# เวฟเล็ตต่อเนื่องที่สัมพันธ์กับกลุ่มที่ก่อกำเนิดด้วยเมทริกซ์สองเมทริกซ์

นางสาวพิสมัย กิตติภูมิ

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# CONTINUOUS WAVELETS ASSOCIATED WITH GROUPS GENERATED BY TWO MATRICES

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# CONTINUOUS WAVELETS ASSOCIATED WITH GROUPS GENERATED BY TWO MATRICES

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for a Master's Degree

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**พิสมัย กิตติภูมิ : เวฟเล็ตต่อเนื่องที่สัมพันธ์กับกลุ่มที่ก่อกำเนิดด้วยเมทริกซ์สองเมทริกซ์ (CONTINUOUS WAVELETS ASSOCIATED WITH GROUPS GENERATED BY TWO MATRICES)**
**อ. ที่ปรึกษา: ASST.PROF.DR. ECKART SCHULZ, 114 หน้า**
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วิทยานิพนธ์นี้ศึกษากลุ่มของเมทริกซ์ที่ก่อกำเนิดด้วยเมทริกซ์สองเมทริกซ์เพื่อหาเงื่อนไขที่ทำให้ยอมรับกลุ่มของเมทริกซ์ดังกล่าวเป็นกลุ่มที่มีการเปลี่ยนขนาดสำหรับการแปลงเวฟเล็ตแบบต่อเนื่อง สำหรับกลุ่มของเมทริกซ์ที่ก่อกำเนิดด้วยเมทริกซ์สองเมทริกซ์ที่สลับที่กันได้ซึ่งมีมิติสองและสามมิติ การศึกษานี้ทำให้ได้เงื่อนไขที่ครบถ้วนที่กลุ่มดังกล่าวสามารถเป็นกลุ่มที่ยอมรับได้ และได้นำเสนอกลุ่มที่ยอมรับได้บางกลุ่มที่มีมิติมากกว่าสามมิติ เช่นเดียวกับกลุ่มของเมทริกซ์ที่ก่อกำเนิดด้วยเมทริกซ์สองเมทริกซ์ที่สลับที่กันไม่ได้ โดยเสนอเพียงบางตัวอย่างบนกลุ่มของเมทริกซ์ที่มีมิติสองมิติ ในท้ายสุดได้สร้างฟังก์ชันเวฟเล็ตที่มีคุณสมบัติปรับเรียบและมีค่าใกล้ศูนย์ที่อนันต์

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา________________

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**PISAMAI KITTIPOOM : CONTINUOUS WAVELETS ASSOCIATED WITH GROUPS GENERATED BY TWO MATRICES**

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ADMISSIBLE GROUP / ADMISSIBLE FUNCTION / CROSS-SECTION / TWO-PARAMETER MATRIX GROUP / WAVELET RECONSTRUCTION / WAVELET TRANSFORM

This thesis discusses what matrix groups with two generators are admissible as dilation groups for the continuous wavelet transform. In two and three dimensions we obtain a complete characterization of the admissible two-parameter matrix groups, and in higher dimensions, we exhibit a large class of admissible two-parameter groups. We then present a large class of admissible, non-commutative groups in two-dimensions. Finally, in two dimensions we construct wavelets with nice smoothness and vanishing properties.

School of Mathematics　　　　Signature of Student ____________

Academic Year 2003　　　　Signature of Advisor ____________

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Pisamai Kittipoom

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# Chapter I

# Introduction

Wavelet theory originated from signal theory which has the goal to represent functions that are local in time and frequency. The classical method used in signal theory is the Fourier transform which transforms a signal function $f(t)$ in the time domain to another function $\hat{f}(\omega)$ in the frequency domain.

Assuming that $f$ is integrable, its Fourier transform is given by

$$Ff(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-2i\pi\omega t}\, dt. \tag{1.1}$$

If $\hat{f}$ is also integrable, then we can reconstruct $f$ from its Fourier transform $\hat{f}$ by

$$\overline{F}\hat{f}(t) = f(t) = \int_{\mathbb{R}} \hat{f}(\omega)e^{2i\pi t\omega} d\omega \tag{1.2}$$

at all points where $f$ is continuous. The integral (1.2) is called the inverse Fourier transform of $\hat{f}$. Additional details and proofs can be founded in Gasquet and Witomski (1998).

The Fourier transform provides information on the frequency content over the whole duration of the signal. It does not tell what frequencies occur when in time. To overcome this disadvantage, Gabor (1946) modified the Fourier transform by multiplying the signal function with a translated window function $\psi$

$$Gf(\omega, b) = \int_{\mathbb{R}} f(t)\overline{\psi(t-b)}e^{-2i\pi\omega t}dt. \tag{1.3}$$

This method is known as the windowed Fourier transform. One can shift this time window to any point $t$ in time by means of the translation parameter $b$.

Gabor showed that if $\psi \in L^2(\mathbb{R})$, then $f$ can be recovered from its windowed Fourier transform by the weak integral

$$f(t) = \frac{1}{||\psi||^2} \iint_{\mathbb{R}^2} Gf(\omega, b)\psi(t-b)e^{2i\pi\omega t} d\omega db. \tag{1.4}$$

Equation (1.4) is called the inversion formula for the windowed Fourier transform.

In general, the window function $\psi$ should be smooth and resemble a characteristic function $\psi := \chi_{[-A,A]}$ closely, so that $f$ is windowed by the time window $b - A \leq t \leq b + A$ and its transform $Gf(\omega, b)$ provides information about the frequency decomposition of $f$ on that time window.

However, this method still has a drawback in that the size of the window is fixed, and since this fixed window is used for all frequencies in the transformation, there is a limit to how well the signal can be localized in time.

In 1984, Grossmann and Morlet defined an integral transform which is now often called as the wavelet transform. It is similar to the windowed Fourier transform but uses a two-parameter family of functions $\psi_{a,b}(t) = |a|^{-1/2}\psi(\frac{t}{a} - b)$, comprising both translations by real numbers $b$ and dilations by positive real numbers $a$. The wavelet transform of a square integrable function $f$ is given by

$$Wf(a, b) = |a|^{-1/2} \int_{\mathbb{R}} f(t)\overline{\psi\left(\frac{t}{a} - b\right)} dt. \tag{1.5}$$

This wavelet transform can be expressed as an inner product $Wf(a, b) = \langle f, \psi_{a,b} \rangle$ in $L^2(\mathbb{R})$, and Grossmann and Morlet (1984) showed that it is directly related with the theory of group representations as follows.

Let $G$ be a locally compact group whose left Haar measure is $\mu$, and let $\pi_g : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Given a vector $\psi \in \mathcal{H}$, the collection $\{\pi_g\psi\}$ of vectors in $\mathcal{H}$ is called a family of wavelets and $\psi$ is called the mother wavelet. The mapping $W_\psi$, taking $f \in \mathcal{H}$ to a continuous function on $G$, defined by

$$(W_\psi f)(g) = \langle f, \pi_g \psi \rangle_{\mathcal{H}} \tag{1.6}$$

is called the wavelet transform of $f$ with respect to $\psi$.

One of the goals in wavelet theory is to investigate under which conditions reconstruction formulas, or inverse wavelet transforms, exist. By using Duflo and Moore's theory of square integrable representations, Grossmann and Morlet (1984) showed that if the representation $\pi$ is square integrable, then there exists a dense set of vectors $\psi$ in $\mathcal{H}$ such that

$$||W_\psi f||_{L^2(G)} = \sqrt{c_\psi}||f||_{\mathcal{H}} \tag{1.7}$$

for some constant $c_\psi$, i.e. the wavelet transform $W_\psi$ associated with $\psi$ is a multiple of an isometry mapping $\mathcal{H}$ into $L^2(G)$. Equality (1.7) then gives a reconstruction formula which is valid as a weak integral, by

$$f = \frac{1}{c_\psi} \int_G (W_\psi f)(g)\, \pi_g \psi \, d\mu(g). \tag{1.8}$$

Initially, numerous authors have studied the theory of the wavelet transform on $L^2(\mathbb{R})$ as defined by (1.5) from the point of view of square integrable group representations [Grossmann and Morlet (1984), Daubechies (1992), Heil and Walnut (1989)]. Later on, the generalization to higher dimensions was explored by many authors [Bernier and Taylor (1996), Führ (1996), Weiss et al. (2002)]. As a generalization of (1.5), these authors considered the affine group $G^\natural$, formed as the semi-direct product of $\mathbb{R}^n$ with $GL_n(\mathbb{R})$ and denoted by $GL_n(\mathbb{R}) \rtimes \mathbb{R}^n$, and the unitary representation $\pi$ of $G^\natural$ on $L^2(\mathbb{R}^n)$ given by translations and dilations:

$$\pi_{(a,\vec{b})}\psi(\vec{x}) = |\det a|^{-1/2}\psi(a^{-1}\vec{x} - \vec{b}) \equiv \psi_{a,\vec{b}}(\vec{x}) \tag{1.9}$$

where $(a,\vec{b}) \in GL_n(\mathbb{R}) \rtimes \mathbb{R}^n$ and $\psi \in L^2(\mathbb{R}^n)$. The wavelet transform of $f \in$

$L^2(\mathbb{R}^n)$ with respect to the mother wavelet $\psi$ becomes

$$(W_\psi f)(a,\vec{b}) = \langle f, \pi_{(a,\vec{b})}\psi\rangle = \frac{1}{\sqrt{|\det a|}} \int\limits_{\mathbb{R}^n} f(\vec{x})\overline{\psi(a^{-1}\vec{x}-\vec{b})}d\vec{x}. \qquad (1.10)$$

The question now is: Given a closed subgroup $D$ of $GL_n(\mathbb{R})$ and the corresponding closed subgroup $G = D \rtimes \mathbb{R}^n$ of $G^\sharp$, is reconstruction formula (1.8) valid?

To answer this question, one usually works with the Fourier transform. Since the Fourier transform $F : f \to \hat{f}$ constitutes a unitary operator on $L^2(\mathbb{R}^n)$, the map

$$\rho = F\pi\overline{F} \qquad (1.11)$$

is also a representation of $G^\sharp$ on $L^2(\mathbb{R}^n)$, and it turns out that

$$\rho_{(a,\vec{b})}\widehat{\psi}(\vec{\gamma}) = |\det\ a|^{1/2}e^{-2i\pi\vec{\gamma}\cdot a\vec{b}}\widehat{\psi}(\vec{\gamma}a) \equiv \widehat{\psi}_{a,\vec{b}}(\vec{\gamma}) \qquad (1.12)$$

for all $\psi \in L^2(\mathbb{R}^n)$, where elements $\vec{\gamma}$ of $\mathbb{R}^n$ are now written as row vectors. Formula (1.5) for the wavelet transform becomes

$$(W_\psi f)(a,\vec{b}) = \langle \hat{f}, \rho_{(a,\vec{b})}\widehat{\psi}\rangle = \int\limits_{\mathbb{R}^n} \hat{f}(\vec{\gamma})\overline{\widehat{\psi}(\vec{\gamma}a)}e^{2i\pi\vec{\gamma}\cdot a\vec{b}}|\det\ a|^{1/2}d\vec{\gamma}. \qquad (1.13)$$

Bernier and Taylor (1996) proved that if there exists an open free $D$-orbit $U$ in $\mathbb{R}^n$, then the representation $\rho_U$ of $G$ obtained by restricting $\rho$ to the subspace $L^2(U)$ of $L^2(\mathbb{R}^n)$ is square integrable. Thus, the restriction $\pi_U$ of $\pi$ to $\mathcal{H}_U = \overline{F}(L^2(U))$ is also square integrable, so that Duflo-Moore's theorem applies. In particular, there exists a dense set of functions $\psi$ in $\mathcal{H}_U$ so that reconstruction formula (1.8) is valid for all functions $f$ in $\mathcal{H}_U$. Note that open free $D$-orbits $U$ in $\mathbb{R}^n$ can only exist if the group $D$ has the structure of an $n$-manifold which limits the choices of $D$.

Führ (1996) generalized this to open orbits $U$ which are not necessary free, and proved that $\rho_U$ is square integrable if and only if the stabilizers associated to points of $U$ are compact.

For a long time it was not clear whether square integrability of the representation $\pi$ was required for the validity of the reconstruction formula. Recently a numbers of authors [Schulz and Taylor (preprint), Weiss et al. (2002)] showed that this is not required and proved the existence of the reconstruction formula in a variety of cases where $\pi$ is not square integrable.

In general, given a dilation subgroup $D$ of $GL_n(\mathbb{R})$, the reconstruction formula (1.8) exists for all $f \in L^2(\mathbb{R}^n)$ if and only if the mean square values of $\widehat{\psi}$ over the orbits are essentially identical, that is,

$$\int_D |\widehat{\psi}(\vec{x}a)|^2 \, d\mu(a) = \text{const} \tag{1.14}$$

for almost all $\vec{x} \in \mathbb{R}^n$. We call such a $\psi$ an admissible function, and we call the dilation group $D$ admissible if there exists $\psi \in L^2(\mathbb{R}^n)$ satisfying this condition.

In the preprint by Schulz and Taylor, the dilation subgroup $D$ of $GL_n(\mathbb{R})$ is of the simplest form, namely the image of a continuous or discrete one-parameter matrix group. That is,

$$D = \{A^t \equiv e^{tB} : t \in \mathbb{R}, B \in M_n(\mathbb{R})\}$$

or

$$D = \{A^k : k \in \mathbb{Z}, A \in GL_n(\mathbb{R})\}$$

respectively. The corresponding representation $\pi$ of $D \rtimes \mathbb{R}^n$ is never square integrable when $n \geq 2$. However, by analyzing the orbits for the discrete action $\vec{x} \mapsto \vec{x}A^k$ on $\mathbb{R}^n$, they have shown that there exists a cross-section $S$ of finite measure if and only if $|\det A| \neq 1$ and that the characteristic function $\chi_S$ of $S$ is admissible for both the discrete and where defined, the continuous one-parameter group. Furthermore, they have shown that $D$ is admissible if and only if $|\det A| \neq 1$.

Weiss et al. (2002) have proved necessary and some sufficient conditions for a general matrix group $D$ to be admissible as follows:

a) If $D$ is admissible, then there exists $a \in D$ such that $\Delta(a) \neq |\det(a)|$ where $\Delta$ denotes the modular function on $D$. Furthermore, the stabilizer of $\vec{x}$ is compact for almost every $\vec{x} \in \mathbb{R}^n$.

b) Conversely, if there exists $a \in D$ such that $\Delta(a) \neq |\det(a)|$ and if there exist compact local stabilizers, that is, for a.e. $\vec{x} \in \mathbb{R}^n$ $\exists \epsilon \geq 0$ such that the $\epsilon$-stabilizer $D^{\epsilon}_{\vec{x}}$ of $\vec{x}$ is compact, then $D$ is admissible.

In this thesis, we extend the results by Schulz and Taylor to groups generated by two matrices. Furthermore, we construct explicit admissible functions with nice smoothness properties.

More precisely, we study the following matrix groups $D$.

**Case 1**: $D$ is the image of a discrete two-parameter group of matrices. That is, there exist commuting invertible matrices $A$ and $B$ such that

$$D = \{A^k B^l : \ k, l \in \mathbb{Z}\}.$$

**Case 2**: $D$ is the image of a continuous two-parameter group of matrices. That is, there exist commuting matrices $A$ and $B$ such that

$$D = \{A^s B^t : \ s, t \in \mathbb{R}\}$$

where $A = e^M$ and $B = e^N$ for some $M, N \in M_n(\mathbb{R})$.

**Case 3**: $D$ is a non-abelian group of $2 \times 2$ matrices.

By using the results by Weiss et al. (2002) we investigate the existence of admissible functions in each of the above cases. In low dimensions we obtain a complete characterization. Moreover, in two dimensions we are able to construct admissible functions which are smooth and vanish at infinity.

This thesis is divided into 4 chapters as follows. Chapter II mainly introduces the notation and provides references to well known facts on the wavelet transform. Chapter III deals with the abstract characterization of admissible groups of Weiss et al. (2002). Chapter IV contains the main results, and discusses the existence of admissible functions for matrix groups generated by two matrices in general, and of smooth and rapidly vanishing admissible functions in two dimensions.

# Chapter II

# The Continuous Wavelet Transform

This chapter is devoted to the theoretical background of the continuous wavelet transform which will be used throughout this thesis. With a few exceptions all results are stated without proof. Additional details and proofs can be found in Apostol (1997), Cohn (1980), Folland (1999), Jones (1993), and Gasquet and Witomski (1998).

## 2.1 Topological Spaces

**Definition 2.1.** Let $X$ be a set. A *topology* on $X$ is a family $\mathcal{O}$ of subsets of $X$ such that

1. $X \in \mathcal{O}$, $\emptyset \in \mathcal{O}$,
2. if $\mathcal{F}$ is an arbitrary collection of sets that belong to $\mathcal{O}$, then $\bigcup \mathcal{F} \in \mathcal{O}$, and
3. if $\mathcal{F}$ is a finite collection of sets that belong to $\mathcal{O}$, then $\bigcap \mathcal{F} \in \mathcal{O}$.

A *topological space* is a pair $(X, \mathcal{O})$, where $X$ is a set and $\mathcal{O}$ is a topology on $X$. The sets belonging to $\mathcal{O}$ are called open sets.

**Definition 2.2.** Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is called *continuous* if $f^{-1}(U)$ is an open subset of $X$ whenever $U$ is an open subset of $Y$.

A function $f : X \to Y$ is called a *homeomorphism* if it is a bijection such that $f$ and $f^{-1}$ are both continuous. The spaces $X$ and $Y$ are *homeomorphic* if there exists a homeomorphism of $X$ onto $Y$.

**Definition 2.3.** A topological space $X$ is *Hausdorff* if for each pair $x, y$ of distinct points in $X$ there exist open sets $U, V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

**Definition 2.4.** Let $A$ be a subset of the topological space $X$. An *open cover* of $A$ is a collection $\mathcal{F}$ of open subsets of $X$ such that $A \subset \bigcup \mathcal{F}$. A *subcover* of an open cover $\mathcal{F}$ is a subfamily of $\mathcal{F}$ that is itself an open cover of $A$. ,

The set $A$ is *compact* if each open cover of $A$ has a finite subcover.

A compact set $A \subset X$ is called a *compact neighborhood* of $x \in X$ if $\exists\ \mathcal{O} \subset X$ open such that $x \in \mathcal{O} \subset A$.

**Definition 2.5.** A topological space $X$ is called *locally compact* if it is Hausdorff, and each of its points has a compact neighborhood.

A topological space is *$\sigma$-compact* if it is the union of a countable collection of compact sets.

Note that $\mathbb{R}^n$ is both locally compact and $\sigma$-compact in the usual topology.

## 2.2 The Lebesgue Integral

**Definition 2.6.** Let $X$ be a set. A collection $\mathcal{M}$ of subsets of $X$ is called a *$\sigma$-algebra* if the following hold:

1. $\emptyset \in \mathcal{M}$, $X \in \mathcal{M}$,

2. $S \in \mathcal{M}$ implies $X \setminus S \in \mathcal{M}$,

3. $S_1, S_2, \cdots \in \mathcal{M}$ implies $\bigcup_{n=1}^{\infty} S_n \in \mathcal{M}$.

The elements of $\mathcal{M}$ are called *measurable sets* and the pair $(X, \mathcal{M})$ is called *a measurable space.*

**Definition 2.7.** Let $\mathcal{F}$ be a collection of subsets of $X$. There exists a smallest $\sigma$-algebra containing $\mathcal{F}$, called the *$\sigma$-algebra generated by $\mathcal{F}$.*

**Definition 2.8.** Let $X$ be a topological space. The $\sigma$-algebra generated by the family of open sets $\mathcal{O}$ is called the *Borel $\sigma$-algebra* on $X$, denoted $\mathcal{B}_X$. Its elements are called *$\mathcal{B}$-measurable sets* or *Borel sets.*

**Definition 2.9.** Let $\mathcal{M}$ be a $\sigma$-algebra of subsets of $X$. *A measure* on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ having the following properties:

1. $\mu(\emptyset) = 0$,

2. if $\{E_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable set, then

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n).$$

The triple $(X, \mathcal{M}, \mu)$ is called *a measure space.*

A measure space $(X, \mathcal{M}, \mu)$ is called *complete* if whenever $E \subset A \in \mathcal{M}$ and $\mu(A) = 0$, then $E \in \mathcal{M}$ (and therefore $\mu(E) = 0$).

$(X, \mathcal{M}, \mu)$ is called a *$\sigma$-finite measure space* if there exists a countable collection $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ such that $\mu(E_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} E_n$.

The measure which we will be working with is the Lebesgue measure on $\mathbb{R}^n$, defined as follows.

**Definition 2.10.** An *n-dimensional interval* in $\mathbb{R}^n$ is defined by

$$I = I_1 \times I_2 \times \cdots \times I_n$$

where $I_1, I_2, \ldots, I_n$ are intervals in $\mathbb{R}$. $I$ is called *open (or closed)* if each $I_i$ is open (or closed) in $\mathbb{R}$. If $I$ is bounded, then its *n-dimensional volume* is defined by

$$\text{vol}(I) = \prod_{i=1}^{n} (b_i - a_i)$$

where $a_i, b_i$ are the left and right end points of $I_i$.

**Definition 2.11.** Let $E \subset \mathbb{R}^n$ be an arbitrary set. Then

$$\lambda^*(E) = \inf\left\{\sum_{i=1}^{\infty} \text{vol}(I_i) : I_i \text{ is an open } n\text{-interval}, E \subset \bigcup_{i=1}^{\infty} I_i\right\}$$

is called the *Lebesgue outer measure of* $E$. A set $A \subset \mathbb{R}^n$ is called $\lambda^*$*-measurable* if for every $E \subset \mathbb{R}^n$,

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^c).$$

**Proposition 2.1.** *Let* $\mathcal{M}_\lambda = \{A \subset \mathbb{R}^n : A \text{ is } \lambda^*\text{-measurable}\}$ *and set* $\lambda(A) = \lambda^*(A)\ \forall A \in \mathcal{M}_\lambda$. *Then*

1. $\mathcal{M}_\lambda$ *is a* $\sigma$*-algebra,*

2. $(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$ *is a complete measure space,*

3. $\mathcal{B}_{\mathbb{R}^n} \subset \mathcal{M}_\lambda$.

We note that $\lambda$ is called the *Lebesgue measure* and $\mathcal{M}_\lambda$ the set of *Lebesgue measurable subsets of* $\mathbb{R}^n$. Next we define the notions of measurable function and Lebesgue integral.

**Definition 2.12.** Let $(X, \mathcal{M}, \mu)$ be a measure space, $S \subset X$ with $S \in \mathcal{M}$. Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. Then a function $f : S \to \mathbb{R}^*$ is called a $\mathcal{M}$*-measurable function* if $\forall\, t \in \mathbb{R}$,

$$\{x \in X : f(x) \leq t\} \in \mathcal{M}.$$

A complex-valued function $f : S \to \mathbb{C}$ is called a $\mathcal{M}$*-measurable function* if $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\mathcal{M}$-measurable.

**Definition 2.13.** (Lebesgue integral): Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. Let $\varphi = \sum\limits_{k=1}^{n} \alpha_k \chi_{A_k}$ where $\chi_{A_k}(x) = \begin{cases} 1 & if\ x \in A_k \\ 0 & if\ x \notin A_k \end{cases}$, $A_k \in \mathcal{M}$ are disjoint, $\alpha_k \geq 0$. $\varphi$ is called a *simple measurable function.* Its integral is defined to be

$$\int\limits_X \varphi \, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k).$$

This integral is independent of the choice of the sets $A_k$.

2. Let $f : X \to [0, \infty]$ be $\mathcal{M}$-measurable. By the structure theorem for measurable functions, there exists an increasing sequence $\{\varphi_n\}$ of non-negative finite-valued measurable simple functions converging pointwise to $f$. We define the integral of $f$ by

$$\int\limits_X f \, d\mu = \lim_{n \to \infty} \int\limits_X \varphi_n d\mu.$$

This integral is independent of the choice of the functions $\varphi_n$.

3. Let $f : X \to \mathbb{R}^*$ be Lebesgue measurable and set $f^+ = \max\{0, f\}, f^- = -\min\{0, f\}$. Then $f^+, f^-$ are measurable and non-negative. The *Lebesgue integral of* $f$ is defined by

$$\int\limits_X f \, d\mu = \int\limits_X f^+ d\mu - \int\limits_X f^- d\mu$$

provided that $\int\limits_X f^+ d\mu, \int\limits_X f^- d\mu$ are not both $\infty$.
We call $f$ *integrable* if $\int\limits_X f^+ \, d\mu < \infty$ and $\int\limits_X f^- \, d\mu < \infty$.

4. A function $f : X \to \mathbb{C}$ is called *integrable* iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are integrable. The integral of $f$ is defined by

$$\int\limits_X f \, d\mu = \int\limits_X \operatorname{Re} f d\mu + i \int\limits_X \operatorname{Im} f d\mu.$$

If $E \subset X \in \mathcal{M}$, then $\int\limits_E f d\mu = \int\limits_X f \chi_E d\mu$.

**Definition 2.14.** A set $E \subset \mathcal{M}$ is called *a set of measure zero*, or *null set* if $\mu(E) = 0$.

Let $P$ be a statement about the elements of $X$, and let $A \in \mathcal{M}$. We say that $P$ holds *$\mu$-almost everywhere* on $A$ if there exists $E \in \mathcal{M}$ so that

1. $\{x \in A : P \text{ does not hold}\} \subset E$.

2. $\mu(E) = 0$.

Note that if $(X, \mathcal{M}, \mu)$ is complete, then this is equivalent to $\mu\{x \in A : P \text{ does not hold}\} = 0$.

Next we define the spaces of functions used in this thesis. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f, g : X \to \mathbb{C}$, define $f \sim g$ iff $f(x) = g(x)$ a.e. on $X$. Then " $\sim$ " defines on equivalence relation on the complex vector space of measurable functions.

**Definition 2.15.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$.Then $L^p(X, \mathcal{M}, \mu)$ is the set of equivalence classes of $\mathcal{M}-$measurable function $f : X \to \mathbb{C}$ such that $|f|^p$ is integrable. If $f \sim g$, then $\int\limits_X |f|^p d\mu = \int\limits_X |g|^p d\mu$. For ease of notation, we usually confuse a function $f$ with its equivalence class in $L^p(X, \mathcal{M}, \mu)$, and simply write

$$L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C} : \int\limits_X |f|^p d\mu < \infty\}.$$

Then the number

$$\|f\|_p = \Big(\int\limits_X |f|^p d\mu\Big)^{1/p}.$$

is a *norm* on $L^p(X, \mathcal{M}, \mu)$.

