

where

$$\varepsilon_m^k(\eta) = \sum_{\substack{l=max\{1,k-m+1\}\\ l\neq k-m-1}}^{\infty} \frac{2a^2 p_k^{ml} e^{(l+m+\frac{3}{2})\eta}}{(m+l+\frac{3}{2})^2-(k+1/2)^2} + \sum_{\substack{l=max\{1,k-m+1\}\\ l=k-m-1}}^{\infty} \frac{a^2 p_k^{ml} \eta e^{(l+m+\frac{3}{2})\eta}}{m+l+\frac{3}{2}},$$

$$\lambda_m^k(\eta) = \sum_{\substack{l=max\{1,k-m+1\}\\ l\neq m-k-1 \text{ or } l\neq k+m}}^{\infty} \frac{2a^2 p_k^{ml} e^{(l-m+\frac{1}{2})\eta}}{(l-m+\frac{1}{2})^2-(k+1/2)^2} + \sum_{\substack{l=max\{1,k-m+1\}\\ l=m-k-1 \text{ or } l=k+m}}^{\infty} \frac{a^2 p_k^{ml} \eta e^{(l-m+\frac{1}{2})\eta}}{l-m+\frac{1}{2}}.$$

**Case 2:** $\eta \geq 0$. In this case, (4.24) adopts to

$$Q_k(\eta) = 2a^2 \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} p_k^{ml} \Big[ L^{(m)} e^{(m-l-\frac{1}{2})\eta} + N^{(m)} e^{-(l+m+\frac{3}{2})\eta} \Big], \tag{4.37}$$

and the particular solution can be determined in the form

$$F_k(\eta) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \Big[ S_k^{(ml)} e^{(m-l-\frac{1}{2})\eta} + T_k^{(ml)} e^{-(l+m+\frac{3}{2})\eta} \Big], \tag{4.38}$$

provided that $k \neq m+l+1, k \neq m-l-1$ and $k \neq l-m$. Proceeding as in Case 1, one obtains

$$S_k^{(ml)} = \frac{2a^2 p_k^{ml} L^{(m)}}{(m-l-\frac{1}{2})^2-(k+\frac{1}{2})^2}, \qquad T_k^{(ml)} = \frac{2a^2 p_k^{ml} N^{(m)}}{(l+m+\frac{3}{2})^2-(k+\frac{1}{2})^2}.$$

By the same reasons as in the case $\eta \leq 0$, we have distinguish two different cases. If $k = l-m$ or $k = m-l-1$, the particular solution is

$$F_k(\eta) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \Big[ S_k^{(ml)} \eta e^{(m-l-\frac{1}{2})\eta} + T_k^{(ml)} e^{-(l+m+\frac{3}{2})\eta} \Big], \tag{4.39}$$

where

$$S_k^{(ml)} = \frac{a^2 p_k^{ml} L^{(m)}}{m-l-\frac{1}{2}}, \qquad T_k^{(ml)} = \frac{2a^2 p_k^{ml} N^{(m)}}{(l+m+\frac{3}{2})^2-(k+\frac{1}{2})^2}.$$

Respectively, if $k = l+m+1$, the particular solution is represented as follow

$$F_k(\eta) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \Big[ S_k^{(ml)} e^{(m-l-\frac{1}{2})\eta} + T_k^{(ml)} \eta e^{-(l+m+\frac{3}{2})\eta} \Big], \tag{4.40}$$

where

$$S_k^{(ml)} = \frac{2a^2 p_k^{ml} L^{(m)}}{(m-l-\frac{1}{2})^2 - (k+\frac{1}{2})^2}, \qquad T_k^{(ml)} = \frac{a^2 p_k^{ml} N^{(m)}}{-(l+m+\frac{3}{2})}.$$

Hence the particular solution becomes

$$f_{k_p}(\eta) = \sum_{m=1}^{\infty} L^{(m)} \omega_m^k(\eta) + \sum_{m=1}^{\infty} N^{(m)} \tau_m^k(\eta), \tag{4.41}$$

where

$$\omega_m^k(\eta) = \sum_{\substack{l=max\{1,k-m+1\} \\ l \neq m-k-1 \text{ or } l \neq k+m}}^{\infty} \frac{2a^2 p_k^{ml} e^{(m-l-\frac{1}{2})\eta}}{(m-l-\frac{1}{2})^2 - (k+1/2)^2}] + \sum_{\substack{l=max\{1,k-m+1\} \\ l=m-k-1 \text{ or } l=k+m}}^{\infty} \frac{a^2 p_k^{ml} \eta e^{(m-l-\frac{1}{2})\eta}}{m-l-\frac{1}{2}},$$

$$\tau_m^k(\eta) = \sum_{\substack{l=max\{1,k-m+1\} \\ l \neq k-m-1}}^{\infty} \frac{2a^2 p_k^{ml} e^{-(m+l+\frac{3}{2})\eta}}{(m+l+\frac{3}{2})^2 - (k+1/2)^2} + \sum_{\substack{l=max\{1,k-m+1\} \\ l=k-m-1}}^{\infty} \frac{a^2 p_k^{ml} \eta e^{-(m+l+\frac{3}{2})\eta}}{-(m+l+\frac{3}{2})}.$$

The unknown coefficients $C_k^{(1)}, C_k^{(2)}, L^{(m)}$ and $N^{(m)}$ are to be determined from boundary conditions. Thus, the expression for the stream function is obtained by

$$\psi(\eta,\mu) = (\cosh\eta - \mu)^{\frac{1}{2}} \sum_{k=1}^{\infty} [f_{k_c}(\eta) + f_{k_p}(\eta)] P_k^{(1)}(\mu). \tag{4.42}$$

## 4.2 Expanding the Boundary Conditions into Series with respect to the Associated Legendre Polynomials

In the previous section, we have obtained the general solution of Eq. (4.42) for the stream function. To determine the unknown coefficients $C_k^{(1)}, C_k^{(2)}, L^{(m)}$ and $N^{(m)}$, we have to expand the boundary conditions for the stream functions into series with respect to the associated Legendre polynomials. Since

$$v_r^{(0)} = \frac{\partial \psi}{\partial z}, \quad v_z^{(0)} = -\frac{1}{r} \frac{\partial r\psi}{\partial r}, \tag{4.43}$$

then the velocity components can be written in terms of bi-spherical coordinates:

$$v_r^{(0)} = \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial z} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial z} = -\Big(\frac{\sin\xi\sinh\eta}{a}\Big)\frac{\partial\psi}{\partial\xi} - \Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)\frac{\partial\psi}{\partial\eta}, \quad (4.44a)$$

$$v_z^{(0)} = -\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial r} - \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial r} - \frac{1}{r}\psi = -\Big(\frac{\cos\xi\cosh\eta - 1}{a}\Big)\frac{\partial\psi}{\partial\xi} + \Big(\frac{\sin\xi\sinh\eta}{a}\Big)\frac{\partial\psi}{\partial\eta} - \Big(\frac{\cosh\eta - \cos\xi}{a\sin\xi}\Big)\psi. \quad (4.44b)$$

Therefore, the boundary conditions for $\psi$ are the following

$$\frac{\partial\psi}{\partial\eta}\Big|_\Gamma - \frac{\sinh\eta}{\cosh\eta - \cos\xi}\psi\Big|_\Gamma = -hv_\xi^{(0)}\Big|_\Gamma, \quad (4.45a)$$

$$\frac{\partial\psi}{\partial\xi}\Big|_\Gamma + \frac{(\cos\xi\cosh\eta - 1)}{\sin\xi(\cosh\eta - \cos\xi)}\psi\Big|_\Gamma = hv_\eta^{(0)}\Big|_\Gamma, \quad (4.45b)$$

where $v_\xi^{(0)}$ and $v_\eta^{(0)}$ are the components of velocity vector with respect to bi-spherical coordinates and $h = a(\cosh\eta - \cos\xi)^{-1}$. The relations between velocity components in cylindrical coordinates (3.8), and bi-polar coordinates (3.28), read

$$v_\xi^{(0)} = \frac{h}{a}(\cos\xi\cosh\eta - 1)v_r^{(0)} - \frac{h}{a}(\sin\xi\sinh\eta)v_z^{(0)}, \quad (4.46a)$$

$$v_\eta^{(0)} = -\frac{h}{a}(\sin\xi\sinh\eta)v_r^{(0)} - \frac{h}{a}(\cos\xi\cosh\eta - 1)v_z^{(0)}. \quad (4.46b)$$

Since we have

$$v_r^{(0)}|_\Gamma = -(G_{11} + G_{22})\frac{r}{2}, \qquad v_z^{(0)}|_\Gamma = -(U^{(z)} + G_{33}z), \quad (4.47)$$

then it follows that

$$v_\xi^{(0)}|_\Gamma = \frac{h}{a}\sin\xi\sinh\eta_i U^{(z)} - \frac{h^2}{2a}(G_{11} + G_{22})(\cos\xi\cosh\eta_i - 1)\sin\xi + \frac{h^2}{a}G_{33}\sin\xi\sinh^2\eta_i, \quad i = 1, 2, \quad (4.48a)$$

$$v_\eta^{(0)}|_\Gamma = \frac{h}{a}(\cos\xi\cosh\eta_i - 1)U^{(z)} + \frac{h^2}{2a}(G_{11} + G_{22})\sin^2\xi\sinh\eta_i + \frac{h^2}{a}G_{33}(\cos\xi\cosh\eta_i - 1)\sinh\eta_i, \quad i = 1, 2. \quad (4.48b)$$

Recall that we have introduced a new variable for the stream function $\psi = \sqrt{\cosh\eta - \mu}\,\widehat{\psi}$, where

$$\widehat{\psi} = \sum_{k=1}^{\infty} Z_k(\eta) P_k^{(1)}(\mu)\,, \quad Z_k(\eta) = f_{k_c}(\eta) + f_{k_p}(\eta). \tag{4.49}$$

