THEORETICAL STUDY OF KAONIC ATOMS

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THEORETICAL STUDY OF KAONIC ATOMS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์นี้นำเสนอการคำนวณเส้นสเปกตรัมของพลังงานของอะตอมเคออนิกไฮโดรเจนที่เลื่อนออกจากเส้นสเปกตรัมเดิมของอะตอมไฮโดรเจนปกติ พร้อมทั้งคำนวณค่าความกว้างของเส้นสเปกตรัมดังกล่าวนั้นซึ่งเป็นผลมาจากแรงนิวเคลียร์แบบเข้มซึ่งเกิดจากอันตรกิริยาระหว่างอนุภาคเคออนกับโปรตอนที่อยู่ในอะตอมเคออนิกไฮโดรเจนนั้น รวมทั้งยังได้คำนวณค่าของฟังก์ชันคลื่นของสถานะพื้นของอะตอมเคออนิกไฮโดรเจนอีกด้วย ซึ่งในการคำนวณปริมาณต่างๆ เหล่านี้เราได้ใช้อันตรกิริยานิวเคลียร์แบบเข้มระหว่างอะตอมของเคออนกับโปรตอนเป็นอันตรกิริยาของศักย์แบบสมจริงในหลายรูปแบบ อีกทั้งได้นำอันตรกิริยาของศักย์แบบคูลอมบ์เข้ามาร่วมในการคำนวณด้วย จากการศึกษาพบว่าฟังก์ชันคลื่นของสถานะพื้นของอะตอมเคออนิกไฮโดรเจนที่คิดอันตรกิริยานิวเคลียร์แบบเข้มเป็นอันตรกิริยาของศักย์แบบสมจริงมีความแตกต่างอย่างเห็นได้ชัดจากฟังก์ชันคลื่นของสถานะพื้นของอะตอมไฮโดรเจนปกติและอะตอมเสมือนไฮโดรเจนที่ในบริเวณระยะใกล้ๆ กับนิวเคลียส นอกจากนี้ ยังพบว่าฟังก์ชันคลื่นของสถานะพื้นของอะตอมเคออนิกไฮโดรเจนได้เกิดบัพขึ้นในบริเวณช่วง 1 ถึง 2 เฟอร์มิ เนื่องจากเกิดสถานะยึดเหนี่ยวแบบลึกขึ้น ซึ่งก็คืออนุภาคแลมด้า (1405) ใกล้ๆ กับค่าพลังงานขีดเริ่มเปลี่ยน

ปัจจุบันเราพบปัญหาเกี่ยวกับค่าความแม่นยำของการคำนวณเส้นสเปกตรัมของพลังงานของอะตอมเคออนิกไฮโดรเจนที่เลื่อนออกจากเส้นสเปกตรัมเดิมของอะตอมไฮโดรเจนปกติ โดยเฉพาะอย่างยิ่งการคำนวณหาค่าฟังก์ชันคลื่นของอะตอมเคออนิกไฮโดรเจน ซึ่งโดยวิธีการมาตรฐานที่เราใช้ในการแก้สมการพลวัตของอะตอมเคออนิกไฮโดรเจน เราจะใช้การกระจายระบบในเซตบริบูรณ์ของฟังก์ชันตั้งฉากปกติซึ่งโดยส่วนใหญ่เราจะประยุกต์ใช้เซตบริบูรณ์ของฟังก์ชันคลื่นตัวแกว่งกวัดแบบฮาร์มอนิกในการแก้ปัญหาของสถานะยึดเหนี่ยวทั้งนี้เนื่องจากเซตบริบูรณ์ของฟังก์ชันคลื่นตัวแกว่งกวัดแบบฮาร์มอนิกมีรูปแบบเชิงวิเคราะห์ทั้งในแบบปริภูมิของตำแหน่งและในแบบปริภูมิของโมเมนตัม แต่วิธีการของฟังก์ชันคลื่นตัวแกว่งกวัดแบบฮาร์มอนิกนี้สามารถใช้อย่างได้ผลเฉพาะปัญหาของสถานะยึดเหนี่ยวในกรณีที่มีเฉพาะอันตรกิริยานิวเคลียร์แบบเข้มหรือในกรณีที่มีเฉพาะอันตรกิริยาแบบคูลอมบ์เท่านั้นไม่สามารถนำใช้ได้กับกรณีอะตอมเคออนิกทั้งนี้ เนื่องจากพิสัยที่ต่างกันมากๆ ของอันตรกิริยาระหว่างอนุภาคเคออนและโปรตอนได้ถูกนำมาใช้ในการคำนวณด้วย ซึ่งในกรณีนี้อันตรกิริยาจะมีทั้งอันตรกิริยาแบบคูลอมบ์ซึ่งเป็นแรงพิสัยไกลและอันตรกิริยานิวเคลียร์แบบเข้มซึ่งเป็นแรงพิสัยใกล้

ในวิทยานิพนธ์นี้ได้ใช้วิธีการศึกษาเชิงตัวเลข ซึ่งมีพื้นฐานมาจากฟังก์ชันของสเตอร์เมียนในการแก้ปัญหาของอะตอมเคออนิกไฮโดรเจน วิธีการเชิงตัวเลขนี้ได้เคยถูกนำมาใช้เพื่อแก้ปัญหา

ของอะตอมโปรตรอนเนียมและไพออนเนียมอย่างประสบผลสำเร็จมาแล้ว วิธีการเชิงตัวเลขนี้เป็นวิธีที่มีประสิทธิภาพ ให้ผลลัพธ์จากการคำนวณที่มีความแม่นยำสูง อีกทั้งยังมีความง่ายต่อการนำไปใช้มากกว่าวิธีการคำนวณแบบอื่นๆ ที่เคยถูกนำมาประยุกต์ใช้กับปัญหาของอะตอมฮาร์ดรอนนิก นอกจากนั้นวิธีการเชิงตัวเลขนี้ยังสามารถนำไปประยุกต์ใช้ในการแก้ปัญหาของอะตอมเอ็กซ็อททิค ซึ่งเป็นอะตอมที่มีทั้งในกรณีศักย์ที่เกิดจากแรงนิวเคลียร์แบบเข้มซึ่งเป็นแรงพิสัยใกล้ (สำหรับทั้งในกรณีที่ศักย์ขึ้นกับตำแหน่งและในกรณีที่ศักย์ไม่ขึ้นกับตำแหน่ง) และในกรณีศักย์ที่เกิดจากแรงคูลอมบ์ซึ่งเป็นแรงพิสัยไกล และยังสามารถคำนวณฟังก์ชันคลื่นและพลังงานยึดเหนี่ยวของอะตอมเอ็กซ็อททิคได้โดยตรง

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ลายมือชื่อนักศึกษา _____
ลายมือชื่ออาจารย์ที่ปรึกษา _____
ลายมือชื่ออาจารย์ที่ปรึกษาร่วม _____

CHAKRIT NUALCHIMPLEE : THEORETICAL STUDY OF KAONIC ATOMS. THESIS ADVISOR : PROF. PRASART SUEBKA, Ph.D. 66 PP.

KAONIC HYDROGEN/ STURMIAN FUNCTIONS/ PIONIUM

In the thesis the energy shift, decay width and wave function of the kaonic hydrogen atom are directly evaluated with various versions of realistic interaction potentials in addition to the Coulomb interaction. It is found that the ground-state wave function of the kaonic hydrogen atoms with realistic strong interactions is considerably different from the hydrogen-like ones at small distances, and also has a node in the region from 1 to 2 fm, because there exists one deep bound state, the $\Lambda(1405)$ near threshold.

It has been a challenge to accurately evaluate the energy shift and especially the wave function of hadronic atoms. One may think that the dynamical equations of the kaonic hydrogen can be solved by simply expanding the system in any complete set of orthonormal functions. The complete set of harmonic oscillator wave functions is widely applied to bound state problems since they have analytical forms both in coordinate and momentum spaces. Bound state problems with only the strong interaction or only the Coulomb force can be well solved in the regime of harmonic oscillator wave functions. However, the harmonic oscillator wave function approach fails to describe hadronic atoms which are dominated by the long-ranged Coulomb force and distorted by the short-ranged strong interaction. The reason is that two very different oscillator lengths are involved to account for the short-ranged strong interaction and the long-ranged Coulomb force.

The protonium, $\overline{p}D$ atom and pionium have been successfully investigated in a numerical approach based on Sturmian functions. The numerical method

is much more powerful, accurate and much easier to use than all other methods applied to the hadronic atoms in history. It can be applied to solve the exotic atom problem for local and non-local potentials, accounting for both the strong short range nuclear interaction and the long range Coulomb force and provides directly the wave function and binding energy of those exotic atoms.

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CHAPTER I

INTRODUCTION

The investigation of exotic atoms opens up new windows in nuclear and particle physics for studying the strong interaction, quantum chromodynamics (QCD), at zero-energy with unprecedented sensitivity. An exotic atom is the analogue of normal atoms in which one or more of the electrons are replaced by other negative particles, such as muon, pion or kaon, or the positively charged nucleus is replaced by other positively charged elementary particles, or both. On the other hand, we can say that an exotic atom is a Coulomb bound state which do not exist in the nature but produced in laboratory. Among them are, for example, positronium (electron-positron bound states), protonium (proton-antiproton atomic states) (Cohen and Padial, 1990), pionium (pion-pion atomic states) (Gall et al., 1999), (Gasser et al., 2001), (Colangelo et al., 2001), (Colangelo et al., 2000), muonic atoms (muon-nucleus bound state) (Hughes, 1966), (Wu and Wilets, 1969), pionic atoms (pion-nucleus atomic state) (Backenstoss, 1970), and kaonic atoms (kaon-nucleus atomic state) (Iwasaki et al., 1997), (Ito et al., 1998).

Exotic atoms was predicted in the 1940s (Tomonaga and Araki, 1940), (Conversi et al., 1945), (Conversi et al., 1947a). The existence of exotic atoms was first established by the observation of Auger electrons in photographic emulsion (Fermi and Teller, 1947). In the early 1950s the characteristic X-rays from pionic and muonic atoms were identified, for the first time. At the present time, X-radiation from exotic atoms with muons (Fry, 1951), pions (Fitch and

Rainwater, 1953), (Rasche and Woolcock, 1982), (Gotta, 2004) kaons (Burleson et al., 1965) antiprotons (Bamberger et al., 1970) and sigma hyperons (Backenstoss, 1970) has been established.

At the first time, strong-interaction was a main reason for studying exotic atoms. Because a binding energy of light system of exotic atoms are in ten keV range, it is far below from the hadronic scale of about 1 GeV. Hence, investigation of exotic atoms provide the unique possibility to perform experiments equivalent to scattering at relative energy. One of the simplest exotic atoms is kaonic hydrogen atoms (Iwasaki et al., 1997), (Ito et al., 1998). Kaonic hydrogen atoms is a bound state of negatively charged kaon K^- and the proton combined mainly by Coulomb interaction and effected also by the short range strong interaction. This strong interaction is the reason for a shift in energy of the lowest-lying level from the purely electromagnetic value and the finite lifetime of the state-corresponding to an increase in the observed level width. This results in the atom being unstable, with a very small lifetime.

The kaonic hydrogen may be formed with a much shorter time than the lifetime of the charged kaons which is 1.24×10^{-8} seconds with the main decay mode (Zmeskal, 2008):

$$K^- \rightarrow \mu^- + \overline{\nu}_\mu \tag{1.1}$$

$$K^- \rightarrow \pi^- + \pi^0 \tag{1.2}$$

$$K^- \rightarrow \pi^- + \pi^- + \pi^+. \tag{1.3}$$

The involved time scales for the formation of the kaonic hydrogen are; first for slowing a kaon down and capturing it into an atomic orbit about 10^{-12} to 10^{-19} s, then for Coulomb de-excitation and Auger processes about 10^{-12} to 10^{-15} s and finally, for radiative transitions about 10^{-15} to 10^{-17} s. Thus the charged kaon in the kaonic hydrogen atom can be considered a practically stable particle. Despite

its short lifetime, kaonic hydrogen can be considered as a quasi-stable bound state, because the charged kaon travels many times around the proton before decaying, as the ratio $\frac{1}{2}\mu_c\alpha^2$ / $\Gamma_1 \sim 10^3$ indicates, where

$$\mu_c = \frac{m_p m_K}{(m_p + m_K)} = 323.9 \text{ MeV/c}^2$$

is the reduced mass of the kaon and proton system which m_p and m_K denote for the masses of the proton and the charged kaon, respectively, and $\alpha \simeq 1/137$ denotes for the fine-structure constant.

As we have briefly mentioned, in comparison with ordinary hydrogen atoms, the ground state of kaonic hydrogen atoms is unstable. The typical size of kaonic hydrogen is characterized by its bohr radius

$$r_B = 1/\alpha\mu_c \simeq 84 \text{ fm} \tag{1.4}$$

Moreover, the typical size of kaonic hydrogen atom is about 630 times smaller than a normal hydrogen atom and the binding energy, proportional with the reduced mass μ_c, for the ground state of kaonic hydrogen is about

$$E_{1s} = \frac{\mu_c\alpha^2}{2} \simeq 8.61 \text{ keV} \tag{1.5}$$

Because both bound states of kaonic hydrogen and normal hydrogen atom are predominantly formed by the Coulomb force, the properties of hadronic atoms are similar to those of the hydrogen atoms in many aspects. The distance $r_B \simeq 84$ fm is much smaller than the hydrogen radius, but still much larger than the range of strong interactions, which is typically of the order of a few fm. This is the reason that strong interactions do not change the structure of the bound-state spectrum in a profound way. At leading order in an expansion in α, the energy of S-wave state of kaonic hydrogen is still given by standard quantum-mechanical formula

$$E_n = m_p + m_K - \frac{\mu_c\alpha^2}{2n^2}, \quad n = 1, 2, ... \tag{1.6}$$

Moreover, Eq. (1.6) describes the bound-state spectrum only approximately, at leading order in an expansion in α. The leading correction to Eq. (1.6), which emerges at order α^3, is due to strong interaction only, this means there is no interference between Coulomb and strong interactions at this order. Consequently, by measuring very accurately of the difference between the true energy levels of kaonic hydrogen and the pure Coulomb value in Eq. (1.6), one can extract information about the strong interactions between the kaon and proton. Because the size of the atom is much larger than the characteristic radius of strong interactions, the bound-state observables can feel only the low-momentum of the strong pion-nucleon S-matrix. Hence, in other words, the energy shift can be expressed in terms of the threshold parameters of kaon-nucleon scattering amplitude, for example, the scattering lengths, effective ranges, etc. In other words, the measurement of the observables of the kaonic hydrogen does not probe the inner region of the kaon-nucleon interaction.

The insensitivity of the kaonic hydrogen observables to the short-range details of the kaon-nucleon interaction is very fortunate, because it provides us with the possibility of directly extracting the values of the kaon-nucleon scattering lengths from atomic experiments. A different method for determining the same quantities is to measure the scattering cross sections at different energies, and to extrapolate the result to threshold. The former method, however, is free from the difficulties which are related to this extrapolation procedure. This property is even more important in other hadronic systems, where the scattering amplitude near threshold is hardly accessible by other experimental technique. Deser, Goldberger, Baumann, and Thirring (Deser et al., 1954) were the first group who derived the model-independent relation between the complex energy shift of the ground state in kaonic hydrogen and the strong kaon-nucleon scattering amplitude at leading

order in α. The result is

$$\Delta E_1 - \frac{i}{2}\Gamma_1 = -\frac{2\pi}{\mu_c}|\tilde{\Psi}_{10}(0)|^2 A(K^- p \to K^- p) + O(\alpha^4), \tag{1.7}$$

where $|\tilde{\Psi}_{10}(0)|^2 = \alpha^3 \mu_c^3/\pi$ denote for the square of the wave function of the atom at the origin and is, therefore, a measure of the probability that the charge kaon and the proton in the atom come very close to each other. Furthermore, $A(K^- p \to K^- p)$ is the $K^- p$ elastic scattering amplitude at threshold, which describes strong interactions between these particles after they come close. The kaonic hydrogen were first seen through their X-ray spectrum, at the KEK in Tsukuba, Japan (Iwasaki et al., 1997), (Ito et al., 1998) where the following results were found :

$$\Delta E_{1s} = -323 \pm 63(\text{statistical}) \pm 13(\text{systematic}) \text{ eV} \tag{1.8}$$

$$\Gamma_{1s} = 407 \pm 208(\text{statistical}) \pm 100(\text{systematic}) \text{ eV} \tag{1.9}$$

More detailed studies have been performed at DAFNE (Beer et al., 2005), kaonic hydrogen has been created in vary low energy collisions of kaons with protons. In the experimental called DAFNE Exotic Atoms Research (DEAR), they measures the X-ray transitions occurring in the cascade processes of kaonic atoms. A kaonic atoms is formed when a negatively charged kaon, coming from the φ -decay which produced at DAFNE, enters a target and loses it kinetic energy through the ionization and excitation of the atoms and molecules of the medium, and is eventually captured, replacing the electron, in an excited orbit. Via different cascade processes (Auger effect, Coulomb de-excitation, scattering) the kaonic atom de-excites to lower states.When a low-n state with small angular momentum is reached, the strong interaction with the nucleus come into play. The DEAR collaboration has performed a measurement of the energy-level shift and decay width of the $\bar{K}N$ ground state with a considerably better accuracy than the earlier KpK

experiment at KEK (Iwasaki et al., 1997). the present experimental values of these quantities are

$$\Delta E_{1s} = -194 \pm 40(\text{statistical}) \pm 6(\text{systematic}) \text{ eV} \tag{1.10}$$

$$\Gamma_{1s} = 249 \pm 111(\text{statistical}) \pm 30(\text{systematic}) \text{ eV} \tag{1.11}$$

As can be seen from the above result, the uncertainty is still ten of eV in the energy shift and more than 100 eV in the width. Now DEAR is being followed by the SIDDHARTA (SIlicon Drift Detector for Hadronic Atom Research by Timing Application) experiment that features new silicon drift detectors. The plans of the SIDDHARTA collaboration include the measurement of both, the energy shift and decay width of $\bar{K}H$, with a precision of several eV, i.e. at the few percent level.