From now on, we will only consider the measure spaces $(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$ and $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \lambda)$. We will call a $\mathcal{B}_{\mathbb{R}^n}$-measurable function. *Borel measurable*, and an

$\mathcal{M}_\lambda$ measurable function simply *measurable.* Note that by proposition 2.1 every Borel measurable function $f$ is Lebesgue measurable. Every Lebesgue measurable function is equal almost everywhere to a Borel measurable function. We often write $\int_{\mathbb{R}^n} f(\vec{x})d\vec{x}$ instead of $\int_{\mathbb{R}^n} f d\lambda$.

**Definition 2.16.** Let $1 \leq p < \infty$. Then $L^p_{loc}(\mathbb{R}^n)$ is the set of Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{C}$ such that $|f|^p$ is integrable on compact sets. That is,

$$\int_K |f|^p d\lambda < \infty$$

for any $K \subset \mathbb{R}^n$ compact.

**Definition 2.17.** Assume that $f : \mathbb{R}^n \to \mathbb{C}$ is continuous. The *support* of $f$, denoted by supp$(f)$, is

$$\text{supp}(f) = \overline{\{\vec{x} \in \mathbb{R}^n : f(\vec{x}) \neq 0\}}.$$

**Definition 2.18.** A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be decay rapidly at infinity, or be rapidly decreasing, if for all $p \in \mathbb{N}$,

$$\lim_{||x|| \to \infty} ||x||^p |f(x)| = 0.$$

We will use the following spaces of continuous functions.

**Definition 2.19.** Let $p \in \{0, 1, 2, ...\}$. Then

1. $C^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is } p \text{ times continuously differentiable}\}$.

2. $C^p_c(\mathbb{R}^n) = \{f \in C^p : f \text{ has compact support}\}$.

3. $C^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is infinitely differentiable}\}$.

4. $C^\infty_c(\mathbb{R}^n) = \{f \in C^\infty : f \text{ has compact support}\}$.

5. $\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : f$ and all of its derivative decay rapidly$\}$.

   The space $\mathcal{S}(\mathbb{R}^n)$ is called the *Schwartz class.*

Next let us introduce some terminology used in product measures. Let $X$ and $Y$ be sets, and let $E \subset X \times Y$. Given $x \in X$ , we set

$$E_x = \{y \in Y : (x,y) \in E\}$$

called the *$x$-section* of $E$. Given $y \in Y$, set

$$E^y = \{x \in X : (x,y) \in E\}$$

called the *$y$-section* of $E$.

Similarly, if $f$ is a function on $X \times Y$, then given $x \in X$, define a function on $Y$ by

$$f_x(y) = f(x,y) \qquad \forall\, y \in Y$$

and given $y \in Y$, define a function on $X$ by

$$f^y(x) = f(x,y) \qquad \forall x \in X.$$

**Theorem 2.2.** *Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be $\sigma$-finite measure spaces. Let $\mathcal{A} \otimes \mathcal{B}$ be the $\sigma$-algebra of subsets of $X \times Y$ generated by rectangles $A \times B$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then there exists a unique measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ such that*

$$(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$$

*for all measurable rectangles $A \times B$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, the measure of any $E \in \mathcal{A} \otimes \mathcal{B}$ is given by*

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y).$$

**Remark 2.1.** *The measure $\mu \times \nu$ is called the* product measure *of $\mu$ and $\nu$, and $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ is called the* product measure space.

**Theorem 2.3.** *(Fubini's Theorem for Nonnegative Functions) Let* $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ *be* $\sigma$*-finite measure spaces, and* $h : X \times Y \to [0, \infty]$ *be* $\mathcal{A} \otimes \mathcal{B}$*-measurable. Then*

1. *The function*

$$f(x) = \int_Y h_x \, d\nu \quad \text{is } \mathcal{A}\text{-measurable, and}$$

$$g(y) = \int_X h^y \, d\mu \quad \text{is } \mathcal{B}\text{-measurable.}$$

2. *Furthermore*

$$\int_{X \times Y} h \, d(\mu \times \nu) = \int_X \int_Y h(x, y) \, d\nu d\mu = \int_Y \int_X h(x, y) \, d\mu d\nu.$$

Note that the above theorem is also called *Tonelli's theorem.* The following theorem is the general *Fubini's theorem.*

**Theorem 2.4.** *(Fubini's Theorem for Integrable Functions) Let* $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ *be* $\sigma$*-finite measure spaces, and* $h \in L^1(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$. *Then*

1. $h_x \in L^1(Y, \mathcal{B}, \nu)$ $\mu$*-almost everywhere on* $X$,

   $h^y \in L^1(X, \mathcal{A}, \mu)$ $\nu$*-almost everywhere on* $Y$.

   *That is*

$$f(x) = \int_Y h_x d\nu \quad \text{is defined } \mu - \text{a.e. on } X,$$

$$g(y) = \int_X h^y d\mu \quad \text{is defined } \nu - \text{a.e. on } Y.$$

2. *Furthermore*

$$\int_{X \times Y} h \, d(\mu \times \nu) = \int_X f(x) \, d\mu = \int_Y g(y) \, d\nu.$$

   *That is*

$$\int_{X \times Y} h \, d(\mu \times \nu) = \int_X \int_Y h(x, y) \, d\nu d\mu = \int_Y \int_X h(x, y) \, d\mu d\nu.$$

Now we discuss the derivative of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.

**Definition 2.20.** Let $U \subset \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^n$. We say that $F$ is *differentiable* at $\vec{x}_0 \in U$ if there exists an $n \times n$ matrix $T$, depending on $\vec{x}_0$ such that

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\|F(\vec{x}) - F(\vec{x}_0) - T(\vec{x} - \vec{x}_0)\|}{\|\vec{x} - \vec{x}_0\|} = 0.$$

It is easily seen that if the components of $F$ are $f_1, ..., f_n$ and if $F$ is differentiable at $\vec{x}_0 \in U$, then the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists for any $i, j \in \{1, ..., n\}$ and are given by the entries of the matrix $T$:

$$F'(\vec{x}_0) \equiv T = \left(\frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right)_{ij}.$$

We call the matrix $F'(\vec{x}_0)$ the *Jacobian matrix.* Its determinant

$$J_F(\vec{x}_0) = \det(F'(\vec{x}_0))$$

is called the *Jacobian of $F$ at $\vec{x}_0$*. $F$ is continuous differentiable on $U$ if the function $F'(\vec{x})$ is continuous.

**Theorem 2.5.** *(Change of Variables): Let $U \subset \mathbb{R}^n$ be open, and $F : U \to \mathbb{R}^n$ be continuously differentiable, injective, with $J_F(\vec{x}) \neq 0$ for all $\vec{x} \in U$. Set $V = F(U)$. If $f : V \to \mathbb{C}$ is Lebesgue measurable, then $f \circ F : U \to \mathbb{C}$ is Lebesgue measurable. Furthermore*

$$\int_V f(\vec{x})\, d\lambda(\vec{x}) = \int_U (f \circ F)(\vec{x}) |J_F(\vec{x})|\, d\lambda(\vec{x}) \tag{2.1}$$

*in the sense that if one of these integral exists, then both exist and are equal.*

Note: If $F$ itself is a linear map, $F : \mathbb{R}^n \to \mathbb{R}^n$, and if $A$ is the matrix associated with $F$, then $F' = A$ and (2.1) becomes

$$\int_{\mathbb{R}^n} f(\vec{x})\, d\lambda(\vec{x}) = \int_{\mathbb{R}^n} f(A\vec{x}) |\det A|\, d\lambda(\vec{x}).$$

**Definition 2.21.** Let $f \in L^1_{loc}(\mathbb{R}^n)$. For $\vec{x}_0 \in \mathbb{R}^n, r > 0$, set

$$A_r f(\vec{x}_0) = \frac{1}{\lambda(B_r(\vec{x}_0))} \int\limits_{B_r(\vec{x}_0)} f(\vec{x}) d\lambda(\vec{x}).$$

$A_r f$ is called the *average value* of $f$ on the open ball $B_r(\vec{x}_0)$ centered at $\vec{x}_0$ with radius $r$.

**Proposition 2.6.** *For each $f \in L^1_{loc}(\mathbb{R}^n)$,*

$$\lim_{r \to 0} A_r f(\vec{x}) = f(\vec{x})$$

*for almost all $\vec{x} \in \mathbb{R}^n$.*

A point $\vec{x}$ where this equation holds is called *a point of differentiability for $f$.*

The proof of these results and more information about differentiation in $\mathbb{R}^n$ can be found in Folland (1999).

## 2.3 The Fourier Transform in Higher Dimensions

**Definition 2.22.** The *Fourier transform* of $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is defined by

$$Ff(\vec{x}) = \hat{f}(\vec{x}) = \int_{\mathbb{R}^n} f(\vec{\gamma}) e^{-2i\pi \vec{x} \cdot \vec{\gamma}} d\vec{\gamma}$$

where $\vec{x}$ denotes an element of $\mathbb{R}^n$ written as a row vector, and $\vec{\gamma}$ an element of $\mathbb{R}^n$ written as a column vector. In this notation, the dot product $\vec{x} \cdot \vec{\gamma}$ is simply multiplication of a row vector with a column vector. Similarly, the *inverse Fourier transform* of $\hat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is given by

$$\overline{F}\hat{f}(\vec{\gamma}) = f(\vec{\gamma}) = \int_{\mathbb{R}^n} \hat{f}(\vec{x}) e^{2i\pi \vec{x} \cdot \vec{\gamma}} d\vec{x}.$$

Since $F$ and $\overline{F}$ preserve the $L^2$-norms, and $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the maps $F$ and $\overline{F}$ extend to unitary operators on $L^2(\mathbb{R}^n)$. Plancherel's theorem says that $\overline{F}$ is the inverse operator of $F$, i.e. $\overline{F}\hat{f} = f$ for all $f \in L^2(\mathbb{R}^n)$.

**Theorem 2.7.** *The Fourier transform $F$ and its inverse Fourier transform $\overline{F}$ are linear one-to-one maps from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$.*

## 2.4 The Real Jordan Normal Form

**Definition 2.23.** A real Jordan block is an upper triangular square matrix $[a_{ij}]$ of one of the following forms

$$A = \begin{pmatrix} \alpha & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \alpha \end{pmatrix} \qquad \text{with } \alpha \in \mathbb{R},$$

or

$$A = \begin{pmatrix} D & I_2 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ (0) & & & D \end{pmatrix} \quad D = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \alpha, \beta \in \mathbb{R} \text{ with } \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By a suitable changing of basis, every matrix can be brought into a block diagonal form with blocks as above, as follows:

**Theorem 2.8.** *Let $A$ be an $n \times n$ real matrix. Then $A$ is similar to a block diagonal matrix of the form $J = Diag(J_1, J_2, \ldots, J_m)$ with each $J_k$ being a real Jordan block.*

*The Jordan blocks are determined by the eigenvalues $\lambda$ of $A$. A real eigenvalue gives rise to a real Jordan block of the first type while a complex pair $\alpha \pm i\beta$ of eigenvalues gives rise to a real Jordan block of the second type. The matrix $J$ is called the* real Jordan normal form of *$A$.*

For the proof of this theorem we refer to Brown (1988).

## 2.5 Exponential Matrices

**Definition 2.24.** Given an $n \times n$ matrix $M$ we define the *exponential* $e^M$ to be the $n \times n$ matrix given by the convergent matrix series

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}.$$

Note that this definition implies $e^O = I$, where $O$ is the zero matrix, and $I$ the identity matrix.

**Proposition 2.9.** *Let $M$ and $N$ be two commuting $n \times n$ matrices: $MN = NM$. Then we have*

$$e^M e^N = e^{M+N}.$$

Note that since the matrices $sM$ and $tN$ commute for all scalars $s$ and $t$, we have

$$e^{sM} e^{tN} = e^{sM+tN}.$$

Obviously, every exponential $e^M$ is invertible, as $e^M e^{-M} = e^0 = I$. It follows immediately from definition 2.24 that if $M = S\widetilde{M}S^{-1}$, then

$$e^M = e^{S\widetilde{M}S^{-1}} = Se^{\widetilde{M}}S^{-1}. \tag{2.2}$$

**Definition 2.25.** Let $M = [a_{ij}]$ be an $n \times n$ matrix. The *trace* of $M$, $\operatorname{tr} M$ is defined by

$$\operatorname{tr} M = \sum_{i=1}^{n} a_{ii}.$$

**Proposition 2.10.** *Let $M$ be an $n \times n$ matrix. Then*

$$\det e^M = e^{trM}. \tag{2.3}$$

For the proof, note that if $M$ is in Jordan normal form, then one easily sees that (2.3) holds. Since the trace of a matrix does not change under a change of basis, it follows that (2.3) holds for general $n \times n$ real matrices.

Note that an invertible matrix whose determinant is negative can not be an exponential. This can be overcome by the next proposition which is probably well known, although we have not found a reference.

**Proposition 2.11.** *Let $A \in GL_n(\mathbb{R})$. Then $A^2$ is an exponential matrix.*

*Proof.* By (2.2) we may assume that $A^2$ is in Jordan Normal Form. That is, $A^2$ is a block diagonal matrix, a block corresponding to a real eigenvalue $\alpha_i$ of $A_i^2$ is of the form

$$(1) \qquad A_i^2 = \begin{pmatrix} \alpha_i & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \alpha_i \end{pmatrix}$$

while a block corresponding to a complex pair of eigenvalues $\alpha_i \pm i\beta_i$ is of the form

$$(2) \qquad A_i^2 = \begin{pmatrix} D_i & I_2 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ (0) & & & D_i \end{pmatrix} \quad D_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix} \text{ with } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now let $M$ be the block-diagonal matrix whose blocks $M_i$ are determined by those of $A_i^2$ as follows:

The block of $M$ corresponding to a block of $A^2$ of type (1) is

$$(3) \qquad M_i = \begin{pmatrix} \mu_i & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \mu_i \end{pmatrix} \text{ with } \mu_i = \ln \alpha_i.$$

Note that this is well defined since the real eigenvalues of $A^2$ 'are all positive. Furthermore, the block of $M$ corresponding to a block of $A$ of type (2) is

$$(4) \qquad M_i = \begin{pmatrix} F_i & I_2 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ (0) & & & F_i \end{pmatrix} \quad F_i = \begin{pmatrix} \mu_i & \gamma_i \\ -\gamma_i & \mu_i \end{pmatrix}$$

where $\mu_i, \gamma_i$ are chosen so that $\alpha_i \pm i\beta_i = e^{\mu_i \pm i\gamma_i}$.

In this basis, $e^M$ is also a block diagonal matrix, and a block corresponding to a block of $M$ of type (3) is

$$e^{M_i} = \begin{pmatrix} \lambda_i & \frac{\lambda_i}{2} & \cdots & \frac{\lambda_i}{(m-1)!} \\ & \lambda_i & \ddots & \frac{\lambda_i}{(m-2)!} \\ & & \ddots & \vdots \\ (0) & & & \lambda_i \end{pmatrix} \quad \text{where } \lambda_i = e^{\mu_i} = e^{\ln \alpha_i} = \alpha_i$$

and a block corresponding to a block of $M$ of type (4) is

$$e^{M_i} = \begin{pmatrix} \lambda_i E & \lambda_i E & \frac{\lambda_i}{2} E & \cdots & \frac{\lambda_i}{(m-1)!} E \\ & \lambda_i E & \lambda_i E & \ddots & \frac{\lambda_i}{(m-2)!} E \\ & & \lambda_i E & \ddots & \vdots \\ & & & \ddots & \lambda_i E \\ (0) & & & & \lambda_i E \end{pmatrix}$$

where $\lambda_i = e^{\mu_i}$ and $E = \begin{pmatrix} \cos \gamma_i & \sin \gamma_i \\ -\sin \gamma_i & \cos \gamma_i \end{pmatrix}$.

So that

$$\lambda_i E = e^{\mu_i} \begin{pmatrix} \cos\gamma_i & \sin\gamma_i \\ -\sin\gamma_i & \cos\gamma_i \end{pmatrix} = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}.$$

By changing the basis through a suitable matrix $S$, $e^M$ can be brought into Jordan normal form again. A simple computation show that a block of $e^M$ corresponding to a real eigenvalue $\lambda_i = \alpha_i$ is then of the form

$$Se^{M_i}S^{-1} = \begin{pmatrix} \alpha_i & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \alpha_i \end{pmatrix}$$

and a block corresponding to a pair of complex eigenvalues $\alpha_i \pm i\beta = e^{\mu_i \pm i\gamma_i}$ is then of the form

$$Se^{M_i}S^{-1} = \begin{pmatrix} D_i & I_2 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ (0) & & & D_i \end{pmatrix} \qquad D_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}.$$

It follows from (2.2) that

$$A^2 = Se^M S^{-1} = e^{SMS^{-1}}.$$

That is, $A^2$ is an exponential matrix. □

## 2.6 Group Theoretical Foundations

In this section, we briefly review the important group-theoretical concepts needed in this thesis. We begin with some definitions.

**Definition 2.26.** A *topological group* is a set $G$ which is both a group and topological space, such that the group operations $(g, h) \mapsto gh$ from $G \times G$ into $G$ and

$g \mapsto g^{-1}$ from $G$ into itself are continuous in this topology. Any subgroup of a topological group $G$ becomes a topological group in the relative topology of $G$.

A *locally compact group* is a topological group whose topology is locally compact and Hausdorff.

Let us consider some examples of topological groups:

1. The set $\mathbb{R}^n$ with its usual topology and with addition as the group operation is a locally compact group.

2. The set $\mathbb{Q}$ of rational numbers, with the subspace topology induced from $\mathbb{R}$ and with addition as the group operation is a topological subgroup of $\mathbb{R}$, but it is not locally compact.

3. $GL_n(\mathbb{R})$, the group of $n \times n$ real-invertible matrices.

   Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{R}$. $M_n(\mathbb{R})$ is a finite dimensional normed linear space, isomorphic to $\mathbb{R}^{n^2}$. Give $GL_n(\mathbb{R})$ the relative topology of $M_n(\mathbb{R})$. Then $GL_n(\mathbb{R})$ is a multiplicative locally compact topological group.

**Definition 2.27.** A homomorphism $\Phi : \mathbb{R}^k \to GL_n(\mathbb{R})$ is called a *k-parameter group.*

**Examples:** Let $N \in M_n(\mathbb{R})$ be fixed. For any $r \in \mathbb{R}$ we define

$$A^r = e^{rN}.$$

Then $\Phi : \mathbb{R} \to GL_n(\mathbb{R})$ is a continuous one-parameter group.

Fix a collection $\{N_i\}_{i=1}^k$ of commuting matrices in $M_n(\mathbb{R})$. Set $A_i = e^{N_i}$. Then

$$\Phi : (r_1, r_2, \ldots, r_k) \mapsto A_1^{r_1} A_2^{r_2} \cdots A_k^{r_k} = e^{r_1 N_1 + r_2 N_2 + \ldots + r_k N_k}.$$

is a continuous *k-parameter group.*

If $\Phi$ is one-to-one, then we often identify $\Phi$ with its image $D = \Phi(\mathbb{R}^k)$.

**Definition 2.28.** Let $G$ be a locally compact group and $\mathcal{H}$ be a Hilbert space. A *representation* $\pi$ of $G$ on $\mathcal{H}$ is a mapping satisfying:

1. $\pi : G \to \mathcal{U}(\mathcal{H})$. ($\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$),
2. $\pi$ is a homomorphism: $\pi_{gh} = \pi_g \pi_h$ for all $g, h \in G$,
3. $\pi$ is continuous with respect to the strong operator topology of $\mathcal{U}(\mathcal{H})$, that is $g \to \pi_g \xi$ is continuous for each $\xi \in \mathcal{H}$.

A representation $\pi$ of a locally compact group $G$ on a Hilbert space $\mathcal{H}$ is called *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ which are invariant under $\pi_g$ for each $g \in G$.

**Definition 2.29.** A representation $\pi$ of a locally compact group $G$ on a Hilbert space $\mathcal{H}$ is called *square integrable* if

1. $\pi$ is irreducible,
2. there exists a vector $\psi \in \mathcal{H}\backslash\{0\}$ such that $\int_G |\langle \psi, \pi_g \psi \rangle|^2 d\mu(g) < \infty$ where $\mu$ is the left Haar measure on $G$. That is, the function $g \to \langle \psi, \pi_g \psi \rangle$ is square integrable. Such a vector $\psi$ is called *admissible.*

In dealing with representations of locally compact groups, measures and integrals are important tools.

**Definition 2.30.** A Borel measure $\mu$ on a locally compact group $G$ is called *left translation invariant* or a *left Haar measure* provided that for every continuous compactly supported function $f$ on $G$ and every $h \in G$ we have

$$\int_G f(hg) d\mu(g) = \int_G f(g) d\mu(g).$$

A *right Haar measure* $\gamma$ is defined similarly. If $G$ is a locally compact group, then the left and right Haar measures exist and are unique up to a constant factor.

Given a left Haar measure on $G$, there exists a unique homomorphism $\Delta : G \to \mathbb{R}^+$ such that for any continuous and compactly supported function $f$ on $G$ and $h \in G$,

$$\int_G f(g)d\mu(g) = \Delta(h) \int_G f(gh)d\mu(g).$$

From this it follows that

$$\int_G f(gh)\Delta^{-1}(g)d\mu(g) = \int_G f(g)\Delta^{-1}(g)d\mu(g)$$

which means that $\gamma = \Delta^{-1}\mu$ is a right Haar measure. $\Delta$ is called the *modular function* on G. Furthermore,

$$\int_G f(g)d\mu(g) = \int_G f(g^{-1})\Delta(g^{-1})d\mu(g).$$

A locally compact group $G$ is *unimodular* if its modular function satisfies $\Delta(g) = 1$ at each $g \in G$. Obviously, every locally compact abelian group is unimodular.

**Definition 2.31.** Let $G$ be a locally compact group, $\mathcal{H}$ Hilbert space, and $F : G \to \mathcal{H}$ continuous. If there exists a vector $f \in \mathcal{H}$ such that

$$\langle f, h\rangle = \int_G \langle F(g), h\rangle \, d\mu(g) \;\; \forall \, h \in \mathcal{H}$$

then we say that $f = \int_G F(g)\, d\mu(g)$ as a *weak integral* in $\mathcal{H}$.

## 2.7 The Abstract Wavelet Transform

**Definition 2.32.** Let $G$ be a locally compact group whose left Haar measure is $\mu$, and let $\pi_g : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Given a vector $\psi \in \mathcal{H}$, the collection $\{\pi_g\psi\}$ of vectors in $\mathcal{H}$ is called the *family of wavelets* generated by $\psi$, and $\psi$ is called the *mother wavelet*. The mapping $W_\psi$, taking $f \in \mathcal{H}$ to a continuous function on $G$ defined by

$$(W_\psi f)(g) = \langle f, \pi_g\psi\rangle_{\mathcal{H}} \tag{2.4}$$

is called the *wavelet transform of $f$ with respect to $\psi$*.

Numerous authors have studied the theory of wavelets from the point of view of square integrable group representations. The following important theorem links the wavelet transform to the theory of square integrable representations:

**Theorem 2.12.** *(Duflo-Moore): If $\pi$ is a square integrable representation of a locally compact group $G$ on $\mathcal{H}$, then there exists a unique densely defined operator $K$ on $\mathcal{H}$, self adjoint and positive which satisfies the following:*

*i) The set of admissible vectors in $\mathcal{H}$ coincides with the domain of $K$, that is dom $K = \{\psi \in \mathcal{H} : \psi$ is admissible $\}$.*

*ii) If $\psi$ is an admissible vector and $f$ is an arbitrary vector in $\mathcal{H}$, then*

$$||W_\psi f||_{L^2(G)} = \sqrt{c_\psi}||f||_{\mathcal{H}}$$

*where $c_\psi = ||K\psi||^2_{\mathcal{H}}$.*

*iii) If the group $G$ is unimodular, then $K$ is a multiple of the identity.*

Thus, if the representation $\pi$ is square integrable, then there exists a dense set of vectors $\psi$ in $\mathcal{H}$ such that

$$||W_\psi f||_{L^2(G)} = \sqrt{c_\psi}||f||_{\mathcal{H}} \quad \forall f \in \mathcal{H} \tag{2.5}$$

for some constant $c_\psi$, i.e. the wavelet transform $W_\psi$ associated with $\psi$ is a multiple of an isometry from $\mathcal{H}$ into $L^2(G)$. It turns out that equality (2.5) is equivalent to the existence of a reconstruction formula as follows:

**Proposition 2.13.** *(Reconstruction Formula for the Wavelet Transform) Let $\pi$ be a representation of $G$ on $\mathcal{H}$, and let $\psi \in \mathcal{H}$. Then (2.5) holds if and only if for all $f \in \mathcal{H}$,*

$$f = \frac{1}{c_\psi}\int_G (W_\psi f)(g)\,\pi_g\psi\,d\mu(g) \tag{2.6}$$

*as a weak integral in $\mathcal{H}$.*

*Proof.* ($\Rightarrow$) Assume that (2.5) holds. By the polarization identity, for all $f, g \in \mathcal{H}$ we have

$$\langle W_\psi f, W_\psi h\rangle_{L^2(G)} = c_\psi \langle f, h\rangle_{\mathcal{H}}. \tag{2.7}$$

Divide by $c_\psi$ and rewrite the inner product in $L^2(G)$ as

$$\begin{aligned}
\langle f, h\rangle &= \frac{1}{c_\psi} \int_G W_\psi f(g) \overline{W_\psi h(g)}\, d\mu(g) \\
&= \frac{1}{c_\psi} \int_G W_\psi f(g) \overline{\langle h, \pi_g \psi\rangle}\, d\mu(g) \\
&= \frac{1}{c_\psi} \int_G W_\psi f(g) \langle \pi_g \psi, h\rangle\, d\mu(g) \\
&= \frac{1}{c_\psi} \int_G \langle W_\psi f(g) \pi_g \psi, h\rangle\, d\mu(g).
\end{aligned}$$

This means that

$$f = \frac{1}{c_\psi} \int_G (W_\psi f)(g) \pi_g \psi\, d\mu(g)$$

as a weak integral. Thus, the reconstruction formula (2.6) is valid.

($\Leftarrow$) Assume that the reconstruction formula (2.6) holds in the weak sense. By going backwards in the above computation, one easily verifies that (2.7) holds, i.e., $W_\psi$ is a multiple of an isometry. □

It is natural to ask whether the reconstruction formula exists if the map $\pi$ is not square integrable.

The traditional wavelet transform operates on functions defined on $\mathbb{R}^n$, thus we will choose $\mathcal{H} = L^2(\mathbb{R}^n)$ from now on.