Thus we need to derive boundary conditions for $\widehat{\psi}$. To this end, Eqs. (4.45) can be transformed to the following

$$(\cosh\eta - \mu)\frac{\partial\widehat{\psi}}{\partial\eta}\Big|_{\Gamma} - \frac{1}{2}\sinh\eta\,\widehat{\psi}\Big|_{\Gamma} = -\frac{a}{(\cosh\eta-\mu)^{\frac{1}{2}}} v_{\xi}^{(0)}\Big|_{\Gamma}, \tag{4.50a}$$

$$\sin\xi(\cosh\eta - \mu)\frac{\partial\widehat{\psi}}{\partial\xi}\Big|_{\Gamma} + [(\mu\cosh\eta - 1) + \frac{1-\mu^2}{2}]\,\widehat{\psi}\Big|_{\Gamma} = \frac{a\sin\xi}{(\cosh\eta-\mu)^{\frac{1}{2}}} v_{\eta}^{(0)}\Big|_{\Gamma}. \tag{4.50b}$$

The stream function given by Eq. (4.42) tends to zero as $\eta \to 0$ and $\mu \to 0$. The boundary conditions given by (4.50) require the representation of both sides in the form of a series with respect to the associated Legendre polynomials. Due to the recursion

$$\mu P_k^{(1)}(\mu) = \frac{k P_{k+1}^{(1)}(\mu) + (k+1) P_{k-1}^{(1)}(\mu)}{2k+1} \tag{4.51}$$

we can show that

$$\begin{aligned}(\cosh\eta - \mu)\frac{\partial\widehat{\psi}}{\partial\eta} - \frac{1}{2}\sinh\eta\,\widehat{\psi} = \sum_{k=1}^{\infty}\cosh\eta Z_k'(\eta) P_k^{(1)}(\mu) - \sum_{k=1}^{\infty}\frac{k}{2k+1} Z_k'(\eta) P_{k+1}^{(1)}(\mu)\\ - \sum_{k=1}^{\infty}\frac{k+1}{2k+1} Z_k'(\eta) P_{k-1}^{(1)}(\mu) - \sum_{k=1}^{\infty}\frac{\sinh\eta}{2} Z_k(\eta) P_k^{(1)}(\mu),\end{aligned}$$

and Eq.(4.50a) adopts the form

$$\begin{aligned}\sum_{k=1}^{\infty}\Big[\Big(\frac{1-k}{2k-1}\Big) Z_{k-1}'(\eta) - \frac{\sinh\eta}{2} Z_k(\eta) + \cosh\eta Z_k'(\eta) - \Big(\frac{k+2}{2k+3}\Big) Z_{k+1}'(\eta)\Big] P_k^{(1)}(\mu)\Big|_{\Gamma}\\ = -\frac{a}{(\cosh\eta-\mu)^{\frac{1}{2}}} v_{\xi}^{(0)}\Big|_{\Gamma}.\end{aligned} \tag{4.52}$$

Next we consider (4.50b). The following relations are used

$$(\mu^2-1)P_k^{(1)'}(\mu)=k\mu P_k^{(1)}(\mu)-(k+1)P_{k-1}^{(1)}(\mu), \tag{4.53}$$

$$\mu P_k^{(1)}(\mu)=\frac{kP_{k+1}^{(1)}(\mu)+(k+1)P_{k-1}^{(1)}(\mu)}{2k+1}, \tag{4.54}$$

to obtain that

$$\begin{aligned}\sin\xi(\cosh\eta-\mu)\frac{\partial}{\partial\xi}\widehat{\psi}&=\sum_{k=0}^{\infty}\Big[\frac{(k+1)^2(k-1)}{(2k+1)(2k-1)}-\frac{k^2(k+2)}{(2k+1)(2k+3)}\Big]Z_k(\eta)P_k^{(1)}(\mu)\\ &\sum_{k=0}^{\infty}\frac{k(k+1)^2}{(2k+1)(2k-1)}Z_k(\eta)P_{k-2}^{(1)}(\mu)-\sum_{k=0}^{\infty}\frac{(k+1)^2}{2k+1}\cosh\eta Z_k(\eta)P_{k-1}^{(1)}(\mu)\\ &+\sum_{k=0}^{\infty}\frac{k^2}{2k+1}\cosh\eta Z_k(\eta)P_{k+1}^{(1)}(\mu)-\sum_{k=0}^{\infty}\frac{k^2(k+1)}{(2k+1)(2k+3)}Z_k(\eta)P_{k+2}^{(1)}(\mu),\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}\Big[(\mu\cosh\eta-1)+(\frac{1-\mu^2}{2})\Big]\widehat{\psi}&=-\sum_{k=1}^{\infty}\frac{1}{2}\Big[1+\frac{k(k+2)}{(2k+1)(2k+3)}+\frac{(k-1)(k+1)}{(2k-1)(2k+1)}\Big]\\ \times Z_k(\eta)P_k^{(1)}(\mu)&+\sum_{k=1}^{\infty}(\frac{k}{2k+1})\cosh\eta Z_k(\eta)P_{k+1}^{(1)}(\mu)+\sum_{k=1}^{\infty}(\frac{k+1}{2k+1})\cosh\eta Z_k(\eta)P_{k-1}^{(1)}(\mu)\\ &-\sum_{k=1}^{\infty}\frac{k(k+1)}{2(2k+1)(2k-1)}Z_k(\eta)P_{k-2}^{(1)}(\mu)-\sum_{k=1}^{\infty}\frac{k(k+1)}{2(2k+1)(2k+3)}Z_k(\eta)P_{k+2}^{(1)}(\mu).\end{aligned} \tag{4.56}$$

Hence Eq. (4.50b) becomes

$$\begin{aligned}&\sum_{k=1}^{\infty}\Big[-\frac{(k-2)(k-1)}{2(2k-1)}Z_{k-2}(\eta)+\cosh\eta\frac{(k-1)k}{2k-1}Z_{k-1}(\eta)-\frac{2k(k+1)}{(2k-1)(2k+3)}Z_k(\eta)\\ &\quad-\cosh\eta\frac{(k+1)(k+2)}{2k+3}Z_{k+1}(\eta)+\frac{(k+2)(k+3)}{2(2k+3)}Z_{k+2}(\eta)\Big]P_k^{(1)}(\mu)\Big|_{\Gamma}\\ &\quad=\frac{a\sin\xi}{(\cosh\eta-\mu)^{\frac{1}{2}}}v_{\eta}^{(0)}\Big|_{\Gamma}.\end{aligned} \tag{4.57}$$

Consider the right hand side of (4.52),

$$\begin{aligned}-\frac{a}{(\cosh\eta-\mu)^{\frac{1}{2}}}v_{\xi}^{(0)}&=-\frac{a\sin\xi\sinh\eta U^{(z)}}{(\cosh\eta-\cos\xi)^{\frac{3}{2}}}+\frac{(G_{11}+G_{22})a^2(\cos\xi\cosh\eta-1)\sin\xi}{2(\cosh\eta-\cos\xi)^{\frac{5}{2}}}\\ &\quad-\frac{G_{33}a^2\sin\xi\sinh^2\eta}{(\cosh\eta-\cos\xi)^{\frac{5}{2}}}.\end{aligned} \tag{4.58}$$

We need to expand each term of the above equation into a series of associated Legendre polynomials by using the generating function method. According to the relations

$$\frac{(1-\mu^2)^{\frac{m}{2}}}{(1-2\mu t+t^2)^{m+1/2}} = \frac{(-1)^m 2^m m!}{(2m)!}\sum_{k=0}^{\infty} t^k P_{k+m}^{(m)}(\mu), \qquad |t|<1, \tag{4.59a}$$

$$\mu P_k^{(m)}(\mu) = \frac{(k-m+1)P_{k+1}^{(m)}(\mu)+(k+m)P_{k-1}^{(m)}(\mu)}{2k+1}. \tag{4.59b}$$

Thus*

$$-\frac{a\sin\xi\sinh\eta U^{(z)}}{(\cosh\eta-\mu)^{\frac{3}{2}}} = a2^{3/2}U^{(z)}\sinh\eta\sum_{k=1}^{\infty} e^{-(k+\frac{1}{2})|\eta|}P_k^{(1)}(\mu), \tag{4.60}$$

$$\frac{(G_{11}+G_{22})a^2(\cos\xi\cosh\eta-1)\sin\xi}{2(\cosh\eta-\cos\xi)^{\frac{5}{2}}} = \frac{(G_{11}+G_{22})a^2\sqrt{2}}{3}\sum_{k=1}^{\infty}\Big[(k+2)e^{-(k+\frac{3}{2})|\eta|} - (k-1)e^{-(k-\frac{1}{2})|\eta|}\Big]P_k^{(1)}(\mu), \tag{4.61}$$

and then

$$-\frac{G_{33}a^2\sin\xi\sinh^2\eta}{(\cosh\eta-\cos\xi)^{\frac{5}{2}}} = sign[\eta]\frac{G_{33}a^2 2^{3/2}\sinh\eta}{3}\sum_{k=1}^{\infty}(2k+1)e^{(k+\frac{1}{2})|\eta|}P_k^{(1)}(\mu). \tag{4.62}$$

Consider now the right hand side of Eq. (4.57):

$$\frac{a\sin\xi}{(\cosh\eta-\mu)^{\frac{1}{2}}}v_\eta^{(0)} = \frac{a\sin\xi(\cos\xi\cosh\eta-1)U^{(z)}}{(\cosh\eta-\cos\xi)^{\frac{3}{2}}} + \frac{(G_{11}+G_{22})a^2\sin^3\xi\sinh\eta}{2(\cosh\eta-\cos\xi)^{\frac{5}{2}}} + \frac{G_{33}a^2(\cos\xi\cosh\eta-1)\sinh\eta\sin\xi}{(\cosh\eta-\cos\xi)^{\frac{5}{2}}}. \tag{4.63}$$

Upon using (4.59), we get the following

$$\frac{a\sin\xi(\cos\xi\cosh\eta-1)U^{(z)}}{(\cosh\eta-\cos\xi)^{\frac{3}{2}}} = -a2^{\frac{3}{2}}U^{(z)}\sum_{k=1}^{\infty}\Big[\cosh\eta\Big(\frac{k-1}{2k-1}e^{-(k-\frac{1}{2})|\eta|} + \frac{k+2}{2k+3}e^{-(k+\frac{3}{2})|\eta|}\Big) - e^{-(k+\frac{1}{2})|\eta|}\Big]P_k^{(1)}(\mu), \tag{4.64}$$

