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EQUIVALENCE OF LINEAR SECOND ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS TO ONE OF THE CANONICAL FORMS

Ekkarath Thailert

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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EQUIVALENCE OF LINEAR SECOND ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS TO ONE OF THE CANONICAL FORMS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์นี้ศึกษาปัญหาสมมูลของสมการเชิงอนุพันธ์ย่อยพาราโบลิกอันดับสองที่มีตัวแปรอิสระสองตัว โดยแบ่งการศึกษาออกเป็นสามส่วนคือ ส่วนแรกทำการหารูปแบบของสมการเชิงอนุพันธ์ย่อยพาราโบลิกอันดับสองที่สมมูลกับสมการเชิงเส้น ซึ่งรูปแบบดังกล่าวนี้ไม่แปรเปลี่ยนภายใต้การเปลี่ยนตัวแปรตามและตัวแปรอิสระใดๆ ส่วนที่สองดำเนินการหายืนยงอันดับหกและเจ็ดด้วยการแปลงแบบจุดของสมการเชิงอนุพันธ์ย่อยพาราโบลิกอันดับสอง

$$u_t + a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u = 0$$

ส่วนที่สามคือการหาผลเฉลยของปัญหาสมมูลสำหรับชั้นแบบบัญญัติของสมการ

$$u_t = u_{xx} + a(x)u$$

และ

$$u_t = u_{xx} + \frac{k}{x^2}u$$

ที่มี k เป็นค่าคงตัว

สาขาวิชาคณิตศาสตร์	ลายมือชื่อนักศึกษา______________________
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EKKARATH THAILERT: EQUIVALENCE OF LINEAR SECOND ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS TO ONE OF THE CANONICAL FORMS. THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D. 81 PP.

PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS / EQUIVALENCE PROBLEM / SYMMETRY GROUP / GROUP ANALYSIS

This thesis is devoted to the study of the equivalence problem of parabolic second-order partial differential equations with two independent variables. The results obtained in the thesis are separated into three parts. The first result describes the form of parabolic second-order partial differential equations which are equivalent to a linear equation. It is proven that this form is an invariant with respect to a change of the dependent and independent variables. The second part of the thesis is related with obtaining invariants with respect to point transformations of linear second-order parabolic partial differential equations

$$u_t + a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u = 0.$$

Differential invariants of sixth and seventh-order are obtained. The third part of the thesis presents the solution of the equivalence problem for the canonical classes of the equations $u_t = u_{xx} + a(x)u$ and $u_t = u_{xx} + \frac{k}{x^2}u$, where k is constant.

School of Mathematics

Academic Year 2008

Student's Signature ____________________

Advisor's Signature ____________________

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Ekkarath Thailert

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CHAPTER I

INTRODUCTION

Mathematical modelling is the basis for analyzing physical phenomena. Many mathematical models are represented by partial differential equations. The equivalence problem is one of the important problems of partial differential equations. Two differential equations are said to be equivalent, if there exists an invertible transformation which transforms one equation into the other. The linearization problem is a particular case of the equivalence problem. In this problem one of equations is a linear equation.

S. Lie (1883) was the first to study the linearization problem of a second-order ordinary differential equation. He found that an equation $y'' = F(x, y, y')$ is linearizable if and only if it has the form

$$y'' + a_1 y'^{\,3} + 3a_2 y'^{\,2} + 3a_3 y' + a_4 = 0, \tag{1.1}$$

and the coefficients $a_i(x, y)$, $(i = 1, 2, 3, 4)$ satisfy conditions

$$\begin{aligned} L_1 = \; & a_{2xx} - 2a_{3xy} + 3a_{4yy} - 6a_{1x}a_4 + a_{2x}a_3 + 3a_{2y}a_4 - 2a_{3y}a_3 - 3a_{4x}a_1 \\ & +3a_{4y}a_2 = 0 \\ L_2 = \; & 3a_{1xx} - 2a_{2xy} + a_{3yy} - 3a_{1x}a_3 + a_{1y}a_4 + 2a_{2x}a_2 - 3a_{3x}a_1 - a_{3y}a_2 \\ & +6a_{4y}a_1 = 0. \end{aligned}$$

Liouville (1889), investigated the invariants of equation (1.1). He found relative invariants of equation (1.1) with respect to a change of the independent and dependent variables:

$$\begin{aligned} v_5 = &\ L_2(L_1L_{2x} - L_2L_{1x}) + L_1(L_2L_{1y} - L_1L_{2y}) - a_1L_1^3 + 3a_2L_1^2L_2 - 3a_3L_1L_2^2 \\ &+ a_4L_2^3, \\ w_1 = &\ L_1^{-4}(-L_1^3\Pi_{12}L_1 - \Pi_{11}L_2) + R_1(L_1^2)_t - L_1^2R_{1t} + L_1R_1(a_3L_1 - a_4L_2)), \end{aligned}$$

where

$$\begin{aligned} R_1 = &\ L_1L_{2t} - L_2L_{1t} + a_2L_1^2 - 2a_3L_1L_2 + a_4L_2^2, \\ \Pi_{11} = &\ 2(a_3^2 - a_2a_4) + a_{3t} - a_{4y}, \\ \Pi_{12} = &\ a_2a_3 - a_1a_4 + a_{2t} - a_{3y}. \end{aligned}$$

If we have $v_5 = 0$ or $w_1 = 0$, then v_5 or w_1 is not changed.

Tresse (1896) applied the Lie approach for finding invariants of equation (1.1) with respect to point transformations. He found the complete set of invariants for equation (1.1). Cartan (1924) used a differential geometry approach for solving the linearization problem of equation (1.1). The linearization of third order differential equations by point and contact transformations was studied by Chern (1940), Grebot (1996), Ibragimov and Meleshko (2005), and Petitot and Neut (2002).

In this thesis, parabolic partial differential equations with two independent variables are studied. There are number of publications which are related to the subject of the thesis. Lie (1881) applied group analysis for solving the linear heat equation $u_t = ku_{xx}$. Ovsiannikov (1959) studied the nonlinear heat equation $u_t = (k(u)u_x)_x$.

Lie (1881) classified linear parabolic partial differential equations with respect to admitted Lie groups. He found canonical forms of all linear second-order partial differential equations

$$u_{xx} = u_t + Hu, \tag{1.2}$$

where $H = H(t, x)$. For arbitrary $H(t, x)$ the symmetry Lie algebra is infinite dimensional and is spanned by

$$X_0 = u\frac{\partial}{\partial u},\ X_\infty = z_0(t,x)\frac{\partial}{\partial u},$$

where $z_0(t,x)$ is any solution of equation (1.2). The algebra is extended in the following cases:

$$\begin{array}{ll} H = 0, & X_1 = \frac{\partial}{\partial x}, X_2 = 2t\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \\ & X_3 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (\frac{1}{4}x^2 - \frac{1}{2}t)u\frac{\partial}{\partial u}, \\ H = H(x), & X_1 = \frac{\partial}{\partial t}, \\ H = \frac{k}{x^2}, k \neq 0, & X_1 = \frac{\partial}{\partial t}, X_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\ & X_3 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (\frac{1}{4}x^2 + \frac{1}{2}t)u\frac{\partial}{\partial u}. \end{array}$$

Ibragimov (2002) found semi-invariants (up to second-order) for a linear parabolic partial differential equation under an action of the equivalence group of point transformations which transform a linear partial differentental equation

$$u_t + a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u = 0, \quad a(t,x) \neq 0 \tag{1.3}$$

into an equation of the same form. He showed that equation (1.3) has the following semi-invariants (up to second order)

$$a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}, K,$$

where

$$K = 2c_xa^2 - b_ta - b_{xx}a^2 - b_xba + b_xa_xa + \frac{1}{2}b^2a_x + ba_t + ba_{xx}a - ba_x^2.$$

Johnpillai and Mahomed (2002) showed that there are no first, second, third and fourth order invariants other than constants and they obtained the relative invariant of fifth-order for equation (1.3):

$$\begin{aligned}\lambda =&4a\,(2aK_{xx}-5a_xK_x)-12K\,(aa_{xx}-2a_x^2)+a_x\,(4aa_{tt}-9a_x^4)\\&-12a_ta_x\,(a_t+2a_x^2)+4a\,(3a_t+6a_x^2-5aa_{xx})a_{tx}\\&+2aa_x\,(16a_ta_{xx}-12aa_{xx}^2+15a_x^2a_{xx})-4a^2a_{ttx}-12a^2a_xa_{txx}\\&-4a^2a_{xxx}\,(2a_t-4aa_{xx}+3a_x^2)+8a^3a_{txxx}-4a^4a_{xxxxx}\end{aligned} \tag{1.4}$$

(see the detail in Chapter V). Morozov (2003) found invariants of contact transformations of equation (1.3). There are also examples of linearization of the nonlinear heat equation by contact transformations (see the discussion in Morozov (2003), page 110).

Part of the thesis is devoted to finding differential invariants of a linear second-order parabolic partial differential equations (1.3). Differential invariants of sixth and seventh-order are found in the thesis. The invariants of eighth and ninth-order are also found, but because of their cumbersome nature, they are not presented in the thesis.

Since in obtaining differential invariants the semi-invariant K is used, we also give a review of this subject here. Ibragimov (2008) is devoted to an extension of Euler's method to linear parabolic equations (1.3) with two independent variables. First at all, these equations are mapped to the form

$$u_t-u_{xx}+a(t,x)u_x+c(t,x)u=0 \tag{1.5}$$

by an appropriate change of the independent variables. The condition of reducibility of equation (1.5) to the heat equation

$$v_t-v_{xx}=0 \tag{1.6}$$

is obtained in the term of the semi-invariant

$$K=aa_x-a_{xx}+a_t+2c_x, \tag{1.7}$$

of equation (1.5). Namely, it is shown that equation (1.5) can be mapped to the heat equation (1.6) by an appropriate change of the dependent variable if and only if the semi-invariant (1.7) vanishes, i.e. $K = 0$. The method developed in the article allows one to derive an explicit formula for the general solution of a wide class of parabolic equations. In particular, the general solution of the Black-Scholes equation is obtained.

Even though, many publications are devoted to equivalence and linearization problems of ordinary differential equations, this problem is less studied for partial differential equations. Equivalence problem for the first canonical form of parabolic linear second-order partial differential equations was studied by Johnpillai and Mahomed (2002). For the second and third canonical forms this problem has not been developed yet. Part of the research of the thesis deals with the equivalence problem of linear second-order parabolic equations to be equivalent to these canonical forms. Moreover, we also found necessary conditions for the linearization problem of a nonlinear second-order parabolic partial differential equation.

The thesis is designed as follows. Chapter II introduces background and notations of the group analysis method. Definitions and theorems of the group analysis are also presented. Chapter III provides an introduction to the concepts of the compatibility theory. General theorems of compatibility and its particular cases are also discussed. Chapter IV deals with obtaining necessary conditions for the linearization problem of a nonlinear second-order parabolic partial differential equation. Chapter V is devoted to finding sixth and seventh-order invariants of linear parabolic differential equations. Equivalence problems of linear second-order parabolic equations to one of the canonical forms are studied in Chapter VI. The conclusion of the thesis is presented in the last chapter.

CHAPTER II

GROUP ANALYSIS METHOD

In this chapter the group analysis method for finding invariants is discussed.

2.1 Local one-parameter Lie groups

Consider transformations

$$\bar{z}^i = g^i(z; a) \tag{2.1}$$

where $i = 1, 2, ..., N$, $z \in V \subset Z = R^N$, $a \in \Delta$ is a parameter and Δ is a symmetric interval of R^1. The set V is an open set in Z.

Definition 2.1. A set of transformations (2.1) is called *a local one-parameter Lie group* G^1 if it has the following properties

1. $g(z; 0) = z$ for all $z \in V$.
2. $g(g(z; a), b) = g(z; a + b)$ for all $a, b, a + b \in \Delta, z \in V$.
3. If for $a \in \Delta$ one have $g(z; a) = z$ for all $z \in V$, then $a = 0$.
4. $g \in C^\infty(V, \Delta)$.

To the group G^1 is associated to the infinitesimal generator

$$X = \sum_{i=1}^{N} \zeta^i(z) \frac{\partial}{\partial z_i}, \text{ (in short, } \mathrm{X} = \zeta^{\mathrm{i}}(\mathrm{z}) \frac{\partial}{\partial \mathrm{z_i}})$$

where

$$\zeta(z) = (\zeta^1(z), \zeta^2(z), ..., \zeta^N(z)) = \frac{dg}{da}(z; 0).$$

Conversel, if one knows a generator X, the one can find the Lie group to which it is associated by the following theorem:

Theorem 2.1. *A local Lie group of transformations (2.1) is completely defined by the solution of the Cauchy problem:*

$$\frac{d\bar{z}_i}{da} = \zeta^i(\bar{z}), \ \ (i = 1, ..., N) \tag{2.2}$$

$$\bar{z} = z \tag{2.3}$$

Here the initial data (2.3) are taken at the point $a = 0$*.*

Equation (2.2) are called Lie equations. Transformations of independent, dependent variables and arbitrary elements, preserving the differential structure of the equations themselves are called equivalence transformations.

Example Let us check whether the transformations $g = (\bar{x}, \bar{y})$ where

$$\bar{x} = e^a x, \ \ \bar{y} = e^{-a} y \tag{2.4}$$

obey the group properties:

1. equation (2.4) becomes the identity transformation when $a = 0$.
2. we have $\bar{\bar{x}} = e^b \bar{x} = e^{a+b} x, \ \ \bar{\bar{y}} = e^{-b} \bar{y} = e^{-(a+b)} y$.
3. if for $a \in \Delta$ one has $\bar{x} = x, \ \ \bar{y} = y$ for all $(x, y) \in V$, then $a = 0$.
4. $(\bar{x}, \bar{y}) \in C^\infty(V, \Delta)$.

Thus (2.4) is a local one-parameter Lie group. From (2.4) one gets

$$\zeta^1(x, y) = \frac{\partial \bar{x}}{\partial a}(x, y; 0) = x, \ \ \zeta^2(x, y) = \frac{\partial \bar{y}}{\partial a}(x, y; 0) = -y.$$

Therefore the group (2.4) is associated to the infinitesimal generator

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{2.5}$$

Applying theorem 2.1, from (2.5) one has the Cauchy problem:

$$\begin{aligned} &\frac{d\bar{x}}{da} = \bar{x}, \ \ \frac{d\bar{y}}{da} = \bar{y} \\ &a = 0, \ \ \bar{x} = x, \ \ \bar{y} = y. \end{aligned} \tag{2.6}$$

Solving the system (2.6) we obtain (2.4).

2.2 Equivalence groups

Definition 2.2. A nondegenerate change of the dependent variable u, independent variables x, and arbitrary elements ϕ, which transfers a system of l-th order differential equations of the given class

$$F^k(x,u,p,\phi)=0,(k=1,2,...,s) \tag{2.7}$$

into a system of equations of the same class is called *an equivalence transformation.* The class is defined by the functions $F^k(x,u,p,\phi)$. Here $(x,u)\in V\subset R^{n+m}$, and $\phi : V\rightarrow R^t$.

Let us consider a one-parameter Lie group of transformations

$$\bar{x}=f^x(x,u,\phi;a),\ \bar{u}=f^u(x,u,\phi;a)\ ,\bar{\phi}=f^{\phi}(x,u,\phi;a). \tag{2.8}$$

The generator of this group has the form

$$X^e=\xi^{x_i}\frac{\partial}{\partial x_i}+\zeta^{u^j}\frac{\partial}{\partial u^j}+\zeta^{\phi^l}\frac{\partial}{\partial \phi^l}, \tag{2.9}$$

where the coordinates are

$$\xi^{x_i}=\xi^{x_i}(x,u,\phi),\ \zeta^{u^j}=\zeta^{u^j}(x,u,\phi),\ \zeta^{\phi^l}=\zeta^{\phi^l}(x,u,\phi),$$

$$i=1,...,n,\ j=1,...,m,\ l=1,...,t.$$

The transformed arbitrary elements are obtained in the following way. Assume that we know $\phi_0(x,u)$, then we have

$$\bar{x}=f^x(x,u,\phi_0(x,u);a),\ \bar{u}=f^u(x,u,\phi_0(x,u);a). \tag{2.10}$$

By the inverse function theorem applied to (2.10), one can find

$$x=\varphi^x(\bar{x},\bar{u};a),\ u=\varphi^u(\bar{x},\bar{u};a) \tag{2.11}$$

Substituting (2.11) into

$$\phi_a(\bar{x}, \bar{u}) = f^{\phi}(x, u, \phi; a), \tag{2.12}$$

one has identity with respect to the space $(\bar{x}, \bar{u})$

$$\phi_a(\bar{x}, \bar{u}) = f^{\phi}((\varphi^x(\bar{x}, \bar{u}; a), \varphi^u(\bar{x}, \bar{u}; a), \phi_0((\varphi^x(\bar{x}, \bar{u}; a), \varphi^u(\bar{x}, \bar{u}; a)); a).$$

The transformed function $u_a(x)$ is obtained as follows. If $u_0(x)$ is a given function, then we define

$$\bar{x} = f^x(x, u_0(x), \phi_0(x, u_0(x)); a). \tag{2.13}$$

By the inverse function theorem, one finds $x = \varphi(\bar{x}; a)$. Substituting $x = \varphi(\bar{x}; a)$ into

$$u_a(\bar{x}) = f^u(x, u_0(x), \phi_0(x, u_0(x)); a), \tag{2.14}$$

one gets the transformed function

$$u_a(\bar{x}) = f^u(\varphi(\bar{x}; a), u_0(\varphi(\bar{x}; a)), \phi_0(\varphi(\bar{x}; a), u_0(\varphi(\bar{x}; a))); a).$$

Using (2.13) and (2.14), there is the identity with respect to x

$$u_a(f^x((x, u_0(x), \phi_0(x, u_0(x)); a)) = f^u(\varphi(\bar{x}; a), u_0(\varphi(\bar{x}; a)), \phi_0(\varphi(\bar{x}; a), u_0(\varphi(\bar{x}; a))); a). \tag{2.15}$$

Formulae for transformations of the partial derivatives $\bar{p}_a = f^p(x, u, p, \phi, ...; a)$ are obtained by differentiating (2.15) with respect to $\bar{x}$.

Note that for constructing the transformations of dependent variables (2.14), x and u are considered to be independent and dependent variables, respectively. For constructing the transformations of arbitrary elements (2.12), (x, u) are considered as the independent variables.

Assuming that $u_0(x)$ is a solution of system (2.7) with $\phi_0(x, u)$, and the transformed function $u_a(\bar{x})$ is a solution of system (2.7) with the transformed arbitrary element $\phi_a(\bar{x}, \bar{u})$, the equations

$$F^k(\bar{x}, u_a(\bar{x}), p_a(\bar{x}), \phi_a(\bar{x}, u_a(\bar{x}))) = 0, (k = 1, 2, ..., s) \tag{2.16}$$

are satisfied for an arbitrary $\bar{x}$. Then one has

$$\begin{aligned} F^k(\quad & f^x(x, u_0(x), \phi_0(x, u_0(x)); a), f^u(x, u_0(x), \phi_0(x, u_0(x)); a), \\ & f^\phi(x, u_0(x), \phi_0(x, u_0(x)); a), f^p(x, u_0(x), \phi_0(x, u_0(x)), p_0(x), ...; a)) \\ & = 0, (k = 1, 2, ..., s) \end{aligned} \tag{2.17}$$

Differentiating (2.17) with respect to group parameter a and substituting $a = 0$, one has

$$\begin{aligned} & F^k_{x_i}(x, u_0(x), p_0(x), \phi_0(x, u_0(x)))\xi^{x_i}(x, u_0(x), \phi_0(x, u_0(x)))+ \\ & F^k_{u^j}(x, u_0(x), p_0(x), \phi_0(x, u_0(x)))\zeta^{u^j}(x, u_0(x), \phi_0(x, u_0(x)))+ \\ & F^k_{\phi^l}(x, u_0(x), p_0(x), \phi_0(x, u_0(x)))\zeta^{\phi^l}(x, u_0(x), \phi_0(x, u_0(x))) + ... = 0. \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} \xi^{x_i}(x, u_0(x), \phi_0(x, u_0(x))) &= \frac{\partial f^{x_i}(x, u_0(x), \phi_0(x, u_0(x)); a)}{\partial a}\Big|_{a=0}, \\ \zeta^{u^j}(x, u_0(x), \phi_0(x, u_0(x))) &= \frac{\partial f^{u^j}(x, u_0(x), \phi_0(x, u_0(x)); a)}{\partial a}\Big|_{a=0}, \\ \zeta^{\phi^l}(x, u_0(x), \phi_0(x, u_0(x))) &= \frac{\partial f^{\phi^l}(x, u_0(x), \phi_0(x, u_0(x)); a)}{\partial a}\Big|_{a=0}. \end{aligned}$$

Since $u_o(x)$ is an arbitrary solution on (2.7), one can write the equations

$$\widetilde{X}^e F^k(x, u, p, \phi)_{|(S)} = 0, \tag{2.19}$$

where

$$\widetilde{X}^e = X^e + \zeta^{u^j_{x_i}} \frac{\partial}{\partial u_{x_i}} + \zeta^{\phi^l_{x_i}} \frac{\partial}{\partial \phi^l_{x_i}} + \zeta^{\phi^l_{u^j}} \frac{\partial}{\partial \phi^l_{u^j}} + ..., \tag{2.20}$$

In (2.19), the sign $|_{(S)}$ means that the equations $\widetilde{X}^e F^k(x, u, p, \phi) = 0$ are consider on the set $(S) = \{(x, u, p, \phi) \mid \quad F^k(x, u, p, \phi) = 0, (k = 1, 2, ..., s)\}$, defined by equations (2.7). Equations (2.19) are equations for the coefficients of the generator X^e. They are called *determining equations.* After solve the determining equation (2.19), one obtains the generator X^e. The set of transformation (2.8) which is generated by one parameter Lie-groups corresponding to the generator X^e, is called an *equivalence group*,

The coefficients of the prolonged operator are defined by the prolongation formulae

$$\zeta^{u^j_{x_i}} = D^e_{x_i}\zeta^{u^j} - u^j_{x_k} D^e_{x_i}\xi^{x_k},$$

$$\zeta^{\phi^l_{x_i}} = D_{x_i}\zeta^{\phi^l} - \phi^l_{x_k} D_{x_i}\xi^{x_k} - \phi^l_{u_k} D_{x_i}\xi^{u_k},$$

$$\zeta^{\phi^l_{u^j}} = D_{u^j}\zeta^{\phi^l} - \phi^l_{x_k} D_{u^j}\xi^{x_k} - \phi^l_{u_k} D_{u^j}\xi^{u_k}, \ldots$$

Here the operators $D^e_{x_i}$ are operators of the total derivatives with respect to x_i, where the space of the independent variables consists of x_i,

$$D^e_{x_i} = \frac{\partial}{\partial x_i} + u_{x_i}\frac{\partial}{\partial u} + (\phi^l_{x_i} + u_{x_i}\phi^l_u)\frac{\partial}{\partial \phi^l}$$

The operators D_{x_i} and D_{u^j} are operators of total derivatives with respect to x_i and u^j, where the space of the independent variables consists of x_i and u^j,

$$D_{x_i} = \frac{\partial}{\partial x_i} + \phi^l_{x_i}\frac{\partial}{\partial \phi^l},\ D_{u^j} = \frac{\partial}{\partial u^j} + \phi^l_{u^j}\frac{\partial}{\partial \phi^l}.$$

2.2.1 Invariant manifolds

This section is devoted to the definitions concerning invariant regularly assigned manifolds. Assume that $\psi : V \to R^s$ is a mapping of the class $C^1(V)$ and V is an open set in R^N.

Definition 2.3. The mapping ψ has a rank on the set V if the rank of the Jacobi matrix $\frac{\partial \psi}{\partial z}$ is constant on V.

Definition 2.4. The set $\Psi = \{z \in V | \psi(z) = 0\}$ is called a *regularly assigned manifold* if Ψ has the rank s.