**Definition 2.33.** Let $G^\sharp$ be the group consisting of pairs $(a, \vec{b}) \in GL_n(\mathbb{R}) \times \mathbb{R}^n$ together with the group operation

$$(\alpha, \vec{\beta}) \cdot (a, \vec{b}) = (\alpha a, a^{-1}\vec{\beta} + \vec{b})$$

and the product topology. $G^\sharp$ is called the *affine group*. This kind of group construction is called a *semi-direct product*, and thus $G^\sharp$ is also called the semi-direct product of $GL_n(\mathbb{R})$ and $\mathbb{R}^n$, written $GL_n(\mathbb{R}) \rtimes \mathbb{R}^n$.

If $D$ is a closed subgroup of $GL_n(\mathbb{R})$ then $G = \{(a, \vec{b}) \in G^\sharp, a \in D, \vec{b} \in \mathbb{R}^n\}$ is a closed subgroup of $G^\sharp$, and $G$ is the semi-direct product $D \rtimes \mathbb{R}^n$. We call $D$ the *dilation subgroup* of $G$ and $\mathbb{R}^n$ the *translation subgroup* of $G$.

Note that the Haar measure $d\nu(a, \vec{b})$ on $G$ is simply the product of the Haar measure $d\mu(a)$ on $D$ with the Lebesgue measure $d\lambda(\vec{b})$ on $\mathbb{R}^n$. In fact, for any $f \in L^1(G)$,

$$\begin{aligned}\int_G f\big((\alpha, \vec{\beta}) \cdot (a, \vec{b})\big) d\nu(a, \vec{b}) &= \int_D \int_{\mathbb{R}^n} f\big(\alpha a, a^{-1}\vec{\beta} + \vec{b}\big) d\lambda(\vec{b}) d\mu(a) \\ &= \int_D \int_{\mathbb{R}^n} f(a, \vec{b}) d\lambda(\vec{b}) d\mu(a) \\ &= \int_G f(a, \vec{b}) d\nu(a, \vec{b}).\end{aligned}$$

This shows that $d\nu(a, \vec{b}) = d\lambda(\vec{b}) d\mu(a)$ is left translation invariant. There is a natural representation $\pi$ of $G^\sharp$ on $L^2(\mathbb{R}^n)$ given by

$$\pi_{(a,\vec{b})}\psi(\vec{\gamma}) = |\det a|^{-1/2}\psi(a^{-1}\vec{\gamma} - \vec{b}) \equiv \psi_{a,\vec{b}}(\vec{\gamma}) \tag{2.8}$$

for $(a, \vec{b}) \in G^\sharp$ and $\psi \in L^2(\mathbb{R}^n)$. If $G$ is a subgroup of $G^\sharp$ as in definition 2.33, then the wavelet transform $W_\psi$ induced by $\psi$ and the representation $\pi$ becomes

$$W_\psi f(a, \vec{b}) = \langle f, \psi_{a,\vec{b}} \rangle = \frac{1}{\sqrt{|\det a|}} \int_{\mathbb{R}^n} f(\vec{\gamma}) \overline{\psi(a^{-1}\vec{\gamma} - \vec{b})} d\vec{\gamma} \tag{2.9}$$

for $f \in L^2(\mathbb{R}^n)$ and $(a, \vec{b}) \in G$.

**Remark 2.2.** *We want to find a reconstruction formula for the wavelet transform (2.9). From the discussion following the Duflo-Moore theorem we know that if (2.5) holds then reconstruction formula (2.6) follows, which here becomes*

$$f(\vec{\gamma}) = \int_D \int_{\mathbb{R}^n} W_\psi f(a, \vec{b})\, \psi_{a,\vec{b}}(\vec{\gamma}) d\lambda(\vec{b})\, d\mu(a) \tag{2.10}$$

*in case* $c_\psi = 1$. *Thus, we need to investigate under what condition (2.5) holds. Note that by scaling the function* $\psi$ *we may always assume that* $c_\psi = 1$.

**Remark 2.3.** *When discussing the wavetet transform, one usually makes use of the tools of the Fourier transform. Since the Fourier transform* $F : f \to \hat{f}$ *constitutes a unitary operator on* $L^2(\mathbb{R}^n)$, $\pi$ *induces a representation*

$$\rho = F\pi\overline{F} \tag{2.11}$$

*of* $G^\sharp$ *on* $L^2(\mathbb{R}^n)$. *Let us compute this representation. For* $\widehat{\psi} = F\psi \in L^2(\mathbb{R}^n)$ *and* $\vec{x} \in \mathbb{R}^n$,

$$\begin{aligned}
\rho_{(a,\vec{b})}\widehat{\psi}(\vec{x}) &= F\pi_{(a,\vec{b})}\overline{F}(F\psi)(\vec{x}) \\
&= \int_{\mathbb{R}^n} (\pi_{(a,\vec{b})}\psi)(\vec{\gamma})e^{-2i\pi\vec{x}\cdot\vec{\gamma}}\, d\vec{\gamma} \\
&= \int_{\mathbb{R}^n} |\det a|^{-1/2}\psi(a^{-1}\vec{\gamma} - \vec{b})e^{-2i\pi\vec{x}\cdot\vec{\gamma}}\, d\vec{\gamma} \\
&= \int_{\mathbb{R}^n} |\det a|^{1/2}\psi(\vec{\gamma} - \vec{b})e^{-2i\pi\vec{x}\cdot(a\vec{\gamma})}\, d\vec{\gamma} \\
&= |\det a|^{1/2}e^{-2i\pi\vec{x}\cdot a\vec{b}}\int_{\mathbb{R}^n} \psi(\vec{\gamma})e^{-2i\pi\vec{x}\cdot a\vec{\gamma}}\, d\vec{\gamma} \\
&= |\det a|^{1/2}e^{-2i\pi\vec{x}\cdot a\vec{b}}\widehat{\psi}(\vec{x}a) \equiv \widehat{\psi}_{a,\vec{b}}(\vec{x})
\end{aligned}$$

*where elements* $\vec{x}$ *of* $\mathbb{R}^n$ *are now written as row vectors, and* $\vec{\gamma}$ *are column vectors. Formula (2.9) for the wavelet transform becomes now*

$$\begin{aligned}
(W_\psi f)(a,\vec{b}) &= \langle f, \pi_{(a,\vec{b})}\psi\rangle = \langle \hat{f}, \rho_{(a,\vec{b})}\widehat{\psi}\rangle = \langle \hat{f}, \widehat{\psi}_{a,\vec{b}}\rangle \\
&= |\det a|^{1/2}\int_{\mathbb{R}^n} \hat{f}(\vec{x})\overline{\widehat{\psi}(\vec{x}a)}e^{2i\pi\vec{x}\cdot a\vec{b}}d\vec{x}.
\end{aligned} \tag{2.12}$$

**Definition 2.34.** Let $D$ be a closed subgroup of $GL_n(\mathbb{R})$, and $\vec{x} \in \mathbb{R}^n$ be fixed.

i) The set $\vec{x}D = \{\vec{x}a : a \in D\}$ is called the$D$-*orbit* of $\vec{x}$.

ii) The orbit $\vec{x}D$ is called *free* if $\vec{x}a = \vec{x}$ implies $a = e$, where $e$ denotes the identity element of $D$.

iii) Given $\epsilon > 0$, the set $D_{\vec{x}}^{\epsilon} = \{a \in D : \|\vec{x}a - \vec{x}\| \leq \epsilon\}$ is called the *$\epsilon$-stabilizer* of $\vec{x}$ in $D$.

iv) The set $D_{\vec{x}} \equiv D_{\vec{x}}^{0} = \{a \in D : \vec{x}a = \vec{x}\}$ is called the *stabilizer* of $\vec{x}$.

We note that $D_{\vec{x}}$ is a closed subgroup of $D$ and $D_{\vec{x}}^{\epsilon}$ is a closed subset of $D_{\vec{x}}$.

**Definition 2.35.** Let $D$ be a closed subgroup of $GL_n(\mathbb{R})$. A Borel set $S \subset \mathbb{R}^n$ is called a Borel *cross-section* for the action $\vec{x} \to \vec{x}a$ $(a \in D)$ on $\mathbb{R}^n$ provided that

i) $\bigcup_{a \in D} Sa = \mathbb{R}^n \backslash N$ for some set $N$ of measure zero,

ii) $Sa_1 \cap Sa_2 = \emptyset$ whenever $a_1 \neq a_2 \in D$.

That is, the cross-section intersects each orbit exactly once, except for some orbits making up a set of measure zero.

# Chapter III

# Characterization of Admissible Groups

One of the goals in wavelet theory is to investigate under which conditions reconstruction formulas, or inverse wavelet transforms, exist. In this chapter, we review the results from Weiss et al. (2002) which produce a nearly complete characterization of those subgroups $D$ of $GL_n(\mathbb{R})$ which allow for reconstruction formula (2.10) to hold.

## 3.1 Admissible Functions

**Definition 3.1.** Let $D$ be a closed subgroup of $GL_n(\mathbb{R})$. We say that $D$ is *admissible* if there exists a Borel measurable $\psi \in L^2(\mathbb{R}^n)$ such that

$$c_\psi(\vec{x}) = \int_D |\widehat{\psi}(\vec{x}a)|^2 d\mu(a) = 1 \tag{3.1}$$

for almost all $\vec{x} \in \mathbb{R}^n$. We call such a $\psi$ an *admissible function* for the group $D$.

Since we will need to work with the Fourier transform $\widehat{\psi}$ of $\psi$, we also call $\widehat{\psi}$ admissible.

The following theorem shows that for a given dilation subgroup $D$ of $GL_n(\mathbb{R})$, the reconstruction formula (2.10) exists for all $f \in L^2(\mathbb{R}^n)$ if and only if the admissibitity condition (3.1) holds. As usual, we let $G$ denote the semi-direct product $G = D \rtimes \mathbb{R}^n$.

**Theorem 3.1.** *The following are equivalent:*

1. $\psi \in L^2(\mathbb{R}^n)$ *is admissible.*

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**Theorem 3.1.** *The following are equivalent:*

1. $\psi \in L^2(\mathbb{R}^n)$ *is admissible.*

2. *The wavelet transform* $W_\psi$ *with respect to* $\psi$ *is a partial isometry from* $L^2(\mathbb{R}^n)$ *into* $L^2(G)$.

3. *The reconstruction formula*

$$f(\vec{\gamma}) = \int_D \int_{\mathbb{R}^n} W_\psi f(a, \vec{b}) \psi_{a,\vec{b}}(\vec{\gamma})\, d\vec{b}\, d\mu(a). \tag{3.2}$$

*holds for all* $f \in L^2(\mathbb{R}^n)$.

*Proof.* The equivalence (2) $\Leftrightarrow$ (3) is precisely proposition 2.13 in chapter II with $c_\psi = 1$.

(1) $\Rightarrow$ (2) Note that for all $\psi$,

$$\begin{aligned}
||W_\psi f||^2_{L^2(G)} &= \int_D \int_{\mathbb{R}^n} |\langle f, \psi_{a,\vec{b}} \rangle|^2 d\vec{b}\, d\mu(a) \\
&= \int_D \int_{\mathbb{R}^n} |\langle \hat{f}, \widehat{\psi}_{a,\vec{b}} \rangle|^2 d\vec{b}\, d\mu(a) \\
&= \int_D \int_{\mathbb{R}^n} \left| |\det a|^{1/2} \int_{\mathbb{R}^n} \hat{f}(\vec{x}) \overline{\widehat{\psi}(\vec{x}a)} e^{2i\pi \vec{x}\cdot a\vec{b}}\, d\vec{x} \right|^2 d\vec{b}\, d\mu(a) \\
&= \int_D \int_{\mathbb{R}^n} |\det a| \left| \int_{\mathbb{R}^n} F_a(\vec{x}) e^{2i\pi \vec{x}\cdot a\vec{b}}\, d\vec{x} \right|^2 d\vec{b}\, d\mu(a)
\end{aligned}$$

where we have set $F_a(\vec{x}) = \hat{f}(\vec{x})\overline{\widehat{\psi}(\vec{x}a)}$. Since both of $\hat{f}(\vec{x})$ and $\overline{\widehat{\psi}(\vec{x}a)}$ are in $L^2(\mathbb{R}^n)$, the product is in $L^1(\mathbb{R}^n)$. The inside integral is precisely the inverse

Fourier transform $\check{F}_a$ of $F_a$, so that

$$\begin{aligned}
\|W_\psi f\|^2_{L^2(G)} &= \int_D \int_{\mathbb{R}^n} |\det a| \left|\check{F}_a(a\vec{b})\right|^2 d\vec{b}\, d\mu(a) \\
&= \int_D \left[\int_{\mathbb{R}^n} \left|\check{F}_a(\vec{b})\right|^2 d\vec{b}\right] d\mu(a) \\
&= \int_D \left[\int_{\mathbb{R}^n} \left|F_a(\vec{x})\right|^2 d\vec{x}\right] d\mu(a) \\
&= \int_D \int_{\mathbb{R}^n} |\hat{f}(\vec{x})|^2 |\widehat{\psi}(\vec{x}a)|^2\, d\vec{x}\, d\mu(a) \\
&= \int_D |\hat{f}(\vec{x})|^2 \int_{\mathbb{R}^n} |\widehat{\psi}(\vec{x}a)|^2\, d\vec{x}\, d\mu(a) \\
&= \int_{\mathbb{R}^n} |\hat{f}(\vec{x})|^2 c_\psi(\vec{x})\, d\vec{x}
\end{aligned}$$

where we have set $c_\psi(\vec{x}) = \int_D |\widehat{\psi}(\vec{x}a)|^2 d\mu(a)$. Now by assumption, $c_\psi = 1$ for almost all $\vec{x}$, so that

$$\|W_\psi f\|^2_{L^2(G)} = \|\hat{f}\|_2^2 = \|f\|_2^2.$$

Hence, $W_\psi$ is a partial isometry.

(2) $\Rightarrow$ (1) Now suppose that $W_\psi$ is a partial isometry. Given $\vec{x}_0 \in \mathbb{R}^n$, let $\hat{f} \in L^2(\mathbb{R}^n)$ be so that $|\hat{f}(\vec{x})|^2 = \lambda(B_r(\vec{x}_0))^{-1}\chi_{B_r(\vec{x}_0)}(\vec{x})$, where $B_r(\vec{x}_0)$ is the open ball of radius $r$ centered at $\vec{x}_0$. Then

$$\|\hat{f}\|_2^2 = 1.$$

We have by the above computations that

$$\begin{aligned}
1 = \|f\|_2^2 = \|W_\psi f\|^2_{L^2(G)} &= \int_{\mathbb{R}^n} |\hat{f}(\vec{x})|^2 c_\psi(\vec{x}) d\vec{x} \\
&= \int_{\mathbb{R}^n} \frac{\chi_{B_r(\vec{x}_0)}(\vec{x})}{\lambda(B_r(\vec{x}_0))} c_\psi(\vec{x})\, d\vec{x} \\
&= \frac{1}{\lambda(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} c_\psi(\vec{x})\, d\vec{x} \qquad \forall\, \vec{x}_0 \in \mathbb{R}^n. \qquad (3.3)
\end{aligned}$$

It follows from (3.3) that $c_\psi$ is locally integrable, Now let $\vec{x}_0$ be a point of differentiability for $c_\psi$. Then

$$c_\psi(\vec{x}_0) = \lim_{r\to 0} \frac{1}{\lambda(B_r(\vec{x}_0))} \int\limits_{B_r(\vec{x}_0)} c_\psi(\vec{x})\, d\vec{x} = 1.$$

It follows that (3.1) holds almost everywhere, i.e. $\psi$ is admissible. Thus, the admissibility condition (3.1) is true and the theorem is proved. □

**Remark 3.1.** *We would like to comment on the relationship between the notions of admissible vector in definition 2.29 and admissible function as defined above. Recall that the representation $\pi$ of $G$ as in (2.8) is called square integrable if it is irreducible, and there exists $\psi \neq 0$ in $L^2(\mathbb{R}^n)$ such that $W_\psi\psi \in L^2(G)$. Such a $\psi$ was called "admissible vector"). Duflo-Moore's theorem then establishes that for each admissible vectors $\psi$,*

$$||W_\psi f||_{L^2(G)} = \sqrt{c_\psi}||f||_{L^2(\mathbb{R}^n)} \tag{3.4}$$

*for some $c_\psi > 0$. If we scale $\psi$ so that $c_\psi = 1$, then by the above theorem, $\psi$ is an admissible function in the sense of (3.1).*

*This shows that the notions of admissible function in definition 3.1 is a generalization of that of an admissible vector in definition 2.29 to representations which are not square integrable, in the context of subgroups of the affine group.*

**Remark 3.2.** *Let $D \subset GL_n(\mathbb{R})$ be a closed subgroup, and let $\widetilde{D} = cDc^{-1}$ where $c \in GL_n(\mathbb{R})$ is fixed. For each $E \subset D$, set*

$$\widetilde{E} = cEc^{-1}.$$

*Since conjugation by $c$ is continuous, it is clear that $E$ is a Borel subset of $D$ iff $\widetilde{E} = cEc^{-1}$ is a Borel subset of $\widetilde{D}$. Furthermore, if $\mu$ is a left Haar measure on $D$, then*

$$\nu(\widetilde{E}) = \mu(E)$$

*defines a Haar measure on $\widetilde{D}$. In fact, for all $\tilde{a} = cac^{-1} \in \widetilde{D}$ we have*

$$\begin{aligned}\nu(\tilde{a}\widetilde{E}) &= \nu\big(cac^{-1}(cEc^{-1})\big) = \nu\big(caEc^{-1}\big)\\ &= \mu(aE) = \mu(E) = \nu(\widetilde{E}).\end{aligned}$$

*Using the definition of measurable functions and integral, one easily verifies that a function $f(\cdot) : \widetilde{D} \to \mathbb{C}$ is $\nu$-measurable iff $f(c \cdot c^{-1}) : D \to \mathbb{C}$ is $\mu$-measurable, that $f$ is integrable on $\widetilde{D}$ iff $f(c \cdot c^{-1})$ is integrable on $D$, and that*

$$\int_{\widetilde{D}} f(\tilde{a})d\nu(\tilde{a}) = \int_{D} f(cac^{-1})d\mu(a). \tag{3.5}$$

The following proposition shows that the admissibility of a group is invariant under a change of basis.

**Proposition 3.2.** *Keep the notations of remark 3.2. Given $\psi \in L^2(\mathbb{R}^n)$, let $\varphi$ be so that $\widehat{\varphi}(\vec{y}) = \widehat{\psi}(\vec{y}c)$. Then $\psi$ is admissible for $D$ iff $\varphi$ is admissible for $\widetilde{D}$.*

*Proof.* ($\Rightarrow$) Suppose $\widehat{\psi}$ is admissible for $D$. Given $\vec{x} \in \mathbb{R}^n$, pick $\vec{y}$ such that $\vec{x} = \vec{y}c^{-1}$. Then by (3.5),

$$\begin{aligned}\int_{\widetilde{D}} |\widehat{\varphi}(\vec{x}\tilde{a})|^2\, d\nu(\tilde{a}) &= \int_{\widetilde{D}} |\widehat{\varphi}(\vec{y}c^{-1}\tilde{a})|^2\, d\nu(\tilde{a})\\ &= \int_{\widetilde{D}} |\widehat{\psi}(\vec{y}c^{-1}\tilde{a}c)|^2\, d\nu(\tilde{a})\\ &= \int_{D} |\widehat{\psi}(\vec{y}a)|^2\, d\mu(a).\end{aligned}$$

Since $\psi$ is admissible, the last integral equals one for almost all $\vec{y} \in \mathbb{R}^n$, so that

$$\int_{\widetilde{D}} |\widehat{\varphi}(\vec{x}\tilde{a})|^2\, d\nu(\tilde{a}) = 1$$

for almost all $\vec{x} = \vec{y}c^{-1} \in \mathbb{R}^n$. Thus, $\varphi$ is admissible for $\widetilde{D}$.

($\Leftarrow$) Since $D = c^{-1}\widetilde{D}c$ , this implication follows by symmetry. □

The following result will be an essential tool in the proof of the main theorem in this chapter.

## 3.2 Characterization of Admissible Groups

**Proposition 3.3.** *If $D$ is admissible, then $\Delta(a) \neq |\det a|$ for at least one $a \in D$.*

*Proof.* Suppose $D$ is admissible. Let $\psi$ be an admissible function, and set $h = |\widehat{\psi}|^2 \in L^1(\mathbb{R}^n)$. By theorem 3.1, for almost all $\vec{x} \in \mathbb{R}^n$

$$\int_D h(\vec{x}a)d\mu(a) = 1. \tag{3.6}$$

We claim that $\int\limits_D h(\vec{y})\, d\vec{y} > 0$. For suppose, $\int\limits_D h(\vec{y})\, d\vec{y} = 0$. Then $h = 0$ almost everywhere. Now if $B_1(\vec{0})$ denotes the unit ball in $\mathbb{R}^n$, then

$$\begin{aligned} 0 < \lambda(B_1(\vec{0})) &\equiv \int\limits_{\mathbb{R}^n} \chi_{B_1(\vec{0})}(\vec{x})\, d\vec{x} \\ &= \int\limits_{\mathbb{R}^n}\int\limits_{D} \chi_{B_1(\vec{0})}(\vec{x})h(\vec{x}a)\, d\mu(a)\, d\vec{x} \\ &= \int\limits_{D}\int\limits_{\mathbb{R}^n} \chi_{B_1(\vec{0})}(\vec{x})h(\vec{x}a)\, d\vec{x}\, d\mu(a) \end{aligned}$$

by Tonelli's theorem. Then

$$\begin{aligned} 0 < \lambda(B_1(\vec{0})) &= \int\limits_{D}\int\limits_{\mathbb{R}^n} \chi_{B_1(\vec{0})}(\vec{x})h(\vec{x}a)\, d\vec{x}\, d\mu(a) \\ &= \int\limits_{D}\int\limits_{\mathbb{R}^n} \chi_{B_1(\vec{0})}(\vec{x}a^{-1})h(\vec{x})|\det a|^{-1}\, d\vec{x}\, d\mu(a) \\ &= \int\limits_{D}\Big[\int\limits_{B_1(\vec{0})a} h(\vec{y})\, d\vec{y}\Big]\, |\det a|^{-1}\, d\mu(a). \end{aligned}$$

Since $h(\vec{y}) = 0$ almost everywhere, then the inner integral will be zero for all $a$, so that the right hand side will be zero, giving a contradiction. Thus, $h(\vec{y})$ cannot be zero almost everywhere. It follows that

$$\int\limits_{\mathbb{R}^n} h(\vec{y})\, d\vec{y} > 0 \tag{3.7}$$

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**Proposition 3.3.** *If $D$ is admissible, then $\Delta(a) \neq |\det\, a|$ for at least one $a \in D$.*

*Proof.* Suppose $D$ is admissible. Let $\psi$ be an admissible function, and set $h = |\widehat{\psi}|^2 \in L^1(\mathbb{R}^n)$. By theorem 3.1, for almost all $\vec{x} \in \mathbb{R}^n$

$$\int_D h(\vec{x}a)d\mu(a) = 1. \tag{3.6}$$

We claim that $\int_D h(\vec{y})\, d\vec{y} > 0$. For suppose, $\int_D h(\vec{y})\, d\vec{y} = 0$. Then $h = 0$ almost everywhere. Now if $B_1(\vec{0})$ denotes the unit ball in $\mathbb{R}^n$, then

$$\begin{aligned} 0 < \lambda(B_1(\vec{0})) &\equiv \int_{\mathbb{R}^n} \chi_{B_1(\vec{0})}(\vec{x})\, d\vec{x} \\ &= \int_{\mathbb{R}^n}\int_D \chi_{B_1(\vec{0})}(\vec{x})h(\vec{x}a)\, d\mu(a)\, d\vec{x} \\ &= \int_D\int_{\mathbb{R}^n} \chi_{B_1(\vec{0})}(\vec{x})h(\vec{x}a)\, d\vec{x}\, d\mu(a) \end{aligned}$$

by Tonelli's theorem. Then

$$\begin{aligned} 0 < \lambda(B_1(\vec{0})) &= \int_D\int_{\mathbb{R}^n} \chi_{B_1(\vec{0})}(\vec{x})h(\vec{x}a)\, d\vec{x}\, d\mu(a) \\ &= \int_D\int_{\mathbb{R}^n} \chi_{B_1(\vec{0})}(\vec{x}a^{-1})h(\vec{x})|\det\, a|^{-1}\, d\vec{x}\, d\mu(a) \\ &= \int_D \left[\int_{B_1(\vec{0})a} h(\vec{y})\, d\vec{y}\right] |\det\, a|^{-1}\, d\mu(a). \end{aligned}$$

Since $h(\vec{y}) = 0$ almost everywhere, then the inner integral will be zero for all $a$, so that the right hand side will be zero, giving a contradiction. Thus, $h(\vec{y})$ cannot be zero almost everywhere. It follows that

$$\int_{\mathbb{R}^n} h(\vec{y})\, d\vec{y} > 0 \tag{3.7}$$

and the claim is proved.

Now let $f(\vec{x}) = 2^n h(2\vec{x})$. Then using Tonelli's theorem again,

$$\begin{aligned}\int\limits_{\mathbb{R}^n} h(\vec{x})\, d\vec{x} &= \int\limits_{\mathbb{R}^n} f(\vec{x})\, d\vec{x} \\ &= \int\limits_{\mathbb{R}^n} \Big[\int_D h(\vec{x}a)\, d\mu(a)\Big] f(\vec{x})\, d\vec{x} \\ &= \int\limits_{D} \Big[\int_{\mathbb{R}^n} f(\vec{x}a^{-1})\, h(\vec{x})\, |\det a|^{-1}\, d\vec{x}\Big] d\mu(a)\end{aligned}$$

By setting $g(a) := \int_{\mathbb{R}^n} f(\vec{x}a^{-1})\, h(\vec{x})\, |\det a|^{-1}\, d\vec{x}$ we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} h(\vec{x})\, d\vec{x} &= \int_D g(a)\, d\mu(a) = \int_D g(a^{-1})\, \Delta(a^{-1})\, d\mu(a) \\ &= \int_D \Big[\int_{\mathbb{R}^n} f(\vec{x}a)\, h(\vec{x})\, |\det a|\, d\vec{x}\Big]\, \Delta(a^{-1}) d\mu(a) \\ &= \int_D \Big[\int_{\mathbb{R}^n} f(\vec{x}a)\, h(\vec{x})\, d\vec{x}\Big] \frac{|\det a|}{\Delta(a)} d\mu(a).\end{aligned}$$

Thus, if $\Delta \equiv |\det|$, the last equality becomes

$$\begin{aligned}\int_{\mathbb{R}^n} h(\vec{x})\, d\vec{x} &= \int_{\mathbb{R}^n} h(\vec{x}) \int_D f(\vec{x}a)\, d\mu(a)\, d\vec{x} \\ &= \int_{\mathbb{R}^n} h(\vec{x}) 2^n \Big[\int_D h(2\vec{x}a)\, d\mu(a)\Big]\, d\vec{x} \\ &= 2^n \int_{\mathbb{R}^n} h(\vec{x}) \cdot 1\, d\vec{x}\end{aligned}$$

since the expression in brackets is 1 almost everywhere. From here it follows that $\int\limits_{\mathbb{R}^n} h(\vec{x})\, d\vec{x} = 0$ which is impossible by (3.7). Hence, $\Delta \neq |\det|$. □

**Proposition 3.4.** *If $D$ is admissible, then $D_{\vec{x}}$ is compact for almost every $\vec{x} \in \mathbb{R}^n$.*

*Proof.* Suppose, $D$ is admissible. Thus there exists $h \in L^1(\mathbb{R}^n), h \geq 0$ such that (3.6) holds for almost all $\vec{x}$. Let $\vec{x}$ be so that (3.6) holds, and suppose to contrary that $D_{\vec{x}}$ is not compact.