*For detail, see appendix B

$$\frac{(G_{11}+G_{22})a^2\sin^3\xi\sinh\eta}{2(\cosh\eta-\cos\xi)^{\frac{5}{2}}}=\frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta}{3}$$
$$\times\sum_{k=1}^{\infty}\Big[\frac{(k-2)(k-1)}{2k-1}e^{-(k-\frac{1}{2})|\eta|}-\frac{(k+2)(k+3)}{2k+3}e^{-(k+\frac{3}{2})|\eta|}\Big]P_k^{(1)}(\mu),\quad(4.65)$$

and

$$\frac{G_{33}a^2(\cos\xi\cosh\eta-1)\sinh\eta\sin\xi}{(\cosh\eta-\cos\xi)^{\frac{5}{2}}}=\frac{G_{33}a^2(2)^{\frac{3}{2}}\sinh\eta}{3}$$
$$\times\sum_{k=1}^{\infty}\Big[(k+2)e^{-(k+\frac{3}{2})|\eta|}-(k-1)e^{-(k-\frac{1}{2})|\eta|}\Big]P_k^{(1)}(\mu).\quad(4.66)$$

As a result, we obtain the following four boundary conditions

$$\sum_{k=1}^{\infty}\Big[\Big(\frac{1-k}{2k-1}\Big)Z'_{k-1}(\eta_1)-\frac{\sinh\eta_1}{2}Z_k(\eta_1)+\cosh\eta_1 Z'_k(\eta_1)-\Big(\frac{k+2}{2k+3}\Big)Z'_{k+1}(\eta_1)\Big]$$
$$\times P_k^{(1)}(\mu)=\sum_{k=1}^{\infty}\Big(2^{3/2}aU^{(z)}\sinh\eta_1 e^{(k+\frac{1}{2})\eta_1}+\frac{(G_{11}+G_{22})\sqrt{2}a^2}{3}[(k+2)e^{(k+\frac{3}{2})\eta_1}$$
$$-(k-1)e^{(k-\frac{1}{2})\eta_1}]-\frac{G_{33}a^2 2^{3/2}\sinh\eta_1}{3}(2k+1)e^{(k+\frac{1}{2})\eta_1}\Big)P_k^{(1)}(\mu)\quad(4.67a)$$

$$\sum_{k=1}^{\infty}\Big[\Big(\frac{1-k}{2k-1}\Big)Z'_{k-1}(\eta_2)-\frac{\sinh\eta_2}{2}Z_k(\eta_2)+\cosh\eta_2 Z'_k(\eta_2)-\Big(\frac{k+2}{2k+3}\Big)Z'_{k+1}(\eta_2)\Big]$$
$$\times P_k^{(1)}(\mu)=\sum_{k=1}^{\infty}\Big(2^{3/2}aU^{(z)}\sinh\eta_2 e^{-(k+\frac{1}{2})\eta_2}+\frac{(G_{11}+G_{22})\sqrt{2}a^2}{3}[(k+2)e^{-(k+\frac{3}{2})\eta_2}$$
$$-(k-1)e^{-(k-\frac{1}{2})\eta_2}]+\frac{G_{33}a^2 2^{3/2}\sinh\eta_2}{3}(2k+1)e^{-(k+\frac{1}{2})\eta_2}\Big)P_k^{(1)}(\mu),\quad(4.67b)$$

$$\sum_{k=1}^{\infty}\Big\{-\frac{(k-2)(k-1)}{2(2k-1)}Z_{k-2}(\eta_1)+\cosh\eta_1\frac{(k-1)k}{2k-1}Z_{k-1}(\eta_1)-\frac{2k(k+1)}{(2k-1)(2k+3)}Z_k(\eta_1)$$
$$-\cosh\eta_1\frac{(k+1)(k+2)}{2k+3}Z_{k+1}(\eta_1)+\frac{(k+2)(k+3)}{2(2k+3)}Z_{k+2}(\eta_1)\Big\}P_k^{(1)}(\mu)$$
$$=\sum_{k=1}^{\infty}\Big\{-a2^{\frac{3}{2}}U^{(z)}\Big[\cosh\eta_1\Big(\frac{k-1}{2k-1}e^{(k-\frac{1}{2})\eta_1}+\frac{k+2}{2k+3}e^{(k+\frac{3}{2})\eta_1}\Big)-e^{(k+\frac{1}{2})\eta_1}\Big]$$
$$+\frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta_1}{3}[\frac{(k-2)(k-1)}{2k-1}e^{(k-\frac{1}{2})\eta_1}-\frac{(k+2)(k+3)}{2k+3}e^{(k+\frac{3}{2})\eta_1}]$$
$$+\frac{G_{33}a^2(2)^{\frac{3}{2}}\sinh\eta_1}{3}\Big[(k+2)e^{(k+\frac{3}{2})\eta_1}-(k-1)e^{(k-\frac{1}{2})\eta_1}\Big]\Big\}P_k^{(1)}(\mu),\quad(4.67c)$$

$$\sum_{k=1}^{\infty}\Big\{-\frac{(k-2)(k-1)}{2(2k-1)}Z_{k-2}(\eta_2)+\cosh\eta_2\frac{(k-1)k}{2k-1}Z_{k-1}(\eta_2)-\frac{2k(k+1)}{(2k-1)(2k+3)}Z_k(\eta_2)$$
$$-\cosh\eta_2\frac{(k+1)(k+2)}{2k+3}Z_{k+1}(\eta_2)+\frac{(k+2)(k+3)}{2(2k+3)}Z_{k+2}(\eta_2)\Big\}P_k^{(1)}(\mu)$$
$$=\sum_{k=1}^{\infty}\Big\{-a2^{\frac{3}{2}}U^{(z)}\Big[\cosh\eta_2\Big(\frac{k-1}{2k-1}e^{-(k-\frac{1}{2})\eta_2}+\frac{k+2}{2k+3}e^{-(k+\frac{3}{2})\eta_2}\Big)-e^{-(k+\frac{1}{2})\eta_2}\Big]$$
$$+\frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta_2}{3}\big[\frac{(k-2)(k-1)}{2k-1}e^{-(k-\frac{1}{2})\eta_2}-\frac{(k+2)(k+3)}{2k+3}e^{-(k+\frac{3}{2})\eta_2}\big]$$
$$+\frac{G_{33}a^2(2)^{\frac{3}{2}}\sinh\eta_2}{3}\Big[(k+2)e^{-(k+\frac{3}{2})\eta_2}-(k-1)e^{-(k-\frac{1}{2})\eta_2}\Big]\Big\}P_k^{(1)}(\mu). \quad (4.67\text{d})$$

The equations defined in (4.67) will be used to derive the algebraic system for finding the unknown coefficients in next chapter.

# CHAPTER V

# NUMERICAL IMPLEMENTATION AND CONVERGENCE

## 5.1 Algebraic system

In this section, we derive the algebraic system for the unknown coefficients $C_k^{(1)}, C_k^{(2)}, L^{(m)}$ and $N^{(m)}$. First we compute the values of the auxiliary function $Z_k(\eta_i), i = 1, 2$ and its derivatives $Z_k'(\eta_i), i = 1, 2$ at the sphere boundaries. Applying the orthogonality of associated Legendre polynomials to equations (4.67a)-(4.67d), gives us the following four equations

$$\left(\frac{1-k}{2k-1}\right)Z_{k-1}'(\eta_1) - \frac{\sinh\eta_1}{2}Z_k(\eta_1) + \cosh\eta_1 Z_k'(\eta_1) - \left(\frac{k+2}{2k+3}\right)Z_{k+1}'(\eta_1) =$$
$$2^{3/2}aU^{(z)}\sinh\eta_1 e^{(k+\frac{1}{2})\eta_1} + \frac{(G_{11}+G_{22})\sqrt{2}a^2}{3}[(k+2)e^{(k+\frac{3}{2})\eta_1} - (k-1)e^{(k-\frac{1}{2})\eta_1}]$$
$$-\frac{G_{33}a^2 2^{3/2}\sinh\eta_1}{3}(2k+1)e^{(k+\frac{1}{2})\eta_1} \tag{5.1}$$

$$\left(\frac{1-k}{2k-1}\right)Z_{k-1}'(\eta_2) - \frac{\sinh\eta_2}{2}Z_k(\eta_2) + \cosh\eta_2 Z_k'(\eta_2) - \left(\frac{k+2}{2k+3}\right)Z_{k+1}'(\eta_2) =$$
$$2^{3/2}aU^{(z)}\sinh\eta_2 e^{-(k+\frac{1}{2})\eta_2} + \frac{(G_{11}+G_{22})\sqrt{2}a^2}{3}[(k+2)e^{-(k+\frac{3}{2})\eta_2} - (k-1)e^{-(k-\frac{1}{2})\eta_2}]$$
$$+\frac{G_{33}a^2 2^{3/2}\sinh\eta_2}{3}(2k+1)e^{-(k+\frac{1}{2})\eta_2} \tag{5.2}$$

$$
\begin{aligned}
&-\frac{(k-2)(k-1)}{2(2k-1)}Z_{k-2}(\eta_1)+\cosh\eta_1\frac{(k-1)k}{2k-1}Z_{k-1}(\eta_1)-\frac{2k(k+1)}{(2k-1)(2k+3)}Z_k(\eta_1)\\
&-\cosh\eta_1\frac{(k+1)(k+2)}{2k+3}Z_{k+1}(\eta_1)+\frac{(k+2)(k+3)}{2(2k+3)}Z_{k+2}(\eta_1)\\
&=-a2^{\frac{3}{2}}U^{(z)}\Big[\cosh\eta_1\Big(\frac{k-1}{2k-1}e^{(k-\frac{1}{2})\eta_1}+\frac{k+2}{2k+3}e^{(k+\frac{3}{2})\eta_1}\Big)-e^{(k+\frac{1}{2})\eta_1}\Big]\\
&+\frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta_1}{3}[\frac{(k-2)(k-1)}{2k-1}e^{(k-\frac{1}{2})\eta_1}-\frac{(k+2)(k+3)}{2k+3}e^{(k+\frac{3}{2})\eta_1}]\\
&+\frac{G_{33}a^2(2)^{\frac{3}{2}}\sinh\eta_1}{3}\Big[(k+2)e^{(k+\frac{3}{2})\eta_1}-(k-1)e^{(k-\frac{1}{2})\eta_1}\Big] \qquad (5.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{(k-2)(k-1)}{2(2k-1)}Z_{k-2}(\eta_2)+\cosh\eta_2\frac{(k-1)k}{2k-1}Z_{k-1}(\eta_2)-\frac{2k(k+1)}{(2k-1)(2k+3)}Z_k(\eta_2)\\
&-\cosh\eta_2\frac{(k+1)(k+2)}{2k+3}Z_{k+1}(\eta_2)+\frac{(k+2)(k+3)}{2(2k+3)}Z_{k+2}(\eta_2)\\
&=-a2^{\frac{3}{2}}U^{(z)}\Big[\cosh\eta_2\Big(\frac{k-1}{2k-1}e^{-(k-\frac{1}{2})\eta_2}+\frac{k+2}{2k+3}e^{-(k+\frac{3}{2})\eta_2}\Big)-e^{-(k+\frac{1}{2})\eta_2}\Big]\\
&+\frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta_2}{3}[\frac{(k-2)(k-1)}{2k-1}e^{-(k-\frac{1}{2})\eta_2}-\frac{(k+2)(k+3)}{2k+3}e^{-(k+\frac{3}{2})\eta_2}]\\
&+\frac{G_{33}a^2(2)^{\frac{3}{2}}\sinh\eta_2}{3}\Big[(k+2)e^{-(k+3/2)\eta_2}-(k-1)e^{-(k-1/2)\eta_2}\Big] \qquad (5.4)
\end{aligned}
$$

