LINEARIZATION OF FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY POINT AND CONTACT TRANSFORMATIONS

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นางสาวสุภาพร สุขเสริญ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์นี้ได้ศึกษาปัญหาการทำให้เป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับสี่โดยใช้การแปลงแบบจุดและคอนแทคท์ ทั้งนี้ได้นำเสนอรูปแบบที่ชัดแจ้งของเงื่อนไขที่จำเป็นและเพียงพอสำหรับการทำให้เป็นเชิงเส้น วิธีการได้มาของการแปลงเชิงเส้น รวมถึงสัมประสิทธิ์ของสมการเชิงเส้น และยังได้มาซึ่งรูปแบบทั่วไปของสมการเชิงอนุพันธ์สามัญที่อันดับสูงกว่าสี่ที่สามารถทำให้เป็นเชิงเส้นได้ด้วยการแปลงแบบจุดและคอนแทคท์ นอกจากนี้ได้นำเกณฑ์ของการทำให้เป็นเชิงเส้นที่ได้มาจากสมการเชิงเส้นอนุพันธ์สามัญอันดับสี่ไปประยุกต์ใช้กับระบบของสมการเชิงอนุพันธ์สามัญอันดับสองที่มีสองสมการ

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LINEARIZATION PROBLEM / POINT TRANSFORMATION / CONTACT TRANSFORMATION / LINEARIZATION TEST / NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

This thesis is devoted to the study of the linearization problem of fourth-order ordinary differential equations by means of point and contact transformations. The necessary and sufficient conditions for linearization, the procedure for obtaining the linearizing transformations as well as the coefficients of the resulting linear equations are provided in explicit forms. The general form of ordinary differential equations of order greater than four linearizable via point and contact transformations are obtained. Moreover, the linearization criteria obtained for fourth-order ordinary differential equations is applied to a system of two second-order ordinary differential equations.

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Supaporn Suksern

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CHAPTER I

INTRODUCTION

Almost all important governing equations in physics take the form of nonlinear differential equations, and, in general, are very difficult to solve explicitly. While solving problems related to nonlinear ordinary differential equations it is often expedient to simplify equations by a suitable change of variables. One of the fundamental methods of solving relies upon the transformation of a given equation to another equation of standard form. The transformation may be to an equation of equal order or of greater or lesser order. In particular, the possibility that a given equation could be linearized, i.e., transformed to a linear equation, was a most attractive proposition due to the special properties of linear differential equations. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows constructing an exact solution of the original equation. Analytical (exact) solution has value, firstly, as an exact description of a real process in the framework of a given model; secondly, as a model to compare various numerical methods; thirdly, as a basis to improve the models used. Therefore, the linearization problem plays a significant role in the nonlinear problem.

Many of the classical methods for solving ordinary differential equations work by applying a change of variables to produce another equation with known solutions. The simplest form of a differential equation is a linear form. It is of interest to provide general criteria for the linearizability of nonlinear ordinary differential equations, as they can then be reduced to easily solvable equations. Linearization criteria via invertible transformations for ordinary differential equations have been of great interest and have been dealt with by many authors over the years.

The linearization problem studied in the thesis can be stated as follows: find a change of variables such that a transformed equation becomes a linear equation. If the change of variables includes derivatives, this change is called a tangent transformation. If the change of variables only depends on the independent and dependent variables, then this change is called a point transformation. A tangent transformation, that is defined by the change of the independent, dependent variables and the first-order partial derivatives, is called a contact transformation. Point transformations are the simplest type of transformations compared with tangent transformations. This thesis studies linearization problem by using point and contact transformations.

1.1 Short Historical Review

The problem of linearization of ordinary differential equations has a long history*. It attracted attention of mathematicians such as S. Lie and E. Cartan. The first linearization problem for ordinary differential equations was solved by Lie (1883). He found the general form of all ordinary differential equations of second order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients†. The linearization criterion is written through relative invariants of the equivalence group. Liouville (1889) and Tresse (1896) treated the equivalence problem for second-order ordinary differential equations in terms of

*Historical review can be found in (Ibragimov, 1999), recent references in (Meleshko, 2005).

†See detail in section 2.5.

relative invariants of the equivalence group of point transformations.

Lie also noted that all second-order ordinary differential equations can be mapped to each other by means of contact transformations[‡], and that this is not so for third-order ordinary differential equations. Hence, the linearization problem using contact transformations becomes interesting for ordinary differential equations of order greater than two.

There are other approaches for solving the linearization problem of a second-order ordinary differential equation. For example, one was developed by Cartan (1924). The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form. Another approach makes use of the generalized Sundman transformation (Durate, Moreira and Santos, 1994).

Cartan's approach was further applied by Chern (1940) to third-order ordinary differential equations. He obtained conditions for a third-order ordinary differential equation to be equivalent to the equations $u''' = 0$ and $u''' + u = 0$. In his work, the conditions for linearization are given in terms of geometric invariants of contact transformations and do not provide practical methods for determining linearizing transformations. In 1993, Bocharov, Sokolov and Svinolupov considered the linearization problem with respect to point transformations. Grebot (1997) studied the linearization of third-order ordinary differential equations by means of a restricted class of point transformations, namely $t = \varphi(x), u = \psi(x, y)$. However, the problem was not completely solved. Complete criteria for linearization by means of point transformations were obtained in (Ibragimov and Meleshko, 2005). Linearization with respect to contact transformations was studied in a series of articles [(Bocharov et al., 1993), (Doubrov, 2001), (Doubrov, Komrakov

[‡]See more detail in Appendix A.

and Morimoto, 1999), (Gusyatnikova and Yumaguzhin, 1999)]. The solutions of the linearization problem were given in (Neut and Petitot, 2002) and (Ibragimov and Meleshko, 2005). Conditions for equivalence with an arbitrary linear equation were announced in (Neut and Petitot, 2002), but the procedure for obtaining linearizing transformations were not given. In (Ibragimov and Meleshko, 2005), the explicit form of the criteria for linearization and the procedure for the construction of the linearizing transformation are presented.

The linearization problem for a third-order ordinary differential equation was also investigated with respect to the generalized Sundman transformation [(Berkovich, 1999), (Euler, Wolf, Leach and Euler, 2003)]:

$$u(t) = F(x, y), \qquad dt = G(x, y)dx.$$

Criteria for a third-order ordinary differential equation to be equivalent to the linear equation

$$u''' = 0$$

with respect to the Sundman transformation were presented in (Euler et al., 2003).

The main difficulty in solving the linearization problem comes from the large number of complicated calculations. Because of this difficulty, there are only a few attempts to solve this problem for equation of orders higher than three. In (Dridi and Neut, 2005) Cartan's method was used for a particular linearization problem of fourth-order ordinary differential equation under contact transformations. As the result, conditions for a fourth-order ordinary differential equation to be equivalent to the trivial equation $u^{(4)} = 0$ were obtained[§]. It is worth noting that application of contact transformations is more complicated than application of point transformations.

[§]See detail in Appendix B.

1.2 Results Obtained in Thesis

The aim of this thesis is to obtain complete criteria for fourth-order ordinary differential equations to be linearizable by point and contact transformations. For solving the problem in thesis, compatibility¶ theory was used. Any study of compatibility requires a large amount of symbolic calculations. These calculations consist of consecutive algebraic operations: prolongation of a system, substitution of some expressions, and the determination of ranks of matrices. Because these operations are very labor intensive, it is necessary to use a computer for symbolic calculations. Here we use symbolic calculation Reduce (Hearn, 1987).

Our motivation for considering the linearization problem is to map a known solution of an ordinary differential equation to solution of a linear ordinary differential equation, thus allowing a systematic use of collections of solved linear ordinary differential equations.

As shown in (Ibragimov and Meleshko, 2005) for third-order ordinary differential equations, two sets (the set of equations linearizable by point transformations and the set of equations linearizable by contact transformations) are complement to each other, but we found that for fourth-order ordinary differential equations, two sets are disjoint. This is one of the interesting results obtained for studying the linearization problem by point and contact transformations.

Other attractions of the study fourth-order ordinary differential equations are the following. Many systems of two second-order ordinary differential equations‖ can be reduced to a fourth-order ordinary differential equation. Hence, the linearization criteria obtained for fourth-order ordinary differential equations can

¶Compatibility means the system has a solution.

‖Short review of results of solving the linearization problem for a system of two second-order ordinary differential equations can be found, for example, in (Wafo Soh and Mahomed, 2000), (Aminova and Aminov, 2006).

be also applied to such type of systems.

It is worth mentioning that among the examples we find well-known equations such as those describing traveling waves of the generalized shallow water wave equation and one class of nonlinear fourth-order partial differential equations.

The study of fourth-order ordinary differential equations allowed us to develop the method for obtaining necessary conditions of linearization of ordinary differential equations of any order greater than four.

The thesis is organized as follows. In chapter II, the background knowledge and the main tools for solving linearization problem are introduced. In chapter III, we consider the criteria for fourth-order ordinary differential equations to be linearizable by point transformations. We show that all fourth-order equations that are linearizable by point transformations are contained in the class of equations which are linear in the third-order derivative. We provide the linearization test and describe the procedure for obtaining the linearizing transformations and the formulae for coefficients of resulting linear equations. For ordinary differential equations of order greater than four we obtain necessary conditions, which separate all linearizable equations into two classes. Illustrative examples and linearization of traveling waves of partial differential equation are provided in the subsequent sections. Application of the linearization theorem to one class of systems with two second-order ordinary differential equations are given. In chapter IV, the linearization via contact transformations for fourth-order ordinary differential equations are presented. We show that all fourth-order ordinary differential equations that are linearizable by contact transformations are contained in the class of equations which are at most quadratic in the third-order derivative. The main results of this chapter are studied in a similar manner as in chapter III. The conclusion of the thesis is presented in the last chapter. For the sake of simplicity of reading, cumbersome formulae, additional calculations and some material for review are presented in the Appendices.

CHAPTER II

PRELIMINARY BACKGROUND

In this chapter, we introduce some elementary knowledge that is used throughout the thesis. The main tools for solving the linearization problem are provided.

2.1 Tangent Transformations

Let us consider the transformations of the independent, dependent variables and their derivatives

$$\tilde{x} = f(x,u,p), \quad \tilde{u} = \phi(x,u,p), \quad \tilde{p} = \psi(x,u,p). \tag{2.1}$$

Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, p is the vector of the partial derivatives $p_\alpha = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}}$. For the multi-index α the following notations are used $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\alpha, j = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_{n)}.$

Remark 2.1. Functions f, ϕ and ψ in equations (2.1) are always assumed to be sufficiently many times continuously differentiable.

The transformations (2.1) are prolonged to the differentials dx, du, dp :

$$\begin{aligned}
d\tilde{x}_i &= \frac{\partial f_i}{\partial x_l}dx_l + \frac{\partial f_i}{\partial u}du + \frac{\partial f_i}{\partial p_\alpha}dp_\alpha,\\
d\tilde{u} &= \frac{\partial \phi}{\partial x_l}dx_l + \frac{\partial \phi}{\partial u}du + \frac{\partial \phi}{\partial p_\alpha}dp_\alpha,\\
d\tilde{p}_\gamma &= \frac{\partial \psi_\gamma}{\partial x_l}dx_l + \frac{\partial \psi_\gamma}{\partial u}du + \frac{\partial \psi_\gamma}{\partial p_\alpha}dp_\alpha,
\end{aligned}$$

where $i = 1, 2, \ldots, n$, the index γ is a multi-index.

Definition 2.1. Transformation (2.1) is called a *tangent transformation* if it preserves the *tangent conditions*

$$du - p_i dx_i = 0, \qquad dp_\gamma - p_{\gamma,i} dx_i = 0.$$

If the functions $\phi(x, u, p)$ and $f_i(x, u, p)$, (i=1,2,...,n) do not depend on the derivatives, then such a transformation is called a point transformation. A tangent transformation which is not a point transformation, that is defined by the transformation* of the independent, dependent variables and the first-order partial derivatives, is called a contact transformation. Point and contact transformations play a special role among all tangent transformations. Their role is explained by the Bäcklund theorem, which states that if in a tangent transformation one can find a closed system†, then such transformation is a prolongation of point or contact transformation.

As in this thesis we apply point and contact transformations to fourth-order ordinary differential equations, let us discuss them in more detail, in the case of ordinary differential equations.

Definition 2.2. A transformation

$$\begin{aligned} t &= \varphi(x, y), \\ u &= \psi(x, y), \end{aligned} \tag{2.2}$$

is called a *point transformation.*

*Transformations of higher order derivatives are defined through the transformations of the independent, dependent variables and first-order partial derivatives by prolongation formulae and tangent conditions.

†Transformations of the independent, dependent variables and derivatives up to some finite order, for example, N, depend on the independent, dependent variables and derivatives up to the order N.

Definition 2.3. A transformation

$$t = \varphi(x, y, p),$$
$$u = \psi(x, y, p), \tag{2.3}$$
$$s = g(x, y, p),$$

where $p = y' = \frac{dy}{dx}$ is called a *contact transformation* if it obeys the *contact condition*

$$s = u' = \frac{du}{dt}.$$

2.2 Mapping of Derivatives in Point Transformations

In general, let us analyze an ith-order ordinary differential equation

$$y^{(i)} = f\left(x, y, y', y'', ..., y^{(i-1)}\right). \tag{2.4}$$

We apply a point transformation (2.2) to equation (2.4). First of all, it has to change $y(x)$ to $u(t)$. Assume that we know the solution of equation (2.4), i.e.,

$$y = y(x).$$

To obtain the transformed function $u(t)$, start with the equation

$$t = \varphi(x, y(x)).$$

Notice that we require the Jacobian

$$\triangle = \frac{\partial(t, u)}{\partial(x, y)} = \frac{\partial(\varphi, \psi)}{\partial(x, y)} = \varphi_x \psi_y - \varphi_y \psi_x \neq 0.$$

Since $\varphi'(x, y(x)) = \varphi_x + y'\varphi_y$ is continuous (as φ is assumed to be continuous differentiable) and $\triangle(\varphi(x, y(x))) = \varphi_x + y'\varphi_y \neq 0$ then by virtue of the Inverse Function Theorem one finds

$$x = \alpha(t).$$

Thus, one obtains

$$u(t)=\psi(\alpha(t),y(\alpha(t))).$$

Now one needs to transform the derivatives. The first-order derivative is transformed by the formula

$$u'(t)=\frac{du}{dt}=\frac{\partial\psi}{\partial x}\frac{d\alpha}{dt}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt}=(\psi_x+y'\psi_y)\frac{d\alpha}{dt}. \tag{2.5}$$

Since $t=\varphi(\alpha(t),y(\alpha(t)))$ then

$$\begin{aligned}\frac{dt}{dt} &= \frac{\partial\varphi}{\partial x}\frac{d\alpha}{dt}+\frac{\partial\varphi}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt}\\ 1 &= (\varphi_x+y'\varphi_y)\frac{d\alpha}{dt}\\ \frac{d\alpha}{dt} &= \frac{1}{(\varphi_x+y'\varphi_y)}. \end{aligned}\tag{2.6}$$

Substituting equation (2.6) into equation (2.5), one obtains

$$u'(t)=\frac{\psi_x+y'\psi_y}{\varphi_x+y'\varphi_y}=\frac{D\psi}{D\varphi}=\psi_1(x,y(x),y'(x)).$$

Notice that $D=\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+y''\frac{\partial}{\partial y'}+\cdots$ is the total derivative with respect to x. So that the first prolongation of transformation (2.2) is $u'=\psi_1(x,y,y')$.

Next, we find the transformation of second-order derivative. Consider

$$\begin{aligned}u''(t) &= \frac{d^2u}{dt^2}\\ &= \frac{\partial\psi_1}{\partial x}\frac{d\alpha}{dt}+\frac{\partial\psi_1}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt}+\frac{\partial\psi_1}{\partial y'}\frac{dy'}{dx}\frac{d\alpha}{dt}\\ &= \left(\psi_{1x}+y'\psi_{1y}+y''\psi_{1y'}\right)\frac{d\alpha}{dt}\\ &= \frac{\psi_{1x}+y'\psi_{1y}+y''\psi_{1y'}}{\varphi_x+y'\varphi_y}\\ &= \frac{D\psi_1}{D\varphi}\\ &= \psi_2(x,y(x),y'(x),y''(x)),\end{aligned}$$

so that the second prolongation of transformation (2.2) is $u''=\psi_2(x,y,y',y'')$. Similarly, one finds

$$u'''(t)=\frac{d^3u}{dt^3}=\frac{D\psi_2}{D\varphi}=\psi_3(x,y,y',y'',y'''),$$

$$u^{(4)}(t) = \frac{d^4u}{dt^4} = \frac{D\psi_3}{D\varphi} = \psi_4(x, y, y', y'', y''', y^{(4)}).$$

In general, one can write

$$u^{(k+1)}(t) = \frac{d^{k+1}u}{dt^{k+1}} = \frac{D\psi_k}{D\varphi} = \psi_{k+1}(x, y, y', y'', y''', ..., y^{(k+1)}), \quad (k = 0, 1, 2, ...).$$

Notice that $\psi_0 = \psi$.

2.3 Mapping of Derivatives in Contact Transformations

Let $y(x)$ be the solution of equation (2.4). Applying a contact transformation (2.3) to equation (2.4), the transformed function $u(t)$ is found from the equations

$$t = \varphi\left(x, y\left(x\right), p\left(x\right)\right),$$

$$u = \psi\left(x, y\left(x\right), p\left(x\right)\right).$$

By virtue of the Inverse Function Theorem, the first equation gives

$$x = \tau(t),$$

and then

$$u\left(t\right) = \psi\left(\tau\left(t\right), y\left(\tau\left(t\right)\right), p\left(\tau\left(t\right)\right)\right).$$

The first-order derivative is transformed by the formula

$$\begin{aligned} u'\left(t\right) &= \frac{du}{dt} \\ &= \frac{\partial\psi}{\partial x}\frac{d\tau}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}\frac{d\tau}{dt} + \frac{\partial\psi}{\partial p}\frac{dp}{dx}\frac{d\tau}{dt} \\ &= \left(\psi_x + p\psi_y + y''\psi_p\right)\frac{d\tau}{dt}. \end{aligned} \tag{2.7}$$

Since $t = \varphi\left(\tau\left(t\right), y\left(\tau\left(t\right)\right), p\left(\tau\left(t\right)\right)\right)$ then

$$\begin{aligned} \frac{dt}{dt} &= \frac{\partial\varphi}{\partial x}\frac{d\tau}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx}\frac{d\tau}{dt} + \frac{\partial\varphi}{\partial p}\frac{dp}{dx}\frac{d\tau}{dt} \\ 1 &= \left(\varphi_x + p\varphi_y + y''\varphi_p\right)\frac{d\tau}{dt} \\ \frac{d\tau}{dt} &= \frac{1}{\left(\varphi_x + p\varphi_y + y''\varphi_p\right)}. \end{aligned} \tag{2.8}$$

Substituting equation (2.8) into equation (2.7), one obtains

$$u'(t) = \frac{\psi_x + p\psi_y + y''\psi_p}{\varphi_x + p\varphi_y + y''\varphi_p} = \frac{D\psi}{D\varphi}\left(\tau\left(t\right), y\left(\tau\left(t\right)\right), p\left(\tau\left(t\right)\right), y''(\tau(t))\right).$$

The contact condition requires

$$g\left(x, y, p\right) = \frac{D\psi}{D\varphi}\left(x, y, p, y''\right). \tag{2.9}$$

Equation (2.9) is rewritten in the form

$$g\left(\varphi_x + p\varphi_y + y''\varphi_p\right) = \psi_x + p\psi_y + y''\psi_p.$$

Since the contact condition is satisfied for any y'', one obtains

$$\begin{gathered} g\left(\varphi_x + p\varphi_y\right) = \psi_x + p\psi_y, \\ g\varphi_p = \psi_p. \end{gathered} \tag{2.10}$$

The second-order derivative is transformed by the formula

$$\begin{aligned} u''\left(t\right) &= \frac{d^2u}{dt^2} \\ &= \frac{\partial g}{\partial x}\frac{d\tau}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dx}\frac{d\tau}{dt} + \frac{\partial g}{\partial p}\frac{dp}{dx}\frac{d\tau}{dt} \\ &= \left(g_x + pg_y + y''g_p\right)\frac{d\tau}{dt} \\ &= \frac{g_x + pg_y + y''g_p}{\varphi_x + p\varphi_y + y''\varphi_p} \\ &= \frac{Dg}{D\varphi} \\ &= g_1(x, y, p, y''). \end{aligned}$$

Similarly, one finds

$$u'''(t) = \frac{d^3u}{dt^3} = \frac{Dg_1}{D\varphi} = g_2(x, y, p, y'', y'''),$$

$$u^{(4)}(t) = \frac{d^4u}{dt^4} = \frac{Dg_2}{D\varphi} = g_3(x, y, p, y'', y''', y^{(4)}).$$

In general, one can write

$$u^{(k+1)}(t) = \frac{d^{(k+1)}u}{dt^{(k+1)}} = \frac{Dg_{(k-1)}}{D\varphi} = g_k(x, y, p, y'', y''', ..., y^{(k+1)}), \quad (k = 1, 2, ...).$$

Notice that $g_0 = g$.

2.4 Equivalent Equations

Definition 2.4. Two equations are called *equivalent* if there is an invertible transformation which transforms one equation into another.

Definition 2.5. The problem of finding all equations which are equivalent to a given equation is called an *equivalence problem.* If the given equation is a linear equation, then the equivalence problem is called a *linearization problem.*

Since all considerations in this thesis are local, we mean local equivalence here.

2.4.1 Linear kth-order Equations

The following properties are well known for point transformations.

- **First-order Equations**

All first-order equations are equivalent to another. In particular, an equation of first-order can be transformed into the simplest one, viz., $y' = 0$.

- **Linear Second-order Equations**

All linear second-order equations are equivalent to another and can, for example, be reduced to the simplest equation $y'' = 0$[‡].

However, a linear equation of order $k \geq 3$ need not be transformable into the simplest form.

- **Laguerre Canonical Form**

The general form of a linear kth-order ordinary differential equation is

$$y^{(k)} + \sum_{i=0}^{k-1} a_i\left(x\right) y^{(i)} = 0.$$

[‡]See detail in Appendix C.

Theorem 2.1. *(Laguerre[§]). Any linear kth-order ordinary differential equation*

$$y^{(k)} + \sum_{i=0}^{k-1} a_i(x)\, y^{(i)} = 0, \quad k \geqslant 3 \tag{2.11}$$

can be transformed by a point transformation to an equation of the form

$$y^{(k)} + \sum_{i=0}^{k-3} a_i(x)\, y^{(i)} = 0. \tag{2.12}$$

Notice that equation (2.12) is called the *Laguerre canonical form* of the linear kth-order ordinary differential equation (2.11).

2.5 The Lie Linearization Test

Since the method used in the thesis is similar to the Lie method, let us consider it in details.

The simplest linear form of a second-order ordinary differential equation with the independent variable t and the dependent variable u is

$$u'' = 0. \tag{2.13}$$

Lie showed that any second-order ordinary differential equation $y'' = f(x, y, y')$ obtained from linear equation (2.13) by a change of the independent and dependent variables,

$$t = \varphi(x, y), \quad u = \psi(x, y), \tag{2.14}$$

is cubic in the first-order derivative:

$$y'' + a(x, y)\, y'^3 + b(x, y)\, y'^2 + c(x, y)\, y' + d(x, y) = 0, \tag{2.15}$$

[§]See proof in Appendix E.

where

$$
\begin{aligned}
a &= \Delta^{-1}\left(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y\right),\\
b &= \Delta^{-1}\left(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\left(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y\right)\right),\\
c &= \Delta^{-1}\left(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2\left(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x\right)\right),\\
d &= \Delta^{-1}\left(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x\right).
\end{aligned}
\tag{2.16}
$$

Here the Jacobian of the change of variables is

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0.$$

Moreover, a second-order ordinary differential equation is linearizable if and only if it has the form (2.15) with the coefficients satisfying the conditions

$$
\begin{aligned}
&3a_{xx} - 2b_{xy} + c_{yy} - 3a_xc + 3a_yd + 2b_xb - 3c_xa - c_yb + 6d_ya = 0,\\
&b_{xx} - 2c_{xy} + 3d_{yy} - 6a_xd + b_xc + 3b_yd - 2c_ya - 3d_xa + 3d_yb = 0.
\end{aligned}
\tag{2.17}
$$

The mapping of equation (2.15) into linear equation (2.13) is reconstituting by finding the functions $\varphi(x, y)$ and $\psi(x, y)$ that satisfy the relations (2.16)¶.