In the thesis we directly evaluated the energy shift, decay width and the wave function of the konic hydrogen atom system, with various versions of realistic interaction potentials, in the framework of the Schrödinger equation involving in addition the Coulomb interaction. In our work we study the konic hydrogen atom problem employing a properly adapted numerical method based on Sturmian functions (Yan et al., 1997). The method accounts for both the strong *short* range nuclear potential (local and non-local) and the *long* range Coulomb force. The protonium, $\overline{p}D$ atom and pionium problems have been successfully investigated (Suebka and Yan, 2004), (Yan et al., 2007) in the numerical approach. The numerical method is much more powerful, accurate and much easier to use than all other methods applied to the exotic atom (hadronic atoms) problem in history.

The thesis is organized as follows. In Chapter II we construct the dynamical equations of kaonic hydrogen atoms system. In Chapter III we describe our numerical procedure based on Sturmian functions. Finally, Chapter IV contains results

for the energy shift, decay width and the wave function from various versions of interaction potentials, discussions and conclusions.

CHAPTER II

DYNAMICAL EQUATIONS

This chapter is devoted to construct the dynamical equations of kaonic hydrogen atoms. Kaonic hydrogen atom is mainly the Coulomb bound state of a K^- and a proton (p) but is affected by the strong interaction at small distances. Basically, the strong interaction couples the K^-p state to the $\bar{K}^0n$, $\pi\Sigma$, $\pi\Lambda$, $\eta\Sigma$ and $\eta\Lambda$ channels, but the decaying of the K^-p are dominated by the $\bar{K}^0n$, $\pi\Sigma$ and $\pi\Lambda$ channels. In our study, we neglect the $\eta\Sigma$ and $\eta\Lambda$ channels since the couplings are believed weak.

2.1 Dynamical equations of kaonic hydrogen atoms

A correct treatment of kaonic hydrogen atomic states must include the coupling of the negatively charge kaon-proton (K^-p), neutral antikaon-neutron ($\bar{K}^0n$), neutral pion-lambda ($\pi^0\Lambda$), neutral pion-sigma ($\pi^0\Sigma$), negatively charge pion-positively charge sigma ($\pi^-\Sigma^+$) and positively charge pion-negatively charge sigma ($\pi^+\Sigma^-$) configurations. We define the Hilbert spaces of K^-p, $\bar{K}^0n$, $\pi^0\Lambda$, $\pi^0\Sigma$, $\pi^-\Sigma^+$ and $\pi^+\Sigma^-$ as P_1, P_2, P_3, P_4, P_5 and P_6 spaces, respectively. The Hilbert space of other channels is defined as Q space. The corresponding projection operators P_1, P_2, P_3, P_4, P_5, P_6 and Q satisfy the completeness relation:

$$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + Q = \mathbb{1} \tag{2.1}$$

$$P_1^2 = P_1, P_2^2 = P_2, P_3^2 = P_3, P_4^2 = P_4, P_5^2 = P_5, P_6^2 = P_6, Q^2 = Q \tag{2.2}$$

as well as orthogonality

$$P_1P_2 = P_1P_3 = P_1P_4 = ... = P_6P_5 = 0 \tag{2.3}$$

$$P_1Q = QP_1 = 0 \tag{2.4}$$

$$P_2Q = QP_2 = 0 \tag{2.5}$$

$$P_3Q = QP_3 = 0 \tag{2.6}$$

$$P_4Q = QP_4 = 0 \tag{2.7}$$

$$P_5Q = QP_5 = 0 \tag{2.8}$$

$$P_6Q = QP_6 = 0. \tag{2.9}$$

The Hamilton operator of the full coupled-channel problem is given by H with the corresponding wave function $|\Psi\rangle$ defined in the complete Hilbert space. To construct the dynamical equations of kaonic hydrogen atoms, we start from the Schrödinger equation

$$(E - H)|\Psi\rangle = 0. \tag{2.10}$$

where E is the energy eigenvalue.

First, we consider the K^-p channel, then Eq. (2.10) becomes

$$EP_1|\Psi\rangle - P_1H\mathbb{1}|\Psi\rangle = 0$$

$$\begin{aligned} EP_1|\Psi\rangle - P_1H(P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + Q)|\Psi\rangle = 0 \\ EP_1|\Psi\rangle - P_1HP_1|\Psi\rangle - P_1HP_2|\Psi\rangle - P_1HP_3|\Psi\rangle - P_1HP_4|\Psi\rangle \\ - P_1HP_5|\Psi\rangle - P_1HP_6|\Psi\rangle - P_1HQ|\Psi\rangle \\ = 0. \end{aligned} \tag{2.11}$$

Next, by using identities from Eq. (2.2), we can write

$$\begin{aligned} EP_1|\Psi\rangle - P_1HP_1P_1|\Psi\rangle - P_1HP_2P_2|\Psi\rangle - P_1HP_3P_3|\Psi\rangle - P_1HP_4P_4|\Psi\rangle \\ - P_1HP_5P_5|\Psi\rangle - P_1HP_6P_6|\Psi\rangle - P_1HQQ|\Psi\rangle \\ = 0. \end{aligned} \tag{2.12}$$

where G is the Greens function for all possible intermediate states, defined as:

$$G = \frac{1}{E - QHQ}. \tag{2.13}$$

Hence,from Eq. (2.12), the dynamical equation of the $K^- p$ channel can be written into

$$\begin{aligned}(E - P_1HP_1)P_1|\Psi\rangle = (P_1HP_2)P_2|\Psi\rangle + (P_1HP_3)P_3|\Psi\rangle + (P_1HP_4)P_4|\Psi\rangle \\ + (P_1HP_5)P_5|\Psi\rangle + (P_1HP_6)P_6|\Psi\rangle + (P_1HQ)Q|\Psi\rangle\end{aligned} \tag{2.14}$$

with $P_1|\Psi\rangle = \Psi_{K^-p}$, $P_2|\Psi\rangle = \Psi_{\bar{K}^0 n}$, $P_3|\Psi\rangle = \Psi_{\pi^0\Lambda}$, $P_4|\Psi\rangle = \Psi_{\pi^0\Sigma}$, $P_5|\Psi\rangle = \Psi_{\pi^-\Sigma^+}$ and $P_6|\Psi\rangle = \Psi_{\pi^+\Sigma^-}$.

By using the same method, the dynamical equation of coupling channel of another channels are

$$\begin{aligned}(E - P_2HP_2)P_2|\Psi\rangle = (P_2HP_1)P_1|\Psi\rangle + (P_2HP_3)P_3|\Psi\rangle + (P_2HP_4)P_4|\Psi\rangle \\ + (P_2HP_5)P_5|\Psi\rangle + (P_2HP_6)P_6|\Psi\rangle + (P_2HQ)Q|\Psi\rangle,\end{aligned} \tag{2.15}$$

for the $\bar{K}^0 n$ channel,

$$\begin{aligned}(E - P_3HP_3)P_3|\Psi\rangle = (P_3HP_1)P_1|\Psi\rangle + (P_3HP_2)P_2|\Psi\rangle + (P_3HP_4)P_4|\Psi\rangle \\ + (P_3HP_5)P_5|\Psi\rangle + (P_3HP_6)P_6|\Psi\rangle + (P_3HQ)Q|\Psi\rangle,\end{aligned} \tag{2.16}$$

for the $\pi^0\Lambda$ channel,

$$\begin{aligned}(E - P_4HP_4)P_4|\Psi\rangle = (P_4HP_1)P_1|\Psi\rangle + (P_4HP_2)P_2|\Psi\rangle + (P_4HP_3)P_3|\Psi\rangle \\ + (P_4HP_5)P_5|\Psi\rangle + (P_4HP_6)P_6|\Psi\rangle + (P_4HQ)Q|\Psi\rangle,\end{aligned} \tag{2.17}$$

for the $\pi^0\Sigma$ channel,

$$\begin{aligned}(E - P_5HP_5)P_5|\Psi\rangle = (P_5HP_1)P_1|\Psi\rangle + (P_5HP_2)P_2|\Psi\rangle + (P_5HP_3)P_3|\Psi\rangle \\ + (P_5HP_4)P_4|\Psi\rangle + (P_5HP_6)P_6|\Psi\rangle + (P_5HQ)Q|\Psi\rangle,\end{aligned} \tag{2.18}$$

for the $\pi^-\Sigma^+$ channel,

$$\begin{aligned}(E - P_6HP_6)P_6|\Psi\rangle = (P_6HP_1)P_1|\Psi\rangle + (P_6HP_2)P_2|\Psi\rangle + (P_6HP_3)P_3|\Psi\rangle \\ + (P_6HP_4)P_4|\Psi\rangle + (P_6HP_5)P_5|\Psi\rangle + (P_6HQ)Q|\Psi\rangle,\end{aligned} \tag{2.19}$$

for the $\pi^+\Sigma^-$ channel and

$$\begin{aligned}(E-QHQ)Q|\Psi\rangle = (QHP_1)P_1|\Psi\rangle + (QHP_2)P_2|\Psi\rangle + (QHP_3)P_3|\Psi\rangle \\ + (QHP_4)P_4|\Psi\rangle + (QHP_5)P_5|\Psi\rangle + (QHP_6)P_6|\Psi\rangle.\end{aligned} \tag{2.20}$$

for all another channels.

Consider Eq. (2.20), we can write

$$\begin{aligned}Q|\Psi\rangle = \frac{1}{E-QHQ}(QHP_1)P_1|\Psi\rangle + \frac{1}{E-QHQ}(QHP_2)P_2|\Psi\rangle \\ + \frac{1}{E-QHQ}(QHP_3)P_3|\Psi\rangle + \frac{1}{E-QHQ}QHP_4)P_4|\Psi\rangle \\ + \frac{1}{E-QHQ}(QHP_5)P_5|\Psi\rangle + \frac{1}{E-QHQ}(QHP_6)P_6|\Psi\rangle\end{aligned} \tag{2.21}$$

by using Eq. (2.13), then we have

$$\begin{aligned}Q|\Psi\rangle = G(QHP_1)P_1|\Psi\rangle + G(QHP_2)P_2|\Psi\rangle \\ + G(QHP_3)P_3|\Psi\rangle + G(QHP_4)P_4|\Psi\rangle \\ + G(QHP_5)P_5|\Psi\rangle + G(QHP_6)P_6|\Psi\rangle\end{aligned} \tag{2.22}$$

Next, inserting Eq. (2.22) into Eq. (2.14–2.19), then the full dynamical equation of all possible coupling channels of kaonic hydrogen atoms become

$$\begin{aligned}(E-P_1HP_1)P_1|\Psi\rangle = (P_1HP_2)P_2|\Psi\rangle + (P_1HP_3)P_3|\Psi\rangle + (P_1HP_4)P_4|\Psi\rangle \\ + (P_1HP_5)P_5|\Psi\rangle + (P_1HP_6)P_6|\Psi\rangle \\ + (P_1HQ)G(QHP_1)P_1|\Psi\rangle + (P_1HQ)G(QHP_2)P_2|\Psi\rangle \\ + (P_1HQ)G(QHP_3)P_3|\Psi\rangle + (P_1HQ)G(QHP_4)P_4|\Psi\rangle \\ + (P_1HQ)G(QHP_5)P_5|\Psi\rangle + (P_1HQ)G(QHP_6)P_6|\Psi\rangle\end{aligned} \tag{2.23}$$

for the $K^- p$ channel,

$$\begin{aligned}(E - P_2HP_2)P_2|\Psi\rangle = (P_2HP_1)P_1|\Psi\rangle + (P_2HP_3)P_3|\Psi\rangle + (P_2HP_4)P_4|\Psi\rangle \\ + (P_2HP_5)P_5|\Psi\rangle + (P_2HP_6)P_6|\Psi\rangle \\ + (P_2HQ)G(QHP_1)P_1|\Psi\rangle + (P_2HQ)G(QHP_2)P_2|\Psi\rangle \\ + (P_2HQ)G(QHP_3)P_3|\Psi\rangle + (P_2HQ)G(QHP_4)P_4|\Psi\rangle \\ + (P_2HQ)G(QHP_5)P_5|\Psi\rangle + (P_2HQ)G(QHP_6)P_6|\Psi\rangle\end{aligned} \tag{2.24}$$

for the $\bar{K}^0 n$ channel,

$$\begin{aligned}(E - P_3HP_3)P_3|\Psi\rangle = (P_3HP_1)P_1|\Psi\rangle + (P_3HP_2)P_2|\Psi\rangle + (P_3HP_4)P_4|\Psi\rangle \\ + (P_3HP_5)P_5|\Psi\rangle + (P_3HP_6)P_6|\Psi\rangle \\ + (P_3HQ)G(QHP_1)P_1|\Psi\rangle + (P_3HQ)G(QHP_2)P_2|\Psi\rangle \\ + (P_3HQ)G(QHP_3)P_3|\Psi\rangle + (P_3HQ)G(QHP_4)P_4|\Psi\rangle \\ + (P_3HQ)G(QHP_5)P_5|\Psi\rangle + (P_3HQ)G(QHP_6)P_6|\Psi\rangle\end{aligned} \tag{2.25}$$

for the $\pi^0 \Lambda$ channel,

$$\begin{aligned}(E - P_4HP_4)P_4|\Psi\rangle = (P_4HP_1)P_1|\Psi\rangle + (P_4HP_2)P_2|\Psi\rangle + (P_4HP_3)P_3|\Psi\rangle \\ + (P_4HP_5)P_5|\Psi\rangle + (P_4HP_6)P_6|\Psi\rangle \\ + (P_4HQ)G(QHP_1)P_1|\Psi\rangle + (P_4HQ)G(QHP_2)P_2|\Psi\rangle \\ + (P_4HQ)G(QHP_3)P_3|\Psi\rangle + (P_4HQ)G(QHP_4)P_4|\Psi\rangle \\ + (P_4HQ)G(QHP_5)P_5|\Psi\rangle + (P_4HQ)G(QHP_6)P_6|\Psi\rangle\end{aligned} \tag{2.26}$$

for the $\pi^0\Sigma$ channel,

$$\begin{aligned}(E - P_5HP_5)P_5|\Psi\rangle = (P_5HP_1)P_1|\Psi\rangle + (P_5HP_2)P_2|\Psi\rangle + (P_5HP_3)P_3|\Psi\rangle \\ + (P_5HP_4)P_4|\Psi\rangle + (P_5HP_6)P_6|\Psi\rangle \\ + (P_5HQ)G(QHP_1)P_1|\Psi\rangle + (P_5HQ)G(QHP_2)P_2|\Psi\rangle \\ + (P_5HQ)G(QHP_3)P_3|\Psi\rangle + (P_5HQ)G(QHP_4)P_4|\Psi\rangle \\ + (P_5HQ)G(QHP_5)P_5|\Psi\rangle + (P_5HQ)G(QHP_6)P_6|\Psi\rangle\end{aligned} \tag{2.27}$$

for the $\pi^-\Sigma^+$ channel and

$$\begin{aligned}(E - P_6HP_6)P_6|\Psi\rangle = (P_6HP_1)P_1|\Psi\rangle + (P_6HP_2)P_2|\Psi\rangle + (P_6HP_3)P_3|\Psi\rangle \\ + (P_6HP_4)P_4|\Psi\rangle + (P_6HP_5)P_5|\Psi\rangle \\ + (P_6HQ)G(QHP_1)P_1|\Psi\rangle + (P_6HQ)G(QHP_2)P_2|\Psi\rangle \\ + (P_6HQ)G(QHP_3)P_3|\Psi\rangle + (P_6HQ)G(QHP_4)P_4|\Psi\rangle \\ + (P_6HQ)G(QHP_5)P_5|\Psi\rangle + (P_6HQ)G(QHP_6)P_6|\Psi\rangle\end{aligned} \tag{2.28}$$

for the $\pi^+\Sigma^-$ channel. $(P_iHQ)G(QHP_j)$ in the equations above are optical potentials, stemming from annihilations to other channels represented by Q. We express the optical potentials for various channels explicitly as follows:

$$W_{K^-p\to K^-p} = (P_1HQ)G(QHP_1) \tag{2.29}$$

$$W_{\bar{K}^0n\to \bar{K}^0n} = (P_2HQ)G(QHP_2) \tag{2.30}$$

$$W_{\pi^0\Lambda\to\pi^0\Lambda} = (P_3HQ)G(QHP_3) \tag{2.31}$$

$$W_{\pi^0\Sigma\to\pi^-\Sigma} = (P_4HQ)G(QHP_4) \tag{2.32}$$

$$W_{\pi^-\Sigma^+\to\pi^-\Sigma^+} = (P_5HQ)G(QHP_5) \tag{2.33}$$

$$W_{\pi^+\Sigma^-\to\pi^+\Sigma^-} = (P_6HQ)G(QHP_6) \tag{2.34}$$

and

$$W_{K^-p\to\bar{K}^0n} = (P_1HQ)G(QHP_2) = (P_2HQ)G(QHP_1) \tag{2.35}$$

$$W_{K^-p\to\pi^0\Lambda} = (P_1HQ)G(QHP_3) = (P_3HQ)G(QHP_1) \tag{2.36}$$

$$W_{K^-p\to\pi^0\Sigma} = (P_1HQ)G(QHP_4) = (P_4HQ)G(QHP_1) \tag{2.37}$$

$$W_{K^-p\to\pi^-\Sigma^+} = (P_1HQ)G(QHP_5) = (P_5HQ)G(QHP_1) \tag{2.38}$$

$$W_{K^-p\to\pi^+\Sigma^-} = (P_1HQ)G(QHP_6) = (P_6HQ)G(QHP_1) \tag{2.39}$$

also

$$W_{\bar{K}^0n\to\pi^0\Lambda} = (P_2HQ)G(QHP_3) = (P_3HQ)G(QHP_2) \tag{2.40}$$

$$W_{\bar{K}^0n\to\pi^0\Sigma} = (P_2HQ)G(QHP_4) = (P_4HQ)G(QHP_2) \tag{2.41}$$

$$W_{\bar{K}^0n\to\pi^-\Sigma^+} = (P_2HQ)G(QHP_5) = (P_5HQ)G(QHP_2) \tag{2.42}$$

$$W_{\bar{K}^0n\to\pi^+\Sigma^-} = (P_2HQ)G(QHP_6) = (P_6HQ)G(QHP_2) \tag{2.43}$$

$$W_{\pi^0\Lambda\to\pi^0\Sigma} = (P_3HQ)G(QHP_4) = (P_4HQ)G(QHP_3) \tag{2.44}$$

$$W_{\pi^0\Lambda\to\pi^-\Sigma^+} = (P_3HQ)G(QHP_5) = (P_5HQ)G(QHP_3) \tag{2.45}$$

$$W_{\pi^0\Lambda\to\pi^+\Sigma^-} = (P_3HQ)G(QHP_6) = (P_6HQ)G(QHP_3) \tag{2.46}$$

$$W_{\pi^0\Sigma\to\pi^-\Sigma^+} = (P_4HQ)G(QHP_5) = (P_5HQ)G(QHP_4) \tag{2.47}$$

$$W_{\pi^0\Sigma\to\pi^+\Sigma^-} = (P_4HQ)G(QHP_6) = (P_6HQ)G(QHP_4) \tag{2.48}$$

$$W_{\pi^-\Sigma^+\to\pi^+\Sigma^-} = (P_5HQ)G(QHP_6) = (P_6HQ)G(QHP_5). \tag{2.49}$$

In case the optical potentials are negligible, the dynamical equation of kaonic hydrogen atoms Eq. (2.23–2.28) reduce to

$$\begin{aligned}(E - P_1HP_1)P_1|\Psi\rangle = (P_1HP_2)P_2|\Psi\rangle + (P_1HP_3)P_3|\Psi\rangle + (P_1HP_4)P_4|\Psi\rangle \\ + (P_1HP_5)P_5|\Psi\rangle + (P_1HP_6)P_6|\Psi\rangle\end{aligned} \tag{2.50}$$

for the $K^- p$ channel,

$$\begin{aligned}(E - P_2HP_2)P_2|\Psi\rangle = (P_2HP_1)P_1|\Psi\rangle + (P_2HP_3)P_3|\Psi\rangle + (P_2HP_4)P_4|\Psi\rangle \\ + (P_2HP_5)P_5|\Psi\rangle + (P_2HP_6)P_6|\Psi\rangle\end{aligned} \tag{2.51}$$

for the $\bar{K}^0 n$ channel,

$$\begin{aligned}(E - P_3HP_3)P_3|\Psi\rangle = (P_3HP_1)P_1|\Psi\rangle + (P_3HP_2)P_2|\Psi\rangle + (P_3HP_4)P_4|\Psi\rangle \\ + (P_3HP_5)P_5|\Psi\rangle + (P_3HP_6)P_6|\Psi\rangle\end{aligned} \tag{2.52}$$

for the $\pi^0\Lambda$ channel,

$$\begin{aligned}(E - P_4HP_4)P_4|\Psi\rangle = (P_4HP_1)P_1|\Psi\rangle + (P_4HP_2)P_2|\Psi\rangle + (P_4HP_3)P_3|\Psi\rangle \\ + (P_4HP_5)P_5|\Psi\rangle + (P_4HP_6)P_6|\Psi\rangle\end{aligned} \tag{2.53}$$

for the $\pi^0\Sigma$ channel,

$$\begin{aligned}(E - P_5HP_5)P_5|\Psi\rangle = (P_5HP_1)P_1|\Psi\rangle + (P_5HP_2)P_2|\Psi\rangle + (P_5HP_3)P_3|\Psi\rangle \\ + (P_5HP_4)P_4|\Psi\rangle + (P_5HP_6)P_6|\Psi\rangle\end{aligned} \tag{2.54}$$

for the $\pi^-\Sigma^+$ channel and

$$\begin{aligned}(E - P_6HP_6)P_6|\Psi\rangle = (P_6HP_1)P_1|\Psi\rangle + (P_6HP_2)P_2|\Psi\rangle + (P_6HP_3)P_3|\Psi\rangle \\ + (P_6HP_4)P_4|\Psi\rangle + (P_6HP_5)P_5|\Psi\rangle\end{aligned} \tag{2.55}$$

for the $\pi^+\Sigma^-$ channel.

Finally, the dynamical equation of kaonic hydrogen atoms Eq. (2.50–2.55), can be rewritten into a matrix form as :

$$\begin{pmatrix} P_1HP_1 & P_1HP_2 & P_1HP_3 & P_1HP_4 & P_1HP_5 & P_1HP_6 \\ P_2HP_1 & P_2HP_2 & P_2HP_3 & P_2HP_4 & P_2HP_5 & P_2HP_6 \\ P_3HP_1 & P_3HP_2 & P_3HP_3 & P_3HP_4 & P_3HP_5 & P_3HP_6 \\ P_4HP_1 & P_4HP_2 & P_4HP_3 & P_4HP_4 & P_4HP_5 & P_4HP_6 \\ P_5HP_1 & P_5HP_2 & P_5HP_3 & P_5HP_4 & P_5HP_5 & P_5HP_6 \\ P_6HP_1 & P_6HP_2 & P_6HP_3 & P_6HP_4 & P_6HP_5 & P_6HP_6 \end{pmatrix} \begin{pmatrix} P_1|\Psi\rangle \\ P_2|\Psi\rangle \\ P_3|\Psi\rangle \\ P_4|\Psi\rangle \\ P_5|\Psi\rangle \\ P_6|\Psi\rangle \end{pmatrix} = E \begin{pmatrix} P_1|\Psi\rangle \\ P_2|\Psi\rangle \\ P_3|\Psi\rangle \\ P_4|\Psi\rangle \\ P_5|\Psi\rangle \\ P_6|\Psi\rangle \end{pmatrix} \tag{2.56}$$

Next, by using relations $P_1|\Psi\rangle = \Psi_{K^-p}$, $P_2|\Psi\rangle = \Psi_{\bar{K}^0n}$, $P_3|\Psi\rangle = \Psi_{\pi^0\Lambda}$, $P_4|\Psi\rangle = \Psi_{\pi^0\Sigma}$, $P_5|\Psi\rangle = \Psi_{\pi^-\Sigma^+}$ and $P_6|\Psi\rangle = \Psi_{\pi^+\Sigma^-}$. Then, Eq. (2.56) becomes

$$\begin{pmatrix} P_1HP_1 & P_1HP_2 & P_1HP_3 & P_1HP_4 & P_1HP_5 & P_1HP_6 \\ P_2HP_1 & P_2HP_2 & P_2HP_3 & P_2HP_4 & P_2HP_5 & P_2HP_6 \\ P_3HP_1 & P_3HP_2 & P_3HP_3 & P_3HP_4 & P_3HP_5 & P_3HP_6 \\ P_4HP_1 & P_4HP_2 & P_4HP_3 & P_4HP_4 & P_4HP_5 & P_4HP_6 \\ P_5HP_1 & P_5HP_2 & P_5HP_3 & P_5HP_4 & P_5HP_5 & P_5HP_6 \\ P_6HP_1 & P_6HP_2 & P_6HP_3 & P_6HP_4 & P_6HP_5 & P_6HP_6 \end{pmatrix} \begin{pmatrix} \Psi_{K^-p} \\ \Psi_{\bar{K}^0n} \\ \Psi_{\pi^0\Lambda} \\ \Psi_{\pi^0\Sigma} \\ \Psi_{\pi^-\Sigma^+} \\ \Psi_{\pi^+\Sigma^-} \end{pmatrix} = E \begin{pmatrix} \Psi_{K^-p} \\ \Psi_{\bar{K}^0n} \\ \Psi_{\pi^0\Lambda} \\ \Psi_{\pi^0\Sigma} \\ \Psi_{\pi^-\Sigma^+} \\ \Psi_{\pi^+\Sigma^-} \end{pmatrix} \tag{2.57}$$

where Ψ_{K^-p}, $\Psi_{\bar{K}^0n}$, $\Psi_{\pi^0\Lambda}$, $\Psi_{\pi^0\Sigma}$, $\Psi_{\pi^-\Sigma^+}$ and $\Psi_{\pi^+\Sigma^-}$ are wave function of the K^-p, $\bar{K}^0n$, $\pi^0\Lambda$, $\pi^0\Sigma$, $\pi^-\Sigma^+$ and $\pi^+\Sigma^-$ channels, respectively.

The interaction terms in Eq. (2.57) are given as:

$$P_1HP_1 = H_0^p + H_0^{K^-} + V_c + V_{K^-p\to K^-p} \tag{2.58}$$

$$P_2HP_2 = H_0^n + H_0^{\bar{K}^0} + V_{\bar{K}^0n\to\bar{K}^0n} \tag{2.59}$$

$$P_3HP_3 = H_0^{\pi^0} + H_0^{\Lambda} + V_{\pi^0\Lambda\to\pi^0\Lambda} \tag{2.60}$$

$$P_4HP_4 = H_0^{\pi^0} + H_0^{\Sigma} + V_{\pi^0\Sigma\to\pi^0\Sigma} \tag{2.61}$$

$$P_5HP_5 = H_0^{\pi^-} + H_0^{\Sigma^+} + V_c + V_{\pi^-\Sigma^+\to\pi^-\Sigma^+} \tag{2.62}$$

$$P_6HP_6 = H_0^{\pi^+} + H_0^{\Sigma^-} + V_c + V_{\pi^+\Sigma^-\to\pi^+\Sigma^-} \tag{2.63}$$

where V_c is the Coulomb interaction.

And the charge-exchange (potential) interaction terms in Eq. (2.57) are given as :

$$P_1HP_2 = P_2HP_1 = V_{K^-p\to\bar{K}^0n} \tag{2.64}$$

$$P_1HP_3 = P_3HP_1 = V_{K^-p\to\pi^0\Lambda} \tag{2.65}$$

$$P_1HP_4 = P_4HP_1 = V_{K^-p\to\pi^0\Sigma} \tag{2.66}$$

$$P_1HP_5 = P_5HP_1 = V_{K^-p\to\pi^-\Sigma^+} \tag{2.67}$$

$$P_1HP_6 = P_6HP_1 = V_{K^-p\to\pi^+\Sigma^-}. \tag{2.68}$$

Also,

$$P_2HP_3 = P_3HP_2 = V_{\bar{K}^0n\to\pi^0\Lambda} \tag{2.69}$$

$$P_2HP_4 = P_4HP_2 = V_{\bar{K}^0n\to\pi^0\Sigma} \tag{2.70}$$

$$P_2HP_5 = P_5HP_2 = V_{\bar{K}^0n\to\pi^-\Sigma^+} \tag{2.71}$$

$$P_2HP_6 = P_6HP_2 = V_{\bar{K}^0n\to\pi^+\Sigma^-} \tag{2.72}$$

$$P_3HP_4 = P_4HP_3 = V_{\pi^0\Lambda\to\pi^0\Sigma} \tag{2.73}$$

$$P_3HP_5 = P_5HP_3 = V_{\pi^0\Lambda\to\pi^-\Sigma^+} \tag{2.74}$$

$$P_3HP_6 = P_6HP_3 = V_{\pi^0\Lambda\to\pi^+\Sigma^-} \tag{2.75}$$

$$P_4HP_5 = P_5HP_4 = V_{\pi^0\Sigma\to\pi^-\Sigma^+} \tag{2.76}$$

$$P_4HP_6 = P_6HP_4 = V_{\pi^0\Sigma\to\pi^+\Sigma^-} \tag{2.77}$$

$$P_5HP_6 = P_6HP_5 = V_{\pi^-\Sigma^+\to\pi^+\Sigma^-} \tag{2.78}$$

and $H_0^p = \sqrt{m_p^2 + \vec{p}^2}$, $H_0^n = \sqrt{m_n^2 + \vec{p}^2}$, $H_0^\Lambda = \sqrt{m_\Lambda^2 + \vec{p}^2}$, $H_0^{K^-} = \sqrt{m_{K^-}^2 + \vec{p}^2}$, $H_0^{\bar{K}^0} = \sqrt{m_{\bar{K}^0}^2 + \vec{p}^2}$, $H_0^{\pi^0} = \sqrt{m_{\pi^0}^2 + \vec{p}^2}$, $H_0^{\pi^-} = \sqrt{m_{\pi^-}^2 + \vec{p}^2}$, $H_0^{\pi^+} = \sqrt{m_{\pi^+}^2 + \vec{p}^2}$, $H_0^\Sigma = \sqrt{m_\Sigma^2 + \vec{p}^2}$, $H_0^{\Sigma^-} = \sqrt{m_{\Sigma^-}^2 + \vec{p}^2}$, $H_0^{\Sigma^+} = \sqrt{m_{\Sigma^+}^2 + \vec{p}^2}$ are the free energies of the proton, neutron, lambda, negatively charge kaon, neutral antikaon, neutral pion, negatively charge pion, sigma, positively charge pion, positively charge sigma and negatively charge sigma , respectively. The masses of the proton, neutron, lambda, negatively charge kaon, neutral antikaon, neutral pion, negatively charge pion, positively charge pion, sigma, positively charge sigma and negatively charge sigma are denoted as m_p, m_n, m_Λ, m_{K^-}, $m_{\bar{K}^0}$, m_{π^0}, m_{π^-}, m_{π^+}, m_Σ, m_{Σ^-} and m_{Σ^+} respectively.

2.2 Strong interactions of kaonic hydrogen atoms

The strong interaction terms, Eq. (2.58–2.78), may be interpreted in terms of interactions in isospin basis since the proton, neutron, neutral antikaon, negatively charge kaon, neutral pion, negatively charge pion, positively charge pion, lambda , sigma, negatively charge sigma and positively charge sigma are all isospin eigenstates.

For the proton, isospin $I = \frac{1}{2}$ and the third component $I_3 = +\frac{1}{2}$,

$$p = \left|\frac{1}{2}, \frac{1}{2}\right\rangle . \tag{2.79}$$

For the neutron, isospin $I = \frac{1}{2}$ and the third component $I_3 = -\frac{1}{2}$,

$$n = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle . \tag{2.80}$$

For the three pions, negatively charge pion, neutral pion and positively charge pion, could be assigned to an isospin triplet with isospin $I = 1$ and the third component $I_3 = -1, 0$ and $+1$, respectively,

$$\pi^- = |1, -1\rangle \tag{2.81}$$

$$\pi^0 = |1, 0\rangle \tag{2.82}$$

$$\pi^+ = |1, 1\rangle . \tag{2.83}$$

For the neutral antikaon and negatively charge kaon, isospin $I = \frac{1}{2}$ and the third component $I_3 = +\frac{1}{2}$ and $-\frac{1}{2}$, respectively,

$$\bar{K}^0 = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \tag{2.84}$$

$$K^- = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle . \tag{2.85}$$

For the lambda, isospin $I = 0$ and the third component $I_3 = 0$,

$$\Lambda^0 = |0, 0\rangle . \tag{2.86}$$

And for the three sigma baryons, negatively charge sigma, neutral sigma and positively charge sigma , could be assigned to an isospin triplet with isospin $I = 1$ and the third component $I_3 = -1, 0$ and $+1$, respectively,

$$\Sigma^- = |1, -1\rangle \tag{2.87}$$

$$\Sigma^0 = |1, 0\rangle \tag{2.88}$$

$$\Sigma^+ = |1, 1\rangle \,. \tag{2.89}$$

Hence, from Eq. (2.79–2.89), the particles basis of each particle-particle scattering process (or each decaying channel) can be written into the linear combination (decompositions) of the isospin basis as :

$$\left|K^- p\right\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}} |1, 0\rangle - \frac{1}{\sqrt{2}} |0, 0\rangle \tag{2.90}$$

$$\left|\bar{K}^0 n\right\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{\sqrt{2}} |0, 0\rangle \tag{2.91}$$

$$\left|\pi^0 \Sigma^0\right\rangle = |1, 0\rangle \, |1, 0\rangle = \sqrt{\frac{2}{3}} |2, 0\rangle - \frac{1}{\sqrt{3}} |0, 0\rangle \tag{2.92}$$

$$\left|\pi^+ \Sigma^-\right\rangle = |1, 1\rangle \, |1, -1\rangle = \frac{1}{\sqrt{3}} |0, 0\rangle + \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{\sqrt{6}} |2, 0\rangle \tag{2.93}$$

$$\left|\pi^- \Sigma^+\right\rangle = |1, -1\rangle \, |1, 1\rangle = \frac{1}{\sqrt{3}} |0, 0\rangle - \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{\sqrt{6}} |2, 0\rangle \tag{2.94}$$

$$\left|\pi^0 \Lambda^0\right\rangle = |1, 0\rangle \, |0, 0\rangle = |1, 0\rangle \tag{2.95}$$

where the coefficients of each isospin basis are called the Clebsch-Gordan coefficients and the method to evaluate the Clebsch-Gordan coefficients will be shown in an appendix A.