Let $C$ be any compact subset of $D$. Then we claim that $\int\limits_C h(\vec{x}a)\, d\mu(a) = 0$.

Since multiplication is continuous, and thus for any $b \in D$, $bCC^{-1}$ is compact. We now construct a collection of disjoint compact set $\{b_k C\}_{k=1}^{\infty}$ where $b_k \in D_{\vec{x}}$, it follows that $CC^{-1}$ is also compact. Pick any $b_1 \in D_{\vec{x}}$. Since $D_{\vec{x}}$ is closed but not compact, and $b_1 CC^{-1}$ is compact then $D_{\vec{x}} \not\subseteq b_1 CC^{-1}$. Thus, we can pick $b_2 \in D_{\vec{x}}$ so that $b_2 \notin b_1 CC^{-1}$. In particular, $\forall c_1, c_2 \in C$, $b_2 \neq b_1 c_1 c_2^{-1}$ and thus $b_2 C \cap b_1 C = \emptyset$. This shows that $b_1 C$ and $b_2 C$ are disjoint. Continuing this process we obtain a collection of compact sets $\{b_k C\}_{k=2}^{\infty}$ with $b_k \in D_{\vec{x}} \setminus \{b_1 CC^{-1} \cup b_2 CC^{-1} \cup \ldots \cup b_{k-1} CC^{-1}\}$, and thus the collection $\{b_k C\}_{k=1}^{\infty}$ is disjoint.

By disjointness,

$$
\begin{aligned}
1 = \int_D h(\vec{x}a)\, d\mu(a) &\geq \sum_{k=1}^{\infty} \int_{b_k C} h(\vec{x}a)\, d\mu(a) \\
&= \sum_{k=1}^{\infty} \int_D h(\vec{x}a)\, \chi_{b_k C}(a)\, d\mu(a) \\
&= \sum_{k=1}^{\infty} \int_D h(\vec{x}b_k a)\, \chi_{b_k C}(b_k a)\, d\mu(a) \\
&= \sum_{k=1}^{\infty} \int_D h(\vec{x}b_k a)\, \chi_C(a)\, d\mu(a) \\
&= \sum_{k=1}^{\infty} \int_C h(\vec{x}b_k a)\, d\mu(a).
\end{aligned}
$$

Since $b_k \in D_{\vec{x}}$ for each $k$, we have $\vec{x}b_k = \vec{x}$ and thus

$$
1 \geq \sum_{k=1}^{\infty} \int_C h(\vec{x}a)\, d\mu(a).
$$

As the terms in this series are all identical, it follows that $\int_C h(\vec{x}a)\, d\mu(a) = 0$. This proves the claim.

Since $D$ is a closed subset of $\mathbb{R}^{n^2}$, it is $\sigma$-compact, that is we can write $D = \bigcup_{k=1}^{\infty} C_k$ where $C_k$ are compact subsets of $D$. Then by the claim, $\int_D h(\vec{x}a)\, d\mu(a) \leq \sum_{k=1}^{\infty} \int_{C_k} h(\vec{x}a)\, d\mu = 0$, so that $\int_D h(\vec{x}a)\, d\mu(a) = 0$, which contradicts the assumption that (3.6) holds.

This show that $D_{\vec{x}}$ is compact for almost all $\vec{x} \in \mathbb{R}^n$. $\square$

The following theorem is almost a complete characterization of admissible groups $D$.

**Theorem 3.5.** *(a) If $D$ is admissible, then there exists $a \in D$ such that $\Delta(a) \neq |\det(a)|$ where $\Delta$ denotes the modular function on $D$. Furthermore, the stabilizer of $\vec{x}$ is compact for almost every $\vec{x} \in \mathbb{R}^n$.*

*(b) If there exists $a \in D$ such that $\Delta(a) \neq |\det(a)|$ and if for a.e. $\vec{x} \in \mathbb{R}^n$ $\exists\, \epsilon = \epsilon(\vec{x}) > 0$ such that the $\epsilon$-stabilizer $D^{\epsilon}_{\vec{x}}$ of $\vec{x}$ is compact, then $D$ is admissible.*

*Proof.* Part a) of the theorem was proved in propositions 3.3 and 3.4. The details of the proof of part b) are quite technical so we only sketch the main idea. The detailed proof can be found in Weiss et al. (2002).

**Step I:** By the second hypothesis, the set

$$\Omega_0 = \{\vec{x} \in \mathbb{R}^n : D^{\epsilon}_{\vec{x}} \text{ is non-compact for all } \epsilon > 0\}$$

has measure zero. Given an open ball $B$, one now defines a function $f_B$ by

$$f_B(\vec{x}) = \mu(\{a \in D : \vec{x}a \in \bar{B}\}) = \int_D \chi_{\bar{B}}(\vec{x}a)\, d\mu(a).$$

Loosely speaking, $f_B$ "measures" how much of the orbit $\mathcal{O}_{\vec{x}}$ of $\vec{x}$ intersects with $\bar{B}$, the measure being the Haar measure of $D$ transfered onto the orbit. Note that $f_B(\vec{x}) = f_B(\vec{x}a_1)\ \ \forall\, a_1 \in D$, that is, $f_B$ is constant on each orbit. One then shows that $\vec{x} \in \Omega_0$ iff each open ball $B$ which intersects the orbit $\mathcal{O}_{\vec{x}}$ intersects it in a set of "infinite measure", i.e.

$$f_B(\vec{x}) = \infty \ \ \forall B \text{ s.t. } B \cap \mathcal{O}_{\vec{x}} \neq \emptyset.$$

Next one chooses a countable family of open balls covering $\mathbb{R}^n$, say $\{B_j\}_{j\in\mathbb{N}}$ ,the collection of open balls in $\mathbb{R}^n$ having rational center and positive rational radius, and sets

$$\Omega = \bigcup_{j\geq 1}\{\vec{x} \in \mathbb{R}^n : 0 < f_{B_j} < \infty\}.$$

Then $\Omega_0 \cup \Omega = \mathbb{R}^n$.

Next one sets

$$\Omega_1 = \{\vec{x} \in \mathbb{R}^n : 0 < f_{B_1}(\vec{x}) < \infty\}, \Omega_2 = \{\vec{x} \in \mathbb{R}^n : 0 < f_{B_2}(\vec{x}) < \infty\} \setminus \Omega_1, ...,$$

$$\Omega_j = \{\vec{x} \in \mathbb{R}^n : 0 < f_{B_j}(\vec{x}) < \infty\} \setminus \{\Omega_1 \cup \Omega_2 ... \cup \Omega_{j-1}\}, ...$$

Then $\{\Omega_j\}_{j=1}^{\infty}$ is a disjoint collection of Borel sets whose union is $\Omega$.

By hypothesis, $\Omega_0$ has measure zero, therefore $\Omega^c \subset \Omega_0$ also has measure zero. We now construct a Borel measurable function $g : \mathbb{R}^n \to [0, \infty)$ which is zero on this set of measure zero by

$$g(\vec{x}) = \sum_{j=1}^{\infty} \chi_{\Omega_j}(\vec{x}) g_j(\vec{x})$$

where $g_j(\vec{x}) = \frac{\chi_{\bar{B}_j}(\vec{x})}{f_{B_j}(\vec{x})}$.

Then

$$\int_D g(\vec{x}a) d\mu(a) = \begin{cases} \int_D \frac{\chi_{\overline{B}_j}(\vec{x}a)}{f_{B_j}(\vec{x}a)} d\mu(a) = \frac{1}{f_{B_j}(\vec{x})} \int_D \chi_{\overline{B}_j}(\vec{x}a)\, d\mu(a) = 1 & \text{if } \vec{x} \in \Omega_j \subset \Omega \\ 0 & \text{if } \vec{x} \in \Omega^c \end{cases}$$

**Step II:** The function $g$ need not be integrable. We now use the first hypothesis in b) to modify $g$ into an integrable function $h$ still satisfying

$$\int_D h(\vec{x}a)\, d\mu(a) = 1 \quad \text{for almost all } \vec{x} \in \mathbb{R}^n. \tag{3.8}$$

The idea is to shift $g$ along orbits, by setting

$$E_{jk} = \{\vec{x} \in \mathbb{R}^n : 2^j < \|\vec{x}\| \le 2^{j+1} \text{ and } 2^k < g(\vec{x}) \le 2^{k+1}\}$$

for $j, k \in \mathbb{Z}$, and decomposing $g$ into functions $g_{jk} = g\chi_{E_{jk}}$, so that

$$g(\vec{x}) = \sum_{j,k \in \mathbb{Z}} g_{jk}(\vec{x}) \quad \text{for } \vec{x} \neq 0.$$

Since $E_{jk}$ has a finite measure and since by the second hypothesis, there exists $b \in D$ such that $\Delta(b) \neq |\det b|$, we can find $l(j,k) \in \mathbb{Z}$ such that

$$\Delta(b)^{l(j,k)} 2^{k+1} \mu(E_{jk} b^{-l(j,k)}) < 2^{-(|j|+|k|)}.$$

Now, we set

$$h(\vec{x}) = \sum_{j,k\in\mathbb{Z}} \Delta(b)^{l(j,k)} g_{jk}(\vec{x}b^{l(j,k)}).$$

Then any function $\psi$ with $|\widehat{\psi}|^2 = h$ is an admissible function, since it is easy to verify that $0 \leq h \in L^1(\mathbb{R}^n)$ and (3.8) holds. □

**Remark 3.3.** *In the proofs of proposition 3.3, 3.4 and theorem 3.5 we have only made use of the fact that $D \subset GL_n(\mathbb{R})$ is $\sigma$-compact, locally compact and acts on $\mathbb{R}^n$ by matrix multiplication.*
*It follows that if $D$ is any $\sigma$-compact, locally compact group, and if $\sigma : D \to GL_n(\mathbb{R})$ is a continuous homomorphism, then the above proofs still apply. In this situation, theorem 3.5 can be restated as follows*

*a) If $D$ is admissible, i.e. if there exists $\widehat{\psi} \in L^2(\mathbb{R}^n)$ such that*

$$\int_D |\widehat{\psi}(\vec{x}\pi(a))|^2 d\mu(a) = 1 \qquad a.e.\ \vec{x} \in \mathbb{R}^n$$

*then there exist $a \in D$ such that $\Delta(a) \neq |\det(\pi(a))|$. Furthermore, the stabilizer $D_{\vec{x}} = \{a \in D : \vec{x}\pi(a) = \vec{x}\}$ is compact for almost every $\vec{x} \in \mathbb{R}^n$.*

*b) If there exists $a \in D$ such that $\Delta(a) \neq |\det(\pi(a))|$, and if for almost all $\vec{x} \in \mathbb{R}^n$ there exists $\epsilon > 0$ such that $D_{\vec{x}}^{\epsilon} = \{a \in D : \|\vec{x} - \vec{x}\pi(a)\| \leq \epsilon\}$ is compact, then $D$ is admissible.*

In the following chapter, $D$ will be a two-parameter group.

**Remark 3.4.** *In practice, the above construction of the admissible function $\psi$ is difficult, because we have to compute countably many functions $g_{i\cdot}$. The question of how to obtain simple admissible functions given an admissible group $D$ still remains, and will be discussed at the end of the next chapter.*

# Chapter IV

# Problem Formulation and Main Results

We now turn to matrix groups generated by two matrices both commutative and non commutative. We study which groups are admissible, and investigate the existence of admissible functions with nice smoothness properties.

## 4.1 Problem Formulation

Generally speaking, we will consider matrix groups which depend on two parameters. We will study the following 3 particular cases:

**Case 1**: $D$ is a discrete abelian group generated by two matrices. That is, there exist commuting invertible matrices $A$ and $B$ such that

$$D = \{A^k B^l : \ k, l \in \mathbb{Z}\}.$$

**Case 2**: $D$ is a continuous abelian group generated by two matrices. That is, there exist commuting matrices $A$ and $B$ such that

$$D = \{A^s B^t : \ s, t \in \mathbb{R}\}$$

where $A = e^M$ and $B = e^N$ for some $M, N \in M_n(\mathbb{R})$.

**Case 3**: $D$ is a non-abelian group depending on two continuous or discrete parameters.

In each of these cases we will investigate the existence of admissible functions for the group $D$. As it is very complex to give a complete characterization of all possible groups, we will focus our attention to subgroups of $GL_2(\mathbb{R})$ and

$GL_3(\mathbb{R})$, and in case 3, we only investigate the particular simple cases of $2 \times 2$ matrix groups.

Let us turn to the first two cases. Note that in the first case, $D$ is the image of a discrete two-parameter group

$$\Psi : \mathbb{Z} \times \mathbb{Z} \to GL_n(\mathbb{R})$$

and in the second case, $D$ is the image of a continuous two-parameter group

$$\Phi : \mathbb{R} \times \mathbb{R} \to GL_n(\mathbb{R}).$$

Thus, one may consider these two-parameter groups instead of their images $D$ in $GL_n(\mathbb{R})$. The corresponding $\epsilon$-stabilizers and stabilizers of a point $\vec{x} \in \mathbb{R}^n$ are

$$\Psi_{\vec{x}}^{\epsilon} = \{(k,l) \in \mathbb{Z} \times \mathbb{Z} : ||\vec{x} - \vec{x}A^k B^l|| \leq \epsilon\}$$

$$\Psi_{\vec{x}} = \{(k,l) \in \mathbb{Z} \times \mathbb{Z} : \vec{x} = \vec{x}A^k B^l\}$$

in the first case, and

$$\Phi_{\vec{x}}^{\epsilon} = \{(s,t) \in \mathbb{R} \times \mathbb{R} : ||\vec{x} - \vec{x}A^s B^t|| \leq \epsilon\}$$

$$\Phi_{\vec{x}} = \{(s,t) \in \mathbb{R} \times \mathbb{R} : \vec{x} = \vec{x}A^s B^t\}$$

in the second case. As discussed at the end of chapter II, the wavelet transform $W_\psi$ associated with $\psi \in L^2(\mathbb{R}^n)$ is

$$W_\psi f(s,t,\vec{b}) = \langle f, \psi_{A^s B^t, \vec{b}} \rangle = \frac{1}{\sqrt{|\det A^s B^t|}} \int_{\mathbb{R}^n} f(\vec{x}) \overline{\psi(A^{-s} B^{-t} \vec{x} - \vec{b})} \, d\vec{x} \qquad (4.1)$$

for $f \in L^2(\mathbb{R}^n)$ with $(s,t) \in \mathbb{Z} \times \mathbb{Z}$ or $(s,t) \in \mathbb{R} \times \mathbb{R}$, respectively. Since $\mathbb{Z} \times \mathbb{Z}$ is discrete, the Haar measure is simply the counting measure, so that the reconstruction formula (2.10) in case 1 becomes

$$f(\vec{x}) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} W_\psi f(k,l,\vec{b}) \, \psi_{(A^k B^l, \vec{b})}(\vec{x}) \, d\vec{b} \qquad (4.2)$$

for all $f \in L^2(\mathbb{R}^n)$. The Haar measure on $\mathbb{R} \times \mathbb{R}$ is simply the Lebesgue measure, so in case 2 the reconstruction formula (2.10) is

$$f(\vec{x}) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} W_\psi f(s,t,\vec{b})\, \psi_{(A^s B^t, \vec{b})}(\vec{x})\, d\vec{b}\, ds\, dt. \tag{4.3}$$

By theorem 3.1 in chapter III, the reconstruction formulas (4.2) and (4.3) are valid if and only if the following admissibility conditions hold,

$$\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} |\widehat{\psi}(\vec{x} A^k B^l)|^2 = 1 \tag{4.4}$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{\psi}(\vec{x} A^s B^t)\, ds\, dt = 1, \tag{4.5}$$

respectively, for almost all $\vec{x} \in \mathbb{R}^n$.

Our question is now, for what choices of $A$ and $B$, respectively $M$ and $N$, are the two-parameter group $\Psi$ and $\Phi$ are admissible.

We know by proposition 3.2 in chapter III that admissibility is invariant under a change of basis. In the discrete case (case 1), we thus may choose a basis so that the matrix $A$ is in real Jordan form. In the continuous case (case 2), we choose a basis so that the matrix $M$ of the exponent is in Jordan form. In order to list all possible groups, we then need to classify all commuting matrix pairs.

Since $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{R} \times \mathbb{R}$ are abelian, the modular function is trivial, and using theorem 3.5 we need only to investigate whether

1. there exist compact $\epsilon$-stabilizers $\Psi^{\epsilon}_{\vec{x}}$ and $\Phi^{\epsilon}_{\vec{x}}$ for almost all $\vec{x}$,

2. there exists $d \in D$ such that $|\det d| \neq 1$.

   Note that $\det(e^M) = e^{\operatorname{tr}(M)}$. Then

$$\det\left(e^{sM+tN}\right) = e^{s\cdot \operatorname{tr}(M) + t\cdot \operatorname{tr}(N)}$$

   Thus in case 2, $|\det d| \neq 1$ for some $d$ iff $\operatorname{tr}(M) \neq 0$ or $\operatorname{tr}(N) \neq 0$.

## 4.2 Admissibility Conditions for Groups Generated by Two Matrices

### 4.2.1 Two-Parameter Groups Generated by Two $2 \times 2$ Commuting Matrices

We start with $2 \times 2$ matrix groups. Let us begin with the case of a continuous two-parameter group, $\Phi : (s,t) \to A^s B^t$ where $s, t \in \mathbb{R}$ and $A = e^M, B = e^N$ and $M, N \in M_2(\mathbb{R})$. As discussed above, we may bring the matrix $M$ into real Jordan normal form, and then we need to find all invertible matrices $N$ which commute with $M$. According to the classification of $2 \times 2$ two commuting matrices in appendix A, there are three distinct cases

**Case 1** Both matrices are simultaneously diagonalizable,

$$M = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

**Case 2** Both matrices have only one real eigenvalue and at least one matrix, say $M$, is not diagonalizable. Then in the Jordan basis of $M$,

$$M = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

**Case 3** At least one matrix, say $M$, has complex eigenvalues,

$$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{where } \beta \neq 0.$$

The corresponding continuous matrix groups $D = \{A^s B^t = e^{sM+tN}\}$ generated by $M, N$ are:

**Case A1:** $D = \left\{ \begin{pmatrix} e^{\alpha s + at} & 0 \\ 0 & e^{\beta s + bt} \end{pmatrix} : s, t \in \mathbb{R} \right\}$

**Case A2:** $D = \left\{ e^{\alpha s + at} \begin{pmatrix} 1 & s + bt \\ 0 & 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}.$

**Case A3:** $D = \left\{ e^{\alpha s+at} \begin{pmatrix} \cos(\beta s+bt) & \sin(\beta s+bt) \\ -\sin(\beta s+bt) & \cos(\beta s+bt) \end{pmatrix} : s,t \in \mathbb{R} \right\}$ where $\beta \neq 0$.

Since the $\epsilon$-stabilizers $\Phi_{\vec{x}}^{\epsilon}$ are closed in $\mathbb{R}^2$, in order to show that $\Phi_{\vec{x}}^{\epsilon}$ is compact, we only need to show that it is bounded.

It turns out that only in the first two case the groups is admissible:

**Theorem 4.1.** *Let $\Phi : \mathbb{R}^2 \to D = \left\{A^s B^t = e^{sM+tN} : s,t \in \mathbb{R}\right\}$ be a continuous two-parameter matrix group, with $A = e^M$ and $B = e^N \in GL_2(\mathbb{R})$. Then $\Phi$ is admissible iff both $M$ and $N$ have real eigenvalues and one of the following conditions holds:*

1. *$D$ is as in case A1, and $\alpha b - a\beta \neq 0$.*

2. *$D$ is as in case A2, and $\alpha b - a \neq 0$.*

*Proof.* The idea of the proof is as follows. In each case where the group is admissible, we prove this by showing that there exist compact $\epsilon$-stabilizers $\Phi_{\vec{x}}^{\epsilon}$. On the other hand, in case where the group is not admissible, we show that $\Phi$ has a non-trivial kernel $\operatorname{Ker}\Phi$. Since every non-trivial subgroup of $\mathbb{R}^2$ is unbounded it follows that $\operatorname{Ker}\Phi$ is non-compact, hence the stabilizer $\Phi_{\vec{x}}$ are never compact. We treat each of the 3 cases separately and start with the two cases where both $M$ and $N$ have real eigenvalues.

**Case A1:** $D = \left\{ \begin{pmatrix} e^{\alpha s+at} & 0 \\ 0 & e^{\beta s+bt} \end{pmatrix} : s,t \in \mathbb{R} \right\}$.

The $\epsilon$-stabilizer of $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$ is

$$\Phi_{\vec{x}}^{\epsilon} = \left\{ (s,t) \in \mathbb{R}^2 : x_1^2(e^{\alpha s+at} - 1)^2 + x_2^2(e^{\beta s+bt} - 1)^2 \leq \epsilon^2 \right\} \tag{4.6}$$

Suppose now that $\alpha b - a\beta \neq 0$. We show that $\Phi_{\vec{x}}^{\epsilon}$ is compact provided that $x_1, x_2 \neq 0$ and $\epsilon \leq \min\left\{\frac{|x_1|}{2}, \frac{|x_2|}{2}\right\}$.

In fact, if $(s,t)$ satisfy (4.6), then in particular

$$|e^{\alpha s+at} - 1| \leq \frac{\epsilon}{|x_1|} \leq \frac{1}{2} \qquad \text{and} \qquad |e^{\beta s+bt} - 1| \leq \frac{\epsilon}{|x_2|} \leq \frac{1}{2}.$$

These 2 inequalities are equivalent to

$$\begin{aligned} -\ln 2 &\leq \alpha s + at \leq \ln 3 - \ln 2 \\ -\ln 2 &\leq \beta s + bt \leq \ln 3 - \ln 2. \end{aligned} \tag{4.7}$$

We thus have the system of equations

$$\begin{aligned} \alpha s + at &= k_1 \\ \beta s + bt &= k_2 \end{aligned} \tag{4.8}$$

where $k_1, k_2 \in [-\ln 2, \ln 3 - \ln 2]$. The set of pairs $(s,t)$ satisfying (4.8) is bounded since $\alpha b - a\beta \neq 0$ and thus, $\Phi_{\vec{x}}^{\epsilon}$ is compact. We have shown that for almost all $\vec{x}$, there exist compact $\epsilon$-stabilizers.

On the other hand, suppose that $\alpha b - a\beta = 0$. Then $(s,t) \in \operatorname{Ker}\Phi$ iff

$$\begin{aligned} \alpha s + at &= 0 \\ \beta s + bt &= 0. \end{aligned} \tag{4.9}$$

But as $\alpha b - a\beta = 0$, the solution set of this system of equations forms a non-zero subspace of $\mathbb{R}^2$. Thus, $\operatorname{Ker}\Phi$ is not compact. Since $\Phi_{\vec{x}} \supset \operatorname{Ker}\Phi$, it follows that $\Phi_{\vec{x}}$ is never compact. Note that $\alpha b - a\beta \neq 0$ implies that at least one of $M, N$ has nonzero trace. We conclude that $\Phi$ is admissible iff $\alpha b - a\beta \neq 0$.

**Case A2:** $D = \left\{ e^{\alpha s+at} \begin{pmatrix} 1 & s+bt \\ 0 & 1 \end{pmatrix} : s,t \in \mathbb{R} \right\}$.

The $\epsilon$-stabilizer of $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$ is

$$\Phi_{\vec{x}}^{\epsilon} = \left\{ (s,t) \in \mathbb{R}^2 : x_1^2 (e^{\alpha s+at} - 1)^2 + \left( x_1 (s+bt) e^{\alpha s+at} + x_2 (e^{\alpha s+at} - 1) \right)^2 \leq \epsilon^2 \right\} \tag{4.10}$$

Suppose that $\alpha b - a \neq 0$. We show that $\Phi^{\epsilon}_{\vec{x}}$ is compact provided that $x_1, x_2 \neq 0$ and $\epsilon \leq \min\left\{\frac{|x_1|}{2}, \frac{|x_2|}{2}\right\}$.

In fact, if $(s,t)$ satisfy (4.10), then in particular,

(1) $|e^{\alpha s+at} - 1| \leq \frac{\epsilon}{|x_1|} \leq \frac{1}{2}$

(2) $|\frac{x_1}{x_2}(s+bt)e^{\alpha s+at} + e^{\alpha s+at} - 1| \leq \frac{\epsilon}{|x_2|} \leq \frac{1}{2}$.

The first inequality is equivalent to

$$-\ln 2 \leq \alpha s + at \leq \ln 3 - \ln 2 \tag{4.11}$$

and the second inequality implies that

$$\frac{|x_1|}{|x_2|}|s+bt|e^{\alpha s+at} \leq \frac{1}{2} + |e^{\alpha s+at} - 1| \leq \frac{1}{2} + \frac{1}{2} = 1$$

$$|s+bt| \leq e^{-(\alpha s+at)}\frac{|x_2|}{|x_1|} \leq 2\frac{|x_2|}{|x_1|} =: L < \infty. \tag{4.12}$$

(4.11) and (4.12) show that every $(s,t) \in \Phi^{\epsilon}_{\vec{x}}$ must be a solution of the system of equations

$$\begin{aligned} \alpha s + at &= k_1 \\ s + bt &= k_2 \end{aligned} \tag{4.13}$$

where $k_1 \in [-\ln 2, \ln 3 - \ln 2]$ and $k_2 \in [-L, L]$. Now the set of pairs $(s,t)$ satisfying (4.13) is bounded since $\alpha b - a \neq 0$, and hence, $\Phi^{\epsilon}_{\vec{x}}$ is compact. We have shown that for almost all $\vec{x}$, there exist compact $\epsilon$-stabilizers.

On the other hand, suppose $\alpha b - a = 0$. Then $(s,t) \in \operatorname{Ker}\Phi$ iff

$$\begin{aligned} \alpha s + at &= 0 \\ s + bt &= 0. \end{aligned} \tag{4.14}$$

But as $\alpha b - a = 0$, the solution set of this system forms a non-zero subspace of $\mathbb{R}^2$, so that $\operatorname{Ker}\Phi$ is not compact. Hence $\Phi_{\vec{x}} \supset \operatorname{Ker}\Phi$ is not compact for all $\vec{x}$. We conclude that $\Phi$ cannot be admissible. Note that $\alpha b - a \neq 0$ implies that at least one of $M, N$ has nonzero trace. We have shown that $\Phi$ is admissible iff $\alpha b - a \neq 0$.