2.6 Theory of Compatibility

There are two approaches for studying compatibility. These approaches are related to the works of E. Cartan and C. H. Riquier.

The Cartan approach is based on the calculus of exterior differential forms. The problem of the compatibility of a system of partial differential equations is then reduced to the problem of the compatibility of a system of exterior differential forms. Cartan studied the formal algebraic properties of systems of exterior forms. For their description he introduced special integer numbers, named characters. With the help of the characters he formulated a criterion for a given system of partial differential equations to be involutive.

¶See compatibility analyze of the system of equations for functions φ and ψ in Appendix F.

The Riquier approach has a different theory of establishing the involution. This method can be found in (Kuranashi, 1967) and (Pommaret, 1978). The main advantage of this approach is that there is no necessity to reduce the system of partial differential equations being studied to exterior differential forms. The calculations in the Riquier approach are shorter than in the Cartan approach. The main operations of the study of compatibility in the Riquier approach are prolongations of a system of partial differential equations and the study of the ranks of some matrices. In this thesis the Riquier approach is used.

2.6.1 Completely Integrable Systems

One class of overdetermined systems, for which the problem of compatibility is solved, is the class of completely integrable systems. The theory of completely integrable systems is developed in the general case.

Definition 2.6. A system

$$\frac{\partial z^i}{\partial a^j} = f^i_j(a,z),\ (i = 1,2,...,N;\ j = 1,2,...,r) \tag{2.18}$$

is called *completely integrable* if it has a solution for any initial values a_0, z_0 in some open domain D.

Theorem 2.2. *A system of the type (2.18) is completely integrable if and only if all of the mixed derivatives equalities*

$$\frac{\partial f^i_j}{\partial a^\beta} + \sum_{\gamma=1}^{N} f^\gamma_\beta \frac{\partial f^i_j}{\partial z^\gamma} = \frac{\partial f^i_\beta}{\partial a^j} + \sum_{\gamma=1}^{N} f^\gamma_j \frac{\partial f^i_\beta}{\partial z^\gamma},\ (i = 1,2,...,N;\ \beta, j = 1,2,...,r) \tag{2.19}$$

are identically satisfied with respect to the variables $(a,z) \in D$.

In practice, sometimes it is enough to use a particular case of the compatibility theorem:

Corollary 2.3. *If in an overdetermined system of partial differential equations all derivatives of order* n *are defined and comparison of all mixed derivatives of order* $n+1$ *does not produce new equations of order less or equal to* n*, then this system is compatible.*

CHAPTER III

LINEARIZATION OF FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY POINT TRANSFORMATIONS

Our starting point is a fourth-order ordinary differential equation

$$y^{(4)} = f(x, y, y', y'', y'''), \tag{3.1}$$

for a real function $y = y(x)$. Here $f = f(x, y, y', y'', y''')$ is a sufficiently many times continuously differentiable function of real variables (x, y, y', y'', y'''). This chapter is devoted to studying the linearization problem of equation (3.1) which is to find an invertible change of independent and dependent variables

$$t = \varphi(x, y), \quad u = \psi(x, y) \tag{3.2}$$

mapping the nonlinear equation (3.1) into a linear equation.

In 1879, E. Laguerre showed that in any linear ordinary differential equation the two terms of orders next below the highest can be simultaneously removed by an equivalence transformation*. Therefore, the general linear ith-order ordinary differential equation in Laguerre's form is

$$u^{(i)} + \alpha_{i-3}(t)u^{(i-3)} + ... + \alpha_0(t)u = 0, \tag{3.3}$$

where t and u are the independent and dependent variables, respectively.

*See detail in section 2.5.

3.1 Necessary Conditions for Linearization

We begin with investigating the necessary conditions for linearization. We consider an ith-order ordinary differential equation

$$y^{(i)} = f\left(x, y, y', y'', ..., y^{(i-1)}\right). \tag{3.4}$$

The general form of equation (3.4) that can be obtained from a linear ordinary differential equation by any point transformation (3.2) is found in this step. Necessary conditions for a linearizable fourth-order ordinary differential equation are studied here in more details.

3.1.1 Necessary Form of a Linearizable ith-order ODE

Applying a point transformation (3.2), the derivatives are changed as follows

$$\frac{du}{dt} = \psi_1 = \frac{D\psi}{D\varphi}, \quad \frac{d^2u}{dt^2} = \psi_2 = \frac{D\psi_1}{D\varphi} = \frac{D^2\psi\, D\varphi - D^2\varphi\, D\psi}{(D\varphi)^3},$$

$$\frac{d^{k+1}u}{dt^{k+1}} = \psi_{k+1} = \frac{D\psi_k}{D\varphi}, \ (k > 1),$$

where

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + y^{(4)}\frac{\partial}{\partial y'''} + \cdots$$

is the operator of total derivative with respect to x. Notice that

$$D^k F = y^{(k)}F_y + ky^{(k-1)}DF_y + h_F(x, y, y', ..., y^{(k-3)}, y^{(k-2)}), \ (k > 2),$$

$$\psi_i = \frac{1}{(D\varphi)^i}[D^i\psi - \frac{i(i-1)}{2}(D^2\varphi)(D\varphi)^{i-2}\psi_{i-1} - i(D^{i-1}\varphi)(D\varphi)\psi_2 - (D^i\varphi)\psi_1] + ...,$$

$$\psi_{i-1} = \frac{\Delta}{(D\varphi)^{i+1}}y^{(i-1)} + ...,$$

where $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ is the Jacobian of the change of variables (3.2), $F = F(x, y)$ is an arbitrary function, and $i > 3$. Here ... means terms with derivatives of order less than $i-1$. Hence,

$$(D\varphi)^i\psi_i = \quad y^{(i)}\tfrac{\Delta}{D\varphi} + iy^{(i-1)}[D\psi_y - \tfrac{D\psi}{D\varphi}D\varphi_y - \tfrac{(i-1)}{2}(D^2\varphi)\tfrac{\Delta}{(D\varphi)^2} - \varphi_y\tfrac{D^2\psi\, D\varphi - D^2\varphi\, D\psi}{(D\varphi)^2}] + \cdots . \tag{3.5}$$

Calculations show that in the right hand side of equation (3.5) the term with the derivative $y^{(i-1)}$ is

$$\begin{aligned}&(D\varphi)^2[D\psi_y - \tfrac{D\psi}{D\varphi}D\varphi_y - \tfrac{(i-1)}{2}(D^2\varphi)\tfrac{\Delta}{(D\varphi)^2} - \varphi_y\tfrac{D^2\psi\, D\varphi - D^2\varphi\, D\psi}{(D\varphi)^2}]\\ &= -y''\varphi_y\tfrac{(i+1)\Delta}{2} + y'^{\,2}(\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\tfrac{i\Delta}{2} - \varphi_{yy}\tfrac{(\varphi_x\psi_y+\varphi_y\psi_x)}{2} - \psi_{xy}\varphi_y^2 + \psi_{yy}\varphi_x\varphi_y)\\ &\quad +y'(-\varphi_{xy}i\Delta + \varphi_{xx}\varphi_y\psi_y - \varphi_{yy}\varphi_x\psi_x - \psi_{xx}\varphi_y^2 + \psi_{yy}\varphi_x^2)\\ &\quad -\varphi_{xy}\varphi_x\psi_x - \varphi_{xx}\tfrac{i\Delta}{2} + \varphi_{xx}\tfrac{(\varphi_x\psi_y+\varphi_y\psi_x)}{2} + \psi_{xy}\varphi_x^2 - \psi_{xx}\varphi_x\varphi_y.\end{aligned}$$

Substituting the resulting expression into the linear equation (3.3), the necessary form of a linearizable ordinary differential equation of ith-order is

$$\begin{aligned}&y^{(i)} + iy^{(i-1)}\tfrac{1}{\Delta D\varphi}[-y''\varphi_y\tfrac{(i+1)\Delta}{2}\\ &+y'^{\,2}(\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\tfrac{i\Delta}{2} - \varphi_{yy}\tfrac{(\varphi_x\psi_y+\varphi_y\psi_x)}{2} - \psi_{xy}\varphi_y^2 + \psi_{yy}\varphi_x\varphi_y)\\ &+y'(-\varphi_{xy}i\Delta + \varphi_{xx}\varphi_y\psi_y - \varphi_{yy}\varphi_x\psi_x - \psi_{xx}\varphi_y^2 + \psi_{yy}\varphi_x^2)\\ &-\varphi_{xy}\varphi_x\psi_x - \varphi_{xx}\tfrac{i\Delta}{2} + \varphi_{xx}\tfrac{(\varphi_x\psi_y+\varphi_y\psi_x)}{2} + \psi_{xy}\varphi_x^2 - \psi_{xx}\varphi_x\varphi_y] + ... = 0.\end{aligned}$$

From this representation we can conclude that for the linearization problem one needs to study two cases: (a) $\varphi_y = 0$, and (b) $\varphi_y \neq 0$. This corresponds to the following two necessary forms of linearizable ordinary differential equations:

$$y^{(i)} + y^{(i-1)}[A_1y' + A_0] + ... = 0, \tag{3.6}$$

and

$$y^{(i)} + y^{(i-1)}\frac{1}{y'+r}[-y''\frac{i(i+1)}{2} + F_2y'^{\,2} + F_1y' + F_0] + ... = 0, \tag{3.7}$$

where $F_j = F_j(x,y),\ A_j = A_j(x,y)$. If $\varphi_y = 0$, in literature this class of transformations is called a *fiber preserving transformations*.

Theorem 3.1. *Any linearizable ith-order ($i \geq 4$) ordinary differential equation has to be either of the form (3.6) or of (3.7).*

3.1.2 Necessary Form of a Linearizable Fourth-order ODE

As was obtained in the previous section, the transformations (3.2) with $\varphi_y = 0$ and $\varphi_y \neq 0$, respectively, provide two distinctly different classes of linearizable equations.

If $\varphi_y = 0$, working out the missing terms in equation (3.6), are substituting the resulting expression into the linear equation

$$u^{(4)} + \alpha(t)u' + \beta(t)u = 0 \tag{3.8}$$

we obtain the following first class for linearization

$$\begin{aligned} y^{(4)} \quad & +(A_1y' + A_0)y''' + B_0y''^2 + (C_2y'^2 + C_1y' + C_0)y'' \\ & +D_4y'^4 + D_3y'^3 + D_2y'^2 + D_1y' + D_0 = 0, \end{aligned} \tag{3.9}$$

where $A_j = A_j(x,y)$, $B_j = B_j(x,y)$, $C_j = C_j(x,y)$ and $D_j = D_j(x,y)$ are arbitrary functions of x, y.

If $\varphi_y \neq 0$, we proceed likewise and setting $r = \frac{\varphi_x}{\varphi_y}$, arrive at the second class for linearization

$$\begin{aligned} y^{(4)} \quad & +\tfrac{1}{y'+r}(-10y'' + F_2y'^2 + F_1y' + F_0)y''' \\ & +\tfrac{1}{(y'+r)^2}\,[15y''^3 + (H_2y'^2 + H_1y' + H_0)y''^2 \\ & +(J_4y'^4 + J_3y'^3 + J_2y'^2 + J_1y' + J_0)y'' \\ & +K_7y'^7 + K_6y'^6 + K_5y'^5 + K_4y'^4 \\ & +K_3y'^3 + K_2y'^2 + K_1y' + K_0] = 0, \end{aligned} \tag{3.10}$$

where $r = r(x,y), F_j = F_j(x,y)$, $H_j = H_j(x,y)$, $J_j = J_j(x,y)$ and $K_j = K_j(x,y)$ are arbitrary functions of x, y.

Thus, we have shown that every linearizable fourth-order ordinary differential equations belongs either to the class of equations (3.9) or to the class of equations (3.10).

3.2 The First Class of Linearizable Equations

3.2.1 The Linearization Test for Equation (3.9)

In this case, the linearizing transformation (3.2) must be a fiber preserving transformation, i.e., it has the form

$$t = \varphi(x), \quad u = \psi(x, y). \tag{3.11}$$

Theorem 3.2. *Equation (3.9) is linearizable if and only if its coefficients obey the following conditions:*

$$A_{0y} - A_{1x} = 0, \tag{3.12}$$

$$4B_0 - 3A_1 = 0, \tag{3.13}$$

$$12A_{1y} + 3A_1^2 - 8C_2 = 0, \tag{3.14}$$

$$12A_{1x} + 3A_0A_1 - 4C_1 = 0, \tag{3.15}$$

$$32C_{0y} + 12A_{0x}A_1 - 16C_{1x} + 3A_0^2A_1 - 4A_0C_1 = 0, \tag{3.16}$$

$$4C_{2y} + A_1C_2 - 24D_4 = 0, \tag{3.17}$$

$$4C_{1y} + A_1C_1 - 12D_3 = 0, \tag{3.18}$$

$$16C_{1x} - 12A_{0x}A_1 - 3A_0^2A_1 + 4A_0C_1 + 8A_1C_0 - 32D_2 = 0, \tag{3.19}$$

$$\begin{aligned} &192D_{2x} + 36A_{0x}A_0A_1 - 48A_{0x}C_1 - 48C_{0x}A_1 - 288D_{1y} + 9A_0^3A_1 \\ &- 12A_0^2C_1 - 36A_0A_1C_0 + 48A_0D_2 + 32C_0C_1 = 0, \end{aligned} \tag{3.20}$$

$$384D_{1xy} - \Big[3((3A_0A_1 - 4C_1)A_0^2 + 16(2A_1D_1 + C_0C_1)$$
$$- 16(A_1C_0 - D_2)A_0)A_0 - 32(4(C_1D_1 - 2C_2D_0 + C_0D_2)$$
$$+ (3A_1D_0 - C_0^2)A_1) - 96D_{1y}A_0 + 384D_{0y}A_1 + 1536D_{0yy}$$
$$- 16(3A_0A_1 - 4C_1)C_{0x} + 12((3A_0A_1 - 4C_1)A_0$$
$$- 4(A_1C_0 - 4D_2))A_{0x}\Big] = 0. \qquad (3.21)$$

Theorem 3.3. *Provided that the conditions (3.12)-(3.21) are satisfied, the linearizing transformation (3.11) is defined by a fourth-order ordinary differential equation for the function $\varphi(x)$, namely by the Riccati equation*

$$40\frac{d\chi}{dx} - 20\chi^2 = 8C_0 - 3A_0^2 - 12A_{0x}, \qquad (3.22)$$

for

$$\chi = \frac{\varphi_{xx}}{\varphi_x}, \qquad (3.23)$$

and by the following integrable system of partial differential equations for the function $\psi(x,y)$

$$4\psi_{yy} = \psi_y A_1, \quad 4\psi_{xy} = \psi_y(A_0 + 6\chi), \qquad (3.24)$$

and

$$1600\psi_{xxxx} = 9600\psi_{xxx}\chi + 160\psi_{xx}(-12A_{0x} - 3A_0^2 - 90\chi^2 + 8C_0)$$
$$+ 40\psi_x(12A_{0x}A_0 + 72A_{0x}\chi - 16C_{0x} + 3A_0^3 + 18A_0^2\chi - 12A_0C_0$$
$$+ 120\chi^3 - 48\chi C_0 + 24D_1 - 8\Omega) + \psi(144A_{0x}^2 + 72A_{0x}A_0^2 - 352A_{0x}C_0$$
$$- 160C_{0xx} - 80C_{0x}A_0 - 1600D_{0y} + 640D_{1x} - 80\Omega_x + 9A_0^4 - 88A_0^2C_0$$
$$+ 160A_0D_1 + 30A_0\Omega - 400A_1D_0 + 300\chi\Omega + 144C_0^2) + 1600\psi_yD_0, \qquad (3.25)$$

where χ is given by equation (3.23) and Ω is the following expression

$$\Omega = A_0^3 - 4A_0C_0 + 8D_1 - 8C_{0x} + 6A_{0x}A_0 + 4A_{0xx}. \qquad (3.26)$$

Finally, the coefficients α and β of the resulting linear equation (3.8) are

$$\alpha = \frac{\Omega}{8\varphi_x^3}\ , \tag{3.27}$$

$$\beta = (1600\varphi_x^4)^{-1}(-144A_{0x}^2 - 72A_{0x}A_0^2 + 352A_{0x}C_0 + 160C_{0xx} + 80C_{0x}A_0$$
$$+ 1600D_{0y} - 640D_{1x} + 80\Omega_x - 9A_0^4 + 88A_0^2C_0 - 160A_0D_1 - 30A_0\Omega$$
$$+ 400A_1D_0 - 300\chi\Omega - 144C_0^2). \tag{3.28}$$

Remark 3.1. Since the system of equations (3.12)-(3.21) provides the necessary and sufficient conditions for linearization, it is invariant with respect to transformations (3.11). It means that the left-hand sides of equations (3.12)-(3.21) are relative invariants of second-order for the equivalence transformations defined by (3.11).

3.2.2 Relations Between Coefficients and Transformations

For proving the linearization theorems we need relations between the functions $\varphi(x), \psi(x,y)$ and the coefficients of equation (3.9).

Lemma 3.4. *The coefficients of equation (3.9) and the functions $\varphi(x)$ and $\psi(x,y)$ in the transformation (3.11) are related by the following equations:*

$$A_1 = 4(\psi_y)^{-1}\psi_{yy}, \tag{3.29}$$

$$A_0 = -2(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy}), \tag{3.30}$$

$$B_0 = 3(\psi_y)^{-1}\psi_{yy}, \tag{3.31}$$

$$C_2 = 6(\psi_y)^{-1}\psi_{yyy}, \tag{3.32}$$

$$C_1 = -6(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_{yy} - 2\varphi_x\psi_{xyy}), \tag{3.33}$$

$$C_0 = -(\varphi_x^2\psi_y)^{-1}\Big[(4\varphi_{xxx}\varphi_x - 15\varphi_{xx}^2)\psi_y + 6(3\varphi_{xx}\psi_{xy} - \varphi_x\psi_{xxy})\varphi_x\Big], \tag{3.34}$$

$$D_4 =(\psi_y)^{-1}\psi_{yyyy}, \tag{3.35}$$

$$D_3 = -2(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_{yyy} - 2\varphi_x\psi_{xyyy}), \tag{3.36}$$

$$D_2 = -(\varphi_x^2\psi_y)^{-1}(4\varphi_{xxx}\varphi_x\psi_{yy} - 15\varphi_{xx}^2\psi_{yy} + 18\varphi_{xx}\varphi_x\psi_{xyy} - 6\varphi_x^2\psi_{xxyy}), \tag{3.37}$$

$$\begin{aligned} D_1 = &-(\varphi_x^3\psi_y)^{-1}\Big[3(5\varphi_{xx}^2\psi_y - 10\varphi_{xx}\varphi_x\psi_{xy} + 6\varphi_x^2\psi_{xxy})\varphi_{xx} \\ &- (\varphi_x^3\psi_y\alpha + 4\psi_{xxxy})\varphi_x^3 - 2(5\varphi_{xx}\psi_y - 4\varphi_x\psi_{xy})\varphi_{xxx}\varphi_x + \varphi_{xxxx}\varphi_x^2\psi_y\Big], \end{aligned} \tag{3.38}$$

$$\begin{aligned} D_0 = &-(\varphi_x^3\psi_y)^{-1}\Big[(15\varphi_{xx}^3 - \varphi_x^6\alpha + \varphi_{xxxx}\varphi_x^2)\psi_x - (10\varphi_{xxx}\varphi_{xx}\psi_x \\ &- 4\varphi_{xxx}\varphi_x\psi_{xx} + 15\varphi_{xx}^2\psi_{xx} - 6\varphi_{xx}\varphi_x\psi_{xxx} + \varphi_x^6\beta\psi + \varphi_x^2\psi_{xxxx})\varphi_x\Big]. \end{aligned} \tag{3.39}$$

3.2.3 Proof of the Linearization Theorems

The proof of the linearization theorems requires the study of integrability conditions for the unknown functions $\varphi(x)$ and $\psi(x, y)$. The functions $\varphi(x)$ and $\psi(x, y)$ satisfy equations (3.29)-(3.39) with given coefficients $A_i(x, y), B_i(x, y), C_i(x, y)$ and $D_i(x, y)$.

We first rewrite the expressions (3.29) and (3.30) for A_1 and A_0 in the following forms

$$\psi_{yy} = \frac{\psi_y A_1}{4}, \quad \psi_{xy} = \frac{(6\varphi_{xx} + \varphi_x A_0)}{4\varphi_x}\psi_y. \tag{3.40}$$

Comparing the mixed derivative $(\psi_{yy})_x = (\psi_{xy})_y$, one arrives at equation (3.12). Then equations (3.31), (3.32) and (3.33) become equations (3.13), (3.14) and (3.15), respectively. Furthermore, equation (3.34) gives

$$\varphi_{xxx} = -\frac{(12A_{0x}\varphi_x^2 - 60\varphi_{xx}^2 + 3\varphi_x^2A_0^2 - 8\varphi_x^2C_0)}{40\varphi_x}. \tag{3.41}$$

Differentiation of equation (3.41) with respect to y yields equation (3.16). Equations (3.35), (3.36) and (3.37) become in the form of equations (3.17), (3.18) and (3.19), respectively.

One can determine α from equation (3.38):

$$\alpha = \frac{4A_{0xx} + 6A_{0x}A_0 - 8C_{0x} + A_0^3 - 4A_0C_0 + 8D_1}{8\varphi_x^3}. \tag{3.42}$$

Since $\varphi = \varphi(x)$, then $\alpha_y = 0$, which yields equation (3.20). From equation (3.39) one finds

$$\begin{aligned} \psi_{xxxx} = \; & -\tfrac{1}{40\varphi_x^3}\Big[32A_{0xx}\varphi_x^3\psi_x - 72A_{0x}\varphi_{xx}\varphi_x^2\psi_x + 48A_{0x}\varphi_x^3\psi_{xx} \\ & +36A_{0x}\varphi_x^3\psi_x A_0 - 48C_{0x}\varphi_x^3\psi_x - 120\varphi_{xx}^3\psi_x + 360\varphi_{xx}^2\varphi_x\psi_{xx} \\ & -240\varphi_{xx}\varphi_x^2\psi_{xxx} - 18\varphi_{xx}\varphi_x^2\psi_x A_0^2 + 48\varphi_{xx}\varphi_x^2\psi_x C_0 + 40\varphi_x^7\beta\psi \\ & +12\varphi_x^3\psi_{xx}A_0^2 - 32\varphi_x^3\psi_{xx}C_0 + 5\varphi_x^3\psi_x A_0^3 - 20\varphi_x^3\psi_x A_0 C_0 \\ & +40\varphi_x^3\psi_x D_1 - 40\varphi_x^3\psi_y D_0\Big]. \end{aligned} \tag{3.43}$$

Forming the mixed derivative $(\psi_{xxxx})_y = (\psi_{xy})_{xxx}$ one obtains

$$\begin{aligned} \beta = \; & \tfrac{1}{1600\varphi_x^5}\Big[320A_{0xxx}\varphi_x - 1200A_{0xx}\varphi_{xx} + 360A_{0xx}\varphi_x A_0 + 336A_{0x}^2\varphi_x \\ & -1800A_{0x}\varphi_{xx}A_0 - 12A_{0x}\varphi_x A_0^2 + 32A_{0x}\varphi_x C_0 - 480C_{0xx}\varphi_x \\ & +2400C_{0x}\varphi_{xx} + 1600D_{0y}\varphi_x - 300\varphi_{xx}A_0^3 + 1200\varphi_{xx}A_0C_0 \\ & -2400\varphi_{xx}D_1 - 39\varphi_x A_0^4 + 208\varphi_x A_0^2 C_0 - 400\varphi_x A_0 D_1 \\ & +400\varphi_x A_1 D_0 - 144\varphi_x C_0^2\Big]. \end{aligned} \tag{3.44}$$

Differentiation of β with respect to y yields equation (3.21).

From equation (3.41) one can rewrite the representation for C_0. Denoting $\chi = \frac{\varphi_{xx}}{\varphi_x}$ leads to equation (3.22) and the representations for ψ_{yy} and ψ_{xy} in the equations (3.40) become equations (3.24). Rewriting the representation for α from equation (3.42) in the following form

$$\alpha = \frac{\Omega}{8\varphi_x^3},$$

where

$$\Omega = A_0^3 - 4A_0C_0 + 8D_1 - 8C_{0x} + 6A_{0x}A_0 + 4A_{0xx},$$

then β of equation (3.44) becomes

$$\beta = (1600\varphi_x^4)^{-1}(-144A_{0x}^2 - 72A_{0x}A_0^2 + 352A_{0x}C_0 + 160C_{0xx} + 80C_{0x}A_0$$

$$+ 1600D_{0y} - 640D_{1x} + 80\Omega_x - 9A_0^4 + 88A_0^2C_0 - 160A_0D_1 - 30A_0\Omega$$

$$+ 400A_1D_0 - 300\chi\Omega - 144C_0^2).$$

Finally, equation (3.43) becomes equation (3.25). This completes the proof of the theorems.