Finally, the strong interaction terms (the strong potentials) in Eq. (2.58–

2.63) can be written in isospin basis as

$$\begin{aligned} V_{K^-p\to K^-p} &= \left\langle K^-p\right| V \left|K^-p\right\rangle \\ &= \frac{1}{\sqrt{2}}\left(\langle 1,0| - \langle 0,0|\right) V \left(|1,0\rangle - |0,0\rangle\right)\frac{1}{\sqrt{2}} \\ &= \frac{1}{2}\langle 1,0| V |1,0\rangle + \frac{1}{2}\langle 0,0| V |0,0)\rangle \\ &= \frac{1}{2}V^{I=1}_{K^-p\to K^-p} + \frac{1}{2}V^{I=0}_{K^-p\to K^-p} \end{aligned} \tag{2.96}$$

$$\begin{aligned} V_{\bar{K}^0n\to\bar{K}^0n} &= \left\langle \bar{K}^0n\right| V \left|\bar{K}^0n\right\rangle \\ &= \frac{1}{\sqrt{2}}\left(\langle 1,0| + \langle 0,0|\right) V \left(|1,0\rangle + |0,0\rangle\right)\frac{1}{\sqrt{2}} \\ &= \frac{1}{2}\langle 1,0| V |1,0\rangle + \frac{1}{2}\langle 0,0| V |0,0)\rangle \\ &= \frac{1}{2}V^{I=1}_{\bar{K}^0n\to\bar{K}^0n} + \frac{1}{2}V^{I=0}_{\bar{K}^0n\to\bar{K}^0n} \end{aligned} \tag{2.97}$$

$$\begin{aligned} V_{\pi^0\Lambda\to\pi^0\Lambda} &= \left\langle \pi^0\Lambda\right| V \left|\pi^0\Lambda\right\rangle \\ &= \langle 1,0| V |1,0\rangle \\ &= V^{I=1}_{\pi^0\Lambda\to\pi^0\Lambda} \end{aligned} \tag{2.98}$$

$$\begin{aligned} V_{\pi^0\Sigma\to\pi^0\Sigma} &= \left\langle \pi^0\Sigma\right| V \left|\pi^0\Sigma\right\rangle \\ &= \left(\sqrt{\frac{2}{3}}\langle 2,0| - \frac{1}{\sqrt{3}}\langle 0,0|\right) V \left(\sqrt{\frac{2}{3}}|2,0\rangle - \frac{1}{\sqrt{3}}|0,0\rangle\right) \\ &= \frac{2}{3}\langle 2,0| V |2,0\rangle + \frac{1}{3}\langle 0,0| V |0,0\rangle \\ &= \frac{2}{3}V^{I=2}_{\pi^0\Sigma\to\pi^0\Sigma} + \frac{1}{3}V^{I=0}_{\pi^0\Sigma\to\pi^0\Sigma} \end{aligned} \tag{2.99}$$

$$\begin{aligned} V_{\pi^-\Sigma^+\to\pi^-\Sigma^+} &= \left\langle \pi^-\Sigma^+\right| V \left|\pi^-\Sigma^+\right\rangle \\ &= \left(\frac{1}{\sqrt{3}}\langle 0,0| - \frac{1}{\sqrt{2}}\langle 1,0| + \frac{1}{\sqrt{6}}\langle 2,0|\right) V \\ &\quad \left(\frac{1}{\sqrt{3}}|0,0\rangle - \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{6}}|2,0\rangle\right) \\ &= \frac{1}{6}\langle 2,0| V |2,0\rangle + \frac{1}{2}\langle 1,0| V |1,0\rangle + \frac{1}{3}\langle 0,0| V |0,0\rangle \\ &= \frac{1}{6}V^{I=2}_{\pi^-\Sigma^+\to\pi^-\Sigma^+} + \frac{1}{2}V^{I=1}_{\pi^-\Sigma^+\to\pi^-\Sigma^+} + \frac{1}{3}V^{I=0}_{\pi^-\Sigma^+\to\pi^-\Sigma^+} \end{aligned} \tag{2.100}$$

$$
\begin{aligned}
V_{\pi^+\Sigma^-\to\pi^+\Sigma^-} &= \left\langle \pi^+\Sigma^- \right| V \left|\pi^+\Sigma^-\right\rangle \\
&= \left(\frac{1}{\sqrt{3}}\left\langle 0,0\right| + \frac{1}{\sqrt{2}}\left\langle 1,0\right| + \frac{1}{\sqrt{6}}\left\langle 2,0\right|\right) V \\
&\quad \left(\frac{1}{\sqrt{3}}\left|0,0\right\rangle + \frac{1}{\sqrt{2}}\left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left|2,0\right\rangle\right) \\
&= \frac{1}{6}\left\langle 2,0\right| V \left|2,0\right\rangle + \frac{1}{2}\left\langle 1,0\right| V \left|1,0\right\rangle + \frac{1}{3}\left\langle 0,0\right| V \left|0,0\right\rangle \\
&= \frac{1}{6}V^{I=2}_{\pi^+\Sigma^-\to\pi^+\Sigma^-} + \frac{1}{2}V^{I=1}_{\pi^+\Sigma^-\to\pi^+\Sigma^-} + \frac{1}{3}V^{I=0}_{\pi^+\Sigma^-\to\pi^+\Sigma^-}.
\end{aligned}
\tag{2.101}
$$

Also, the charge-exchange interaction terms Eq. (2.64–2.78) can be written in isospin basis as

$$
\begin{aligned}
V_{K^-p\to\bar{K}^0n} &= \left\langle K^-p\right| V \left|\bar{K}^0n\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\left\langle 1,0\right| - \left\langle 0,0\right|\right) V \left(\left|1,0\right\rangle + \left|0,0\right\rangle\right)\frac{1}{\sqrt{2}} \\
&= \frac{1}{2}\left\langle 1,0\right| V \left|1,0\right\rangle - \frac{1}{2}\left\langle 0,0\right| V \left|0,0\right)\right\rangle \\
&= \frac{1}{2}V^{I=1}_{K^-p\to\bar{K}^0n} - \frac{1}{2}V^{I=0}_{K^-p\to\bar{K}^0n}
\end{aligned}
\tag{2.102}
$$

$$
\begin{aligned}
V_{K^-p\to\pi^0\Lambda} &= \left\langle K^-p\right| V \left|\pi^0\Lambda\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\left\langle 1,0\right| - \left\langle 0,0\right|\right) V \left|1,0\right\rangle \\
&= \frac{1}{\sqrt{2}}\left\langle 1,0\right| V \left|1,0\right\rangle \\
&= \frac{1}{\sqrt{2}}V^{I=1}_{K^-p\to\pi^0\Lambda}
\end{aligned}
\tag{2.103}
$$

$$
\begin{aligned}
V_{K^-p\to\pi^0\Sigma} &= \left\langle K^-p\right| V \left|\pi^0\Sigma\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\left\langle 1,0\right| - \left\langle 0,0\right|\right) V \left(\sqrt{\frac{2}{3}}\left|2,0\right\rangle - \frac{1}{\sqrt{3}}\left|0,0\right\rangle\right) \\
&= \frac{1}{\sqrt{6}}\left\langle 0,0\right| V \left|0,0\right\rangle \\
&= \frac{1}{\sqrt{6}}V^{I=0}_{K^-p\to\pi^0\Sigma}
\end{aligned}
\tag{2.104}
$$

$$
\begin{aligned}
V_{K^-p\to\pi^-\Sigma^+} &= \left\langle K^- p\right| V \left|\pi^-\Sigma^+\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\langle 1,0| - \langle 0,0|\right) V \left(\frac{1}{\sqrt{3}}\left|0,0\right\rangle - \frac{1}{\sqrt{2}}\left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left|2,0\right\rangle\right) \\
&= -\frac{1}{2}\left\langle 1,0\right| V \left|1,0\right\rangle - \frac{1}{\sqrt{6}}\left\langle 0,0\right| V \left|0,0\right)\rangle \\
&= -\frac{1}{2} V^{I=1}_{K^-p\to\pi^-\Sigma^+} - \frac{1}{\sqrt{6}} V^{I=0}_{K^-p\to\pi^-\Sigma^+}
\end{aligned}
\tag{2.105}
$$

$$
\begin{aligned}
V_{K^-p\to\pi^+\Sigma^-} &= \left\langle K^- p\right| V \left|\pi^+\Sigma^-\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\langle 1,0| - \langle 0,0|\right) V \left(\frac{1}{\sqrt{3}}\left|0,0\right\rangle + \frac{1}{\sqrt{2}}\left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left|2,0\right\rangle\right) \\
&= \frac{1}{2}\left\langle 1,0\right| V \left|1,0\right\rangle - \frac{1}{\sqrt{6}}\left\langle 0,0\right| V \left|0,0\right)\rangle \\
&= \frac{1}{2} V^{I=1}_{K^-p\to\pi^+\Sigma^-} - \frac{1}{\sqrt{6}} V^{I=0}_{K^-p\to\pi^+\Sigma^-}
\end{aligned}
\tag{2.106}
$$

$$
\begin{aligned}
V_{\bar{K}^0n\to\pi^0\Lambda} &= \left\langle \bar{K}^0 n\right| V \left|\pi^0\Lambda\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\langle 1,0| + \langle 0,0|\right) V \left|1,0\right\rangle \\
&= \frac{1}{\sqrt{2}}\left\langle 1,0\right| V \left|1,0\right\rangle \\
&= \frac{1}{\sqrt{2}} V^{I=1}_{\bar{K}^0n\to\pi^0\Lambda}
\end{aligned}
\tag{2.107}
$$

$$
\begin{aligned}
V_{\bar{K}^0n\to\pi^0\Sigma} &= \left\langle \bar{K}^0 n\right| V \left|\pi^0\Sigma\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\langle 1,0| + \langle 0,0|\right) V \left(\sqrt{\frac{2}{3}}\left|2,0\right\rangle - \frac{1}{\sqrt{3}}\left|0,0\right\rangle\right) \\
&= \frac{1}{\sqrt{6}}\left\langle 0,0\right| V \left|0,0\right\rangle \\
&= -\frac{1}{\sqrt{6}} V^{I=0}_{\bar{K}^0n\to\pi^0\Sigma}
\end{aligned}
\tag{2.108}
$$

$$
\begin{aligned}
V_{\bar{K}^0 n\to\pi^-\Sigma^+} &= \left\langle \bar{K}^0 n\right| V \left|\pi^-\Sigma^+\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\langle 1,0| + \langle 0,0|\right) V \left(\frac{1}{\sqrt{3}}\left|0,0\right\rangle - \frac{1}{\sqrt{2}}\left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left|2,0\right\rangle\right) \\
&= -\frac{1}{2}\left\langle 1,0\right| V \left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left\langle 0,0\right| V \left|0,0\right)\rangle \\
&= -\frac{1}{2} V^{I=1}_{\bar{K}^0 n\to\pi^-\Sigma^+} + \frac{1}{\sqrt{6}} V^{I=0}_{\bar{K}^0 n\to\pi^-\Sigma^+}
\end{aligned}
\tag{2.109}
$$

$$
\begin{aligned}
V_{\bar{K}^0 n\to\pi^+\Sigma^-} &= \left\langle \bar{K}^0 n\right| V \left|\pi^+\Sigma^-\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\langle 1,0| + \langle 0,0|\right) V \left(\frac{1}{\sqrt{3}}\left|0,0\right\rangle + \frac{1}{\sqrt{2}}\left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left|2,0\right\rangle\right) \\
&= \frac{1}{2}\left\langle 1,0\right| V \left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left\langle 0,0\right| V \left|0,0\right)\rangle \\
&= \frac{1}{2} V^{I=1}_{\bar{K}^0 n\to\pi^+\Sigma^-} + \frac{1}{\sqrt{6}} V^{I=0}_{\bar{K}^0 n\to\pi^+\Sigma^-}
\end{aligned}
\tag{2.110}
$$

$$
\begin{aligned}
V_{\pi^0\Lambda\to\pi^0\Sigma} &= \left\langle \pi^0\Lambda\right| V \left|\pi^0\Sigma\right\rangle \\
&= \langle 1,0| V \left(\sqrt{\frac{2}{3}}\left|2,0\right\rangle - \frac{1}{\sqrt{3}}\left|0,0\right\rangle\right) \\
&= 0
\end{aligned}
\tag{2.111}
$$

$$
\begin{aligned}
V_{\pi^0\Lambda\to\pi^-\Sigma^+} &= \left\langle \pi^0\Lambda\right| V \left|\pi^-\Sigma^+\right\rangle \\
&= \langle 1,0| V \left(\frac{1}{\sqrt{3}}\left|0,0\right\rangle - \frac{1}{\sqrt{2}}\left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left|2,0\right\rangle\right) \\
&= -\frac{1}{\sqrt{2}}\left\langle 1,0\right| V \left|1,0\right\rangle \\
&= -\frac{1}{\sqrt{2}} V^{I=1}_{\pi^0\Lambda\to\pi^-\Sigma^+}
\end{aligned}
\tag{2.112}
$$

$$
\begin{aligned}
V_{\pi^0\Lambda\to\pi^+\Sigma^-} &= \left\langle \pi^0\Lambda\right| V \left|\pi^+\Sigma^-\right\rangle \\
&= \langle 1,0| V \left(\frac{1}{\sqrt{3}}\left|0,0\right\rangle + \frac{1}{\sqrt{2}}\left|1,0\right\rangle + \frac{1}{\sqrt{6}}\left|2,0\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left\langle 1,0\right| V \left|1,0\right\rangle \\
&= \frac{1}{\sqrt{2}} V^{I=1}_{\pi^0\Lambda\to\pi^+\Sigma^-}
\end{aligned}
\tag{2.113}
$$

$$
\begin{aligned}
V_{\pi^0\Sigma\to\pi^-\Sigma^+} &= \left\langle \pi^0\Sigma \right| V \left| \pi^-\Sigma^+ \right\rangle \\
&= \left(\sqrt{\frac{2}{3}} \left\langle 2,0 \right| - \frac{1}{\sqrt{3}} \left\langle 0,0 \right| \right) V \left(\frac{1}{\sqrt{3}} \left| 0,0 \right\rangle - \frac{1}{\sqrt{2}} \left| 1,0 \right\rangle + \frac{1}{\sqrt{6}} \left| 2,0 \right\rangle \right) \\
&= \frac{1}{3} \left\langle 2,0 \right| V \left| 2,0 \right\rangle - \frac{1}{3} \left\langle 0,0 \right| V \left| 0,0 \right\rangle \\
&= \frac{1}{3} V^{I=2}_{\pi^0\Sigma\to\pi^-\Sigma^+} - \frac{1}{3} V^{I=0}_{\pi^0\Sigma\to\pi^-\Sigma^+}
\end{aligned}
\tag{2.114}
$$

$$
\begin{aligned}
V_{\pi^0\Sigma\to\pi^+\Sigma^-} &= \left\langle \pi^0\Sigma \right| V \left| \pi^+\Sigma^- \right\rangle \\
&= \left(\sqrt{\frac{2}{3}} \left\langle 2,0 \right| - \frac{1}{\sqrt{3}} \left\langle 0,0 \right| \right) V \left(\frac{1}{\sqrt{3}} \left| 0,0 \right\rangle + \frac{1}{\sqrt{2}} \left| 1,0 \right\rangle + \frac{1}{\sqrt{6}} \left| 2,0 \right\rangle \right) \\
&= \frac{1}{3} \left\langle 2,0 \right| V \left| 2,0 \right\rangle - \frac{1}{3} \left\langle 0,0 \right| V \left| 0,0 \right\rangle \\
&= \frac{1}{3} V^{I=2}_{\pi^0\Sigma\to\pi^+\Sigma^-} - \frac{1}{3} V^{I=0}_{\pi^0\Sigma\to\pi^+\Sigma^-}
\end{aligned}
\tag{2.115}
$$

$$
\begin{aligned}
V_{\pi^-\Sigma^+\to\pi^+\Sigma^-} &= \left\langle \pi^-\Sigma^+ \right| V \left| \pi^+\Sigma^- \right\rangle \\
&= \left(\frac{1}{\sqrt{3}} \left\langle 0,0 \right| - \frac{1}{\sqrt{2}} \left\langle 1,0 \right| + \frac{1}{\sqrt{6}} \left\langle 2,0 \right| \right) V \\
&\quad \left(\frac{1}{\sqrt{3}} \left| 0,0 \right\rangle + \frac{1}{\sqrt{2}} \left| 1,0 \right\rangle + \frac{1}{\sqrt{6}} \left| 2,0 \right\rangle \right) \\
&= \frac{1}{6} \left\langle 2,0 \right| V \left| 2,0 \right\rangle - \frac{1}{2} \left\langle 1,0 \right| V \left| 1,0 \right\rangle + \frac{1}{3} \left\langle 0,0 \right| V \left| 0,0 \right\rangle \\
&= \frac{1}{6} V^{I=2}_{\pi^-\Sigma^+\to\pi^+\Sigma^-} - \frac{1}{2} V^{I=1}_{\pi^-\Sigma^+\to\pi^+\Sigma^-} + \frac{1}{3} V^{I=0}_{\pi^-\Sigma^+\to\pi^+\Sigma^-}.
\end{aligned}
\tag{2.116}
$$