Now let us consider the case where $M$ has complex eigenvalues.

**Case A3:** $D = \left\{ e^{\alpha s + at} \begin{pmatrix} \cos(\beta s + bt) & \sin(\beta s + bt) \\ -\sin(\beta s + bt) & \cos(\beta s + bt) \end{pmatrix} : s, t \in \mathbb{R} \right\}$ where $\beta \neq 0$.

It is easy to see that $(s,t) \in \operatorname{Ker} \Phi$ iff

$$e^{\alpha s + at} \cos(\beta s + bt) = 1$$

$$e^{\alpha s + at} \sin(\beta s + bt) = 0.$$

This certainly holds if

$$\alpha s + at = 0 \tag{4.15}$$

$$\text{and} \qquad \beta s + bt = 2k\pi, \quad k \in \mathbb{Z}. \tag{4.16}$$

If $\alpha b - a\beta = 0$, then the solution set of (4.15) and (4.16) forms a non-zero linear subspace of $\mathbb{R}^2$. If $\alpha b - a\beta \neq 0$, then the solution set of (4.15) and (4.16) is

$$\left\{ (s,t) : s = \frac{2ka\pi}{a\beta - \alpha b}, t = -\frac{\alpha}{a} s : k \in \mathbb{Z} \right\}$$

which is unbounded. Thus, $\operatorname{Ker} \Phi$ is unbounded. Since $\operatorname{Ker} \Phi \subset \Phi_{\vec{x}}$, $\Phi_{\vec{x}}$ is never compact for all $\vec{x}$. We conclude that $\Phi$ is not admissible.

This shows that if one of $M$ or $N$ has complex eigenvalues, then $\Phi$ is not admissible. □

**Remark 4.1.** *The proof of the above theorem gives us a complete characterization of* $2 \times 2$ *continuous two-parameter groups that are admissible. We would like to have a basis free characterization for the matrices* $M$ *and* $N$. *A straightforward computation shows that the conditions* $\alpha b - a\beta = 0$, *respectively* $\alpha b - a = 0$ *are equivalent to* $M$ *and* $N$ *being scalar multiples of another.*

*Thus we can rephrase the theorem as follows*

**Corollary 4.2.** *Let $\Phi : \mathbb{R}^2 \to D = \{A^s B^t = e^{sM+tN} : s,t \in \mathbb{R}\}$ be a continuous two-parameter matrix group, with $A = e^M$ and $B = e^N$. Then $\Phi$ is admissible iff both $M$ and $N$ have real eigenvalues and are not constant multiples of each other.*

**Corollary 4.3.** *Let $\Phi : \mathbb{R}^2 \to D = \{A^s B^t = e^{sM+tN} : s,t \in \mathbb{R}\}$ be a continuous two-parameter matrix group generated by two $2 \times 2$ commuting matrices. Then the following are equivalent:*

1. *$\Phi$ is admissible*

2. *The stabilizer $\Phi_{\vec{x}}$ is compact for almost all $\vec{x} \in \mathbb{R}^n$.*

3. *$\Phi$ is one-to-one.*

*Proof.* (1) $\Rightarrow$ (2) This follows from proposition 3.4.

(2) $\Rightarrow$ (3) Pick an $\vec{x}$ such that $\Phi_{\vec{x}}$ is compact. Since $\operatorname{Ker}\Phi \subset \Phi_{\vec{x}}$ and $\operatorname{Ker}\Phi$ is closed, it follows that $\operatorname{Ker}\Phi$ is compact also. But any non-trivial subgroup of $\mathbb{R}^2$ is unbounded, thus $\operatorname{Ker}\Phi = \{0\}$.

(3) $\Rightarrow$ (1) In the above proof we have shown that whenever $\Phi$ is not admissible, then $\operatorname{Ker}\Phi$ is non-trivial. Thus, if $\operatorname{Ker}\Phi$ is trivial, then $\Phi$ must be admissible. □

**Remark 4.2.** *In corollary 4.2, we have formulated admissibility conditions in terms of the matrices $M$ and $N$. We would like to formulate conditions in term of the matrices $A$ and $B$. In general there is the difficulty that the exponential map is not one-to-one, so that one cannot always recover $M$ and $N$ from $A$ and $B$. However, we will show now that this is not a problem.*

*Note that in case A1,*

$$A = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{\beta} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e^{a} & 0 \\ 0 & e^{b} \end{pmatrix}.$$

*The conditions that $\alpha b - a\beta \neq 0$ now becomes*

$$e^{\alpha} e^{b} \neq e^{a} e^{\beta}.$$

*Since the real exponential function is one-to-one, we can easily recover $\alpha, \beta$ from $A$ and $a, b$ from $B$, that is, there exists a unique pair of matrices $M, N$ with real eigenvalues such that $A = e^M, B = e^N$.*

*In case A2,*

$$A = \begin{pmatrix} e^\alpha & e^\alpha \\ 0 & e^\alpha \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e^a & be^a \\ 0 & e^a \end{pmatrix}$$

*and after changing to the Jordan basis of $A$,*

$$A = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} y & z \\ 0 & y \end{pmatrix}$$

*where $u = e^\alpha, y = e^a$ and $z = \frac{be^a}{u}$.*

*Since $b = \frac{z \cdot u}{e^a} = \frac{z \cdot u}{y}$, the condition that $\alpha b - a \neq 0$ now becomes*

$$\alpha b \neq a$$

$$(\ln u) \cdot \frac{z \cdot u}{y} \neq \ln y$$

$$y \cdot \ln y \neq z \cdot u \cdot \ln u.$$

*Again, we can recover $a, b, \alpha$ from $u, y$ and $z$, that is, there exists a unique pair of matrices $M, N$ with real eigenvalues such that $A = e^M$, $B = e^N$.*

*In case A3,*

$$A = e^\alpha \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \quad \text{and} \quad B = e^a \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}$$

*with $\beta \neq 0$. Note that we cannot recover $\beta$ and $b$ from $A$ and $B$ since the trigonometric functions are periodic.*

*The eigenvalues of $A$ and $B$ are $e^\alpha(\cos\beta \pm i \sin\beta)$ and $B = e^a(\cos b \pm i \sin b)$, respectively. Note that $\beta$ and $b$ are both multiples of $2k\pi$ iff $A$ and $B$ have positive* *eigenvalues, in which case A and B can be considered matrices as in case A1. Otherwise, if at least one of A and B has negative or complex eigenvalues, then any pair of matrices M, N such that* $A = e^M, B = e^N$ *must be as in case A3.*

The above discussion shows that case 1 and case 2 hold precisely if both $A$ and $B$ have both positive eigenvalues. We summarize this as follows.

**Proposition 4.4.** *Let A and B be the commuting invertible* $2 \times 2$ *matrices. Then there exists matrices M and N such that* $A = e^M$ *and* $B = e^N$*, and the two-parameter group*

$$\Phi : (s,t) \to A^s B^t = e^{sM} e^{tN}$$

*is admissible iff both A and B have only positive eigenvalues, and one of the following is true:*

1. *A and B are simultaneously diagonalizable,*

$$A = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}$$

   *with* $u \cdot z \neq v \cdot y$.

2. *A and B have each only one eigenvalue, and one of the two matrices, say A is not diagonalizable,*

$$A = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} y & z \\ 0 & y \end{pmatrix}$$

   *and* $y \cdot \ln y \neq z \cdot u \cdot \ln u$.

   *Furthermore, M and N are unique.*

Results for discrete two-parameter groups can be obtained in essentially the same way as above. We will not do this in order to avoid a repetition of the arguments. Instead, the next proposition shows how the results for the continuous case can be used to discuss the discrete case.

**Proposition 4.5.** *Let $A, B$ be in the exponential group of $GL_n(\mathbb{R})$ with $AB = BA$ and let $\psi \in L^2(\mathbb{R}^n)$.*

*1. If $\psi$ is an admissible function for the discrete two-parameter group generated by $A$ and $B$, then it is also an admissible function for the continuous two-parameter group generated by $A$ and $B$.*

*2. If $\psi$ is an admissible function for the continuous two-parameter group generated by $A$ and $B$, then there exists $\widetilde{\psi} \in L^2(\mathbb{R}^n)$ which is admissible for the discrete two-parameter group generated by $A$ and $B$.*

*Proof.* 1. Assume that $\psi$ is an admissible function for the discrete two-parameter group. Then by theorem 3.1, $E_1 = \left\{\vec{y} \in \mathbb{R}^n : \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\vec{y}A^kB^l)|^2 \neq 1\right\}$ is a set of measure zero.

We claim that for almost all $\vec{x} \in \mathbb{R}^n$, $\sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\vec{x}A^sB^tA^kB^l)|^2 = 1$ for almost all $(s,t) \in [0,1] \times [0,1] =: [0,1]^2$. Set

$$E = \{(\vec{x},(s,t)) \in \mathbb{R}^n \times [0,1]^2 : \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\vec{x}A^sB^tA^kB^l)|^2 \neq 1\}.$$

Then $E$ is $\lambda_n \times \lambda_2$-measurable, where $\lambda_n$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $\lambda_2$ the Lebesgue measure on $[0,1]^2$. It follows from the definition of the product measure and theorem 2.2 that

$$(\lambda_n \times \lambda_2)(E) = \int_{[0,1]^2} \lambda_n(E^{(s,t)})\, d\lambda_2(s,t) \tag{4.17}$$

where

$$\begin{aligned} E^{(s,t)} &= \{\vec{x} \in \mathbb{R}^n : (\vec{x}, s, t) \in E\} \\ &= \{\vec{x} \in \mathbb{R}^n : \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\vec{x}A^sB^tA^kB^l)|^2 \neq 1\}. \\ &= E_1 A^{-s} B^{-t} \end{aligned}$$

Hence, $\lambda_n(E^{(s,t)}) = \lambda_n(E_1)|\det A|^{-s}|\det B|^{-t} = 0$. It follows from (4.17) that

$(\lambda_n \times \lambda_2)(E) = 0$. By theorem 2.2 again,

$$\int_{\mathbb{R}^n} \lambda_2(E_{\vec{x}})\, d\lambda_n(\vec{x}) = (\lambda_n \times \lambda_2)(E) = 0,$$

so that $\lambda_2(E_{\vec{x}}) = 0$ for almost all $\vec{x}$. That is, for almost all $\vec{x} \in \mathbb{R}^n$,

$$\sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\vec{x}A^s B^t A^k B^l)|^2 = 1 \quad \text{for almost all } (s,t) \in [0,1]^2. \tag{4.18}$$

This proves the claim.

Now if $\vec{x}$ is so that $\lambda_2(E_{\vec{x}}) = 0$, then using Tonelli's theorem

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\psi}(\vec{x}A^s B^t)|^2 dsdt &= \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \int_l^{l+1} \int_k^{k+1} |\widehat{\psi}(\vec{x}A^s B^t)|^2 dsdt \\
&= \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \int_0^1 \int_0^1 |\widehat{\psi}(\vec{x}A^{s+k} B^{t+l})|^2 dsdt \\
&= \int_0^1 \int_0^1 \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\vec{x}A^{s+k} B^{t+l})|^2 dsdt \\
&= \int_0^1 \int_0^1 \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\vec{x}A^s B^t A^k B^l)|^2 dsdt \\
&= \int_0^1 \int_0^1 1\, dsdt = 1.
\end{aligned}$$

The last equality is valid since (4.18) holds.

2. Assume that $\psi \in L^2(\mathbb{R}^n)$ is an admissible function for the continuous two-parameter group. Then $\int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\psi}(\vec{x}A^s B^t)|^2 dsdt = 1$ for almost all $\vec{x} \in \mathbb{R}^n$. Define a function $\widetilde{\psi}$ by

$$\widetilde{\psi}(\vec{x}) = \Big(\int_0^1 \int_0^1 |\widehat{\psi}(\vec{x}A^s B^t)|^2 dsdt\Big)^{1/2}.$$

Then $\widetilde{\psi}(\vec{x}) \geq 0$ and using Tonelli's theorem again,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\widetilde{\psi}(\vec{x})|^2\, d\vec{x} &= \int_{\mathbb{R}^n} \Big[\int_0^1 \int_0^1 |\widehat{\psi}(\vec{x}A^s B^t)|^2\, dsdt\Big]\, d\vec{x} \\
&= \int_0^1 \int_0^1 \Big[\int_{\mathbb{R}^n} |\widehat{\psi}(\vec{x}A^s B^t)|^2\, d\vec{x}\Big]\, dsdt \\
&= \int_0^1 \int_0^1 |\det A|^{-s} |\det B|^{-t} \Big[\int_{\mathbb{R}^n} |\widehat{\psi}(\vec{x})|^2\, d\vec{x}\Big]\, dsdt < \infty.
\end{aligned}$$

Hence, $\widetilde{\psi} \in L^2(\mathbb{R}^n)$. Furthermore,

$$\begin{aligned}
\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\widetilde{\psi}(\vec{x}A^kB^l)|^2 &= \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_0^1\int_0^1 |\widehat{\psi}(\vec{x}A^kB^lA^sB^t)|^2 dsdt \\
&= \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_0^1\int_0^1 |\widehat{\psi}(\vec{x}A^{s+k}B^{t+l})|^2 dsdt \\
&= \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_l^{l+1}\int_k^{k+1} |\widehat{\psi}(\vec{x}A^sB^t)|^2 dsdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\psi}(\vec{x}A^sB^t)|^2 dsdt = 1.
\end{aligned}$$

Thus, the inverse Fourier transform of $\widetilde{\psi}$ is admissible for the discrete two-parameter group. □

In the discrete case, the matrices $A$ and $B$ need not be exponentials. However, proposition 2.11 shows that the matrices $A^2$ and $B^2$ are always exponential matrices. It follows from proposition 4.5 that the result of proposition 4.4 also applies to the discrete two-parameter group generated by $A^2$ and $B^2$.

We now show that this is sufficient to determine whether the discrete parameter group generated by $A$ and $B$ is admissible.

**Proposition 4.6.** *Let $A, B$ be commuting matrices in $GL_n(\mathbb{R})$. Then the discrete two-parameter group*

$$\Psi : \mathbb{Z}^2 \to \{A^kB^l : k, l \in \mathbb{Z}\}$$

*is admissible iff*

$$\Psi_1 : \mathbb{Z}^2 \to \{(A^2)^k(B^2)^l : k, l \in \mathbb{Z}\}$$

*is admissible.*

*Proof.* ($\Rightarrow$) Let $\psi \in L^2(\mathbb{R}^n)$ be admissible for $\Psi$. Let $\psi_1$ be the inverse Fourier transform of $\widehat{\psi}_1(\vec{x}) = \left|\widehat{\psi}(\vec{x})\right|^2 + \left|\widehat{\psi}(\vec{x}A)\right|^2 + \left|\widehat{\psi}(\vec{x}B)\right|^2 + \left|\widehat{\psi}(\vec{x}AB)\right|^2$. Then

$$\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\widehat{\psi}_1(\vec{x}(A^2)^k(B^2)^l)\right|^2 = \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\Big(\left|\widehat{\psi}(\vec{x}A^{2k}B^{2l})\right|^2 + \left|\widehat{\psi}(\vec{x}A^{2k+1}B^{2l})\right|^2$$
$$+ \left|\widehat{\psi}(\vec{x}A^{2k}B^{2l+1})\right|^2 + \left|\widehat{\psi}(\vec{x}A^{2k+1}B^{2l+1})\right|^2\Big)$$
$$= \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\widehat{\psi}(\vec{x}A^kB^l)\right|^2 = 1$$

for almost all $\vec{x} \in \mathbb{R}^n$. This shows that $\psi_1$ is admissible for the group $\Psi_1$.

($\Leftarrow$) Let $\psi$ be admissible for the group $\Psi_1$. Set

$$E_0 = \{\vec{x} \in \mathbb{R}^n : \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(\vec{x}(A^2)^k(B^2)^l)|^2 \neq 1\}$$

so that $\lambda(E_0) = 0$. Then set

$$E = E_0 \cup E_0A^{-1} \cup E_0B^{-1} \cup E_0A^{-1}B^{-1}$$

so that $\lambda(E) = 0$ also. If $\vec{x} \notin E$, then

$$\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(\vec{x}A^iB^j(A^2)^k(B^2)^l)|^2 = 1 \;\; \forall i = 0,1, \; j = 0,1$$

so that

$$\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\frac{1}{2}\widehat{\psi}(\vec{x}A^kB^l)\right|^2 = \frac{1}{4}\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\left|\widehat{\psi}(\vec{x}A^{2k}B^{2l})\right|^2 + \left|\widehat{\psi}(\vec{x}A^{2k+1}B^{2l})\right|^2\right.$$
$$\left.+ \left|\widehat{\psi}(\vec{x}A^{2k}B^{2l+1})\right|^2 + \left|\widehat{\psi}(\vec{x}A^{2k+1}B^{2l+1})\right|^2\right)$$
$$= \frac{1}{4}(1+1+1+1) = 1.$$

Hence, $\frac{1}{2}\psi$ is admissible for $\Psi$. □

In proposition 4.4 we have formulated admissibility conditions for the continuous two-parameter group generated by commuting invertible matrices $A$ and $B$. Now we would like to formulate conditions for discrete two-parameter group.

Proposition 4.5 state that if $\widetilde{A}$ and $\widetilde{B}$ are exponential matrices, then the continuous two-parameter group generated by $\widetilde{A}$ and $\widetilde{B}$ is admissible iff the discrete two-parameter group generated by $\widetilde{A}$ and $\widetilde{B}$ is. Since $A^2$ and $B^2$ are exponentials, and it follows that, the continuous two-parameter group generated by $\widetilde{A} = A^2$ and $\widetilde{B} = B^2$ is admissible iff the discrete two-parameter group generated by $A$ and $B$ is admissible. Let us investigate how $A$ and $B$ must look like. By proposition 4.4, $\widetilde{A}$ and $\widetilde{B}$ must be as follows:

1. $\widetilde{A}$ and $\widetilde{B}$ are simultaneously diagonalizable,

$$\widetilde{A} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \quad \textit{and} \quad \widetilde{B} = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}$$

with $u \cdot z \neq v \cdot y$. Since $u, v, y, z > 0$, one easily verifies that $A$ and $B$ must be of the form

$$A = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$$

where $c = \pm\sqrt{u}, d = \pm\sqrt{v}, e = \pm\sqrt{y}$, and $f = \pm\sqrt{z}$.

The condition $u \cdot z \neq v \cdot y$ is equivalent to $|c \cdot f| \neq |e \cdot d|$.

2. $\widetilde{A}$ and $\widetilde{B}$ have each only one eigenvalue, and one of the two matrices, say $\widetilde{A}$ is not diagonalizable,

$$\widetilde{A} = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} \quad \textit{and} \quad \widetilde{B} = \begin{pmatrix} y & z \\ 0 & y \end{pmatrix}$$

and $y \cdot \ln y \neq z \cdot u \cdot \ln u$. It is easy to verify that $A$ and $B$ must be of the form

$$A = \begin{pmatrix} c & \frac{1}{2c} \\ 0 & c \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} d & \frac{z}{2d} \\ 0 & d \end{pmatrix}$$

where $c = \pm\sqrt{u}, d = \pm\sqrt{y}$. After a change of basis,

$$A = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} d & w \\ 0 & d \end{pmatrix}$$

where $w = \frac{cz}{d}$. The above condition now becomes

$$d^2 \cdot 2\ln|d| \neq \frac{dw}{c} \cdot c^2 \cdot 2\ln|c|$$

$$d \cdot \ln|d| \neq w \cdot c \cdot \ln|c|.$$

**Corollary 4.7.** *Let $A$ and $B$ be the commuting invertible $2 \times 2$ matrices. Then two-parameter group*

$$\Psi : (k,l) \to A^k B^l$$

*is admissible iff both $A$ and $B$ have only real eigenvalues, and one of the following is true:*

1. *$A$ and $B$ are simultaneously diagonalizable,*

$$A = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}$$

   *with $|u \cdot z| \neq |v \cdot y|$.*

2. *$A$ and $B$ have each only one eigenvalue, and one of the two matrices, say $A$ is not diagonalizable,*

$$A = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} y & z \\ 0 & y \end{pmatrix}$$

   *and $y \cdot \ln|y| \neq z \cdot u \cdot \ln|u|$.*

### 4.2.2 Two-Parameter Groups Generated by Two $3 \times 3$ Commuting Matrices

Let $M, N \in M_3(\mathbb{R})$ be non-zero matrices with $MN = NM$ and suppose, at least one of $M, N$ has non-zero trace. As before, we bring $M$ into real Jordan normal form and classify all matrices $N$ commuting with $M$. After exchanging $M$ and $N$ if necessary, there are six cases as listed in appendix B:

**Case 1**: $M = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ commutes with $N = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

**Case 2**: $M = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$ commutes with $N = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$.

**Case 3**: $M = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ commutes with $N = \begin{pmatrix} a & b & d \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$ where $a \neq c$ and $d \neq 0$.

**Case 4**: $M = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ commutes with $N = \begin{pmatrix} a & b & d_2 \\ 0 & a & 0 \\ 0 & d_1 & a \end{pmatrix}$

**Case 5**: $M = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$ commutes with $N = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$ where $\alpha \neq \beta$.

**Case 6**: $M = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ commutes with $N = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}$ where $\beta \neq 0$.

We start with the continuous two-parameters group $\Phi : \mathbb{R}^2 \to GL_3(\mathbb{R})$ determined by matrices $A = e^M$ and $B = e^N$ where $M, N$ are as above. Then the corresponding images, $D = \{A^s B^t = e^{sM+tN}\}$ in $GL_3(\mathbb{R})$ are

**Case B1**: $D = \left\{ \begin{pmatrix} e^{\alpha s+at} & 0 & 0 \\ 0 & e^{\beta s+bt} & 0 \\ 0 & 0 & e^{\gamma s+ct} \end{pmatrix} : s, t \in \mathbb{R} \right\}$

**Case B2:** $D = \left\{ e^{\alpha s+at} \begin{pmatrix} 1 & s+bt & \frac{(s+bt)^2}{2}+ct \\ 0 & 1 & s+bt \\ 0 & 0 & 1 \end{pmatrix} : s,t \in \mathbb{R} \right\}$

**Case B3:** $D = \left\{ e^{\alpha s+at} \begin{pmatrix} 1 & s+bt & \frac{d(1-e^{ct-at})}{a-c} \\ 0 & 1 & 0 \\ 0 & 0 & e^{ct-at} \end{pmatrix} : s,t \in \mathbb{R} \right\}$ where $a \neq c$ and $d \neq 0$.

**Case B4:** $D = \left\{ e^{\alpha s+at} \begin{pmatrix} 1 & bt+\frac{d_1d_2t^2}{2}+s & d_2t \\ 0 & 1 & 0 \\ 0 & d_1t & 1 \end{pmatrix} : s,t \in \mathbb{R} \right\}$

**Case B5:** $D = \left\{ \begin{pmatrix} e^{\alpha s+at} & e^{\alpha s+at}(s+bt) & 0 \\ 0 & e^{\alpha s+at} & 0 \\ 0 & 0 & e^{\beta s+ct} \end{pmatrix} : s,t \in \mathbb{R} \right\}$

where $\alpha \neq \beta$.

**Case B6:** $D = \left\{ \begin{pmatrix} e^{\alpha s+at}\cos(\beta s+bt) & e^{\alpha s+at}\sin(\beta s+bt) & 0 \\ -e^{\alpha s+at}\sin(\beta s+bt) & e^{\alpha s+at}\cos(\beta s+bt) & 0 \\ 0 & 0 & e^{\gamma s+ct} \end{pmatrix} : s,t \in \mathbb{R} \right\}$

where $\beta \neq 0$.

**Theorem 4.8.** *Let $\Phi$ be a continuous two-parameter matrix group generated by two commuting matrices $A = e^M$ and $B = e^N$ as above, and suppose that at least one of $M$ and $N$ has non-zero trace. $\Phi$ is admissible if and only if one of the following holds*

1. *$D$ is as in case B1 and at least one of the following holds:*

   - $\alpha b - \beta a \neq 0$.
   - $\alpha c - \gamma a \neq 0$.

- $\beta c - \gamma b \neq 0$.

2. *D is as in case B2 satisfying the condition:* $c \neq 0$ *or* $\alpha b - a \neq 0$.

3. *D is as in case B3.*

4. *D is as in case B4 and at least one of the following holds:*

   *i)* $d_2 \neq 0$

   *ii)* $\alpha b - a \neq 0$.

   *iii)* $\alpha \neq 0$ *and* $d_1 \neq 0$.

5. *D is as in case B5 and one of the following holds:*

   - $\alpha b - a \neq 0$.
   - $\beta a - \alpha c \neq 0$.
   - $\beta b - c \neq 0$.

6. *D is as in case B6 satisfying the condition:* $\gamma a - \alpha c \neq 0$.

*Proof.* The proof proceeds similar to the $2 \times 2$ case. We show that under the above assumptions, there exist bounded, and hence compact $\epsilon$-stabilizers for almost all $\vec{x} \in \mathbb{R}^3$, which implies that $\Phi$ is admissible. On the other hand, if the above assumptions do not hold, then the stabilizers $\Phi_{\vec{x}}$ are not compact for almost all $\vec{x}$, so that $\Phi$ can not be admissible.

**Case B1**: $D = \left\{ \begin{pmatrix} e^{\alpha s+at} & 0 & 0 \\ 0 & e^{\beta s+bt} & 0 \\ 0 & 0 & e^{\gamma s+ct} \end{pmatrix} : s,t \in \mathbb{R} \right\}$.

The $\epsilon$-stabilizer of $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is

$$\Phi_{\vec{x}}^{\epsilon} = \left\{ (s,t) \in \mathbb{R} \times \mathbb{R} : x_1^2(e^{\alpha s+at} - 1)^2 + x_2^2(e^{\beta s+bt} - 1)^2 + x_3^2(e^{\gamma s+ct} - 1)^2 \leq \epsilon^2 \right\}.$$

Suppose, $x_1, x_2, x_3 \neq 0$, choose $\epsilon > 0$ such that $\epsilon \leq \min\left\{\frac{|x_1|}{2}, \frac{|x_2|}{2}, \frac{|x_3|}{2}\right\}$. We claim that $\Phi_{\bar{x}}^{\epsilon}$ is compact.

In fact, if $(s,t) \in \Phi_{\bar{x}}^{\epsilon}$, then we obtain the three inequalities

(1) $|e^{\alpha s + at} - 1| \leq \frac{\epsilon}{|x_1|} \leq \frac{1}{2}$

(2) $|e^{\beta s + bt} - 1| \leq \frac{\epsilon}{|x_2|} \leq \frac{1}{2}$

(3) $|e^{\gamma s + ct} - 1| \leq \frac{\epsilon}{|x_3|} \leq \frac{1}{2}$.