3.2.4 Illustration of the Linearization Theorems

Example 3.1. Consider the nonlinear ordinary differential equation

$$x^2y(2y^{(4)} + y) + 8x^2y'y''' + 16xyy''' + 6x^2y''^2 + 48xy'y'' + 24yy'' + 24y'^2 = 0. \quad (3.45)$$

It is an equation of the form (3.9) with coefficients

$$\begin{gathered} A_1 = \frac{4}{y}\,,\ A_0 = \frac{8}{x}\,,\ B_0 = \frac{3}{y}\,,\ C_2 = 0\,,\ C_1 = \frac{24}{xy}\,,\ C_0 = \frac{12}{x^2}\,, \\ D_4 = 0\,,\ D_3 = 0\,,\ D_2 = \frac{12}{x^2y}\,,\ D_1 = 0\,,\ D_0 = \frac{y}{2}\,\cdot \end{gathered} \quad (3.46)$$

One can check that the coefficients (3.46) obey the conditions (3.12)-(3.21). Thus, equation (3.45) is linearizable. We have

$$8C_0 - 3A_0^2 - 12A_{0x} = 0 \quad (3.47)$$

and the equation (3.22) is written as

$$2\frac{d\chi}{dx} - \chi^2 = 0.$$

Let us take its simplest solution $\chi = 0$. Then invoking equation (3.23), we let

$$\varphi = x.$$

Now equations (3.24) are rewritten as

$$\frac{\psi_{yy}}{\psi_y} = \frac{1}{y}, \quad \frac{\psi_{xy}}{\psi_y} = \frac{2}{x}$$

and yield

$$\psi_y = Kx^2y, \qquad K = const.$$

Hence,

$$\psi = K\frac{x^2y^2}{2} + f(x).$$

Since one can use any particular solution, we set $K = 2$, $f(x) = 0$ and take

$$\psi = x^2y^2.$$

Invoking equation (3.47) and noting that equation (3.26) yields $\Omega = 0$, one can readily verify that the function $\psi = x^2y^2$ solves equation (3.25) as well. Hence, one obtains the following transformations

$$t = x, \quad u = x^2y^2. \tag{3.48}$$

Since $\Omega = 0$, equations (3.27) and (3.28) give

$$\alpha = 0, \quad \beta = \frac{1}{\varphi_x^4} = 1.$$

Hence, the equation (3.45) is mapped by the transformation (3.48) to the linear equation

$$u^{(4)} + u = 0.$$

Example 3.2. The third-order member of the Riccati Hierarchy is given by Euler, Euler and Leach (2007) as

$$y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 = 0. \tag{3.49}$$

Applying (Ibragimov and Meleshko, 2005) and (Euler et al., (2003)) one checks that this equation cannot be linearized by a point transformation or contact transformation or generalized Sundman transformation. Under the Riccati transformation $y = \frac{a\omega'}{\omega}$ the equation (3.49) becomes (Andriopoulos and Leach, 2007)

$$\begin{aligned} &\omega^3\omega^{(4)} + 4(a-1)\omega^2\omega'\omega''' + 3(a-1)\omega^2\omega''^2 \\ &+6(a-1)(a-2)\omega\omega'^2\omega'' + (a-1)(a-2)(a-3)\omega'^4 = 0. \end{aligned} \tag{3.50}$$

It is an equation of the form (3.9) with the coefficients

$$\begin{gathered} A_1 = \tfrac{4(a-1)}{\omega}\,,\ A_0 = 0\,,\ B_0 = \tfrac{3(a-1)}{\omega}\,, \\ C_2 = \tfrac{6(a^2-3a+2)}{\omega^2}\,,\ C_1 = 0\,,\ C_0 = 0\,, \\ D_4 = \tfrac{a^3-6a^2+11a-6}{\omega^3}\,,\ D_3 = 0\,,\ D_2 = 0\,,\ D_1 = 0\,,\ D_0 = 0. \end{gathered} \tag{3.51}$$

One can verify that the coefficients (3.51) obey the linearization conditions (3.12)-(3.21). Furthermore,

$$8C_0 - 3A_0^2 - 12A_{0x} = 0 \tag{3.52}$$

and the equation (3.22) is written as

$$2\frac{d\chi}{dx} - \chi^2 = 0.$$

We take its simplest solution $\chi = 0$ and obtain from equation (3.23) the equation $\varphi'' = 0$, whence

$$\varphi = x.$$

Equations (3.24) have the form

$$\frac{\psi_{\omega\omega}}{\psi_\omega} = \frac{a-1}{\omega}, \qquad \psi_{x\omega} = 0$$

and yield

$$\psi_\omega = K\omega^{(a-1)}, \qquad K = const.$$

Hence

$$\psi = K\frac{\omega^a}{a} + f\,(x)\,.$$

Since one can use any particular solution, we set $K = a,\ \ f\,(x) = 0$ and take

$$\psi = \omega^a.$$

Invoking equation (3.52) and noting that equation (3.26) yields $\Omega = 0$, one can readily verifies that the function $\psi = \omega^a$ solves equation (3.25) as well. One obtains the following transformation

$$t = x, \quad u = \omega^a. \tag{3.53}$$

Since $\Omega = 0$, equations (3.27) and (3.28) give

$$\alpha = 0, \quad \beta = 0.$$

Thus, the equation (3.50) is mapped by the transformation (3.53) to the linear equation

$$u^{(4)} = 0.$$

Furthermore, one can transform the solution $u(t)$ to the Riccati substitution $y = \frac{a\omega'}{\omega}$. The solution of the linear equation is

$$u(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3,$$

where $c_i, (i = 0, 1, 2, 3)$ are arbitrary constants. By using the transformation (3.53), one finds

$$\omega^a = c_0 + c_1 x + c_2 x^2 + c_3 x^3.$$

Hence,

$$\begin{aligned} y &= (\ln \omega^a)' \\ &= \left(\ln\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3\right)\right)' \\ &= \frac{c_1 + 2c_2 x + 3c_3 x^2}{c_0 + c_1 x + c_2 x^2 + c_3 x^3}. \end{aligned}$$

This example shows that as for second-order ordinary differential equations (Ibragimov and Meleshko, 2007)[†] the Riccati substitution can map a third-order ordinary differential equation into a linearizable fourth-order ordinary differential equation. Using the criteria of linearization obtained in this thesis, one can obtain complete criteria for third-order ordinary differential equations linearizable by Riccati substitution.

[†]In (Ibragimov and Meleshko, 2007) the complete study of second-order ordinary differential equations linearizable by the Riccati substitution is presented.

3.2.5 Linearization of Traveling Waves of PDEs

Solutions of many partial differential equations were obtained by assuming that a solution is a traveling wave type.

- **One Class of Fourth-order Partial Differential Equations**

Let us consider the nonlinear fourth-order partial differential equation (Clarkson and Priestley, 1999)

$$u_{tt} = \left(\kappa u + \gamma u^2\right)_{xx} + \nu u u_{xxxx} + \mu u_{xxtt} + \alpha u_x u_{xxx} + \beta u_{xx}^2, \tag{3.54}$$

where $\alpha, \beta, \gamma, \mu, \nu$ and κ are arbitrary constants. This equation may be thought of as a fourth-order analogue of a generalization of the Camassa-Holm equation, about which there has been considerable recent interest. Furthermore, equation (3.54) is a Boussinesq-type equation which arises as a model of vibrations of harmonic mass-spring chain and admits both compacton and conventional solitons.

Of particular interest among solutions of equation (3.54) are traveling wave solutions:

$$u(x,t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave. Substituting the representation of a solution into equation (3.54), one finds

$$(\nu H + \mu D^2)H^{(4)} + \alpha H' H''' + \beta H''^2 + (2\gamma H + \kappa - D^2)H'' + 2\gamma H'^2 = 0. \tag{3.55}$$

This is an equation of the form (3.9) with coefficients

$$A_1 = \frac{\alpha}{\nu H + \mu D^2}, \quad A_0 = 0, \quad B_0 = \frac{\beta}{\nu H + \mu D^2},$$

$$C_2 = C_1 = 0, \quad C_0 = \frac{2\gamma H + \kappa - D^2}{\nu H + \mu D^2},$$

$$D_4 = D_3 = 0, \quad D_2 = \frac{2\gamma}{\nu H + \mu D^2}, \quad D_1 = D_0 = 0.$$

It is assumed that $\nu \neq 0$ and $\gamma \neq 0$.

Equation (3.55) is linearizable if and only if

$$\alpha = 4\nu, \ \ \beta = 3\nu, \ \ \kappa = \frac{(2\gamma\mu + \nu)D^2}{\nu}.$$

- **The Shallow Water Wave Equation**

In this topic we discuss the generalized shallow water wave (GSWW) equation (Clarkson and Mansfield, 1994)

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \tag{3.56}$$

where α and β are arbitrary, non-zero, constants. This equation, together with several variants, can be derived from the classical shallow water theory in the so-called Boussinesq approximation (Whitham, 1998).

Substituting the traveling wave representation of a solution into equation (3.56), one gets

$$-DH^{(4)} - (D(\alpha + \beta)H' + (1 - D))H'' = 0. \tag{3.57}$$

It has the form of equation (3.9) with the following coefficients

$$A_1 = A_0 = B_0 = C_2 = 0, \ \ C_1 = \alpha + \beta, \ \ C_0 = \frac{1 - D}{D},$$

$$D_4 = D_3 = D_2 = D_1 = D_0 = 0.$$

Equation (3.57) is linearizable if and only if

$$\alpha = -\beta.$$

- **Boussinesq Equation**

Let us consider the Boussinesq equation

$$u_{tt} - uu_{xx} - {u_x}^2 - u_{xxxx} = 0. \tag{3.58}$$

Substituting the traveling wave representation of a solution into equation (3.58), one finds

$$H^{(4)} + (H - D^2)H'' + H'^2 = 0. \tag{3.59}$$

It is an equation of the form (3.9) with the coefficients

$$\begin{gathered} A_1 = 0,\ A_0 = 0,\ B_0 = 0,\ C_2 = 0,\ C_1 = 0,\ C_0 = -D^2 + H, \\ D_4 = 0,\ D_3 = 0,\ D_2 = 1,\ D_1 = 0,\ D_0 = 0. \end{gathered} \tag{3.60}$$

Since the coefficients (3.60) do not satisfy the linearization conditions (3.16), (3.19) and (3.21), hence, the equation (3.59) is not linearizable.

3.2.6 Application of the Linearization Theorems to a System of Two Second-order ODEs

In this section we give some necessary and sufficient conditions of linearization for a system of two second-order ordinary differential equations with two dependent variables y, z and one independent variable x of the form

$$y'' = f_1(x, y, y', z), \quad z'' = f_2(x, y, y', z). \tag{3.61}$$

Assuming that $f_{1z} \neq 0$, by virtue of the Inverse Function Theorem the first equation of (3.61) can be solved with respect to $z = g(x, y, y', y'')$. Substituting this into the second equation of (3.61), one obtains that system (3.61) is equivalent to the fourth-order ordinary differential equation

$$\begin{aligned} y^{(4)} g_{y''} & + y'''^2 g_{y''y''} + y'''(2g_{y'y''}y'' + 2g_{y''y}y' + 2g_{y''x} + g_{y'}) + g_{y'y'}y''^2 \\ & + (2g_{y'y}y' + 2g_{y'x} + g_y)y'' + g_{yy}y'^2 + 2g_{xy}y' + g_{xx} - f_2 = 0. \end{aligned} \tag{3.62}$$

Applying linearization theorems to equation (3.62) one can obtain conditions for the functions $f_2(x, y, y', z)$ and $g(x, y, y', z)$ which are necessary and sufficient for equation (3.62) to be linearizable. It is worth noting that, in general, these linearizing transformations, which are point transformations for equation (3.62), are not point transformations for system of equations (3.61).

Since one of the necessary conditions for linearization of equation (3.62) requires that this equation has to be a linear equation with respect to the third-order derivative y''', one obtains that $g_{y''y''} = 0$, i.e., $g = g_0 + g_1 y''$, where $g_i = g_i(x, y, y'), (i = 0, 1)$. Since $g_{y''} \neq 0$, the function $g_1 \neq 0$. Equation (3.62) becomes

$$\begin{aligned} y^{(4)} &+ \Big[(3g_{1y'}y'' + 2g_{1y}y' + g_{0y'} + 2g_{1x})y''' + g_{1y'y'}y''^3 \\ &+ (2g_{1y'y}y' + g_{0y'y'} + 2g_{1y'x} + g_{1y})y''^2 \\ &+ (g_{1yy}y'^2 + 2(g_{0y'y} + g_{1xy})y' + 2g_{0y'x} + g_{1xx} + g_{0y})y'' \\ &+ g_{0yy}y'^2 + 2g_{0xy}y' + g_{0xx} - f_2\Big]/g_1 = 0. \end{aligned}$$

Considering the coefficient related with the product $y''y'''$, for a linearizable equation one obtains either $g_{1y'} = 0$ or $3(y' + r)g_{1y'} + 10g_1 = 0$, where $r = r(x, y)$. In this section we study the case $g_{1y'} = 0$. Since the coefficients with the derivative y''' has to be linear with respect to the first-order derivative y', one obtains $g_{0y'y'y'} = 0$, that is

$$g_0 = g_{00} + g_{01}y' + g_{02}y'^2,$$

where $g_{0i} = g_{0i}(x, y), (i = 0, 1, 2)$. Hence, the coefficients A_1 and A_0 in equation (3.9) are

$$A_1 = 2(g_{1y} + g_{02})/g_1, \quad A_0 = (2g_{1x} + g_{01})/g_1.$$

Proceeding to compare coefficients of equation (3.62) with equation (3.9) we obtain that[‡]

$$\begin{aligned} f_2 = \; & f_{22}z^2 + (f_{210} + f_{211}y' + f_{212}y'^2)\, z \\ & + f_{200} + f_{201}y' + f_{202}y'^2 + f_{203}y'^3 + f_{204}y'^4, \end{aligned}$$

‡See more calculations in Appendix H.

where $f_{22} = f_{22}(x,y)$, $f_{21i} = f_{21i}(x,y), (i = 0,1,2)$, $f_{20i} = f_{20i}(x,y), (i = 0,1,2,3,4)$ and

$$
\begin{aligned}
B_0 &= (g_{1y} - f_{22}g_1^2 + 2g_{02})/g_1,\\
C_2 &= (5g_{02y} + g_{1yy} - f_{212}g_1 - 2f_{22}g_{02}g_1)/g_1,\\
C_1 &= (3g_{01y} + 4g_{02x} + 2g_{1xy} - f_{211}g_1 - 2f_{22}g_{01}g_1)/g_1,\\
C_0 &= (g_{00y} + 2g_{01x} + g_{1xx} - f_{210}g_1 - 2f_{22}g_{00}g_1)/g_1,\\
D_4 &= (g_{02yy} - f_{204} - f_{212}g_{02} - f_{22}g_{02}^2)/g_1,\\
D_3 &= (g_{01yy} + 2g_{02xy} - f_{203} - f_{211}g_{02} - f_{212}g_{01} - 2f_{22}g_{01}g_{02})/g_1,\\
D_2 &= (g_{00yy} + 2g_{01xy} + g_{02xx} - f_{202} - f_{210}g_{02} - f_{211}g_{01} - f_{212}g_{00}\\
&\quad -2f_{22}g_{00}g_{02} - f_{22}g_{01}^2)/g_1,\\
D_1 &= (2g_{00xy} + g_{01xx} - f_{201} - f_{210}g_{01} - f_{211}g_{00} - 2f_{22}g_{00}g_{01})/g_1,\\
D_0 &= (g_{00xx} - f_{200} - f_{210}g_{00} - f_{22}g_{00}^2)/g_1.
\end{aligned}
$$

For the sake of simplicity we present here the linearization conditions for the case $f_{22} = 0$. One can verify that in the case $f_{22} = 0$ the found coefficients A_i, B_i, C_i and D_i satisfy the linearization conditions (3.12)-(3.21) if and only if

$$g_{01y} = (2g_{02x}g_1 - 2g_{1x}g_{02} + g_{01}g_{02})/g_1, \tag{3.63}$$

$$g_{1y} = g_{02}, \tag{3.64}$$

$$
\begin{aligned}
g_{00yy} &= (f_{210y}g_1^3 + g_{00y}g_{02}g_1 + g_{02xx}g_1^2 - 2g_{02x}g_{1x}g_1 + g_{02x}g_{01}g_1\\
&\quad -g_{1xx}g_{02}g_1 + 2g_{1x}^2g_{02} - g_{1x}g_{01}g_{02})/g_1^2,
\end{aligned} \tag{3.65}
$$

$$f_{210y} = f_{202}/g_1, \tag{3.66}$$

$$f_{201y} = (2f_{202x}g_1 + f_{201}g_{02} - f_{202}g_{01})/g_1, \tag{3.67}$$

$$
\begin{aligned}
f_{200yy} &= (f_{200y}g_{02}g_1 + f_{202xx}g_1^2 - f_{202y}g_{00}g_1 - 2g_{00y}f_{202}g_1\\
&\quad +g_{02x}f_{201}g_1 - g_{1x}f_{201}g_{02} - f_{202}f_{210}g_1^2 + 2f_{202}g_{00}g_{02})/g_1^2
\end{aligned} \tag{3.68}
$$

and

$$f_{203} = f_{204} = f_{211} = f_{212} = 0. \tag{3.69}$$

One type of the functions $f_2(x, y, y', z)$ and $g(x, y, y', y'')$ satisfying conditions (3.63)-(3.69) is[§]

$$f_2 = z\mu_2 + \mu_3 H + \mu_5, \quad g = y'' H_y + y'^2 H_{yy} + 2y' H_{xy} + \mu_1 H + \mu_4,$$

where $\mu_i = \mu_i(x)$, $(i = 1, 2, 3, 4, 5)$ are arbitrary functions, and the function $H(x, y)$ satisfies the equation

$$\left(\left(\frac{H_{xy}}{H_y}\right)_x + \left(\frac{H_{xy}}{H_y}\right)^2\right)_y = 0. \tag{3.70}$$

System (3.61) corresponding to these functions is

$$y'' = z/H_y - (y'^2 H_{yy} + 2y' H_{xy} + \mu_4)/H - \mu_1, \quad z'' = z\mu_2 + \mu_3 H + \mu_5. \tag{3.71}$$

Hence, we can conclude that a system (3.61) is linearizable if it has the form (3.71) where the function H satisfies the equation (3.70).

3.3 The Second Class of Linearizable Equations

3.3.1 The Linearization Test for Equation (3.10)

The following theorems provide the test for linearization of the second class.

Theorem 3.5. *Equation (3.10) is linearizable if and only if its coefficients obey conditions:*[¶] *(G.1)-(G.18).*

The necessary and sufficient conditions comprise eighteen differential equations (G.1)-(G.18) for twenty one coefficients of equation (3.10).

Theorem 3.6. *Provided that the conditions (G.1)-(G.18) are satisfied, the transformation (3.2) mapping equation (3.10) into a linear equation (3.8) is obtained*

[§]See more calculations in Appendix H.

[¶]Since equations (G.1)-(G.18) are cumbersome, they are presented in Appendix G.

by solving the compatible system of equations for the functions $\varphi(x,y)$ and $\psi(x,y)$: (G.19)-(G.22)[‖]*. The coefficients α and β are given by equations (G.23) and (G.24).*

Remark 3.2. Equations (G.1)-(G.18) define eighteen relative invariants of third-order of point transformations (3.2).

3.3.2 Relations Between Coefficients and Transformations

Lemma 3.7. *The coefficients of equation (3.10) and the functions $\varphi(x,y)$ and $\psi(x,y)$ in the transformation (3.2) are related by equations* [**] *(G.26)-(G.44).*

3.3.3 Proof of the Linearization Theorems

The problem is: for the given coefficients $F_i(x,y), H_i(x,y), J_i(x,y), K_i(x,y)$ of equation (3.10), find the integrability conditions for the functions $\varphi(x,y)$ and $\psi(x,y)$.

Recall that, according to our notations, the following equations hold

$$\alpha_x = r\,\alpha_y, \quad \beta_x = r\,\beta_y$$

and

$$\varphi_x = r\varphi_y, \quad \psi_x = \frac{\psi_y\varphi_x - \Delta}{\varphi_y}. \tag{3.72}$$

From equations (G.26) and (G.27) one finds

$$\varphi_{yy} = \Big[(4\Delta_y - F_2\Delta)\varphi_y\Big]/(10\Delta), \tag{3.73}$$

$$\Delta_x = (20r_y\Delta + 4\Delta_y r + F_1\Delta - 2F_2 r\Delta)/4.$$

[‖]Equations (G.19)-(G.22) and (G.23)-(G.24) are presented in Appendix G

[**]Equations (G.26)-(G.44) are presented in Appendix G.

Comparison of the mixed derivative $(\varphi_x)_{yy} = (\varphi_{yy})_x$ gives equation (G.1). Then equations (G.28)-(G.31) become equations (G.2)-(G.5) and equation (G.32) gives

$$\Delta_{yy} = -(20F_{2y}\Delta^2 - 48\Delta_y^2 + 4\Delta_y F_2\Delta + 7F_2^2\Delta^2 - 20J_4\Delta^2)/(40\Delta).$$

The equation $(\Delta_{yy})_x = (\Delta_x)_{yy}$ leads to equation (G.6). Equations (G.33)-(G.36) yield equations (G.7)-(G.10), and from equation (G.37) one finds

$$\begin{aligned}\psi_{yyyy} = \ & \Big[300\psi_{yyy}\varphi_y\Delta^2(4\Delta_y - F_2\Delta) + 5\psi_{yy}\varphi_y\Delta(-120F_{2y}\Delta^2 - 144\Delta_y^2 \\ & +72\Delta_y F_2\Delta - 39F_2^2\Delta^2 + 80J_4\Delta^2) + \psi_y\varphi_y(-500\varphi_y^3\alpha\Delta^3 \\ & -150F_{2yy}\Delta^3 + 360F_{2y}\Delta_y\Delta^2 - 165F_{2y}F_2\Delta^3 + 100J_{4y}\Delta^3 \\ & +96\Delta_y^3 - 72\Delta_y^2F_2\Delta + 108\Delta_yF_2^2\Delta^2 - 240\Delta_yJ_4\Delta^2 - 24F_2^3\Delta^3 \\ & +60F_2J_4\Delta^3) - 500\psi\varphi_y^5\beta\Delta^3 + 500K_7\Delta^4\Big]/(500\varphi_y\Delta^3).\end{aligned} \tag{3.74}$$

Equation (G.38) defines α:

$$\alpha = (4F_{2yy} + 6F_{2y}F_2 - 8J_{4y} + F_2^3 - 4F_2J_4 - 8K_6 + 56K_7r)/(8\varphi_y^3). \tag{3.75}$$

The relation $\alpha_x - r\alpha_y = 0$ leads to equation (G.11). Furthermore, considering $(\psi_x)_{yyyy} - (\psi_{yyyy})_x = 0$, one obtains

$$\begin{aligned}\beta = \ & 120\Delta_y(-4F_{2yy} - 6F_{2y}F_2 + 8J_{4y} - F_2^3 + 4F_2J_4 + 8K_6 - 56K_7r) \\ & +\Delta(320F_{2yyy} + 480F_{2yy}F_2 + 336F_{2y}^2 + 168F_{2y}F_2^2 + 32F_{2y}J_4 \\ & -480J_{4yy} - 240J_{4y}F_2 - 1600K_{7x} + 1600K_{7y}r - 400F_1K_7 \\ & -9F_2^4 + 88F_2^2J_4 + 160F_2K_6 - 320F_2K_7r - 144J_4^2)/(1600\Delta\varphi_y^4).\end{aligned} \tag{3.76}$$

The relation $\beta_x - r\beta_y = 0$ leads to equation (G.12). Equations (G.39)-(G.44) become equations (G.13)-(G.18), respectively.