Also, the optical potentials in Eq. (2.29–2.49) can be written in isospin basis as

$$
W_{K^-p\to K^-p} = \frac{1}{2} W^{I=1}_{K^-p\to K^-p} + \frac{1}{2} W^{I=0}_{K^-p\to K^-p} \tag{2.117}
$$

$$
W_{\bar{K}^0n\to \bar{K}^0n} = \frac{1}{2} W^{I=1}_{\bar{K}^0n\to \bar{K}^0n} + \frac{1}{2} W^{I=0}_{\bar{K}^0n\to \bar{K}^0n} \tag{2.118}
$$

$$
W_{\pi^0\Lambda\to\pi^0\Lambda} = W^{I=1}_{\pi^0\Lambda\to\pi^0\Lambda} \tag{2.119}
$$

$$
W_{\pi^0\Sigma\to\pi^0\Sigma} = \frac{2}{3} W^{I=2}_{\pi^0\Sigma\to\pi^0\Sigma} + \frac{1}{3} W^{I=0}_{\pi^0\Sigma\to\pi^0\Sigma} \tag{2.120}
$$

$$W_{\pi^-\Sigma^+\to\pi^-\Sigma^+} = \frac{1}{6}W^{I=2}_{\pi^-\Sigma^+\to\pi^-\Sigma^+} + \frac{1}{2}W^{I=1}_{\pi^-\Sigma^+\to\pi^-\Sigma^+} + \frac{1}{3}W^{I=0}_{\pi^-\Sigma^+\to\pi^-\Sigma^+} \tag{2.121}$$

$$W_{\pi^+\Sigma^-\to\pi^+\Sigma^-} = \frac{1}{6}W^{I=2}_{\pi^+\Sigma^-\to\pi^+\Sigma^-} + \frac{1}{2}W^{I=1}_{\pi^+\Sigma^-\to\pi^+\Sigma^-} + \frac{1}{3}W^{I=0}_{\pi^+\Sigma^-\to\pi^+\Sigma^-} \tag{2.122}$$

$$W_{K^-p\to\bar{K}^0n} = \frac{1}{2}W^{I=1}_{K^-p\to\bar{K}^0n} - \frac{1}{2}W^{I=0}_{K^-p\to\bar{K}^0n} \tag{2.123}$$

$$W_{K^-p\to\pi^0\Lambda} = \frac{1}{\sqrt{2}}W^{I=1}_{K^-p\to\pi^0\Lambda} \tag{2.124}$$

$$W_{K^-p\to\pi^0\Sigma} = \frac{1}{\sqrt{6}}W^{I=0}_{K^-p\to\pi^0\Sigma} \tag{2.125}$$

$$W_{K^-p\to\pi^-\Sigma^+} = -\frac{1}{2}W^{I=1}_{K^-p\to\pi^-\Sigma^+} - \frac{1}{\sqrt{6}}W^{I=0}_{K^-p\to\pi^-\Sigma^+} \tag{2.126}$$

$$W_{K^-p\to\pi^+\Sigma^-} = \frac{1}{2}W^{I=1}_{K^-p\to\pi^+\Sigma^-} - \frac{1}{\sqrt{6}}W^{I=0}_{K^-p\to\pi^+\Sigma^-} \tag{2.127}$$

And,

$$W_{\bar{K}^0n\to\pi^0\Lambda} = \frac{1}{\sqrt{2}}W^{I=1}_{\bar{K}^0n\to\pi^0\Lambda} \tag{2.128}$$

$$W_{\bar{K}^0n\to\pi^0\Sigma} = -\frac{1}{\sqrt{6}}W^{I=0}_{\bar{K}^0n\to\pi^0\Sigma} \tag{2.129}$$

$$W_{\bar{K}^0n\to\pi^-\Sigma^+} = -\frac{1}{2}W^{I=1}_{\bar{K}^0n\to\pi^-\Sigma^+} + \frac{1}{\sqrt{6}}W^{I=0}_{\bar{K}^0n\to\pi^-\Sigma^+} \tag{2.130}$$

$$W_{\bar{K}^0n\to\pi^+\Sigma^-} = \frac{1}{2}W^{I=1}_{\bar{K}^0n\to\pi^+\Sigma^-} + \frac{1}{\sqrt{6}}W^{I=0}_{\bar{K}^0n\to\pi^+\Sigma^-} \tag{2.131}$$

$$W_{\pi^0\Lambda\to\pi^0\Sigma} = 0 \tag{2.132}$$

$$W_{\pi^0\Lambda\to\pi^-\Sigma^+} = -\frac{1}{\sqrt{2}}W^{I=1}_{\pi^0\Lambda\to\pi^-\Sigma^+} \tag{2.133}$$

$$W_{\pi^0\Lambda\to\pi^+\Sigma^-} = \frac{1}{\sqrt{2}}W^{I=1}_{\pi^0\Lambda\to\pi^+\Sigma^-} \tag{2.134}$$

$$W_{\pi^0\Sigma\to\pi^-\Sigma^+} = \frac{1}{3}W^{I=2}_{\pi^0\Sigma\to\pi^-\Sigma^+} - \frac{1}{3}W^{I=0}_{\pi^0\Sigma\to\pi^-\Sigma^+} \tag{2.135}$$

$$W_{\pi^0\Sigma\to\pi^+\Sigma^-} = \frac{1}{3}W^{I=2}_{\pi^0\Sigma\to\pi^+\Sigma^-} - \frac{1}{3}W^{I=0}_{\pi^0\Sigma\to\pi^+\Sigma^-} \tag{2.136}$$

$$W_{\pi^-\Sigma^+\to\pi^+\Sigma^-} = \frac{1}{6}W^{I=2}_{\pi^-\Sigma^+\to\pi^+\Sigma^-} - \frac{1}{2}W^{I=1}_{\pi^-\Sigma^+\to\pi^+\Sigma^-} + \frac{1}{3}W^{I=0}_{\pi^-\Sigma^+\to\pi^+\Sigma^-}. \tag{2.137}$$

Now we have the dynamical equations of the kaonic hydrogen system, Eq. (2.57). The dynamical equation will be solved in a numerical method base on the Sturmian function. Detail of our calculations and numerical method will be shown in Chapter III.

CHAPTER III

NUMERICAL METHOD BASE ON STURMIAN FUNCTION

This chapter we describe the numerical procedure to solve our dynamical equation. First, we start from the theoretical background of Sturmian function. Later we will explain how to use numerical method base on Sturmian function to solve our dynamical equation. In this work we study the kaonic hydrogen atom problem employing a properly adapted numerical method based on Sturmian functions (Suebka and Yan, 2004), (Yan et al., 2007). The method accounts for both the strong *short* range nuclear potential (local and non-local) and the *long* range Coulomb force and provides directly the wave function of the the kaonic hydrogen atom system with complex eigenvalues $E = E_R - i\frac{\Gamma}{2}$. The protonium and pionium problems have been successfully investigated (Suebka and Yan, 2004), (Yan et al., 2007) in the numerical approach. The numerical method is much more powerful, accurate and much easier to use than all other methods applied to the exotic atom problem in history.

3.1 Theoretical background of Sturmian function

In principle, one could solve Eq. (2.57) through expanding the wave functions Ψ_{K^-p},$\Psi_{\bar{K}^0n}$,$\Psi_{\pi^0\Lambda}$,$\Psi_{\pi^0\Sigma}$,$\Psi_{\pi^-\Sigma^+}$ and $\Psi_{\pi^+\Sigma^-}$ in any complete set. The complete set of harmonic oscillator wave functions is widely applied to bound state problems since they have analytical forms both in coordinate and momentum spaces. Bound

state problems with only the strong interaction or only the Coulomb force can be well solved in the regime of harmonic oscillator wave functions, by choosing the oscillator length being of order 1 fm or 100 fm, respectively. Detailed investigations, however, have shown that the harmonic oscillator wave function approach fails to describe exotic atoms which are dominated by the long-ranged Coulomb force and influenced by the short-ranged strong interaction. The reason is that two very different oscillator lengths are involved to describe the $\bar{N}N$ deep bound state *and* the atomic state. The Sturmian function method was first used in atomic physics to evaluate the binding energy and wave function of atoms. It was pointed out that the method is more powerful than the approach using harmonic oscillator and hydrogen wave functions. Subsequently, the method was applied to various physical problems such as electromagnetic collisions, binding energies of nuclei and bound and resonance states in special potentials. The Sturmian functions are very similar to the hydrogen wave functions, and are therefore, also named Coulomb - Sturmian functions. In coordinate state space the Sturmians $S_{nl}(r)$, which are used in the present work, satisfy the second order differential equation

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2b(n+l+1)}{r} - b^2\right) S_{nl}(r) = 0. \tag{3.1}$$

By solving Eq. (3.1), one finds

$$S_{nl}(r) = \left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}} (2br)^{l+1} exp(-br) L_n^{2l+1}(2br) \tag{3.2}$$

where $L_n^{2l+1}(x)$ are associated Laguerre polynomials defined as

$$L_n^k(x) = (-1)^k \frac{d^k}{dx^k} \left[L_{n+k}(x)\right] \tag{3.3}$$

that is

$$L_n^{2l+1}(2br) = \sum_{m=0}^{n} (-1)^m \frac{(n+2l+1)!}{(n-m)!(2l+1+m)!m!} (2br)^m. \tag{3.4}$$

The Sturmians are orthogonal and form a completes set with respect to the weight function $1/r$, which follows from the corresponding $1/r$ potential term in Eq. (3.1),

$$\int_0^\infty r^2 dr \frac{S_{nl}(r)}{r} \frac{1}{r} \frac{S_{n'l}(r)}{r} = \delta_{nn'}. \tag{3.5}$$

Thus radial function $R_l(r)$ can be expanded in the complete set of the Sturmian functions $S_{nl}(r)$,

$$R_l(r) = \sum_n a_{nl} \frac{S_{nl}(r)}{r}. \tag{3.6}$$

The Sturmian functions can be defined in momentum space as

$$\begin{aligned} S_{nlm}(\vec{p}) &\equiv S_{nl}(p) Y_{lm}(\theta_p, \phi_p) \\ &= \frac{1}{(2\pi)^{3/2}} \int dr d\Omega S_{nl}(r) Y_{lm}(\theta, \phi) \, e^{-i\vec{p}.\vec{r}}. \end{aligned} \tag{3.7}$$

One can derive the momentum from analytically

$$\begin{aligned} S_{nl}(p) = &\left[\frac{2^{4l+3}(n+l+1)n!(l!)^2}{b(n+2l+1)!} \right]^{1/2} \\ &\frac{(p/b)^l}{[(p/b)^2+1]^{l+1}} C_n^{l+1} \left[\frac{(p/b)^2-1}{(p/b)^2+1} \right] \end{aligned} \tag{3.8}$$

where $C_t^s(x)$ are the Gegenbauer polynomials. It is very convenient to have such a complete set to study interactions in momentum space.

Because almost all bound-state hydrogenic wave functions are close to zero energy, the innermost zeros of the functions are sensitive to the principle quantum number. This accounts for the fact that the bound hydrogen functions do not form a complete set ; the continuum is needed to analyze the region between the original and the limiting first zero. Unlike hydrogen functions, the first node of the Sturmian functions continues to move closer to the origin with increasing the principle number n. This is the key point why a short-ranged nuclear force can easily be taken into account for the $\bar{N}N$ atomic state problem by using complete set of the Sturmian functions.

The parameter b is the length scale entering the Sturmian functions in Eq. (3.1) and Eq. (3.2), in the same way as the corresponding parameter enters the harmonic oscillator functions. For $\bar{N}N$ deep bound states one should use $1/b$ of order 1 fm while the atomic states without strong interactions require $1/b$ of order 10^2 fm. However, for protonium accounting for both the strong interaction and the Coulomb force, one must use a $1/b$ between the two values used for the above cases. Using a complete basis of, for example, 200 Sturmian functions (100 for the $L = J - 1$ wave, and another 100 for the $L = J + 1$) with $1/b = 5 - 500$ fm, one can precisely reproduce the analytical $1s$ and $2p$ wave functions of the $\bar{N}N$ systems subject to only the Coulomb interaction. Using the same basis with $1/b = 0.1 - 30$ fm, the wave functions of $\bar{N}N$ deep bound states can be precisely evaluated. The $\bar{N}N$ deep bound states can be evaluated in the complete set of the harmonic oscillator wave functions, and also in the complete set of Sturmian functions with a more suitable length parameter, for example, $1/b = 1$ fm. From the above investigation, a length parameter $1/b$ around 20 fm is suitable for the protonium problem.

In principle, there is no limit to the accuracy in the evaluation of the $\bar{N}N$ atomic states in the Sturmian functions approach. One is allowed to use larger and larger complete basis of the Sturmian functions until the theoretical results converge. And the $\bar{N}N$ atomic states with higher angular momenta can be easily handled in the approach.

3.2 Sturmian function approach to kaonic atoms

Inserting Eq. (3.6) into Eq. (2.57) does not lead to a diagonal form on the right hand side of Eq. (2.57) unlike the case of the harmonic oscillator wave functions. The matrices on both sides of Eq. (2.57) must be simultaneously

diagonalized. Note that the Sturmian functions have analytical form in momentum space. One is allowed to deal with strong interactions in momentum space with the complete set of the Sturmian as easily as with the set of the harmonic oscillator wave functions. The matrix elements of the Coulomb interaction as well as the kinetic term can be evaluated analytically according to Eq. (3.1) and Eq. (3.5). To evaluate the matrix elements of the Coulomb interaction as well as the kinetic term, we consider

$$\int_0^\infty S_{n'l}(r)S_{nl}(r)dr = \int_0^\infty S_{n'l}(r)dr \left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}} (2br)^{l+1} exp(-br) L_n^{2l+1}(2br) \tag{3.9}$$

using identity

$$(n+1)L_{n+1}^{\alpha}(x) = -(\alpha+n)L_{n-1}^{\alpha}(x) + (2n+\alpha+1-x)L_n^{\alpha}(x) \tag{3.10}$$

then, we can write

$$\begin{aligned}(2br)L_n^{2l+1}(2br) = (2n+2l+1+1)L_n^{2l+1}(2br) \\ - (n+2l+1)L_{n-1}^{2l+1}(2br) - (n+1)L_{n+1}^{2l+1}(2br)\end{aligned}$$

or we have

$$\begin{aligned}L_n^{2l+1}(2br) = \frac{1}{2br}[(2n+2l+1+1)L_n^{2l+1}(2br) \\ - (n+2l+1)L_{n-1}^{2l+1}(2br) - (n+1)L_{n+1}^{2l+1}(2br)]\end{aligned} \tag{3.11}$$

hence, by inserting Eq. (3.11) into Eq. (3.9) we have

$$\begin{aligned}
&\int_0^\infty S_{n'l}(r)S_{nl}(r)dr \\
&= \int_0^\infty S_{n'l}(r)dr \left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}} (2br)^{l+1}exp(-br)\frac{1}{2br}[(2n+2l+1+1)L_n^{2l+1}(2br) \\
&\quad - (n+2l+1)L_{n-1}^{2l+1}(2br) - (n+1)L_{n+1}^{2l+1}(2br)] \\
&= \int_0^\infty drS_{n'l}(r)\left(\frac{2n+2l+2}{2br}\right)\left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}} (2br)^{l+1}exp(-br)L_n^{2l+1}(2br) \\
&\quad - \int_0^\infty drS_{n'l}(r)\left(\frac{n+2l+1}{2br}\right)\left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}} (2br)^{l+1}exp(-br)L_{n-1}^{2l+1}(2br) \\
&\quad - \int_0^\infty drS_{n'l}(r)\left(\frac{n+1}{2br}\right)\left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}} (2br)^{l+1}exp(-br)L_{n+1}^{2l+1}(2br)
\end{aligned} \tag{3.12}$$

the first term on the right hand side of Eq. (3.12) can be written as

$$\begin{aligned}
&\int_0^\infty drS_{n'l}(r)\left(\frac{2n+2l+2}{2br}\right)\left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}} (2br)^{l+1}exp(-br)L_n^{2l+1}(2br) \\
&= \left(\frac{n+l+1}{b}\right)\int_0^\infty drS_{n'l}(r)\left(\frac{1}{r}\right)\left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}} (2br)^{l+1}exp(-br)L_n^{2l+1}(2br) \\
&= \left(\frac{n+l+1}{b}\right)\int_0^\infty drS_{n'l}(r)\left(\frac{1}{r}\right)S_{nl}(r) \\
&= \left(\frac{n+l+1}{b}\right)\int_0^\infty r^2dr\frac{S_{n'l}(r)}{r}\left(\frac{1}{r}\right)\frac{S_{nl}(r)}{r} \\
&= \frac{n+l+1}{b}\delta_{n'n}
\end{aligned} \tag{3.13}$$