These inequalities are equivalent to

$$\begin{array}{rcccl} -\ln 2 & \leq & \alpha s + at & \leq & \ln 3 - \ln 2 \\ -\ln 2 & \leq & \beta s + bt & \leq & \ln 3 - \ln 2 \\ -\ln 2 & \leq & \gamma s + ct & \leq & \ln 3 - \ln 2 \end{array} \tag{4.19}$$

The system of inequalities (4.19) can be written as

$$\begin{aligned} \alpha s + at &= k_1 \\ \beta s + bt &= k_2 \\ \gamma s + ct &= k_3 \end{aligned} \tag{4.20}$$

where $k_1, k_2$ and $k_3 \in [-\ln 2, \ln 3 - \ln 2]$. The pairs $(s,t)$ which satisfy system (4.20) remain in some bounded set provided that at least one of the following holds:

$$\begin{aligned} \alpha b - \beta a &\neq 0 \\ \alpha c - \gamma a &\neq 0. \\ \beta c - \gamma b &\neq 0. \end{aligned} \tag{4.21}$$

Thus, $\Phi_{\bar{x}}^{\epsilon}$ is bounded if at least one of the inequalities in (4.21) holds. This proves the claim.

On the other hand, suppose that none of the three conditions in (4.21) hold. Note that the kernel of $\Phi$ is the solution set of the system

$$\begin{aligned} \alpha s + at &= 0 \\ \beta s + bt &= 0 \\ \gamma s + ct &= 0. \end{aligned} \tag{4.22}$$

Since none of the conditions in (4.21) holds, the set of pairs $(s,t)$ solving (4.22) forms a non-zero subspace of $\mathbb{R}^2$. That is, $\operatorname{Ker}\Phi$ is unbounded. Since $\operatorname{Ker}\Phi \subset \Phi_{\vec{x}}$, it follows that $\Phi_{\vec{x}}$ is unbounded for all $\vec{x} \in \mathbb{R}^3$. Thus, $\Phi$ can not be admissible.

**Case B2:** $D = \left\{ e^{\alpha s+at} \begin{pmatrix} 1 & s+bt & \frac{(s+bt)^2}{2}+ct \\ 0 & 1 & s+bt \\ 0 & 0 & 1 \end{pmatrix} : s,t \in \mathbb{R} \right\}$

The $\epsilon$-stabilizer of $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is

$$\Phi^{\epsilon}_{\vec{x}} = \Bigg\{ (s,t) \in \mathbb{R}\times\mathbb{R} : x_1^2(e^{\alpha s+at}-1)^2 + [x_1 e^{\alpha s+at}(s+bt) + x_2(e^{\alpha s+at}-1)]^2 + \left[x_1 e^{\alpha s+at}\left(\frac{(s+bt)^2}{2}+ct\right) + x_2 e^{\alpha s+at}(s+bt) + x_3(e^{\alpha s+at}-1)\right]^2 \le \epsilon^2 \Bigg\}.$$

Suppose, $x_1, x_2, x_3 \neq 0$, and choose $\epsilon > 0$ such that $\epsilon \le \min\left\{\frac{|x_1|}{2}, \frac{|x_2|}{2}, \frac{|x_3|}{2}\right\}$. We claim that $\Phi^{\epsilon}_{\vec{x}}$ is compact.

In fact, if $(s,t) \in \Phi^{\epsilon}_{\vec{x}}$, then we obtain the three inequalities

(1) $|e^{\alpha s+at}-1| \le \frac{\epsilon}{|x_1|} \le \frac{1}{2} \quad \Rightarrow \quad \frac{1}{2} \le e^{\alpha s+at} \le \frac{3}{2}$

(2) $\left|\frac{x_1}{x_2} e^{\alpha s+at}(s+bt) + e^{\alpha s+at} - 1\right| \le \frac{\epsilon}{|x_2|} \le \frac{1}{2}$

(3) $\left|\frac{x_1}{x_3} e^{\alpha s+at}\left(\frac{(s+bt)^2}{2}+ct\right) + \frac{x_2}{x_3} e^{\alpha s+at}(s+bt) + e^{\alpha s+at} - 1\right| \le \frac{\epsilon}{|x_3|} \le \frac{1}{2}$

By calculation, inequality (1) is equivalent to

$$-\ln 2 \le \alpha s + at \le \ln 3 - \ln 2. \tag{4.23}$$

Then inequality (2) implies that

$$\left|\frac{x_1}{x_2}\right| e^{\alpha s+at} |s+bt| \le \frac{1}{2} + |e^{\alpha s+at} - 1| \le \frac{1}{2} + \frac{1}{2} = 1$$

$$|s+bt| \le e^{-(\alpha s+at)}\left|\frac{x_2}{x_1}\right| \le 2\left|\frac{x_2}{x_1}\right| =: L. \tag{4.24}$$

From inequalities (4.23) and (4.24) we obtain the system of equations

$$\begin{aligned} \alpha s + at &= k_1 \\ s + bt &= k_2 \end{aligned} \tag{4.25}$$

where $k_1 \in [-\ln 2, \ln 3 - \ln 2]$ and $k_2 \in [-L, L]$.

The third inequality implies that

$$\begin{aligned} \left|\frac{x_1}{x_3} e^{\alpha s+at}\left(\frac{(s+bt)^2}{2} + ct\right)\right| &\le \frac{1}{2} + \left|\frac{x_2}{x_3}\right| e^{\alpha s+at}|s+bt| + |e^{\alpha s+at} - 1| \\ &\le \frac{1}{2} + \frac{3}{2}\left|\frac{x_2}{x_3}\right| L + \frac{1}{2} \end{aligned}$$

and thus

$$\begin{aligned} \left|\frac{(s+bt)^2}{2} + ct\right| &\le \left(1 + \frac{3}{2}\left|\frac{x_2}{x_3}\right| L\right)\left|\frac{x_3}{x_1}\right| e^{-(\alpha s+at)} \\ &\le \left(1 + \frac{3}{2}\left|\frac{x_2}{x_3}\right| L\right)\left|\frac{x_3}{x_2}\right| \cdot 2 =: K_1 \\ |ct| \le K_1 + \frac{(s+bt)^2}{2} &\le K_1 + \frac{L^2}{2} := K. \end{aligned}$$

Hence, $t$ must lie in some bounded set if $c \neq 0$. On the other hand, if $c = 0$, we come back to system (4.25), and see that $t$ is in some bounded set if $\alpha b - a \neq 0$. In both cases it follows from (4.25) that $s$ remains in some bounded set also, so that $\Phi_{\bar{x}}^{\epsilon}$ is bounded. This proves the claim.

On the other hand, suppose that $c = 0$ and $\alpha b - a = 0$. Note that the kernel of $\Phi$ is the solution set of the system

$$\begin{aligned} \alpha s + at &= 0 \\ s + bt &= 0 \\ \frac{(s+bt)^2}{2} + ct &= 0 \end{aligned} \tag{4.26}$$

Since $c = 0$ and $\alpha b - a = 0$, the set of pairs $(s,t)$ solving (4.26) forms a non-zero subspace of $\mathbb{R}^2$. Thus $\operatorname{Ker}\Phi$ is unbounded and then $\Phi_{\vec{x}}$ is unbounded $\forall \vec{x} \in \mathbb{R}^3$. Hence, $\Phi$ is not admissible.

**Case B3:** $D = \left\{ e^{\alpha s + at} \begin{pmatrix} 1 & s+bt & \frac{d(1-e^{ct-at})}{a-c} \\ 0 & 1 & 0 \\ 0 & 0 & e^{ct-at} \end{pmatrix} : s,t \in \mathbb{R} \right\}$ where $a \neq c$ and $d \neq 0$.

The epsilon stabilizer of $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is

$$\Phi_{\vec{x}}^{\epsilon} = \Bigg\{ (s,t) \in \mathbb{R} \times \mathbb{R} : x_1^2 (e^{\alpha s + at} - 1)^2 + \left[ x_1 e^{\alpha s + at}(s + bt) + x_2(e^{\alpha s + at} - 1) \right]^2 + \left[ x_1 e^{\alpha s + at} \left( \frac{d(1 - e^{ct - at})}{a - c} \right) + x_3 (e^{\alpha s + ct} - 1) \right]^2 \leq \epsilon^2 \Bigg\}.$$

Suppose, $x_1, x_2, x_3 \neq 0$ and $(c-a)x_3 + dx_1 \neq 0$. For any $\epsilon > 0$, if $(s,t) \in \Phi_{\vec{x}}^{\epsilon}$, we have in particular,

(1) $|x_1(e^{\alpha s + at} - 1)| \leq \epsilon$

(2) $\left| (s+bt) e^{\alpha s + at} \frac{x_1}{x_2} + e^{\alpha s + at} - 1 \right| \leq \frac{\epsilon}{|x_2|}$

(3) $\left| \frac{x_1}{x_3} \left( \frac{d(1 - e^{ct-at})}{a-c} \right) e^{\alpha s + at} + (e^{\alpha s + ct} - 1) \right| \leq \frac{\epsilon}{|x_3|}$.

Since $d \neq 0$, given $\delta > 0$ we can choose $\epsilon = \epsilon(\delta) > 0$ such that $\epsilon \leq \min\left\{\frac{\delta|x_3(a-c)|}{2|d|}, \frac{\delta|x_3|}{2}\right\}$.

By the first inequality, we get for $(s,t) \in \Phi^{\epsilon}_{\bar{x}}$,

$$|x_1(e^{\alpha s+at} - 1)| \leq \epsilon \leq \frac{\delta|x_3(a-c)|}{2|d|}$$

or

$$\left|\frac{x_1 d}{x_3(a-c)}(e^{\alpha s+at} - 1)\right| \leq \frac{\delta}{2}. \tag{4.27}$$

It follows from the third inequality that

$$\left|\frac{x_1 d}{x_3(a-c)}e^{\alpha s+at}(1 - e^{ct-at}) + e^{\alpha s+ct} - 1\right| \leq \frac{\epsilon}{|x_3|} \leq \frac{\delta}{2}$$

or

$$\left|\frac{x_1 d}{x_3(a-c)}e^{\alpha s+at} + e^{\alpha s+ct}\left(1 - \frac{x_1 d}{x_3(a-c)}\right) - 1\right| \leq \frac{\delta}{2}$$

$$\left|\frac{x_1 d}{x_3(a-c)}e^{\alpha s+at} + e^{\alpha s+ct}\left(\frac{x_3(a-c) - x_1 d}{x_3(a-c)}\right) - 1\right| \leq \frac{\delta}{2}. \tag{4.28}$$

Consider

$$\begin{aligned}
&\left|\frac{x_1 d}{x_3(a-c)} - 1 + e^{\alpha s+ct}\left(\frac{x_3(a-c) - x_1 d}{x_3(a-c)}\right)\right| \\
&\quad \leq \left|\frac{x_1 d}{x_3(a-c)} + \frac{x_1 d}{x_3(a-c)}e^{\alpha s+at} - \frac{x_1 d}{x_3(a-c)}e^{\alpha s+at}\right. \\
&\qquad \left. + e^{\alpha s+ct}\left(\frac{x_3(a-c) - x_1 d}{x_3(a-c)}\right) - 1\right| \\
&\quad \leq \left|\frac{x_1 d}{x_3(a-c)}e^{\alpha s+at} + e^{\alpha s+ct}\left(\frac{x_3(a-c) - x_1 d}{x_3(a-c)}\right) - 1\right| \\
&\qquad + \left|\frac{x_1 d}{x_3(a-c)}(1 - e^{\alpha s+at})\right| \\
&\quad \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad \text{by (4.27) and (4.28).}
\end{aligned}$$

Rewriting the left hand side,

$$\left|\frac{x_3(a-c) - x_1 d}{x_3(a-c)}\right| |e^{\alpha s+ct} - 1| \leq \delta$$

$$|e^{\alpha s+ct} - 1| \leq \delta\left|\frac{x_3(a-c)}{x_3(a-c) - x_1 d}\right|. \tag{4.29}$$

We now choose $\delta$ so that $0 < \delta \leq \frac{1}{2} \min \left\{ \left| \frac{x_3(a-c)-x_1 d}{x_3(a-c)} \right|, \left| \frac{x_1 d}{2x_3(a-c)} \right| \right\}$. By (4.27) and (4.29) we get

$$|e^{\alpha s + at} - 1| \leq \frac{1}{2}$$
$$|e^{\alpha s + ct} - 1| \leq \frac{1}{2},$$

so that $s$ and $t$ must solve the system of equations

$$\begin{aligned} \alpha s + at &= k_1 \\ \alpha s + ct &= k_2 \end{aligned} \tag{4.30}$$

where $k_1, k_2 \in [-\ln 2, \ln 3 - \ln 2]$. By the assumption $a \neq c$, the set of pairs $(s,t)$ satisfying (4.30) is bounded. We have shown that for almost all $\vec{x} \in \mathbb{R}^3$, there exists $\epsilon > 0$ such that $\Phi^{\epsilon}_{\vec{x}}$ is compact, so that $\Phi$ is admissible.

**Case B4:** $D = \left\{ e^{\alpha s + at} \begin{pmatrix} 1 & bt + \frac{d_1 d_2 t^2}{2} + s & d_2 t \\ 0 & 1 & 0 \\ 0 & d_1 t & 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$

The $\epsilon$-stabilizer of $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is

$$\Phi^{\epsilon}_{\vec{x}} = \Big\{ (s,t) \in \mathbb{R} \times \mathbb{R} : x_1^2 (e^{\alpha s + at} - 1)^2 + \Big[ x_1 e^{\alpha s + at} \Big( bt + \frac{d_1 d_2 t^2}{2} + s \Big) + x_2 (e^{\alpha s + at} - 1) + x_3 d_1 t e^{\alpha s + at} \Big]^2 + \Big[ x_1 d_2 t e^{\alpha s + at} + x_3 (e^{\alpha s + at} - 1) \Big]^2 \leq \epsilon^2 \Big\}.$$

Suppose, $x_1, x_2, x_3 \neq 0$ and choose $\epsilon > 0$ such that $\epsilon \leq \min \left\{ \frac{|x_1|}{2}, \frac{|x_2|}{2}, \frac{|x_3|}{2} \right\}$. We claim that $\Phi^{\epsilon}_{\vec{x}}$ is compact for almost all such $\vec{x}$, under the given assumptions on the entries of the matrices.

In fact, if $(s,t) \in \Phi^{\epsilon}_{\vec{x}}$, then we obtain the three inequalities

(1) $|e^{\alpha s + at} - 1| \leq \frac{\epsilon}{|x_1|} \leq \frac{1}{2} \quad \Rightarrow \quad \frac{1}{2} \leq e^{\alpha s + at} \leq \frac{3}{2}$

(2) $\left| \frac{x_1}{x_2} e^{\alpha s + at} \left( bt + \frac{d_1 d_2 t^2}{2} + s \right) + (e^{\alpha s + at} - 1) + \frac{x_3}{x_2} d_1 t e^{\alpha s + at} \right| \leq \frac{\epsilon}{|x_2|} \leq \frac{1}{2}$

(3) $\left| \frac{x_1}{x_3} d_2 t \, e^{\alpha s + at} + e^{\alpha s + at} - 1 \right| \leq \frac{\epsilon}{|x_3|} \leq \frac{1}{2}$

By calculation, inequality (1) is equivalent to

$$-\ln 2 \leq \alpha s + at \leq \ln 3 - \ln 2. \tag{4.31}$$

Inequality (2) together with (1) implies that

$$\left|\frac{x_1}{x_2} e^{\alpha s+at}\left(bt + \frac{d_1 d_2 t^2}{2} + s\right) + \frac{x_3}{x_2} d_1 t e^{\alpha s+at}\right| \leq \frac{1}{2} + |e^{\alpha s+at} - 1| \leq 1$$

$$\left|x_1\left(bt + \frac{d_1 d_2 t^2}{2} + s\right) + x_3 d_1 t\right| \leq |x_2| e^{-(\alpha s+at)} \leq |x_2| 2 =: K \tag{4.32}$$

and inequality (3) together with (1) implies that,

$$\left|\frac{x_1}{x_3} d_2 t e^{\alpha s+at}\right| \leq \frac{1}{2} + |e^{\alpha s+at} - 1| \leq 1.$$

If $d_2 \neq 0$, we have

$$|t| \leq \left|\frac{x_3}{d_2 x_1}\right| e^{-(\alpha s+at)} \leq 2\left|\frac{x_3}{d_2 x_1}\right| =: L.$$

Thus, $t$ is bounded and by inequality (4.32), $s$ is also bounded.

If $d_2 = 0$, then inequality (4.32) becomes

$$|(bx_1 + d_1 x_3)t + x_1 s| \leq K. \tag{4.33}$$

Hence, by inequalities (4.31) and (4.33) we get the system of equations

$$\begin{aligned} \alpha s + at &= k_1 \\ x_1 s + (bx_1 + d_1 x_3)t &= k_2 \end{aligned} \tag{4.34}$$

where $k_1 \in [-\ln 2, \ln 3 - \ln 2]$ and $k_2 \in [-K, K]$. The set of pairs $(s, t)$ which satisfy (4.34) is bounded if $\alpha(bx_1 + d_1 x_3) - ax_1 \neq 0$, or equivalently, $(\alpha b - a)x_1 + \alpha d_1 x_3 \neq 0$. Now the set of vectors $\vec{x}$ such that $(\alpha b - a)x_1 + \alpha d_1 x_3 = 0$ is a set of measure zero iff $\alpha b - a \neq 0$ or $\alpha d_1 \neq 0$. Hence, the set of pairs $(s, t)$ satisfying (4.34) is bounded for almost all $\vec{x}$ if one of the following holds:

i) $d_2 \neq 0$

ii) $d_2 = 0$ and $\alpha b - a \neq 0$.

iii) $d_2 = 0$ and $\alpha \neq 0$ and $d_1 \neq 0$.

That is, $\Phi^{\epsilon}_{\vec{x}}$ is compact for almost all $\vec{x}$ in these 3 cases. This proves the claim.

On the other hand, suppose that none of conditions in i)-iii) holds. Let $\vec{x} \in \mathbb{R}^3$ be arbitrary. Note that

$$\Phi_{\vec{x}} = \{(s,t) \in \mathbb{R} \times \mathbb{R} : \vec{x}A^sB^t = e^{\alpha s + at}\Big(x_1, \big(bt + \frac{d_1 d_2 t^2}{2} + s\big)x_1 + x_2 + d_1 t x_3, \\ d_2 t x_1 + x_3\Big) = (x_1, x_2, x_3)\}.$$

Thus, if

$$\begin{aligned} \alpha s + at &= 0 \\ \big(bt + \frac{d_1 d_2 t^2}{2} + s\big)x_1 + d_1 t x_3 &= 0 \\ d_2 t x_1 &= 0 \end{aligned} \tag{4.35}$$

then $(s,t) \in \Phi_{\vec{x}}$. But since $d_2 = 0$ and $\alpha b - a = 0$ and either $\alpha = 0$ or $d_1 = 0$, the set of pairs $(s,t)$ satisfying (4.35) forms a non-zero subspace of $\mathbb{R}^2$. Thus, $\Phi_{\vec{x}}$ is not compact for all $\vec{x} \in \mathbb{R}^3$.

**Case B5:** $D = \left\{ \begin{pmatrix} e^{\alpha s + at} & e^{\alpha s + at}(s + bt) & 0 \\ 0 & e^{\alpha s + at} & 0 \\ 0 & 0 & e^{\beta s + ct} \end{pmatrix} : s,t \in \mathbb{R} \right\}$ where $\alpha \neq \beta$.

Let $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. The epsilon stabilizer is

$$\Phi^{\epsilon}_{\vec{x}} = \Big\{(s,t) \in \mathbb{R} \times \mathbb{R} : x_1^2(e^{\alpha s + at} - 1)^2 + [x_1 e^{\alpha s + at}(s + bt) + x_2(e^{\alpha s + at} - 1)]^2 \\ + x_3^2(e^{\beta s + ct} - 1)^2 \leq \epsilon^2\Big\}.$$

Suppose $x_1, x_2, x_3 \neq 0$ and choose $\epsilon > 0$ such that $\epsilon \leq \min\left\{\frac{|x_1|}{2}, \frac{|x_2|}{2}, \frac{|x_3|}{2}\right\}$. We claim that $\Phi^{\epsilon}_{\vec{x}}$ is compact. In fact, if $(s,t) \in \Phi^{\epsilon}_{\vec{x}}$ then in particular

(1) $|e^{\alpha s+at}-1| \leq \frac{\epsilon}{|x_1|} \leq \frac{1}{2} \quad \Rightarrow \quad \frac{1}{2} \leq e^{\alpha s+at} \leq \frac{3}{2}$

(2) $\left|\frac{x_1}{x_2} e^{\alpha s+at}(s+bt) + e^{\alpha s+at} - 1\right| \leq \frac{\epsilon}{|x_2|} \leq \frac{1}{2}$

(3) $|e^{\beta s+ct}-1| \leq \frac{\epsilon}{|x_3|} \leq \frac{1}{2} \quad \Rightarrow \quad \frac{1}{2} \leq e^{\beta s+ct} \leq \frac{3}{2}.$

Inequalities (1) and (3) are equivalent to

$$\begin{aligned} -\ln 2 \;&\leq\; \alpha s + at \;\leq\; \ln 3 - \ln 2 \\ -\ln 2 \;&\leq\; \beta s + ct \;\leq\; \ln 3 - \ln 2 \end{aligned} \tag{4.36}$$

Inequality (2) together with (1) implies that

$$\begin{aligned} \left|\frac{x_1}{x_2}\right| e^{\alpha s+at}\, |s+bt| &\leq \frac{1}{2} + |e^{\alpha s+at}-1| \leq \frac{1}{2}+\frac{1}{2} = 1 \\ |s+bt| &\leq e^{-(\alpha s+at)}\left|\frac{x_2}{x_1}\right| \leq 2\left|\frac{x_2}{x_1}\right| =: L. \end{aligned} \tag{4.37}$$

From inequalities (4.36) and (4.37) we obtain the system of equations

$$\begin{aligned} \alpha s + at &= k_1 \\ \beta s + ct &= k_2 \\ s + bt &= k_3 \end{aligned} \tag{4.38}$$

where $k_1, k_2 \in [-\ln 2, \ln 3 - \ln 2]$ and $k_3 \in [-L, L]$. The set of pairs $(s,t)$ which satisfy system of equation (4.38) is bounded iff at least one of the following holds:

i) $\alpha b - a \neq 0$.

ii) $\beta a - \alpha c \neq 0$.

iii) $\beta b - c \neq 0$.

Thus, $\Phi_{\bar{x}}^{\epsilon}$ is compact if at least one of i)-iii) holds. This proves the claim.

On the other hand, suppose that none of conditions in i)-iii) holds. Note that the kernel of $\Phi$ is the solution set of the system

$$\begin{aligned} \alpha s + at &= 0 \\ s + bt &= 0 \\ \beta s + ct &= 0. \end{aligned} \tag{4.39}$$

Since none of conditions in i)-iii) holds, the set of pairs $(s,t)$ satisfying (4.39) forms a non-zero subspace of $\mathbb{R}^2$. Thus, $\Phi_{\vec{x}} \supset \operatorname{Ker}\Phi$ is not compact.

**Case B6:**

$$D=\left\{\begin{pmatrix} e^{\alpha s+at}\cos(\beta s+bt) & e^{\alpha s+at}\sin(\beta s+bt) & 0 \\ -e^{\alpha s+at}\sin(\beta s+bt) & e^{\alpha s+at}\cos(\beta s+bt) & 0 \\ 0 & 0 & e^{\gamma s+ct} \end{pmatrix} : s,t \in \mathbb{R}\right\}$$

where $\beta \neq 0$.

Let $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. The epsilon stabilizer is

$$\begin{aligned} \Phi_{\vec{x}}^{\epsilon} = \Big\{ (s,t) \in \mathbb{R}\times\mathbb{R} : [x_1(e^{\alpha s+at}\cos(\beta s+bt) - 1) - x_2 e^{\alpha s+at}\sin(\beta s+bt)]^2 \\ + [x_1 e^{\alpha s+at}\sin(\beta s+bt) + x_2(e^{\alpha s+at}\cos(\beta s+bt) - 1)]^2 \\ + x_3^2(e^{\gamma s+ct} - 1)^2 \leq \epsilon^2 \Big\}. \end{aligned}$$

Suppose, $x_1, x_2, x_3 \neq 0$ and choose $\epsilon > 0$ such that $\epsilon \leq \min\left\{\frac{|x_1|}{2}, \frac{|x_2|}{2}, \frac{|x_3|}{2}\right\}$. We claim that $\Phi_{\vec{x}}^{\epsilon}$ is compact.

In fact, if $(s,t) \in \Phi_{\vec{x}}^{\epsilon}$, then we obtain 3 inequalities

(1) $|\frac{x_1}{x_2}(e^{\alpha s+at}\cos(\beta s+bt) - 1) - e^{\alpha s+at}\sin(\beta s+bt)| \leq \frac{\epsilon}{|x_2|} \leq \frac{1}{2}$

(2) $|e^{\alpha s+at}\sin(\beta s+bt) + \frac{x_2}{x_1}(e^{\alpha s+at}\cos(\beta s+bt) - 1)| \leq \frac{\epsilon}{|x_1|} \leq \frac{1}{2}$

(3) $|e^{\gamma s+ct} - 1| \leq \frac{\epsilon}{|x_3|} \leq \frac{1}{2} \quad \Rightarrow \quad \frac{1}{2} \leq e^{\gamma s+ct} \leq \frac{3}{2}.$

Inequality (3) is equivalent to

$$-\ln 2 \leq \gamma s + ct \leq \ln 3 - \ln 2. \tag{4.40}$$

By combining inequalities (1) and (2) we get

$$\begin{aligned}\left|\frac{x_1}{x_2} + \frac{x_2}{x_1}\right| |e^{\alpha s + at}\cos(\beta s + bt) - 1| &\leq 1 \\ |e^{\alpha s + at}\cos(\beta s + bt) - 1| &\leq \left|\frac{x_1 x_2}{x_1^2 + x_2^2}\right| =: K\end{aligned} \tag{4.41}$$

Clearly, $K$ is strictly less than 1, and thus

$$0 < 1 - K \leq e^{\alpha s + at}\cos(\beta s + bt) \leq 1 + K. \tag{4.42}$$

By inequality (2) and (4.41) we have

$$\begin{aligned}|e^{\alpha s + at}\sin(\beta s + bt)| &\leq \frac{1}{2} + \left|\frac{x_2}{x_1}\right| |e^{\alpha s + at}\cos(\beta s + bt) - 1| \\ &\leq \frac{1}{2} + \left|\frac{x_2}{x_1}\right| K =: L\end{aligned} \tag{4.43}$$

Squaring inequalities (4.42) and (4.43) we obtain

$$(1 - K)^2 \leq e^{2(\alpha s + at)}\cos^2(\beta s + bt) \leq (1 + K)^2 \tag{4.44}$$

$$0 \leq e^{2(\alpha s + at)}\sin^2(\beta s + bt) \leq L^2. \tag{4.45}$$

By combining (4.44) and (4.45), we get

$$\begin{aligned}0 < (1 - K)^2 \leq e^{2(\alpha s + at)} &\leq L^2 + (1 + K)^2 =: K^{*2} \\ 2\ln(1 - K) \leq 2(\alpha s + at) &\leq 2\ln K^*\end{aligned} \tag{4.46}$$

From inequalities(4.40) and (4.46) we obtain the system of equations

$$\begin{aligned}\gamma s + ct &= k_1 \\ \alpha s + at &= k_2\end{aligned} \tag{4.47}$$

where $k_1 \in [-\ln 2, \ln 3 - \ln 2]$ and $k_2 \in [\ln(1 - K). \ln K^*]$.