Let us turn now to the integrability problem. One can find all second-order derivatives of function φ and all fourth-order derivatives of the function ψ by using equations (3.72), (3.73) and (3.74). So that one obtains at equations

(G.19)-(G.22). Finally, the coefficients α and β of the resulting linear equations (3.75) and (3.76) are given by

$$\alpha = \Theta/(8\varphi_y^3),$$

$$\begin{aligned}\beta = \;& (-144F_{2y}^2\Delta - 72F_{2y}F_2^2\Delta + 352F_{2y}J_4\Delta + 160J_{4yy}\Delta + 80J_{4y}F_2\Delta \\ & +640K_{6y}\Delta - 1600K_{7x}\Delta - 2880K_{7y}r\Delta - 4480r_yK_7\Delta + 80\Theta_y\Delta \\ & -120\Delta_y\Theta - 400F_1K_7\Delta - 9F_2^4\Delta + 88F_2^2J_4\Delta + 160F_2K_6\Delta \\ & -320F_2K_7r\Delta - 144J_4^2\Delta)/(1600\varphi_y^4\Delta),\end{aligned}$$

where

$$\Theta = (F_2^2 - 4J_4)F_2 - 8(K_6 - 7K_7r) - 8J_{4y} + 6F_{2y}F_2 + 4F_{2yy}.$$

Hence, we complete the proof of theorems.

3.3.4 Illustration of the Linearization Theorems

Example 3.3. Consider the non-linear equation

$$y^{(4)} - \frac{10}{y'}y''y''' + \frac{1}{y'^2}\left(15y''^3 - xy'^7 - y'^6\right) = 0. \tag{3.77}$$

It has the form of equation (3.10) with the following coefficients:

$$\begin{gathered} r = 0\,,\ F_2 = 0\,,\ F_1 = 0\,,\ F_0 = 0\,,\ H_2 = 0\,,\ H_1 = 0\,,\ H_0 = 0\,, \\ J_4 = 0\,,\ J_3 = 0\,,\ J_2 = 0\,,\ J_1 = 0\,,\ J_0 = 0\,,\ K_7 = -x\,, \\ K_6 = -1\,,\ K_5 = 0\,,\ K_4 = 0\,,\ K_3 = 0\,,\ K_2 = 0\,,\ K_1 = 0\,,\ K_0 = 0. \end{gathered} \tag{3.78}$$

Let us test the equation (3.77) for linearization by using Theorem 3.5. It is manifest that the equations (G.1)-(G.18) are satisfied by the coefficients (3.78). Thus, the equation (3.77) is linearizable, and we can proceed further.

Let us take its simplest solution $\varphi = y$ and $\psi = x$ which satisfy the compatible system of equations (G.19)-(G.22). So that one obtains the following transformations

$$t = y, \quad u = x. \tag{3.79}$$

Since $\Theta = 8$, equations (G.23) and (G.24) give

$$\alpha = 1, \quad \beta = 1.$$

Hence, the equation (3.77) is mapped by the transformation (3.79) into the linear equation

$$u^{(4)} + u' + u = 0.$$

CHAPTER IV

LINEARIZATION OF FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY CONTACT TRANSFORMATIONS

Recall that a transformation

$$t = \varphi(x, y, p), \quad u = \psi(x, y, p), \quad s = g(x, y, p), \tag{4.1}$$

of the variables x, y and $p = y' = dy/dx$ is called a *contact transformation* if it obeys the *contact condition*

$$s = u' = \frac{du}{dt}. \tag{4.2}$$

This chapter deals with the linearization of fourth-order ordinary differential equations (3.1) by means of contact transformations (4.1). A contact transformation (4.1) preserves the contact condition (4.2) if

$$g\,(x, y, p) = \frac{D\psi\,(x, y, p)}{D\varphi\,(x, y, p)} = \frac{\psi_x + p\psi_y + y''\psi_p}{\varphi_x + p\varphi_y + y''\varphi_p}. \tag{4.3}$$

Splitting equation (4.3), it implies that the functions φ, ψ and g are related by

$$\psi_p = g\varphi_p, \quad \psi_x + p\psi_y = (\varphi_x + p\varphi_y)\, g. \tag{4.4}$$

In particular, if $\varphi_p = 0$, then $\psi_p = 0$, and hence, the transformation (4.1) becomes a point transformation. Since the linearization problem using point transformations was solved in the previous chapter, we further assume that $\varphi_p \neq 0$. Moreover, by virtue of equations (4.4), the Jacobian of the contact transformation (4.1) is

$$(\varphi_y g - \psi_y)((g_x + g_y p)\varphi_p - (\varphi_x + \varphi_y p)g_p) \neq 0.$$

Applying a contact transformation (4.1), the derivatives are changed as follows. The tangent conditions $du' = u''dt$, $du'' = u'''dt$, $du''' = u^{(4)}dt, ...$ give the representation of the transformed derivatives

$$\begin{aligned}
\tfrac{d^2u}{dt^2} = \; g_1 = \; & \tfrac{Dg}{D\varphi},\\
\tfrac{d^3u}{dt^3} = \; g_2 = \; & \tfrac{D^2g\, D\varphi - D^2\varphi\, Dg}{(D\varphi)^3} = \tfrac{1}{(D\varphi)^2}(D^2g - D^2\varphi\, g_1),\\
\tfrac{d^4u}{dt^4} = \; g_3 = \; & \tfrac{1}{(D\varphi)^3}[D^3g - 3(D^2\varphi)D\varphi\, g_2 - (D^3\varphi)g_1],\\
\tfrac{d^{(k+1)}u}{dt^{(k+1)}} = \; g_k = \; & \tfrac{1}{(D\varphi)^k}[D^kg - \tfrac{k(k-1)}{2}(D^2\varphi)(D\varphi)^{k-2}g_{k-1}\\
& -k(D^{k-1}\varphi)(D\varphi)g_2 - (D^k\varphi)g_1] + h_k(x, y, y', ..., y^{(k-1)}), \quad (k > 3).
\end{aligned} \tag{4.5}$$

Notice that for any function $F = F(x, y, p)$:

$$\begin{aligned}
DF = \; & y''F_p + pF_y + F_x,\\
D^2F = \; & y'''F_p + y''\left(y''F_{pp} + 2\widetilde{D}F_p + F_y\right) + p\widetilde{D}F_y + \widetilde{D}F_x,\\
= \; & y'''F_p + y''\left(2DF_p + F_y - y''F_{pp}\right) + p\widetilde{D}F_y + \widetilde{D}F_x,\\
D^{k-1}F = \; & y^{(k)}F_p + y^{(k-1)}((k-1)DF_p + F_y - \delta_{k,3}y''F_{pp})\\
& + f_F(y^{(k-2)}, y^{(k-3)}, ..., y', y, x), \quad (k \geq 3),
\end{aligned}$$

where δ_{kj} is the Kronecker symbol, and

$$\widetilde{D} = \partial_x + p\partial_y.$$

4.1 Necessary Conditions for Linearization

Here we consider ith-order ordinary differential equation (3.4). Our goal is to describe all equations (3.4), which are equivalent with respect to contact transformations (4.1) to a linear equation.

We start with investigating the necessary conditions for linearization. The general form of equation (3.4) that can be obtained from a linear ordinary differential equation by a contact transformation (4.1) is found on this step. Necessary conditions for a linearizable fourth-order ordinary differential equation are studied here in more details.

Because of the formulae of changing the derivatives (4.5), for obtaining necessary conditions one has to study separately equations of order two, three, four and orders greater than four. Here we present necessary conditions for equations of fourth-order ($i = 4$) and higher ($i > 4$).

4.1.1 Necessary Form of a Linearizable Fourth-order ODE

As was obtained in the previous section, the transformation (4.1) provides the change of derivatives (4.5). Substituting u, u' and $u^{(4)}$ into the linear equation (3.8), and setting $a(x, y, p) = (\varphi_x + p\varphi_y)/\varphi_p$, we arrive at the following equation

$$\begin{gathered} y^{(4)} + \tfrac{1}{y''+a}\Big[-3y'''^2 + (A_2 y''^2 + A_1 y'' + A_0)\, y''' \\ + B_5 y''^5 + B_4 y''^4 + B_3 y''^3 + B_2 y''^2 + B_1 y'' + B_0 \Big] = 0, \end{gathered} \tag{4.6}$$

where $A_i = A_i(x, y, p)$ and $B_i = B_i(x, y, p)$ are some functions of x, y, p. Thus, we have proved the theorem.

Theorem 4.1. *Any fourth-order ordinary differential equation linearizable by a contact transformation belongs to the class of equations (4.6).*

Remark 4.1. Comparing the general forms of fourth-order ordinary differential equations linearizable by point transformations and contact transformations, we can conclude that in contrast to third-order ordinary differential equations, the sets linearizable by these two types of transformations are disjoint.

4.1.2 Necessary Form of a Linearizable ith-order ODE

In this section necessary conditions for linearizable ordinary differential equations of order $i > 4$ are obtained.

Calculations show that

$$\begin{aligned} g_{i-2} = & \tfrac{1}{(D\varphi)^{i-2}}[D^{i-2}g - \tfrac{(i-2)(i-3)}{2}(D^2\varphi)(D\varphi)^{i-4}g_{i-3} \\ & -(i-2)(D^{i-3}\varphi)(D\varphi)g_2 - (D^{i-2}\varphi)g_1] + \cdots \\ = & \tfrac{1}{(D\varphi)^{i-2}}[D^{i-2}g - (D^{i-2}\varphi)g_1] + \cdots \\ = & \tfrac{1}{(D\varphi)^{i-2}}y^{(i-1)}[g_p - \varphi_p g_1] + \cdots . \end{aligned}$$

Here ... means terms with derivatives of order less than $i-1$. Hence, the function g_{i-1} has the representation

$$\begin{aligned} g_{i-1} = & \tfrac{1}{(D\varphi)^{i-1}}[y^{(i)}g_p + y^{(i-1)}((i-1)Dg_p + g_y) - \tfrac{(i-1)(i-2)}{2}\tfrac{D^2\varphi}{D\varphi}y^{(i-1)}\left(g_p - \varphi_p g_1\right) \\ & -(i-1)(D\varphi)g_2 y^{(i-1)}\varphi_p - g_1\left(y^{(i)}\varphi_p + y^{(i-1)}((i-1)D\varphi_p + \varphi_y)\right)] + \cdots \\ = & \tfrac{1}{(D\varphi)^{i-1}}[y^{(i)}\left(g_p - \varphi_p g_1\right) + y^{(i-1)}[((i-1)Dg_p + g_y) - \tfrac{(i-1)(i-2)}{2}\tfrac{D^2\varphi}{D\varphi}\left(g_p - \varphi_p g_1\right) \\ & -(i-1)(D\varphi)g_2\varphi_p - g_1((i-1)D\varphi_p + \varphi_y)]] + \cdots . \end{aligned}$$

Substituting derivatives of the function $u(t)$ into a linear equation (3.3), one obtains

$$y^{(i)} + y^{(i-1)}\lambda_i + ... = 0, \qquad (i > 4),$$

where

$$\begin{aligned} \lambda_i(x,y,y',y'',y''') = & \Delta^{-1}[(i-1)Dg_p + g_y - \tfrac{(i-1)(i-2)}{2}\tfrac{D^2\varphi}{D\varphi}\left(g_p - \varphi_p g_1\right) \\ & -(i-1)\varphi_p\tfrac{D^2g\,D\varphi - D^2\varphi\,Dg}{(D\varphi)^2} - \tfrac{Dg}{D\varphi}((i-1)D\varphi_p + \varphi_y)] \end{aligned}$$

and

$$\Delta(x,y,p) = \frac{g_p D\varphi - \varphi_p Dg}{D\varphi} = \frac{g_p\widetilde{D}\varphi - \varphi_p\widetilde{D}g}{D\varphi} \neq 0.$$

The function $\lambda_i(x, y, y', y'', y''')$ has the form

$$\lambda_i = \frac{1}{y''+a}\Big[-\frac{i(i-1)}{2}y''' + (A_2y''^2 + A_1y'' + A_0)\Big]$$

with some functions $A_i = A_i(x, y, p)$, $(i = 0, 1, 2)$. Thus, the necessary form of a linearizable ordinary differential equation of ith-order is

$$y^{(i)} + y^{(i-1)}\frac{1}{y''+a}\Big[-\frac{i(i-1)}{2}y''' + \big(A_2y''^2 + A_1y'' + A_0\big)\Big] + \ldots. \tag{4.7}$$

Theorem 4.2. *Any ith-order ($i > 4$) ordinary differential equation linearizable by a contact transformation belongs to the class of equations (4.7).*

4.2 Formulation of the Linearization Theorems

We have shown in the previous section that every linearizable fourth-order ordinary differential equation belongs to the class of equations (4.6) that are at most quadratic in y'''. In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization, the methods for constructing the linearizing transformations as well as the coefficients of the resulting linear equations.

Theorem 4.3. *Equation (4.6) is linearizable if and only if its coefficients obey the following conditions:*

$$-4f_qf_r + 6f_rDf_r - {f_r}^3 - 8f_p - 4D^2f_r + 8Df_q = \lambda_1(q+a)^3, \tag{4.8}$$

$$\begin{aligned}
&-1440f_rD^2f_r - 1600f_y + 832f_qDf_r - 144{f_q}^2 + 1512Df_r{f_r}^2 - 808f_q{f_r}^2\\
&+480D^3f_r - 1600f_pf_r - 189{f_r}^4 + 2000Df_qf_r - 864D{f_r}^2 - 1120D^2f_q\\
&+1600Df_p = \tfrac{(q+a)^2}{3}\Big[-600\lambda_{1p}(q+a)^2 + 1800\lambda_1 r + 30\lambda_1(-24a_pq\\
&-24a_pa - 10A_0 - 12A_1q - 2A_1a - 15A_2q^2 - 6A_2qa - A_2a^2)\\
&+\lambda_2(q+a)^2\Big],
\end{aligned} \tag{4.9}$$

$$\lambda_{1x} = (-12\lambda_{1p}a_p p + \lambda_{1p}(-A_1 p + 2A_2 pa + 10a) \\ -6a_p\lambda_1 + 3\lambda_1(-A_1 + 2A_2 a - \mu_1 p))/10, \tag{4.10}$$

$$\lambda_{1y} = (12\lambda_{1p}a_p + \lambda_{1p}(A_1 - 2A_2 a) + 3\lambda_1\mu_1)/10, \tag{4.11}$$

$$\lambda_{2y} = (12\lambda_{2p}a_p + \lambda_{2p}(A_1 - 2A_2 a) + 1800\lambda_1\mu_2 p + 4\lambda_2\mu_1)/10, \tag{4.12}$$

where $p = y', q = y''$ *and* $r = y'''$.

The following notations are used :

$$\begin{aligned}
\lambda_1 &= 4A_{2pp} + 6A_{2p}A_2 + 8B_{4p} - 32B_{5p}a - 8B_{5x} - 8B_{5y}p - 56a_pB_5 \\
&\quad -4A_1B_5 + A_2^3 + 4A_2B_4 - 12A_2B_5a, \\
\lambda_2 &= 960A_{1p}B_5 - 432A_{2p}^2 - 216A_{2p}A_2^2 - 1056A_{2p}B_4 + 5040A_{2p}B_5a \\
&\quad -1680A_{2x}B_5 - 1680A_{2y}B_5p - 480B_{4pp} - 240B_{4p}A_2 + 1920B_{5px} \\
&\quad +1920B_{5py}p + 480B_{5pp}a + 960B_{5p}a_p + 480B_{5p}A_1 - 240B_{5p}A_2a \\
&\quad +480B_{5x}A_2 + 480B_{5y}A_2p + 6720B_{5y} + 240\lambda_{1p} - 480a_pA_2B_5 \\
&\quad -120A_1A_2B_5 - 27A_2^4 - 264A_2^2B_4 + 1560A_2^2B_5a - 90A_2\lambda_1 \\
&\quad -640B_3B_5 - 432B_4^2 + 6880B_4B_5a - 17200B_5^2a^2, \\
\mu_1 &= -8A_{2p}a + 6A_{2y}p + 4a_pA_2 + A_{1p} + 6A_{2x} + 3A_1A_2 - 6A_2^2a \\
&\quad +8B_3 - 32B_4a + 80B_5a^2, \\
\mu_2 &= (156A_{2p}a_p + 3A_{2p}(A_1 - 52A_2a) + 150A_{2y}(A_2p - 1) \\
&\quad +2a_p(57A_2^2 + 92B_4 - 460B_5a) + 150A_{2x}A_2 - 40B_{3p} + 100B_{4p}a \\
&\quad +60B_{4x} + 60B_{4y}p - 100B_{5p}a^2 - 300B_{5x}a - 300B_{5y}pa + 50A_0B_5 \\
&\quad +57A_1A_2^2 + 12A_1B_4 - 110A_1B_5a - 114A_2^3a + 140A_2B_3 \\
&\quad -584A_2B_4a + 1570A_2B_5a^2 - 15A_2\mu_1)/(60p).
\end{aligned}$$

Theorem 4.4. *Provided that the conditions (4.8)-(4.12) are satisfied, the transformation (4.1) mapping equation (4.6) into a linear equation (3.8) is obtained by* *solving the following compatible system of equations for the functions* $\varphi(x,y,p)$, $\psi(x,y,p)$, $g(x,y,p)$ *and* $k(x,y,p)$*:*

$$\varphi_x = a\varphi_p - p\varphi_y, \quad \varphi_y = \varphi_p(12a_p + A_1 - 2A_2a)/10,$$

$$\varphi_{ppp} = (60\varphi_{pp}^2 + \varphi_p^2(-12A_{2p} - 3A_2^2 - 8B_4 + 40B_5a))/(40\varphi_p),$$

$$\psi_x = -p\psi_y + g(\varphi_x + p\varphi_y), \quad \psi_y = -\varphi_p k + \varphi_y g, \quad \psi_p = \varphi_p g,$$

$$g_x = g_p a - g_y p + k, \qquad g_y = (-6\varphi_{pp}k - \varphi_p A_2 k + 4\varphi_y g_p)/(4\varphi_p),$$

$$g_{ppp} = (-7200\varphi_{pp}^2 g_p + 14400\varphi_{pp}\varphi_p g_{pp} + 900\varphi_{pp}\lambda_1\psi + 120\varphi_p^2 g_p(-12A_{2p} - 3A_2^2 - 8B_4 + 40B_5a) + 600\varphi_p^2(-8B_5k - g\lambda_1) - \varphi_p\lambda_2\psi)/(4800\varphi_p^2),$$

$$k_x = (2\varphi_{pp}k(-12a_p p - A_1 p + 2A_2 pa + 10a) + \varphi_p k(56a_p + 8A_1 - 6A_2a - \omega p))/(40\varphi_p),$$

$$k_y = (2\varphi_{pp}k(12a_p + A_1 - 2A_2a) + \varphi_p k\omega)/(40\varphi_p), \qquad k_p = (2\varphi_{pp}k + \varphi_p A_2 k)/(4\varphi_p),$$

where

$$k = g_x + g_y p - g_p a \neq 0, \tag{4.13}$$

$$\omega = 12A_{1p} - 56A_{2p}a + 32A_{2x} + 32A_{2y}p + 28a_pA_2 + 21A_1A_2 - 42A_2^2a + 56B_3 - 224B_4a + 560B_5a^2.$$

The coefficients α *and* β *of the resulting linear equation (3.8) are given by*

$$\alpha = \frac{\lambda_1}{8\varphi_p^3}, \qquad \beta = \frac{-900\varphi_{pp}\lambda_1 + \varphi_p\lambda_2}{4800\varphi_p^5}.$$

Remark 4.2. If the left hand sides of equations (4.8), (4.9) are equal to zero, and equation (4.6) is linearizable by contact transformations, then $\lambda_1 = 0$ and $\lambda_2 = 0$. In this case equations (4.10)-(4.12) are satisfied, and $\alpha = 0$, $\beta = 0$. This particular case was studied in (Dridi and Neut, 2005)*. Conversely, if an equation (4.6) can

*There are two more equations in (Dridi and Neut, 2005): $f_{rrr} = 0$ and $6f_{qrr} + {f_{rr}}^2 = 0$. These equations are equivalent to the form (4.6).

be mapped into the trivial equation $u^{(4)} = 0$, then $\alpha = 0$, $\beta = 0$. This leads to $\lambda_1 = 0$ and $\lambda_2 = 0$. Hence, the left hand side of equations (4.8), (4.9) are equal to zero. Thus, the result obtained in the thesis extends linearization conditions obtained in (Dridi and Neut, 2005) for the most general case of linear equations.

4.3 Relations Between Coefficients and Transformations

Lemma 4.5. *The coefficients of equation (4.6) and the functions $\varphi(x, y, p)$, $\psi(x, y, p)$ and $g(x, y, p)$ in the transformation (4.1) mapping linear equation (3.8) into equation (4.6) are related by equations*[†] *(I.1)-(I.9).*

4.4 Proof of the Linearization Theorems

Proof : For given coefficients $A_i(x, y, p)$ and $B_i(x, y, p)$ of equation (4.6), we have to find the necessary and sufficient conditions for integrability of the overdetermined system of equations (I.1)-(I.9) for the unknown functions $\varphi(x, y, p)$, $\psi(x, y, p)$, $g(x, y, p)$ and $k(x, y, p)$.

Defining the derivatives ψ_p and ψ_x from equation (4.4), and equating the mixed derivative $(\psi_p)_x = (\psi_x)_p$, one finds from this equation the derivative ψ_y. Recall that, according to our notation $\varphi_x = a\varphi_p - p\varphi_y$ and for simplicity of calculation, we introduce the function k as in equation (4.13). From equation (4.13) one finds the derivative g_x. The equations $(\psi_x)_y = (\psi_y)_x$ and $(\psi_y)_p = (\psi_p)_y$ can be solved with respect to the derivatives k_x and k_p, respectively. Equations (I.1)-(I.5) give g_y, k_y, a_x, α and φ_{ppp}. Thus, equations $(g_x)_y = (g_y)_x$, $(\varphi_x)_y = (\varphi_y)_x$ and $(k_x)_p = (k_p)_x$ can be solved with respect to the derivatives φ_y, A_{1x} and A_{2px}, respectively. The equation (I.6) defines the derivative a_{ppp}.

[†]Since equations (I.1)-(I.9) are cumbersome, they are presented in Appendix I.

Since $\alpha(x,y,p) = \alpha \circ \varphi(x,y,p)$, one obtains the relations

$$\alpha_x\varphi_y - \alpha_y\varphi_x = 0, \quad \alpha_x\varphi_p - \alpha_p\varphi_x = 0, \quad \alpha_y\varphi_p - \alpha_p\varphi_y = 0.$$

From these relations, one finds β and g_{ppp}. Equations (I.8)-(I.9) serve for finding the derivative A_{0x} and the coefficient B_0. Notice that the following derivatives

$$\varphi_x,\ \varphi_y,\ \varphi_{ppp},\ \psi_x,\ \psi_y,\ \psi_p,\ g_x,\ g_y,\ g_{ppp},\ k_x,\ k_y,\ k_p$$

are found through

$$\varphi_p,\ \varphi_{pp},\ g_p,\ g_{pp},\ \psi,\ g,\ k.$$

Thus, one has found all third-order derivatives of the function φ, all first-order derivatives of the function ψ, all third-order derivatives of the function g and all first-order derivatives of the function k. The remaining compatibility conditions are obtained by equating the mixed derivatives (with corresponding orders) of the functions $\varphi(x,y,p), \psi(x,y,p), g(x,y,p)$ and $k(x,y,p)$.

Since $\beta(x,y,p) = \beta \circ \varphi(x,y,p)$, one obtains the relations

$$\beta_x\varphi_y - \beta_y\varphi_x = 0, \tag{4.14}$$

$$\beta_x\varphi_p - \beta_p\varphi_x = 0, \tag{4.15}$$

$$\beta_y\varphi_p - \beta_p\varphi_y = 0. \tag{4.16}$$

Comparing the mixed derivatives $(\varphi_x)_{ppp} = (\varphi_{ppp})_x$ and $(\varphi_y)_{ppp} = (\varphi_{ppp})_y$, one obtains the derivatives A_{1ppp} and a_{pp}. The equation $a_{ppp} = (a_{pp})_p$ gives the derivative A_{1pp}. Substituting A_{1pp} into the relation $A_{1ppp} = (A_{1pp})_p$, one finds the derivative A_{2py}. Setting $(k_x)_y = (k_y)_x$ and $(k_y)_p = (k_p)_y$, one gets only the derivative A_{0pp}. The derivative B_{4pxx} is found from equation (I.7). The relations (4.14)-(4.16) and the mixed derivatives $(g_x)_{ppp} = (g_{ppp})_x$, $(g_y)_{ppp} = (g_{ppp})_y$ give the conditions for B_{4ppx}, B_{4ppy} and B_{4px}. The equation $(a_x)_{pp} = (a_{pp})_x$ provides the expression for A_{2xx}. The equations $B_{4ppx} = (B_{4px})_p$, $(A_{1x})_{pp} = (A_{1pp})_x$, $(A_{2xx})_p = (A_{2px})_x$, $(A_{0x})_{pp} = (A_{0pp})_x$ and $B_{4pxx} = (B_{4px})_x$ give, respectively: $B_{3ppp}, B_{4xx}, B_{3ppx}, B_{3xx}$ and A_{0p}. The derivative a_{py} can be found from the equation $A_{0pp} = (A_{0p})_p$. Equation $(a_{py})_p = (a_{pp})_y$ yields the derivative A_{1py}. One can find the derivative A_{0y} from the equation $(A_{0p})_x = (A_{0x})_p$. Comparing the mixed derivatives $(A_{0p})_y = (A_{0y})_p$ and $(A_{1py})_p = (A_{1pp})_y$, one arrives to formulae for the derivatives B_{2px} and B_{2ppp}, respectively. The equations $(A_{0x})_y = (A_{0y})_x$ and $(B_{3xx})_{pp} = (B_{3ppx})_x$ give the derivatives B_{1px} and B_{5xxxx}, respectively.