Since equations (5.3) and (5.4) are uncoupled from each other, we can begin the computations with just $Z_k(\eta_i)$ by solving (5.3) and (5.4), respectively. For large $k$, $Z_k(\eta_i)$ become quite small. Thus the recursion equations for the first $K$ sets give $K$ coefficients. The matrix for these equations is a five-diagonal matrix

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & * & * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & * & * & * \end{pmatrix} \begin{pmatrix} Z_1(\eta_i) \\ Z_2(\eta_i) \\ Z_3(\eta_i) \\ . \\ . \\ . \\ Z_{K-2}(\eta_i) \\ Z_{K-1}(\eta_i) \\ Z_K(\eta_i) \end{pmatrix} = H(\eta_i),$$

which make the computations very fast.

The notation $*$ replaces constant multiples of $Z_k(\eta_i)$ which are not zero. The right-hand-side matrix is given by $H(\eta_i) = (H_1(\eta_i), \ldots, H_k(\eta_i), \ldots, H_K(\eta_i))^T$, where the elements $H_k(\eta_i)$ are expressed as follows

$$H_k(\eta_i) = -a2^{\frac{3}{2}}U^{(z)}\Big[\cosh\eta_i\Big(\frac{k-1}{2k-1}e^{-(k-\frac{1}{2})|\eta_i|} + \frac{k+2}{2k+3}e^{-(k+\frac{3}{2})|\eta_i|}\Big) - e^{-(k+\frac{1}{2})|\eta_i|}\Big]$$
$$+ \frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta_i}{3}\Big[\frac{(k-2)(k-1)}{2k-1}e^{-(k-\frac{1}{2})|\eta_i|} - \frac{(k+2)(k+3)}{2k+3}e^{-(k+\frac{3}{2})|\eta_i|}\Big]$$
$$+ \frac{G_{33}a^2(2)^{\frac{3}{2}}\sinh\eta_i}{3}\Big[(k+2)e^{-(k+\frac{3}{2})|\eta_i|} - (k-1)e^{-(k-\frac{1}{2})|\eta_i|}\Big],\ k = 1, ..., K. \qquad (5.5)$$

After obtaining the values of $Z_k(\eta_i)$, we substitute them into the equations (5.1), (5.2) and the latter become the definitive equations for finding $Z'_k(\eta_i)$. Because the equations (5.1) and (5.2) are uncoupled, we can compute $Z'_k(\eta_i), i = 1, 2$ by solving (5.1) and (5.2), respectively. The values $Z'_k(\eta_i)$ are also quite small for large $k$. Therefore, the recursion equations for the first $K$ set yield $K$ coefficients.

The matrix for these equations are given by in the following tri-diagonal matrix

$$\begin{pmatrix} * & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & * & * \end{pmatrix} \begin{pmatrix} Z'_1(\eta_i) \\ Z'_2(\eta_i) \\ . \\ . \\ . \\ Z'_{K-1}(\eta_i) \\ Z'_K(\eta_i) \end{pmatrix} = G(\eta_i)$$

The right hand side matrix $G(\eta_i) = (G_1(\eta_i), \ldots, G_k(\eta_i), \ldots, G_K(\eta_i))^T$ is given by

$$G_k(\eta_i) = 2^{3/2} a U^{(z)} \sinh \eta_i e^{-(k+\frac{1}{2})|\eta_i|} + \frac{(G_{11}+G_{22})\sqrt{2}a^2}{3}\Big[(k+2)e^{-(k+\frac{3}{2})|\eta_i|}$$

$$- (k-1)e^{-(k-\frac{1}{2})|\eta_i|}\Big] + sign[\eta_i]\frac{G_{33}a^2 2^{3/2}\sinh\eta_i}{3}(2k+1)e^{-(k+\frac{1}{2})|\eta_i|}$$

$$+ \frac{\sinh \eta_i}{2} Z_k(\eta_i), \quad k = 1, ..., K \tag{5.6}$$

Using the definition of $Z_k(\eta)$ and the values of $Z_k(\eta_i), Z'_k(\eta_i)$, the following four linear algebraic recursion formulas can be derived

$$Z_k(\eta_1) = e^{(k+\frac{1}{2})\eta_1} C_k^{(1)} + e^{-(k+\frac{1}{2})\eta_1} C_k^{(2)} + \sum_{m=1}^{\infty} L^{(m)} \varepsilon_m^k(\eta_1) + \sum_{m=1}^{\infty} N^{(m)} \lambda_m^k(\eta_1), \tag{5.7a}$$

$$Z_k(\eta_2) = e^{(k+\frac{1}{2})\eta_2} C_k^{(1)} + e^{-(k+\frac{1}{2})\eta_2} C_k^{(2)} + \sum_{m=1}^{\infty} L^{(m)} \omega_m^k(\eta_2) + \sum_{m=1}^{\infty} N^{(m)} \tau_m^k(\eta_2), \tag{5.7b}$$

and

$$Z'_k(\eta_1) = (k+\frac{1}{2})e^{(k+\frac{1}{2})\eta_1} C_k^{(1)} - (k+\frac{1}{2})e^{-(k+\frac{1}{2})\eta_1} C_k^{(2)} + \sum_{m=1}^{\infty} L^{(m)} \varepsilon_m'^k(\eta_1)$$

$$+ \sum_{m=1}^{\infty} N^{(m)} \lambda_m'^k(\eta_1), \tag{5.8a}$$

$$Z'_k(\eta_2) = (k+\frac{1}{2})e^{(k+\frac{1}{2})\eta_2}C_k^{(1)} - (k+\frac{1}{2})e^{-(k+\frac{1}{2})\eta_2}C_k^{(2)} + \sum_{m=1}^{\infty} L^{(m)}\omega'^{k}_{m}(\eta_2)$$
$$+\sum_{m=1}^{\infty} N^{(m)}\tau'^{k}_{m}(\eta_2). \tag{5.8b}$$

Because the coefficients $C_k^{(1)}, C_k^{(2)}, L^{(m)}$ and $N^{(m)}$ become small with large $k$, the simultaneous solution of these four recursion equations for the first $K$ sets yields $4K$ coefficients. Note that the matrix for these equations is represented as follows

$$\begin{pmatrix}
* & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & + & \times & \cdots & + & \times \\
* & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & + & \times & \cdots & + & \times \\
* & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & + & \times & \cdots & + & \times \\
* & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & + & \times & \cdots & + & \times \\
0 & 0 & * & * & 0 & \cdots & 0 & 0 & 0 & + & \times & \cdots & + & \times \\
0 & 0 & * & * & 0 & \cdots & 0 & 0 & 0 & + & \times & \cdots & + & \times \\
0 & 0 & * & * & 0 & \cdots & 0 & 0 & 0 & + & \times & \cdots & + & \times \\
0 & 0 & * & * & 0 & \cdots & 0 & 0 & 0 & + & \times & \cdots & + & \times \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & + & \times & \cdots & + & \times \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & + & \times & \cdots & + & \times \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & + & \times & \cdots & + & \times \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & + & \times & \cdots & + & \times
\end{pmatrix} \cdot F = B$$

where

$$F = \begin{pmatrix} C_1^{(1)}, & C_1^{(2)}, ..., & C_K^{(1)}, & C_K^{(2)}, & L^{(1)}, & N^{(1)}, ..., & L^{(K)}, & N^{(K)} \end{pmatrix}^T,$$

$$B = \begin{pmatrix} Z_1(\eta_1), & Z_1(\eta_2), & Z'_1(\eta_1), & Z'_1(\eta_2), ..., Z_K(\eta_1), & Z_K(\eta_2), & Z'_K(\eta_1), & Z'_K(\eta_2) \end{pmatrix}^T.$$

The notations $*, +, \times$ denote the constant multiples of $C_k^{(i)}, L^{(k)}$ and $N^{(k)}$, respectively.

## 5.2 Verification of Exponential Convergence

It is important to verify that the results obtained here are compatible with the theoretical exponential convergence of the Legendre series. The examples that we consider in this work include four main cases involving equal and unequal spheres and small and large distances between the spheres. Specifically,

1. Large distance between the spheres:
   (a) $r_1 = 2$, $r_2 = 2$, $d = 15$
   (b) $r_1 = 3$, $r_2 = 2$, $d = 15$
2. Small distance between the spheres:
   (a) $r_1 = 2$, $r_2 = 2$, $d = 5$
   (b) $r_1 = 3$, $r_2 = 2$, $d = 6$

In the figures which follow, we present the computer coefficients $C_k^{(1)}$, $C_k^{(2)}$, $L^{(k)}$, $N^{(k)}$, as functions of their number $k$, alongside with a best fit approximation of exponential type.

We begin with the case when the spheres are separated far from each-other. Fig. 5.1 shows the case of two equal spheres of radii $r_1 = r_2 = 2$ when their centers are separated by a distance of $d = 15$ calibers. The exponential convergence of the series is superbly demonstrated in the figure. The best fit produces an exponential function with exponent $\lambda = -4$, which is a very fast decaying function. This tells us that in this case, a mere 10 terms in the Legendre series produce truncation error $10^{-15}$ which is the round-off error of computations with double precision. We were able to get result down to $10^{-35}$, because Mathematica software can be used with arbitrary number of significant digits. The results presented in Fig. 5.1

outline the effective number of terms to be used on computers with finite number of digits.