Definition 2.5. A function $J(z),\ z \in V$ is called an *invariant* of a Lie group G if

$$J(g(z;a)) = J(z),\ \forall z \in V, \forall a \in \Delta$$

and the function $J(z)$ is called a *relative invariant* if $\forall z \in V$ such that $J(z) = 0$, then

$$J(g(z;a)) = 0, \forall a \in \Delta.$$

Theorem 2.2. *A function $J(z)$ is an invariant of a Lie group $G(X)$ with generator X if and only if it satisfies the infinitesimal test*

$$XJ(z) = 0.$$

Definition 2.6. A manifold Ψ is invariant with respect to a Lie group G if after its transformation, any point $z \in \Psi$ belongs to the same mainifold Ψ.

Theorem 2.3. *A regularly assigned manifold Ψ is an invariant manifold with respect to a Lie group $G(X)$ if and only if*

$$X\psi^k(z)_{|_{(\Psi)}} = 0, (k = 1, 2, ..., s).$$

Here $|_{(\Psi)}$ means that the equations are considered on the mainfold Ψ. For applications the following theorem plays a very important role.

Theorem 2.4. *Any regularly assigned by $\psi^k(z_1, z_2, ..., z_N) = 0,\ (k = 1, 2, ..., s)$ invariant manifold Ψ can be represented by the formula*

$$\Phi^i(J^1(z), J^2(z), ..., J^{N-1}(z)) = 0, (i = 1, 2, ..., s),$$

where Φ^i are arbitrary functions of $N-1$ functionally invariants $J^l(z), (l = 1, 2, ..., N-1)$ for a Lie group $G(X)$.

Notice that definitions and theorems above can apply to an equivalence group, because an equivalence group is a special case of a Lie group.

It is assumed that (S) is a regularly assigned manifold. That is for differential equations it is also assumed

$$rank\left(\frac{\partial F}{\partial(x,u,p,\phi)}\right)=s.$$

By virtue of the inverse function theorem, it enough to assume

$$rank\left(\frac{\partial F}{\partial(p,\phi)}\right)=s.$$

This assumption allows system (S) to be solved with respect to s terms, which are derivatives of the dependent variables u and arbitrary elements ϕ. These terms are called main variables and the other variables are called parametric variables.

Now, one can notice that equations (2.19) mean that the regularly assigned manifold (S) is invariant with respect to the prolonged equivalence group $\underset{l}{G^e}$. This means that the point $(x, u_0(x), p_0(x), \phi_0(x,u))$ belonging to the manifold (S) is transformed to the point $(\bar{x}, u_a(\bar{x}), p_a(\bar{x}), \phi_0(\bar{x}, \bar{u}))$ which also belongs to the manifold (S). Such as it is the main feature of the equivalence group G^e, any solution $u_0(x)$ with $\phi_0(x,u)$ of the system (S) is transformed to the solution $u_a(\bar{x})$ with $\phi_0(\bar{x}, \bar{u})$ of the same system.

CHAPTER III

THEORY OF COMPATIBILITY

This section gives the necessary knowledge of involutive systems. Because this theory is very specialized subject of mathematical analysis, the statements are given without proofs. Detailed theory of involutive systems can be found in Cartan (1946), Finikov (1948), Kuranashi (1967), and Pommaret (1978). A short history of the theory can be found in Pommaret (1978).

There are two approaches for studying compatibility. These approaches are related to the works of E.Cartan and C.H.Riquier.

The Cartan approach is based on the calculus of exterior differential forms. The problem of the compatibility of a system of partial differential equations is then reduced to the problem of the compatibility of a system of exterior differential forms. Cartan studied the formal algebraic properties of systems of exterior forms. For their description he introduced special integer numbers, named characters. With the help of the characters he formulated a criterion for a given system of partial differential equations to be involutive.

The Riquier approach has a different theory of establishing the involution. This method can be found in Kuranashi (1967) and Pommaret (1978). The main advantage of this approach is that there is no necessity to reduce the system of partial differential equations being studied to exterior differential forms. Calculations in the Riquier approach are shorter than in the Cartan approach. The main operations of the study of compatibility in the Riquier approach are prolongations of a system of partial differential equations and the study of the ranks of some matrices. First we consider a simple case of compatibility, system of linear homogeneous equations with one dependent variable. In the next case we give a solution of first-order partial differential equation where all derivatives are defined. Then the general approach is discussed.

3.1 Complete systems partial differential equations

This section is devoted to solving a linear system of homogeneous first-order partial differential equations with one unknown function $u(x)$, $(x \in R^n)$:

$$X_i(u) \equiv \xi^i_j(x)\frac{\partial u}{\partial x_j} = 0 \ (i = 1, 2, ..., m). \tag{3.1}$$

Here, the function $u(x)$ and the coefficients $\xi^i_j(x)$ are assumed to be sufficiently many times continuously differentiable. For the sake of simplicity it is assumed that $rank(\xi^i_j(x)) = m$. This means that $m \leq n$. Notice that, because for $m = n$, the determinant $det((\xi^i_j(x))) \neq 0$, hence, the linear homogeneous (3.1) yields $u_{x_i} = 0$. In this case there is only the trivial solution $u = constant$. Thus, a necessary condition for the existence of nontrival solution is $m < n$.

3.1.1 Homogeneous linear equation

Let $x = (x_1, ..., x_n)$ be $n \geq 2$ independent variables and u a dependent variable. Consider the linear partial differential operator of the first order

$$X = \xi^i(x)\frac{\partial}{\partial x_i}. \tag{3.2}$$

In terms of this operator, the homogeneous linear partial differential equation is written as follow:

$$X(u) \equiv \xi^i(x)\frac{\partial u}{\partial x_i} = 0. \tag{3.3}$$

Theorem 3.1. *The general solution of equation (3.3) has the form*

$$u = F(\psi_1(x), ..., \psi_{n-1}(x))$$

where F is an arbitrary function of $n-1$ variables and

$$\psi_1(x), \psi_2(x), ..., \psi_{n-1}(x) \tag{3.4}$$

are functionally independent solutions of (3.3).

Notice that, functions (3.4) are said to be functionally dependent if there exists a function W such that $W(\psi_1(x), ..., \psi_{n-1}(x)) = 0$ for all x, and functionally independent otherwise. The solutions (3.4) are obtained by solving the *characteristic system* of equations

$$\frac{dx_1}{\xi_1(x)} = \frac{dx_2}{\xi_2(x)} = ... = \frac{dx_n}{\xi_n(x)}.$$

3.1.2 Poisson bracket

The linear operators X_i have the properties

$$X_i(u_1 + u_2) = X_i(u_1) + X_i(u_2),\ X_i(u_1 u_2) = u_2 X_i(u_1) + u_1 X_i(u_2),$$

$$X_i(X_j(u)) - X_j(X_i(u)) = [X_i(\xi^j_\alpha(x)) - X_j(\xi^i_\alpha(x))]\frac{\partial u}{\partial x_\alpha}.$$

Definition 3.1. The operator $(X_i, X_j)(u) \equiv [X_i(\xi^j_\alpha(x)) - X_j(\xi^i_\alpha(x))]\frac{\partial u}{\partial x_\alpha}$ is called a *Poisson bracket* with X_i and X_j.

Definition 3.2. The equations (3.1) are said to be *linearly dependent* if there exist functions $\lambda^\alpha(x)$, not all zero, such that

$$\lambda^1(x)X_1(u) + ... + \lambda^m(x)X_m(u) = 0, \tag{3.5}$$

in some neighborhood of x. If the relation (3.5) implies $\lambda^1 = ... = \lambda^m = 0$, we say that the equations (3.1) are *linearly independent* .

It is obvious that if $u(x)$ is a solution of the equations $X_i(u) = 0$ and $X_j(u) = 0$, then it also a solution of the equation $(X_i, X_j)(u) = 0$. Hence, new linear homogeneous equations can be produced by means of Poisson brackets. If the new equations $(X_i, X_j)(u) = 0$ are linearly independent of the equations of system (3.1), then one can append them to the initial system. Let m' be the number of the equations after append equations produced by Poisson brackets to the initial system. There are only two possibilities either $m' = n$ and $m' < n$. In the first case one has only trivial solution. The second case leads to the following definition.

Definition 3.3. System (3.1) is called a *complete system* if any Poisson bracket is linearly dependent on the equations of the initial system (3.1).

That is, every system of homogeneous linear partial differential equations can be converted into a complete system by adding all equations which produced by Poisson brackets. We can solve the complete system by solving equation by equation.

3.2 General theory of compatibility

Let a system of q-th order differential equations (S) be defined by the equations

$$(S) \qquad \Phi^i(x, u, p) = 0, \ (i = 1, 2, \dots, s). \tag{3.6}$$

Here $x = (x_1, x_2, \dots, x_n)$ are the independent variables, $u = (u^1, u^2, \dots, u^m)$ are the dependent variables, $p = (p^j_\alpha)$ is the set of the derivatives $p^j_\alpha = \frac{\partial^{|\alpha|} u^j}{\partial x^\alpha}$, $(j = 1,\ 2,\ \dots,\ m;\ |\alpha| \leqslant q)$, $\alpha = (\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. All constructions are considered in some neighborhood of the point $X_0 = (x_0, u_0, p_0) \in (S)$. First the algebraic properties of a symbol of the system (S) are studied. The symbol G_q of the system (S) at the point X_0 is defined as the vector space of vectors with the coordinates $(\xi^j_\alpha)\,,(j = 1,\ 2,\ \ldots,\ m;\ |\alpha| = q)$, where the coordinates (ξ^j_α) satisfy the algebraic equations

$$\sum_{j=1}^{m}\sum_{|\alpha|=q}\xi^j_\alpha\frac{\partial\Phi^i}{\partial p^j_\alpha}(X_0) = 0,\ (i = 1,\ 2,\ \ldots,\ s)\,.$$

The subspace of the symbol G_q composed by the vectors with

$$\xi^j_{\beta,\,l} = 0,\ (|\beta| = q-1;\ \ l = 1,\ 2,\ \ldots,\ k;\ \ j = 1,\ 2,\ \ldots,\ m)$$

is denoted by $(G_q)^k$, $(k = 1,\ 2,\ \ldots,\ n-1)$. Here $\beta,\ \ l = (\beta_1,\ \beta_2,\ \ldots,\ \beta_{l-1},\ \beta_l + 1,\ \beta_{l+1},\ \ldots,\ \beta_n)$, $(G_q)^0 = G_q$, and $(G_q)^n = \{0\}$. Let the dimensions of the vector spaces $(G_q)^k$ be τ_k. For example,

$$\tau_0 = m\begin{pmatrix} n+q-1 \\ q \end{pmatrix} - rank\left(\frac{\partial\Phi^i}{\partial p^j_\alpha}(X_0)\right),\quad \tau_n = 0.$$

The number

$$\sum_{k=0}^{n-1}\tau_k$$

is called the *Cartan number*. With the help of the numbers $\tau_k,\ \ (k = 0,\ 1,\ \ldots,\ n)$ the *Cartan characters* are defined by the formulae

$$\sigma_{k+1} = \tau_k - \tau_{k+1},\ \ (k = 1,\ \ldots,\ n-1)\,.$$

Note that $\tau_0 = \sum\limits_{k=1}^{n}\sigma_k$ and the Cartan number can be expressed through the Cartan characters

$$\sum_{k=0}^{n-1}\tau_k = \sum_{k=1}^{n}k\sigma_k.$$

Let G_{q+1} be the symbol of the prolonged system (DS):

$$(DS)\qquad D_l\Phi^i(x,u,p) = 0,\ \ (l = 1,\ 2,\ \ldots,\ n;\ \ i = 1,\ 2,\ \ldots,\ s)\,. \tag{3.7}$$

Here the operator D_l is the total derivative with respect to x_l

$$D_l = \frac{\partial}{\partial x_l} + \sum_{|\alpha|}\sum_{j=1}^{m}p^j_{\alpha,l}\frac{\partial}{\partial p^j_\alpha}.$$

Definition 3.4. The system of differential equations composed by the system (S) and (DS) is called the first prolongation of the system (S).

The theory of compatibility is a local theory, i.e., all properties are considered in some neighborhood of a point X_0, and all manifolds and functions are assumed to be the necessary number of times continuously differentiable. Moreover, the Cartan theorem works only for analytical functions.

Note that Cartan characters depend on the order of the independent variables $(x_1, x_2, x_3, \ldots, x_n)$: so any change of the order can change the Cartan characters. There is the estimate

$$\dim(G_{q+1}) \leqslant \sum_{k=1}^{n} k\sigma_k.$$

Definition 3.5. A coordinate system of the independent variables in which there is the equality

$$\dim(G_{q+1}) = \sum_{k=1}^{n} k\sigma_k$$

is called a *quasiregular coordinate system.*

Definition 3.6. If there exists a quasiregular coordinate system, then a symbol G_q is called an *involutive symbol.*

After studying the algebraic properties of the system (S) one has to analyze the differential structure of the manifold defined by the equations (DS). From the system (DS) one can find $N = \dim(G_{q+1})$ derivatives of the highest $q+1$ order. These derivatives are called the main derivatives of the system (DS) of order $q+1$.

Definition 3.7. If a system (S) with an involutive symbol posseses the property that after substituting the main derivatives of the prolonged system (DS) of order $q+1$, the remaining equations of the system (DS) are identities because of the system (S), then system (S) is called *involutive.*

Theorem 3.2 (Cartan)**.** *Any analytic system of partial differential equations after a finite number of prolongations becomes either involutive or incompatible.*

Theorem 3.3 (Cartan – Köhler)**.** *If a system (S) of order q is involutive and analytic, there exists one and only one analytic solution of the Cauchy problem with given σ_k functions of k arguments* $(k = 1, 2, \ldots, n-1)$.

The property of analyticity of an involutive system is not a necessary condition for the existence of a solution. There are theorems of existence of involutive systems of the class C^1 (Meleshko (1980)).

Any study of compatibility requires a large amount of symbolic calculations. These calculations consist of consecutive algebraic operations: prolongation of a system, substitution of some expressions (transition onto manifold), and the determination of ranks of matrices (for obtaining the Cartan characters). Because these operations are very labor intensive, it is necessary to use a computer for symbolic calculations.

In practice, sometimes it is enough to use the particular case of the compatibility theorem.

Corollary 3.4 (completely integrable systems)**.** *If in an overdetermined system of partial differential equations all derivatives of order n are defined and comparison of all mixed derivatives of order $n+1$ does not produce new equations of order less or equal to n, then this system is compatible.*

3.3 Completely integrable systems

One class of overdetermined systems, for which the problem of compatibility is solved, is the class of completely integrable systems. The theory of completely integrable systems is developed in the general case.

Definition 3.8. A system

$$\frac{\partial z^i}{\partial a^j} = f^i_j(a, z), \ (i = 1, 2, ..., N; \ j = 1, 2, ..., r) \tag{3.8}$$

is called a *completely integrable* if it has a solution for any initial values a_0, z_0 in some open domain D.

Theorem 3.5. *Any system of the type (3.8) is completely integrable if and only if all of the mixed derivatives equalities*

$$\frac{\partial f^i_j}{\partial a^\beta} + \sum_{\gamma=1}^{N} f^\gamma_\beta \frac{\partial f^i_j}{\partial z^\gamma} = \frac{\partial f^i_\beta}{\partial a^j} + \sum_{\gamma=1}^{N} f^\gamma_j \frac{\partial f^i_\beta}{\partial z^\gamma}, \ (i = 1, 2, ..., N; \ \beta, j = 1, 2, ..., r) \tag{3.9}$$

are identically satisfied with respect to the variables $(a, z) \in D$.

CHAPTER IV

NECESSARY CONDITIONS FOR THE LINEARIZATION PROBLEM

In this thesis a nonlinear parabolic partial differential equation

$$F(t, x, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0 \tag{4.1}$$

is studied. The problem considered in this chapter is related with the linearization problem. The linearization problem is to find an invertible change of the independent and dependent variables

$$\tau = H(t, x, u),\ \ y = Y(t, x, u),\ \ v = V(t, x, u) \tag{4.2}$$

which transforms the nonlinear equation (4.1) into a linear second-order parabolic partial differential equation

$$v_\tau + b_1 v_{yy} + b_2 v_y + b_3 v = 0. \tag{4.3}$$

Here the functions b_1, b_2, b_3 depend on the independent variables τ, y.

This chapter is devoted to obtaining a form of parabolic second-order partial differential equation which is necessary for equation (4.1) to be linearizable via point transformations (4.2).

4.1 Obtaining necessary conditions

We assume that the equation (4.1) is obtained from a linear equation (4.3) by an invertible change of variables (4.2). Let us obtain change of derivatives. For this we suppose that $u_0(t,x)$ is a given function. Substituting $u_0(t,x)$ into (4.2), one obtains

$$\tau = H(t,x,u_0(t,x)),\ y = Y(t,x,u_0(t,x)). \tag{4.4}$$

By virtue of the inverse function theorem, there exist functions $T(\tau,y)$, $X(\tau,y)$ such that

$$t = T(\tau,y),\ x = X(\tau,y). \tag{4.5}$$

After substituting (4.5) into the third equation of (4.2), one obtains the transformation of the function $u_0(t,x)$:

$$v_0(\tau,y) = V(t,x,u_0(t,x)),$$

where t and x are defined by (4.5). Notice that the function $v_0(\tau,y)$ satisfies the relation

$$\begin{aligned} &v_0(H(t,x,u_0(t,x)),Y(t,x,u_0(t,x))) = V(t,x,u_0(t,x)),\\ &v_0(\tau,y) = V(T(\tau,y),X(\tau,y),u_0(T(\tau,y),X(\tau,y))). \end{aligned} \tag{4.6}$$

Differentiating the first equation of (4.6) with respect to t and x, and using the chain rule one gets

$$v_{0\tau}D_tH + v_{0y}D_tY = D_tV, \quad v_{0\tau}D_xH + v_{0y}D_xY = D_xV, \tag{4.7}$$

where

$$D_t = \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + u_{tt}\partial_{u_t}, \quad D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t}.$$

Solving a linear system of algebraic equations (4.7) with respect to the derivatives $v_{0\tau}$ and v_{0y}, one has

$$\begin{aligned} v_{0\tau}(\tau,y) &= \triangle^{-1}(D_xYD_tV - D_tYD_xV),\\ v_{0y}(\tau,y) &= -\triangle^{-1}(D_xHD_tV - D_tHD_xV), \end{aligned} \tag{4.8}$$

where $\triangle = (D_tH)(D_xY) - (D_xH)(D_tY) \neq 0$ is the Jacobian of the change of variables. Differentiating the second equation in (4.8) with respect to t and x, one obtains

$$v_{0y\tau}D_tH + v_{0yy}D_tY = A_1, \quad v_{0y\tau}D_xH + v_{0yy}D_xY = A_2, \tag{4.9}$$

where

$$A_1 = (\triangle D_t(D_xHD_tV - D_tHD_xV) - (D_xHD_tV - D_tHD_xV)D_t\triangle)/\triangle^2,$$

$$A_2 = (\triangle D_x(D_xHD_tV - D_tHD_xV) - (D_xHD_tV - D_tHD_xV)D_x\triangle)/\triangle^2$$

Hence, the derivative v_{yy} is

$$v_{0yy} = \triangle^{-1}(-A_1D_xH + A_2D_tH). \tag{4.10}$$

Substituting (4.8) and (4.10) into (4.3), one obtains a nonlinear equation of the form

$$\begin{aligned}&Au_{tt} + Bu_{tx} + Cu_{xx} + a_1u_t^3 + a_2u_x^3 + a_3u_t^2u_x + a_4u_tu_x^2 + a_5u_t^2 + a_6u_x^2\\&+a_7u_tu_x + a_8u_t + a_9u_x + a_{10} = 0,\end{aligned} \tag{4.11}$$

where

$$A = a_{11} + a_{12}u_x^2 - a_{13}u_x,$$

$$B = a_{15} - 2a_{12}u_tu_x + a_{13}u_t - a_{14}u_x,$$

$$C = a_{16} + a_{12}u_t^2 + a_{14}u_t,$$

and $a_i,\ (i = 1, ..., 16)$ are some functions which depend on t, x and u *. For nonlinear parabolic partial differential equation, one requires the condition $B^2 - 4AC = 0$, i.e.,

$$\begin{aligned}&(a_{13}^2 - 4a_{11}a_{12})u_t^2 + 2(a_{13}a_{14} - 2a_{15}a_{12})u_tu_x + 2(a_{13}a_{15} - 2a_{11}a_{14})u_t\\&+(a_{14}^2 - 4a_{12}a_{16})u_x^2 + 2(2a_{13}a_{16} - a_{15}a_{14})u_x - 4a_{16}a_{11} + a_{15}^2 = 0.\end{aligned} \tag{4.12}$$

Equation (4.12) implies that

$$\begin{aligned}&a_{13}^2 - 4a_{11}a_{12} = 0,\ a_{13}a_{14} - 2a_{15}a_{12} = 0,\ a_{13}a_{15} - 2a_{11}a_{14} = 0,\\&a_{14}^2 - 4a_{12}a_{16} = 0,\ 2a_{13}a_{16} - a_{15}a_{14} = 0,\ a_{15}^2 - 4a_{16}a_{11} = 0.\end{aligned} \tag{4.13}$$

*The representations of $a_i,\ (i = 1, 2, ..., 16)$ are in Appendix A

The equation in the form (4.11) is a necessary condition for the linearization problem by means of point transformations (4.3) which transforms the nonlinear equation (4.1) into a linear second-order parabolic partial differential equation (4.3).

Let us transform (4.11) by an invertible change of the independent and dependent variables

$$t = \Phi(\bar{t}, \bar{x}, \bar{u}),\ x = \Psi(\bar{t}, \bar{x}, \bar{u}),\ u = \Omega(\bar{t}, \bar{x}, \bar{u}). \tag{4.14}$$

The change of derivatives of the dependent variable u is similar to the change of derivatives of the dependent variable v given in the beginning of this section. Substituting them into (4.11), one gets

$$\begin{aligned}&\bar{A}\bar{u}_{\bar{t}\bar{t}} + \bar{B}\bar{u}_{\bar{t}\bar{x}} + \bar{C}\bar{u}_{\bar{x}\bar{x}} + \bar{a}_1\bar{u}_{\bar{t}}^3 + \bar{a}_2\bar{u}_{\bar{x}}^3 + \bar{a}_3\bar{u}_{\bar{t}}^2\bar{u}_{\bar{x}} + \bar{a}_4\bar{u}_{\bar{t}}\bar{u}_{\bar{x}}^2 + \bar{a}_5\bar{u}_{\bar{t}}^2 + \bar{a}_6\bar{u}_{\bar{x}}^2\\ &+\bar{a}_7\bar{u}_{\bar{t}}\bar{u}_{\bar{x}} + \bar{a}_8\bar{u}_{\bar{t}} + \bar{a}_9\bar{u}_{\bar{x}} + \bar{a}_{10} = 0,\end{aligned} \tag{4.11$'$}$$

where

$$\begin{aligned}\bar{A} &= \bar{a}_{11} + \bar{a}_{12}\bar{u}_{\bar{x}}^2 - \bar{a}_{13}\bar{u}_{\bar{x}},\\ \bar{B} &= \bar{a}_{15} - 2\bar{a}_{12}\bar{u}_{\bar{t}}\bar{u}_{\bar{x}} + \bar{a}_{13}\bar{u}_{\bar{t}} - \bar{a}_{14}\bar{u}_{\bar{x}},\\ \bar{C} &= \bar{a}_{16} + \bar{a}_{12}\bar{u}_{\bar{t}}^2 + \bar{a}_{14}\bar{u}_{\bar{t}},\end{aligned}$$

and $\bar{a}_i,\ i = 1, 2, ..., 16$ are some functions which depend on $\bar{t}, \bar{x}$ and $\bar{u}$ and the coefficients $a_i,\ (i = 1, ..., 16)$. Direct checking shows that the following conditions are satisfied

$$\begin{aligned}&\bar{a}_{13}^2 - 4\bar{a}_{11}\bar{a}_{12} = 0,\ \bar{a}_{13}\bar{a}_{14} - 2\bar{a}_{15}\bar{a}_{12} = 0,\ \bar{a}_{13}\bar{a}_{15} - 2\bar{a}_{11}\bar{a}_{14} = 0,\\ &\bar{a}_{14}^2 - 4\bar{a}_{12}\bar{a}_{16} = 0,\ 2\bar{a}_{13}\bar{a}_{16} - \bar{a}_{15}\bar{a}_{14} = 0,\ \bar{a}_{15}^2 - 4\bar{a}_{16}\bar{a}_{11} = 0.\end{aligned} \tag{4.13$'$}$$

These conditions guarantee that $\bar{B}^2 - 4\bar{A}\bar{C} = 0$, which means that (4.11$'$) is a nonlinear parabolic differential equations. From (4.11$'$) and (4.13$'$), we see that the form and type of parabolic second-order partial differential equations (4.11) are not changed by a change of the dependent and independent variables (4.14).