Analyzing the results of (Dridi and Neut, 2005), and recalculating the left hand sides of equations (4.8)-(4.12), we could represent the obtained conditions in the final criteria of linearization of a fourth-order ordinary differential equation in the form presented in (4.8)-(4.12).

4.5 Illustration of the Linearization Theorems

Consider the nonlinear ordinary differential equation

$$-16y'^2y''y^{(4)} + 48y'^2y'''^2 + y'y''^5x - 48y'y''^2y''' - y''^5y + 12y''^4 = 0. \tag{4.17}$$

It is an equation of the form (4.6) with coefficients

$$a = 0,\ \ A_2 = \tfrac{3}{p},\ \ A_1 = 0,\ \ A_0 = 0,$$
$$B_5 = \tfrac{-px+y}{16p^2},\ \ B_4 = \tfrac{-3}{4p^2},\ \ B_3 = B_2 = B_1 = B_0 = 0\,,$$
$$\lambda_1 = 0,\ \ \lambda_2 = \tfrac{300}{p^2},\ \ \mu_1 = \mu_2 = \omega = 0.$$

These coefficients obey the linearization conditions (4.8)-(4.12). Thus, equation (4.17) is linearizable. The linearizing transformation is found as follows. The equations for the function φ are

$$\varphi_y = 0, \qquad \varphi_x = 0, \qquad \varphi_{ppp} = \frac{12p^2\varphi_{pp}^2 + 3\varphi_p^2}{8p^2\varphi_p}. \tag{4.18}$$

The function $\varphi = \sqrt{p}$ is a particular solution of the equations (4.18). In this case $g_y = 0$. Then the function $k(x, y, p)$ has to satisfy the equations

$$k_x = 0, \qquad k_y = 0, \qquad k_p = \frac{g_x}{2p}.$$

Since $g_x = k$ and $k = k(p)$, the general solution is $k = k_0\sqrt{p}$, where k_0 is constant. So that $g = k_0 x\sqrt{p} + f(p)$, in particular, one can consider

$$g = x\sqrt{p}.$$

One can readily verify that the function $g = x\sqrt{p}$ solves equation for g_{ppp} as well. Solving system of equations $\psi_x = -p\psi_y$, $\psi_y = -1/2$ and $\psi_p = x/2$, one finds

$$\psi = \frac{xp - (y - C)}{2},$$

where C is constant. Taking for example $C = 0$, one obtains the transformation

$$\varphi = \sqrt{p}, \qquad \psi = \frac{xp - y}{2}, \qquad g = x\sqrt{p}. \tag{4.19}$$

The coefficients α and β of the resulting linear equation (3.8) are

$$\alpha = 0, \qquad \beta = \frac{1}{16p^2\varphi_p^4} = 1.$$

Hence, the nonlinear equation (4.17) is mapped by the transformation (4.19) into the linear equation

$$u^{(4)} + u = 0.$$

4.6 Application of the Linearization Theorems to a System of Two Second-order ODEs

In this section we give some conditions for linearization of a system of two second-order ordinary differential equations with two dependent variables y, z and one independent variable x

$$y'' = f_1(x, y, y', z), \quad z'' = f_2(x, y, y', z). \tag{4.20}$$

Assuming that $f_{1z} \neq 0$, by virtue of the Inverse Function Theorem the first equation of (4.20) can be solved with respect to $z = h(x, y, y', y'')$. Substituting this into the second equation of (4.20), one obtains that system (4.20) is equivalent to the fourth-order ordinary differential equation

$$\begin{aligned} y^{(4)}h_{y''} \quad & + y'''^2 h_{y''y''} + y'''(2h_{y'y''}y'' + 2h_{y''y}y' + 2h_{y''x} + h_{y'}) + h_{y'y'}y''^2 \\ & + (2h_{y'y}y' + 2h_{y'x} + h_y)y'' + h_{yy}y'^2 + 2h_{xy}y' + h_{xx} - f_2 = 0. \end{aligned} \tag{4.21}$$

Applying linearization theorems to equation (4.21) one can obtain conditions for the functions $f_2(x, y, y', z)$ and $h(x, y, y', y'')$ which are necessary and sufficient for equation (4.21) to be linearizable.

Since one of the necessary conditions for linearization of equation (4.21) requires that this equation has to be a quadratic equation with respect to the third-order derivative y''' with the coefficient $-\frac{3}{y''+a}$, one obtains that the general form of the function h is

$$h = h_0 + \frac{h_1}{(y'' + a)^2},$$

where $h_i = h_i(x, y, y'), (i = 0, 1)$. Since $h_{y''} \neq 0$, the function $h_1 \neq 0$. Because of the coefficients with the derivative y''' have to be quadratic with respect to the second-order derivative y'', one obtains $h_{0y'} = 0$, which means that $h_0 = h_0(x, y)$. Hence, the coefficients A_2, A_1 and A_0 in equation (4.6) are

$$\begin{aligned} A_2 &= 3h_{1p}/(2h_1), \\ A_1 &= (2(h_{1x} + h_{1y}p) + h_{1p}a - 5a_p h_1)/h_1, \\ A_0 &= ((4(h_{1x} + h_{1y}p) - h_{1p}a)a - 12a_y h_1 p - 12a_x h_1 + 2a_p a h_1)/(2h_1), \end{aligned}$$

where $p = y'$.

Since the coefficients with the derivative y''^6 have to be zero, one arrives at equation

$$(y''+a)^4 f_{2y''y''y''y''y''y''} + 24(y''+a)^3 f_{2y''y''y''y''y''}$$

$$+18(y''+a)^2 f_{2y''y''y''y''} + 480(y''+a) f_{2y''y''y''} + 360 f_{2y''y''} = 0.$$

The general solution of this equation is

$$f_2 = \frac{c_1}{y''+a} + \frac{c_2}{(y''+a)^2} + \frac{c_3}{(y''+a)^3} + \frac{c_4}{(y''+a)^4} + c_5 y'' + c_6,$$

where $c_i = c_i(x, y, y'), (i = 1, 2, \ldots, 6)$ and one obtains

$$\begin{aligned}
B_5 &= -(h_{0y} - c_5)/(2h_1), \\
B_4 &= -(2h_{0xy}p + h_{0xx} + h_{0yy}p^2 + 4h_{0y}a + h_{1pp} - 4ac_5 - c_6)/(2h_1), \\
B_3 &= (2a_{pp}h_1 + 4a_p h_{1p} - 8h_{0xy}ap - 4h_{0xx}a - 4h_{0yy}ap^2 - 6h_{0y}a^2 - 2h_{1px} \\
&\quad -2h_{1py}p - 2h_{1pp}a - h_{1y} + 6a^2c_5 + 4ac_6 + c_1)/(2h_1), \\
B_2 &= (4a_{px}h_1 + 4a_{py}h_1p + 2a_{pp}ah_1 - 6a_p^2h_1 + 4a_ph_{1p}a + 4a_ph_{1x} + 4a_ph_{1y}p \\
&\quad +4a_xh_{1p} + 4a_yh_{1p}p + 2a_yh_1 - 12h_{0xy}a^2p - 6h_{0xx}a^2 - 6h_{0yy}a^2p^2 \\
&\quad -4h_{0y}a^3 - 4h_{1px}a - 4h_{1py}ap - h_{1pp}a^2 - 2h_{1xy}p - h_{1xx} - h_{1yy}p^2 \\
&\quad -2h_{1y}a + 4a^3c_5 + 6a^2c_6 + 3ac_1 + c_2)/(2h_1), \\
B_1 &= (4a_{px}ah_1 + 4a_{py}ah_1p - 12a_pa_xh_1 - 12a_pa_yh_1p + 4a_ph_{1x}a + 4a_ph_{1y}ap \\
&\quad +4a_{xy}h_1p + 2a_{xx}h_1 + 4a_xh_{1p}a + 4a_xh_{1x} + 4a_xh_{1y}p + 2a_{yy}h_1p^2 \\
&\quad +4a_yh_{1p}ap + 4a_yh_{1x}p + 4a_yh_{1y}p^2 + 2a_yah_1 - 8h_{0xy}a^3p - 4h_{0xx}a^3 \\
&\quad -4h_{0yy}a^3p^2 - h_{0y}a^4 - 2df(h_1, p, x)a^2 - 2h_{1py}a^2p - 4h_{1xy}ap - 2h_{1xx}a \\
&\quad -2h_{1yy}ap^2 - h_{1y}a^2 + a^4c_5 + 4a^3c_6 + 3a^2c_1 + 2ac_2 + c_3)/(2h_1), \\
B_0 &= (4a_{xy}ah_1p + 2a_{xx}ah_1 - 6a_x^2h_1 - 12a_xa_yh_1p + 4a_xh_{1x}a + 4a_xh_{1y}ap \\
&\quad +2a_{yy}ah_1p^2 - 6a_y^2h_1p^2 + 4a_yh_{1x}ap + 4a_yh_{1y}ap^2 - 2h_{0xy}a^4p - h_{0xx}a^4 \\
&\quad -h_{0yy}a^4p^2 - 2h_{1xy}a^2p - h_{1xx}a^2 - h_{1yy}a^2p^2 + a^4c_6 + a^3c_1 + a^2c_2 \\
&\quad +ac_3 + c_4)/(2h_1).
\end{aligned}$$

For the sake of simplicity here we consider a particular case where $a = 0, h_0 = 0$ and $h_{1x} = 0$. We present the linearization conditions for the case $h_{1y} = 0$ and $h_{1y} \neq 0$.

4.6.1 Case $h_{1y} = 0$

One can verify that in this case the coefficients A_i and B_i found satisfy the linearization conditions (4.8)-(4.12) if and only if

$$c_1 = c_2 = c_3 = c_4 = 0, \quad c_{6x} = c_{6y} = 0, \quad c_{5xx} = c_{5xy} = c_{5yy} = 0. \tag{4.22}$$

In this case the functions $f_2(x, y, y', z)$ and $h(x, y, y', y'')$ satisfying conditions (4.22) are

$$f_2 = (\nu_1 + \nu_2 x + \nu_3 y)y'' + \nu_4, \quad h = \frac{{\nu_5}^2}{y''^2}, \tag{4.23}$$

where $\nu_i = \nu_i(y'), (i = 1, \ldots, 5)$ are arbitrary functions[‡]. System (4.20) corresponding to these functions is

$$y'' = \frac{\nu_5}{\sqrt{z}}, \quad z'' = (\nu_1 + \nu_2 x + \nu_3 y)\frac{\nu_5}{\sqrt{z}} + \nu_4. \tag{4.24}$$

Hence, we can conclude that a system (4.20) is linearizable if it has the form (4.24).

4.6.2 Case $h_{1y} \neq 0$

In this case setting $c_1 = 0$, the found coefficients A_i and B_i obey the linearization conditions (4.8)-(4.12) if and only if

$$c_2 = c_3 = c_4 = 0, \quad h_{1p} = 4h_1/p, \quad h_{1yy} = 6h_{1y}^2/(5h_1), \tag{4.25}$$

$$c_{6x} = -(c_{6p}p^3 - 6h_1)h_{1y}/(5ph_1), \quad c_{6y} = (c_{6p}p + 3c_6)h_{1y}/(5h_1), \tag{4.26}$$

[‡]See more calculations in Appendix J.

$$\begin{aligned}
c_{5xx} &= -(20c_{5px}h_1p^4 + 2c_{5pp}h_{1y}p^6 + 15c_{5p}h_{1y}p^5 + 60c_{5x}h_1p^3 \\
&\quad +5c_{5y}h_1p^4 + 20h_{1y}c_5p^4 + 10c_6h_1p^2 - 120h_1^2)h_{1y}/(50p^2h_1^2), \quad (4.27) \\
c_{5xy} &= (10(c_{5py}p^2 + 4c_{5x} + c_{5px}p)h_1 - (20c_{5py}h_1p - 2c_{5pp}h_{1y}p^2 \\
&\quad -15c_{5p}h_{1y}p + 25c_{5y}h_1 - 20h_{1y}c_5)p)h_{1y}/(50h_1^2), \quad (4.28) \\
c_{5yy} &= -(2c_{5pp}h_{1y}p^2 + 15c_{5p}h_{1y}p - 75c_{5y}h_1 + 20h_{1y}c_5 \\
&\quad -20c_{5py}h_1p)h_{1y}/(50h_1^2). \quad (4.29)
\end{aligned}$$

One type of the functions $f_2(x, y, y', z)$ and $h(x, y, y', y'')$ satisfying conditions (4.25)-(4.29) is

$$f_2 = \frac{\sigma_1}{y^3} + \frac{3y'^2}{\kappa y^5} + y''\left(-\frac{3}{4\kappa y^4} + \frac{\sigma_1}{2y^2y'^2} + \frac{\sigma_2}{\sqrt{y'}y'^2} + \frac{\sigma_3}{yy'^3} + \frac{\sigma_4}{y'^4}\right), \quad h = \frac{y'^4}{\kappa y^5 y''^2}, \quad (4.30)$$

where $\sigma_i = \sigma_i(\frac{y}{p} - x)$, $(i = 1, 2, 3, 4)$ are arbitrary functions and κ is a constant[§]. System (4.20) corresponding to these functions is

$$y'' = \frac{y'^2}{\sqrt{\kappa y^5 z}}, \quad z'' = \frac{\sigma_1}{y^3} + \frac{3y'^2}{\kappa y^5} + \frac{y'^2}{\sqrt{\kappa y^5 z}}\left(-\frac{3}{4\kappa y^4} + \frac{\sigma_1}{2y^2y'^2} + \frac{\sigma_2}{\sqrt{y'}y'^2} + \frac{\sigma_3}{yy'^3} + \frac{\sigma_4}{y'^4}\right). \quad (4.31)$$

Hence, we can conclude that a system (4.20) is linearizable if it has the form (4.31).

[§]See detail in Appendix J.

CHAPTER V

CONCLUSIONS

This thesis is devoted to the study of the linearization problem of fourth-order ordinary differential equations via point and contact transformations. The results obtained are separated into two parts.

In the first part, the criteria for fourth-order ordinary differential equations to be linearizable by point transformations are given. Two distinctly different classes for linearization are provided: the sets of all fourth-order ordinary differential equations that are linearizable by point transformations are contained either in the class of equations (3.9) or in the class of equations (3.10).

- The main Theorem 3.2 for the first class, provides necessary and sufficient conditions for linearization. The explicit procedure for constructing the linearizing point transformations and the formulae for the coefficients of the resulting linear equations are summarized in Theorem 3.3. An example of a third-order member of Riccati Hierarchy equation which is not linearizable by a point transformation or contact transformation or generalized Sundman transformation, but is linearizable by our method, is given. Linearization of traveling waves of partial differential equation are applied. Applications of how one can effect linearization for a system of two second-order ordinary differential equations are presented.

- The main Theorem 3.5 for the second class, provides necessary and sufficient conditions for linearization. The procedure for obtaining the linearizing point transformations and the coefficients of the resulting linear equations is summarized in Theorem 3.6.

Moreover, the general form of ordinary differential equations of order greater than four linearizable via point transformations are obtained.

The second part deals with the linearization of fourth-order ordinary differential equations by contact transformations. The general form of fourth-order ordinary differential equations that are linearizable via contact transformations is (4.6). Conditions which guarantee that equations (4.6) can be linearizable are provided Theorem 4.3. The explicit procedure for obtaining the linearizing contact transformations and coefficients of the resulting linear equations are presented in Theorem 4.4. The linearization criteria obtained for fourth-order ordinary differential equations are applied to a system of two second-order ordinary differential equations. The general form of ordinary differential equations of order greater than four linearizable via contact transformations are provided.

Furthermore, it is proven that the set of fourth-order ordinary differential equations linearizable by point transformations and the set of fourth-order ordinary differential equations linearizable by contact transformations are disjoint.

We can conclude that the criteria for fourth-order ordinary differential equations to be linearizable via point and contact transformations are completed. Program for checking the linearizable criteria have also been developed.

In the future work I will analyze the conditions for fourth-order ordinary differential equations to be linearizable by tangent transformations.

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APPENDICES

APPENDIX A

REMARKS ON CONTACT TRANSFORMATIONS OF SECOND-ORDER ODEs

The contact condition has the following meaning. Let the function $y(x)$ be given. The transformed function $u(t)$ is found from the equations

$$t = \varphi\left(x, y\left(x\right), y'\left(x\right)\right),$$

$$u = \psi\left(x, y\left(x\right), y'\left(x\right)\right).$$

By virtue of the Inverse Function Theorem, the first equation gives

$$x = \tau(t)$$

and then

$$u\left(t\right) = \psi\left(\tau\left(t\right), y\left(\tau\left(t\right)\right), y'\left(\tau\left(t\right)\right)\right).$$

Hence, it is assumed $D\varphi \neq 0$. The derivative is

$$u'\left(t\right) = \frac{D\psi}{D\varphi}\left(\tau\left(t\right), y\left(\tau\left(t\right)\right), y'\left(\tau\left(t\right)\right), y''\left(\tau\left(t\right)\right)\right).$$

The contact condition requires

$$g\left(x, y, p\right) = \frac{D\psi}{D\varphi}\left(x, y, p, w\right), \tag{A.1}$$

where $p = y'$ and $w = y''$. Equation (A.1) is rewritten in the form

$$g\left(\varphi_x + p\varphi_y + w\varphi_p\right) = \psi_x + p\psi_y + w\psi_p.$$

Since the contact condition is satisfied for any $w = y''$, one obtains

$$g\left(\varphi_x + p\varphi_y\right) = \psi_x + p\psi_y,$$
$$g\varphi_p = \psi_p. \tag{A.2}$$

S. Lie showed that all second-order ordinary differential equations are equivalent with respect to the set of contact transformations. In fact, let us prove that any equation

$$y'' = f(x, y, p)$$

is equivalent with respect to the set of contact transformations to the equation

$$u'' = 0.$$

Since $u'' = \frac{Dg}{D\varphi}$, one needs to find functions $\varphi(x, y, p)$, $\psi(x, y, p)$ and $g(x, y, p)$ which satisfy (A.2) and the equation $Dg = 0$, which is

$$g_x + pg_y + fg_p = 0. \tag{A.3}$$

Notice that the Jacobian of the transformation is

$$\triangle = \left(\psi_y - g\varphi_y\right)\left(g_p\left(\varphi_x + p\varphi_y\right) - \varphi_p\left(g_x + pg_y\right)\right) \neq 0,$$

or we can write

$$\triangle = \left(\psi_y - g\varphi_y\right) g_p \left(\varphi_x + p\varphi_y + f\varphi_p\right) \neq 0.$$

Without loss of generality it is assumed that $f \neq 0$.

Assume that $g(x, y, p)$ is some solution of (A.3) such that $g_p \neq 0$. Since $f \neq 0$, then the value

$$g_x + pg_y \neq 0.$$

Let us denote

$$\alpha = \psi - \varphi g.$$

Equations (A.2) become

$$\alpha_x + p\alpha_y = \varphi f g_p\,,$$

$$\alpha_p + \varphi g_p = 0.$$

Thus, the function $\alpha(x, y, p)$, has to satisfy the equation

$$\alpha_x + p\alpha_y + f\alpha_p = 0. \tag{A.4}$$

Notice that the requirement $\Delta \neq 0$ leads to

$$\alpha_y g_p - \alpha_p g_y \neq 0. \tag{A.5}$$

Since $g_p \neq 0$, then for solving equation (A.4) one can change the independent variables (x, y, p) into (x, y, g).

Let $p = h\,(x, y, g)\,, \alpha = H\,(x, y, g)\,.$ Since

$$\alpha_x = H_x + H_g g_x, \qquad \alpha_y = H_y + H_g g_y, \qquad \alpha_p = H_g g_p$$

then the function $H(x, y, g)$ has to satisfy the equation

$$H_x + hH_y = 0. \tag{A.6}$$

The condition (A.5) becomes

$$H_y \neq 0.$$

In equation (A.6) the variable g plays the role of a parameter. Finding any solution $H(x, y, g)$ of equation (A.6) satisfying this condition one finds the transformation of the equation $y'' = f(x, y, p)$ to the equation $u'' = 0$.

APPENDIX B

A PARTICULAR LINEARIZATION PROBLEM OF FOURTH-ORDER ODEs UNDER CONTACT TRANSFORMATIONS

In 2005, Dridi and Neut used Cartan's method to study the equivalence problem of fourth-order ordinary differential equation with the flat model under contact transformations. As a result, they obtained that the following propositions are equivalent:

(i) the equation $y^{(4)} = f(x, y, y', y'', y''')$ is equivalent to the equation $u^{(4)} = 0$ under a contact transformation,

(ii) the equation $y^{(4)} = f(x, y, y', y'', y''')$ admits a contact symmetry group of 8 parameters,

(iii) f satisfies

$$f_{rrr} = 0, \quad {f_{rr}}^2 + 6f_{qrr} = 0,$$

$$-4f_q f_r + 6f_r Df_r - {f_r}^3 - 8f_q - 4D^2 f_r + 8Df_q = 0,$$

$$-1440 f_r D^2 f_r - 1600 f_y + 832 f_q Df_r - 144{f_q}^2 + 1512 Df_r {f_r}^2$$

$$-808 f_q f_r^2 + 480 D^3 f_r - 1600 f_p f_r - 189{f_r}^4 + 2000 Df_q f_r$$

$$-864\,(Df_r)^2 - 1120 D^2 f_q + 1600 Df_q = 0,$$

where $p = y',\ q = y'',\ r = y'''$ and $D = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + r\frac{\partial}{\partial q} + f(x, y, p, q, r)\frac{\partial}{\partial r}$.

APPENDIX C

LINEAR SECOND-ORDER ODEs

The general form of a linear second-order ordinary differential equation is

$$y''(x) + a(x)\,y'(x) + b(x)\,y(x) = c(x)\,. \tag{C.1}$$

Any linear second-order ordinary differential equation (C.1) is equivalent to the equation

$$u'' = 0.$$

In fact, because a solution of equation (C.1) is represented as

$$y = y_h + y_p,$$

where y_h is a solution of the homogeneous equation

$$y'' + ay' + by = 0$$

and y_p is a particular solution of equation (C.1), one can construct the transformation

$$t = x, \;\; v = y - y_p.$$

Then,

$$v' = y' - y_p',$$

$$v'' = y'' - y_p''.$$

Substituting these expressions into equation (C.1), one gets

$$\begin{aligned} \left(v'' + y_p''\right) + a\left(v' + y_p'\right) + b\left(v + y_p\right) &= c \\ \left(v'' + av' + bv\right) + \left(y_p'' + ay_p' + by_p\right) &= c \\ \left(v'' + av' + bv\right) + c &= c \\ v'' + av' + bv &= 0. \end{aligned}$$

That is we can exclude the coefficient c from equation (C.1).

Let us exclude the coefficients a and b in the equation

$$y'' + ay' + by = 0. \tag{C.2}$$

Assume that a solution of equation (C.2) has the form

$$y(x) = u(t)\, q(x).$$

Consider the transformation

$$t = x, \quad u = \frac{y}{q}; \quad q \neq 0.$$

Then,

$$y' = u'q + uq',$$

$$y'' = u''q + 2u'q' + uq''.$$

Substituting these expressions into equation (C.2), one has

$$\begin{aligned} (u''q + 2u'q' + uq'') + a(u'q + uq') + buq &= 0 \\ u''q + u'(2q' + aq) + u(q'' + aq' + bq) &= 0. \end{aligned}$$

Choosing $q(x)$ which satisfies equation (C.2), one obtains

$$u'' + u'\left(\frac{2q'}{q} + a\right) = 0. \tag{C.3}$$

Thus, we can exclude the coefficient b in equation (C.2).