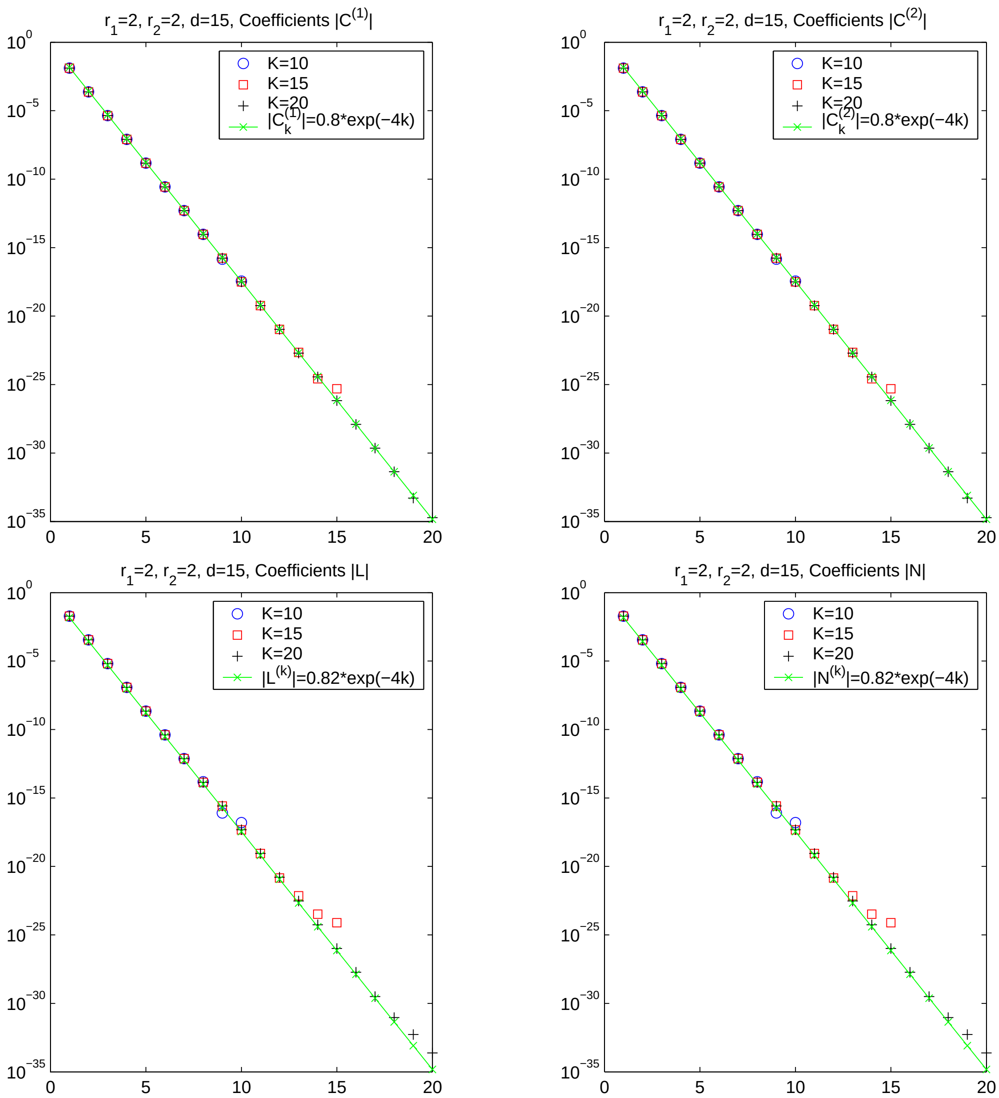



**Figure 5.1** Case 1a: $r_1 = 2$, $r_2 = 2$, $d = 15$

The convergence pattern is rather similar for the case of two unequal spheres situated far from each other, as in Fig. 5.2. An interesting observation is that the coefficients related with the smaller sphere decay slightly slower than the coefficients relevant to the bigger sphere.

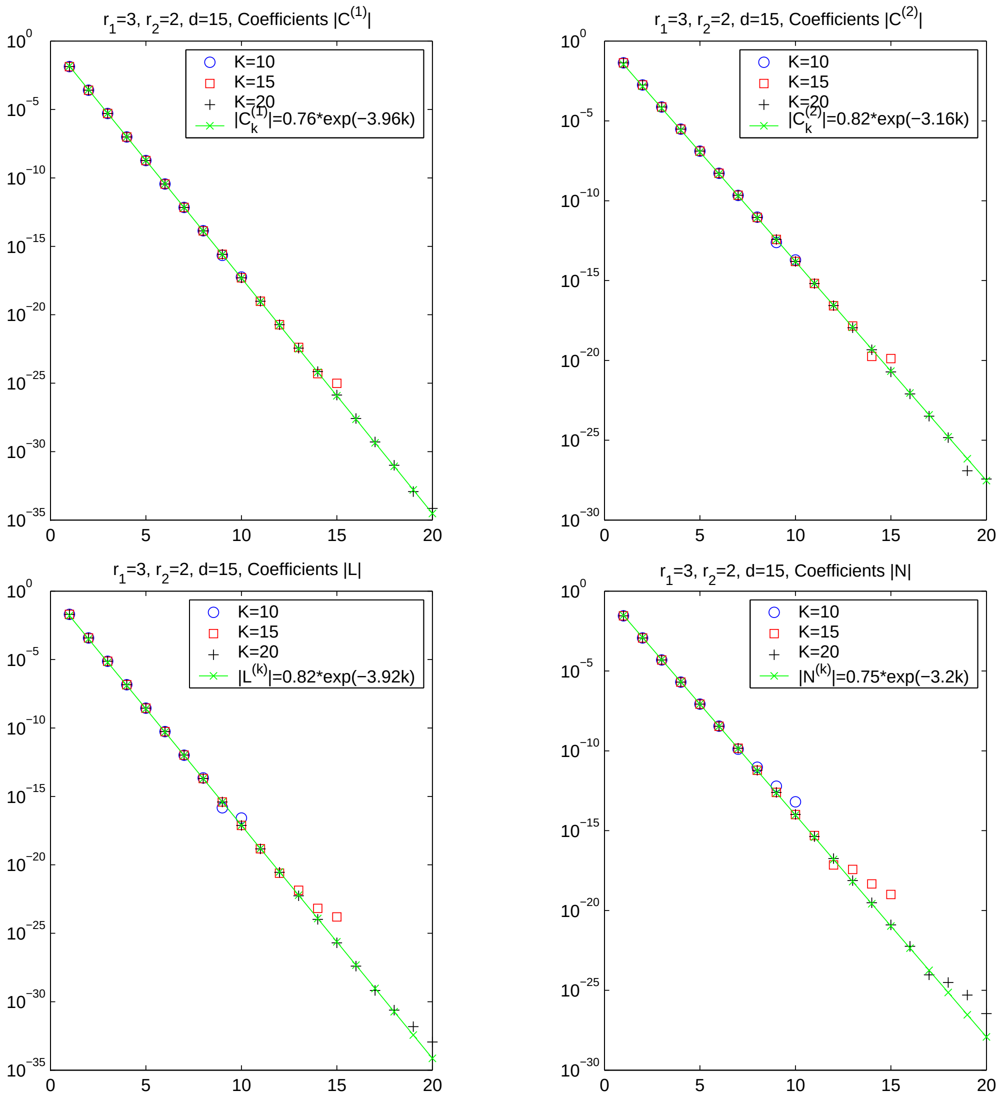



**Figure 5.2** Case 1b: $r_1 = 3$, $r_2 = 2$, $d = 15$

The behavior of the coefficients is rather different when the two spheres are situated at a distance of one caliber from each other. For the two equal spheres this means $d = 2 + 2 + 1 = 5$, which is shown in Fig. 5.3. It is seen now that the convergence is still exponential, but it is much slower, in the sense that the exponent is $\lambda = -1.4$ which is almost 3 times smaller than for the case of well separated spheres. Similarly to the case of well separated spheres, the departure from exponential curve is much stronger for the coefficients $L^{(k)}$ and $N^{(k)}$, rather