4.2 Particular forms of linearlizable parabolic partial differential equations

4.2.1 Case $a_{11} \neq 0$

One can assume that $a_{11} = 1$. Then (4.13) give

$$a_{12} = a_{13}^2/4,\ a_{14} = a_{13}a_{15}/2,\ a_{16} = a_{15}^2/4. \tag{4.15}$$

Substituting a_{12}, a_{14} and a_{16} from (4.15) into (4.11), one has

$$\begin{aligned}&(1 + \tfrac{a_{13}^2}{4}u_x^2 - a_{13}u_x)u_{tt} + (a_{15} - \tfrac{a_{13}^2}{2}u_tu_x + a_{13}u_t - \tfrac{a_{13}a_{15}}{2}u_x)u_{tx}\\ &+(\tfrac{a_{15}^2}{4} + \tfrac{a_{13}^2}{4}u_t^2 + \tfrac{a_{13}a_{15}}{2}u_t)u_{xx} + a_1u_t^3 + a_2u_x^3 + a_3u_t^2u_x\\ &+a_4u_tu_x^2 + a_5u_t^2 + a_6u_x^2 + a_7u_tu_x + a_8u_t + a_9u_x + a_{10} = 0.\end{aligned} \tag{4.16}$$

4.2.2 Case $a_{11} = 0$, $a_{16} \neq 0$

Assume $a_{16} = 1$, from (4.13), one obtains

$$a_{13} = 0,\ a_{15} = 0,\ a_{12} = a_{14}^2/4. \tag{4.17}$$

Substituting a_1, a_2 and a_{12} into (4.11), one gets

$$\begin{aligned}&\tfrac{a_{14}^2}{4}u_{tt} - (\tfrac{a_{14}^2}{4}u_tu_x + a_{14}u_x)u_{tx} + (1 + \tfrac{a_{14}^2}{4}u_t^2 + a_{14}u_t)u_{xx} + a_1u_t^3 + a_2u_x^3\\ &+a_3u_t^2u_x + a_4u_tu_x^2 + a_5u_t^2 + a_6u_x^2 + a_7u_tu_x + a_8u_t + a_9u_x + a_{10} = 0.\end{aligned} \tag{4.18}$$

4.2.3 Case $a_{11} = 0$, $a_{16} = 0$

From (4.13), one obtains

$$a_{13} = 0,\ a_{15} = 0,\ a_{14} = 0. \tag{4.19}$$

Substituting a_{13}, a_{15} and a_{14} from (4.19) into (4.11), one gets

$$\begin{aligned}&a_{12}(u_x^2u_{tt} - 2u_tu_xu_{tx} + u_t^2u_{xx}) + a_1u_t^3 + a_2u_x^3 + a_3u_t^2u_x + a_4u_tu_x^2 + a_5u_t^2\\ &+a_6u_x^2 + a_7u_tu_x + a_8u_t + a_9u_x + a_{10} = 0.\end{aligned} \tag{4.20}$$

CHAPTER V

INVARIANTS OF LINEAR PARABOLIC DIFFERENTIAL EQUATIONS

5.1 Introduction

We consider linear second-order parabolic partial differential equations in two independent variables:

$$u_t + a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u = 0, \quad a(t,x) \neq 0. \tag{5.1}$$

Recall that the well-known group of equivalence transformations for equation (5.1) (Lie (1881)), i.e. the changes of variables t, x and u that do not change the form of equation (5.1), is composed of the linear transformation of the dependent variable

$$\overline{u} = \sigma(t,x)\,u, \tag{5.2}$$

and the following change of the independent variables:

$$\overline{t} = \phi(t), \quad \overline{x} = \psi(t,x), \tag{5.3}$$

where $\sigma(t,x), \phi(t)$ and $\psi(t,x)$ are arbitrary functions obeying the invertibility conditions, $\sigma(t,x) \neq 0,\ \phi'(t) \neq 0$ and $\psi_x(t,x) \neq 0$. Invariance of the form of Equation (5.1) means that the transformations (5.2)–(5.3) map equation (5.1) into an equation of the same form:

$$\overline{u}_{\bar{t}} + \bar{a}(\bar{t},\bar{x})\,\overline{u}_{\bar{x}\bar{x}} + \bar{b}(\bar{t},\bar{x})\,\overline{u}_{\bar{x}} + \bar{c}(\bar{t},\bar{x})\,\overline{u} = 0. \tag{5.1$'$}$$

Equations (5.1) and (5.1′) connected by an equivalence transformation are called *equivalent equations.* We leave the proof that any transformation

$$\overline{u} = U(t,x,u),\ \ \overline{x} = X(t,x,u),\ \ \bar{t} = T(t,x,u) \tag{5.4}$$

which maps equation (5.1) into an equation of the same form has the representation (5.2) and (5.3) to chapter VI.

An *invariant of Equation (5.1)* is a function

$$J(a,b,c,a_t,a_x,b_t,b_x,c_t,c_x,a_{tt},a_{tx},a_{xx},\ldots,c_{xx},\ldots)$$

that remains unaltered under the equivalence transformations (5.2)–(5.3). It means that J has the same value for equivalent equations (5.1) and (5.1′):

$$J(a,b,c,a_t,\ldots,c_{xx},\ldots) = J(\bar{a},\bar{b},\bar{c},\bar{a}_{\bar{t}},\ldots,\bar{c}_{\bar{x}\bar{x}},\ldots).$$

If J is invariant only under the transformation (5.2) it is termed a *semi-invariant* (Ibragimov N.H. (2002)). The *order* of an invariant (or semi-invariant) J is identified with the highest order of derivatives of a, b, c involved in J.

Semi-invariants of hyperbolic equations (termed the Laplace invariants) have been known since the 1770s. Recently there has been considerable interest in invariants of parabolic equations. The first step toward solving the problem of invariants for parabolic equations was made in (Ibragimov N.H. (2002)) where the semi-invariant of the second order

$$K = 2c_xa^2 - b_ta - b_{xx}a^2 - b_xba + b_xa_xa + \frac{1}{2}b^2a_x + ba_t + ba_{xx}a - ba_x^2 \tag{5.5}$$

was found. It was also shown that K and the coefficient $a(t,x)$ provide a basis of semi-invariants. This solves the problem of semi-invariants. Namely, any semi-invariant J of arbitrary order involves only a and K together with their derivatives of appropriate order, i.e.

$$J = J(a,a_t,a_x,a_{tt},a_{tx},a_{xx},\ldots,K,K_t,K_x,K_{tt},K_{tx},K_{xx},\ldots). \tag{5.6}$$

Furthermore, it follows from this result that the invariants of Equation (5.1) with respect to the general equivalence group can be obtained by subjecting the functions (5.6) to the condition of invariance under the change (5.3) of the independent variables.

The method and result of (Ibragimov N.H. (2002)) were used in (Johnpillai I.K. and Mahomed F.M. (2001)) for investigating invariants and invariant equations up to fifth-order with respect to the joint transformations (5.2) and (5.3). It has been shown in (Johnpillai I.K. and Mahomed F.M. (2001)) that Equation (5.1) has no invariants up to fifth-order and that it has precisely one invariant equation of the fifth-order, namely the equation

$$\lambda = 0 \tag{5.7}$$

where the quantity λ is defined by

$$\begin{aligned}\lambda =& 4a\left(2aK_{xx} - 5a_xK_x\right) - 12K\left(aa_{xx} - 2a_x^2\right) + a_x\left(4aa_{tt} - 9a_x^4\right)\\ &- 12a_ta_x\left(a_t + 2a_x^2\right) + 4a\left(3a_t + 6a_x^2 - 5aa_{xx}\right)a_{tx}\\ &+ 2aa_x\left(16a_ta_{xx} - 12aa_{xx}^2 + 15a_x^2a_{xx}\right) - 4a^2a_{ttx} - 12a^2a_xa_{txx}\\ &- 4a^2a_{xxx}\left(2a_t - 4aa_{xx} + 3a_x^2\right) + 8a^3a_{txxx} - 4a^4a_{xxxxx}\end{aligned} \tag{5.8}$$

and is termed a *relative invariant* due to the invariance of equation (5.7) with respect to the equivalence transformations (5.2)–(5.3). The function λ becomes an invariant if it restricted to equation (5.1) satisfying $\lambda = 0$. It is demonstrated in (Johnpillai I.K. and Mahomed F.M. (2001)) that equation (5.7) provides a necessary and sufficient condition for Equation (5.1) to be equivalent to the heat equation.

In the thesis, we find all invariants and invariant equations of the sixth and seventh orders. Since $\lambda = 0$ singles out the heat equation and all equations equivalent to the heat equation, we exclude these equations and assume in what follows that $\lambda \neq 0$. Under this assumption, we prove the following result.

Theorem 5.1. *An arbitrary equation (5.1) with $\lambda \neq 0$ has one invariant of the 6-th order:*

$$\Lambda_1 = \frac{2a\lambda_x - 5\lambda a_x}{\lambda^{6/5}} \tag{5.9}$$

and one invariant of the 7th order:

$$\Lambda_2 = \frac{2a^2\lambda_{xx} - 9aa_x\lambda_x + 5(3a_x^2 - aa_{xx})\lambda}{\lambda^{7/5}}\,. \tag{5.10}$$

Furthermore, there are additional invariants of the 7-th order in the following particular cases.

(A) The family of Equations (5.1) obeying the invariant conditions

$$5\Lambda_2 - 3\Lambda_1^2 = 0, \quad \Lambda_1 \neq 0 \tag{5.11}$$

has the invariant

$$\Lambda_3 = \frac{a}{\lambda^{8/5}}\left[a_x\lambda_t + 2a_t\lambda_x - \frac{12}{5\lambda}\,a\lambda_t\lambda_x + 2a\lambda_{tx} - 5\lambda a_{tx}\right]. \tag{5.12}$$

(B) The family of Equations (5.1) defined by two invariant equations

$$\Lambda_1 = 0, \quad \Lambda_2 = 0, \quad \Lambda_3 = 0 \tag{5.13}$$

has the invariant

$$\begin{aligned}\Lambda_4 = \frac{1}{4\lambda^{9/5}}\Big[&10\lambda a^2(3a_xa_{xxx} - 2aa_{xxxx} + 3a_{xx}^2 - 4a_{txx}) + 5\lambda a(8a_ta_{xx}\\ &- 8a_{tt} + 16a_xa_{tx} - 15a_x^2a_{xx} - 8K_x) + 2\lambda(50a_t^2 - 4a_ta_x^2 + 15a_x^4\\ &+ 40a_xK) + a\lambda_x(8a_xa_t + 6aa_xa_{xx} - 4a^2a_{xxx} - 8aa_{tx} - 3a_x^3 - 8K)\\ &- 40aa_t\lambda_t + 8a^2\lambda_{tt}\Big] - \frac{3}{5\lambda^{14/5}}\left(2a\lambda_t - 5\lambda a_t\right)^2.\end{aligned} \tag{5.14}$$

(C) The family of Equations (5.1) obeying the invariant conditions

$$\Lambda_1 = 0, \quad \Lambda_2 \neq 0 \tag{5.15}$$

has the invariant

$$\Lambda_5 = 4\Lambda_2\Lambda_4 - 3\Lambda_3^2. \tag{5.16}$$

5.2 Equivalence Lie group

For obtaining invariants we use the Lie approach. This approach consists of finding an equivalence group of point transformations, and finding its invariants by solving a system of homogeneous linear equations. Let us recall the method for obtaining an equivalence group. Consider a parabolic equation (5.1). Since the functions a, b, c depend on the independent variables t, x only, the equivalence group should leave invariant the equations

$$a_u = 0, \quad b_u = 0, \quad c_u = 0. \tag{5.17}$$

Let the generator of a one-parameter equivalence Lie group be

$$X^e = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \zeta^u \frac{\partial}{\partial u} + \zeta^a \frac{\partial}{\partial a} + \zeta^b \frac{\partial}{\partial b} + \zeta^c \frac{\partial}{\partial c} \tag{5.18}$$

where the coefficients $\xi^t, \ldots, \zeta^c$ may, in general, depend on the variables t, x, u, a, b, c. The coefficients of the prolonged operator

$$\widetilde{X}^e = X^e + \zeta^{u_t} \frac{\partial}{\partial u_t} + \zeta^{u_x} \frac{\partial}{\partial u_x} + \zeta^{u_{xx}} \frac{\partial}{\partial u_{xx}} + \zeta^{a_u} \frac{\partial}{\partial a_u} + \zeta^{b_u} \frac{\partial}{\partial b_u} + \zeta^{c_u} \frac{\partial}{\partial c_u}$$

are defined by the prolongation formulae

$$\zeta^{u_t} = D_t^e \zeta^u - u_t D_t^e \xi^t - u_x D_t^e \xi^x, \quad \zeta^{u_x} = D_x^e \zeta^u - u_t D_x^e \xi^t - u_x D_x^e \xi^x,$$

$$\zeta^{u_{xx}} = D_x^e \zeta^{u_x} - u_{xt} D_x^e \xi^t - u_{xx} D_x^e \xi^x, \quad \zeta^{a_u} = D_u \zeta^a - a_t D_u \xi^t - a_x D_u \xi^x,$$

$$\zeta^{b_u} = D_u \zeta^b - b_t D_u \xi^t - b_x D_u \xi^x, \quad \zeta^{c_u} = D_u \zeta^b - c_t D_u \xi^t - c_x D_u \xi^x,$$

Here the operators $D_t^e,\ D_x^e$ are operators of the total derivatives with respect to t and x, respectively, where the space of the independent variables consists of t and x,

$$D_t^e = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + (a_t + u_t a_u)\frac{\partial}{\partial a} + (b_t + u_t b_u)\frac{\partial}{\partial b} + (c_t + u_t c_u)\frac{\partial}{\partial c} + \cdots ,$$

$$D_x^e = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + (a_x + u_x a_u)\frac{\partial}{\partial a} + (b_x + u_x b_u)\frac{\partial}{\partial b} + (c_x + u_x c_u)\frac{\partial}{\partial c} + \cdots .$$

The operators $D_t,\ D_x$ and D_u are operators of total derivatives with respect to t, x and u, where the space of the independent variables consists of t, x, and u,

$$D_t = \frac{\partial}{\partial t} + a_t\frac{\partial}{\partial a} + b_t\frac{\partial}{\partial b} + c_t\frac{\partial}{\partial c} + \cdots ,$$

$$D_x = \frac{\partial}{\partial x} + a_x\frac{\partial}{\partial a} + b_x\frac{\partial}{\partial b} + c_x\frac{\partial}{\partial c} + \cdots ,$$

$$D_u = \frac{\partial}{\partial u} + a_u\frac{\partial}{\partial a} + b_u\frac{\partial}{\partial b} + c_u\frac{\partial}{\partial c} + \cdots .$$

Because of (5.17) and the definitions of $\zeta^{a_u},\ \zeta^{b_u},\ \zeta^{c_u}$, one can split the part of determining equations

$$\zeta^{a_u} = 0, \quad \zeta^{b_u} = 0, \quad \zeta^{c_u} = 0$$

with respect to $a_t,\ a_x,\ b_t,\ b_x,\ c_t,\ c_x$. Consequently the coefficients $\xi^t, \xi^x, \zeta^a, \zeta^b$ and ζ^c do not depend on u.

Solving the determining equations

$$\widetilde{X}^e F\big|_{(1),(2)} = 0,$$

one finds

$$\xi^t = p,\ \xi^x = q,\ \zeta^u = u\sigma(t,x),\ \ \zeta^a = 2aq_t - ap_t,$$

$$\zeta^b = aq_{xx} + bq_x - bp_t + q_t - 2a\sigma_x, \quad \zeta^c = -cp_t - \sigma_t - a\sigma_{xx} - b\sigma_x,$$

with arbitrary functions $p = p(t),\ q = q(t,x),\ \sigma = \sigma(t,x)$. Hence, we arrive at the following generator of the equivalence Lie group:

$$\begin{aligned} X^e = \ & p\tfrac{\partial}{\partial t} + q\tfrac{\partial}{\partial x} + u\sigma\tfrac{\partial}{\partial u} + a(2q_x - p_t)\tfrac{\partial}{\partial a} \\ & + (aq_{xx} + bq_x - bp_t + q_t - 2a\sigma_x)\tfrac{\partial}{\partial b} - (cp_t + \sigma_t + a\sigma_{xx} + b\sigma_x)\tfrac{\partial}{\partial c}. \end{aligned} \tag{5.19}$$

5.3 Representation of invariants

For obtaining n-th order invariants we use the infinitesimal test

$$\widetilde{X}^e(J) = 0,$$

where J depends on $a,\ b,\ c$ and their derivatives up to order n. Notice that for relative invariants the infinitesimal test is

$$\widetilde{X}^e(J_k)\big|_{(S)} = 0, \quad k = 1, ..., s,$$

where (S) is the manifold defined by equations $J_k = 0,\ k = 1, ..., s$.

Recall that the generator for finding semi-invariants is (see (Ibragimov N.H. (2002))

$$X^e = \quad u\sigma\tfrac{\partial}{\partial u} - (2a\sigma_x)\tfrac{\partial}{\partial b} - (\sigma_t + a\sigma_{xx} + b\sigma_x)\tfrac{\partial}{\partial c} \tag{5.20}$$

and that Equation (5.1) has the following semi-invariants up to the second order (see Introduction)

$$a, \quad a_t, \quad a_x, \quad a_{tt}, \quad a_{tx}, \quad a_{xx}, \quad K,$$

where K is given by Equation (5.5):

$$K = 2c_x a^2 - b_t a - b_{xx} a^2 - b_x b a + b_x a_x a + \frac{1}{2} b^2 a_x + b a_t + b a_{xx} a - b a_x^2.$$

Furthermore, the invariants of the equivalence group defined by the generator (5.19) are in the class of functions J of the form (5.5) involving, in general, the derivatives of a up to the order n, and derivatives of the function $K(t, x)$ are up to the order $n - 2$. Accordingly, the generator (5.19) is rewritten in the form

$$X^e = a(2q_x - p_t)\frac{\partial}{\partial a} + \zeta^K \frac{\partial}{\partial K},$$

where

$$\begin{aligned} \zeta^K = \quad & q_{tx} a a_x - q_{xxxx} a^3 - q_{xxx} a_x a^2 - 2q_{txx} a^2 + 3q_x K \\ & - q_{tt} a + q_t \left(a_t + a_{xx} a - a_x^2\right) - 3p_t K. \end{aligned}$$

The coefficients of the prolonged operator

$$\widetilde{X}^e = X^e + \zeta^{a_t}\frac{\partial}{\partial a_t} + \zeta^{a_x}\frac{\partial}{\partial a_x} + ... + \zeta^{K_t}\frac{\partial}{\partial K_t} + \zeta^{K_x}\frac{\partial}{\partial K_x} + ..., \tag{5.21}$$

are defined by the prolongation formulae, e.g.

$$\zeta^{a_t} = D_t\zeta^a - a_t D_t\xi^t - a_x D_t\xi^x, \quad \zeta^{a_x} = D_x\zeta^a - a_t D_x\xi^t - a_x D_x\xi^x,$$
$$\zeta^{K_t} = D_t\zeta^K - K_t D_t\xi^t - K_x D_t\xi^x, \quad \zeta^{K_x} = D_x\zeta^K - K_t D_x\xi^t - K_x D_x\xi^x,$$

where

$$D_t = \frac{\partial}{\partial t} + a_t\frac{\partial}{\partial a} + K_t\partial_K + ..., \quad D_x = \frac{\partial}{\partial x} + a_x\frac{\partial}{\partial a} + K_x\partial_K + ... \, .$$

For finding invariants one has to apply the following procedure. Let us consider an invariant of order n, where it is assumed that J depends on the variable a, its derivatives up to n^{th} order, the function K and its derivatives up to $(n-2)$ order. Invariants can be obtained by solving the equations

$$\widetilde{X}^e(J) = 0,$$

and relative invariants by solving the equations

$$\widetilde{X}^e(J_k)_{|(S)} = 0.$$

5.4 Method of solving

This section is devoted to finding sixth-order differential invariants. Let

$$J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}, ..., a_{xxxxxx}, K, K_t, K_x, K_{tt}, K_{tx}, K_{xx}, ..., K_{xxxx})$$

be a sixth-order differential invariant.

The prolonged operator $\widetilde{X}^e$ is defined by (5.21). Splitting the equations $\widetilde{X}^e(J) = 0$ with respect to p, q and its derivatives, one obtains a system of 43 linear homogeneous equations. Some of these equations are of the following two types. The first type is

$$J_x + \sum_{i=1}^{n} a_i J_{y_i} = 0, \tag{5.22}$$

where $J = J\left(x, y_1, y_2, ..., y_n\right)$, and the coefficients a_i $(i = 1, 2, ..., n)$ are linear functions of the independent variables $y_1, y_2, ..., y_{i-1}$ which have the form

$$a_i = \sum_{k=1}^{i-1} \beta_{i,k}\left(x\right) y_k + \gamma_i\left(x\right).$$

The characteristic system for equation (5.22) is

$$\frac{dx}{1} = \frac{dy_1}{\gamma_1\left(x\right)} = \frac{dy_2}{\beta_{2,1}\left(x\right) y_1 + \gamma_2\left(x\right)} = \frac{dy_3}{\beta_{3,1}\left(x\right) y_1 + \beta_{3,2}\left(x\right) y_2 + \gamma_3\left(x\right)} = \cdots.$$

From the characteristic system one can obtain the general solution of (5.22).

The second type of equations is

$$xJ_x + \sum_{i=1}^{n} k_i y_i J_{y_i} = 0, \tag{5.23}$$

where k_i $(i = 1, 2, ..., n)$ are constant. The general solution of (5.23) is

$$J = J(J_1, J_2, ..., J_n), \quad \text{where} \quad J_i = \frac{y_i}{x^{k_i}}, \ (i = 1, 2, ..., n).$$

The calculations for obtaining the system of equations for finding invariants and solving its equations are cumbersome. For these calculations we therefore used the Reduce programs developed for solving the linearization problem of third-order ordinary differential equation (Ibragimov, N.H. and Meleshko, S.V. (2005)).

5.5 Sixth-order invariants

After solving the equations of the first and second types the system is reduced to the following system of equations

$$\frac{\partial J}{\partial J_2} J_1 = 0, \ 6\frac{\partial J}{\partial J_3} J_3 + 5\frac{\partial J}{\partial J_1} J_1 + 7\frac{\partial J}{\partial J_2} J_2 = 0, \ \frac{\partial J}{\partial J_2} J_3 = 0, \tag{5.24}$$

where

$$J_1 = \lambda/(8a^5), \quad J_2 = (2\frac{\partial \lambda}{\partial t}a - 5\frac{\partial a}{\partial t}\lambda)/(16a^7), \quad J_3 = (2\frac{\partial \lambda}{\partial x}a - 5\frac{\partial a}{\partial x}\lambda)/(8a^6)$$

and $J = (J_1, J_2, J_3)$.