Next, let the function $h(x)$ such that

$$h' = \frac{2q'}{q} + a.$$

Then, equation (C.3) becomes

$$u'' + u'h' = 0.$$

Because of

$$e^{\int_{t_0}^{t} h'(s)ds}\,(u'' + u'h') = 0$$

so that,

$$e^{h}\,(u'' + u'h') = 0.$$

Setting $v' = u'e^{h}$. Hence, one obtains

$$v'' = 0.$$

APPENDIX D

SOME MATERIAL FOR REVIEW AND REFERENCE

Theorem 4.1. *(Inverse Function Theorem). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on some open set containing a, and suppose $\det Jf(a) \neq 0$, where J is the Jacobian matrix. Then there is some open set V containing a and an open W containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable for all $y \in W$.*

Theorem 4.2. *(Faa de Bruno Formula). If g and f are functions with a sufficient number of derivatives, then*

$$\frac{d^k}{dt^k} g\left(f\left(t\right)\right) = \sum \frac{k!}{l_1! l_2! \cdots l_k!} g^{(m)}\left(f\left(t\right)\right) \left(\frac{f'\left(t\right)}{1!}\right)^{l_1} \left(\frac{f''\left(t\right)}{2!}\right)^{l_2} \cdots \left(\frac{f^{(k)}\left(t\right)}{k!}\right)^{l_k},$$

where the sum is over all different solutions in nonnegative integers $l_1, l_2, \ldots, l_k$ of $l_1 + 2l_2 + \cdots + kl_k = k$, and $m = l_1 + l_2 + \cdots + l_k$.

Theorem 4.3. *(Leibnitz Formula for the n-th Derivative of a Product). Let $u(x)$ and $v(x)$ be functions of class C^n, i.e., functions with continuous n-th derivative. Then their product is also of class C^n, and*

$$\frac{d^n}{dx^n}\left[u\left(x\right) v\left(x\right)\right] = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \frac{d^r}{dx^r}\left[u\left(x\right)\right] \frac{d^{n-r}}{dx^{n-r}}\left[v\left(x\right)\right],$$

where

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!\left(n-r\right)!}$$

is the usual binomial coefficient.

APPENDIX E

PROOF OF THEOREM 2.1

Proof : The substitution $y = vq$ leads equation (2.11) to the equation

$$v^{(k)} + \overline{a}_{k-2}v^{(k-2)} + \cdots + \overline{a}_1 v' + \overline{a}_0 v = 0. \tag{E.1}$$

In fact, the Leibnitz formula for the derivative of the product of the functions v and q is

$$\begin{aligned}(vq)^{(k)} &= \sum_{r=0}^{k} \frac{k!}{(k-r)!r!} v^{(k-r)} q^{(r)} \\ &= \sum_{r=2}^{k} \frac{k!}{(k-r)!r!} v^{(k-r)} q^{(r)} + v^{(k)} q + k v^{(k-1)} q'.\end{aligned}$$

Hence, equation (2.11) becomes

$$v^{(k)} q + v^{(k-1)} \left(kq' + a_{k-1} q\right) + \cdots = 0.$$

Choosing the function q such that

$$kq' + a_{k-1} q = 0$$

one obtains equation (E.1).

Let us exclude the coefficient a_{k-2} in the equation

$$y^{(k)} + a_{k-2} y^{(k-2)} + a_{k-3} y^{(k-3)} + \cdots + a_1 y' + a_0 y = 0. \tag{E.2}$$

For this purpose, one can use the change

$$t = \varphi\left(x\right), \quad y = uq\left(x\right).$$

Consider

$$y^{(k)} + a_{k-2}y^{(k-2)} = \left(\frac{d^k}{dt^k}u\right)q + k\left(\frac{d^{k-1}}{dt^{k-1}}u\right)q' + \frac{k(k-1)}{2}\left(\frac{d^{k-2}}{dt^{k-2}}u\right)q'' + \cdots + a_{k-2}\left[\left(\frac{d^{k-2}}{dt^{k-2}}u\right)q + \cdots\right]. \tag{E.3}$$

Using the Faa de Bruno formula, one has

$$\frac{d^k}{dt^k}u = \sum \frac{k!}{l_1!l_2!\cdots l_k!}u^{(l_1+l_2+\cdots+l_k)}\left(\frac{\varphi'}{1!}\right)^{l_1}\left(\frac{\varphi''}{2!}\right)^{l_2}\cdots\left(\frac{\varphi^{(k)}}{k!}\right)^{l_k} = b_0u^{(k)} + b_1u^{(k-1)} + b_2u^{(k-2)} + \cdots .$$

For finding the coefficient b_0 one has to solve the equation

$$l_1 + 2l_2 + \cdots + kl_k = k,$$
$$l_1 + l_2 + \cdots + l_k = k.$$

Thus,

$$l_2 + 2l_3 + \cdots + (k-1)\,l_k = 0.$$

This means that $l_2 = l_3 = \cdots = l_k = 0$ and $l_1 = k$.

Similar for the coefficient b_1:

$$l_1 + 2l_2 + \cdots + kl_k = k,$$
$$l_1 + l_2 + \cdots + l_k = k-1.$$

Elimination of l_1 gives

$$l_2 + 2l_3 + \cdots + (k-1)\,l_k = 1,$$

whence $l_2 = 1,\ l_3 = \cdots = l_k = 0$ and $l_1 = k-2$.

For b_2:

$$l_1 + 2l_2 + \cdots + kl_k = k,$$
$$l_1 + l_2 + \cdots + l_k = k-2,$$

or

$$l_2 + 2l_3 + \cdots + (k-1)\, l_k = 2,$$

which gives case $l_2 = 2,\ l_3 = \cdots = l_k = 0$ and then $l_1 = k-4$ and case $l_2 = 0,\ l_3 = 1,\ l_4 = \cdots = l_k = 0,\ l_1 = k-3$.

Notice that for $k = 3$ the first case is not involved in calculations.

Thus, one obtains

$$\begin{aligned}
b_0 &= (\varphi')^k, \\
b_1 &= \frac{k(k-1)}{2} (\varphi')^{k-2} \varphi'', \\
b_2 &= \alpha_k \frac{k!}{(k-4)!2} (\varphi')^{k-4} \left(\frac{\varphi''}{2}\right)^2 + \frac{k!}{(k-3)!} (\varphi')^{k-3} \left(\frac{\varphi'''}{3!}\right),
\end{aligned}$$

where $\alpha_k = \begin{cases} 0, & k = 3 \\ 1, & k \neq 3 \end{cases} = 1 - \delta_{k3}.$

Therefore,

$$\begin{aligned}
\frac{d^k u}{dt^k} &= u^{(k)} (\varphi')^k + \frac{k(k-1)}{2} (\varphi')^{k-2} \varphi'' u^{(k-1)} \\
&\quad + \left[\alpha_k \frac{k!}{(k-4)!2} (\varphi')^{k-4} \left(\frac{\varphi''}{2}\right)^2 + \frac{k!}{(k-3)!} (\varphi')^{k-3} \left(\frac{\varphi'''}{3!}\right)\right] u^{(k-2)} + \cdots, \\
\frac{d^{k-1} u}{dt^{k-1}} &= u^{(k-1)} (\varphi')^{k-1} + \frac{(k-1)(k-2)}{2} (\varphi')^{k-3} \varphi'' u^{(k-2)} + \cdots, \\
\frac{d^{k-2} u}{dt^{k-2}} &= u^{(k-2)} (\varphi')^{k-2} + \cdots.
\end{aligned}$$

Substituting these expressions into equation (E.3), one arrives at equation

$$\begin{aligned}
y^{(k)} + a_{k-2} y^{(k-2)} &= q u^{(k)} (\varphi')^k + u^{(k-1)} \left[\frac{k(k-1)}{2} \varphi'' (\varphi')^{k-2} q + k q' (\varphi')^{k-1}\right] \\
&\quad + u^{(k-2)} \left\{ q \left(\frac{k!}{(k-3)!} (\varphi')^{k-3} \left(\frac{\varphi'''}{3!}\right) \right.\right. \\
&\quad \left. + \alpha_k \frac{k!}{(k-4)!2} (\varphi')^{k-4} \left(\frac{\varphi''}{2}\right)^2 \right) \\
&\quad + k q' \left(\frac{(k-1)(k-2)}{2} (\varphi')^{k-3} \varphi'' \right) + \frac{k(k-1)}{2} q'' (\varphi')^{k-2} \\
&\quad \left. + a_{k-2} q (\varphi')^{k-2} \right\} + \cdots.
\end{aligned}$$

The functions $q(x)$ and $\varphi(x)$ satisfy the equations

$$\frac{(k-1)}{2}\varphi''q + q'\varphi' = 0, \tag{E.4}$$

$$\begin{aligned} q\frac{k!}{(k-3)!}\frac{\varphi'''}{3!} + \alpha_k\frac{k!}{(k-4)!2}\frac{1}{\varphi'}\left(\frac{\varphi''}{2}\right)^2 q + kq'\frac{(k-1)(k-2)}{2}\varphi'' \\ +\frac{k(k-1)}{2}q''\varphi' + a_{k-2}q\varphi' = 0. \end{aligned} \tag{E.5}$$

From equation (E.4) one obtains

$$q' = q\frac{(1-k)}{2}\frac{\varphi''}{\varphi'}$$

so that,

$$\begin{aligned} q'' &= \frac{(1-k)}{2}\left[\left(\frac{\varphi'''}{\varphi'} - \left(\frac{\varphi''}{\varphi'}\right)^2\right) + \frac{(1-k)}{2}\left(\frac{\varphi''}{\varphi'}\right)^2\right] q \\ &= \frac{(1-k)}{2}\left[\frac{\varphi'''}{\varphi'} - \frac{(k+1)}{2}\left(\frac{\varphi''}{\varphi'}\right)^2\right] q. \end{aligned}$$

Substituting these expressions into equation (E.5), one gets

$$\begin{aligned} \frac{k!}{(k-3)!}\frac{\varphi'''}{3!} + \alpha_k\frac{k!}{(k-4)!2}\frac{1}{\varphi'}\left(\frac{\varphi''}{2}\right)^2 + \frac{k(k-1)(k-2)}{2}\frac{(1-k)}{2}\frac{(\varphi'')^2}{\varphi'} \\ +\frac{k(k-1)}{2}\frac{(1-k)}{2}\left(\varphi''' - \frac{(k+1)}{2}\frac{(\varphi'')^2}{\varphi'}\right) + a_{k-2}\varphi' = 0. \end{aligned}$$

Hence, equation (E.2) is equivalent to equation (2.12). The proof is thus complete.

APPENDIX F

COMPATIBILITY ANALYZE OF THE SYSTEM OF EQUATIONS FOR FUNCTIONS φ AND ψ IN SECTION 2.5

Since for a given equation there are only two unknown functions $\varphi(x,y)$ and $\psi(x,y)$, equations (2.16) form an overdetermined system of partial differential equations. Let us analyze the compatibility of this system.

First assume that $\varphi_y = 0$. From relations (2.16) one defines

$$a = 0, \quad \psi_{yy} = \psi_y b, \quad 2\psi_{xy} = \varphi_x^{-1}\psi_y\varphi_{xx} + c\psi_y, \quad \psi_{xx} = \varphi_x^{-1}\psi_x\varphi_{xx} + \psi_y d. \quad \text{(F.1)}$$

Comparing the mixed derivatives $(\psi_{xy})_y = (\psi_{yy})_x$ and $(\psi_{xy})_x = (\psi_{xx})_y$, one finds

$$c_y = 2b_x, \quad \varphi_x^{-2}\left(2\varphi_x\varphi_{xx} - 3\varphi_{xx}^2\right) = 4\left(d_y + bd\right) - \left(2c_x + c^2\right). \quad \text{(F.2)}$$

Because $\varphi_y = 0$, differentiating the last equation with respect to y , one obtains

$$d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0.$$

Thus, a second-order ordinary differential equation of the form (2.15) is linearizable with the function $\varphi = \varphi(x)$ if the coefficients of this equation satisfy the conditions

$$a = 0, \quad c_y = 2b_x, \quad d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0. \quad \text{(F.3)}$$

The functions $\varphi(x)$ and $\psi(x,y)$ are restituted by solving the involutive overdetermined system of equations (F.1) and (F.2).

Relations (2.16) in the case $\varphi(y) \neq 0$ are analyzed similarly, but the process is more cumbersome. In fact, from equations (2.16) one finds

$$\varphi_y \psi_{yy} = (\varphi_{yy}\psi_y + a\, \triangle),$$

$$2\varphi_y^2 \psi_{xy} = 2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\, \triangle - (a\varphi_x - b\varphi_y)\, \triangle,$$

$$\varphi_y^2 \psi_{xx} = 2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2(\psi_y d - \psi_x c),$$

$$\varphi_y^2 \varphi_{xx} = 2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d.$$

From the equations $(\psi_{xy})_y = (\psi_{yy})_x$ and $(\psi_{xy})_x = (\psi_{xx})_y$, one gets

$$\begin{aligned} 2\varphi_y\varphi_{yyy} = {} & 3\left(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy} a + \varphi_x^2 a^2\right) - 2\varphi_x\varphi_y (a_y + ab) \\ & + \varphi_y^2\left(2b_y - 4a_x + 4ac - b^2\right), \\ 6\varphi_y^2\varphi_{xyy} = {} & 3\left(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b\right) \\ & + 3\varphi_x^3 a^2 + 3\varphi_x\varphi_y^2\left(-2a_x + 2ac - b^2\right) + 2\varphi_y^3(-b_x + 2c_y + 3ad). \end{aligned}$$

Forming the mixed derivatives $(\varphi_{xyy})_y = (\varphi_{yyy})_x$ and $(\varphi_{xx})_{yy} = (\varphi_{xyy})_x$, one obtains equations (2.17). Conditions (F.3) form a particular case of the relations (2.17): they are selected by the way of finding a linearizing transformation.

APPENDIX G

EQUATIONS FOR THE SECOND CLASS OF LINEARIZABLE EQUATIONS IN SECTION 3.3

G.1 Equations for Theorem 3.5 in Section 3.3.1

In this section we present equations which were used in previous sections.

$$10r_{yy} = -(F_{1y} + F_{2x} + F_{2y}r + r_yF_2), \tag{G.1}$$

$$10r_x = 10r_yr - F_0 + F_1r - F_2r^2, \tag{G.2}$$

$$H_2 = -3F_2, \tag{G.3}$$

$$4H_1 = -3(5F_1 - 2F_2r), \tag{G.4}$$

$$4H_0 = -3(6F_0 - F_1r), \tag{G.5}$$

$$\begin{aligned}10F_{1yy} = &-(F_{1y}F_2 - 40F_{2xy} - 16F_{2x}F_2 + 20F_{2yy}r + 40F_{2y}r_y \\ &+ 14F_{2y}F_2r + 20J_{4x} - 20J_{4y}r + 14r_yF_2^2 - 40r_yJ_4),\end{aligned} \tag{G.6}$$

$$12F_{2x} = 12F_{2y}r - 3F_1F_2 + 6F_2^2r + 4J_3 - 16J_4r, \tag{G.7}$$

$$\begin{aligned}60F_{1x} = &60F_{1y}r - 36F_0F_2 - 15F_1^2 + 66F_1F_2r - 36F_2^2r^2 + 40J_2 \\ &- 80J_3r + 80J_4r^2,\end{aligned} \tag{G.8}$$

$$\begin{aligned}60F_{0x} = &60F_{0y}r - 51F_0F_1 + 66F_0F_2r + 36F_1^2r - 72F_1F_2r^2 + 36F_2^2r^3 \\ &+ 60J_1 - 80J_2r + 80J_3r^2 - 80J_4r^3,\end{aligned} \tag{G.9}$$

$$20J_0 = 9F_0^2 - 18F_0F_1r + 18F_0F_2r^2 + 9F_1^2r^2 - 18F_1F_2r^3 + 9F_2^2r^4 + 20J_1r - 20J_2r^2 + 20J_3r^3 - 20J_4r^4, \tag{G.10}$$

$$\begin{aligned}120J_{3yy} = {} & 216F_{1y}F_{2y} + 54F_{1y}F_2^2 - 48F_{1y}J_4 + 360F_{2yy}r_y + 90F_{2yy}F_1 \\ & - 180F_{2yy}F_2r - 432F_{2y}^2r + 324F_{2y}r_yF_2 + 189F_{2y}F_1F_2 - 486F_{2y}F_2^2r \\ & - 192F_{2y}J_3 + 864F_{2y}J_4r - 60J_{3y}F_2 + 720J_{4xy} + 180J_{4x}F_2 - 240J_{4yy}r \\ & - 1200J_{4y}r_y + 60J_{4y}F_2r + 720K_{6x} - 720K_{6y}r - 5040K_{7x}r + 5040K_{7y}r^2 \\ & + 36r_yF_2^3 - 432r_yF_2J_4 - 2160r_yK_6 + 15120r_yK_7r + 504F_0K_7 + 36F_1F_2^3 \\ & - 102F_1F_2J_4 - 504F_1K_7r - 72F_2^4r - 48F_2^2J_3 + 396F_2^2J_4r + 504F_2K_7r^2 \\ & + 136J_3J_4 - 544J_4^2r,\end{aligned} \tag{G.11}$$

$$
\begin{aligned}
240J_{4xyy} = &-(36F_{1y}F_{2yy} + 162F_{1y}F_{2y}F_2 - 72F_{1y}J_{4y} + 36F_{1y}F_2^3 \\
&- 168F_{1y}F_2J_4 - 72F_{1y}K_6 - 168F_{1y}K_7r - 72F_{2yy}F_{2y}r + 144F_{2yy}r_yF_2 \\
&+ 54F_{2yy}F_1F_2 - 108F_{2yy}F_2^2r - 72F_{2yy}J_3 + 288F_{2yy}J_4r + 432F_{2y}^2r_y \\
&+ 108F_{2y}^2F_1 - 540F_{2y}^2F_2r - 144F_{2y}J_{3y} + 528F_{2y}J_{4x} + 192F_{2y}J_{4y}r \\
&+ 324F_{2y}r_yF_2^2 - 1008F_{2y}r_yJ_4 + 162F_{2y}F_1F_2^2 - 132F_{2y}F_1J_4 - 396F_{2y}F_2^3r \\
&- 180F_{2y}F_2J_3 + 1320F_{2y}F_2J_4r + 144F_{2y}K_6r - 336F_{2y}K_7r^2 - 36J_{3y}F_2^2 \\
&+ 176J_{3y}J_4 + 120J_{4xy}F_2 + 132J_{4x}F_2^2 - 432J_{4x}J_4 - 240J_{4yyy}r - 960J_{4yy}r_y \\
&- 120J_{4yy}F_2r - 768J_{4y}r_yF_2 - 138J_{4y}F_1F_2 + 288J_{4y}F_2^2r + 184J_{4y}J_3 \\
&- 1008J_{4y}J_4r + 960K_{6xy} + 240K_{6x}F_2 - 960K_{6yy}r - 3840K_{6y}r_y \\
&- 240K_{6y}F_2r - 1920K_{7xy}r - 2400K_{7xx} + 2880K_{7x}r_y - 600K_{7x}F_1 \\
&- 480K_{7x}F_2r + 4320K_{7yy}r^2 + 24000K_{7y}r_yr + 432K_{7y}F_0 + 168K_{7y}F_1r \\
&+ 912K_{7y}F_2r^2 + 20160r_y^2K_7 + 1728r_yF_1K_7 + 36r_yF_2^4 - 264r_yF_2^2J_4 \\
&- 1248r_yF_2K_6 + 5280r_yF_2K_7r + 160r_yJ_4^2 + 408F_0F_2K_7 + 150F_1^2K_7 \\
&+ 27F_1F_2^4 - 120F_1F_2^2J_4 - 168F_1F_2K_6 + 168F_1F_2K_7r - 54F_2^5r - 36F_2^3J_3 \\
&+ 384F_2^3J_4r + 336F_2^2K_6r - 1344F_2^2K_7r^2 + 160F_2J_3J_4 - 640F_2J_4^2r \\
&- 400J_2K_7 + 224J_3K_6 - 368J_3K_7r - 896J_4K_6r + 3872J_4K_7r^2 \\
&+ 672F_{0y}K_7),
\end{aligned} \tag{G.12}
$$

$$
4J_{4x} = 4J_{4y}r - F_1J_4 + 2F_2J_4r - 4K_5 + 24K_6r - 84K_7r^2, \tag{G.13}
$$

$$60F_{0yy} = -(30F_{0y}F_2 + 36F_{1y}F_1 - 36F_{1y}F_2r - 60F_{2yy}r^2 + 24F_{2y}F_0$$
$$- 36F_{2y}F_1r - 54F_{2y}F_2r^2 - 40J_{2y} + 40J_{3y}r + 80J_{4y}r^2$$
$$- 36r_yF_1F_2 + 36r_yF_2^2r + 40r_yJ_3 - 80r_yJ_4r + 6F_0F_2^2 - 6F_0J_4$$
$$+ 9F_1^2F_2 - 18F_1F_2^2r - 12F_1J_3 + 24F_1J_4r - 6F_2^3r^2 - 10F_2J_2$$
$$+ 22F_2J_3r + 26F_2J_4r^2 - 60K_4 + 180K_5r - 180K_6r^2 - 420K_7r^3), \qquad \text{(G.14)}$$

$$20J_{2x} = 20J_{2y}r + 20J_{3x}r - 20J_{3y}r^2 - 14F_0J_3 + 28F_0J_4r - 5F_1J_2$$
$$+ 19F_1J_3r - 28F_1J_4r^2 + 10F_2J_2r - 24F_2J_3r^2 + 28F_2J_4r^3$$
$$- 120K_3 + 360K_4r - 640K_5r^2 + 840K_6r^3 - 840K_7r^4, \qquad \text{(G.15)}$$

$$60J_{1x} = 60J_{1y}r - 40J_{3x}r^2 + 40J_{3y}r^3 - 42F_0J_2 + 42F_0J_3r - 70F_0J_4r^2$$
$$- 15F_1J_1 + 42F_1J_2r - 52F_1J_3r^2 + 70F_1J_4r^3 + 30F_2J_1r$$
$$- 42F_2J_2r^2 + 62F_2J_3r^3 - 70F_2J_4r^4 - 600K_2 + 1080K_3r - 1380K_4r^2$$
$$+ 1700K_5r^3 - 2100K_6r^4 + 2100K_7r^5, \qquad \text{(G.16)}$$

$$80K_1 = 3F_0^2F_1 - 6F_0^2F_2r - 6F_0F_1^2r + 18F_0F_1F_2r^2 - 12F_0F_2^2r^3 - 8F_0J_1$$
$$+ 16F_0J_2r - 24F_0J_3r^2 + 32F_0J_4r^3 + 3F_1^3r^2 - 12F_1^2F_2r^3 + 15F_1F_2^2r^4$$
$$+ 8F_1J_1r - 16F_1J_2r^2 + 24F_1J_3r^3 - 32F_1J_4r^4 - 6F_2^3r^5 - 8F_2J_1r^2$$
$$+ 16F_2J_2r^3 - 24F_2J_3r^4 + 32F_2J_4r^5 + 160K_2r - 240K_3r^2 + 320K_4r^3$$
$$- 400K_5r^4 + 480K_6r^5 - 560K_7r^6, \qquad \text{(G.17)}$$

$$400K_0 = -(6F_0^3 - 33F_0^2F_1r + 48F_0^2F_2r^2 + 48F_0F_1^2r^2 - 126F_0F_1F_2r^3$$
$$+ 78F_0F_2^2r^4 + 40F_0J_1r - 80F_0J_2r^2 + 120F_0J_3r^3 - 160F_0J_4r^4 - 21F_1^3r^3$$
$$+ 78F_1^2F_2r^4 - 93F_1F_2^2r^5 - 40F_1J_1r^2 + 80F_1J_2r^3 - 120F_1J_3r^4 + 160F_1J_4r^5$$
$$+ 36F_2^3r^6 + 40F_2J_1r^3 - 80F_2J_2r^4 + 120F_2J_3r^5 - 160F_2J_4r^6 - 400K_2r^2$$
$$+ 800K_3r^3 - 1200K_4r^4 + 1600K_5r^5 - 2000K_6r^6 + 2400K_7r^7). \qquad \text{(G.18)}$$