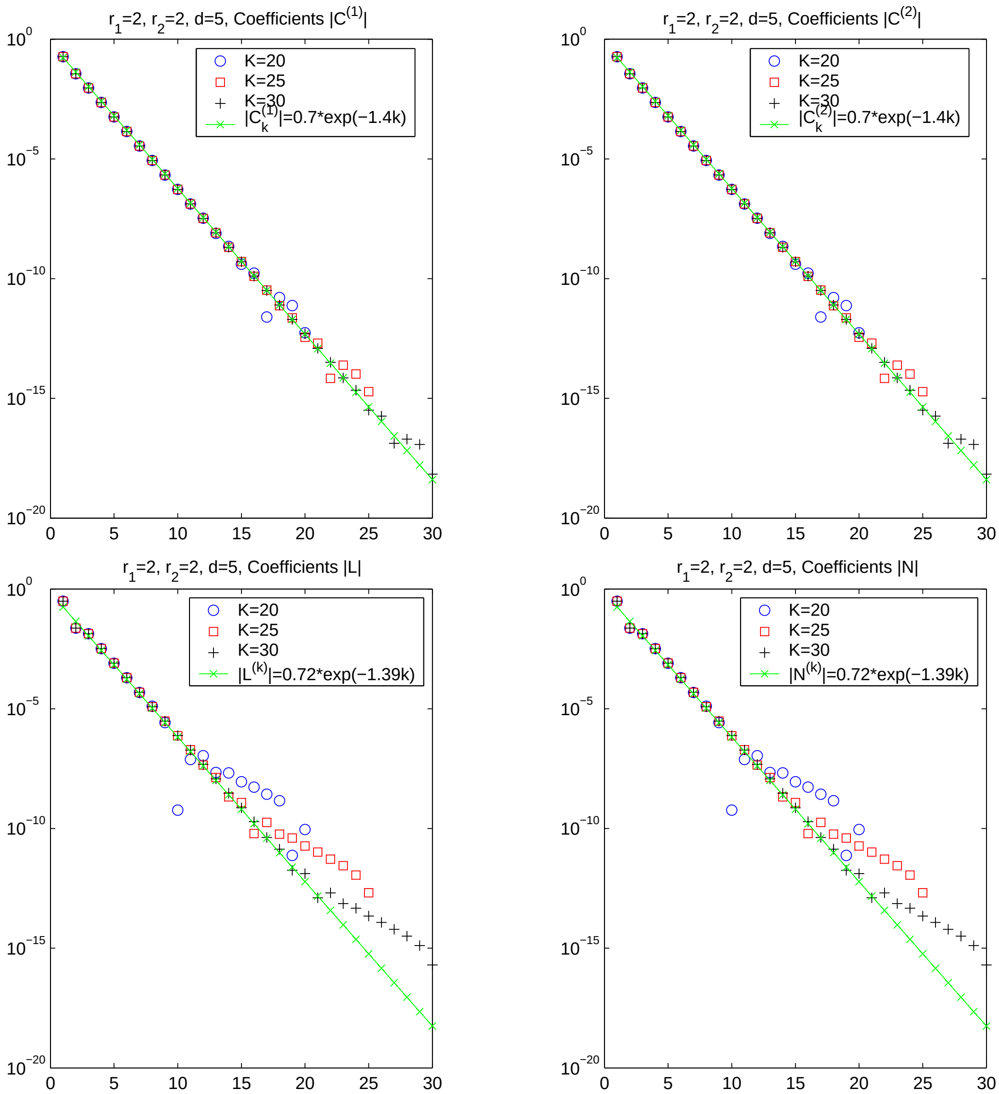



**Figure 5.3** Case 2a: $r_1 = 2$, $r_2 = 2$, $d = 5$

than $C_k^{(1)}$, $C_k^{(2)}$. This is because they enter the algebraic system multiplied by one more matrix, and are thus susceptible to more round-off errors.

Finally, we present in Fig. 5.4 the convergence results for two unequal closely separated spheres. In this case $d = 3 + 2 + 1 = 6$. Once again, the $L^{(k)}$ and $N^{(k)}$ coefficients are much more susceptible to errors. As in the case of well separated spheres, the coefficients relevant to the smaller sphere have a smaller exponent (converge slower).

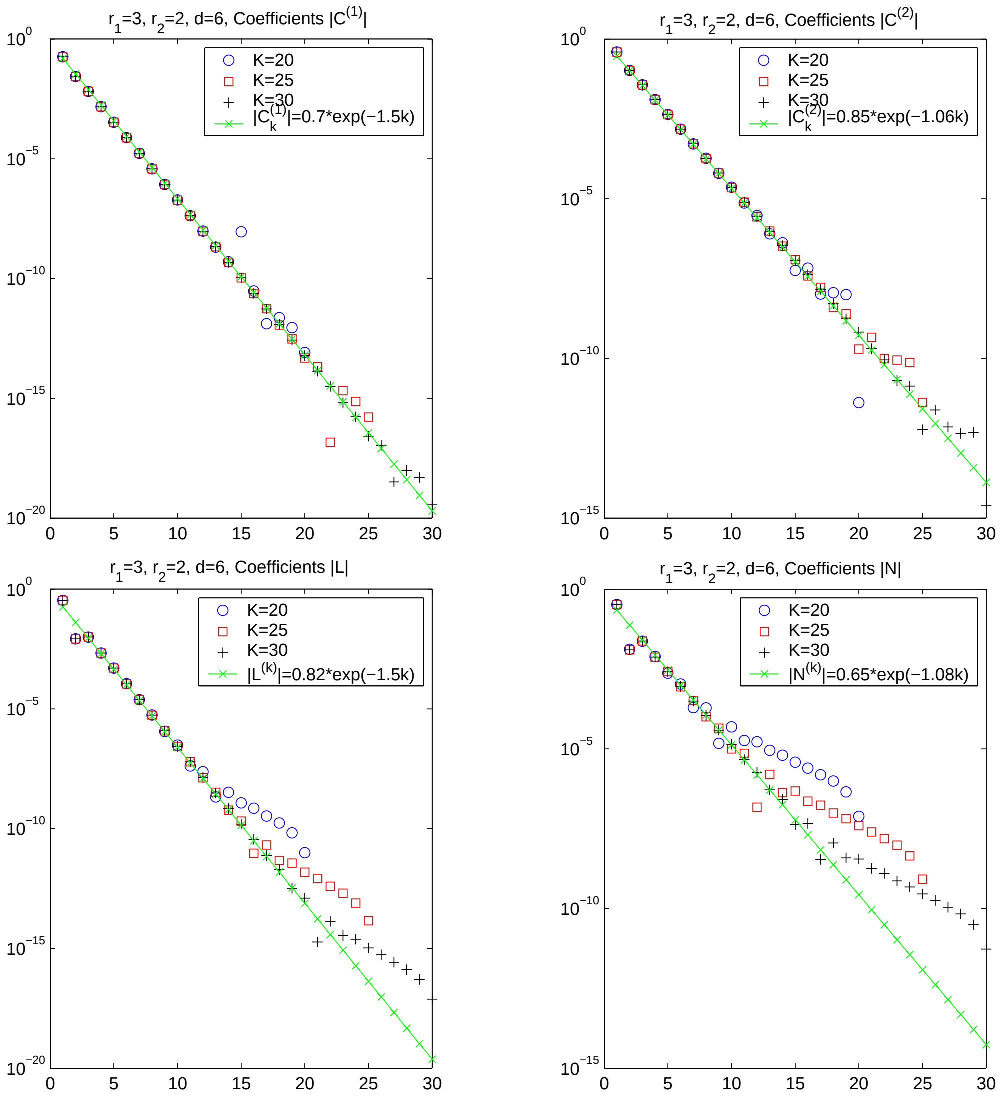



**Figure 5.4** Case 2b: $r_1 = 3$, $r_2 = 2$, $d = 6$

The conclusion of this chapter is that the presented here Legendre-series method is highly efficient, having a very fast exponential convergence. Even for the tough cases of closely situated spheres, results with accuracy better than the rounding-off error of the computer can be obtained with as few as 30 terms in the series. A similar accuracy can be achieved by a difference method with at least 400 points along the two spatial dimensions. In addition, the spectral method has a global convergence, i.e. the accuracy is very well controlled.

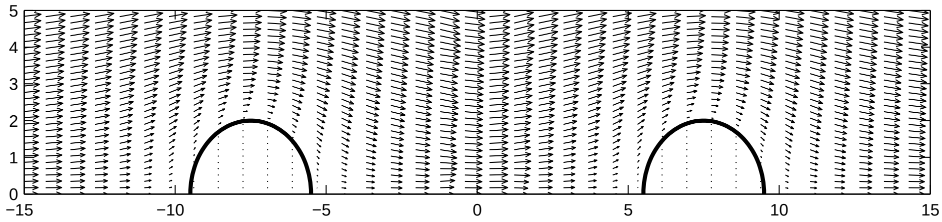
# CHAPTER VI

# SOME NUMERICAL RESULTS AND DISCUSSIONS

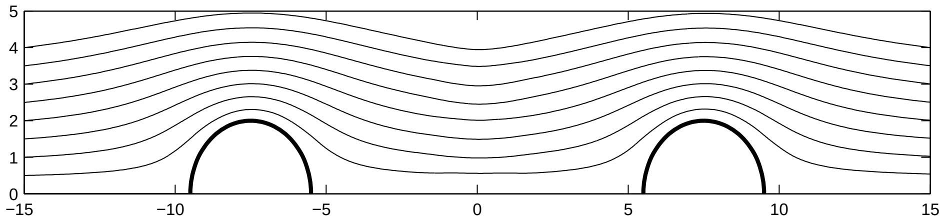
Based on the numerical method and algorithm developed in the previous chapters, we obtained results for the couple of cases already mentioned. One should be reminded that all these cases are only for a uniform flow with no gradient at infinity. To include the gradient effects, one has to create similar algorithms like the one in the previous chapter for the other four systems. This goes beyond the scope of the present dissertation which focuses on the creation of the new type of mixed Legendre-Chebyshev spectral method.

When presenting the results, one has to return to the absolute coordinate system. Our results have been obtained for the relative stream function. While, for the velocity components, the returning requires merely the addition of the velocity at infinity, for the stream function, the process requires integration, in order to find the stream function that is related to the uniform stream. In order to find the stream function in the absolute coordinate system, we have created an integration procedure, but it is far from a truly efficient one, and for this reason we did not treat many different cases. One should understand the results presented here as merely preliminary computations that are aimed to prove the concept. In addition, the main results are for the velocity components, while the stream function is computed only for the reason that people are adapted in discussing stream lines.

Following the convention from the previous chapter about the cases under consideration, we present first the cases of well separated spheres. In Fig. 6.1 we show the velocity components and the stream lines for $r_1 = 2$, $r_2 = 2$ and distance between the spheres $d = 15$. It is well seen that for this case, the flow resembles a

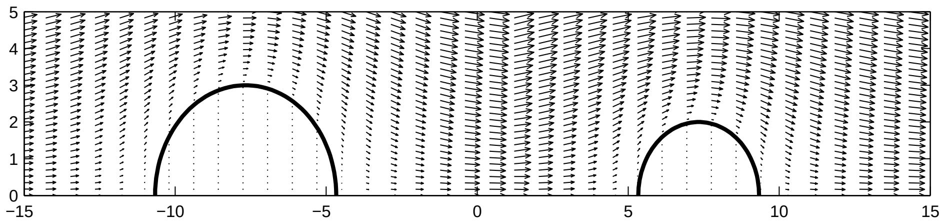



(a) velocity vector

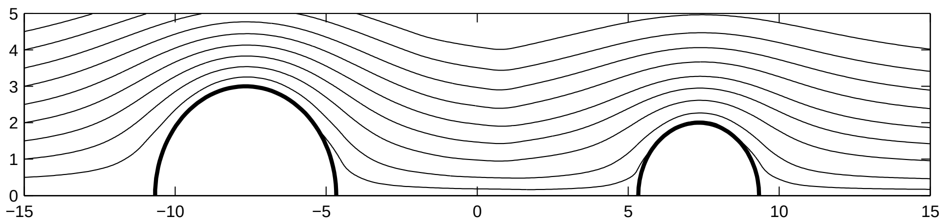



(b) stream function

**Figure 6.1** Flow around two equal well separated spheres: $r_1 = 2, r_2 = 2, d = 15$.




(a) velocity vector

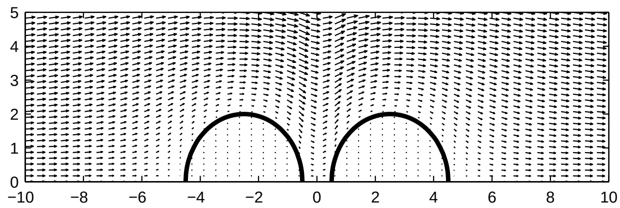



(b) stream function

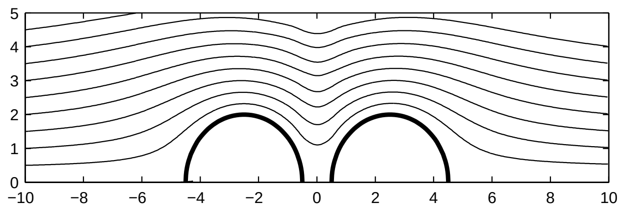
**Figure 6.2** Flow around two unequal well separated spheres: $r_1 = 3, r_2 = 2$, $d = 15$.

superposition of flows around single spheres. This is a good piece of evidence about the reliability of the proposed technique. The case of two unequal well separated spheres is depicted in Fig. 6.2. The conclusions are similar as to the previous case. The above graphs compare very well with the available experimental observations.