If $\lambda = 0$, then $J_1 = J_2 = J_3 = 0$. This case was studied in (Johnpillai I.K. and Mahomed F.M. (2001)). If $\lambda \neq 0$, then $J_1 \neq 0$. Because of the first equation of (5.24), J does not depend on J_2. Solving the second equation of (5.24), one obtains the only invariant J_3^5/J_1^6. This invariant was also obtained in (Morozov O.I. (2003)) as an invariant with respect to contact transformations.

5.6 Seventh-order invariants

Similar to the previous section the system for finding invariants of seventh-order is reduced to the following equations

$$24\frac{\partial J}{\partial J_6}J_2 + 6\frac{\partial J}{\partial J_4}J_3 + 5\frac{\partial J}{\partial J_2}J_1 = 0, \tag{5.25}$$

$$9\tfrac{\partial J}{\partial J_6}J_6 + 7\tfrac{\partial J}{\partial J_5}J_5 + 6\tfrac{\partial J}{\partial J_3}J_3 + 8\tfrac{\partial J}{\partial J_4}J_4 + 5\tfrac{\partial J}{\partial J_1}J_1 + 7\tfrac{\partial J}{\partial J_2}J_2 = 0, \tag{5.26}$$

$$3\frac{\partial J}{\partial J_6}J_4 + 2\frac{\partial J}{\partial J_4}J_5 + \frac{\partial J}{\partial J_2}J_3 = 0, \tag{5.27}$$

where

$$J_4 = (-5\lambda_t a_x a - 4\lambda_x a_t a - 5a_{tx}a\lambda + 15a_t a_x \lambda + 2\lambda_{tx}a^2)/(8a^8),$$

$$J_5 = (-9\lambda_x a_x a - 5a_{xx}a\lambda + 15{a_x}^2\lambda + 2\lambda_{xx}a^2)/(8a^7),$$

$$\begin{gathered} J_6 = (-40a_{txx}a^2\lambda - 4a_{xxx}\lambda_x a^3 + 30a_{xxx}a_x a^2\lambda - 40\lambda_t a_t a + \lambda_x a(-8a_{tx}a + 8a_t a_x \\ +6a_{xx}a_x a - 3{a_x}^3 - 8K) + 80a_{tx}a_x a\lambda - 40a_{tt}a\lambda + 100{a_t}^2\lambda \\ +40a_t a_{xx}a\lambda - 80a_t{a_x}^2\lambda - 20a_{xxxx}a^3\lambda + 30{a_{xx}}^2a^2\lambda \\ -75a_{xx}{a_x}^2a\lambda + 30{a_x}^4\lambda + 80a_x K\lambda - 40K_x a\lambda + 8\lambda_{tt}a^2)/(32a^9). \end{gathered}$$

Taking the Poisson bracket of equation (5.25) and (5.27), one obtains the equation

$$\frac{\partial J}{\partial J_6} J_3 = 0. \tag{5.28}$$

Assuming $\lambda \neq 0$, equations (5.25) and (5.26) can be solved. The remaining equation (5.27), (5.28) are reduced to the equations

$$2\frac{\partial J}{\partial J_{10}}(5J_8 - 3{J_7}^2) + 15\frac{\partial J}{\partial J_9} J_{10} = 0, \tag{5.29}$$

$$\frac{\partial J}{\partial J_9} J_7 = 0, \tag{5.30}$$

where $J = J(J_7, J_8, J_9, J_{10})$, and

$$J_7 = \frac{J_3}{{J_1}^{6/5}}, \quad J_8 = \frac{J_5}{{J_1}^{7/5}}, \quad J_9 = \frac{5J_1J_6 - 12{J_2}^2}{5{J_1}^{14/5}}, \quad J_{10} = \frac{5J_1J_4 - 6J_2J_3}{5{J_1}^{13/5}}.$$

Since equations (5.29), (5.30) contain no derivatives with respect to J_7 and J_8, the variables J_7 and J_8 are invariants.

If $J_7 \neq 0$, then J does not depend on J_9, and equation (5.29) becomes

$$2\frac{\partial J}{\partial J_{10}}(5J_8 - 3{J_7}^2) = 0.$$

This equation shows that there is the additional invariant J_{10} which is obtained for $(5J_8 - 3{J_7}^2) = 0$.

If $J_7 = 0$, then one needs only to solve equation (5.29) which becomes

$$10\frac{\partial J}{\partial J_{10}} J_8 + 15\frac{\partial J}{\partial J_9} J_{10} = 0.$$

If $J_8 \neq 0$, this equation yields the invariant

$$J_{11} = J_9 - \frac{3}{4}\frac{{J_{10}}^2}{J_8}.$$

The assumption $J_8 = 0$ leads to the analysis of the equation

$$J_{10}\frac{\partial J}{\partial J_9} = 0.$$

If $J_{10} = 0$ then one only obtains the invariant J_9.

Conditions			Additional invariant
$J_7 \neq 0$	$5J_8 - 3{J_7}^2 \neq 0$		no
	$5J_8 - 3{J_7}^2 = 0$		J_{10}
$J_7 = 0$	$J_8 \neq 0$		J_{11}
	$J_8 = 0$	$J_{10} \neq 0$	no
		$J_{10} = 0$	J_9

Table 5.1 Invariants for particular case

Remark. The invariants J_7, J_8, J_{10} and J_{11} are equal, up to immaterial constant factors, to the invariants (5.9), (5.10), (5.12), (5.14) and (5.16) respectively, i.e., $J_7 = 8^{1/5}\Lambda_1,\ J_8 = 8^{2/5}\Lambda_2,\ J_{10} = 8^{3/5}\Lambda_3,\ J_9 = 8^{4/5}\Lambda_4,\ J_{11} = 8^{4/5}\,\frac{\Lambda_5}{4\Lambda_2}.$

CHAPTER VI

EQUIVALENCE OF LINEAR SECOND ORDER PARABOLIC EQUATIONS TO CANONICAL FORMS

6.1 Introduction

The equivalence problem of a linear second-order parabolic partial differential equations in two independent variables

$$a_1(t,x)u_t + a_2(t,x)u_x + a_3(t,x)u + u_{xx} = 0 \tag{6.1}$$

is considered in the thesis. Notice that equation as in (5.1) can be rewritten in form (6.1). Recall that the well-known group of equivalence transformations for equation (6.1) (given in Lie (1881)), i.e. the changes of the independent variables t, x and the dependent variable u that do not change the form of equation (6.1), is composed of the linear transformation of the dependent variable

$$v = uV(t,x) \tag{6.2}$$

and the following change of the independent variables:

$$\tau = H(t), \quad y = Y(t,x), \tag{6.3}$$

where $V(t,x), H(t)$ and $Y(t,x)$ are arbitrary functions obeying the invertibility conditions, $V(t,x) \neq 0$, $H'(t) \neq 0$ and $Y_x(t,x) \neq 0$. Invariance of the form of equation (6.1) means that the transformations (6.2)–(6.3) map equation (6.1) into an equation of the same form:

$$\beta_1(\tau, y)v_\tau + \beta_2(\tau, y)v_y + \beta_3(\tau, y)v + v_{yy} = 0. \tag{6.4}$$

Equations (6.1) and (6.4), related by an equivalence transformation, are called equivalent equations. Notice that in our calculation we let $\beta_1 = -b_1, \beta_2 = -b_2, \beta_3 = -b_3$.

Let us show that any transformation

$$\tau = H(t, x, u),\ y = Y(t, x, u),\ v = V(t, x, u). \tag{6.5}$$

which maps equation (6.1) into an equation of the same form has the representation (6.2) and (6.3).

Using (4.11), we have that equation (6.1) is mapped into the equation of the same form if the following relations are satisfied

$$\begin{gathered} a_1 = 0,\quad a_2 = 0,\quad a_3 = 0,\quad a_4 = 0,\quad a_5 = 0,\quad a_6 = 0,\quad a_7 = 0,\quad a_{8u} = 0, \\ a_{9u} = 0,\quad a_{10u} = 0,\quad a_{11} = 0,\quad a_{12} = 0,\quad a_{13} = 0,\quad a_{14} = 0,\quad a_{15} = 0,\quad a_{16u} = 0, \end{gathered} \tag{6.6}$$

where the representation of a_i, $(i = 1, 2, ..., 16)$ are given in Appendix A. From the relations $a_{11} = 0,\ a_{12} = 0$, one has $H_x^2 b_1 = 0, H_u^2 b_1 = 0$. This implies that

$$H_x = 0,\ H_u = 0. \tag{6.7}$$

Thus

$$H_t \neq 0. \tag{6.8}$$

From $a_4 = 0$, one obtains $H_t Y_u^2 = 0$, which means that

$$Y_u = 0. \tag{6.9}$$

The equation $a_6 = 0$ gives

$$V_{uu} = 0. \tag{6.10}$$

From equation (6.7)-(6.10), one gets

$$H = H(t),\ Y = Y(t,x),\ V = \alpha(t,x)u + \beta(t,x). \tag{6.11}$$

Substituting (6.11) into (4.11), one obtains the nonhomogenous linear equation

$$A_1(t,x)u_t + A_2(t,x)u_x + A_3(t,x)u + u_{xx} = A_4, \tag{6.12}$$

where

$$\begin{aligned}
A_1 = {} & Y_x^2/(H'b_1),\\
A_2 = {} & (2\alpha_x H'Y_x b_1 - H'Y_{xx}\alpha b_1 + H'Y_x^2\alpha b_2 - \alpha_t Y_x^2\alpha)/(Y_x\alpha H'b_1),\\
A_3 = {} & (\alpha_t Y_x^3 + \alpha_{xx}H'Y_x b_1 - \alpha_x H'Y_{xx}b_1 + \alpha_x H'Y_x^2 b_2 - \alpha_x\alpha_t Y_x^2\\
& + H'Y_x^3\alpha b_3)/(Y_x\alpha H'b_1),\\
A_4 = {} & (\beta_x H'Y_{xx}b_1 - \beta_t Y_x^3 - \beta_{xx}H'Y_x b_1 - \beta_x H'Y_x^2 b_2 + \beta_x\alpha_t Y_x^2\\
& - H'Y_x^3 b_3\beta)/(Y_x\alpha H'b_1).
\end{aligned}$$

Equation (6.12) is the same form as (6.1) if and only if the function $A_4 = 0$. This condition can be considered as equation for the functions β. A particular solution of this equation is $\beta = 0$. That is, any transformation which maps equation (6.1) into an equation of the same form has the representation (6.2) and (6.3).

Lie (1881) obtained the classification of linear second-order partial differential equations (6.1). Ovsiannikov (1978) studied the group classification of a nonlinear parabolic equation. Ibragimov (2002) found first and second order semi-invariants of a parabolic partial differential equation (6.1). Johnpillai and Mahomed (2001) showed that there are no first, second, third and fourth order invariants other than constant and they obtained one relative invariant. Sixth and seventh-order differential invariants of linear second-order parabolic partial differential equation (6.1) under an action of the equivalence group of point transformations (6.2)–(6.3) were found in chapter V. The paper (Ibragimov (2008)) gives an extension of Euler's method to linear parabolic equations with two independent variables. The new method allowed deriving an explicit formula for the general solution of a wide class of parabolic equations. Morozov (2003) studied invariants of contact transformations for linear parabolic equations.

6.1.1 Canonical parabolic equations

According to Lie's classification (Lie (1881)), the canonical forms of linear second-order parabolic partial differential equations (6.4) are the heat equation

$$u_t = u_{xx}, \tag{6.13}$$

the equation

$$u_t = u_{xx} + a(x)u, \tag{6.14}$$

with arbitrary function $a(x)$, and the equation

$$u_t = u_{xx} + \frac{k}{x^2}u, \tag{6.15}$$

where k is a nonzero constant. These equations posses additional symmetry properties compared with the general case of linear parabolic equations. These properties allow constructing additional exact solutions. Notice that a change of the dependent and independent variables conserves symmetry properties. Hence equivalent equations to (6.13),(6.14) also posses additional invariant solutions. For example, the Block-Scholes equation is a linear parabolic equation which is equivalent to the heat equation. Invariant solutions of the heat equation can be used for Block-Scholes equation. Thus, there is interest to finding equations which are equivalent to equation (6.13),(6.14) and (6.15).

As explained in chapter V, equation (6.4) is equivalent to the heat equation (6.13) if and only if

$$\lambda = 0$$

because we have changed the form of the equation to (6.4), λ and K are new of the form

$$\begin{aligned}\lambda = \quad & (-8b_{1\tau yyy}b_1^5 + 36b_{1\tau yy}b_{1y}b_1^4 - 4b_{1\tau y}b_{1\tau}b_1^5 + 28b_{1\tau y}b_{1yy}b_1^4 \\ & -80b_{1\tau y}b_{1y}^2b_1^3 + 4b_{1\tau\tau y}b_1^6 - 4b_{1\tau\tau}b_{1y}b_1^5 + 4b_{1\tau}^2b_{1y}b_1^4 + 8b_{1\tau}b_{1yyy}b_1^4 \\ & -64b_{1\tau}b_{1yy}b_{1y}b_1^3 + 80b_{1\tau}b_{1y}^3b_1^2 + 4b_{1yyyyy}b_1^4 - 40b_{1yyyy}b_{1y}b_1^3 \\ & -64b_{1yyy}b_{1yy}b_1^3 + 220b_{1yyy}b_{1y}^2b_1^2 + 288b_{1yy}^2b_{1y}b_1^2 - 810b_{1yy}b_{1y}^3b_1 \\ & +12b_{1yy}b_1^7k + 405b_{1y}^5 + 20b_{1y}K_yb_1^7 + 8K_{yy}b_1^8)/b_1^{10},\end{aligned} \tag{6.16}$$

$$K = (2b_{1y}b_{2y} - b_{1y}b_2^2 - 4b_{1y}b_3 + 2b_{2\tau}b_1^2 - 2b_{2yy}b_1 + 2b_{2y}b_1b_2 + 4b_{3y}b_1)/(2b_1^4).$$

The present chapter is devoted to obtaining conditions for equation (6.4) to be equivalent to (6.14) or (6.15). The chapter is organized as follows.

6.2 Statement of the problem

Let us obtain a representation of the changed equations. For this we suppose that $u_0(t,x)$ is a given function. Applying the inverse function theorem to (6.3), there exist functions $t = T(\tau),\ x = X(\tau, y)$ such that

$$t = H(T(\tau)),\ y = Y(X(\tau, y)). \tag{6.17}$$

After substituting $T(\tau),\ X(\tau, y)$ into equation (6.2), one obtains the transformation of the function $u_0(t,x)$:

$$v_0(\tau, y) = u_0(T(\tau), X(\tau, y))V(T(\tau), X(\tau, y)),$$

Notice that the function $v_0(\tau, y)$ satisfies the relation

$$v_0(H(t), Y(t,x)) = u_0(t,x)/V(t,x). \tag{6.18}$$

Differentiating equation (6.18) with respect to t and x, one gets

$$v_{0\tau}H' + v_{0y}Y_t = (u_0/V)_t, \quad v_{0y}Y_x = (u_0/V)_x. \tag{6.19}$$

Solving linear system (6.19) with respect to the derivatives $v_{0\tau}$ and v_{0y}, one has

$$\begin{aligned} v_{0\tau}(\tau,y) &= \triangle^{-1}(Y_t(u_0/V)_x - Y_x(u_0/V)_t), \\ v_{0y}(\tau,y) &= -\triangle^{-1}H'(u_0/V)_x, \end{aligned} \tag{6.20}$$

where it assumed that $\triangle = -H'Y_x \neq 0$. Differentiating second equation (6.20) with respect to x, one obtains

$$v_{0yy} = \triangle^{-3}H'((H'(u_0/V)_x)\triangle_x - \triangle(H'(u_0/V)_x)_x). \tag{6.21}$$

Thus equation (6.4) becomes (6.1), where

$$\begin{aligned} a_1 =& \ \Delta^{-1}Y_x^3 b_1, \\ a_2 =& \ -\Delta^{-1}V^{-1}(2H'V_xY_x - H'Y_{xx}V - H'Y_x^2Vb_2 + Y_tY_x^2Vb_1), \\ a_3 =& \ -\Delta^{-1}V^{-1}(H'V_{xx}Y_x - H'V_xY_{xx} - H'V_xY_x^2b_2 - H'Y_x^3b_3V \\ & -V_tY_x^3b_1 + V_xY_tY_x^2b_1). \end{aligned} \tag{6.22}$$

6.3 Equivalence problem for equation (6.13)

This section studies equations (6.4) which are equivalent to equation (6.13). Since for equation (6.13)

$$a_1 = -1, \quad a_2 = 0, \quad a_3 = 0, \tag{6.23}$$

equation (6.22) becomes

$$\begin{aligned} 1 =& \ -\Delta^{-1}Y_x^3 b_1, \\ 0 =& \ 2H'V_xY_x - H'Y_{xx}V - H'Y_x^2Vb_2 + Y_tY_x^2Vb_1, \\ 0 =& \ -\Delta^{-1}V^{-1}(H'V_{xx}Y_x - H'V_xY_{xx} - H'V_xY_x^2b_2 - H'Y_x^3b_3V \\ & -V_tY_x^3b_1 + V_xY_tY_x^2b_1). \end{aligned} \tag{6.24}$$

The problem is to find conditions for the coefficients $b_1(\tau,y)$, $b_2(\tau,y)$, $b_3(\tau,y)$ which guarantee existence of the functions $H(t)$, $Y(t,x)$, $V(t,x)$ transforming the coefficients of (6.4) into (6.23). Solution of this problem consists of the analysis of compatibility of (6.24).

From the first equation of (6.24) one has

$$H' = b_1 Y_x^2. \tag{6.25}$$

Differentiating equation (6.25) with respect to x, one finds

$$Y_{xx} = \ -b_{1y} Y_x^2/(2b_1). \tag{6.26}$$

The second and the third equation of (6.24) become

$$Y_t = (-4V_x Y_x b_1 + Y_x^2 V(-b_{1y} + 2b_1 b_2))/(2b_1 V), \tag{6.27}$$

$$\begin{aligned} V_{xx} = \ & (-b_{1y} V_x Y_x^2 + 2V_t Y_x b_1 - 2V_x Y_t b_1 + 2V_x Y_x^2 b_1 b_2 \\ & +2Y_x^3 b_1 b_3 V)/(2Y_x b_1). \end{aligned} \tag{6.28}$$

Comparing the mixed derivatives $(Y_t)_{xx} - (Y_{xx})_t = 0$, one finds

$$\begin{aligned} V_{tx} = \ & (4V_t Y_t Y_x b_1^3 + 2V_t Y_x^3 b_1^2 (b_{1y} - 2b_1 b_2) - 4V_x Y_t^2 b_1^3 + 4V_x Y_t Y_x^2 b_1^2 (-b_{1y} \\ & +2b_1 b_2) + V_x Y_x^4 b_1 (-4b_{1yy} b_1 + 7b_{1y}^2 + 4b_{1y} b_1 b_2 - 4b_1^2 b_2^2 - 8b_1^2 b_3) \\ & +2Y_t Y_x^3 b_1 V(-b_{1yy} b_1 + 2b_{1y}^2 + 2b_1^2 b_3) + Y_x^5 V(-2b_{1yyy} b_1^2 + 9b_{1yy} b_{1y} b_1 \\ & -7b_{1y}^3 - 6b_{1y} b_{2y} b_1^2 + 10b_{1y} b_1^2 b_3 + 4b_{2yy} b_1^3 - 8b_{3y} b_1^3 - 4b_1^3 b_2 b_3) \\ & +2Y_x^3 H' b_1 V(b_{1\tau y} b_1 - b_{1\tau} b_{1y}))/(8Y_x^2 b_1^3). \end{aligned} \tag{6.29}$$

The equation $(V_{xx})_t - (V_{tx})_x = 0$ gives

$$\begin{aligned} V_{tt} = \ & (-32V_t^2 Y_x^3 b_1^4 V + 128V_t V_x^2 Y_x^3 b_1^4 + 80V_t V_x Y_t Y_x^2 b_1^4 V \\ & +8V_t V_x Y_x^4 b_1^3 V(3b_{1y} - 10b_1 b_2) + 8V_t Y_t^2 Y_x b_1^4 V^2 - 16V_t Y_t Y_x^3 b_1^4 b_2 V^2 \\ & +2V_t Y_x^5 b_1^2 V^2 (-b_{1y}^2 + 4b_1^2 b_2^2) - 64V_x^3 Y_t Y_x^2 b_1^4 + 32V_x^3 Y_x^4 b_1^3 (b_{1y} + 2b_1 b_2) \\ & -32V_x^2 Y_t^2 Y_x b_1^4 V + 32V_x^2 Y_t Y_x^3 b_1^3 V(b_{1y} + 2b_1 b_2) + 8V_x^2 Y_x^5 b_1^2 V(8b_{1yy} b_1 \\ & -9b_{1y}^2 - 4b_{1y} b_1 b_2 - 8b_{2y} b_1^2 - 4b_1^2 b_2^2 + 24b_1^2 b_3) - 64V_x^2 Y_x^3 b_{1\tau} H' b_1^3 V \end{aligned}$$

$$-8V_xY_t^3b_1^4V^2+4V_xY_t^2Y_x^2b_1^3V^2(b_{1y}+6b_1b_2)+2V_xY_tY_x^4b_1^2V^2(4b_{1yy}b_1$$
$$+5b_{1y}^2-4b_{1y}b_1b_2-16b_{2y}b_1^2-12b_1^2b_2^2+24b_1^2b_3)-32V_xY_tY_x^2b_{1\tau}H'b_1^3V^2$$
$$+V_xY_x^6b_1V^2(-8b_{1yyy}b_1^2+68b_{1yy}b_{1y}b_1-73b_{1y}^3-26b_{1y}^2b_1b_2+8b_{1y}b_{2y}b_1^2$$
$$+4b_{1y}b_1^2b_2^2+72b_{1y}b_1^2b_3-16b_{2yy}b_1^3+32b_{2y}b_1^3b_2+8b_1^3b_2^3-48b_1^3b_2b_3)$$
$$+8V_xY_x^4H'b_1^2V^2(-b_{1\tau y}b_1-b_{1\tau}b_{1y}+4b_{1\tau}b_1b_2)+4Y_t^2Y_x^3b_1^2V^3(-3b_{1yy}b_1$$
$$+6b_{1y}^2+2b_1^2b_3)+2Y_tY_x^5b_1V^3(-10b_{1yyy}b_1^2+50b_{1yy}b_{1y}b_1+6b_{1yy}b_1^2b_2$$
$$-43b_{1y}^3-12b_{1y}^2b_1b_2-18b_{1y}b_{2y}b_1^2+48b_{1y}b_1^2b_3+12b_{2yy}b_1^3-32b_{3y}b_1^3$$
$$-8b_1^3b_2b_3)+12Y_tY_x^3H'b_1^2V^3(b_{1\tau y}b_1-b_{1\tau}b_{1y})+Y_x^7V^3(-8b_{1yyyy}b_1^3$$
$$+50b_{1yyy}b_{1y}b_1^2+12b_{1yyy}b_1^3b_2+40b_{1yy}^2b_1^2-195b_{1yy}b_{1y}^2b_1-54b_{1yy}b_{1y}b_1^2b_2$$
$$-32b_{1yy}b_{2y}b_1^3+64b_{1yy}b_1^3b_3+113b_{1y}^4+42b_{1y}^3b_1b_2+58b_{1y}^2b_{2y}b_1^2-98b_{1y}^2b_1^2b_3$$
$$-36b_{1y}b_{2yy}b_1^3+36b_{1y}b_{2y}b_1^3b_2+64b_{1y}b_{3y}b_1^3-32b_{1y}b_1^3b_2b_3+16b_{2yyy}b_1^4$$
$$-24b_{2yy}b_1^4b_2-64b_{2y}b_1^4b_3-32b_{3yy}b_1^4+32b_{3y}b_1^4b_2+8b_1^4b_2^2b_3+64b_1^4b_3^2)$$
$$+2Y_x^5H'b_1V^3(4b_{1\tau yy}b_1^2-7b_{1\tau y}b_{1y}b_1-6b_{1\tau y}b_1^2b_2-4b_{1\tau}b_{1yy}b_1+7b_{1\tau}b_{1y}^2$$
$$+6b_{1\tau}b_{1y}b_1b_2-16b_{3\tau}b_1^3))/(32Y_x^3b_1^4V^2). \tag{6.30}$$

Comparing the mixed derivatives $(V_{tt})_x-(V_{tx})_t=0$, one obtains

$$\lambda=0, \tag{6.31}$$

where λ is defined in (6.16). This condition guarantees that the overdetermined system (6.24) is compatible. Moreover, the overdetermined system of equations (6.25)-(6.39) is involutive under the condition (6.31). The condition (6.31) was obtained in Johnpillai and Mahomed (2001).