G.2 Equations for Theorem 3.6 in Section 3.3.1

$$\varphi_x = r\varphi_y, \tag{G.19}$$

$$\varphi_y\psi_x = r\varphi_y\psi_y - \Delta, \tag{G.20}$$

$$10\Delta\varphi_{yy} = \varphi_y(4\Delta_y - F_2\Delta), \tag{G.21}$$

$$\begin{aligned}
&500\varphi_y\psi_{yyyy}\Delta^3 = 300\psi_{yyy}\varphi_y\Delta^2(4\Delta_y - F_2\Delta) + 5\psi_{yy}\varphi_y\Delta(-120F_{2y}\Delta^2 \\
&- 144\Delta_y^2 + 72\Delta_y F_2\Delta - 39F_2^2\Delta^2 + 80J_4\Delta^2) + \psi_y\varphi_y(-500\varphi_y^3\alpha\Delta^3 \\
&- 150F_{2yy}\Delta^3 + 360F_{2y}\Delta_y\Delta^2 - 165F_{2y}F_2\Delta^3 + 100J_{4y}\Delta^3 + 96\Delta_y^3 \\
&- 72\Delta_y^2F_2\Delta + 108\Delta_yF_2^2\Delta^2 - 240\Delta_yJ_4\Delta^2 - 24F_2^3\Delta^3 + 60F_2J_4\Delta^3) \\
&- 500\psi\varphi_y^5\beta\Delta^3 + 500K_7\Delta^4,
\end{aligned} \tag{G.22}$$

$$\alpha = \frac{\Theta}{8\varphi_y^3}, \tag{G.23}$$

$$\begin{aligned}
\beta = {} & (1600\Delta\varphi_y^4)^{-1}\Big[\Delta(-144F_{2y}^2 - 72F_{2y}F_2^2 + 352F_{2y}J_4 + 160J_{4yy} \\
& + 80J_{4y}F_2 + 640K_{6y} - 1600K_{7x} - 2880K_{7y}r + 80\Theta_y - 4480r_yK_7 \\
& - 400F_1K_7 - 9F_2^4 + 88F_2^2J_4 + 160F_2K_6 - 320F_2K_7r - 144J_4^2) \\
& - 120\Delta_y\Theta\Big],
\end{aligned} \tag{G.24}$$

where Θ is the following expression

$$\Theta = (F_2^2 - 4J_4)F_2 - 8(K_6 - 7K_7r) - 8J_{4y} + 6F_{2y}F_2 + 4F_{2yy}. \tag{G.25}$$

G.3 Equations for Lemma 3.7 in Section 3.3.2

$$F_2 = -2(\varphi_y\Delta)^{-1}(5\varphi_{yy}\Delta - 2\varphi_y\Delta_y), \tag{G.26}$$

$$F_1 = 4(\varphi_y\Delta)^{-1}\Big[(\Delta_x + \Delta_y r - 5r_y\Delta)\varphi_y - 5\varphi_{yy}r\Delta\Big], \tag{G.27}$$

$$F_0 = -2(\varphi_y\Delta)^{-1}\Big[((5r_y\Delta - 2\Delta_x)r + 5r_x\Delta)\varphi_y + 5\varphi_{yy}r^2\Delta\Big], \tag{G.28}$$

$$H_2 = 6(\varphi_y\Delta)^{-1}(5\varphi_{yy}\Delta - 2\varphi_y\Delta_y), \tag{G.29}$$

$$H_1 = -3(\varphi_y\Delta)^{-1}\Big[(5\Delta_x + 3\Delta_y r - 25r_y\Delta)\varphi_y - 20\varphi_{yy}r\Delta\Big], \tag{G.30}$$

$$H_0 = 3(\varphi_y\Delta)^{-1}\Big[(5(3r_x + 2r_y r)\Delta - (5\Delta_x - \Delta_y r)r)\varphi_y + 10\varphi_{yy}r^2\Delta\Big], \tag{G.31}$$

$$J_4 = -(\varphi_y^2\Delta)^{-1}(10\varphi_{yyy}\varphi_y\Delta - 45\varphi_{yy}^2\Delta + 30\varphi_{yy}\varphi_y\Delta_y - 6\varphi_y^2\Delta_{yy}), \tag{G.32}$$

$$\begin{aligned} J_3 = {} & 2(\varphi_y^2\Delta)^{-1}\Big[3((2(\Delta_{xy} + \Delta_{yy}r - 5r_y\Delta_y) - 5r_{yy}\Delta)\varphi_y^2 \\ & - 5((\Delta_x + 3\Delta_y r - 4r_y\Delta)\varphi_y - 6\varphi_{yy}r\Delta)\varphi_{yy}) - 20\varphi_{yyy}\varphi_y r\Delta\Big], \end{aligned} \tag{G.33}$$

$$\begin{aligned} J_2 = {} & 6(\varphi_y^2\Delta)^{-1}\Big[(\Delta_{xx} + \Delta_{yy}r^2 + 4\Delta_{xy}r - 5(2\Delta_x + 3\Delta_y r - 5r_y\Delta)r_y \\ & - 10r_{yy}r\Delta - 5r_x\Delta_y - 5r_{xy}\Delta)\varphi_y^2 - 5(((3(\Delta_x + \Delta_y r) - 10r_y\Delta)r \\ & - 2r_x\Delta)\varphi_y - 9\varphi_{yy}r^2\Delta)\varphi_{yy} - 10\varphi_{yyy}\varphi_y r^2\Delta\Big], \end{aligned} \tag{G.34}$$

$$\begin{aligned} J_1 = {} & -2(\varphi_y^2\Delta)^{-1}\Big[((5(3(3\Delta_x + \Delta_y r) - 14r_y\Delta)r_y - 6(\Delta_{xy}r + \Delta_{xx}) \\ & + 20r_{yy}r\Delta)r + 5(3(\Delta_x + \Delta_y r) - 16r_y\Delta)r_x + 5r_{xx}\Delta + 20r_{xy}r\Delta)\varphi_y^2 \\ & + 15(((3\Delta_x + \Delta_y r - 8r_y\Delta)r - 4r_x\Delta)\varphi_y - 6\varphi_{yy}r^2\Delta)\varphi_{yy}r \\ & + 20\varphi_{yyy}\varphi_y r^3\Delta\Big], \end{aligned} \tag{G.35}$$

$$J_0 = -(\varphi_y^2\Delta)^{-1}\Big[((2((5r_{yy}r\Delta - 3\Delta_{xx})r + 5r_{xx}\Delta + 5r_{xy}r\Delta)$$
$$- 5(7r_y\Delta - 6\Delta_x)r_yr)r - 5(2(7r_y\Delta - 3\Delta_x)r + 9r_x\Delta)r_x)\varphi_y^2$$
$$- 5(3(2((2r_y\Delta - \Delta_x)r + 2r_x\Delta)\varphi_y + 3\varphi_{yy}r^2\Delta)\varphi_{yy}$$
$$- 2\varphi_{yyy}\varphi_y r^2\Delta)r^2\Big], \tag{G.36}$$

$$K_7 = -(\varphi_y^2\Delta)^{-1}\Big[\varphi_{yyyy}\varphi_y^2\psi_y - 10\varphi_{yyy}\varphi_{yy}\varphi_y\psi_y + 4\varphi_{yyy}\varphi_y^2\psi_{yy} + 15\varphi_{yy}^3\psi_y$$
$$- 15\varphi_{yy}^2\varphi_y\psi_{yy} + 6\varphi_{yy}\varphi_y^2\psi_{yyy} - \varphi_y^7\beta\psi - \varphi_y^6\psi_y\alpha - \varphi_y^3\psi_{yyyy}\Big], \tag{G.37}$$

$$K_6 = (\varphi_y^3\Delta)^{-1}\Big[3(5((7\varphi_y\psi_{yy}r - 6\Delta_y)\varphi_y - 7(\varphi_y\psi_y r - \Delta)\varphi_{yy})\varphi_{yy}$$
$$- 2(7\varphi_y\psi_{yyy}r - 5\Delta_{yy})\varphi_y^2)\varphi_{yy} + (7\varphi_y^5\beta\psi r + 7\varphi_y^4\psi_y\alpha r - \varphi_y^3\alpha\Delta$$
$$+ 7\varphi_y\psi_{yyyy}r - 4\Delta_{yyy})\varphi_y^3 + 2(35\varphi_{yy}\varphi_y\psi_y r - 30\varphi_{yy}\Delta - 14\varphi_y^2\psi_{yy}r$$
$$+ 10\varphi_y\Delta_y)\varphi_{yyy}\varphi_y - (7\varphi_y\psi_y r - 5\Delta)\varphi_{yyyy}\varphi_y^2\Big], \tag{G.38}$$

$$K_5 = -(\varphi_y^3\Delta)^{-1}\Big[(2(3(\Delta_{xyy} + 3\Delta_{yyy}r - 5r_y\Delta_{yy} - 5r_{yy}\Delta_y) - 5r_{yyy}\Delta)$$
$$- 3(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 2\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r)\varphi_y r)\varphi_y^3$$
$$- 3(2(5(\Delta_{xy} + 5\Delta_{yy}r - 4r_y\Delta_y - 2r_{yy}\Delta) - 21\varphi_y\psi_{yyy}r^2)\varphi_y^2$$
$$- 15((\Delta_x + 11\Delta_y r - 3r_y\Delta - 7\varphi_y\psi_{yy}r^2)\varphi_y$$
$$+ 7(\varphi_y\psi_y r - 2\Delta)\varphi_{yy}r)\varphi_{yy})\varphi_{yy} - 2((5(\Delta_x + 11\Delta_y r - 3r_y\Delta)$$
$$- 42\varphi_y\psi_{yy}r^2)\varphi_y + 15(7\varphi_y\psi_y r - 12\Delta)\varphi_{yy}r)\varphi_{yyy}\varphi_y$$
$$+ 3(7\varphi_y\psi_y r - 10\Delta)\varphi_{yyyy}\varphi_y^2 r\Big], \tag{G.39}$$

$$
\begin{aligned}
K_4 = &-(\varphi_y^3\Delta)^{-1}\Big[(2(45r_{yy}r_y\Delta - 10r_{yy}\Delta_x - 55r_{yy}\Delta_y r + 50r_y^2\Delta_y\\
&- 20r_y\Delta_{xy} - 50r_y\Delta_{yy}r + 11\Delta_{xyy}r + 2\Delta_{xxy} + 17\Delta_{yyy}r^2\\
&- 20r_{yyy}r\Delta - 5r_x\Delta_{yy} - 10r_{xy}\Delta_y - 5r_{xyy}\Delta)\\
&- 5(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 3\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r)\varphi_y r^2)\varphi_y^3\\
&+ 15((3((5(\Delta_x + 5\Delta_y r) - 14r_y\Delta)r - r_x\Delta) - 35\varphi_y\psi_{yy}r^3)\varphi_y\\
&+ 35(\varphi_y\psi_y r - 3\Delta)\varphi_{yy}r^2)\varphi_{yy}^2 - 10(\Delta_{xx} + 31\Delta_{yy}r^2 + 13\Delta_{xy}r\\
&- 8(\Delta_x + 6\Delta_y r - 2r_y\Delta)r_y - 26r_{yy}r\Delta - 4r_x\Delta_y - 4r_{xy}\Delta\\
&- 21\varphi_y\psi_{yyy}r^3)\varphi_{yy}\varphi_y^2 - 10(((5(\Delta_x + 5\Delta_y r) - 14r_y\Delta)r - r_x\Delta\\
&- 14\varphi_y\psi_{yy}r^3)\varphi_y + 5(7\varphi_y\psi_y r - 18\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y\\
&+ 5(7\varphi_y\psi_y r - 15\Delta)\varphi_{yyyy}\varphi_y^2 r^2\Big],
\end{aligned} \tag{G.40}
$$

$$
\begin{aligned}
K_3 = &-(\varphi_y^3\Delta)^{-1}\Big[((13\Delta_{xxy} + 35\Delta_{yyy}r^2)r + \Delta_{xxx} + 31\Delta_{xyy}r^2\\
&- 5(3\Delta_{xx} + 26\Delta_{yy}r^2 + 23\Delta_{xy}r - (15\Delta_x + 49\Delta_y r - 25r_y\Delta)r_y)r_y\\
&- 5(13\Delta_x + 32\Delta_y r - 50r_y\Delta)r_{yy}r - 65r_{yyy}r^2\Delta - 5(3\Delta_{xy} + 5\Delta_{yy}r\\
&- 16r_y\Delta_y - 7r_{yy}\Delta)r_x - 5r_{xx}\Delta_y - 5r_{xxy}\Delta\\
&- 5(3\Delta_x + 11\Delta_y r - 15r_y\Delta)r_{xy} - 30r_{xyy}r\Delta - 5(7\varphi_y^4\beta\psi r + 7\varphi_y^3\psi_y\alpha r\\
&- 4\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r)\varphi_y r^3)\varphi_y^3 - 5(2((2(2\Delta_{xx} + 17\Delta_{yy}r^2 + 11\Delta_{xy}r)\\
&- (29\Delta_x + 75\Delta_y r - 51r_y\Delta)r_y - 45r_{yy}r\Delta)r - (3\Delta_x + 13\Delta_y r - 13r_y\Delta)r_x\\
&- r_{xx}\Delta - 14r_{xy}r\Delta - 21\varphi_y\psi_{yyy}r^4)\varphi_y^2 - 3((6((5(\Delta_x + 3\Delta_y r) - 13r_y\Delta)r\\
&- 2r_x\Delta) - 35\varphi_y\psi_{yy}r^3)\varphi_y + 35(\varphi_y\psi_y r - 4\Delta)\varphi_{yy}r^2)\varphi_{yy}r)\varphi_{yy}\\
&- 10(2((5(\Delta_x + 3\Delta_y r) - 13r_y\Delta)r - 2r_x\Delta - 7\varphi_y\psi_{yy}r^3)\varphi_y\\
&+ 5(7\varphi_y\psi_y r - 24\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r + 5(7\varphi_y\psi_y r - 20\Delta)\varphi_{yyyy}\varphi_y^2 r^3\Big],
\end{aligned} \tag{G.41}
$$

$$
\begin{aligned}
K_2 =& -(\varphi_y^3\Delta)^{-1}\Big[((3((5\Delta_{xxy}+7\Delta_{yyy}r^2)r+\Delta_{xxx}+7\Delta_{xyy}r^2) \\
&-(3(13\Delta_{xx}+28\Delta_{yy}r^2+39\Delta_{xy}r)+(204r_y\Delta-161\Delta_x-217\Delta_y r)r_y)r_y \\
&-(79\Delta_x+116\Delta_y r-264r_y\Delta)r_{yy}r-54r_{yyy}r^2\Delta)r \\
&-(3(2\Delta_{xx}+7\Delta_{yy}r^2+11\Delta_{xy}r)+(171r_y\Delta-64\Delta_x-140\Delta_y r)r_y \\
&-72r_{yy}r\Delta-18r_x\Delta_y)r_x-(4\Delta_x+11\Delta_y r-21r_y\Delta)r_{xx}-12r_{xxy}r\Delta \\
&-r_{xxx}\Delta-((37\Delta_x+53\Delta_y r-150r_y\Delta)r-33r_x\Delta)r_{xy}-33r_{xyy}r^2\Delta \\
&-3(7\varphi_y^4\beta\psi r+7\varphi_y^3\psi_y\alpha r-5\varphi_y^2\alpha\Delta+7\psi_{yyyy}r)\varphi_y r^4)\varphi_y^3 \\
&-3(2(5((2\Delta_{xx}+7\Delta_{yy}r^2+6\Delta_{xy}r-(13\Delta_x+19\Delta_y r-20r_y\Delta)r_y \\
&-13r_{yy}r\Delta)r^2-((3\Delta_x+5\Delta_y r-11r_y\Delta)r-r_x\Delta)r_x-r_{xx}r\Delta \\
&-6r_{xy}r^2\Delta)-21\varphi_y\psi_{yyy}r^5)\varphi_y^2-15((2((5(\Delta_x+2\Delta_y r)-12r_y\Delta)r \\
&-3r_x\Delta)-7\varphi_y\psi_{yy}r^3)\varphi_y+7(\varphi_y\psi_y r-5\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy} \\
&-2(2(5((5(\Delta_x+2\Delta_y r)-12r_y\Delta)r-3r_x\Delta)-21\varphi_y\psi_{yy}r^3)\varphi_y \\
&+15(7\varphi_y\psi_y r-30\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^2+3(7\varphi_y\psi_y r-25\Delta)\varphi_{yyyy}\varphi_y^2 r^4\Big],
\end{aligned}
\tag{G.42}
$$

$$
\begin{aligned}
K_1 =& -(\varphi_y^3\Delta)^{-1}\Big[((7(\Delta_{xxy}+\Delta_{yyy}r^2)r+3\Delta_{xxx}+7\Delta_{xyy}r^2 \\
&-(33\Delta_{xx}+28\Delta_{yy}r^2+49\Delta_{xy}r+2(59r_y\Delta-56\Delta_x-42\Delta_y r)r_y)r_y \\
&-(43\Delta_x+42\Delta_y r-128r_y\Delta)r_{yy}r-23r_{yyy}r^2\Delta)r^2 \\
&-((12\Delta_{xx}+7\Delta_{yy}r^2+21\Delta_{xy}r+2(86r_y\Delta-49\Delta_x-35\Delta_y r)r_y \\
&-49r_{yy}r\Delta)r+(85r_y\Delta-15\Delta_x-21\Delta_y r)r_x)r_x \\
&-((8\Delta_x+7\Delta_y r-32r_y\Delta)r-10r_x\Delta)r_{xx}-9r_{xxy}r^2\Delta-2r_{xxx}r\Delta \\
&-((29\Delta_x+21\Delta_y r-95r_y\Delta)r-46r_x\Delta)r_{xy}r-16r_{xyy}r^3\Delta \\
&-(7\varphi_y^4\beta\psi r+7\varphi_y^3\psi_y\alpha r-6\varphi_y^2\alpha\Delta+7\psi_{yyyy}r)\varphi_y r^5)\varphi_y^3 \\
&-(2(5((4\Delta_{xx}+7\Delta_{yy}r^2+7\Delta_{xy}r-(23\Delta_x+21\Delta_y r-31r_y\Delta)r_y \\
&-17r_{yy}r\Delta)r^2-((9\Delta_x+7\Delta_y r-27r_y\Delta)r-6r_x\Delta)r_x-3r_{xx}r\Delta \\
&-10r_{xy}r^2\Delta)-21\varphi_y\psi_{yyy}r^5)\varphi_y^2-15((3((5\Delta_x+7\Delta_y r-11r_y\Delta)r \\
&-4r_x\Delta)-7\varphi_y\psi_{yy}r^3)\varphi_y+7(\varphi_y\psi_y r-6\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}r \\
&-2((5((5\Delta_x+7\Delta_y r-11r_y\Delta)r-4r_x\Delta)-14\varphi_y\psi_{yy}r^3)\varphi_y \\
&+5(7\varphi_y\psi_y r-36\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^3+(7\varphi_y\psi_y r-30\Delta)\varphi_{yyyy}\varphi_y^2 r^5\Big], \quad \text{(G.43)}
\end{aligned}
$$

$$
\begin{aligned}
K_0 =&(\varphi_y^3\Delta)^{-1}\Big[((((2(r_{xxy}+2r_{yyy}r^2)r+r_{xxx}+3r_{xyy}r^2)\Delta \\
&+3(3\Delta_x+2\Delta_y r-8r_y\Delta)r_{yy}r^2)r-((10r_x+11r_y r)\Delta \\
&-(4\Delta_x+\Delta_y r)r)r_{xx}-((13r_x+20r_y r)\Delta-(7\Delta_x+3\Delta_y r)r)r_{xy}r \\
&+((\varphi_y^4\beta\psi+\varphi_y^3\psi_y\alpha+\psi_{yyyy})r-\varphi_y^2\alpha\Delta)\varphi_y r^5+(9\Delta_{xx}+4\Delta_{yy}r^2 \\
&+7\Delta_{xy}r-2(13\Delta_x+6\Delta_y r-12r_y\Delta)r_y)r_y r^2-((\Delta_{xxy}+\Delta_{yyy}r^2)r \\
&+\Delta_{xxx}+\Delta_{xyy}r^2)r^2)r-((2((17\Delta_x+5\Delta_y r-23r_y\Delta)r_y+6r_{yy}r\Delta) \\
&-(6\Delta_{xx}+\Delta_{yy}r^2+3\Delta_{xy}r))r^2-(5(3r_x+8r_y r)\Delta \\
&-3(5\Delta_x+\Delta_y r)r)r_x)r_x)\varphi_y^3-((2((5(r_{xx}+3r_{yy}r^2+2r_{xy}r)\Delta \\
&+3\varphi_y\psi_{yyy}r^4+5(5\Delta_x+3\Delta_y r-6r_y\Delta)r_y r-5(\Delta_{xx}+\Delta_{yy}r^2+\Delta_{xy}r)r)r \\
&-5((3r_x+7r_y r)\Delta-(3\Delta_x+\Delta_y r)r)r_x)\varphi_y^2-15((3(r_x+2r_y r)\Delta \\
&+\varphi_y\psi_{yy}r^3-3(\Delta_x+\Delta_y r)r)\varphi_y-(\varphi_y\psi_y r-7\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy} \\
&+(2((5(r_x+2r_y r)\Delta+2\varphi_y\psi_{yy}r^3-5(\Delta_x+\Delta_y r)r)\varphi_y \\
&-5(\varphi_y\psi_y r-6\Delta)\varphi_{yy}r^2)\varphi_{yyy}+(\varphi_y\psi_y r-5\Delta)\varphi_{yyyy}\varphi_y r^2)\varphi_y r^2)r^2\Big].
\end{aligned}
\tag{G.44}
$$

APPENDIX H

MORE CALCULATIONS IN SECTION 3.2.6

H.1 Obtaining the Coefficients B_i, C_i and D_i

Since the coefficients with the derivative y''^3 have to be zero, one obtains $f_{2y''y''y''} = 0$, that is

$$f_2 = f_{20} + f_{21}z + f_{22}z^2,$$

where $f_{2i} = f_{2i}(x, y, y'), (i = 0, 1, 2)$. Because the coefficients with the derivative y''^2 have to depend on x and y, then

$$\left(\frac{g_{1y} - f_{22}{g_1}^2 + 2g_{02}}{g_1}\right)_{y'} = 0$$

i.e., $f_{22} = f_{22}(x, y)$, so that

$$B_0 = (g_{1y} - f_{22}g_1^2 + 2g_{02})/g_1.$$

Because of the necessary form (3.9), the coefficient related to the product of y'^3y'' has to be zero, so that f_{21} has the form

$$f_{21} = f_{210} + f_{211}y' + f_{212}y'^2,$$

where $f_{21i} = f_{21i}(x, y), (i = 0, 1, 2)$. Thus, one obtains that

$$\begin{aligned}
C_2 &= (5g_{02y} + g_{1yy} - f_{212}g_1 - 2f_{22}g_{02}g_1)/g_1, \\
C_1 &= (3g_{01y} + 4g_{02x} + 2g_{1xy} - f_{211}g_1 - 2f_{22}g_{01}g_1)/g_1, \\
C_0 &= (g_{00y} + 2g_{01x} + g_{1xx} - f_{210}g_1 - 2f_{22}g_{00}g_1)/g_1.
\end{aligned}$$

Since the coefficients with the derivative y'^5 have to be zero, one obtains $f_{20y'y'y'y'y'} = 0$, that is

$$f_{20} = f_{200} + f_{201}y' + f_{202}y'^2 + f_{203}y'^3 + f_{204}y'^4,$$

where $f_{20i} = f_{20i}(x,y), (i = 0,1,...,4)$. Hence, the coefficients $D_i, (i = 0,1,...,4)$ in equations (3.35)-(3.39) are in the following forms

$$\begin{aligned}
D_4 &= (g_{02yy} - f_{204} - f_{212}g_{02} - f_{22}g_{02}^2)/g_1, \\
D_3 &= (g_{01yy} + 2g_{02xy} - f_{203} - f_{211}g_{02} - f_{212}g_{01} - 2f_{22}g_{01}g_{02})/g_1, \\
D_2 &= (g_{00yy} + 2g_{01xy} + g_{02xx} - f_{202} - f_{210}g_{02} - f_{211}g_{01} - f_{212}g_{00} \\
&\quad -2f_{22}g_{00}g_{02} - f_{22}g_{01}^2)/g_1, \\
D_1 &= (2g_{00xy} + g_{01xx} - f_{201} - f_{210}g_{01} - f_{211}g_{00} - 2f_{22}g_{00}g_{01})/g_1, \\
D_0 &= (g_{00xx} - f_{200} - f_{210}g_{00} - f_{22}g_{00}^2)/g_1.
\end{aligned}$$