In the end, we show for completeness the flow patterns for the two cases when the spheres are closely situated to each other. In Fig. 6.3 we present the

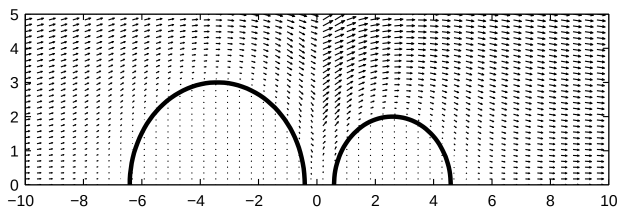


(a) velocity vector

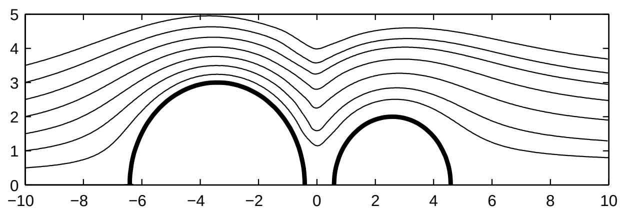


(b) stream function

**Figure 6.3** Flow around two equal closely situated spheres: $r_1 = 2, r_2 = 2, d = 5$.



(a) velocity vector



(b) stream function

**Figure 6.4** Flow around two unequal closely situated spheres: $r_1 = 3, r_2 = 2$, $d = 6$.

case of two equal and closely situated spheres. In this case, it is hard to obtain a quantitative intuition from the case of a single sphere, because in real flows, the Reynolds number is never equal to zero. This means that for even small but nontrivial Reynolds numbers, some kind of separation of the flow will take place between the spheres, which will change qualitatively the stream lines. If there is no separation, the presented patterns are the ones that should be expected. The case of two unequal closed spheres is shown in Fig. 6.4. The conclusions are similar to the previous case. Because of lack of experiments we should not go deeper at this stage, and will leave the more detailed investigation of the physical characteristics for a different specialized work.

The important conclusion is that a very efficient and reliable numerical tool has been developed in the present dissertation.

# CHAPTER VII

# CONCLUSIONS

In this thesis, we have studied the problem of Stokes flow around two nonintersecting, unequal spheres. The flow field at infinity is subject to constant velocity gradient. The following contribution to the field have been made.

Under the assumption of a constant velocity gradient at infinity, the problem is three-dimensional and therefore difficult and expensive to solve numerically or analytically. We make use of the presence of a cyclic variable and propose a special representation of the solution compatible with the form of the boundary conditions. As a result, we have succeeded in reducing the original 3D problem to five partially coupled 2D problems.

The main contribution of the present work is in creating a spectral method based on Legendre polynomials for solving the boundary-value problems of what can be called the bi-Stokesian equations which arise for the stream function for the first of the above described systems. The crucial difference between the second order equation (Laplace equation) and the fourth order equation (the bi-Stokesian equation, in our case) is that the separation of variables is not complete. When the fourth-order equation is rearranged as a system of two second order equations, one of them is homogeneous, and the other is nonhomogeneous. The method of making the homogeneous equation separable is similar to that of the Laplace equation, but for the inhomogeneous one a different approach is needed. We developed here, for the first time in the literature, a method based on hybrid cross expansion involving Chebyshev polynomials of the second kind and associated Legendre polynomials and derived the necessary formulas for the its application.

The new hybrid expansion is completed with a method based on the generating function to acknowledge the boundary conditions. Using the generating function, the boundary conditions that involve the projections of the gradient field on the spheres are expanded into associated Legendre series and a closed linear algebraic system for the unknown coefficients is derived, completing thus the mathematical model of the gradient viscous flow around two unequal spheres.

The Legendre series are known to converge exponentially. We thoroughly validated the proposed numerical procedure and demonstrated that indeed the convergence is exponential. We have found the exponent of the convergence for several important cases: equal and unequal spheres, and short and large distances between the centers of spheres. We have found that although the convergence is exponential in all cases, the absolute value of the actual exponent is much smaller (the series converges slower) for the case of closely situated spheres.

The numerical results for small and large distances between sphere surfaces are shown and discussed. For the large distance between sphere, the flow patterns resemble a superposition of flows around single spheres. In case of closely situated spheres, the computed patterns are the one that have to be expected.

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# REFERENCES


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# APPENDICES

# APPENDIX A

# CYLINDRICAL COORDINATES

The cylindrical coordinates $(r, \phi, z)$ can be expressed by

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z \quad (0 \le \phi \le 2\pi). \tag{A.1}$$

Coordinate surface $r = const > 0$ are circular cylinders (coaxial to $z-$axis), $\phi = const$ are half-planes passing through $z-$axis and $z = const$ are planes perpendicular to $z-$axis. Coordinate curves are : $l_1$ (intersection of $\phi = const$ and $z = const$) are straight rays going from $z-$axis and perpendicular to it; $l_2$ (intersection of $r = const$ and $z = const$) are circles (these circles lie on the planes, which are perpendicular to $z-$axis with a center in the $z-$axis with a center in the $z-$axis); $l_3$ (intersection of $r = const$ and $\phi = const$) are straight lines that are parallel to $z-$axis.

The basis and cobasis of the cylindrical coordinate system are orthogonal and consist of the vectors

$$\begin{aligned}
\bar{e}_1 &= \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right) = (\cos\phi, \sin\phi, 0), \\
\bar{e}_2 &= \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi}\right) = (-r\sin\phi, r\cos\phi, 0), \\
\bar{e}_3 &= \left(\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}\right) = (0, 0, 1), \\
\bar{e}^1 &= \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = (\cos\phi, \sin\phi, 0), \\
\bar{e}^2 &= \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = (-\frac{1}{r}\sin\phi, \frac{1}{r}\cos\phi, 0), \\
\bar{e}^3 &= \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial z}{\partial z}\right) = (0, 0, 1).
\end{aligned}$$

The fundamental tensor and its inverse are defined by

$$(g_{ij}) = (\bar{e}_i \cdot \bar{e}_j)\,, \quad (g^{ij}) = (\bar{e}^i \cdot \bar{e}^j), \tag{A.2}$$

i.e.,

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad |g| = r^2. \tag{A.3}$$

The Christoffel symbols which are related to the derivatives of the fundamental tensor are

$$\Gamma^l_{ij} = \frac{1}{2}\Big(\frac{\partial g_{is}}{\partial K^j} + \frac{\partial g_{js}}{\partial K^i} - \frac{\partial g_{ij}}{\partial K^s}\Big), \quad \Longrightarrow \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^1_{22} = -r, \tag{A.4}$$

and all others are equal to zero.

Since cylindrical coordinates are orthogonal, then all physical components of any type coincide. Let $(u, v, w)$ be physical components of a vector $\bar{v}$. Then the tensor components in the covariant and contravariant form of $\bar{v}$ are

$$(v_1, v_2, v_3) = (u, rv, w), \quad (v^1, v^2, v^3) = (u, \frac{1}{r}v, w).$$

The coordinates of the gradient of the scalar function $p$ are

$$(\nabla p)_1 = (\nabla p)^1 = \frac{\partial p}{\partial r}, \quad (\nabla p)_2 = \frac{\partial p}{\partial \phi}, \, (\nabla p)^2 = \frac{1}{r^2}\frac{\partial p}{\partial \phi}, \quad (\nabla p)_3 = (\nabla p)^3 = \frac{\partial p}{\partial z}.$$

The Stokes equations can be written as

$$\begin{aligned}\frac{\partial u^i}{\partial t} = -\frac{1}{\rho} g^{ij}\frac{\partial p}{\partial K^j} + \nu g^{in}\Big[&\frac{\partial^2 u^i}{\partial K^j \partial K^n} + \Gamma^i_{ln}\frac{\partial u^l}{\partial K^j} + \Gamma^i_{lj}\frac{\partial u^l}{\partial K^n} - \Gamma^l_{jn}\frac{\partial u^i}{\partial K^l} \\ &+ \Big(\frac{\partial \Gamma^i_{lj}}{\partial K^n} + \Gamma^i_{mn}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{jn}\Big)u^l\Big].\end{aligned} \tag{A.5}$$

Hence the Stokes equations and continuity equation in cylindrical coordinates read

$$\frac{\partial u_r}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r u_r + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial^2 u_r}{\partial z^2}\Big], \tag{A.6a}$$

$$\frac{\partial u_\phi}{\partial t} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r u_\phi + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} + \frac{\partial^2 u_\phi}{\partial z^2}\Big], \tag{A.6b}$$

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Big[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2}\Big], \tag{A.6c}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r u_r + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0. \tag{A.6d}$$

# APPENDIX B

# MORE CALCULATIONS OF EQUATIONS (4.60)-(4.66)

We use the following relations to compute (4.60)-(4.66).