The set of pairs $(s, t)$ which satisfy the system of equation (4.47) is bounded if $\gamma a - \alpha c \neq 0$. Thus, $\Phi^{\epsilon}_{\bar{x}}$ is compact if $\gamma a - \alpha c \neq 0$. This proves the claim.

On the other hand, suppose that $\gamma a - \alpha c = 0$. Note that the kernel of $\Phi$ is the solution set of the system

$$\alpha s + at = 0 \tag{4.48}$$

$$\beta s + bt = 2k\pi \quad \text{where } k \in \mathbb{Z} \tag{4.49}$$

$$\gamma s + ct = 0 \tag{4.50}$$

Since $\gamma a - \alpha c = 0$, the solution set of (4.48) and (4.50) forms a non-zero linear subspace of $\mathbb{R}^2$, that is, either a line passing through the origin or $\mathbb{R}^2$ itself. Since $\beta \neq 0$, the solution of (4.49) is a countable family of parallel lines $l_k$.
Now if $\alpha b - a\beta = 0$, then the line $l_0$ coincides with the solution set of (4.48) and (4.50), so the solution set of (4.48) - (4.50) is a one-dimensional linear subspace. If $\alpha b - a\beta \neq 0$, then each line $l_k$ intersects the line (or plane) determined by (4.48) and (4.50). Thus, the solution set of (4.48) - (4.50) is at least countably infinite and unbounded. This show that $\operatorname{Ker}\Phi$ is an unbounded subset of $\mathbb{R}^2$ and that $\Phi_{\bar{x}} \supset \operatorname{Ker}\Phi$ is never compact. ☐

**Corollary 4.9.** *Let $\Phi$ be a continuous two-parameter matrix group generated by two $3 \times 3$ commuting matrices. Then $\Phi$ is admissible iff $\Phi_{\bar{x}}$ is compact.*

*Proof.* ($\Rightarrow$) This follows from theorem 3.5.
($\Leftarrow$) In the above proof, whenever $\Phi$ is not admissible, we have shown this by showing that $\Phi_{\bar{x}}$ is not compact. It follows that if $\Phi_{\bar{x}}$ is compact, then $\Phi$ must be admissible. ☐

**Remark 4.3.** *In the $2 \times 2$ case we could strengthen this statement to "$\Phi$ is admissible iff it is one-to-one". In the $3 \times 3$ case this is no longer true, because of the case B4.*

*Consider the situation in case B4 where $\Phi$ is not admissible because $d_2 = 0, \alpha b - a = 0$ and $\alpha = 0$. Then*

$$D = \left\{ A^s B^t = \begin{pmatrix} 1 & s+bt & 0 \\ 0 & 1 & 0 \\ 0 & d_1 t & 1 \end{pmatrix} : s,t \in \mathbb{R} \right\}.$$

*Hence, $Ker\,\Phi$ is the solution of the system*

$$s + bt = 0$$
$$d_1 t = 0.$$

*Now if $d_1 \neq 0$ then this system has only the trivial solution, so in this case $\Phi$ is one-to-one, but the stabilizers $\Phi_{\bar{x}}$ are non-compact.*

Discrete two-parameter groups can now be discussed just as in the $2 \times 2$ case, using proposition 4.5 and proposition 4.6.

### 4.2.3 Two-Parameter Groups Generated by Two Commuting Matrices in Higher Dimensions

In dimension greater than 3, there is a great variety of commuting matrix pairs $M$ and $N$, and so one can not hope to obtain a nice classification of all two-parameter groups.

However, if there exists a Jordan basis of $M$ in which $M$ and $N$ are decomposed into blocks of equal dimensions,

$$M = \begin{pmatrix} M_1 & & & (0) \\ & M_2 & & \\ & & \ddots & \\ (0) & & & M_k \end{pmatrix} \quad N = \begin{pmatrix} N_1 & & & (0) \\ & N_2 & & \\ & & \ddots & \\ (0) & & & N_k \end{pmatrix}$$

with $M_i$ being a Jordan block of $M$ and $N_i$ a corresponding block of $N$, then we can apply the idea of the earlier sections to discuss admissibility of the two-parameter group generated by $M$ and $N$.

It turns out that these ideas can be used even if $M$ and $N$ are not in block-diagonal form. All that is needed is that the first few columns resemble those of an upper triangular matrix. In fact, we suppose that there exists a basis in which $M$ and $N$ are commuting $n \times n$ matrices as follows:

**Case H1:**

$$M = \begin{pmatrix} \alpha & 0 & (*) & \cdots & (*) \\ 0 & \beta & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}, \quad N = \begin{pmatrix} a & 0 & (*) & \cdots & (*) \\ 0 & b & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}.$$

**Case H2:**

$$M = \begin{pmatrix} \alpha & 1 & (*) & \cdots & (*) \\ 0 & \alpha & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}, \quad N = \begin{pmatrix} a & b & (*) & \cdots & (*) \\ 0 & a & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}$$

**Case H3:**

$$M = \begin{pmatrix} D_1 & 0 & (*) & \cdots & (*) \\ 0 & D_2 & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}, \quad N = \begin{pmatrix} E_1 & 0 & (*) & \cdots & (*) \\ 0 & E_2 & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}$$

where $D_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}$, $\beta_1 \neq 0$, and $E_1 = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}$,

$D_2$ and $E_2$ are either scalars, i.e., $D_2 = \alpha_2$ and $E_2 = a_2$, or $2 \times 2$ matrices of the form, $D_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{pmatrix}$, $E_2 = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$.

**Case H4:**

$$M = \begin{pmatrix} D & I_2 & (*) & \cdots & (*) \\ 0 & D & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix} \qquad N = \begin{pmatrix} E_1 & E_2 & (*) & \cdots & (*) \\ 0 & E_1 & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}.$$

where $D = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $\beta \neq 0$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$, $i = 1, 2$.

So, let $A = e^M$ and $B = e^N$ where $M, N \in M_n(\mathbb{R})$ are commuting matrices as above. If one of $M$ and $N$ has non-zero trace, then by theorem 3.5, in order to discuss admissibility of the continuous two-parameter group $\Phi : \mathbb{R}^2 \to D = \{A^s B^t = e^{sM+tN} : s, t \in \mathbb{R}\}$ generated by $M, N$, we only need to investigate the existence of compact local stabilizers. First let us consider some special situations of case H3 and H4.

**Lemma 4.10.** *Let $M, N$ be commuting matrices of the form*

$$M = \begin{pmatrix} D_1 & (0) \\ (0) & D_2 \end{pmatrix} \quad \text{and } N = \begin{pmatrix} E_1 & (0) \\ (0) & E_2 \end{pmatrix}$$

*where* $D_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}$, $\beta_1 \neq 0$, *and* $E_1 = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}$,

*$D_2$ and $E_2$ are either scalars, i.e., $D_2 = \alpha_2$ and $E_2 = a_2$, or $2 \times 2$ matrices of the form,* $D_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{pmatrix}$, $E_2 = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$.

*Then*

$$\Phi : \mathbb{R}^2 \to D = \{A^s B^t = e^{sM+tN} : s, t \in \mathbb{R}\}$$

*is admissible if*

1. $D_2 = \alpha_2$ *and* $E_2 = a_2$ *are scalar and* $\alpha_1 a_2 - \alpha_2 a_1 \neq 0$.

2. $D_2$ *and* $E_2$ *are* $2 \times 2$ *matrices and* $\alpha_1 a_2 - \alpha_2 a_1 \neq 0$.

*Proof.* In case where $D_2 = \alpha_2$ and $E_2 = a_2$ are scalar, we refer to the proof of theorem 4.8, case B6 which shows that $\Phi$ is admissible iff $\alpha_1 a_2 - \alpha_2 a_1 \neq 0$.

Now assume that $D_2$ and $E_2$ are $2 \times 2$ matrices and $\alpha_1 a_2 - \alpha_2 a_1 \neq 0$. Note that the condition $\alpha_1 a_2 - \alpha_2 a_1$ implies that one of $M$ or $N$ has non-zero trace. Direct computation shows that $A^s B^t$ is of the form

$$\begin{pmatrix} e^{\alpha_1 s + a_1 t}\cos\theta_1 & e^{\alpha_1 s + a_1 t}\sin\theta_1 & 0 & 0 \\ -e^{\alpha_1 s + a_1 t}\sin\theta_1 & e^{\alpha_1 s + a_1 t}\cos\theta_1 & 0 & 0 \\ 0 & 0 & e^{\alpha_2 s + a_2 t}\cos\theta_2 & e^{\alpha_2 s + a_2 t}\sin\theta_2 \\ 0 & 0 & -e^{\alpha_2 s + a_2 t}\sin\theta_2 & e^{\alpha_2 s + a_2 t}\cos\theta_2 \end{pmatrix} \tag{4.51}$$

where $\theta_i = \beta_i s + b_i t, i = 1, 2$.

Let $\vec{x} \in \mathbb{R}^n$ with $x_i \neq 0 \; \forall i = 1, ..., 4$. Set $\epsilon < \min\left\{\frac{|x_i|}{2} : i = 1, ..., 4\right\}$. If $(s, t)$ lies in the $\epsilon$-stabilizer

$$\Phi_{\vec{x}}^{\epsilon} = \{(s, t) \in \mathbb{R} \times \mathbb{R} : ||\vec{x} A^s B^t - \vec{x}|| \leq \epsilon\} \tag{4.52}$$

then similar to the proofs in the $2 \times 2$ and $3 \times 3$ cases we obtain the following four conditions on $(s, t)$:

(1) $\left|(e^{\alpha_1 s + a_1 t}\cos\theta_1 - 1)\frac{x_1}{x_2} - e^{\alpha_1 s + a_1 t}\sin\theta_1\right| \leq \frac{\epsilon}{|x_2|} \leq \frac{1}{2}$

(2) $\left|e^{\alpha_1 s + a_1 t}\sin\theta_1 + (e^{\alpha_1 s + a_1 t}\cos\theta_1 - 1)\frac{x_2}{x_1}\right| \leq \frac{\epsilon}{|x_1|} \leq \frac{1}{2}$

(3) $\left|(e^{\alpha_2 s + a_2 t}\cos\theta_2 - 1)\frac{x_3}{x_4} - e^{\alpha_2 s + a_2 t}\sin\theta_2\right| \leq \frac{\epsilon}{|x_4|} \leq \frac{1}{2}$

(4) $\left|e^{\alpha_2 s+a_2 t}\sin\theta_2 + (e^{\alpha_2 s+a_2 t}\cos\theta_2 - 1)\frac{x_4}{x_3}\right| \leq \frac{\epsilon}{|x_3|} \leq \frac{1}{2}$

In the proof of theorem 4.8, case B5, we showed that the first two inequalities give us

$$0 < K_1 \leq e^{2(\alpha_1 s+a_1 t)} \leq K_2 \tag{4.53}$$

and the last two inequalities give us

$$0 < L_1 \leq e^{2(\alpha_2 s+a_2 t)} \leq L_2, \tag{4.54}$$

for some constants $K_1, K_2, L_1$ and $L_2$.

From (4.53) and (4.54) we get the system of equations

$$\begin{aligned} \alpha_1 s + a_1 t &= M \\ \alpha_2 s + a_2 t &= N \end{aligned} \tag{4.55}$$

where $M \in \left[\ln\frac{K_1}{2}, \ln\frac{K_2}{2}\right]$ and $N \in \left[\ln\frac{L_1}{2}, \ln\frac{L_2}{2}\right]$.

The set of pairs $(s,t)$ satisfying (4.55) is bounded since $\alpha_1 a_2 - \alpha_2 a_1 \neq 0$. Therefore $\Phi^{\epsilon}_{\vec{x}}$ is compact for almost all $\vec{x}$ and small $\epsilon$, so that $\Phi$ is admissible.

$\square$

**Lemma 4.11.** *Let $M, N \in M_4(\mathbb{R})$ be commuting matrices of the form*

$$M = \begin{pmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix} \quad \textit{and } N = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & -b_2 & a_2 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & -b_1 & a_1 \end{pmatrix}$$

*with $\alpha a_2 - a_1 \neq 0$. Then*

$$\Phi : \mathbb{R}^2 \to D = \{A^s B^t = e^{sM+tN} : s,t \in \mathbb{R}\}$$

*is admissible.*

*Proof.* Note that the condition $\alpha a_2 - a_1 \neq 0$ implies that one of $M$ and $N$ has non-zero trace. Direct computation shows that $A^sB^t$ is of the form

$$e^{\alpha s+a_1t}\begin{pmatrix} \cos\theta & \sin\theta & (s+a_2t)\cos\theta - b_2t\sin\theta & (s+a_2t)\sin\theta + b_2t\cos\theta \\ -\sin\theta & \cos\theta & -[(s+a_2t)\sin\theta + b_2t\cos\theta] & (s+a_2t)\cos\theta - b_2t\sin\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{4.56}$$

where $\theta = \beta s + b_1 t$.

Let $\vec{x} \in \mathbb{R}^n$ with $x_1, x_2 \neq 0$. Set $\epsilon < \min\left\{\frac{|x_i|}{2} : i = 1, 2\right\}$. If $(s,t)$ lies in the $\epsilon$-stabilizer

$$\Phi_{\vec{x}}^{\epsilon} = \{(s,t) \in \mathbb{R} \times \mathbb{R} : \|\vec{x}A^sB^t - \vec{x}\| \leq \epsilon\} \tag{4.57}$$

then similar to the proofs in the $2 \times 2$ and $3 \times 3$ cases we obtain the following four conditions on $(s,t)$:

(1) $\left|(e^{\alpha s+a_1t}\cos\theta - 1)\frac{x_1}{x_2} - e^{\alpha s+a_1t}\sin\theta\right| \leq \frac{\epsilon}{|x_2|} \leq \frac{1}{2}$

(2) $\left|e^{\alpha s+a_1t}\sin\theta + (e^{\alpha s+a_1t}\cos\theta - 1)\frac{x_2}{x_1}\right| \leq \frac{\epsilon}{|x_1|} \leq \frac{1}{2}$

(3) $\big|e^{\alpha s+a_1t}[(s+a_2t)\cos\theta - b_2t\sin\theta]x_1 - e^{\alpha s+a_1t}[(s+a_2t)\sin\theta + b_2t\cos\theta]x_2 +$
$(e^{\alpha s+a_1t}\cos\theta - 1)x_3 - e^{\alpha s+a_1t}\sin\theta x_4\big| \leq \epsilon$

(4) $\big|e^{\alpha s+a_1t}[(s+a_2t)\sin\theta + b_2t\cos\theta]x_1 + e^{\alpha s+a_1t}[(s+a_2t)\cos\theta - b_2t\sin\theta]x_2 +$
$e^{\alpha s+a_1t}\sin\theta x_3 + (e^{\alpha s+a_1t}\cos\theta - 1)x_4\big| \leq \epsilon$

In the proof of theorem 4.8, case B5, we showed that the first two inequalities give us

$$0 < K_1 \leq e^{2(\alpha s+a_1t)} \leq K_2, \tag{4.58}$$

for some constants $K_1, K_2$.

By taking $\frac{(3)}{|x_2|} + \frac{(4)}{|x_1|}$ and using (4.58) we obtain

$$\Big| e^{\alpha s + a_1 t}[(s + a_2 t)\cos\theta - b_2 t \sin\theta]\big(\frac{x_1}{x_2} + \frac{x_2}{x_1}\big) + (e^{\alpha s + a_1 t}\cos\theta - 1)\big(\frac{x_3}{x_2} + \frac{x_4}{x_1}\big)$$
$$+ e^{\alpha s + a_1 t}\sin\theta\big( - \frac{x_4}{x_2} + \frac{x_3}{x_1}\big)\Big| \leq \frac{\epsilon}{|x_2|} + \frac{\epsilon}{|x_1|} \leq 1$$

or using (4.58),

$$|e^{\alpha s + a_1 t}|(s + a_2 t)\cos\theta - b_2 t \sin\theta|\Big|\frac{x_1^2 + x_2^2}{x_1 x_2}\Big| \leq 1 + |e^{\alpha s + a_1 t}\cos\theta - 1|\Big|\frac{x_3}{x_2} + \frac{x_4}{x_1}\Big|$$
$$+ e^{\alpha s + a_1 t}|\sin\theta|\Big| - \frac{x_4}{x_2} + \frac{x_3}{x_1}\Big| \leq L_1$$
$$|(s + a_2 t)\cos\theta - b_2 t \sin\theta| \leq L_1\Big|\frac{x_1 x_2}{x_1^2 + x_2^2}\Big|e^{-(\alpha s + a_1 t)} =: \widetilde{L}_1. \tag{4.59}$$

Again, taking $\frac{(3)}{|x_1|} + \frac{(4)}{|-x_2|}$, we get

$$\Big| - e^{\alpha s + a_1 t}[(s + a_2 t)\sin\theta + b_2 t \cos\theta]\big(\frac{x_2}{x_1} + \frac{x_1}{x_2}\big) + (e^{\alpha s + a_1 t}\cos\theta - 1)\big(\frac{x_3}{x_1} - \frac{x_4}{x_2}\big)$$
$$- e^{\alpha s + a_1 t}\sin\theta\big(\frac{x_4}{x_1} + \frac{x_3}{x_2}\big)\Big| \leq \frac{\epsilon}{|x_1|} + \frac{\epsilon}{|-x_2|} \leq 1.$$

Similar to the above calculation, we get

$$|(s + a_2 t)\sin\theta + b_2 t \cos\theta| \leq \widetilde{L}_2. \tag{4.60}$$

By multiplying (4.59) by $|\cos\theta|$ and (4.60) by $|\sin\theta|$, we get

$$|(s + a_2 t)\cos^2\theta - b_2 t \sin\theta\cos\theta| \leq \widetilde{L}_1|\cos\theta| \leq \widetilde{L}_1 \tag{4.61}$$

$$|(s + a_2 t)\sin^2\theta + b_2 t \cos\theta\sin\theta| \leq \widetilde{L}_2|\sin\theta| \leq \widetilde{L}_2 \tag{4.62}$$

Combining (4.61) and (4.62),

$$|s + a_2 t| \leq \widetilde{L}_1 + \widetilde{L}_2 =: L. \tag{4.63}$$

Together with (4.58) we get the system of equations

$$\begin{aligned} \alpha s + a_1 t &= M \\ s + a_2 t &= N \end{aligned} \tag{4.64}$$

where $M \in \left[\ln \frac{K_1}{2}, \ln \frac{K_2}{2}\right]$ and $N \in [-L, L]$.

The set of pairs $(s,t)$ satisfying (4.64) is bounded since $\alpha a_2 - a_1 \neq 0$. Therefore $\Phi^\epsilon_{\vec{x}}$ is compact for almost all $\vec{x}$ and small $\epsilon$, so that $\Phi$ is admissible. □

**Theorem 4.12.** *Let $\Phi : \mathbb{R}^2 \to D = \{A^s B^t = e^{sM+tN} : s,t \in \mathbb{R}\}$ where $M, N$ are commuting matrices as in cases H1 to H4. If at least one of $M$ and $N$ has non-zero trace, and*

1. *$M, N$ are as in case H1 with $\alpha b - a\beta \neq 0$, or*

2. *$M, N$ are as in case H2 with $\alpha b - a \neq 0$, or*

3. *$M, N$ are as in case H3 with $\alpha_1 a_2 - \alpha_2 a_1 \neq 0$, or*

4. *$M, N$ are as in case H4 with $\alpha a_2 - a_1 \neq 0$,*

*then $\Phi$ is admissible.*

*Proof.* **Case H1**: Direct computation shows that

$$A^s B^t = \begin{pmatrix} e^{\alpha s + at} & 0 & (*) & \cdots & (*) \\ 0 & e^{\beta s + bt} & (*) & \ddots & (*) \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}$$

Let $\vec{x} \in \mathbb{R}^n$ with $x_1, x_2 \neq 0$. Pick $\epsilon < \min\left\{\frac{|x_i|}{2} : i = 1,2\right\}$. Proceeding exactly as in the proof of theorem 4.1, case A1, one shows that if $(s,t)$ lies in the $\epsilon$-stabilizer

$$\Phi^\epsilon_{\vec{x}} = \{(s,t) \in \mathbb{R} \times \mathbb{R} : \|\vec{x} A^s B^t - \vec{x}\| \leq \epsilon\} \tag{4.65}$$

then $(s,t)$ must solve the system of equations

$$\begin{aligned} \alpha s + at &= k_1 \\ \beta s + bt &= k_2 \end{aligned} \tag{4.66}$$

where $k_1, k_2 \in [-\ln 2, \ln 3 - \ln 2]$. The set of pairs $(s,t)$ satisfying (4.66) is bounded since $\alpha b - a\beta \neq 0$. Therefore, $\Phi^{\epsilon}_{\vec{x}}$ is compact for almost all $\vec{x}$ and small $\epsilon$, so that $\Phi$ is admissible.

**Case H2**: Direct computation shows that

$$A^s B^t = e^{\alpha s + at} \begin{pmatrix} 1 & bt+s & (*) & \cdots & (*) \\ 0 & 1 & (*) & \ddots & (*) \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (*) & \cdots & (*) \end{pmatrix}.$$

Let $\vec{x} \in \mathbb{R}^n$ with $x_1, x_2 \neq 0$. Set $\epsilon < \min\left\{\frac{|x_i|}{2} : i = 1,2\right\}$. Proceeding exactly as in the proof of theorem 4.1, case A2, one shows that if $(s,t)$ lies in the $\epsilon$-stabilizer

$$\Phi^{\epsilon}_{\vec{x}} = \{(s,t) \in \mathbb{R} \times \mathbb{R} : \|\vec{x}A^s B^t - \vec{x}\| \leq \epsilon\} \tag{4.67}$$

then $(s,t)$ must solve the system of equations

$$\begin{aligned} \alpha s + at &= k_1 \\ s + bt &= k_2 \end{aligned} \tag{4.68}$$

where $k_1 \in [-\ln 2, \ln 3 - \ln 2]$ and $k_2 \in \left[-2\frac{|x_2|}{|x_1|}, 2\frac{|x_2|}{|x_1|}\right]$ The set of pairs $(s,t)$ satisfying (4.68) is bounded since $\alpha b - a \neq 0$. Therefore, $\Phi^{\epsilon}_{\vec{x}}$ is compact for almost all $\vec{x}$ and small $\epsilon$, so that $\Phi$ is admissible.

**Case H3a**: If $D_2$ and $E_2$ are scalar, direct computation shows that $A^s B^t$ equals

$$\begin{pmatrix} e^{\alpha_1 s + a_1 t}\cos(\theta_1) & e^{\alpha_1 s + a_1 t}\sin(\theta_1) & 0 & (*) & \cdots & (*) \\ -e^{\alpha_1 s + a_1 t}\sin(\theta_1) & e^{\alpha_1 s + a_1 t}\cos(\theta_1) & 0 & (*) & \ddots & (*) \\ 0 & 0 & e^{\alpha_2 s + a_2 t} & (*) & \ddots & (*) \\ \vdots & \vdots & (*) & (*) & \ddots & (*) \\ 0 & 0 & (*) & (*) & \cdots & (*) \end{pmatrix}$$

Let $\vec{x} \in \mathbb{R}^n$ with $x_1, x_2, x_3 \neq 0$. Pick $\epsilon < \min\left\{\frac{|x_i|}{2} : i = 1,2,3\right\}$. Proceeding exactly as in the proof of theorem 4.8, case B6, one show that if $(s,t)$ lies in the $\epsilon$-stabilizer

$$\Phi^{\epsilon}_{\vec{x}} = \{(s,t) \in \mathbb{R} \times \mathbb{R} : \|\vec{x}A^sB^t - \vec{x}\| \leq \epsilon\} \tag{4.69}$$

then $(s,t)$ must solve the system of equations

$$\begin{aligned} \alpha_1 s + a_1 t &= k_1 \\ \alpha_2 s + a_2 t &= k_2 \end{aligned} \tag{4.70}$$

where $k_1 \in [-\ln 2, \ln 3 - \ln 2]$ and $k_2 \in \left[\ln(1-K), \frac{1}{2}\ln(L^2 + (1+K)^2)\right]$, $K = \left|\frac{x_1 x_2}{x_1^2 + x_2^2}\right|$, $L = \frac{1}{2} + \frac{|x_2|}{|x_1|}K$.

The set of pairs $(s,t)$ satisfying (4.70) is bounded since $\alpha_1 a_2 - \alpha_2 a_1 \neq 0$. Therefore, $\Phi^{\epsilon}_{\vec{x}}$ is compact for almost all $\vec{x}$ and small $\epsilon$, so that $\Phi$ is admissible.

**Case H3b**: If both $D_2$ and $E_2$ are $2 \times 2$ matrices, then direct computation shows that $A^sB^t$ equals

$$\begin{pmatrix} K & (*) \\ (0) & (*) \end{pmatrix}$$

where $K$ is as in (4.51).

Let $\vec{x} \in \mathbb{R}^n$ with $x_1, x_2 \neq 0$. Pick $\epsilon < \min\left\{\frac{|x_i|}{2} : i = 1,2\right\}$. If $(s,t)$ lies in the $\epsilon$-stabilizer

$$\Phi^{\epsilon}_{\vec{x}} = \{(s,t) \in \mathbb{R} \times \mathbb{R} : \|\vec{x}A^sB^t - \vec{x}\| \leq \epsilon\} \tag{4.71}$$

then considering the first four components $x_1, ..., x_4$ of $\vec{x}$, we see that the four conditions (1)-(4) of lemma 4.10 must hold

It follows from the proof of lemma 4.10 that the set of pairs $(s,t)$ is bounded since $\alpha_1 a_2 - \alpha_2 a_1 \neq 0$. Therefore, $\Phi^{\epsilon}_{\vec{x}}$ is compact for almost all $\vec{x}$ and small $\epsilon$ , so that $\Phi$ is admissible.