H.2 Obtaining the Form of Functions f_2 and g

One can rewrite equations (3.63) and (3.64) in the following forms

$$\frac{g_{01y}g_1 - g_{01}g_{1y}}{g_1^2} = \frac{2(g_{02x}g_1 - g_{02}g_{1x})}{g_1^2},$$

or

$$(\frac{g_{01}}{g_1})_y = (\frac{2g_{02}}{g_1})_x,$$

or

$$(2\lambda_x)_y = (2\lambda_y)_x,$$

where $\lambda = \lambda(x,y)$. That is

$$g_{01} = 2g_1\lambda_x, \qquad g_{02} = g_1\lambda_y.$$

Since $g_{02} = g_{1y}$, then $g_{1y} = g_1\lambda_y$. The general solution of this equation is

$$g_1 = e^{\lambda + k(x)}.$$

One can use any particular solution $k(x) = 0$, so that

$$g_1 = e^{\lambda}.$$

Therefore equation (3.66) becomes

$$f_{202} = e^{\lambda} f_{210y}.$$

Thus equations (3.65), (3.67) and (3.68) are written in the following form

$$\begin{aligned} g_{00yy} &= g_{00y}\lambda_y + e^{\lambda}(f_{210y} + 2\lambda_{xy}\lambda_x + \lambda_{xxy}), && \text{(H.1)} \\ f_{201y} &= f_{201}\lambda_y + 2e^{\lambda} f_{210xy}, && \text{(H.2)} \\ f_{200yy} &= f_{200y}\lambda_y - e^{\lambda}(f_{210}f_{210y} + 2f_{210xy}\lambda_x + f_{210xxy} + f_{210y}\lambda_{xx} \\ &\quad + f_{210y}\lambda_x^2) - f_{210yy}g_{00} - 2f_{210y}g_{00y} + f_{210y}\lambda_y g_{00} + \lambda_{xy} f_{201}. && \text{(H.3)} \end{aligned}$$

Thus, functions f_2 and g have the following form

$$\begin{aligned} f_2 &= f_{210}g + e^{\lambda} f_{210y} y'^2 + f_{201}y' + f_{200}, \\ g &= e^{\lambda}(2\lambda_x y' + \lambda_y y'^2 + y'') + g_{00}. \end{aligned}$$

One solution of equation (H.1) is $g_{00y} = \eta_0(x)e^{\lambda}$, then equation (H.1) becomes

$$e^{\lambda}(f_{210y} + 2\lambda_{xy}\lambda_x + \lambda_{xxy}) = 0.$$

Since $e^{\lambda} = g_1 \neq 0$, then $f_{210y} + 2\lambda_{xy}\lambda_x + \lambda_{xxy} = 0$ i.e.,

$$f_{210} = -\lambda_{xx} - \lambda_x^2 + \eta_1(x).$$

One solution of equation (H.2) is $f_{201} = \eta_2(x)e^{\lambda}$, so that equation (H.2) is

$$2e^{\lambda}(-2\lambda_{xy}\lambda_{xx} - \lambda_{xxxy} - 2\lambda_{xxy}\lambda_x) = 0.$$

One arrives at

$$\lambda_{xx} = -\lambda_x^2 + \eta_3(x) + \eta_4(y). \qquad \text{(H.4)}$$

One solution of equation (H.3) is $f_{200} = g_{00}\eta_4 + f_{200k}(x, y)$, so that equation (H.3) becomes

$$e^{\lambda}[\eta_{4y}(\eta_1 - 2\eta_3 - 2\eta_4) + \lambda_{xy}\eta_2] - f_{200kyy} + f_{200ky}\lambda_y = 0. \tag{H.5}$$

One solution of equation (H.5) is $f_{200ky} = \eta_5(x)e^{\lambda}$. Hence equation (H.5) becomes

$$\lambda_{xy}\eta_2 + \eta_{4y}(\eta_1 - 2\eta_3 - 2\eta_4) = 0. \tag{H.6}$$

Considering case $\eta_2 = 0$ and $\eta_{4y} = 0$ (i.e., $\eta_4 = 0$), thus equation (H.4) becomes

$$\lambda_{xx} + \lambda_x^2 = \eta_3. \tag{H.7}$$

Let

$$e^{\lambda} = H_y. \tag{H.8}$$

So that $f_{200ky} = H_y\eta_5(x)$ and $g_{00_y} = H_y\eta_0(x)$. The general solution of these two equations are $f_{200k} = H\eta_5(x) + \eta_6(x)$ and $g_{00} = H\eta_0(x) + \eta_7(x)$, respectively. Differentiating equation (H.8) with respect to x, one arrives at

$$\lambda_x H_y = H_{xy}. \tag{H.9}$$

Substituting equation (H.9) into equation (H.7), one gets

$$\left(\left(\frac{H_{xy}}{H_y}\right)_x + \left(\frac{H_{xy}}{H_y}\right)^2\right)_y = 0.$$

Differentiating equation (H.8) with respect to y, one obtains

$$\lambda_y H_y = H_{yy}.$$

Therefore, $f_2(x, y, y', z)$ and $g(x, y, y', y'')$ become

$$f_2 = z\eta_8 + \eta_5 H + \eta_6, \quad g = y''H_y + y'^2 H_{yy} + 2y'H_{xy} + \eta_0 H + \eta_7,$$

where $\eta_8 = \eta_1 - \eta_3$. One can change the coefficients $\eta_i(x)$ to $\mu_i(x)$ as in section 3.2.6. Hence,

$$f_2 = z\mu_2 + \mu_3 H + \mu_5, \quad g = y''H_y + y'^2 H_{yy} + 2y'H_{xy} + \mu_1 H + \mu_4.$$

APPENDIX I

EQUATIONS FOR LEMMA 4.5 IN SECTION 4.3

For proving theorems we need the relations between $\varphi(x,y,p)$, $\psi(x,y,p)$, $g(x,y,p)$ and the coefficients of equation (4.6). These relations are presented here.

$$\begin{aligned} A_2 =& -((g_x + g_y p - g_p a)\varphi_p)^{-1}\Big[(3g_p a_p - 2g_y + 3g_{pp}a - 3g_{py}p - 3g_{px})\varphi_p \\ &+ 3(g_x + g_y p - g_p a)\varphi_{pp} - \varphi_y g_p\Big], \end{aligned} \tag{I.1}$$

$$\begin{aligned} A_1 =& -((g_x + g_y p - g_p a)\varphi_p)^{-1}\Big[(3(3g_x a_p - g_{yy}p^2 - g_{xx} - 2g_{xy}p + g_{pp}a^2) \\ &+ (9a_p p - a)g_y + 3(a_x + a_y p - 2a_p a)g_p)\varphi_p - 2((2(g_x + g_y p) - g_p a)\varphi_y \\ &- 3(g_x + g_y p - g_p a)\varphi_{pp}a)\Big], \end{aligned} \tag{I.2}$$

$$\begin{aligned} A_0 =& ((g_x + g_y p - g_p a)\varphi_p)^{-1}\Big[(3((g_{xx} + g_{yy}p^2 + 2g_{xy}p - g_{py}pa - g_{px}a \\ &+ (a_x + a_y p + a_p a)g_p)a - (2(a_x + a_y p) + a_p a)g_x) \\ &- (6a_y p^2 + a^2 + 6a_x p + 3a_p pa)g_y)\varphi_p \\ &+ ((4(g_x + g_y p) - 3g_p a)\varphi_y - 3(g_x + g_y p - g_p a)\varphi_{pp}a)a\Big], \end{aligned} \tag{I.3}$$

$$\begin{aligned} B_5 =& -((g_x + g_y p - g_p a)\varphi_p^2)^{-1}\Big[((\alpha g + \beta\psi)\varphi_p^4 - \varphi_{ppp}g_p)\varphi_p \\ &+ 3\varphi_{pp}^2 g_p - 3\varphi_{pp}\varphi_p g_{pp} + \varphi_p^2 g_{ppp}\Big], \end{aligned} \tag{I.4}$$

$$
\begin{aligned}
B_4 =& -((g_x + g_y p - g_p a)\varphi_p^2)^{-1}\Big[3((2g_p a_p - g_y - 3g_{pp}a - 2g_{py}p - 2g_{px})\varphi_p \\
& - 2\varphi_y g_p)\varphi_{pp} + (3(\varphi_{py}g_p + \varphi_y g_{pp}) + 5(\alpha g + \beta\psi)\varphi_p^4 a)\varphi_p \\
& - (g_x + g_y p + 4g_p a)(\varphi_{ppp}\varphi_p - 3\varphi_{pp}^2) \\
& - (3(2g_{pp}a_p + g_p a_{pp} - g_{ppy}p - g_{ppx}) - 2g_{ppp}a - 3g_{py})\varphi_p^2\Big],
\end{aligned} \tag{I.5}
$$

$$
\begin{aligned}
B_3 =& ((g_x + g_y p - g_p a)\varphi_p^2)^{-1}\Big[(3((2a_{pp}a - 3a_p^2 + a_{py}p + a_{px})g_p \\
& + (a_{pp}p + 2a_p)g_y + g_x a_{pp} + (a_x + a_y p + 3a_p a)g_{pp} - 2g_{ppy}pa - 2g_{ppx}a) \\
& - g_{ppp}a^2 + 6(2a_p p - a)g_{py} + 12g_{px}a_p)\varphi_p^2 \\
& + 3(3g_p a_p - g_y - 2g_{pp}a - 2g_{py}p - 2g_{px})\varphi_p\varphi_y \\
& - (10(\alpha g + \beta\psi)\varphi_p^5 a^2 + 3\varphi_y^2 g_p) - 3(((2a_p p - 3a)g_y - g_{yy}p^2 + 2g_x a_p \\
& - g_{xx} - 2g_{xy}p + (a_x + a_y p + 5a_p a)g_p - 3g_{pp}a^2 - 6g_{py}pa - 6g_{px}a)\varphi_p \\
& - 2(g_x + g_y p + 3g_p a)\varphi_y + 2(2(g_x + g_y p) + 3g_p a)\varphi_{pp}a)\varphi_{pp} \\
& + 2(2(g_x + g_y p) + 3g_p a)\varphi_{ppp}\varphi_p a - 3((g_{xy} + g_{yy}p + g_{pyy}p^2 + g_{pxx} \\
& + 2g_{pxy}p)\varphi_p + (g_x + g_y p + 3g_p a)\varphi_{py})\varphi_p\Big],
\end{aligned} \tag{I.6}
$$

$$
\begin{aligned}
B_2 =&((g_x + g_y p - g_p a)\varphi_p^2)^{-1}\Big[(6g_{xx}a_p - g_{yyy}p^3 - 3g_{xxy}p - g_{xxx} - 3g_{xyy}p^2 \\
&- 3g_{ppy}pa^2 - 3g_{ppx}a^2 - 6g_{pyy}p^2a - 6g_{pxx}a - 12g_{pxy}pa \\
&+ 6(a_p p - a)g_{yy}p + 6(2a_p p - a)g_{xy} + 3(a_x + a_y p + a_p a)g_{pp}a \\
&+ 6(a_x + a_y p + 3a_p a)g_{px} + 3(2a_y p^2 - a^2 + 2a_x p + 6a_p pa)g_{py} \\
&+ 3(2a_{pp}a - 3a_p^2 + a_{py}p + a_{px})g_x - 3(3(a_p p - a)a_p - (a_x + a_y p) \\
&- 2a_{pp}pa - a_{py}p^2 - a_{px}p)g_y + (a_{yy}p^2 - a_y a + a_{xx} + 2a_{xy}p - 8a_p^2 a \\
&+ 4a_{pp}a^2 + 4a_{py}pa + 4a_{px}a - 10(a_x + a_y p)a_p)g_p)\varphi_p^2 \\
&+ 3(((3(g_{xx} + g_{yy}p^2 + 2g_{xy}p) + g_{pp}a^2 + 6g_{py}pa + 6g_{px}a \\
&- 2(a_x + a_y p + 2a_p a)g_p)a - (a_x + a_y p + 5a_p a)g_x \\
&- (a_y p^2 - 3a^2 + a_x p + 5a_p pa)g_y)\varphi_p + 6(g_x + g_y p + g_p a)\varphi_y a)\varphi_{pp} \\
&+ (3(3g_x a_p - g_{yy}p^2 - g_{xx} - 2g_{xy}p - g_{pp}a^2 - 4g_{py}pa - 4g_{px}a \\
&+ (3a_p p - 2a)g_y) + 2(2(a_x + a_y p) + 7a_p a)g_p)\varphi_p\varphi_y \\
&- ((9(g_x + g_y p + g_p a)\varphi_{py} + 10(\alpha g + \beta\psi)\varphi_p^4 a^2)\varphi_p a \\
&+ 3(g_x + g_y p + 2g_p a)\varphi_y^2 - 2(3(g_x + g_y p) + 2g_p a)(\varphi_{ppp}\varphi_p - 3\varphi_{pp}^2)a^2)\Big], \quad \text{(I.7)}
\end{aligned}
$$

$$
\begin{aligned}
B_1 = &-((g_x + g_y p - g_p a)\varphi_p^2)^{-1}\Big[((2((3g_{xxy} + g_{yyy}p^2)p + g_{xxx} + 3g_{xyy}p^2) \\
&+ 3g_{pyy}p^2 a + 3g_{pxx}a + 6g_{pxy}pa - 6(a_x + a_y p + a_p a)(g_{px} + g_{py}p))a \\
&- 3(a_x + a_y p + 3a_p a)g_{xx} - 3(a_y p^2 - a^2 + a_x p + 3a_p pa)g_{yy}p \\
&- 3(2a_y p^2 - a^2 + 2a_x p + 6a_p pa)g_{xy} - (a_{yy}p^2 - a_y a + a_{xx} + 2a_{xy}p \\
&- 8a_p^2 a + 4a_{pp}a^2 + 4a_{py}pa + 4a_{px}a - 10(a_x + a_y p)a_p)g_x \\
&- ((a_{yy}p^2 + 2a_y a)p + 3a_x a + a_{xx}p + 2a_{xy}p^2 - 8a_p^2 pa + 4a_{pp}pa^2 \\
&+ 4a_{py}p^2 a + 4a_{px}pa - (10a_y p^2 - 3a^2 + 10a_x p)a_p)g_y + ((3a_y p^2 + a^2)a_y \\
&- a_{yy}p^2 a + 3(a_x + 2a_y p)a_x - a_{xx}a - 2a_{xy}pa + 2a_p^2 a^2 - a_{pp}a^3 - a_{py}pa^2 \\
&- a_{px}a^2 + 4(a_x + a_y p)a_p a)g_p)\varphi_p^2 - (3(((3(g_{xx} + g_{yy}p^2 + 2g_{xy}p) \\
&+ 2g_{py}pa + 2g_{px}a - (a_x + a_y p + a_p a)g_p)a - 2(a_x + a_y p + 2a_p a)g_x \\
&- (2a_y p^2 - a^2 + 2a_x p + 4a_p pa)g_y)\varphi_p + 2(3(g_x + g_y p) + g_p a)\varphi_y a)\varphi_{pp}a \\
&- ((2(3(g_{xx} + g_{yy}p^2 + 2g_{xy}p + g_{py}pa + g_{px}a)a - (2(a_x + a_y p) + 7a_p a)g_x) \\
&- (4(a_x + a_y p) + 5a_p a)g_p a - (4a_y p^2 - 3a^2 + 4a_x p + 14a_p pa)g_y)\varphi_p\varphi_y \\
&+ (3(2(g_x + g_y p) + g_p a)\varphi_y^2 + 5(\alpha g + \beta\psi)\varphi_p^5 a^3 \\
&+ 3(3(g_x + g_y p) + g_p a)\varphi_{py}\varphi_p a \\
&- (4(g_x + g_y p) + g_p a)(\varphi_{ppp}\varphi_p - 3\varphi_{pp}^2)a^2)a))\Big], \qquad \text{(I.8)}
\end{aligned}
$$

$$
\begin{aligned}
B_0 =&((g_x + g_y p - g_p a)\varphi_p^2)^{-1}\Big[((a_{yy}p^2 - a_y a + a_{xx} + 2a_{xy}p \\
&+ (2(a_x + a_y p) + a_p a)a_p + a_{pp}a^2 + a_{py}pa + a_{px}a)\varphi_p \\
&- ((2(a_x + a_y p) + a_p a)\varphi_y + \varphi_{yyy}p^3) + 3(a_x + a_y p + a_p a)\varphi_{pp}a \\
&+ \varphi_{ppp}a^3 - 3\varphi_{py}a^2)(g_x + g_y p)\varphi_p a + 3((\varphi_p a_p - \varphi_y + \varphi_{pp}a - \varphi_{py}p)a \\
&- ((\varphi_p a_y - \varphi_{yy}p + \varphi_{py}a)p - \varphi_p a_x))((g_{xx} + g_{yy}p^2 + 2g_{xy}p)\varphi_p a \\
&- 2(g_x + g_y p)\varphi_{yy}p^2) - (\varphi_p^5 \alpha g a^5 + \varphi_p^5 \beta \psi a^5 + 3\varphi_p^2 g_{xyy}p^2 a^2 + \varphi_p^2 g_{xxx}a^2 \\
&+ 3\varphi_p^2 g_{xxy}pa^2 + \varphi_p^2 g_{yyy}p^3 a^2 - \varphi_p \varphi_{yyy} g_x p^3 a - \varphi_p \varphi_{yyy} g_y p^4 a \\
&- 6\varphi_p \varphi_{yy} g_{xy} p^3 a - 3\varphi_p \varphi_{yy} g_{xx} p^2 a - 3\varphi_p \varphi_{yy} g_{yy} p^4 a + \varphi_{yyy}\varphi_y g_x p^4 \\
&+ \varphi_{yyy}\varphi_y g_y p^5) - 6(2((\varphi_p a_p - \varphi_y + \varphi_{pp}a - \varphi_{py}p)a + (a_x - a_y p)\varphi_p \\
&+ 2\varphi_{yy}p^2 - \varphi_{py}pa)(g_x + g_y p) - (g_{xx} + g_{yy}p^2 + 2g_{xy}p)\varphi_p a)(\varphi_p a_y \\
&- \varphi_{yy}p + \varphi_{py}a)p - (3((\varphi_p a_p - \varphi_y + \varphi_{pp}a - \varphi_{py}p)a \\
&- ((\varphi_p a_y - \varphi_{yy}p + \varphi_{py}a)p - \varphi_p a_x))^2 + (12(\varphi_p a_y - \varphi_{yy}p + \varphi_{py}a)^2 \\
&- (\varphi_{yyy}\varphi_y - 3\varphi_{yy}^2)p^2)p^2)(g_x + g_y p)\Big].
\end{aligned}
\tag{I.9}
$$

APPENDIX J

MORE CALCULATIONS IN SECTION 4.6

J.1 Obtaining the Form of Functions f_2 and g in Section 4.6.1

Considering equation $c_{5yy} = 0$ of equations (4.22), one obtains that the general solution of this equation is

$$c_5 = \tilde{f}y + \tilde{k}, \tag{J.1}$$

where $\tilde{f} = \tilde{f}(x, p)$ and $\tilde{k} = \tilde{k}(x, p)$ are arbitrary functions. So that

$$c_{5xx} = \tilde{f}_{xx}y + \tilde{k}_{xx}. \tag{J.2}$$

From equations (4.22) one has $c_{5xx} = 0$, then $c_{5xxy} = 0$. Differentiating equation (J.2) with respect to y yields $\tilde{f}_{xx} = 0$, moreover one finds $\tilde{k}_{xx} = 0$. Therefore, the forms of $\tilde{f}$ and $\tilde{k}$ are

$$\tilde{f} = l_1x + l_0, \qquad \tilde{k} = \tilde{k}_1x + \tilde{k}_0,$$

where $l_i = l_i(p)$, and $\tilde{k}_i = \tilde{k}_i(p)$, $(i = 0, 1)$ are arbitrary functions. Differentiating equation (J.1) with respect to y, one gets

$$c_{5y} = \tilde{f}.$$

Substituting into $c_{5xy} = 0$, one obtains $l_1 = 0$. That is $\tilde{f} = l_0$. Hence, the general form of c_5 is

$$c_5 = l_0y + \tilde{k}_1x + \tilde{k}_0.$$

From equations (4.22), one has $c_1 = c_2 = c_3 = c_4 = 0$ and $c_{6x} = c_{6y} = 0$ (i.e., $c_6 = c_{6p}$). So that f_2 has the following form

$$f_2 = (\tilde{k}_0 + \tilde{k}_1 x + l_0 y)y'' + c_6.$$

One can rewrite f_2 in the form of equations (4.23).

By setting $a = 0, h_0 = 0, h_{1x} = 0$, function h in this case has the form

$$h = \frac{h_1}{y''^2},$$

where $h_1 = h_1(p)$. One can rewrite it in the form of equations (4.23).

J.2 Obtaining the Form of Functions f_2 and g in Section 4.6.2

Because of $h_{1x} = 0$, that is h_1 is functions of y and p. By consideration of the second equation of equations (4.25), one obtains that

$$h_1 = \bar{k}p^4,$$

where $\bar{k} = \bar{k}(y)$ is arbitrary function. Substituting into the third equation of equations (4.25), one finds that

$$\bar{k} = \frac{1}{(\bar{k}_0 + \bar{k}_1 y)^5},$$

where $\bar{k}_0$ and $\bar{k}_1$ are arbitrary constants. One can choose any particular solution $\bar{k}_0 = 0$, thus $\bar{k} = \frac{1}{(\bar{k}_1 y)^5}$. So that equations (4.26) become

$$c_{6x} = \frac{(c_{6p}\bar{k}_1^5 y^5 - 6p)p^2}{\bar{k}_1^5 y^6}, \qquad c_{6y} = -\frac{(c_{6p}p + 3c_6)}{y}. \tag{J.3}$$

By using Cauchy method the second equation of equations (J.3) gives

$$c_6 = \frac{\bar{f}}{y^3},$$

where $\bar{f} = \bar{f}(x, \frac{p}{y})$ is arbitrary function. Substituting c_6 into the first equation of equations (J.3), one arrives at equation

$$-\bar{f}_x + z^2\bar{f}_z - \frac{6z^3}{\bar{k}_1^5} = 0, \tag{J.4}$$

here $z = \frac{p}{y}$. By virtue of Cauchy method, the general solution of equation (J.4) is

$$\bar{f} = \frac{3z^2}{\bar{k}_1^5} + \bar{f}_0,$$

where $\bar{f}_0 = \bar{f}_0(\frac{1}{z} - x)$. Thus c_6 becomes

$$c_6 = \frac{3p^2}{\bar{k}_1^5 y^5} + \frac{\bar{f}_0}{y^3}.$$

Setting $S_k = c_{5x} + pc_{5y}$, one arrives at equation

$$yS_{ky} + pS_{kp} + 4S_k = 0.$$

Solving by Cauchy method, one obtains

$$S_k = \frac{w}{y^4}, \tag{J.5}$$

where $w = w(x, \frac{p}{y})$ is arbitrary function. Hence,

$$c_{5x} = \frac{w}{y^4} - pc_{5y}.$$

This solves equation (4.28) as well. Substituting the value of c_{5x} into equation (4.27), one obtains

$$w_x - z^2 w_z = z(2w - \frac{9z}{\bar{k}_1^5}) + \bar{f}_0. \tag{J.6}$$

By using Cauchy method, the general solution of equation (J.6) is

$$w = \frac{3z}{\bar{k}_1^5} - \frac{\bar{f}_0}{z} + \frac{m}{z^2},$$

where $m = m(\frac{1}{z} - x)$ is arbitrary function. One can rewrite w in the following form

$$w = \frac{3p}{\bar{k}_1^5 y} - \frac{\bar{f}_0 y}{p} + \frac{my^2}{p^2},$$

where $m = m(\frac{y}{p} - x)$ is arbitrary function. Therefore S_k of equation (J.5) becomes

$$c_{5x} + pc_{5y} + \frac{(\bar{f}_0 \bar{k}_1^5 y^2 - 3p^2)p - \bar{k}_1^5 m y^3}{\bar{k}_1^5 p^2 y^5} = 0.$$

The general solution of this equation is

$$c_5 = -\frac{3}{4\bar{k}_1^5 y^4} + \frac{\bar{f}_0}{2y^2p^2} - \frac{m}{p^3 y} + d,$$

where $d = d(p, \frac{y}{p} - x)$ is arbitrary function. This solution solves equations (4.27)-(4.29) as well with

$$d = \frac{\lambda_1}{p^{5/2}} + \frac{\lambda_2}{p^4},$$

where $\lambda_i = \lambda_i(\frac{y}{p} - x),\ i = (1, 2)$ are arbitrary functions. Hence, functions h and f_2 become

$$h = \frac{p^4}{\bar{k}_1^5 y^5 y''^2},$$

$$f_2 = \frac{\bar{f}_0}{y^3} + \frac{3p^2}{\bar{k}_1^5 y^5} + y''(\frac{\bar{f}_0}{2y^2p^2} - \frac{3}{4\bar{k}_1^5 y^4} - \frac{m}{yp^3} + \frac{\lambda_1}{p^{5/2}} + \frac{\lambda_2}{p^4}).$$

One can rewrite these equations as in the form of equations (4.30).

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