the second term on the right hand side of Eq. (3.12) can be written as

$$\begin{aligned}
&\int_0^\infty dr S_{n'l}(r)\left(\frac{n+2l+2}{2br}\right)\left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}}(2br)^{l+1}exp(-br)L_{n-1}^{2l+1}(2br)\\
&=\left(\frac{n+2l+1}{2b}\right)\left[\frac{n}{(n+2l+1)}\right]^{\frac{1}{2}}\int_0^\infty dr S_{n'l}(r)\left(\frac{1}{r}\right)\left[\frac{(n-1)!}{((n-1)+2l+1)!}\right]^{\frac{1}{2}}\\
&\quad\cdot(2br)^{l+1}exp(-br)L_{n-1}^{2l+1}(2br)\\
&=\frac{\sqrt{n(n+2l+1)}}{2b}\int_0^\infty dr S_{n'l}(r)\left(\frac{1}{r}\right)S_{(n-1)l}(r)\\
&=\frac{\sqrt{n(n+2l+1)}}{2b}\int_0^\infty r^2dr\frac{S_{n'l}(r)}{r}\left(\frac{1}{r}\right)\frac{S_{(n-1)l}(r)}{r}\\
&=\frac{\sqrt{n(n+2l+1)}}{2b}\delta_{n'(n-1)}
\end{aligned}\tag{3.14}$$

and the third term on the right hand side of Eq. (3.12) can be written as

$$\begin{aligned}
&\int_0^\infty dr S_{n'l}(r)\left(\frac{n+1}{2br}\right)\left[\frac{n!}{(n+2l+1)!}\right]^{\frac{1}{2}}(2br)^{l+1}exp(-br)L_{n+1}^{2l+1}(2br)\\
&=\left(\frac{n+1}{2b}\right)\left[\frac{((n+1)+2l+1)}{(n+1)}\right]^{\frac{1}{2}}\int_0^\infty dr S_{n'l}(r)\left(\frac{1}{r}\right)\left[\frac{(n+1)!}{((n+1)+2l+1)!}\right]^{\frac{1}{2}}\\
&\quad\cdot(2br)^{l+1}exp(-br)L_{n+1}^{2l+1}(2br)\\
&=\frac{\sqrt{(n+1)((n+1)+2l+1)}}{2b}\int_0^\infty dr S_{n'l}(r)\left(\frac{1}{r}\right)S_{(n+1)l}(r)\\
&=\frac{\sqrt{(n+1)((n+1)+2l+1)}}{2b}\int_0^\infty r^2dr\frac{S_{n'l}(r)}{r}\left(\frac{1}{r}\right)\frac{S_{(n+1)l}(r)}{r}\\
&=\frac{\sqrt{(n+1)((n+1)+2l+1)}}{2b}\delta_{n'(n+1)}
\end{aligned}\tag{3.15}$$

inserting Eq. (3.13–3.15) into Eq. (3.12), then we have

$$\begin{aligned}
\int_0^\infty S_{n'l}(r)S_{nl}(r)dr=&\frac{n+l+1}{b}\delta_{n'n}-\frac{\sqrt{n(n+2l+1)}}{2b}\delta_{n'(n-1)}\\
&-\frac{\sqrt{(n+1)((n+1)+2l+1)}}{2b}\delta_{n'(n+1)}
\end{aligned}\tag{3.16}$$

or we can write Eq. (3.16) as

$$\int_0^\infty S_{n'l}(r)S_{nl}(r)dr = \begin{cases} \frac{n+l+1}{b}, & n' = n \\ -\frac{\sqrt{n(n+2l+1)}}{2b}, & n' = n-1 \\ -\frac{\sqrt{(n+1)((n+1)+2l+1)}}{2b}, & n' = n+1. \end{cases} \tag{3.17}$$

Next we will evaluate the matrix elements of the Coulomb interaction and the kinetic term. First we consider the first element of the matrix in Eq. (2.57), the term P_1HP_1. From Eq. (2.57), we have

$$\begin{aligned} P_1HP_1 &= H_0^p + H_0^{K^-} + V_c + V_{K^-p\to K^-p} \\ &= H_0^p + H_0^{K^-} - \frac{e^2}{r} + V_N(r) \end{aligned} \tag{3.18}$$

where $-e^2/r$ is the Coulomb interaction, $V_N(r)$ is the strong interaction, $H_0^p = \sqrt{m_p^2 + p_1^2}$ and $H_0^{K^-} = \sqrt{m_{K^-}^2 + p_2^2}$ with p_1 and p_2 being momentum of proton and negatively charged kaon, respectively.

If we evaluate only K^-p channel, the dynamical equation, Eq. (2.57), was reduced into the from

$$(P_1HP_1)\Psi_{K^-p} = E\Psi_{K^-p}. \tag{3.19}$$

From Eq. (3.18) we consider

$$\begin{aligned} H_0^p &= \sqrt{m_p^2 + p_1^2} \\ &= m_p\left(1 + \frac{p_1^2}{m_p^2}\right)^{\frac{1}{2}} \\ &= m_p\left(1 + \frac{p_1^2}{2m_p^2} + ...\right) \\ &= m_p + \frac{p_1^2}{2m_p} \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} H_0^{K^-} &= \sqrt{m_{K^-}^2 + p_2^2} \\ &= m_{K^-}\left(1 + \frac{p_2^2}{m_{K^-}^2}\right)^{\frac{1}{2}} \\ &= m_{K^-}\left(1 + \frac{p_2^2}{2m_{K^-}^2} + ...\right) \\ &= m_{K^-} + \frac{p_2^2}{2m_{K^-}} \end{aligned} \tag{3.21}$$

then

$$\begin{aligned} H_0^p + H_0^{K^-} &= m_p + \frac{p_1^2}{2m_p} + m_{K^-} + \frac{p_2^2}{2m_{K^-}} \\ &= \left(\frac{p_1^2}{2m_p} + \frac{p_2^2}{2m_{K^-}}\right) + m_p + m_{K^-} \\ &= \frac{p^2}{2\mu} + \frac{P^2}{2M} + m_p + m_{K^-}. \end{aligned} \tag{3.22}$$

where $\mu = (m_p m_{K^-})/(m_p + m_{K^-})$ and $M = m_p + m_{K^-}$ are respectively the reduced and total masses of the proton (p) and the negatively charge kaon (K^-) system. Inserting Eq. (3.22) into Eq. (3.18), we have

$$\begin{aligned} P_1 H P_1 &= H_0^p + H_0^{K^-} - \frac{e^2}{r} + V_N(r) \\ &= \frac{p^2}{2\mu} + \frac{P^2}{2M} + m_p + m_{K^-} - \frac{e^2}{r} + V_N(r). \end{aligned} \tag{3.23}$$

After the center-of-mass motion removed in Eq. (3.23), the dynamical equation, Eq. (3.19), becomes

$$\left[\frac{p^2}{2\mu} - \frac{e^2}{r} + V_N(r)\right] \Psi_{K^-p}(\vec{r}) = E_b \Psi_{K^-p}(\vec{r}). \tag{3.24}$$

with

$$E_b = E - m_p - m_{K^-} \tag{3.25}$$

being the binding energy.

If the $V_N(r)$ is a center potential, one may separate the variables to get the dynamical equation for the radial part of the kaonic hydrogen wave function,

$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} - \frac{e^2}{r} + V_N(r)\right) R_l(r) = E_b R_l(r). \tag{3.26}$$

We expand the radial wave function $R_l(r)$ in the complete set of Sturmian functions

$$R_l(r) = \sum_n A_n \frac{S_{nl}(r)}{r} \tag{3.27}$$

and derive finally the coupled matrix equations as follows

$$(P^{\alpha}_{n'n} + V^{\alpha\beta}_{n'n} + \Delta m_{1,\alpha}) A^{\beta}_n = E_b \, B^{\alpha}_{n'n} A^{\alpha}_n \tag{3.28}$$

where "1" labels for $K^- p$ channel and α for other channels, and the indices β and n are summed over.

The matrix elements in the above equation are evaluated as below

$$\begin{aligned}
P_{n'n} &= \langle S_{n'l}|P|S_{nl}\rangle \\
&= \int_0^\infty S_{n'l}(r) \left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2\mu e^2}{\hbar^2}\frac{1}{r} \right) S_{nl}(r) dr \\
&= \int_0^\infty S_{n'l}(r) \left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2b(n+l+1)}{r} + b^2 \right) S_{nl}(r) dr \\
&\quad - \int_0^\infty S_{n'l}(r) \left(-\frac{2b(n+l+1)}{r} + b^2 + \frac{2\mu e^2}{\hbar^2}\frac{1}{r} \right) S_{nl}(r) dr
\end{aligned} \tag{3.29}$$

using identity from Eq. (3.1), the first term on the right hand side of above equation is removed, then Eq. (3.29) becomes

$$\begin{aligned}
P_{n'n} &= -\int_0^\infty S_{n'l}(r) \left(-\frac{2b(n+l+1)}{r} + b^2 + \frac{2\mu e^2}{\hbar^2}\frac{1}{r} \right) S_{nl}(r) dr \\
&= \int_0^\infty S_{n'l}(r) \left(\frac{2b(n+l+1)}{r} - b^2 - \frac{2\mu e^2}{\hbar^2}\frac{1}{r} \right) S_{nl}(r) dr \\
&= 2b(n+l+1) \int_0^\infty S_{n'l}(r) \left(\frac{1}{r} \right) S_{nl}(r) dr \\
&\quad - \frac{2\mu e^2}{\hbar^2} \int_0^\infty S_{n'l}(r) \left(\frac{1}{r} \right) S_{nl}(r) dr \\
&\quad - b^2 \int_0^\infty S_{n'l}(r) S_{nl}(r) dr
\end{aligned}$$

from Eq. (3.5) and Eq. (3.17) we can write

$$\begin{aligned}
P_{n'n} &= 2b(n+l+1)\delta_{n'n} - \frac{2\mu e^2}{\hbar^2}\delta_{n'n} - b^2 \frac{n+l+1}{b}\delta_{n'n} \\
&\quad + b^2 \frac{\sqrt{n(n+2l+1)}}{2b}\delta_{n'(n-1)} + b^2 \frac{\sqrt{(n+1)((n+1)+2l+1)}}{2b}\delta_{n'(n+1)},
\end{aligned} \tag{3.30}$$

the quantity $2\mu e^2/\hbar^2$ can be written as

$$\frac{2\mu e^2}{\hbar^2} = \frac{2\mu c^2 e^2}{\hbar^2 c^2} = 2\mu c^2 \frac{e^2}{\hbar^2 c^2} = 2\mu c^2 \alpha^2$$

where α is the fine structure constant and it is a dimensionless quantity, the value of α is

$$\alpha \simeq 1/137 \simeq 7.297 \times 10^{-3}.$$

Also

$$\begin{aligned} V_{n'n} &= \langle S_{n'l}|V|S_{nl}\rangle \\ &= \int_0^\infty S_{n'l}(r)\left(\frac{2\mu}{\hbar^2}V_N(r)\right)S_{nl}(r)dr \\ &= \frac{2\mu}{\hbar^2}\int_0^\infty S_{n'l}(r)V_N(r)S_{nl}(r)dr. \end{aligned} \tag{3.31}$$

And

$$\begin{aligned} B_{n'n} &= \langle S_{n'l}|B|S_{nl}\rangle \\ &= \int_0^\infty S_{n'l}(r)\left(\frac{2\mu}{\hbar^2}\right)S_{nl}(r)dr \\ &= \frac{2\mu}{\hbar^2}\int_0^\infty S_{n'l}(r)S_{nl}(r)dr \\ &= \frac{2\mu}{\hbar^2}\frac{n+l+1}{b}\delta_{n'n} - \frac{2\mu}{\hbar^2}\frac{\sqrt{n(n+2l+1)}}{2b}\delta_{n'(n-1)} \\ &\quad - \frac{2\mu}{\hbar^2}\frac{\sqrt{(n+1)((n+1)+2l+1)}}{2b}\delta_{n'(n+1)}. \end{aligned} \tag{3.32}$$

CHAPTER IV

RESULTS, DISCUSSIONS AND CONCLUSIONS

This chapter is devoted to report our numerical results of the kaonic hydrogen atoms with realistic interactions and compare our theoretical results with the experimental data. In the study both real and complex interaction potentials are considered. Finally, the discussion and conclusions are also given.

4.1 Results from the real interaction potentials

In this thesis, we study kaonic hydrogen first with the interaction taken from the work of Akaishi Y. and Yamazaki T. (Akaishi and Yamazaki, 2002). The interactions are in the form :

$$V^I_{\bar{K}N}(r) = V^I_D exp[-(r/b)^2], \tag{4.1}$$

$$V^I_{\bar{K}N,\pi\Sigma}(r) = V^I_{C_1} exp[-(r/b)^2], \tag{4.2}$$

$$V^I_{\bar{K}N,\pi\Lambda}(r) = V^I_{C_2} exp[-(r/b)^2], \tag{4.3}$$

where

$$b = 0.66 \tag{4.4}$$

in unit of fermi (fm) with

$V^{I=0}_D = -436$ MeV, $V^{I=0}_{C_1} = -412$ MeV, $V^{I=0}_{C_2} =$ none, $V^{I=1}_D = -62$ MeV, $V^{I=1}_{C_1} = -285$ MeV and $V^{I=1}_{C_2} =$ none.

The two interactions $V^I_{\bar{K}N,\pi\Sigma}(r)$ and $V^I_{\bar{K}N,\pi\Lambda}(r)$, are taken to be vanishing to simply reduce the number of parameter. This justified because they are almost irrelevant in describing the $\bar{K}$ bound states.

The interaction, Eq. (4.1–4.3), is constructed by fitting the free $\bar{K}N$ scattering data (Martin, 1981), the KpK data of kaonic hydrogen by the KEK Collaboration (Iwasaki et al., 1997) and the binding energy, which is regarded as an isospin $I = 0$ bound state of $\bar{K}N$. Since the interaction gives one molecular state $\Lambda(1405)$, it must be much stronger than the strong pion-pion interaction.

Shown in Fig. 4.1 are both the K^-p and $\bar{K}^0n$ components of the kaonic hydrogen

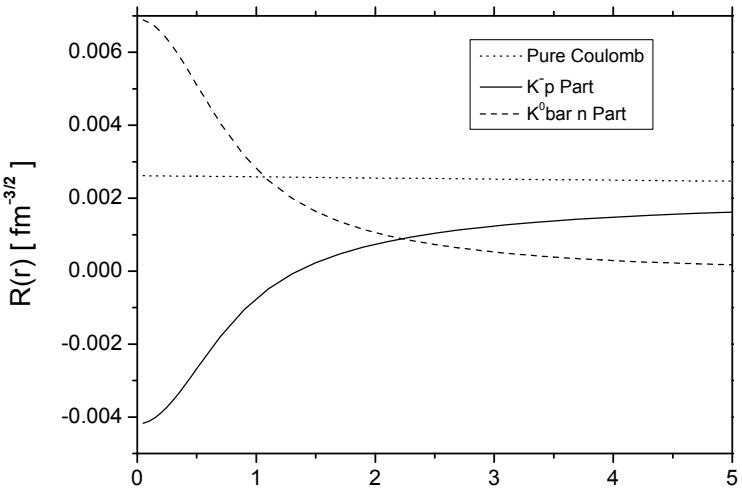

Figure 4.1 1s radial wave functions of kaonic hydrogen with the interaction of (Akaishi and Yamazaki, 2002). The pure Coulomb kaonic hydrogen wave function is plotted as a dotted curve.

with the interaction, Eq. (4.1–4.3), It is found that the K^-p part of the kaonic hydrogen wave function differs considerably from the hydrogen-like one at small distances, and also has a node at $r \approx 1.5$ fm because there exists one deep bound state. Also at small distances the $\bar{K}n$ part of the kaonic hydrogen wave function

is not negligible.

4.2 Results from the complex interaction potentials

Based on the coupled-channel interaction, Eq. (4.1–4.3), equivalent single-channel $\bar{K}N$ potentials are derived with imaginary parts in energy-independent forms which have been also constructed by Akaishi Y. and Yamazaki T. (Yamazaki and Akaishi, 2007). With this single-channel interaction, the kaonic hydrogen is investigated in our study. The obtained single-channel interaction or the complex potentials are :

$$V_{\bar{K}N}^{I=0}(r) = (-595 - i83)exp[-(r/0.66)^2], \tag{4.5}$$

$$V_{\bar{K}N}^{I=1}(r) = (-175 - i105)exp[-(r/0.66)^2]. \tag{4.6}$$

The potentials are in unit of MeV with the range parameter equal to 0.66 fm. Fig. 4.2 (lower panel) we plot the various components of the kaonic hydrogen wave function with the complex $\bar{K}N$ potentials (Yamazaki and Akaishi, 2007). One find again that the K^-p component differs considerably from the hydrogen-like kaonic hydrogen wave function at small distance and, again, the $\bar{K}^0n$ component is not negligible.