6.4 Equivalence problem for equation (6.14)

This section studies equations (6.4) which are equivalent to equation (6.14). Since for equation (6.14)

$$a_1 = -1, \quad a_2 = 0, \quad a_3 = a(x), \tag{6.32}$$

equations (6.22) become

$$\begin{aligned} 1 = & \ -\Delta^{-1} Y_x^3 b_1, \\ 0 = & \ 2H' V_x Y_x - H' Y_{xx} V - H' Y_x^2 V b_2 + Y_t Y_x^2 V b_1, \\ a = & \ -\Delta^{-1} V^{-1} (H' V_{xx} Y_x - H' V_x Y_{xx} - H' V_x Y_x^2 b_2 - H' Y_x^3 b_3 V \\ & -V_t Y_x^3 b_1 + V_x Y_t Y_x^2 b_1). \end{aligned} \tag{6.33}$$

The problem is to find conditions for the coefficients $b_1(\tau, y)$, $b_2(\tau, y)$, $b_3(\tau, y)$ which guarantee existence of the functions $H(t)$, $Y(t, x)$, $V(t, x)$ transforming the coefficients of (6.4) into (6.32). Solution of this problem consists of the analysis of compatibility of (6.33).

From the first equation of equation (6.33), one has

$$H' = b_1 Y_x^2. \tag{6.34}$$

Then

$$Y_{xx} = -b_{1y} Y_x^2 / (2b_1). \tag{6.35}$$

The second equation and the third equation of equation (6.33) can be solved with respect to Y_t and V_{xx}:

$$Y_t = (-4V_x Y_x b_1 + Y_x^2 V(-b_{1y} + 2b_1 b_2))/(2b_1 V), \tag{6.36}$$

$$\begin{aligned} V_{xx} = & \ (-b_{1y} V_x Y_x^2 + 2V_t Y_x b_1 - 2V_x Y_t b_1 + 2V_x Y_x^2 b_1 b_2 \\ & +2Y_x^3 b_1 b_3 V + 2Y_x a b_1 V)/(2Y_x b_1). \end{aligned} \tag{6.37}$$

Comparing the mixed derivatives $(Y_t)_{xx} - (Y_{xx})_t = 0$, one finds

$$
\begin{aligned}
V_{tx} = \ & (4V_tY_tY_xb_1^3 + 2V_tY_x^3b_1^2(b_{1y} - 2b_1b_2) - 4V_xY_t^2b_1^3 + 4V_xY_tY_x^2b_1^2(-b_{1y} \\
& +2b_1b_2) + V_xY_x^4b_1(-4b_{1yy}b_1 + 7b_{1y}^2 + 4b_{1y}b_1b_2 - 4b_1^2b_2^2 - 8b_1^2b_3) \\
& -8V_xY_x^2ab_1^3 + 2Y_tY_x^3b_1V(-b_{1yy}b_1 + 2b_{1y}^2 + 2b_1^2b_3) + 4Y_tY_xab_1^3V \\
& +Y_x^5V(-2b_{1yyy}b_1^2 + 9b_{1yy}b_{1y}b_1 - 7b_{1y}^3 - 6b_{1y}b_{2y}b_1^2 + 10b_{1y}b_1^2b_3 \\
& +4b_{2yy}b_1^3 - 8b_{3y}b_1^3 - 4b_1^3b_2b_3) + 2Y_x^3b_1V(b_{1\tau y}H'b_1 - b_{1\tau}b_{1y}H' \\
& +b_{1y}ab_1 - 2ab_1^2b_2) - 8Y_x^2a_xb_1^3V)/(8Y_x^2b_1^3).
\end{aligned}
\tag{6.38}
$$

The equation $(V_{xx})_t - (V_{tx})_x = 0$ gives

$$
\begin{aligned}
V_{tt} = \ & (-32V_t^2Y_x^3b_1^4V + 128V_tV_x^2Y_x^3b_1^4 + 80V_tV_xY_tY_x^2b_1^4V + 8V_tV_xY_x^4b_1^3V(3b_{1y} \\
& -10b_1b_2) + 8V_tY_t^2Y_xb_1^4V^2 - 16V_tY_tY_x^3b_1^4b_2V^2 + 2V_tY_x^5b_1^2V^2(-b_{1y}^2 \\
& +4b_1^2b_2^2) - 128V_tY_x^3ab_1^4V^2 - 64V_x^3Y_tY_x^2b_1^4 + 32V_x^3Y_x^4b_1^3(b_{1y} + 2b_1b_2) \\
& -32V_x^2Y_t^2Y_xb_1^4V + 32V_x^2Y_tY_x^3b_1^3V(b_{1y} + 2b_1b_2) + 8V_x^2Y_x^5b_1^2V(8b_{1yy}b_1 \\
& -9b_{1y}^2 - 4b_{1y}b_1b_2 - 8b_{2y}b_1^2 - 4b_1^2b_2^2 + 24b_1^2b_3) + 64V_x^2Y_x^3b_1^3V(-b_{1\tau}H' \\
& +3ab_1) - 8V_xY_t^3b_1^4V^2 + 4V_xY_t^2Y_x^2b_1^3V^2(b_{1y} + 6b_1b_2) + 2V_xY_tY_x^4b_1^2V^2 \\
& (4b_{1yy}b_1 + 5b_{1y}^2 - 4b_{1y}b_1b_2 - 16b_{2y}b_1^2 - 12b_1^2b_2^2 + 24b_1^2b_3) + 16V_xY_tY_x^2b_1^3V^2 \\
& (-2b_{1\tau}H' + 7ab_1) + V_xY_x^6b_1V^2(-8b_{1yyy}b_1^2 + 68b_{1yy}b_{1y}b_1 - 73b_{1y}^3 \\
& -26b_{1y}^2b_1b_2 + 8b_{1y}b_{2y}b_1^2 + 4b_{1y}b_1^2b_2^2 + 72b_{1y}b_1^2b_3 - 16b_{2yy}b_1^3 + 32b_{2y}b_1^3b_2 \\
& +8b_1^3b_2^3 - 48b_1^3b_2b_3) + 8V_xY_x^4b_1^2V^2(-b_{1\tau y}H'b_1 - b_{1\tau}b_{1y}H' + 4b_{1\tau}H'b_1b_2 \\
& +5b_{1y}ab_1 - 14ab_1^2b_2) + 4Y_t^2Y_x^3b_1^2V^3(-3b_{1yy}b_1 + 6b_{1y}^2 + 2b_1^2b_3) \\
& +8Y_t^2Y_xab_1^4V^3 + 2Y_tY_x^5b_1V^3(-10b_{1yyy}b_1^2 + 50b_{1yy}b_{1y}b_1 + 6b_{1yy}b_1^2b_2 \\
& -43b_{1y}^3 - 12b_{1y}^2b_1b_2 - 18b_{1y}b_{2y}b_1^2 + 48b_{1y}b_1^2b_3 + 12b_{2yy}b_1^3 - 32b_{3y}b_1^3 \\
& -8b_1^3b_2b_3) + 4Y_tY_x^3b_1^2V^3(3b_{1\tau y}H'b_1 - 3b_{1\tau}b_{1y}H' - 4ab_1^2b_2) - 32Y_tY_x^2a_xb_1^4V^3 \\
& +Y_x^7V^3(-8b_{1yyyy}b_1^3 + 50b_{1yyy}b_{1y}b_1^2 + 12b_{1yyy}b_1^3b_2 + 40b_{1yy}^2b_1^2 - 195b_{1yy}b_{1y}^2b_1 \\
& -54b_{1yy}b_{1y}b_1^2b_2 - 32b_{1yy}b_{2y}b_1^3 + 64b_{1yy}b_1^3b_3 + 113b_{1y}^4 + 42b_{1y}^3b_1b_2 \\
& +58b_{1y}^2b_{2y}b_1^2 - 98b_{1y}^2b_1^2b_3 - 36b_{1y}b_{2yy}b_1^3 + 36b_{1y}b_{2y}b_1^3b_2 + 64b_{1y}b_{3y}b_1^3 \\
& -32b_{1y}b_1^3b_2b_3 + 16b_{2yyy}b_1^4 - 24b_{2yy}b_1^4b_2 - 64b_{2y}b_1^4b_3 - 32b_{3yy}b_1^4
\end{aligned}
\tag{6.39}
$$

$$+32b_{3y}b_1^4b_2+8b_1^4b_2^2b_3+64b_1^4b_3^2)+2Y_x^5b_1V^3(4b_{1\tau yy}H'b_1^2-7b_{1\tau y}b_{1y}H'b_1$$
$$-6b_{1\tau y}H'b_1^2b_2-4b_{1\tau}b_{1yy}H'b_1+7b_{1\tau}b_{1y}^2H'+6b_{1\tau}b_{1y}H'b_1b_2-b_{1y}^2ab_1$$
$$-16b_{3\tau}H'b_1^3+4ab_1^3b_2^2)+16Y_x^4a_xb_1^3V^3(-b_{1y}+2b_1b_2)$$
$$+32Y_x^3b_1^4V^3(-a_{xx}-2a^2))/(32Y_x^3b_1^4V^2).$$

Equating $(V_{tt})_x=(V_{tx})_t$, one obtains

$$a_{xxx}=(Y_x^5\lambda_3)/(16b_1^5), \tag{6.40}$$

where $\lambda_3=-b_1^{10}\lambda$. Notice that by virtue of $\lambda\neq 0$, one has $\lambda_3\neq 0$. Because a does not depend on t, differentiating (6.40) with respect to t, one has

$$\begin{aligned} V_t= \; & (-20V_x^2b_1^2\lambda_3+2V_xY_xb_1V(15b_{1y}\lambda_3-2\lambda_{3y}b_1)+Y_x^2\lambda_4V^2 \\ & -20ab_1^2\lambda_3V^2)/(20b_1^2\lambda_3V), \end{aligned} \tag{6.41}$$

where

$$\begin{aligned} \lambda_2= \; & -b_{1yy}b_1+2b_{1y}^2-2b_{1y}b_1b_2+2b_{2y}b_1^2-4b_1^2b_3, \\ \lambda_4= \; & -10b_{1\tau}b_1^2\lambda_3+5b_{1y}^2\lambda_3-b_{1y}\lambda_{3y}b_1-10b_{1y}b_1b_2\lambda_3+2\lambda_{3\tau}b_1^3 \\ & +2\lambda_{3y}b_1^2b_2+5\lambda_2\lambda_3. \end{aligned}$$

Substitution of V_t into (6.38) and (6.39) gives

$$\begin{aligned} a_{xx}= \; & (-16V_x^2Y_x^2b_1^2\lambda_5-8V_xY_x^3b_1\lambda_6V+Y_x^4\lambda_7V^2+40Y_xa_xb_1^3\lambda_3V^2 \\ & (15b_{1y}\lambda_3-2\lambda_{3y}b_1))/(400b_1^4\lambda_3^2V^2), \end{aligned} \tag{6.42}$$

$$4V_xb_1\lambda_5+Y_x\lambda_6V=0, \tag{6.43}$$

where

$$\begin{aligned} \lambda_1= \; & 2b_{1\tau y}b_1^3-2b_{1\tau}b_{1y}b_1^2-2b_{1yyy}b_1^2+10b_{1yy}b_{1y}b_1-2b_{1yy}b_1^2b_2-9b_{1y}^3 \\ & +4b_{1y}^2b_1b_2-6b_{1y}b_{2y}b_1^2+8b_{1y}b_1^2b_3+4b_{2yy}b_1^3-8b_{3y}b_1^3, \\ \lambda_5= \; & -150b_{1y}^2\lambda_3^2+25b_{1y}\lambda_{3y}b_1\lambda_3+150b_{1y}b_1b_2\lambda_3^2-150b_{2y}b_1^2\lambda_3^2 \\ & +10\lambda_{3yy}b_1^2\lambda_3-12\lambda_{3y}^2b_1^2+300b_1^2b_3\lambda_3^2+75\lambda_2\lambda_3^2, \end{aligned}$$

$$\begin{aligned}
\lambda_6 = \;& -300b_{1y}b_1^2b_3\lambda_3^2 + 15b_{1y}\lambda_3\lambda_4 + 40\lambda_{3y}b_1^3b_3\lambda_3 + 12\lambda_{3y}b_1\lambda_4,\\
& -10\lambda_{4y}b_1\lambda_3 + 25\lambda_1\lambda_3^2,\\
\lambda_7 = \;& 140b_{1\tau}b_1^2\lambda_3\lambda_4 - 300b_{1y}^2b_1^2b_3\lambda_3^2 - 55b_{1y}^2\lambda_3\lambda_4 + 200b_{1y}b_{3y}b_1^3\lambda_3^2,\\
& +40b_{1y}\lambda_{3y}b_1^3b_3\lambda_3 + 12b_{1y}\lambda_{3y}b_1\lambda_4 + 600b_{1y}b_1^3b_2b_3\lambda_3^2\\
& +110b_{1y}b_1b_2\lambda_3\lambda_4 - 275b_{1y}\lambda_1\lambda_3^2 - b_{1y}\lambda_6 - 400b_{3\tau}b_1^5\lambda_3^2\\
& -400b_{3y}b_1^4b_2\lambda_3^2 + 50\lambda_{1y}b_1\lambda_3^2 - 80\lambda_{3y}b_1^4b_2b_3\lambda_3 - 24\lambda_{3y}b_1^2b_2\lambda_4\\
& +10\lambda_{3y}b_1\lambda_1\lambda_3 - 20\lambda_{4\tau}b_1^3\lambda_3 - 800b_1^4b_3^2\lambda_3^2 - 400b_1^2b_3\lambda_2\lambda_3^2\\
& -50b_1b_2\lambda_1\lambda_3^2 + 2b_1b_2\lambda_6 - 70\lambda_2\lambda_3\lambda_4 + 12\lambda_4^2.
\end{aligned}$$

Differentiating (6.42) with respect to t, one has

$$\begin{aligned}
& 64V_x^3Y_x^3b_1^3(15b_{1y}\lambda_3\lambda_5 - 14\lambda_{3y}b_1\lambda_5 + 5\lambda_{5y}b_1\lambda_3)\\
& +16V_x^2Y_x^4b_1^2V(120b_{1\tau}b_1^2\lambda_3\lambda_5 - 60b_{1y}^2\lambda_3\lambda_5 + 5b_{1y}\lambda_{5y}b_1\lambda_3\\
& +120b_{1y}b_1b_2\lambda_3\lambda_5 + 15b_{1y}\lambda_3\lambda_6 - 28\lambda_{3y}b_1\lambda_6 - 10\lambda_{5\tau}b_1^3\lambda_3\\
& -10\lambda_{5y}b_1^2b_2\lambda_3 + 10\lambda_{6y}b_1\lambda_3 + 40b_1^2b3\lambda_3\lambda_5 - 60\lambda_2\lambda_3\lambda_5 + 14\lambda_4\lambda_5)\\
& +4V_xY_x^5b_1V^2(260b_{1\tau}b_1^2\lambda_3\lambda_6 - 130b_{1y}^2\lambda_3\lambda_6 + 10b_{1y}\lambda_{6y}b_1\lambda_3\\
& +260b_{1y}b_1b_2\lambda_3\lambda_6 + 14\lambda_{3y}b_1\lambda_7 - 20\lambda_{6\tau}b_1^3\lambda_3 - 20\lambda_{6y}b_1^2b_2\lambda_3 - 5\lambda_{7y}b_1\lambda_3\\
& +40b_1^2b3\lambda_3\lambda_6 - 10\lambda_1\lambda_3\lambda_5 - 130\lambda_2\lambda_3\lambda_6 + 28\lambda_4\lambda_6) + 480V_xY_x^2a_xb_1^4\lambda_3\lambda_5V^2\\
& +Y_x^6V^3(-140b_{1\tau}b_1^2\lambda_3\lambda_7 + 70b_{1y}^2\lambda_3\lambda_7 - 5b_{1y}\lambda_{7y}b_1\lambda_3 - 140b_{1y}b_1b_2\lambda_3\lambda_7\\
& +10\lambda_{7\tau}b_1^3\lambda_3 + 10\lambda_{7y}b_1^2b_2\lambda_3 - 10\lambda_1\lambda_3\lambda_6 + 70\lambda_2\lambda_3\lambda_7 - 14\lambda_4\lambda_7)\\
& +120Y_x^3a_xb_1^3\lambda_3\lambda_6V^3 = 0.
\end{aligned} \tag{6.44}$$

Substituting a_{xx} into (6.40), one finds

$$\begin{aligned}
& 16V_x^2Y_xb_1^2(15b_{1y}\lambda_3\lambda_5 - 26\lambda_{3y}b_1\lambda_5 + 10\lambda_{5y}b_1\lambda_3) + 8V_xY_x^2b_1V\\
& (-15b_{1y}\lambda_3\lambda_6 - 24\lambda_{3y}b_1\lambda_6 + 10\lambda_{6y}b_1\lambda_3 + 40b_1^2b_3\lambda_3\lambda_5 + 2\lambda_4\lambda_5)\\
& +Y_x^3V^2(45b_{1y}\lambda_3\lambda_7 + 22\lambda_{3y}b_1\lambda_7 - 10\lambda_{7y}b_1\lambda_3 + 80b_1^2b_3\lambda_3\lambda_6\\
& +250\lambda_3^4 + 4\lambda_4\lambda_6) + 80a_xb_1^3\lambda_3\lambda_5V^2 = 0.
\end{aligned} \tag{6.45}$$

Further study depends on the value of λ_5.

Case 1 : $\lambda_5 \neq 0$.

From equations (6.43) and (6.45), one finds

$$V_x = -Y_x \lambda_6 V/(4b_1\lambda_5) \tag{6.46}$$

$$a_x = Y_x^3 \lambda_8/(80b_1^3\lambda_3\lambda_5^3), \tag{6.47}$$

where

$$\begin{aligned}\lambda_8 = \;& 10\lambda_{7y}b_1\lambda_3\lambda_5^2 - 45b_{1y}\lambda_3\lambda_5\lambda_6^2 - 22\lambda_{3y}b_1\lambda_5^2\lambda_7 - 22\lambda_{3y}b_1\lambda_5\lambda_6^2\\ & -10\lambda_{5y}b_1\lambda_3\lambda_6^2 - 45b_{1y}\lambda_3\lambda_5^2\lambda_7 + 20\lambda_{6y}b_1\lambda_3\lambda_5\lambda_6 - 250\lambda_3^4\lambda_5^2.\end{aligned}$$

Equation (6.44) becomes

$$\begin{aligned}& 95b_{1y}^2\lambda_3\lambda_5^3\lambda_7 - 280b_{1\tau}b_1^2\lambda_3\lambda_5^3\lambda_7 - 280b_{1\tau}b_1^2\lambda_3\lambda_5^2\lambda_6^2 + 95b_{1y}^2\lambda_3\lambda_5^2\lambda_6^2\\ & -22b_{1y}\lambda_{3y}b_1\lambda_5^3\lambda_7 - 22b_{1y}\lambda_{3y}b_1\lambda_5^2\lambda_6^2 - 190b_{1y}b_1b_2\lambda_3\lambda_5^3\lambda_7\\ & -190b_{1y}b_1b_2\lambda_3\lambda_5^2\lambda_6^2 - 250b_{1y}\lambda_3^4\lambda_5^3 + 45b_{1y}\lambda_3\lambda_5^2\lambda_6\lambda_7 + 45b_{1y}\lambda_3\lambda_5\lambda_6^3\\ & -b_{1y}\lambda_5\lambda_8 + 44\lambda_{3y}b_1^2b_2\lambda_5^3\lambda_7 + 44\lambda_{3y}b_1^2b_2\lambda_5^2\lambda_6^2 - 6\lambda_{3y}b_1\lambda_5^2\lambda_6\lambda_7\\ & -6\lambda_{3y}b_1\lambda_5\lambda_6^3 - 20\lambda_{5\tau}b_1^3\lambda_3\lambda_5\lambda_6^2 + 40\lambda_{6\tau}b_1^3\lambda_3\lambda_5^2\lambda_6 + 20\lambda_{7\tau}b_1^3\lambda_3\lambda_5^3\\ & +500b_1b_2\lambda_3^4\lambda_5^3 + 2b_1b_2\lambda_5\lambda_8 + 140\lambda_2\lambda_3\lambda_5^3\lambda_7 + 140\lambda_2\lambda_3\lambda_5^2\lambda_6^2\\ & +250\lambda_3^4\lambda_5^2\lambda_6 - 28\lambda_4\lambda_5^3\lambda_7 - 28\lambda_4\lambda_5^2\lambda_6^2 + \lambda_6\lambda_8 = 0.\end{aligned} \tag{6.48}$$

Substituting V_x into (6.37), one has

$$\begin{aligned}\lambda_{6y} = \;& (15b_{1y}\lambda_3\lambda_5\lambda_6 - \lambda_{3y}b_1\lambda_5\lambda_6 + 5\lambda_{5y}b_1\lambda_3\lambda_6 - 20b_1^2b_3\lambda_3\lambda_5^2\\ & -\lambda_4\lambda_5^2)/(5b_1\lambda_3\lambda_5).\end{aligned} \tag{6.49}$$

Comparing the mixed derivatives $(V_t)_x - (V_x)_t = 0$, one gets

$$\begin{aligned}& 40b_{1y}b_1^2b_3\lambda_3\lambda_5^3 - 20b_{1\tau}b_1^2\lambda_3\lambda_5^2\lambda_6 - 20b_{1y}^2\lambda_3\lambda_5^2\lambda_6 + 2b_{1y}\lambda_{3y}b_1\lambda_5^2\lambda_6\\ & +40b_{1y}b_1b_2\lambda_3\lambda_5^2\lambda_6 + 15b_{1y}\lambda_3\lambda_5\lambda_6^2 + 2b_{1y}\lambda_4\lambda_5^3 - 4\lambda_{3y}b_1^2b_2\lambda_5^2\lambda_6\\ & -2\lambda_{3y}b_1\lambda_5\lambda_6^2 - 20\lambda_{5\tau}b_1^3\lambda_3\lambda_5\lambda_6 + 20\lambda_{6\tau}b_1^3\lambda_3\lambda_5^2 - 80b_1^3b_2b_3\lambda_3\lambda_5^3\\ & -4b_1b_2\lambda_4\lambda_5^3 + 10\lambda_1\lambda_3\lambda_5^3 + 10\lambda_2\lambda_3\lambda_5^2\lambda_6 - 2\lambda_4\lambda_5^2\lambda_6 - \lambda_8 = 0.\end{aligned} \tag{6.50}$$

Substituting a_x into (6.42), one obtains

$$15b_{1y}\lambda_3\lambda_5\lambda_8 - 2\lambda_{3y}b_1\lambda_5\lambda_8 + 5\lambda_{12} + 2\lambda_5^4\lambda_7 + 2\lambda_5^3\lambda_6^2 = 0, \tag{6.51}$$

where

$$\lambda_{12} = 9b_{1y}\lambda_3\lambda_5\lambda_8 + 2\lambda_{3y}b_1\lambda_5\lambda_8 + 6\lambda_{5y}b_1\lambda_3\lambda_8 - 2\lambda_{8y}b_1\lambda_3\lambda_5.$$

Since a does not depend on t, the equation $(a_x)_t = 0$ gives

$$\begin{aligned}
&35b_{1y}^2\lambda_3\lambda_5^2\lambda_8 - 10b_{1y}\lambda_{3y}b_1\lambda_5^2\lambda_8 - 160b_{1\tau}b_1^2\lambda_3\lambda_5^2\lambda_8 \\
&-70b_{1y}b_1b_2\lambda_3\lambda_5^2\lambda_8 + 5b_{1y}\lambda_{12}\lambda_5 + 45b_{1y}\lambda_3\lambda_5\lambda_6\lambda_8 + 20\lambda_{3y}b_1^2b_2\lambda_5^2\lambda_8 \\
&-6\lambda_{3y}b_1\lambda_5\lambda_6\lambda_8 - 60\lambda_{5\tau}b_1^3\lambda_3\lambda_5\lambda_8 + 20\lambda_{8\tau}b_1^3\lambda_3\lambda_5^2 - 10b_1b_2\lambda_{12}\lambda_5 \\
&-5\lambda_{12}\lambda_6 + 80\lambda_2\lambda_3\lambda_5^2\lambda_8 - 16\lambda_4\lambda_5^2\lambda_8 = 0.
\end{aligned} \tag{6.52}$$

If conditions (6.48)-(6.52) are satisfied, then the system of equation (6.33) is compatible. Thus, we have obtained that conditions (6.48)-(6.52) guarantee that the parabolic equation (6.4) is equivalent to (6.14).