$$\frac{(1-\mu^2)^{\frac{m}{2}}}{(1-2\mu t+t^2)^{m+1/2}} = \frac{(-1)^m 2^m m!}{(2m)!}\sum_{k=0}^{\infty} t^k P_{k+m}^{(m)}(\mu) \quad , \quad |t|<1 \tag{B.1a}$$

$$\mu P_k^{(m)}(\mu) = \frac{(k-m+1)P_{k+1}^{(m)}(\mu)+(k+m)P_{k-1}^{(m)}(\mu)}{2k+1} \tag{B.1b}$$

$$(1-\mu^2)^{\frac{1}{2}} P_k^{(m+1)}(\mu) = (k-m)\mu P_k^{(m)}(\mu)-(k+m)P_{k-1}^{(m)}(\mu). \tag{B.1c}$$

**Equation 4.60** Let $e^{-|\eta|}=t$.

$$-\frac{a\sin\xi\sinh\eta U^{(z)}}{(\cosh\eta-\mu)^{\frac{3}{2}}} = -\frac{a\sinh\eta(2t)^{\frac{3}{2}}(1-\mu^2)^{\frac{1}{2}}U^{(z)}}{(1-2\mu t+t^2)^{\frac{3}{2}}}$$

$$\overset{\text{by (B.1a)}}{=} a2^{3/2}U^{(z)}\sinh\eta t^{3/2}\sum_{k=1}^{\infty} t^{k-1}P_k^{(1)}(\mu) = a2^{3/2}U^{(z)}\sinh\eta\sum_{k=1}^{\infty} t^{k+1/2}P_k^{(1)}(\mu).$$

**Equation 4.61**

$$\frac{(G_{11}+G_{22})a^2(\mu\cosh\eta-1)\sin\xi}{2(\cosh\eta-\mu)^{\frac{5}{2}}} = \frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}} t^{\frac{5}{2}}(\mu\cosh\eta-1)(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}}.$$

Let

$$G^1(t,\mu) \equiv \frac{(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{3}{2}}} = -\sum_{k=1}^{\infty} t^{k-1}P_k^{(1)}(\mu). \tag{B.2}$$

Then

$$\frac{\partial G^1}{\partial t} = \frac{3(\mu-t)(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}} = -\sum_{k=1}^{\infty}(k-1)t^{k-2}P_k^{(1)}(\mu). \tag{B.3}$$

Hence

$$\frac{\mu(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}} = \frac{1}{3}\frac{\partial G^1}{\partial t} + \frac{t(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}}.$$

It follows that

$$\frac{(\mu\cosh\eta-1)(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}} = \frac{1}{2}(t+t^{-1})\Big[\frac{1}{3}\frac{\partial G^1}{\partial t} + \frac{t(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}}\Big]$$
$$- \frac{(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}} = \frac{1}{6}(t+t^{-1})\frac{\partial G^1}{\partial t} + \frac{(t^2-1)(1-\mu^2)^{\frac{1}{2}}}{2(1-2\mu t+t^2)^{\frac{5}{2}}}. \quad \text{(B.4)}$$

Consider $(B.2) + \frac{2}{3}t \times (B.3)$;

$$\frac{(t^2-1)(1-\mu^2)^{\frac{1}{2}}}{2(1-2\mu t+t^2)^{\frac{5}{2}}} = \sum_{k=1}^{\infty}[\frac{1}{2} + \frac{(k-1)}{3}]t^{k-1}P_k^{(1)}(\mu). \quad \text{(B.5)}$$

Substitute into the above, we see that

$$\frac{(\mu\cosh\eta-1)(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}} = -\frac{1}{6}(t+t^{-1})\sum_{k=1}^{\infty}(k-1)t^{k-2}P_k^{(1)}(\mu)$$
$$+\sum_{k=1}^{\infty}[\frac{1}{2} + \frac{(k-1)}{3}]t^{k-1}P_k^{(1)}(\mu) = \frac{1}{6}\sum_{k=1}^{\infty}\Big[(k+2)t^{k-1} - (k-1)t^{k-3}\Big]P_k^{(1)}(\mu).$$
$$\text{(B.6)}$$

Therefore,

$$\frac{(G_{11}+G_{22})a^2(\mu\cosh\eta-1)\sin\xi}{2(\cosh\eta-\mu)^{\frac{5}{2}}}$$
$$= \frac{(G_{11}+G_{22})a^2\sqrt{2}}{3}\sum_{k=1}^{\infty}\Big[(k+2)t^{(k+3/2)} - (k-1)t^{(k-1/2)}\Big]P_k^{(1)}(\mu). \quad \text{(B.7)}$$

**Equation 4.62**

$$-\frac{G_{33}a^2\sin\xi\sinh^2\eta}{(\cosh\eta-\mu)^{\frac{5}{2}}} = -\frac{G_{33}a^2\sinh^2\eta(2t)^{\frac{5}{2}}(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}}$$
$$= sign[\eta]\frac{G_{33}a^2\sinh\eta(t-t^{-1})(2t)^{\frac{5}{2}}(1-\mu^2)^{\frac{1}{2}}}{2(1-2\mu t+t^2)^{\frac{5}{2}}}$$
$$= sign[\eta]\frac{G_{33}a^2\sinh\eta 2^{\frac{5}{2}}t^{\frac{3}{2}}(t^2-1)(1-\mu^2)^{\frac{1}{2}}}{2(1-2\mu t+t^2)^{\frac{5}{2}}}$$
$$\overset{\text{by (B.5)}}{=} sign[\eta]\frac{G_{33}a^2 2^{3/2}\sinh\eta}{3}\sum_{k=1}^{\infty}[3+2(k-1)]t^{k+1/2}P_k^{(1)}(\mu)$$
$$= sign[\eta]\frac{G_{33}a^2 2^{3/2}\sinh\eta}{3}\sum_{k=1}^{\infty}(2k+1)t^{k+1/2}P_k^{(1)}(\mu). \quad \text{(B.8)}$$

**Equation 4.64**

$$\frac{a\sin\xi(\mu\cosh\eta-1)U^{(z)}}{(\cosh\eta-\mu)^{\frac{3}{2}}}=\frac{a(2t)^{\frac{3}{2}}U^{(z)}(\mu\cosh\eta-1)(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{3}{2}}}$$

$$\stackrel{\text{by (B.1a)}}{=}-a(2t)^{\frac{3}{2}}U^{(z)}(\mu\cosh\eta-1)\sum_{k=1}^{\infty}t^{(k-1)}P_k^{(1)}(\mu)$$

$$=-a(2)^{\frac{3}{2}}U^{(z)}(\mu\cosh\eta-1)\sum_{k=1}^{\infty}t^{(k+1/2)}P_k^{(1)}(\mu)$$

$$=-a2^{\frac{3}{2}}U^{(z)}\Big[\cosh\eta\sum_{k=1}^{\infty}t^{(k+\frac{1}{2})}\mu\,P_k^{(1)}(\mu)-\sum_{k=1}^{\infty}t^{(k+\frac{1}{2})}P_k^{(1)}(\mu)\Big]$$

$$=-a2^{\frac{3}{2}}U^{(z)}\Big[\cosh\eta\sum_{k=1}^{\infty}t^{(k+\frac{1}{2})}\Big(\frac{kP_{k+1}^{(1)}(\mu)+(k+1)P_{k-1}^{(1)}(\mu)}{2k+1}\Big)-\sum_{k=1}^{\infty}t^{(k+\frac{1}{2})}P_k^{(1)}(\mu)\Big]$$

$$=-a2^{\frac{3}{2}}U^{(z)}\sum_{k=1}^{\infty}\Big[\cosh\eta\Big(\frac{k-1}{2k-1}t^{(k-\frac{1}{2})}+\frac{k+2}{2k+3}t^{(k+\frac{3}{2})}\Big)-t^{(k+\frac{1}{2})}\Big]P_k^{(1)}(\mu). \qquad \text{(B.9)}$$

**Equation 4.65**

$$\frac{(G_{11}+G_{22})a^2\sin^3\xi\sinh\eta}{2(\cosh\eta-\mu)^{\frac{5}{2}}}=\frac{(G_{11}+G_{22})a^2(2t)^{\frac{5}{2}}(1-\mu^2)^{\frac{1}{2}}\sinh\eta(1-\mu^2)}{2(1-2\mu t+t^2)^{\frac{5}{2}}}$$

$$\stackrel{\text{by (B.1a)}}{=}\frac{(G_{11}+G_{22})a^2(2t)^{\frac{5}{2}}\sinh\eta(1-\mu^2)^{\frac{1}{2}}}{2}\frac{2^2 2!}{4!}\sum_{k=1}^{\infty}t^{k-1}P_{k+1}^{(2)}(\mu)$$

$$=\frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta}{3}\sum_{k=1}^{\infty}t^{(k+\frac{3}{2})}(1-\mu^2)^{\frac{1}{2}}P_{k+1}^{(2)}(\mu)$$

$$\stackrel{\text{by (B.1c)}}{=}\frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta}{3}\sum_{k=1}^{\infty}\Big[\frac{(k-2)(k-1)}{2k-1}t^{(k-\frac{1}{2})}$$

$$-\frac{(k+2)(k+3)}{2k+3}t^{(k+\frac{3}{2})}\Big]P_k^{(1)}(\mu). \qquad \text{(B.10)}$$

**Equation 4.66**

$$\frac{G_{33}a^2(\mu\cosh\eta-1)\sinh\eta\sin\xi}{(\cosh\eta-\mu)^{\frac{5}{2}}}=\frac{G_{33}a^2\sinh\eta(2t)^{\frac{5}{2}}(\mu\cosh\eta-1)(1-\mu^2)^{\frac{1}{2}}}{(1-2\mu t+t^2)^{\frac{5}{2}}}$$

$$\stackrel{\text{by (B.6)}}{=}\frac{G_{33}a^2(2t)^{\frac{5}{2}}\sinh\eta}{6}\sum_{k=1}^{\infty}\Big[(k+2)t^{k-1}-(k-1)t^{k-3}\Big]P_k^{(1)}(\mu)$$

$$=\frac{G_{33}a^2(2)^{\frac{3}{2}}\sinh\eta}{3}\sum_{k=1}^{\infty}\Big[(k+2)t^{(k+3/2)}-(k-1)t^{(k-1/2)}\Big]P_k^{(1)}(\mu). \qquad \text{(B.11)}$$

# CURRICULUM VITAE

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