The kaonic hydrogen is also studied in the interactions taken from the work of Hyodo T. and Weise W. (Hyodo and Weise, 2008), which are constructed such as to reproduce the full scattering amplitude of the chiral SU(3) coupled-channel framework. The interaction from this chiral SU(3) coupled-channel dynamics is implemented by introducing a third order polynomial in $\sqrt{s}$,

$$U(r = 0, E) = K_0 + K_1\sqrt{s} + K_2(\sqrt{s})^2 + K_3(\sqrt{s})^3 \tag{4.7}$$

where, 1300 MeV $\leq \sqrt{s} \leq$ 1450 MeV. For our case $\sqrt{s} = m_p + m_{K^-} = 1431.94$ MeV/c^2.

The coefficients K_i take the values, for example, for the so-called HNJH model (Hyodo and Weise, 2008) as follows:

For $I = 0$:

$$K_0[10^5\text{MeV}] = -5.1020 - 4.3660\text{i} \tag{4.8}$$

$$K_1[10^2\text{MeV}^0] = 11.453 + 9.6378\text{i} \tag{4.9}$$

$$K_2[10^{-1}\text{MeV}^{-1}] = -8.5527 - 7.0773\text{i} \tag{4.10}$$

$$K_3[10^{-4}\text{MeV}^{-2}] = 2.1218 - 1.7285\text{i} \tag{4.11}$$

For $I = 1$:

$$K_0[10^5\text{MeV}] = -4.4348 - 0.67630\text{i} \tag{4.12}$$

$$K_1[10^2\text{MeV}^0] = 9.8340 + 1.4675\text{i} \tag{4.13}$$

$$K_2[10^{-1}\text{MeV}^{-1}] = -7.2582 - 1.0532\text{i} \tag{4.14}$$

$$K_3[10^{-4}\text{MeV}^{-2}] = 1.7818 - 0.24953\text{i} \tag{4.15}$$

It is found that the different versions of the potential (Hyodo and Weise, 2008) give quite similar results. The wave functions derived with the effective potentials (Hyodo and Weise, 2008) are also similar to the one derived with the interaction of Ref. (Yamazaki and Akaishi, 2007). As an example, in Fig. 4.2 (upper panel), we plot the various components of the kaonic hydrogen wave function evaluated with the equivalent local HNJH potential of (Hyodo and Weise, 2008)

In summary, the kaonic hydrogen wave functions derived with the interaction of (Akaishi and Yamazaki, 2002), (Yamazaki and Akaishi, 2007) and (Hyodo and Weise, 2008) are featured by that the K^-p component is also largely different from the hydrogen-like one and has a node in the region from 1 to 2 fm, and the K^0n component is also rather large at small distances.

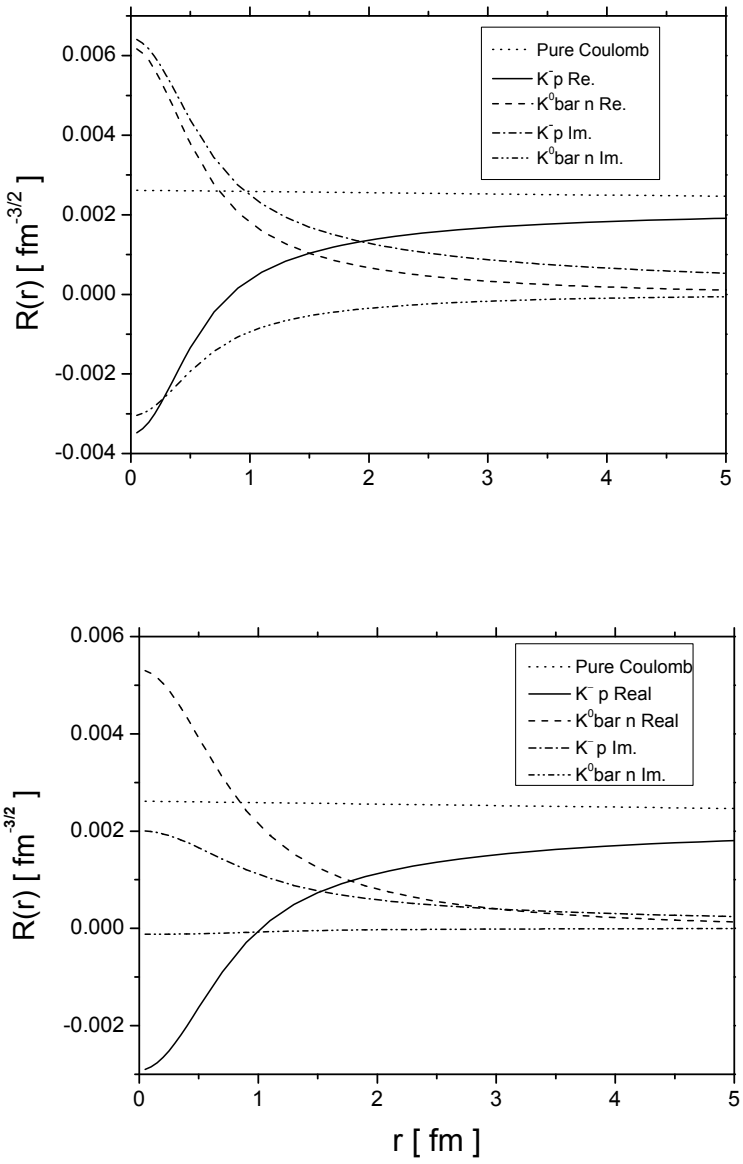

Figure 4.2 1s radial wave functions of kaonic hydrogen : Upper panel with the HNJH potential of (Hyodo and Weise, 2008) and lower panel with the interaction of (Yamazaki and Akaishi, 2007). The pure Coulomb kaonic hydrogen wave function is plotted as a dotted curve.

The theoretical results for energy shifts and decay widths of the kaonic hydrogen are shown in Table 4.1, the first, second and third rows for the single-channel real $\bar{K}N$ potential (Akaishi and Yamazaki, 2002), singer-channel complex

Table 4.1 $1s$ kaonic hydrogen energy shifts and decay width. Theoretical results are listed in the first, second and third rows while experimental data are listed in the fourth and fifth row.

	ΔE_{1s}[eV]	$\Gamma_{1s}/2$[eV]
Real Yamazaki	-472	$-$
Complex Yamazaki	-375	160
HNJH model	-328	352
DEAR Lab	-194 ± 40	125 ± 56
KEK Lab	-323 ± 64	204 ± 115

$\bar{K}N$ potential (Yamazaki and Akaishi, 2007) and the equivalent local HNJH potential of (Hyodo and Weise, 2008), respectively. In Table 4.1, the energy shift ΔE_{1s}, and decay width Γ_{1s} are derived directly by solving the Schrödinger equation with the interactions (Yamazaki and Akaishi, 2007), and the equivalent local HNJH potential of (Hyodo and Weise, 2008). The negative energy shifts in Table 4.1 mean that the $1s$ energy level is pushed up by the strong interaction since there exist one deep bound state, the $\Lambda(1405)$. For comparison, experimental data (Beer et al., 2005), (Martin, 1981) are shown in table 4.1 for the fourth and the fifth row with the energy shift and decay width determined by the DEAR (Beer et al., 2005) and the the KEK experiment (Iwasaki et al., 1997)

4.3 Discussions and conclusions

Kaonic hydrogen is studied with various versions of realistic interaction potentials in a numerical approach based on Sturmian functions. It is found that: The theoretical results for the energy shifts and decay widths of kaonic hydrogen

with those realistic interactions are in line with experimental data.

The ground-state wave function of kaonic hydrogen, derived for various $\bar{K}N$ interactions, is shown to be largely different from the hydrogen-like one at small distances.

The considerable discrepancy between the kaonic hydrogen wave function and the hydrogen-like one indicates that a perturbation method may NOT apply to the kaon-nucleon system at low energies. That is, the kaon-nucleon scattering lengths may not be safely extracted, using the Deser-type or corrected Deser-type relations, from the kaonic hydrogen data.

Though the theoretical prediction of various employed potentials for the energy shifts and decays of kaonic hydrogen are in line with experimental data, the discrepancy between the theoretical results and data is obvious. A potential, which is able to understand both the low-energy kaon-nucleon scattering data and kaonic hydrogen data, is in demand.

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APPENDICES

APPENDIX A

THE ADDITION OF ANGULAR MOMENTA OR ISOSPINS

Suppose we have two particles of angular momenta $\mathbf{j}_1$ and $\mathbf{j}_2$ with z-components m_1 and m_2. The total z-component is

$$m = m_1 + m_2.$$

The total angular momentum is

$$\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$$

and may therefore lie anywhere inside the limits

$$|j_1 - j_2| \leq j \leq |j_1 + j_2|$$

We wish to find the weights of the various allowed j-values contributing to the two-particle state, i.e.

$$\phi_1(j_1 m_1)\phi_2(j_2 m_2) = \sum_j C_j \psi(j, m), \tag{A.1}$$

with $m = m_1 + m_2$

The C_j are called Clebsch-Gordan coefficients (or Wigner, or vector addition, coefficients). Alternatively, we may want to express $\psi(j, m)$ as a sum of terms of different j_1 and j_2 combinations. We can do this by the use of angular-momentum(or isospin) *shift operators* (also known as "raising" and "lowering" operators).

First let us recall the definition of the $x-, y-$ and $z-$ component angular-momentum operators, in terms of the differential Cartesian operators

$$\begin{aligned} J_x &= -\frac{ih}{2\pi}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ J_y &= -\frac{ih}{2\pi}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ J_z &= -\frac{ih}{2\pi}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{aligned} \tag{A.2}$$

These Cartesian operations can also be interpreted in term of rotations. A rotation in azimuthal angle in the xy-plane has Cartesian components

$$\delta y = rcos\phi\delta\phi = x\delta\phi$$

$$\delta x = -rsin\phi\delta\phi = -y\delta\phi$$

Thus, the effect of a small rotation on a function $\psi(x, y, z)$ will be

$$\begin{aligned} R(\phi, \delta\phi)\psi(x, y, z) &= \psi(x + \delta x, y + \delta y, z) \\ &= \psi(x, y, z) + \delta x\frac{\partial\psi}{\partial x} + \delta y\frac{\partial\psi}{\partial y} \\ &= \psi\left[1 + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\delta\phi\right] \\ &= \psi\left(1 + \delta\phi\frac{\partial}{\partial\phi}\right) \end{aligned}$$

and, from Eq. (A.2)

$$J_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial\phi} \tag{A.3}$$

So

$$R = 1 + \delta\phi\frac{\partial}{\partial\phi} = 1 + \frac{iJ_z}{\hbar}\delta\phi \tag{A.4}$$

It is readily verified that the operators

$$J_x, J_y, J_z$$

and

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

obey the commutation rules

$$J^2 J_x + J_x J^2 = 0 \quad \text{etc},$$

and

$$\begin{aligned} J_x J_y - J_y J_x &= i J_z \\ J_y J_z - J_z J_y &= i J_x \\ J_z J_x - J_x J_z &= i J_y \end{aligned} \tag{A.5}$$

where we have used units $\hbar = c = 1$ for brevity. The eigenvalues of the operators J^2 and J_z are given in Eq. (A.6–A.7) below.

The *shift operators* are defined as

$$\begin{aligned} J_+ &= J_x + i J_x \\ J_- &= J_x - i J_x \end{aligned} \tag{A.6}$$

whence

$$\begin{aligned} J_z J_+ - J_+ J_z &= J_+ \\ J_z J_- - J_- J_z &= -J_-. \end{aligned} \tag{A.7}$$

Thus

$$J_z(J_-\phi) = J_z J_-\phi = J_-(J_z - 1)\phi = (m-1)J_-\phi.$$

Similarly,

$$J_z(J_+\phi) = (m+1)J_+\phi.$$

This last equation shows that the wavefunction $J_+\phi$ is an eigenstate of J_z with eigenvalue $m+1$. We can therefore write it as

$$J_+\phi(j,m) = C_+\phi(j,m+1)$$

where C_+ is an unknown (and generally complex) constant. If we multiply both sides of this equation by $\phi^*(j,m+1)$, and integrate over volume, we get

$$\int \phi^*(j,m+1) J_+\phi(j,m) dV = C_+ \int \phi^*(j,m+1)\phi(j,m+1) dV$$

where * indicates complex conjugation.

We choose the normalization of ϕ so that the last integral is unity, and all allowed m-values have unit weight. So

$$C_+ = \int \phi^*(j, m+1) J_+ \phi(j, m) dV.$$

Similarly,

$$\begin{aligned} C_- &= \int \phi^*(j, m) J_- \phi(j, m+1) dV \\ &= \int \phi^*(j, m) J_+^* \phi(j, m+1) dV \\ &= C_+^* \end{aligned}$$

from Eq. (A.6) and Eq. (A.7).

If we neglect arbitrary and unobservable phase, we must have

$$C_+ = C_- = C \quad \text{(a real number)}.$$

Also, from Eq. (A.6) and Eq. (A.7),

$$J_+ J_- = J_x^2 + J_y^2 - i(J_x J_y - J_y J_x) = J_x^2 + J_y^2 + J_z = J^2 - J_z^2 + J_z$$

Then

$$J_+ J_- \phi(j, m+1) = [j(j+1) - m^2 - m]\phi(j, m+1) = C^2 \phi(j, m+1).$$

So

$$C = \sqrt{j(j+1) - m(m+1)}$$

is the coefficient connecting states (j, m) and $(j, m+1)$.

To summarize, the angular-momentum operators have the following properties:

$$J_z \phi(j, m) = m\phi(j, m) \tag{A.8}$$

$$J_z^2 \phi(j, m) = j(j+1)\phi(j, m) \tag{A.9}$$

$$J_+\phi(j,m) = \sqrt{j(j+1) - m(m+1)}\phi(j,m+1) \tag{A.10}$$

$$J_-\phi(j,m) = \sqrt{j(j+1) - m(m+1)}\phi(j,m-1) \tag{A.11}$$

Example

As and example, we consider two particles, j_1, m_1 and j_2, m_2, forming the combined tate $\psi(j,m)$, and we take the case where $j_1 = 1, j_2 = \frac{1}{2}$ and $j = \frac{3}{2}$ or $\frac{1}{2}$.

Obviously the states with $m = \pm\frac{3}{2}$ can be formed in only one way :

$$\psi\left(\frac{3}{2},\frac{3}{2}\right) = \phi(1,1)\phi\left(\frac{1}{2},\frac{1}{2}\right) \tag{A.12}$$

$$\psi\left(\frac{3}{2},-\frac{3}{2}\right) = \phi(1,-1)\phi\left(\frac{1}{2},-\frac{1}{2}\right) \tag{A.13}$$

Now we use the operators $j_\pm$ to form the relations

$$J_-\phi\left(\frac{1}{2},\frac{1}{2}\right) = \phi\left(\frac{1}{2},-\frac{1}{2}\right), \quad J_-\phi\left(\frac{1}{2},-\frac{1}{2}\right) = 0$$

$$J_-\phi(1,1) = \sqrt{2}\phi(1,0) \quad J_-\phi(1,0) = \sqrt{2}\phi(1,-1), \quad J_-\phi(1,-1) = 0,$$

using Eq. (A.10) and Eq. (A.11)

Now operate on Eq. (A.12) with J_- on both sides:

$$\begin{aligned} J_-\psi\left(\frac{3}{2},\frac{3}{2}\right) &= \sqrt{3}\psi\left(\frac{3}{2},\frac{1}{2}\right) = J_-\phi(1,1)\phi\left(\frac{1}{2},\frac{1}{2}\right) \\ &= \sqrt{2}\phi(1,0)\phi\left(\frac{1}{2},\frac{1}{2}\right) + \phi(1,1)\phi\left(\frac{1}{2},-\frac{1}{2}\right) \end{aligned}$$

so

$$\psi\left(\frac{3}{2},\frac{1}{2}\right) = \sqrt{\frac{2}{3}}\phi(1,0)\phi\left(\frac{1}{2},\frac{1}{2}\right) + \sqrt{\frac{1}{3}}\phi(1,1)\phi\left(\frac{1}{2},-\frac{1}{2}\right). \tag{A.14}$$

Similarly, for Eq. (A.13),

$$\psi\left(\frac{3}{2},-\frac{1}{2}\right) = \sqrt{\frac{2}{3}}\phi(1,0)\phi\left(\frac{1}{2},-\frac{1}{2}\right) + \sqrt{\frac{1}{3}}\phi(1,-1)\phi\left(\frac{1}{2},\frac{1}{2}\right). \tag{A.15}$$

The $j = \frac{1}{2}$ state can be expressed as a linear sum:

$$\psi\left(\frac{1}{2},\frac{1}{2}\right) = a\phi(1,1)\phi\left(\frac{1}{2},-\frac{1}{2}\right) + b\phi(1,0)\phi\left(\frac{1}{2},\frac{1}{2}\right)$$

with $a^2 + b^2 = 1$. Then

$$J_+\psi\left(\frac{1}{2},\frac{1}{2}\right) = 0 = a\phi(1,1)\phi\left(\frac{1}{2},\frac{1}{2}\right) + b\sqrt{2}\phi(1,0)\phi\left(\frac{1}{2},\frac{1}{2}\right)$$

Thus, $a = \sqrt{\frac{2}{3}}$, $b = -\frac{1}{3}$, and so

$$\psi\left(\frac{1}{2},\frac{1}{2}\right) = \sqrt{\frac{2}{3}}\phi(1,1)\phi\left(\frac{1}{2},-\frac{1}{2}\right) + \sqrt{\frac{1}{3}}\phi(1,0)\phi\left(\frac{1}{2},\frac{1}{2}\right) \tag{A.16}$$

Similarly,

$$\psi\left(\frac{1}{2},-\frac{1}{2}\right) = \sqrt{\frac{1}{3}}\phi(1,0)\phi\left(\frac{1}{2},-\frac{1}{2}\right) - \sqrt{\frac{2}{3}}\phi(1,-1)\phi\left(\frac{1}{2},\frac{1}{2}\right). \tag{A.17}$$

APPENDIX B

CLEBSCH-GORDAN COEFFICIENTS

As an example of the use of the following table, take the case of combining two angular momenta $j_1 = 1$, $m_1 = 1$ and $j_2 = 1$, $m_1 = 1$. We look up the entry for combining angular momenta 1×1, and the fourth line gives for the coefficients C_j in Eq. (A.1) of Appendix A

$$\phi_1(1,1)\phi_2(1,-1) = \sqrt{\frac{1}{6}}\psi(2,0) + \sqrt{\frac{1}{2}}\psi(1,0) + \sqrt{\frac{1}{3}}\psi(0,0)$$

This tell us how two particles of angular momentum (or isospin) unity combine to form states of angular momentum $j = 0, 1$ or 2. Alternatively, a state of particular j, m can be decomposed into constituents. Thus $j = 2, m = 0$ can be decomposed into products of state with $j = 1$ and $m_1 + m_2 = m = 0$. The fourth column of the 1×1 table gives

$$\psi(2,0) = \sqrt{\frac{1}{6}}\phi_1(1,1)\phi_2(1,-1) + \sqrt{\frac{2}{3}}\phi_1(1,0)\phi_(1,0) + \sqrt{\frac{1}{6}}\phi_1(1,-1)\phi_(1,1)$$

The sign convention in the table follows that of Condon and Shortley (1951).