Case $2: \ \lambda_5 = 0$.

From (6.43), one has that, $\lambda_6 = 0$, and equation (6.45) becomes

$$\lambda_{7y} = (45b_{1y}\lambda_3\lambda_7 + 22\lambda_{3y}b_1\lambda_7 + 250\lambda_3^4)/(10b_1\lambda_3). \tag{6.53}$$

Notice that the condition $\lambda_3 \neq 0$ implies $\lambda_7 \neq 0$. Differentiating a_{xx} in (6.42) with respect to t, one gets

$$12V_xb_1\lambda_9 + Y_x\lambda_{10}V = 0, \tag{6.54}$$

where

$$\begin{aligned}
\lambda_9 = \ & 6\lambda_{3y}b_1\lambda_7 - 45b_{1y}\lambda_3\lambda_7 - 250\lambda_3^4, \\
\lambda_{10} = \ & 420b_{1y}b_1b_2\lambda_3\lambda_7 - 840b_{1\tau}b_1^2\lambda_3\lambda_7 - 210b_{1y}^2\lambda_3\lambda_7 - 3500b_{1y}\lambda_3^4 \\
& -11b_{1y}\lambda_9 + 60\lambda_{7\tau}b_1^3\lambda_3 + 7000b_1b_2\lambda_3^4 + 22b_1b_2\lambda_9 + 420\lambda_2\lambda_3\lambda_7 \\
& -84\lambda_4\lambda_7.
\end{aligned}$$

From definition of λ_9, one finds λ_{3y}. Then (6.53) becomes

$$\lambda_{7y} = (630b_{1y}\lambda_3\lambda_7 + 3500\lambda_3^4 + 11\lambda_9)/(30b_1\lambda_3). \tag{6.55}$$

Case 2.1: $\lambda_9 \neq 0$.

From (6.54), one finds

$$V_x = -Y_x\lambda_{10}V/(12b_1\lambda_9). \tag{6.56}$$

Substituting V_x into (6.37) and (6.38), one has

$$\begin{aligned}\lambda_{10y} = \; & (45b_{1y}\lambda_{10}\lambda_3\lambda_7\lambda_9 + 30\lambda_{9y}b_1\lambda_{10}\lambda_3\lambda_7 - 360b_1^2b_3\lambda_3\lambda_7\lambda_9^2 \\ & -250\lambda_{10}\lambda_3^4\lambda_9 - \lambda_{10}\lambda_9^2 - 18\lambda_4\lambda_7\lambda_9^2)/(30b_1\lambda_3\lambda_7\lambda_9),\end{aligned} \tag{6.57}$$

$$a_x = Y_x^3\lambda_{11}, \tag{6.58}$$

where

$$\begin{aligned}\lambda_{11} = \; & (-12b_{1\tau}b_1^2\lambda_{10}\lambda_9^2 + 6b_{1y}^2\lambda_{10}\lambda_9^2 - 6b_{1y}\lambda_{10y}b_1\lambda_9^2 + 6b_{1y}\lambda_{9y}b_1\lambda_{10}\lambda_9 \\ & -12b_{1y}b_1b_2\lambda_{10}\lambda_9^2 - 3b_{1y}\lambda_{10}^2\lambda_9 + 12\lambda_{10\tau}b_1^3\lambda_9^2 + 12\lambda_{10y}b_1^2b_2\lambda_9^2 \\ & +2\lambda_{10y}b_1\lambda_{10}\lambda_9 - 12\lambda_{9\tau}b_1^3\lambda_{10}\lambda_9 - 12\lambda_{9y}b_1^2b_2\lambda_{10}\lambda_9 - 2\lambda_{9y}b_1\lambda_{10}^2 \\ & +24b_1^2b_3\lambda_{10}\lambda_9^2 + 18\lambda_1\lambda_9^3 + 6\lambda_{10}\lambda_2\lambda_9^2)/(144b_1^3\lambda_9^3).\end{aligned}$$

Differentiating (6.58) with respect to t, and substituting a_x into (6.42), one gets

$$\begin{aligned} & 5400b_{1\tau}b_{1y}b_1^2\lambda_{10}\lambda_3\lambda_7\lambda_9^3 - 10800b_{1\tau}b_1^3b_2\lambda_{10}\lambda_3\lambda_7\lambda_9^3 \\ & -1800b_{1\tau}b_1^2\lambda_{10}^2\lambda_3\lambda_7\lambda_9^2 - 2700b_{1y}^3\lambda_{10}\lambda_3\lambda_7\lambda_9^3 \\ & +135b_{1y}^2\lambda_{9y}b_1\lambda_{10}\lambda_3\lambda_7\lambda_9^2 - 145800b_{1y}^2b_1^3\lambda_{11}\lambda_3^2\lambda_7^2\lambda_9^3 \\ & +10800b_{1y}^2b_1b_2\lambda_{10}\lambda_3\lambda_7\lambda_9^3 + 2250b_{1y}^2\lambda_{10}^2\lambda_3\lambda_7\lambda_9^2 \\ & -270b_{1y}\lambda_{9\tau}b_1^3\lambda_{10}\lambda_3\lambda_7\lambda_9^2 - 540b_{1y}\lambda_{9y}b_1^2b_2\lambda_{10}\lambda_3\lambda_7\lambda_9^2 \\ & -90b_{1y}\lambda_{9y}b_1\lambda_{10}^2\lambda_3\lambda_7\lambda_9 + 291600b_{1y}b_1^4b_2\lambda_{11}\lambda_3^2\lambda_7^2\lambda_9^3 \\ & -810000b_{1y}b_1^3\lambda_{11}\lambda_3^5\lambda_7\lambda_9^3 - 3240b_{1y}b_1^3\lambda_{11}\lambda_3\lambda_7\lambda_9^4 \\ & +97200b_{1y}b_1^3\lambda_{13}\lambda_3^2\lambda_7^2\lambda_9^3 - 10800b_{1y}b_1^2b_2^2\lambda_{10}\lambda_3\lambda_7\lambda_9^3 \\ & -4500b_{1y}b_1b_2\lambda_{10}^2\lambda_3\lambda_7\lambda_9^2 - 450b_{1y}\lambda_{10}^3\lambda_3\lambda_7\lambda_9\end{aligned}$$

$$
\begin{aligned}
&+7500b_{1y}\lambda_{10}^2\lambda_3^4\lambda_9^2+30b_{1y}\lambda_{10}^2\lambda_9^3-2700b_{1y}\lambda_{10}\lambda_2\lambda_3\lambda_7\lambda_9^3\\
&+540b_{1y}\lambda_{10}\lambda_4\lambda_7\lambda_9^3+243b_{1y}\lambda_7^3\lambda_9^3\\
&+194400\lambda_{11\tau}b_1^6\lambda_3^2\lambda_7^2\lambda_9^3+540\lambda_{9\tau}b_1^4b_2\lambda_{10}\lambda_3\lambda_7\lambda_9^2\\
&+90\lambda_{9\tau}b_1^3\lambda_{10}^2\lambda_3\lambda_7\lambda_9+540\lambda_{9y}b_1^3b_2^2\lambda_{10}\lambda_3\lambda_7\lambda_9^2\\
&+180\lambda_{9y}b_1^2b_2\lambda_{10}^2\lambda_3\lambda_7\lambda_9+15\lambda_{9y}b_1\lambda_{10}^3\lambda_3\lambda_7\\
&+1620000b_1^4b_2\lambda_{11}\lambda_3^5\lambda_7\lambda_9^3+6480b_1^4b_2\lambda_{11}\lambda_3\lambda_7\lambda_9^4\\
&-194400b_1^4b_2\lambda_{13}\lambda_3^2\lambda_7^2\lambda_9^3-540000b_1^3\lambda_{10}\lambda_{11}\lambda_3^5\lambda_7\lambda_9^2\\
&-2160b_1^3\lambda_{10}\lambda_{11}\lambda_3\lambda_7\lambda_9^3-32400b_1^3\lambda_{10}\lambda_{13}\lambda_3^2\lambda_7^2\lambda_9^2\\
&+291600b_1^3\lambda_{11}\lambda_2\lambda_3^2\lambda_7^2\lambda_9^3-58320b_1^3\lambda_{11}\lambda_3\lambda_4\lambda_7^2\lambda_9^3\\
&-15000b_1b_2\lambda_{10}^2\lambda_3^4\lambda_9^2-60b_1b_2\lambda_{10}^2\lambda_9^3\\
&+5400b_1b_2\lambda_{10}\lambda_2\lambda_3\lambda_7\lambda_9^3-1080b_1b_2\lambda_{10}\lambda_4\lambda_7\lambda_9^3\\
&-486b_1b_2\lambda_7^3\lambda_9^3-2500\lambda_{10}^3\lambda_3^4\lambda_9-10\lambda_{10}^3\lambda_9^2\\
&+900\lambda_{10}^2\lambda_2\lambda_3\lambda_7\lambda_9^2-180\lambda_{10}^2\lambda_4\lambda_7\lambda_9^2-81\lambda_{10}\lambda_7^3\lambda_9^2=0,
\end{aligned} \tag{6.59}
$$

$$
600b_1^3\lambda_3^2\lambda_7\lambda_{13}-10000b_1^3\lambda_{11}\lambda_3^5-40b_1^3\lambda_{11}\lambda_3\lambda_9+3\lambda_7^2=0, \tag{6.60}
$$

where

$$
\lambda_{13}=3b_{1y}\lambda_{11}-2\lambda_{11y}b_1.
$$

Therefore the conditions $\lambda_5 = 0$, $\lambda_6 = 0$, (6.55), (6.57), (6.59) and (6.60) guarantee that the parabolic equation (6.4) equivalent to equation (6.14).

Case 2.2 : $\lambda_9 = 0$.

From (6.54) and (6.55), it follow that

$$
\lambda_{10}=0,
$$

$$
\lambda_{7y}=7(9b_{1y}\lambda_7+50\lambda_3^3)/(3b_1). \tag{6.61}
$$

Therefore the conditions $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_9 = 0$, $\lambda_{10} = 0$ and (6.61) guarantee that the parabolic equation (6.4) equivalent to equation (6.14).

Theorem 6.1. *The parabolic equation (6.4) is equivalent to equation (6.14) if and only if the coefficients of (6.4) obey one of the following conditions:*
(A) equations (6.48)-(6.52), in this case the functions $H(t), Y(t,x), V(t,x)$ *and* $a(x)$ *are obtained by solving involutive system of equations (6.34)-(6.36), (6.41), (6.46), (6.47);*
(B) equations $\lambda_5 = 0$, $\lambda_6 = 0$, *(6.55), (6.57), (6.59), (6.60), in this case the functions* $H(t), Y(t,x), V(t,x)$ *and* $a(x)$ *are obtained by solving involutive system of equations (6.34)-(6.36), (6.41), (6.56), (6.58);*
(C) equations $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_9 = 0$, $\lambda_{10} = 0$, *(6.61), in this case the functions* $H(t), Y(t,x), V(t,x)$ *and* $a(x)$ *are obtained by solving involutive system of equations (6.34)-(6.37), (6.41), (6.42).*

6.5 Equivalence problem for equation (6.15)

This section studies equations (6.4) which are equivalent to equation (6.15). Since for equation (6.15), the coefficient is

$$a(x) = k/x^2, (k \neq 0), \tag{6.62}$$

we continue studying the various cases from the previous section.

Case $1:\ \lambda_5 \neq 0$.

Substituting a in (6.62) into (6.47), one has

$$k = -Y_x^3 \lambda_8 x^3 / (160 b_1^3 \lambda_3 \lambda_5^3). \tag{6.63}$$

Since k is constant, differentiating (6.63) with respect to x, one obtains

$$Y_x \lambda_{12} x - 6 b_1 \lambda_3 \lambda_5 \lambda_8 = 0. \tag{6.64}$$

Case 1.1 : $\lambda_{12} \neq 0$.

In this case, one can find

$$Y_x = 6b_1\lambda_3\lambda_5\lambda_8/(\lambda_{12}x). \tag{6.65}$$

Substituting Y_x into (6.35), one has

$$\begin{aligned}&3\lambda_{12y}b_1\lambda_3\lambda_5\lambda_8 - 18b_{1y}\lambda_{12}\lambda_3\lambda_5\lambda_8 - 6\lambda_{3y}b_1\lambda_{12}\lambda_5\lambda_8\\&-12\lambda_{5y}b_1\lambda_{12}\lambda_3\lambda_8 + 2\lambda_{12}^2 = 0.\end{aligned} \tag{6.66}$$

Comparing the mixed derivatives $(Y_t)_x - (Y_x)_t = 0$, one obtains

$$\begin{aligned}&60\lambda_{12}\lambda_2\lambda_3\lambda_5^2\lambda_8 - 120b_{1\tau}b_1^2\lambda_{12}\lambda_3\lambda_5^2\lambda_8 + 105b_{1y}^2\lambda_{12}\lambda_3\lambda_5^2\lambda_8\\&-10b_{1y}\lambda_{12y}b_1\lambda_3\lambda_5^2\lambda_8 + 10b_{1y}\lambda_{3y}b_1\lambda_{12}\lambda_5^2\lambda_8 + 40b_{1y}\lambda_{5y}b_1\lambda_{12}\lambda_3\lambda_5\lambda_8\\&-210b_{1y}b_1b_2\lambda_{12}\lambda_3\lambda_5^2\lambda_8 - 5b_{1y}\lambda_{12}^2\lambda_5 - 45b_{1y}\lambda_{12}\lambda_3\lambda_5\lambda_6\lambda_8\\&+20\lambda_{12\tau}b_1^3\lambda_3\lambda_5^2\lambda_8 + 20\lambda_{12y}b_1^2b_2\lambda_3\lambda_5^2\lambda_8 + 10\lambda_{12y}b_1\lambda_3\lambda_5\lambda_6\lambda_8\\&-20\lambda_{3y}b_1^2b_2\lambda_{12}\lambda_5^2\lambda_8 - 22\lambda_{3y}b_1\lambda_{12}\lambda_5\lambda_6\lambda_8 - 20\lambda_{5\tau}b_1^3\lambda_{12}\lambda_3\lambda_5\lambda_8\\&-80\lambda_{5y}b_1^2b_2\lambda_{12}\lambda_3\lambda_5\lambda_8 - 40\lambda_{5y}b_1\lambda_{12}\lambda_3\lambda_6\lambda_8 - 20\lambda_{8\tau}b_1^3\lambda_{12}\lambda_3\lambda_5^2\\&+10b_1b_2\lambda_{12}^2\lambda_5 + 5\lambda_{12}^2\lambda_6 - 12\lambda_{12}\lambda_4\lambda_5^2\lambda_8 = 0.\end{aligned} \tag{6.67}$$

Therefore the condition (6.48)-(6.52) and (6.66)-(6.67) guarantee that the parabolic equation (6.4) equivalent to equation (6.15).

Case 1.2 : $\lambda_{12} = 0$.

Equation (6.64) implies $\lambda_8 = 0$. Then (6.52) is the identity and equation (6.51) and (6.48) become

$$\lambda_7 = -\lambda_6^2/\lambda_5, \tag{6.68}$$

$$\lambda_6 = \lambda_5(b_{1y} - 2b_1b_2). \tag{6.69}$$

Substituting λ_6 into (6.49) and (6.50), one gets

$$\begin{aligned}\lambda_{6y} = \;&(15b_{1y}^2\lambda_3\lambda_5 - b_{1y}\lambda_{3y}b_1\lambda_5 + 5b_{1y}\lambda_{5y}b_1\lambda_3 - 30b_{1y}b_1b_2\lambda_3\lambda_5\\&+2\lambda_{3y}b_1^2b_2\lambda_5 - 10\lambda_{5y}b_1^2b_2\lambda_3 - 20b_1^2b_3\lambda_3\lambda_5 - \lambda_4\lambda_5)/(5b_1\lambda_3),\end{aligned} \tag{6.70}$$

$$\begin{aligned}&b_{1y}^3-4b_{1y}^2b_1b_2+4b_{1y}b_{2y}b_1^2+4b_{1y}b_1^2b_2^2-8b_{1y}b_1^2b_3+10b_{1y}\lambda_2-8b_{2\tau}b_1^4\\&-8b_{2y}b_1^3b_2-4\lambda_{2y}b_1+4\lambda_1=0.\end{aligned} \tag{6.71}$$

Therefore the conditions $\lambda_8=0$, $\lambda_{12}=0$, (6.68)-(6.70) and (6.71) guarantee that the parabolic equation (6.4) equivalent to equation (6.15).

Case 2 : $\lambda_5=0$.

Case 2.1 : $\lambda_9\neq 0$.

Substituting a from (6.62) into (6.58), one has

$$k=-Y_x^3\lambda_{11}x^3/2. \tag{6.72}$$

Differentiating k with respect to t and x, one gets

$$\begin{aligned}&45b_{1y}^2\lambda_{11}\lambda_3\lambda_7\lambda_9-90b_{1y}b_1b_2\lambda_{11}\lambda_3\lambda_7\lambda_9-15b_{1y}\lambda_{13}\lambda_3\lambda_7\lambda_9\\&-60\lambda_{11\tau}b_1^3\lambda_3\lambda_7\lambda_9+30b_1b_2\lambda_{13}\lambda_3\lambda_7\lambda_9+250\lambda_{10}\lambda_{11}\lambda_3^4\\&+\lambda_{10}\lambda_{11}\lambda_9+5\lambda_{10}\lambda_{13}\lambda_3\lambda_7-90\lambda_{11}\lambda_2\lambda_3\lambda_7\lambda_9+18\lambda_{11}\lambda_4\lambda_7\lambda_9=0,\end{aligned} \tag{6.73}$$

$$xY_x\lambda_{13}-6b_1\lambda_{11}=0, \tag{6.74}$$

Case 2.1.1 : $\lambda_{13}\neq 0$.

Solving (6.74) with respect to Y_x, one obtains

$$Y_x=6b_1\lambda_{11}/(\lambda_{13}x). \tag{6.75}$$

Substituting Y_x into (6.35), one has

$$0=3\lambda_{13y}b_1\lambda_{11}-9b_{1y}\lambda_{11}\lambda_{13}+2\lambda_{13}^2. \tag{6.76}$$

The requirement $(Y_t)_x-(Y_x)_t=0$, leads to condition

$$\begin{aligned}&225b_{1y}^2\lambda_{11}\lambda_{13}\lambda_3\lambda_7\lambda_9-180b_{1\tau}b_1^2\lambda_{11}\lambda_{13}\lambda_3\lambda_7\lambda_9\\&-90b_{1y}\lambda_{13y}b_1\lambda_{11}\lambda_3\lambda_7\lambda_9-450b_{1y}b_1b_2\lambda_{11}\lambda_{13}\lambda_3\lambda_7\lambda_9\\&-90b_{1y}\lambda_{10}\lambda_{11}\lambda_{13}\lambda_3\lambda_7-45b_{1y}\lambda_{13}^2\lambda_3\lambda_7\lambda_9-180\lambda_{11\tau}b_1^3\lambda_{13}\lambda_3\lambda_7\lambda_9\end{aligned}$$

$$
\begin{aligned}
&+180\lambda_{13\tau}b_1^3\lambda_{11}\lambda_3\lambda_7\lambda_9+180\lambda_{13y}b_1^2b_2\lambda_{11}\lambda_3\lambda_7\lambda_9\\
&+30\lambda_{13y}b_1\lambda_{10}\lambda_{11}\lambda_3\lambda_7+90b_1b_2\lambda_{13}^2\lambda_3\lambda_7\lambda_9-250\lambda_{10}\lambda_{11}\lambda_{13}\lambda_3^4\\
&-\lambda_{10}\lambda_{11}\lambda_{13}\lambda_9+15\lambda_{10}\lambda_{13}^2\lambda_3\lambda_7+90\lambda_{11}\lambda_{13}\lambda_2\lambda_3\lambda_7\lambda_9\\
&-18\lambda_{11}\lambda_{13}\lambda_4\lambda_7\lambda_9=0.
\end{aligned}
\tag{6.77}
$$

Therefore the conditions (6.55), (6.57), (6.59), (6.60), (6.73), (6.76) and (6.77) guarantee that the parabolic equation (6.4) equivalent to equation (6.15).

Case 2.1.2 : $\lambda_{13}=0$.

Equation (6.74) and (6.60) imply $\lambda_{11}=0$ and $\lambda_7=0$. This contradicts $\lambda_7\neq 0$.

Case 2.2 : $\lambda_9=0$.

Equation (6.42) is

$$3Y_x^4\lambda_7^2x^4+800b_1^3\lambda_3^2k(25Y_x\lambda_3^3x-9b_1\lambda_7)=0. \tag{6.78}$$

Analyzing equation (6.78) one obtains that $25Y_x\lambda_3^3x-9b_1\lambda_7\neq 0$. Hence,

$$k=-3Y_x^4\lambda_7^2x^4/(800b_1^3\lambda_3^2(25Y_x\lambda_3^3x-9b_1\lambda_7)). \tag{6.79}$$

Differentiating k with respect to x, one has

$$675Y_xb_1\lambda_3^3\lambda_7^2x-1875Y_x^2\lambda_3^6\lambda_7x^2+108b_1^2\lambda_7^3=0. \tag{6.80}$$

Differentiating (6.80) with respect to t and x, one gets

$$
\begin{aligned}
&93750V_xY_x^3b_1\lambda_3^7x^2(13b_{1y}\lambda_7+70\lambda_3^3)+33750V_xY_x^2b_1^2\lambda_3^4\lambda_7x\\
&(-13b_{1y}\lambda_7-70\lambda_3^3)+5400V_xY_xb_1^3\lambda_3\lambda_7^2(-13b_{1y}\lambda_7-70\lambda_3^3)\\
&+1875Y_x^4\lambda_3^6\lambda_7Vx^2(-220b_{1\tau}b_1^2\lambda_3+110b_{1y}^2\lambda_3-220b_{1y}b_1b_2\lambda_3\\
&+105\lambda_2\lambda_3-21\lambda_4)+675Y_x^3b_1\lambda_3^3\lambda_7^2Vx(220b_{1\tau}b_1^2\lambda_3-110b_{1y}^2\lambda_3\\
&+220b_{1y}b_1b_2\lambda_3-105\lambda_2\lambda_3+21\lambda_4)+108Y_x^2b_1^2\lambda_7^3V(220b_{1\tau}b_1^2\lambda_3\\
&-110b_{1y}^2\lambda_3+220b_{1y}b_1b_2\lambda_3-105\lambda_2\lambda_3+21\lambda_4)=0,
\end{aligned}
\tag{6.81}
$$

$$3125Y_x^2\lambda_3^6x^2(-39b_{1y}\lambda_7 - 220\lambda_3^3) + 375Y_xb_1\lambda_3^3\lambda_7x(117b_{1y}\lambda_7 \\ +635\lambda_3^3) + 135b_1^2\lambda_7^2(52b_{1y}\lambda_7 + 285\lambda_3^3) = 0. \tag{6.82}$$

Using (6.80) and (6.82), one finds

$$Y_x = -3b_1\lambda_7/(25\lambda_3^3x). \tag{6.83}$$

Substituting Y_x into (6.81) and equations $Y_{xx} = (Y_x)_x$, $Y_{tx} = (Y_x)_t$, $(Y_t)_x = (Y_x)_t$, one obtains identities. Therefore the conditions $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_9 = 0$, $\lambda_{10} = 0$ and (6.61) guarantee that the parabolic equation (6.4) equivalent to equation (6.15).

Theorem 6.2. *The parabolic equation (6.4) is equivalent to equation (6.15) if and only if the coefficients of (6.4) obey one of the following conditions:*

(A) (6.48)-(6.52), (6.66)-(6.67), in this case the functions $H(t), Y(t,x), V(t,x)$ and k are obtained by solving involutive system of equations (6.34), (6.36), (6.41), (6.46), (6.63), (6.65);

(B) $\lambda_8 = 0$, $\lambda_{12} = 0$, (6.68)-(6.70), (6.71), in this case the functions $H(t), Y(t,x), V(t,x)$ and k are obtained by solving involutive system of equations (6.34)-(6.36), (6.41), (6.46), (6.63);

(C) (6.55), (6.57), (6.59), (6.60), (6.73), (6.76), (6.77), in this case the functions $H(t), Y(t,x), V(t,x)$ and k are obtained by solving involutive system of equations (6.34), (6.36), (6.41), (6.56), (6.72), (6.75);

(D) $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_9 = 0$, $\lambda_{10} = 0$, (6.61), in this case the functions $H(t), Y(t,x), V(t,x)$ and k are obtained by solving involutive system of equations (6.34), (6.36), (6.37), (6.41), (6.79), (6.83).

Remark. Using an invariant of sixth-order one can conclude that if equation

(6.4) is equivalent to (6.15) then it satisfies the following relation

$$\begin{aligned} 3125/(3k) = \ & 4(506250 b_{1y}^4 \lambda_{3y} b_1 \lambda_3^4 - 759375 b_{1y}^5 \lambda_3^5 - 135000 b_{1y}^3 \lambda_{3y}^2 b_1^2 \lambda_3^3 \\ & + 18000 b_{1y}^2 \lambda_{3y}^3 b_1^3 \lambda_3^2 - 1200 b_{1y} \lambda_{3y}^4 b_1^4 \lambda_3 + 32 \lambda_{3y}^5 b_1^5)/\lambda_3^6. \end{aligned} \tag{6.84}$$

Example 1. Consider the Black-Scholes equation

$$v_\tau + \frac{1}{2} A^2 y^2 v_{yy} + B y v_y - C v = 0, \tag{6.85}$$

where A, B, C are constants. It has the form of equation (6.4) with the following coefficients

$$b_1 = -2/(A^2 y^2),\ b_2 = -2B/(A^2 y),\ b_3 = 2C/(A^2 y^2). \tag{6.86}$$

For the coefficients (6.95) one obtains that $\lambda = 0$. Hence equation (6.94) is equivalent to the heat equation. For finding a transformation mapping equation (6.94) into the heat equation, one need to solve equations (6.25)-(6.28) for $H(t)$, $Y(t,x)$, $V(t,x)$.

Let us find the transformations which maps equation (6.94) into equation (6.13). Substituting b_1 into equation (6.26), one has

$$Y Y_{xx} - Y_x^2 = 0.$$

Thus the general solution is

$$Y = c_1(t) e^{c_2(t) A x},$$

where c_1, c_2 are arbitrary functions. Substituting Y into equation (6.25), one gets

$$H' = -2 c_2^2. \tag{6.87}$$

From equations (6.27) and (6.28), one finds

$$V_x = \ V(-c_1' - c_2' A c_1 x + A^2 c_1 c_2^2 - 2 B c_1 c_2^2)/(2 A c_1 c_2), \tag{6.88}$$

$$\begin{aligned} V_t = \;& V(-c_1'^2 - 2c_1'c_2'Ac_1x + 2c_1'A^2c_1c_2^2 - 4c_1'Bc_1c_2^2 - c_2'^2A^2c_1^2x^2 \\ & +2c_2'A^3c_1^2c_2^2x - 2c_2'A^2c_1^2c_2 - 4c_2'ABc_1^2c_2^2x - A^4c_1^2c_2^4 + 4A^2Bc_1^2c_2^4 \\ & -8A^2Cc_1^2c_2^4 - 4B^2c_1^2c_2^4)/(4A^2c_1^2c_2^2). \end{aligned} \tag{6.89}$$

Comparing the mixed derivatives $(V_t)_x - (V_x)_t = 0$, one gets

$$Ac_1^2(c_2''c_2 - 2c_2'^2)x + (c_1''c_1c_2 - c_1'^2c_2 - 2c_1'c_2'c_1) = 0.$$

Splitting this equation with respect to x, we have

$$c_2''c_2 - 2c_2'^2 = 0, \tag{6.90}$$

$$c_1''c_1c_2 - c_1'^2c_2 - 2c_1'c_2'c_1 = 0. \tag{6.91}$$

The general solution of equation (6.90) is

$$c_2 = 1/(k_1t + k_0),$$

where k_0 and k_1 are arbitrary constants. Setting $k_0 = \sqrt{2}$, $k_1 = 0$, one has $c_2 = 1/\sqrt{2}$. Substituting c_2 into (6.91), one has

$$c_1''c_1 = c_1'^2.$$

Hence,

$$c_1 = k_4e^{k_3t},$$

where k_3 and k_4 are arbitrary constants. Substituting c_1, c_2 into equation (6.87) and (6.88), one gives

$$H' = -1, \tag{6.92}$$

$$V_x = -V(\frac{B}{\sqrt{2}A} - \frac{A}{2\sqrt{2}} + \frac{k_3}{\sqrt{2}A}). \tag{6.93}$$

The general solution of equation (6.92) and (6.93) are

$$H = -t + k_5,$$

$$V = e^{-(\frac{B}{\sqrt{2}A} - \frac{A}{2\sqrt{2}} + \frac{k_3}{\sqrt{2}A})x + c_3(t)},$$

where k_5 is arbitrary constant and c_3 is arbitrary function. Substituting c_1, c_2 and V into (6.89), one obtains

$$c_3' = -(C + \frac{A^2}{8} - \frac{B}{2} + \frac{B^2}{2A^2} - \frac{k_3}{2} + \frac{k_3 B}{A^2} + \frac{k_3^2}{4A^2}).$$

That is

$$c_3 = -(C + \frac{A^2}{8} - \frac{B}{2} + \frac{B^2}{2A^2} - \frac{k_3}{2} + \frac{k_3 B}{A^2} + \frac{k_3^2}{4A^2})t + k_6,$$

where k_6 is arbitrary constant. Setting $k_3 = 0$, $k_4 = 1$, $k_5 = 0$, $k_6 = 0$. Thus, the transformation which maps the Black-Scholes equation (6.94) into the heat equation $u_t = u_{xx}$ is

$$\tau = -t,\ y = e^{(Ax/\sqrt{2})},\ v = ue^{-\rho(t,x)},$$

where

$$\rho(t,x) = (\frac{B}{\sqrt{2}A} - \frac{A}{2\sqrt{2}})x + (C + \frac{A^2}{8} - \frac{B}{2} + \frac{B^2}{2A^2})t.$$

Example 2. Consider the linear second-order parabolic partial differential equation

$$\tau v_\tau + y v_y - (e^{y/\tau}\tau + 1)v - \tau^3 v_{yy} = 0. \tag{6.94}$$

It has the form of equation (6.4) with the following coefficients

$$b_1 = 1/\tau^2,\ b_2 = y/\tau^3,\ b_3 = -(e^{y/\tau}\tau + 1)/\tau^3. \tag{6.95}$$

Coefficients of equation (6.95) satisfy conditions (A) in Theorem 1. Hence, the parabolic equation (6.94) is equivalent to equation (6.14). For finding transformation $H(t), Y(t,x), V(t,x)$ which mapping equation (6.94) into equation (6.14), and $a(x)$ one needs to solve equations (6.34)-(6.36), (6.41), (6.46) and (6.47).

Substituting b_1, b_2, b_3 into λ_6, one has $\lambda_6 = 0$. Thus equation (6.46) becomes $V_x = 0$, i.e.,

$$V = V(t).$$

Substituting b_1 into equation (6.35), one has $Y_{xx} = 0$. Hence,

$$Y = H(t)(\alpha(t)x + \beta(t)),$$

where $\alpha(t)$ and $\beta(t)$ are arbitrary functions. From equation (6.47), one finds

$$a = \alpha^2 e^{\alpha x + \beta} + C,$$

where C is arbitrary constant, which can chosen, for example $C = 0$. Substituting V, Y, and a into (6.34), (6.41), (6.36), one obtains

$$H' = \alpha^2, \tag{6.96}$$

$$V' = \alpha^2 V/H, \tag{6.97}$$

$$\alpha' = 0, \ \beta' = 0. \tag{6.98}$$

Since any particular solution for equations (6.96)-(6.98) can be used, we set $\alpha = 1$, $\beta = 0$. Hence, $H = t$, $Y = tx$, $V = t$ and $a(x) = e^x$. Therefore, one obtains the following transformations

$$\tau = t, \ y = tx, \ v = tu,$$

mapping equation (6.94) into the equation

$$u_t = u_{xx} + e^x u.$$

Example 3. Consider the linear second-order parabolic partial differential equation

$$\tau y^2 v_\tau + y(2\tau^3 + y^2)v_y - 3\tau^3 v - \tau^3 y^2 v_{yy} = 0. \tag{6.99}$$

It is an equation of the form (6.4) with the coefficients

$$b_1 = 1/\tau^2,\ b_2 = (2\tau^3 + y^2)/(\tau^3 y),\ b_3 = -3/y^2. \tag{6.100}$$

One can check that the coefficients (6.100) obey the conditions (D) in Theorem 2. Furthermore, they also satisfy condition (6.84). Thus, the parabolic equation (6.99) is equivalent to equation (6.15). For finding a transformation mapping equation (6.99) into equation (6.15), one need to solve equations (6.34), (6.36), (6.37), (6.41), (6.79), (6.83) for $H(t), Y(t,x)$, $V(t,x)$ and k.

Substituting b_1, b_2, b_3 into equation (6.83), one has

$$xY_x - Y = 0.$$

Substituting the general solution of this equation

$$Y = c_1(t)H(t)x \tag{6.101}$$

into equation (6.79), one has $k = 1$, where $c_1(t)$ is an arbitrary function. From equations (6.34) and (6.36), one obtains

$$H' = c_1^2, \tag{6.102}$$

$$2c_1 x V_x + (c_1' x^2 - 2c_1)V = 0.$$

The general solution of the last equation is

$$V = c_2(t) x e^{-(c_1'/4c_1)x^2}.$$

Substituting the function V into (6.37), one gets

$$(c_1'' c_1 c_2 - 2c_1'^2 c_2)x^2 + (2c_1' c_1 c_2 - 4c_2' c_1^2) = 0.$$

Splitting this equation with respect to x, we have

$$\begin{aligned} c_1 c_1'' &= 2c_1'^2, \\ 2c_1 c_2' &= c_1' c_2. \end{aligned} \tag{6.103}$$

The general solution of the system of ordinary differential equations (6.103) is

$$c_1 = 1/(k_1 t + k_2),$$
$$c_2^2 = (1/(k_3(k_1 t + k_2))).$$

Substituting c_1 into (6.102), one has

$$H' = 1/(k_1 t + k_2)^2. \tag{6.104}$$

Further study depends on quantity of k_1.

Case $k_1 \neq 0$

The general solution of (6.104) is

$$H = k_4 - 1/(k_1(k_1 t + k_2)).$$

Because of $k_1 \neq 0$, $k_3 \neq 0$ and k_2, k_4 are arbitrary constant. Setting $k_1 = 1$, $k_2 = 0$, $k_3 = 1$, $k_4 = 0$. Therefore, one obtains the following transformations

$$\tau = -\frac{1}{t},\ y = -\frac{x}{t^2},\ v = \frac{x^2}{\sqrt{t}} e^{4x^2/t^3}.$$

mapping equation (6.99) into the equation

$$u_t = u_{xx} + \frac{1}{x^2} u.$$

Case $k_1 = 0$

For this case we have

$$c_1 = 1/k_2,\ c_2^2 = 1/k_2,$$

where $k_2 \neq 0$. The general solution of (6.104) is

$$H = \frac{t}{k_2^2} + k_5.$$

where k_5 is arbitrary constant. Setting $k_2 = 1$, $k_5 = 0$. Therefore, one obtains the following transformations

$$\tau = t,\ y = tx,\ v = xu,$$

mapping equation (6.99) into the equation

$$u_t = u_{xx} + \frac{1}{x^2}u.$$

CHAPTER VII

CONCLUSIONS

This thesis was devoted to the study of the equivalence problem for parabolic second-order partial differential equations with two independent variables. The results obtained are separated into three parts.

The first result is the form of parabolic second-order partial differential equations which are equivalent to a linear equation. It is proven that this form is an invariant with respect to a change of the dependent and independent variables.

The second result of the thesis is related with obtaining invariants of the linear second-order parabolic partial differential equations with respect to point transformations. Differential invariants of sixth and seventh-order are obtained.

The third result is devoted to the equivalence problem for a linear second-order parabolic partial differential equations to be equivalent to one of the canonical equations. Conditions which guarantee that the second-order parabolic differential equations is equivalent to one of the canonical forms are found.

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APPENDICES

APPENDIX A

COEFFICIENTS OF EQUATION (4.11)

$$
\begin{aligned}
a_1 = \; & \Lambda^{-1}(2H_{xu}H_uH_xV_uY_xb_1 - 2H_{xu}H_uH_xV_xY_ub_1 - H_{uu}H_x^2V_uY_xb_1 + H_{uu}H_x^2V_xY_ub_1 \\
& +H_u^3V_{xx}Y_xb_1 - H_u^3V_xY_{xx}b_1 + H_u^3V_xY_x^2b_2 + H_u^3Y_x^3b_3V - H_u^2H_{xx}V_uY_xb_1 \\
& +H_u^2H_{xx}V_xY_ub_1 - 2H_u^2H_xV_{xu}Y_xb_1 + H_u^2H_xV_uY_{xx}b_1 - H_u^2H_xV_uY_x^2b_2 \\
& -H_u^2H_xV_{xx}Y_ub_1 + 2H_u^2H_xV_xY_{xu}b_1 - 2H_u^2H_xV_xY_uY_xb_2 - 3H_u^2H_xY_uY_x^2b_3V \\
& +H_u^2V_uY_x^3 - H_u^2V_xY_uY_x^2 + 2H_uH_x^2V_{xu}Y_ub_1 + H_uH_x^2V_{uu}Y_xb_1 \\
& -2H_uH_x^2V_uY_{xu}b_1 + 2H_uH_x^2V_uY_uY_xb_2 - H_uH_x^2V_xY_{uu}b_1 + H_uH_x^2V_xY_u^2b_2 \\
& +3H_uH_x^2Y_u^2Y_xb_3V - 2H_uH_xV_uY_uY_x^2 + 2H_uH_xV_xY_u^2Y_x - H_x^3V_{uu}Y_ub_1 \\
& +H_x^3V_uY_{uu}b_1 - H_x^3V_uY_u^2b_2 - H_x^3Y_u^3b_3V + H_x^2V_uY_u^2Y_x - H_x^2V_xY_u^3), \\
a_2 = \; & \Lambda^{-1}(2H_{tu}H_tH_uV_tY_ub_1 - 2H_{tu}H_tH_uV_uY_tb_1 - H_{tt}H_u^2V_tY_ub_1 + H_{tt}H_u^2V_uY_tb_1 \\
& +H_t^3V_{uu}Y_ub_1 - H_t^3V_uY_{uu}b_1 + H_t^3V_uY_u^2b_2 + H_t^3Y_u^3b_3V - H_t^2H_{uu}V_tY_ub_1 \\
& +H_t^2H_{uu}V_uY_tb_1 - 2H_t^2H_uV_{tu}Y_ub_1 + H_t^2H_uV_tY_{uu}b_1 - H_t^2H_uV_tY_u^2b_2 \\
& -H_t^2H_uV_{uu}Y_tb_1 + 2H_t^2H_uV_uY_{tu}b_1 - 2H_t^2H_uV_uY_tY_ub_2 - 3H_t^2H_uY_tY_u^2b_3V \\
& +H_t^2V_tY_u^3 - H_t^2V_uY_tY_u^2 + 2H_tH_u^2V_{tu}Y_tb_1 + H_tH_u^2V_{tt}Y_ub_1 - 2H_tH_u^2V_tY_{tu}b_1 \\
& +2H_tH_u^2V_tY_tY_ub_2 - H_tH_u^2V_uY_{tt}b_1 + H_tH_u^2V_uY_t^2b_2 + 3H_tH_u^2Y_t^2Y_ub_3V \\
& -2H_tH_uV_tY_tY_u^2 + 2H_tH_uV_uY_t^2Y_u - H_u^3V_{tt}Y_tb_1 + H_u^3V_tY_{tt}b_1 - H_u^3V_tY_t^2b_2 \\
& -H_u^3Y_t^3b_3V + H_u^2V_tY_t^2Y_u - H_u^2V_uY_t^3), \\
a_3 = \; & \Lambda^{-1}(-2H_{tu}H_uH_xV_uY_xb_1 + 2H_{tu}H_uH_xV_xY_ub_1 + 2H_{tx}H_u^2V_uY_xb_1 \\
& -2H_{tx}H_u^2V_xY_ub_1 - 2H_tH_{xu}H_uV_uY_xb_1 + 2H_tH_{xu}H_uV_xY_ub_1 \\
& +2H_tH_{uu}H_xV_uY_xb_1 - 2H_tH_{uu}H_xV_xY_ub_1 + 2H_tH_u^2V_{xu}Y_xb_1 \\
& -H_tH_u^2V_uY_{xx}b_1 + H_tH_u^2V_uY_x^2b_2 + H_tH_u^2V_{xx}Y_ub_1 - 2H_tH_u^2V_xY_{xu}b_1
\end{aligned}
$$

$$
\begin{aligned}
& +2H_tH_u^2V_xY_uY_xb+3H_tH_u^2Y_uY_x^2b_3V-4H_tH_uH_xV_{xu}Y_ub_1 \\
& -2H_tH_uH_xV_{uu}Y_xb_1+4H_tH_uH_xV_uY_{xu}b_1-4H_tH_uH_xV_uY_uY_xb_2 \\
& +2H_tH_uH_xV_xY_{uu}b_1-2H_tH_uH_xV_xY_u^2b_2-6H_tH_uH_xY_u^2Y_xb_3V \\
& +2H_tH_uV_uY_uY_x^2-2H_tH_uV_xY_u^2Y_x+3H_tH_x^2V_{uu}Y_ub_1-3H_tH_x^2V_uY_{uu}b_1 \\
& +3H_tH_x^2V_uY_u^2b_2+3H_tH_x^2Y_u^3b_3V-2H_tH_xV_uY_u^2Y_x+2H_tH_xV_xY_u^3 \\
& +2H_{xu}H_uH_xV_tY_ub_1-2H_{xu}H_uH_xV_uY_tb_1-H_{uu}H_x^2V_tY_ub_1 \\
& +H_{uu}H_x^2V_uY_tb_1-2H_u^3V_{tx}Y_xb_1+H_u^3V_tY_{xx}b_1-H_u^3V_tY_x^2b_2-H_u^3V_{xx}Y_tb_1 \\
& +2H_u^3V_xY_{tx}b_1-2H_u^3V_xY_tY_xb_2-3H_u^3Y_tY_x^2b_3V-H_u^2H_{xx}V_tY_ub_1 \\
& +H_u^2H_{xx}V_uY_tb_1+2H_u^2H_xV_{tu}Y_xb_1+2H_u^2H_xV_{tx}Y_ub_1-2H_u^2H_xV_tY_{xu}b_1 \\
& +2H_u^2H_xV_tY_uY_xb_2+2H_u^2H_xV_{xu}Y_tb_1-2H_u^2H_xV_uY_{tx}b_1+2H_u^2H_xV_uY_tY_xb_2 \\
& -2H_u^2H_xV_xY_{tu}b_1+2H_u^2H_xV_xY_tY_ub_2+6H_u^2H_xY_tY_uY_xb_3V+H_u^2V_tY_uY_x^2 \\
& -3H_u^2V_uY_tY_x^2+2H_u^2V_xY_tY_uY_x-2H_uH_x^2V_{tu}Y_ub_1+H_uH_x^2V_tY_{uu}b_1 \\
& -H_uH_x^2V_tY_u^2b_2-H_uH_x^2V_{uu}Y_tb_1+2H_uH_x^2V_uY_{tu}b_1-2H_uH_x^2V_uY_tY_ub_2 \\
& -3H_uH_x^2Y_tY_u^2b_3V-2H_uH_xV_tY_u^2Y_x+4H_uH_xV_uY_tY_uY_x-2H_uH_xV_xY_tY_u^2 \\
& +H_x^2V_tY_u^3-H_x^2V_uY_tY_u^2), \\
a_4 = {} & \Lambda^{-1}(2H_{tu}H_tH_uV_uY_xb_1-2H_{tu}H_tH_uV_xY_ub_1-2H_{tu}H_uH_xV_tY_ub_1 \\
& +2H_{tu}H_uH_xV_uY_tb_1+2H_{tx}H_u^2V_tY_ub_1-2H_{tx}H_u^2V_uY_tb_1-H_{tt}H_u^2V_uY_xb_1 \\
& +H_{tt}H_u^2V_xY_ub_1-H_t^2H_{uu}V_uY_xb_1+H_t^2H_{uu}V_xY_ub_1+2H_t^2H_uV_{xu}Y_ub_1 \\
& +H_t^2H_uV_{uu}Y_xb_1-2H_t^2H_uV_uY_{xu}b_1+2H_t^2H_uV_uY_uY_xb_2-H_t^2H_uV_xY_{uu}b_1 \\
& +H_t^2H_uV_xY_u^2b_2+3H_t^2H_uY_u^2Y_xb_3V-3H_t^2H_xV_{uu}Y_ub_1+3H_t^2H_xV_uY_{uu}b_1 \\
& -3H_t^2H_xV_uY_u^2b_2-3H_t^2H_xY_u^3b_3V+H_t^2V_uY_u^2Y_x-H_t^2V_xY_u^3 \\
& -2H_tH_{xu}H_uV_tY_ub_1+2H_tH_{xu}H_uV_uY_tb_1+2H_tH_{uu}H_xV_tY_ub_1 \\
& -2H_tH_{uu}H_xV_uY_tb_1-2H_tH_u^2V_{tu}Y_xb_1-2H_tH_u^2V_{tx}Y_ub_1+2H_tH_u^2V_tY_{xu}b_1 \\
& -2H_tH_u^2V_tY_uY_xb_2-2H_tH_u^2V_{xu}Y_tb_1+2H_tH_u^2V_uY_{tx}b_1-2H_tH_u^2V_uY_tY_xb_2 \\
& +2H_tH_u^2V_xY_{tu}b_1-2H_tH_u^2V_xY_tY_ub_2-6H_tH_u^2Y_tY_uY_xb_3V \\
& -2H_tH_uH_xV_tY_{uu}b_1+2H_tH_uH_xV_tY_u^2b_2+2H_tH_uH_xV_{uu}Y_tb_1
\end{aligned}
$$