Table B.1 Clebsch-Gordan coefficients for the addition of $\mathbf{J}_1 = \frac{1}{2}$ and $\mathbf{J}_2 = \frac{1}{2}$

		$J =$	1	1	0	1
m_1	m_2	$M =$	+1	0	0	-1
$+\frac{1}{2}$	$+\frac{1}{2}$		1			
$+\frac{1}{2}$	$-\frac{1}{2}$			$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
$-\frac{1}{2}$	$+\frac{1}{2}$			$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
$-\frac{1}{2}$	$-\frac{1}{2}$					1

Table B.2 Clebsch-Gordan coefficients for the addition of $\mathbf{J}_1 = 1$ and $\mathbf{J}_2 = \frac{1}{2}$

		$J =$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
m_1	m_1	$M =$	$+\frac{3}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
+1	$+\frac{1}{2}$		1					
+1	$-\frac{1}{2}$			$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$			
0	$+\frac{1}{2}$			$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$			
0	$-\frac{1}{2}$					$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	
-1	$+\frac{1}{2}$					$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	
-1	$-\frac{1}{2}$							1

Table B.3 Clebsch-Gordan coefficients for the addition of $\mathbf{J}_1 = 1$ and $\mathbf{J}_2 = 1$

		$J =$	2	2	1	2	1	0	2	1	2
m_1	m_2	$M =$	+2	+1	+1	0	0	0	-1	-1	-2
+1	+1		1								
+1	0			$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$						
0	+1			$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$						
+1	-1					$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
0	0					$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$			
-1	+1					$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
0	-1								$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
-1	0								$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
-1	-1										1

APPENDIX C

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ACCURATE EVALUATION OF WAVE FUNCTIONS OF PIONIUM AND KAONIUM

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Pionium and kaonium are studied in an accurate numerical approach based on Sturmian functions. It is found that the ground-state wave functions of the exotic atoms in realistic strong interactions, particularly for kaonium, are considerably different from the hydrogen-like ones at small distances. The kaon-kaon scattering length derived from the $1s$ energy shift of kaonium by applying the Deser-Trueman formula is strongly inconsistent with the one derived directly by solving the Schödinger equation. The theoretical results indicate that it is arguable to treat kaonium perturbatively.

Keywords: Pionium; kaonium; Sturmian functions; low energy strong interaction.

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1. Introduction

Hadronic exotic atoms are bound mainly by the Coulomb force, but the strong interaction also plays a role, leading to an energy shift from the pure Coulomb energy and distorting the hydrogen-like wave function at short distance (a few fm). Pionium and kaonium are among the simplest hadronic exotic atoms since they couple to only few other channels. Pionium decays into only the $\pi^0\pi^0$ pair via the strong interaction while kaonium decays to the $\pi\pi$ and $\eta\pi$ channels. One may link, after a simple calculation in the quantum field theory, the decay branching ratio of pionium and kaonium to the corresponding scattering amplitude. For pionium decaying to the $\pi^0\pi^0$ pair, for example, we have

$$\Gamma = \frac{64\pi}{M^3} p \left| \int \frac{d\vec{k}}{(2\pi)^3} \, \psi_{1s}(\vec{k}) \, f_0(k,p) \right|^2 \tag{1}$$

where M is the mass of the exotic atom, p is the momentum of the final π_0, f_0 is the S-wave scattering amplitude of the process $\pi^+\pi^- \to \pi^0\pi^0$ at zero energy, and $\psi_{1s}(\vec{k})$ is the $1s$ wave function of pionium in momentum space and normalized according to $\int \frac{d\vec{k}}{(2\pi)^3} \, |\psi_{1s}(\vec{k})|^2 = 1$. In the approximation that the scattering amplitude $f_0(k,p)$

is estimated by its on-shell form $f_0(0,p)$, one derives

$$\Gamma = \frac{64\,\pi}{M^3}\,p\,|\psi_{1s}(0)|^2\,|f(0,p)|^2 \tag{2}$$

where $\psi_{1s}(0)$ is the $1s$ pionium wave function at the origin. The above equation is just the widely referred Trueman formula.[1]

It is clear that the wave function of hadronic exotic atoms plays a crucial role in linking the life time of the atoms to the scattering lengths of the corresponding systems. For pionium, its wave function might be reasonably approximated by the hydrogen-like one since the pion-pion strong interaction is believed to be relatively weak, compared to other hadron-hadron interactions. But for kaonium, it could be another story since the kaon-kaon strong interaction can be strong enough to support deep bound states. It is arguable that the wave function of kaonium can be well approximated by the hydrogen-like one.

The evaluation of wave functions of exotic atoms has been a challenge to numerical methods. Required is an approach, which is able to account accurately for both the strong short-range interaction and the long-range Coulomb force. The numerical approach based on Sturmian functions has been found effective and accurate. In this work we use the numerical method which has been carefully studied and discussed in the work[2] to study pionium and kaonium. The paper is arrange as follows: Pionium and kaonium are studied in Section 2 and 3, respectively. Discussion and conclusions are given in Section 4.

2. Pionium

Pionium is mainly a Coulomb bound state of π^+ and π^-, coupled strongly with $\pi^0\pi^0$ due to the strong interaction at small distance. The strong interaction between the two pions leads to an energy shift from the Coulomb energy (E = -1.86 keV) and a distortion to the hydrogen-like wave function at short distance (a few fm). Pionium decays predominantly into $\pi^0\pi^0$ via strong interaction, which probes the low energy interactions of the pions, especially at zero-energy.

Among all hadronic exotic atoms, pionium is the simplest and has been studied the best up to now. The DIRAC experiment at CERN has been commissioned since 1998 to measure the pionium lifetime and the first results have been published recently based on part of the collected data. The result of the pionium lifetime is $\tau_{1S} = 2.91^{+0.49}_{-0.62}\times 10^{-15}$ seconds.[3] In the theoretical sector, pionium has been studied extensively in various models. As expected, the results of the chiral perturbation theory is in line with the experimental data.

In the approximation of the pionium wave function ψ_{1s} in Eq. (2) to the hydrogen-like wave function, one derives the chiral perturbation result at leading order

$$\Gamma = \frac{2}{9}\alpha^3\,p\,|a_0 - a_2|^2 \tag{3}$$

where α is the fine structure constant, and a_0 and a_2 are respectively the isospin $I = 0$ and $I = 2$ S-wave scattering lengths of the pion-pion reaction. The chiral perturbation theory has a NLO prediction for the pionium lifetime,

$$\Gamma = \frac{2}{9}\alpha^3 \, p \, |a_0 - a_2|^2 (1 + \delta) \, , \tag{4}$$

with $\delta = 0.058 \pm 0.012$.[4] Inserting into Eq. (4) $|a_0 - a_2| = 0.265 \pm 0.004$, the $O(p^6)$ result of the chiral perturbation theory,[5] one gets the pionium lifetime $\tau = (2.9 \pm 0.1) \cdot 10^{-15}$ s.

That the chiral perturbation theory reproduces the pionium lifetime data perfectly implies that a pion-pion strong interaction applicable to the pion-pion dynamics equation in the quantum mechanism regime must give a pionium wave function which differs not much from the hydrogen-like one. In another word, the pion-pion strong interaction is rather weak, compared with strong interactions for other hadronic systems, for example, the $\overline{N}N$ system.

As mentioned in the works,[6,7] the evaluation of pionium wave functions is not an easy task. It is more difficult than other exotic problems, for example, the protonium problem since the Bohr radius of pionium is much larger than the one of protonium. Employed here is a numerical approach based on Sturmian functions.[8] The numerical method is much more powerful, accurate and much easier to use than all other methods applied to exotic atom problems in the quantum mechanism regime in history. For the details of the numerical method and the accuracy in the hadronic exotic atom problem, we refer to the works.[2,8,9]

In this work we have no intention to study various versions of pion-pion strong interactions, but instead just to demonstrate the problem with one of the simplest forms of pion-pion strong interactions. The investigation of pionium with various pion-pion strong interaction models, in both local and nonlocal forms, may be found in the work.[10] Employed here for the purpose of demonstration is the pion-pion strong interaction which has been widely employed for calculating of the electromagnetic corrections in low energy pion-nucleon scattering and for studying the influence of the hadronic interaction on pionium wave functions.[6] The potential is independent of both the energy of the pionium system and pion masses, and reproduce very well the phase shifts given by two-loop chiral perturbation theory.

Shown in Fig. 1 as the solid line is the $1s$ radial wave function for the $\pi^+\pi^-$ component of the pionium in the pion-pion strong interaction taken from the works.[6] In the calculation we have employed the non-relativistic Schödinger equation for the $(\pi^+\pi^-, \pi^0\pi^0)$ system where the mass difference has been considered between the $\pi^+\pi^-$ pair and the $\pi^0\pi^0$ pair. It is found that the difference is not much between the full pionium wave function (solid line) and the hydrogen-like one (dashed line). The energy shift is derived as $\Delta E_{1s} = 3.04$ eV, indicating that the energy level is pulled down by the strong interaction, compared to the pure Coulomb interaction. The lifetime of the pionium is estimated in the potential model to be 3.15×10^{-15} and 2.35×10^{-15} seconds, where the hydrogen-like and full pionium wave functions

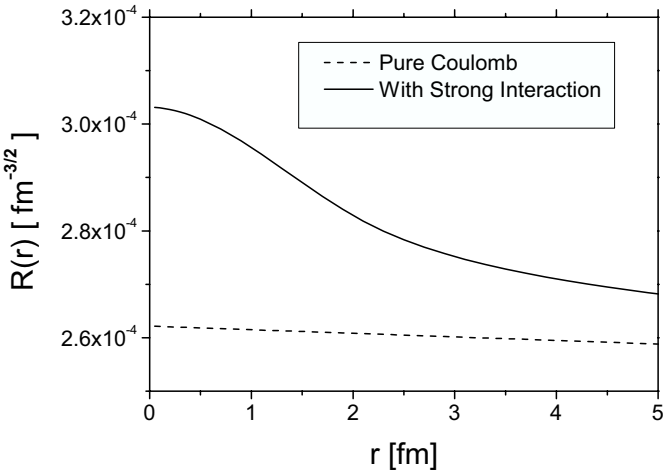

Fig. 1. $\pi^+\pi^-$ component of full and pure-Coulomb 1s radial pionium wave functions.

are applied to Eq. (3), respectively. The theoretical results of the pionium lifetime shows the importance of the pionium wave function in potential models or quantum mechanics regime. It is believed that the application of Eq. (1) with the full pionium wave function and full scattering amplitude in the pion-pion interaction[6] will lead to a pionium lifetime close to the DIRAC data.

3. Kaonium

Kaonium is the hadronic atom of K^+ and K^- mixed with the $\overline{K^0}K^0$ component at small distance. It is bound mainly by the Coulomb force, but affected by the strong interaction at small distance. The kaonium can not decay into a K^0K^0 pair due to the kinetic reason, but may decay into $\pi\pi$ and $\eta\pi$ via strong interaction. Unlike the pionium, there are few works[12–15] on this exotic atom.

We study the kaonium first in the K^-K^+ interaction taken from the work.[14] The interaction is derived under the assumption that K^+K^- forms quasi-bound states in $I = 0$ and $I = 1$, which correspond to $f_0(980)$ and $a_0(980)$, respectively. Since the interaction gives two molecular states $f_0(980)$ and $a_0(980)$, it must be much stronger than the pion-pion strong interaction. Shown in Fig. 2 (left panel) are both the real and imaginary parts of the K^-K^+ component of the kaonium in the interaction[14] (Model A). It is found that the K^+K^- real part of the kaonium wave function differs considerably from the hydrogen-like one at small distance, and also has a node at $r \approx 1.5$ fm since there exist deep bound states. At small distance the imaginary part of the pionium wave function is not negligible.

Recently there has been a work studying kaonium in the strong interaction generated by vector meson exchange within the framework of the standard $SU(3)_V \otimes SU(3)_A$ invariant effective Lagrangian.[15] Since the imaginary part of the interaction is in the δ-function form which is not suitable for quantum mechanics

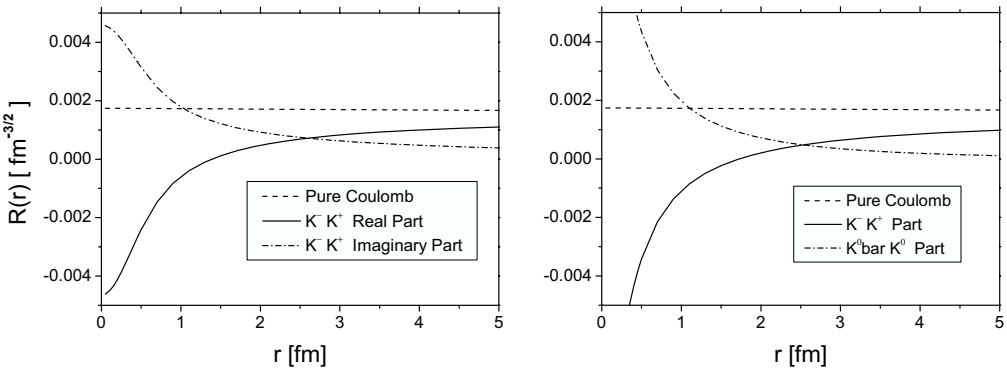

Fig. 2. $1s$ radial wave functions of kaonium in Model A (left panel) and Model B (right panel).

calculations, we apply only the real part of the interaction to evaluate the kaonium wave function. Shown in Fig. 2 (right panel) are the derived kaonium wave functions, with the K^+K^- component largely different from the hydrogen-like one and having a node at $r \approx 1.5$ fm and the K^0K^0 part also rather large.

To see further how safe it is to approximate the kaonium wave function to the hydrogen-like one, we compare the scattering length evaluated directly by solving the Schrödinger equation with the one derived from the kaonium energy shifts by applying the Deser-Trueman formula,[1,11]

$$-\Delta E_{1s} + i\frac{\Gamma_{1s}}{2} = 2\alpha^3\,\mu^2\,f_0^{K^+K^-}(0) \tag{5}$$

where μ is the reduced mass of the K^-K^+ pair, ΔE_{1s} and Γ_{1s} are respectively the energy shift and decay width of the 1s kaonium state due to strong interaction, and $f_0^{K^+K^-}(0)$ is the S-wave K^-K^+ scattering amplitude at zero energy.

Shown in Table 1 are the energy shifts (the second column) evaluated with only the real part of the interactions in both Model A and B, the scattering lengths (the third column) derived from the energy shifts listed in the second column by applying the Deser-Trueman formula, and the scattering lengths (the fourth column) derived directly[14,15] by solving the Schrödinger equation. The negative energy shifts in Table 1 mean that the $1s$ energy level is pushed up by the strong interactions. It is found that in the same interaction the scattering length derived directly by solving the Schrödinger equation is considerably different from the one derived from the $1s$ energy shift of the kaonium by applying the Deser-Trueman formula.

Table 1. $1s$ kaonium energy shifts and kaon-kaon scattering lengths in unit of M_K^{-1}.

	$\Delta E_{1s}(eV)$	$a^{K^+K^-}(M_K^{-1})$	$a^{K^+K^-}(M_K^{-1})$
Model A	-548	5.7	7.8
Model B	-354	3.69	2.72

4. Discussion and Conclusions

Pionium and kaonium are studied in an accurate numerical approach based on Sturmian functions. It is found that the ground-state wave functions of the exotic atoms in realistic strong interactions, particularly for the kaonium, are considerably different from the hydrogen-like ones at small distances. The kaon-kaon scattering length derived from the $1s$ kaonium energy shift by applying the Deser-Trueman formula is strongly inconsistent with the one derived directly by solving the Schrödinger equation. The theoretical results indicate that it might not be safe to treat the kaonium perturbatively.

Acknowledgments

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