$$
\begin{aligned}
& +4H_tH_uH_xV_uY_tY_ub_2+6H_tH_uH_xY_tY_u^2b_3V+2H_tH_uV_tY_u^2Y_x \\
& +2H_tH_uV_xY_tY_u^2-2H_tH_xV_tY_u^3+2H_tH_xV_uY_tY_u^2+2H_u^3V_{tx}Y_tb_1+H_u^3V_{tt}Y_xb_1 \\
& -2H_u^3V_tY_{tx}b_1+2H_u^3V_tY_tY_xb_2-H_u^3V_xY_{tt}b_1+H_u^3V_xY_t^2b+3H_u^3Y_t^2Y_xb_3V \\
& -2H_u^2H_xV_{tu}Y_tb_1-H_u^2H_xV_{tt}Y_ub_1+2H_u^2H_xV_tY_{tu}b_1-2H_u^2H_xV_tY_tY_ub_2 \\
& +H_u^2H_xV_uY_{tt}b_1-H_u^2H_xV_uY_t^2b_2-3H_u^2H_xY_t^2Y_ub_3V-2H_u^2V_tY_tY_uY_x \\
& +3H_u^2V_uY_t^2Y_x-H_u^2V_xY_t^2Y_u+2H_uH_xV_tY_tY_u^2-2H_uH_xV_uY_t^2Y_u \\
& -4H_tH_uH_xV_uY_{tu}b_1-4H_tH_uV_uY_tY_uY_x+4H_tH_uH_xV_{tu}Y_ub_1), \\
a_5 = \;& \Lambda^{-1}(-2H_{tu}H_x^2V_uY_xb_1+2H_{tu}H_x^2V_xY_ub_1+2H_{tx}H_uH_xV_uY_xb_1 \\
& +2H_tH_{xu}H_xV_uY_xb_1-2H_tH_{xu}H_xV_xY_ub_1+3H_tH_u^2V_{xx}Y_xb_1-3H_tH_u^2V_xY_{xx}b_1 \\
& +3H_tH_u^2V_xY_x^2b_2+3H_tH_u^2Y_x^3b_3V-2H_tH_uH_{xx}V_uY_xb_1+2H_tH_uH_{xx}V_xY_ub_1 \\
& -4H_tH_uH_xV_{xu}Y_xb_1+2H_tH_uH_xV_uY_{xx}b_1-2H_tH_uH_xV_uY_x^2b_2 \\
& +4H_tH_uH_xV_xY_{xu}b_1-4H_tH_uH_xV_xY_uY_xb_2-6H_tH_uH_xY_uY_x^2b_3V \\
& -2H_tH_uV_xY_uY_x^2+2H_tH_x^2V_{xu}Y_ub_1+H_tH_x^2V_{uu}Y_xb_1-2H_tH_x^2V_uY_{xu}b_1 \\
& +2H_tH_x^2V_uY_uY_xb_2-H_tH_x^2V_xY_{uu}b_1+H_tH_x^2V_xY_u^2b_2+3H_tH_x^2Y_u^2Y_xb_3V \\
& -2H_tH_xV_uY_uY_x^2+2H_tH_xV_xY_u^2Y_x+2H_{xu}H_uH_xV_tY_xb_1-2H_{xu}H_uH_xV_xY_tb_1 \\
& -H_{uu}H_x^2V_tY_xb_1+H_{uu}H_x^2V_xY_tb_1-H_u^2H_{xx}V_tY_xb_1+H_u^2H_{xx}V_xY_tb_1 \\
& -2H_u^2H_xV_{tx}Y_xb_1+H_u^2H_xV_tY_{xx}b_1-H_u^2H_xV_tY_x^2b_2-H_u^2H_xV_{xx}Y_tb_1 \\
& +2H_u^2H_xV_xY_{tx}b_1-2H_u^2H_xV_xY_tY_xb_2-3H_u^2H_xY_tY_x^2b_3V+H_u^2V_tY_x^3 \\
& -H_u^2V_xY_tY_x^2+2H_uH_x^2V_{tu}Y_xb_1+2H_uH_x^2V_{tx}Y_ub_1-2H_uH_x^2V_tY_{xu}b_1 \\
& +2H_uH_x^2V_tY_uY_xb_2+2H_uH_x^2V_{xu}Y_tb_1-2H_uH_x^2V_uY_{tx}b_1+2H_uH_x^2V_uY_tY_xb_2 \\
& -2H_uH_x^2V_xY_{tu}b_1+2H_uH_x^2V_xY_tY_ub_2+6H_uH_x^2Y_tY_uY_xb_3V-2H_uH_xV_tY_uY_x^2 \\
& -2H_uH_xV_uY_tY_x^2+4H_uH_xV_xY_tY_uY_x-2H_x^3V_{tu}Y_ub_1+H_x^3V_tY_{uu}b_1 \\
& -H_x^3V_tY_u^2b_2-H_x^3V_{uu}Y_tb_1+2H_x^3V_uY_{tu}b_1-2H_x^3V_uY_tY_ub_2-3H_x^3Y_tY_u^2b_3V \\
& +H_x^2V_tY_u^2Y_x+2H_x^2V_uY_tY_uY_x-3H_x^2V_xY_tY_u^2-2H_{tx}H_uH_xV_xY_ub_1 \\
& -2H_tH_uH_xV_{xx}Y_ub_1+2H_tH_uV_uY_x^3),
\end{aligned}
$$

$$
\begin{aligned}
a_6 = \; & \Lambda^{-1}(2H_{tu}H_tH_uV_tY_xb_1 - 2H_{tu}H_tH_uV_xY_tb_1 + 2H_{tu}H_tH_xV_tY_ub_1 \\
& +2H_{tx}H_tH_uV_tY_ub_1 - 2H_{tx}H_tH_uV_uY_tb_1 - H_{tt}H_u^2V_tY_xb_1 + H_{tt}H_u^2V_xY_tb_1 \\
& -2H_{tt}H_uH_xV_tY_ub_1 + 2H_{tt}H_uH_xV_uY_tb_1 + 2H_t^3V_{xu}Y_ub_1 + H_t^3V_{uu}Y_xb_1 \\
& -2H_t^3V_uY_{xu}b_1 + 2H_t^3V_uY_uY_xb_2 - H_t^3V_xY_{uu}b_1 + H_t^3V_xY_u^2b_2 + 3H_t^3Y_u^2Y_xb_3V \\
& -2H_t^2H_{xu}V_tY_ub_1 + 2H_t^2H_{xu}V_uY_tb_1 - H_t^2H_{uu}V_tY_xb_1 + H_t^2H_{uu}V_xY_tb_1 \\
& -2H_t^2H_uV_{tu}Y_xb_1 - 2H_t^2H_uV_{tx}Y_ub_1 + 2H_t^2H_uV_tY_{xu}b_1 - 2H_t^2H_uV_tY_uY_xb_2 \\
& -2H_t^2H_uV_{xu}Y_tb_1 + 2H_t^2H_uV_uY_{tx}b_1 - 2H_t^2H_uV_uY_tY_xb_2 + 2H_t^2H_uV_xY_{tu}b_1 \\
& -2H_t^2H_uV_xY_tY_ub_2 - 6H_t^2H_uY_tY_uY_xb_3V - 2H_t^2H_xV_{tu}Y_ub_1 + H_t^2H_xV_tY_{uu}b_1 \\
& -H_t^2H_xV_tY_u^2b_2 - H_t^2H_xV_{uu}Y_tb_1 + 2H_t^2H_xV_uY_{tu}b_1 - 2H_t^2H_xV_uY_tY_ub_2 \\
& -3H_t^2H_xY_tY_u^2b_3V + 3H_t^2V_tY_u^2Y_x - 2H_t^2V_uY_tY_uY_x - H_t^2V_xY_tY_u^2 \\
& +2H_tH_u^2V_{tx}Y_tb_1 + H_tH_u^2V_{tt}Y_xb_1 - 2H_tH_u^2V_tY_{tx}b_1 + 2H_tH_u^2V_tY_tY_xb_2 \\
& -H_tH_u^2V_xY_{tt}b_1 + H_tH_u^2V_xY_t^2b_2 + 3H_tH_u^2Y_t^2Y_xb_3V + 4H_tH_uH_xV_{tu}Y_tb_1 \\
& +2H_tH_uH_xV_{tt}Y_ub_1 - 4H_tH_uH_xV_tY_{tu}b_1 + 4H_tH_uH_xV_tY_tY_ub_2 \\
& +2H_tH_uH_xV_uY_t^2b_2 + 6H_tH_uH_xY_t^2Y_ub_3V - 4H_tH_uV_tY_tY_uY_x \\
& +2H_tH_uV_xY_t^2Y_u - 2H_tH_xV_tY_tY_u^2 + 2H_tH_xV_uY_t^2Y_u - 3H_u^2H_xV_{tt}Y_tb_1 \\
& +3H_u^2H_xV_tY_{tt}b_1 - 3H_u^2H_xV_tY_t^2b_2 - 3H_u^2H_xY_t^3b_3V + H_u^2V_tY_t^2Y_x - H_u^2V_xY_t^3 \\
& +2H_uH_xV_tY_t^2Y_u - 2H_uH_xV_uY_t^3 - 2H_{tu}H_tH_xV_uY_tb_1 - 2H_tH_uH_xV_uY_{tt}b_1 \\
& +2H_tH_uV_uY_t^2Y_x), \\
a_7 = \; & 2\Lambda^{-1}(H_{tu}H_tH_xV_uY_xb_1 - H_{tu}H_tH_xV_xY_ub_1 - H_{tu}H_uH_xV_tY_xb_1 \\
& -H_{tu}H_x^2V_tY_ub_1 + H_{tu}H_x^2V_uY_tb_1 + H_{tx}H_tH_uV_uY_xb_1 - H_{tx}H_tH_uV_xY_ub_1 \\
& +H_{tx}H_u^2V_tY_xb_1 - H_{tx}H_u^2V_xY_tb_1 + H_{tx}H_uH_xV_tY_ub_1 - H_{tx}H_uH_xV_uY_tb_1 \\
& -H_{tt}H_uH_xV_uY_xb_1 + H_{tt}H_uH_xV_xY_ub_1 - H_t^2H_{xu}V_uY_xb_1 + H_t^2H_{xu}V_xY_ub_1 \\
& +2H_t^2H_uV_{xu}Y_xb_1 - H_t^2H_uV_uY_{xx}b_1 + H_t^2H_uV_uY_x^2b_2 + H_t^2H_uV_{xx}Y_ub_1- \\
& 2H_t^2H_uV_xY_{xu}b_1 + 2H_t^2H_uV_xY_uY_xb_2 + 3H_t^2H_uY_uY_x^2b_3V - 2H_t^2H_xV_{xu}Y_ub_1 \\
& -H_t^2H_xV_{uu}Y_xb_1 + 2H_t^2H_xV_uY_{xu}b_1 - 2H_t^2H_xV_uY_uY_xb_2 + H_t^2H_xV_xY_{uu}b_1 \\
& -H_t^2H_xV_xY_u^2b_2 - 3H_t^2H_xY_u^2Y_xb_3V + H_t^2V_uY_uY_x^2 - H_t^2V_xY_u^2Y_x
\end{aligned}
$$

$$-H_tH_{xu}H_uV_tY_xb_1 + H_tH_{xu}H_uV_xY_tb_1 + H_tH_{xu}H_xV_tY_ub_1 - H_tH_{xu}H_xV_uY_tb_1$$
$$+H_tH_{uu}H_xV_tY_xb_1 - H_tH_{uu}H_xV_xY_tb_1 - 2H_tH_u^2V_{tx}Y_xb_1 + H_tH_u^2V_tY_{xx}b_1$$
$$-H_tH_u^2V_tY_x^2b_2 - H_tH_u^2V_{xx}Y_tb_1 + 2H_tH_u^2V_xY_{tx}b_1 - 2H_tH_u^2V_xY_tY_xb_2$$
$$-3H_tH_u^2Y_tY_x^2b_3V - H_tH_uH_{xx}V_tY_ub_1 + H_tH_uH_{xx}V_uY_tb_1 + 2H_tH_uV_tY_uY_x^2$$
$$-2H_tH_uV_uY_tY_x^2 + 2H_tH_x^2V_{tu}Y_ub_1 - H_tH_x^2V_tY_{uu}b_1 + H_tH_x^2V_tY_u^2b_2$$
$$+H_tH_x^2V_{uu}Y_tb_1 - 2H_tH_x^2V_uY_{tu}b_1 + 2H_tH_x^2V_uY_tY_ub_2 + 3H_tH_x^2Y_tY_u^2b_3V$$
$$-2H_tH_xV_tY_u^2Y_x + 2H_tH_xV_xY_tY_u^2 + 2H_u^2H_xV_{tx}Y_tb_1 + H_u^2H_xV_{tt}Y_xb_1$$
$$-2H_u^2H_xV_tY_{tx}b_1 + 2H_u^2H_xV_tY_tY_xb_2 - H_u^2H_xV_xY_{tt}b_1 + H_u^2H_xV_xY_t^2b_2$$
$$+3H_u^2H_xY_t^2Y_xb_3V - H_u^2V_tY_tY_x^2 + H_u^2V_xY_t^2Y_x - 2H_uH_x^2V_{tu}Y_tb_1$$
$$-H_uH_x^2V_{tt}Y_ub_1 + 2H_uH_x^2V_tY_{tu}b_1 - 2H_uH_x^2V_tY_tY_ub_2 + H_uH_x^2V_uY_{tt}b_1$$
$$-H_uH_x^2V_uY_t^2b_2 - 3H_uH_x^2Y_t^2Y_ub_3V + 2H_uH_xV_uY_t^2Y_x - 2H_uH_xV_xY_t^2Y_u$$
$$+H_x^2V_tY_tY_u^2 - H_x^2V_uY_t^2Y_u + H_{tu}H_uH_xV_xY_tb_1),$$

$$a_8 = \Lambda^{-1}(-2H_{tu}H_x^2V_tY_xb_1 + 2H_{tu}H_x^2V_xY_tb_1 + 2H_{tx}H_tH_xV_uY_xb_1$$
$$+2H_{tx}H_uH_xV_tY_xb_1 - 2H_{tx}H_uH_xV_xY_tb_1 - H_{tt}H_x^2V_uY_xb_1 + H_{tt}H_x^2V_xY_ub_1$$
$$+3H_t^2H_uV_{xx}Y_xb_1 - 3H_t^2H_uV_xY_{xx}b_1 + 3H_t^2H_uV_xY_x^2b_2 + 3H_t^2H_uY_x^3b_3V$$
$$-H_t^2H_{xx}V_uY_xb_1 + H_t^2H_{xx}V_xY_ub_1 - 2H_t^2H_xV_{xu}Y_xb_1 + H_t^2H_xV_uY_{xx}b_1$$
$$-H_t^2H_xV_uY_x^2b_2 - H_t^2H_xV_{xx}Y_ub_1 + 2H_t^2H_xV_xY_{xu}b_1 - 2H_t^2H_xV_xY_uY_xb_2$$
$$-3H_t^2H_xY_uY_x^2b_3V + H_t^2V_uY_x^3 - H_t^2V_xY_uY_x^2 + 2H_tH_{xu}H_xV_tY_xb_1$$
$$-2H_tH_{xu}H_xV_xY_tb_1 - 2H_tH_uH_{xx}V_tY_xb_1 + 2H_tH_uH_{xx}V_xY_tb_1$$
$$+2H_tH_uH_xV_tY_{xx}b_1 - 2H_tH_uH_xV_tY_x^2b_2 - 2H_tH_uH_xV_{xx}Y_tb_1$$
$$-4H_tH_uH_xV_xY_tY_xb_2 - 6H_tH_uH_xY_tY_x^2b_3V + 2H_tH_uV_tY_x^3 - 2H_tH_uV_xY_tY_x^2$$
$$+2H_tH_x^2V_{tu}Y_xb_1 + 2H_tH_x^2V_{tx}Y_ub_1 - 2H_tH_x^2V_tY_{xu}b_1 + 2H_tH_x^2V_tY_uY_xb_2$$
$$+2H_tH_x^2V_{xu}Y_tb_1 - 2H_tH_x^2V_uY_{tx}b_1 + 2H_tH_x^2V_uY_tY_xb_2 - 2H_tH_x^2V_xY_{tu}b_1$$
$$+2H_tH_x^2V_xY_tY_ub_2 + 6H_tH_x^2Y_tY_uY_xb_3V - 2H_tH_xV_tY_uY_x^2 - 2H_tH_xV_uY_tY_x^2$$
$$+4H_tH_xV_xY_tY_uY_x + 2H_uH_x^2V_{tx}Y_tb_1 + H_uH_x^2V_{tt}Y_xb_1 - 2H_uH_x^2V_tY_{tx}b_1$$
$$+2H_uH_x^2V_tY_tY_xb - H_uH_x^2V_xY_{tt}b_1 + H_uH_x^2V_xY_t^2b_2 + 3H_uH_x^2Y_t^2Y_xb_3V$$

$$
\begin{aligned}
& -2H_uH_xV_tY_tY_x^2 + 2H_uH_xV_xY_t^2Y_x - 2H_x^3V_{tu}Y_tb_1 - H_x^3V_{tt}Y_ub_1 + 2H_x^3V_tY_{tu}b_1 \\
& -2H_x^3V_tY_tY_ub_2 + H_x^3V_uY_{tt}b_1 - H_x^3V_uY_t^2b_2 - 3H_x^3Y_t^2Y_ub_3V + 2H_x^2V_tY_tY_uY_x \\
& +H_x^2V_uY_t^2Y_x - 3H_x^2V_xY_t^2Y_u - 2H_{tx}H_tH_xV_xY_ub_1 - 4H_tH_uH_xV_{tx}Y_xb_1 \\
& +4H_tH_uH_xV_xY_{tx}b_1), \\
a_9 = \ & \Lambda^{-1}(2H_{tu}H_tH_xV_tY_xb_1 - 2H_{tu}H_tH_xV_xY_tb_1 + 2H_{tx}H_tH_uV_tY_xb_1 \\
& +2H_{tx}H_tH_xV_tY_ub_1 - 2H_{tx}H_tH_xV_uY_tb_1 - 2H_{tt}H_uH_xV_tY_xb_1 \\
& -H_{tt}H_x^2V_tY_ub_1 + H_{tt}H_x^2V_uY_tb_1 + 2H_t^3V_{xu}Y_xb_1 - H_t^3V_uY_{xx}b_1 + H_t^3V_uY_x^2b_2 \\
& +H_t^3V_{xx}Y_ub_1 - 2H_t^3V_xY_{xu}b_1 + 2H_t^3V_xY_uY_xb_2 + 3H_t^3Y_uY_x^2b_3v \\
& +2H_t^2H_{xu}V_xY_tb_1 - 2H_t^2H_uV_{tx}Y_xb_1 + H_t^2H_uV_tY_{xx}b_1 - H_t^2H_uV_tY_x^2b_2 \\
& -H_t^2H_uV_{xx}Y_tb_1 + 2H_t^2H_uV_xY_{tx}b_1 - 2H_t^2H_uV_xY_tY_xb_2 - 3H_t^2H_uY_tY_x^2b_3V \\
& -H_t^2H_{xx}V_tY_ub_1 + H_t^2H_{xx}V_uY_tb_1 - 2H_t^2H_xV_{tu}Y_xb_1 - 2H_t^2H_xV_{tx}Y_ub_1 \\
& +2H_t^2H_xV_tY_{xu}b_1 - 2H_t^2H_xV_tY_uY_xb_2 - 2H_t^2H_xV_{xu}Y_tb_1 + 2H_t^2H_xV_uY_{tx}b_1 \\
& -2H_t^2H_xV_uY_tY_xb_2 + 2H_t^2H_xV_xY_{tu}b_1 - 2H_t^2H_xV_xY_tY_ub_2 - 6H_t^2H_xY_tY_uY_xb_3V \\
& +3H_t^2V_tY_uY_x^2 - H_t^2V_uY_tY_x^2 - 2H_t^2V_xY_tY_uY_x + 4H_tH_uH_xV_{tx}Y_tb_1 \\
& +2H_tH_uH_xV_{tt}Y_xb_1 - 4H_tH_uH_xV_tY_{tx}b_1 + 4H_tH_uH_xV_tY_tY_xb_2 \\
& +2H_tH_uH_xV_xY_t^2b_2 + 6H_tH_uH_xY_t^2Y_xb_3V - 2H_tH_uV_tY_tY_x^2 + 2H_tH_uV_xY_t^2Y_x \\
& +2H_tH_x^2V_{tu}Y_tb_1 + H_tH_x^2V_{tt}Y_ub_1 - 2H_tH_x^2V_tY_{tu}b_1 + 2H_tH_x^2V_tY_tY_ub_2 \\
& -H_tH_x^2V_uY_{tt}b_1 + H_tH_x^2V_uY_t^2b_2 + 3H_tH_x^2Y_t^2Y_ub_3V - 4H_tH_xV_tY_tY_uY_x \\
& +2H_tH_xV_uY_t^2Y_x + 2H_tH_xV_xY_t^2Y_u - 3H_uH_x^2V_{tt}Y_tb_1 + 3H_uH_x^2V_tY_{tt}b_1 \\
& -3H_uH_x^2V_tY_t^2b_2 - 3H_uH_x^2Y_t^3b_3V + 2H_uH_xV_tY_t^2Y_x - 2H_uH_xV_xY_t^3 \\
& +H_x^2V_tY_t^2Y_u - H_x^2V_uY_t^3 - 2H_{tx}H_tH_uV_xY_tb_1 + 2H_{tt}H_uH_xV_xY_tb_1 \\
& -2H_t^2H_{xu}V_tY_xb_1 - 2H_tH_uH_xV_xY_{tt}b_1),
\end{aligned}
$$

$$\begin{aligned}
a_{10} = \quad & \Lambda^{-1}(2H_{tx}H_tH_xV_tY_xb_1 - 2H_{tx}H_tH_xV_xY_tb_1 - H_{tt}H_x^2V_tY_xb_1 + H_{tt}H_x^2V_xY_tb_1 \\
& +H_t^3V_{xx}Y_xb_1 - H_t^3V_xY_{xx}b_1 + H_t^3V_xY_x^2b_2 + H_t^3Y_x^3b_3V - H_t^2H_{xx}V_tY_xb_1 \\
& +H_t^2H_{xx}V_xY_tb_1 - 2H_t^2H_xV_{tx}Y_xb_1 + H_t^2H_xV_tY_{xx}b_1 - H_t^2H_xV_tY_x^2b_2 \\
& -H_t^2H_xV_{xx}Y_tb_1 + 2H_t^2H_xV_xY_{tx}b_1 - 2H_t^2H_xV_xY_tY_xb_2 - 3H_t^2H_xY_tY_x^2b_3V \\
& +H_t^2V_tY_x^3 - H_t^2V_xY_tY_x^2 + 2H_tH_x^2V_{tx}Y_tb_1 + H_tH_x^2V_{tt}Y_xb_1 - 2H_tH_x^2V_tY_{tx}b_1 \\
& +2H_tH_x^2V_tY_tY_xb_2 - H_tH_x^2V_xY_{tt}b_1 + H_tH_x^2V_xY_t^2b_2 + 3H_tH_x^2Y_t^2Y_xb_3V \\
& -2H_tH_xV_tY_tY_x^2 + 2H_tH_xV_xY_t^2Y_x - H_x^3V_{tt}Y_tb_1 + H_x^3V_tY_{tt}b_1 - H_x^3V_tY_t^2b_2 \\
& -H_x^3Y_t^3b_3V + H_x^2V_tY_t^2Y_x - H_x^2V_xY_t^3),
\end{aligned}$$

$$\begin{aligned}
a_{11} &= H_x^2b_1, & a_{12} &= H_u^2b_1, & a_{13} &= -2H_uH_xb_1, \\
a_{14} &= 2H_tH_ub_1, & a_{15} &= -2H_tH_xb_1, & a_{16} &= H_t^2b_1.
\end{aligned}$$

where it assumed the jacobian

$$\Lambda = (H_tV_uY_x - H_tV_xY_u - H_uV_tY_x + H_uV_xY_t + H_xV_tY_u - H_xV_uY_t).$$

of (4.2) is not equal to zero.

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