**Case H4**: Direct computation shows that

$$A^sB^t = e^{\alpha s+a_1 t}\begin{pmatrix} E & KE & (*) & \dots & (*) \\ (0) & E & (*) & \ddots & (*) \\ (0) & (0) & (*) & \ddots & (*) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (0) & (0) & (*) & \dots & (*) \end{pmatrix},$$

where $K = \begin{pmatrix} s+a_2t & b_2t \\ -b_2t & s+a_2t \end{pmatrix}$, $E = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $\theta = \beta s + bt$.

That is, the top left block $\begin{pmatrix} E & KE \\ (0) & E \end{pmatrix}$ is exactly as in (4.56).

Let $\vec{x} \in \mathbb{R}^n$ with $x_1, x_2 \neq 0$. Pick $\epsilon < \min\left\{\frac{|x_i|}{2} : i = 1, 2\right\}$. If $(s,t)$ lies in the $\epsilon$-stabilizer

$$\Phi^\epsilon_{\vec{x}} = \{(s,t) \in \mathbb{R}\times\mathbb{R} : \|\vec{x}A^sB^t - \vec{x}\| \leq \epsilon\} \tag{4.72}$$

then looking at the first four components $x_1, ..., x_4$ of $\vec{x}$, one sees that inequalities (1)-(4) of lemma 4.11 must be satisfied.

It follows from the proof of lemma 4.11 that the set of pairs $(s,t)$ is bounded since $\alpha a_2 - a_1 \neq 0$. Therefore, $\Phi^\epsilon_{\vec{x}}$ is compact for almost all $\vec{x}$ and small $\epsilon$, so that $\Phi$ is admissible. □

**Remark 4.4.** *In section 4.2.2 regarding* $3\times 3$ *matrix groups, we discussed one possibility where the matrix* $N$ *does not appear to be of the above form, namely case B4. Note, however, that after a change of basis, the matrices* $M$ *and* $N$ *in case B4 can be brought into the above form, namely to*

$$M = \begin{pmatrix} \alpha & 0 & \frac{1}{d_2} \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad N = \begin{pmatrix} a & 1 & \frac{b}{d_2} \\ 0 & a & d_1 \\ 0 & 0 & a \end{pmatrix} \qquad (\text{if } d_2 \neq 0)$$

*or*

$$M = \begin{pmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad N = \begin{pmatrix} a & 1 & b \\ 0 & a & d_1 \\ 0 & 0 & a \end{pmatrix} \qquad (\textit{if } d_2 = 0).$$

*Thus, all of the admissible matrices* $2\times2$ *or* $3\times3$ *matrix pairs discussed in theorem 4.1 and 4.8 fall into the classes of matrices discussed in this theorem.*

### 4.2.4 Non-Commuting $2 \times 2$ Matrix Groups

In this section, we present some examples of two dimensional non-commuting matrix subgroups of $GL_2(\mathbb{R})$ determined by two parameters. First, we start with subgroups determined by continuous parameters.

**Example 1:** Let $D = \left\{ d = \begin{pmatrix} a & t \\ 0 & a^r \end{pmatrix} : a \in \mathbb{R}^+, t \in \mathbb{R} \right\}$ where $r \in \mathbb{R}$ is fixed. If $r = 1$, this group is abelian and falls into the types of groups discussed in section 4.2.1, so we will assume that $r \neq 1$. Note that in general, setting $a = e^s$ we can consider $D$ the image of a two-parameter continuous map $\Phi : (s,t) \to \begin{pmatrix} e^s & t \\ 0 & e^{rs} \end{pmatrix}$, so this is the non-commutative analogue of the two-parameter groups discussed in section 4.2.1.

We can identify $D$ topologically with $\mathbb{R}^+ \times \mathbb{R}$ through the map $d \mapsto (a,t)$. The group operation on $D$ is then

$$d_1 \cdot d_2 = (a_1, t_1) \cdot (a_2, t_2) = (a_1 a_2, a_1 t_2 + a_2^r t_1).$$

We note that the topology on $D$ is equivalent to that of $\mathbb{R}^+ \times \mathbb{R}$. The left and right-Haar measures are as follows:

1. $d\mu_L(d) = \frac{1}{a^2} da\, dt$ is a left Haar measure on $D$ where $da\, dt$ denotes the product Lebesgue measure on $\mathbb{R}^+ \times \mathbb{R}$. To see this, note that for $f \in C_c(D)$

and every $d_1 \in D$ we have

$$\int_D f(d_1 \cdot d_2)\, d\mu_L(d_2) = \int_{\mathbb{R}} \int_{\mathbb{R}^+} f(a_1 a_2, a_1 t_2 + a_2^r t_1) \frac{1}{a_2^2}\, da_2\, dt_2.$$

By theorem 2.5 of changing variables, setting $u(a_2, t_2) = a_1 a_2$, and $v(a_2, t_2) = a_1 t_2 + a_2^r t_1$, we have

$$\int_D f(d_1 \cdot d_2)\, d\mu_L(d_2) = \int_{\mathbb{R}} \int_{\mathbb{R}^+} f(u, v) \frac{1}{u^2}\, du\, dv.$$

2. In a similar way, one shows that $d\mu_R(d) = \frac{1}{a^{r+1}} da\, dt$ is a right Haar measure on $D$.

3. $\Delta(d) = a^{r-1}$ is the modular function. In fact,

$$\Delta(d_2) \int_D f(d_1 \cdot d_2)\, d\mu_L(d_1) = a_2^{r-1} \int_{\mathbb{R}} \int_{\mathbb{R}^+} f(a_1 a_2, a_1 t_2 + a_2^r t_1) \frac{1}{a_1^2}\, da_1\, dt_1.$$

By changing variables, $u(a_1, t_1) = a_1 a_2, v(a_1, t_1) = a_1 t_2 + a_2^r t_1$, we have

$$\Delta(d_2) \int_D f(d_1 \cdot d_2)\, d\mu_L(d_1) = \int_{\mathbb{R}} \int_{\mathbb{R}^+} f(u, v) \frac{1}{u^2}\, du\, dv = \int_D f(d_1)\, d\mu_L(d_1).$$

Since $|\det d| = a^{r+1}$, it follows that $|\det d| \neq \Delta(d)$ for all $a \neq 1$. So if we want to use theorem 3.5 to establish admissibility for this group, we only need to show that $D_{\vec{x}}^{\epsilon}$ is compact for almost all $\vec{x} \in \mathbb{R}^2$ and for some $\epsilon > 0$.

Given $\vec{x} \in \mathbb{R}^n$, the $\epsilon$-stabilizer is the subset

$$D_{\vec{x}}^{\epsilon} = \left\{(a, t) \in \mathbb{R}^+ \times \mathbb{R} : (a-1)^2 x_1^2 + [t x_1 + (a^r - 1)]^2 \leq \epsilon^2\right\} \tag{4.73}$$

of $\mathbb{R}^+ \times \mathbb{R}$. To show that it is compact we need to show that $D_{\vec{x}}^{\epsilon} \subset [\alpha, \beta] \times [c, d]$ for some closed interval in $\mathbb{R}^+ \times \mathbb{R}$. So let $\vec{x} \in \mathbb{R}^2$ with $x_1 \neq 0$. Choose $\epsilon$ so that $0 < \epsilon < |x_1|$. If $(a, t)$ satisfy (4.73), then in particular

(1) $|a - 1| \leq \frac{\epsilon}{|x_1|} < 1$

(2) $|t x_1 + (a^r - 1) x_2| \leq \epsilon$.

From the first inequality, we obtain that

$$0 < 1 - \frac{\epsilon}{|x_1|} \leq a \leq 1 + \frac{\epsilon}{|x_1|}$$

so that $a \in [\alpha, \beta]$ with $\alpha = 1 - \frac{\epsilon}{|x_1|}$, $\beta = 1 + \frac{\epsilon}{|x_1|}$.

From the second inequality it follows that

$$|t| \leq \frac{\epsilon}{|x_1|} + \left|\frac{x_2}{x_1}\right| |a^r - 1|,$$

so that $t$ also lies in the closed interval $[-c, c], c = \frac{\epsilon}{|x_1|} + \left|\frac{x_2}{x_1}\right| \cdot \beta^r$. Therefore $D_{\vec{x}}^{\epsilon}$ is compact. We have shown that $D$ is admissible.

**Note:** If $r = 0$, then $D = \left\{ d = \begin{pmatrix} a & t \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^+, t \in \mathbb{R} \right\}$ is called the first Galilei group.

**Example 2:** Let $D = \left\{ d = \begin{pmatrix} a & 0 \\ t & a^r \end{pmatrix} : a \in \mathbb{R}^+, t \in \mathbb{R} \right\}$ where $r \in \mathbb{R}$ is fixed.

Computations as in example 1 show that the left Haar measure is $d\mu_L(d) = \frac{1}{a^{r+1}} da\, dt$ while the right Haar measure is $d\mu_R(d) = \frac{1}{a^2} da\, dt$, and the modular function is $\Delta(d) = a^{1-r}$. Hence $|\det d| = a^{r+1} \neq a^{1-r}$ for all $a \neq 1$ unless $r = 0$. In case $r = 0$, $\Delta(d) = |\det d|$ for all $d \in D$ and we conclude that $D$ is not admissible. Thus we will assume that $r \neq 0$. If $r = 1$, then $D$ is abelian and falls into the groups discussed in section 4.2.1.

Similar to example 1, we need only investigate whether compact local stabilizer $D_{\vec{x}}^{\epsilon}$ exist for almost all $\vec{x}$. Given $\vec{x} \in \mathbb{R}^2$ with $x_2 \neq 0$, pick $\epsilon$ such that $0 < \epsilon < |x_2|$. Then

$$D_{\vec{x}}^{\epsilon} = \left\{(a, t) \in \mathbb{R}^+ \times \mathbb{R} : [(a-1)x_1 + tx_2]^2 + (a^r - 1)^2 x_2^2 \leq \epsilon^2\right\}. \tag{4.74}$$

If $(a, t) \in D_{\vec{x}}^{\epsilon}$ then in particular,

(1) $|(a-1)x_1 + tx_2| \leq \epsilon$

(2) $|a^r - 1| \leq \frac{\epsilon}{|x_2|} < 1$.

Since $r \neq 0$, the second inequality implies that

$$0 < \left(1 - \frac{\epsilon}{|x_2|}\right)^{1/r} < a < \left(1 + \frac{\epsilon}{|x_1|}\right)^{1/r}.$$

So that $a$ lies in the closed interval $[\alpha, \beta] \subset \mathbb{R}^+$. with $\alpha = \left(1 - \frac{\epsilon}{|x_2|}\right)^{1/2}$ and $\beta = \left(1 + \frac{\epsilon}{|x_1|}\right)^{1/2}$. From the first inequality it follows that

$$|t| \leq \frac{\epsilon}{|x_2|} + \left|\frac{x_1}{x_2}\right||a - 1| < \frac{\epsilon + |x_1| \cdot \beta^{1/r}}{|x_2|}$$

so that $t$ also lies in some closed interval. Therefore $D^{\epsilon}_{\vec{x}}$ is compact. It follows that $D$ is admissible iff $r \neq 0$.

The remaining two examples are devoted to semi-discrete subgroups of $GL_2(\mathbb{R})$. That is, we consider images of continuous maps $\Phi : \mathbb{Z} \times \mathbb{R} \to GL_2(\mathbb{R})$. As for the discrete abelian case, it would be natural to consider groups such as for example,

$$D = \left\{ \begin{pmatrix} 2^n & t \\ 0 & 4^n \end{pmatrix} \right\}$$

with $n \in \mathbb{Z}$ and $t$ a dyadic rational, $t = \frac{m}{2^k}$ for some $k \in \mathbb{Z}$ and $m$ an odd integer or $m = 0$, since this is the smallest discrete analogue to the case discussed in example 1. However, this group $D$ is not closed in $GL_2(\mathbb{R})$, so that theorem 3.5 does not apply.

Thus, we still require that the parameter $t$ be continuous.

**Example 3:** Fix $a > 0$, $a \neq 1$, and $r \in \mathbb{Z}$. Consider the subgroup $D$ of $GL_2(\mathbb{R})$

$$D = \left\{ d = \begin{pmatrix} a^k & t \\ 0 & a^{rk} \end{pmatrix} : k \in \mathbb{Z}, t \in \mathbb{R} \right\}.$$

We can identify this group topologically with $\mathbb{Z} \times \mathbb{R}$, where group operation is given by

$$d_1 \cdot d_2 = (k_1, t_1) \cdot (k_2, t_2) = (k_1 + k_2, t_2 a^{k_1} + t_1 a^{rk_2}).$$

Note that $D$ is a closed subgroup of the group considered in example 1.

The left Haar integral is given by $\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} f(k,t)\frac{1}{a^k}dt$, and the right Haar integral is $\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} f(k,t)\frac{1}{a^{rk}}dt$, and the modular function $\Delta(d) = a^{k(r-1)}$. Since $a \neq 1$ we have $|\det d| = a^{k(r+1)} \neq a^{k(r-1)} = \Delta(d)$ for $k \neq 0$. It is left to show that $D_{\vec{x}}^{\epsilon}$ is compact for almost all $\vec{x}$. Note that $D_{\vec{x}}^{\epsilon} = \{(k,t) \in \mathbb{Z}\times\mathbb{R} : (a^k-1)^2x_1^2 + [tx_1 + (a^{rk}-1)x_2]^2 \le \epsilon^2\}$ is homeomorphic with $\widetilde{D} = \{(a^k,t) \in \mathbb{R}^+\times\mathbb{R} : (a^k-1)^2x_1^2 + [tx_1 + (a^{rk}-1)x_2]^2 \le \epsilon^2\}$. As shown in example 1, $\widetilde{D}$ is compact in $\mathbb{R}^+\times\mathbb{R}$ for almost all $\vec{x}$ provided that $\epsilon$ is sufficiently small. Hence, $D_{\vec{x}}^{\epsilon}$ is compact in $\mathbb{Z}\times\mathbb{R}$, so that $D$ is admissible.

In a similar way, one can discretize the parameter $a$ in example 2:

**Example 4:** Fix $a > 0$, $a \neq 1$ and $r \in \mathbb{Z}$. Consider the subgroup $D$ of $GL_2(\mathbb{R})$

$$D = \left\{ d = \begin{pmatrix} a^k & 0 \\ t & a^{rk} \end{pmatrix} : k \in \mathbb{Z}, t \in \mathbb{R} \right\}.$$

This group carries the topology of $\mathbb{Z}\times\mathbb{R}$.

The left Haar integral is $\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} f(k,t)\frac{1}{a^{rk}}dt$, and the right Haar integral is $d\mu_R(d) = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} f(k,t)\frac{1}{a^k}dt$, and the modular function $\Delta(d) = a^{k(1-r)}$. Thus, if $r = 0$, then $|\det d| = a^{k(r+1)} = a^{k(1-r)} = \Delta(d)$ and we conclude that $D$ is not admissible. Suppose that $r \neq 0$. Note that $D_{\vec{x}}^{\epsilon} = \{(k,t) \in \mathbb{Z}\times\mathbb{R} : [(a^k-1)x_1 + tx_2]^2 + (a^{rk}-1)^2x_2^2 \le \epsilon^2\}$ is homeomorphic with $\widetilde{D} = \{(a^k,t) \in \mathbb{R}^+\times\mathbb{R} : [(a^k-1)x_1 + tx_2]^2 + (a^{rk}-1)^2x_2^2 \le \epsilon^2\}$. As shown in example 2, $\widetilde{D}$ is compact in $\mathbb{R}^+\times\mathbb{R}$ for almost all $\vec{x}$ provided that $\epsilon$ is sufficiently small. Hence, $D_{\vec{x}}^{\epsilon}$ is compact in $\mathbb{Z}\times\mathbb{R}$, so that $D$ is admissible iff $r \neq 0$.

We note that it is not difficult to verify that in all the above examples, we may also permit $a < 0$, that is $a \in \mathbb{R}\setminus\{0\}$. The Haar measures are identical, with $a$ replaced by $|a|$.

## 4.3 Admissible Functions

In section 4.2, we obtained admissibility conditions for two-parameter groups generated by two-commuting matrices $A$ and $B$, and for groups in two and three dimensions we had a complete characterization. Let us now construct admissible functions for each admissible group in two dimensions explicitly.

We will consider both the continuous and discrete case. Note that by proposition 2.11, $A^2$ and $B^2$ are exponentials, and by proposition 4.6 an admissible function for the discrete group generated by $A$ and $B$ can be constructed from an admissible function for the discrete group generated by $A^2$ and $B^2$. Thus, we only need to consider matrix groups generated by commuting exponential matrices which we will do from now on.

**Proposition 4.13.** *Let $\Phi : \mathbb{R}^2 \to D = \{A^sB^t : s,t \in \mathbb{R}\}$ be an admissible two-parameter group where $A = e^M, B = e^N$. So its range $D = \Phi(s,t)$ is one of the following:*

*Case A1:* $D = \left\{ \begin{bmatrix} e^{\alpha s+at} & 0 \\ 0 & e^{\beta s+bt} \end{bmatrix} s,t \in \mathbb{R} \right\}$ *where* $\alpha b - a\beta \neq 0$.

*Case A2:* $D = \left\{ e^{\alpha s+at} \begin{bmatrix} 1 & s+bt \\ 0 & 1 \end{bmatrix} s,t \in \mathbb{R} \right\}$ *where* $\alpha b - a \neq 0$.

*Then*

$S_1' = \{(\pm 1, \pm 1)\}$ *is a cross-section for* $\Phi$ *in case A1.*

$S_2' = \{(\pm 1, 1)\}$ *is a cross-section for* $\Phi$ *in case A2.*

*Proof.* Note that clearly $S_1'$ and $S_2'$ are Borel sets.

**Case A1**: Set $P = \{\vec{x} \in \mathbb{R}^2 : x_1, x_2 \neq 0\}$. Then

$$\vec{x}A^sB^t = (e^{\alpha s+at}x_1, e^{\beta s+bt}x_2).$$

Given $\vec{x} \in P$ we want to find a unique pair $(s,t) \in \mathbb{R} \times \mathbb{R}$ so that $\vec{x}A^sB^t \in S_1'$.

Obviously, $\vec{x}A^sB^t \in S_1'$ iff

$$e^{\alpha s+at} = \frac{1}{|x_1|}$$
$$e^{\beta s+bt} = \frac{1}{|x_2|}$$

or equivalently,

$$\alpha s + at = -\ln|x_1|$$

$$\beta s + bt = -\ln|x_2|.$$

By the assumption that $\alpha b - a\beta \neq 0$, this system has the unique solution

$$s = \frac{a\ln|x_2| - b\ln|x_1|}{\alpha b - a\beta} \tag{4.75}$$
$$t = \frac{\beta\ln|x_1| - \alpha\ln|x_2|}{\alpha b - a\beta}, \tag{4.76}$$

that is, there exists a unique pair $(s,t)$ such that $\vec{x}A^sB^t \in S_1'$. Conversely, if $\vec{x} = (\pm 1, \pm 1) \in S_1'$, then $\vec{x}A^sB^t = (\pm e^{\alpha s+at}, \pm e^{\beta s+bt}) \in P$. Since $P^c$ has measure zero, we conclude that $S_1'$ is a Borel cross-section.

**Case A2**: Set $P = \{\vec{x} \in \mathbb{R}^2 : x_1 \neq 0\}$. Then

$$\vec{x}A^sB^t = e^{\alpha s+at}(x_1, (s+bt)x_1 + x_2).$$

Given $\vec{x} \in P$ we want to find a unique pair $(s,t) \in \mathbb{R} \times \mathbb{R}$ so that $\vec{x}A^sB^t \in S_2'$. Obviously, $\vec{x}A^sB^t \in S_2'$ iff

$$e^{\alpha s+at} = \frac{1}{|x_1|}$$

$$(s+bt)x_1 + x_2 = |x_1|$$

or equivalently,

$$\alpha s + at = -\ln|x_1|$$
$$s + bt = \frac{|x_1| - x_2}{x_1}.$$

By the assumption that $\alpha b - a \neq 0$, this system the unique solution

$$s = \frac{-\left(b \ln|x_1| + \frac{a}{x_1}(|x_1| - x_2)\right)}{\alpha b - a} \tag{4.77}$$

$$t = \frac{\ln|x_1| + \frac{\alpha}{x_1}(|x_1| - x_2)}{\alpha b - a}, \tag{4.78}$$

that is, there exists a unique pair $(s,t)$ such that $\vec{x}A^sB^t \in S_2'$. Conversely, if $\vec{x} = (\pm 1, 1) \in S_2'$, then $\vec{x}A^sB^t = \left(\pm e^{\alpha s + at}, \left(1 \pm (s + bt)\right)e^{\alpha s + at}\right) \in P$. Since $P^c$ has measure zero, we conclude that $S_2'$ is a Borel cross-section. □

Next we consider cross-sections for discrete admissible two-parameter groups $\Psi : \mathbb{Z}^2 \to \widetilde{D} = \{A^kB^l : k, l \in \mathbb{Z}\}$ where $A = e^M$ and $B = e^N$. Then $\widetilde{D}$ is in one of the following:

**Case A1:** $\widetilde{D} = \left\{ A^kB^l = \begin{bmatrix} e^{\alpha k + al} & 0 \\ 0 & e^{\beta k + bl} \end{bmatrix} : k, l \in \mathbb{Z} \right\}$ where $\alpha b - a\beta \neq 0$.

**Case A2:** $\widetilde{D} = \left\{ A^kB^l = e^{\alpha k + al} \begin{bmatrix} 1 & k + bl \\ 0 & 1 \end{bmatrix} : k, l \in \mathbb{Z} \right\}$ where $\alpha b - a \neq 0$.

**Proposition 4.14.** *Let $S'$ denote the cross-section for the continuous two-parameter group $\Phi : \mathbb{R}^2 \to \{A^sB^t : s, t \in \mathbb{R}\}$ as in proposition 4.13. Set $S = \{\vec{y_0}A^sB^t : \vec{y_0} \in S', s, t \in [0, 1)\}$. Then $S$ is a cross-section for the discrete two-parameter group $\Psi : \mathbb{Z}^2 \to \{A^kB^l : k, l \in \mathbb{Z}\}$.*

*Proof.* Keep the notation of the proof of the above proposition. Clearly, if $\vec{y} = \vec{y_0}A^sB^t \in S$, then $\vec{y}A^kB^l = \vec{y_0}A^{s+k}B^{t+l} \in P$. Now let $\vec{x} \in P$. Since $S'$ is a cross-section for the continuous action, there exist unique $r, q \in \mathbb{R}$ and $\vec{y_0} \in S'$ such that

$$\vec{x} = \vec{y_0}A^rB^q.$$

Now there exist unique $k, l, s, t$ with

$$r = k + s \quad \text{where} \quad k \in \mathbb{Z} \quad \text{and} \quad s \in [0, 1)$$

$$q = l + t \quad \text{where} \quad l \in \mathbb{Z} \quad \text{and} \quad t \in [0, 1).$$

Set $\vec{y}_1 = \vec{y}_0 A^s B^t \in S$. Then $\vec{y}_1 A^k B^l = \vec{x}$. Since $\vec{y}_1, k, l$ are unique, we conclude that $S$ is a cross-section for the discrete action. □

**Remark 4.5.** *The cross-section $S$ gives rise to an admissible function $\psi$ in a natural way. Choose $\psi \in L^2(\mathbb{R}^2)$ so that $\widehat{\psi} = \chi_S$ . Then for all $\vec{x} \in P, \vec{x}A^k B^l \in S$ for exactly one pair $(k, l)$. Thus for almost all $\vec{x}$,*

$$\sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\vec{x}A^k B^l)|^2 = \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|\chi_S(\vec{x}A^k B^l)\right|^2 = 1$$

*so that $\psi$ is an admissible function.*

It follows from propositions 4.6 and 4.14 that we have an easily constructed admissible function for both the discrete and continuous actions. Unfortunately, since $\widehat{\psi}$ is the characteristic function of some Borel set $S$, $\psi$ is not smooth. Our next goal is thus to construct an admissible functions $\psi$ which is smooth and has compact support. Our construction is adapted from a preprint by Schulz and Taylor. Since any admissible function for the discrete action is also admissible for the continuous action, we need only consider the discrete case. We start with the cross-section in proposition, 4.14.

$$S = \left\{\vec{y}_0 A^s B^t : \vec{y}_0 \in S', s, t \in [0, 1)\right\},$$

and modify $S$ to a slightly larger set

$$S_\delta = \left\{\vec{y}A^s B^t : \vec{y} \in S, |s|, |t| \le \delta\right\}$$

so that

$$S_\delta = \left\{\vec{y}_0 A^s B^t : \vec{y}_0 \in S', s, t \in [-\delta, 1 + \delta]\right\}$$

where we chose $\delta$ so that $0 < \delta < \frac{1}{2}$.

**Lemma 4.15.** *Let $\Psi : \mathbb{Z}^2 \to D = \{A^k B^l : k, l \in \mathbb{Z}\} \subset GL_2(\mathbb{R})$ be admissible as in the proof of proposition 4.14. Then for all $\vec{x}_0 \in P$, there exist a neighborhood $\mathcal{U}$ of $\vec{x}$ such that*

$$\vec{x} A^k B^l \in S_\delta$$

*for at most 4 pairs $(k, l)$ and all $\vec{x} \in \mathcal{U}$.*

*Proof.* Keep the notation of the previous propositions. Let $S' = S_1'$, or $S_2'$, depending on the types of matrices $A$ and $B$.

Note that if $\vec{x} = (x_1, x_2) \in P$, then as shown in the proof of proposition 4.13, there exist unique real numbers $s = s(\vec{x})$ and $t = t(\vec{x})$ such that $\vec{x} A^s B^t \in S'$. Then

$$\vec{x} = \vec{y}_0 A^{-s} B^{-t} \qquad \text{for some } \vec{y}_0 \in S'$$

$$\vec{x} A^k B^l = \vec{y}_0 A^{k-s} B^{l-t}$$

So if $\vec{x} A^k B^l \in S_\delta$ then

$$-\delta \le k - s \le 1 + \delta$$

$$-\delta \le l - t \le 1 + \delta,$$

or equivalently

$$\begin{aligned} s(\vec{x}) - \delta \;&\le\; k \;\le\; s(\vec{x}) + (1 + \delta) \\ t(\vec{x}) - \delta \;&\le\; l \;\le\; t(\vec{x}) + (1 + \delta). \end{aligned} \tag{4.79}$$

Now formulas (4.75) and (4.76), respectively (4.77) and (4.78) for the functions $s, t$ show that the map

$$\vec{x} \in P \mapsto (s(\vec{x}), t(\vec{x}))$$

is continuous. Thus, given a fixed $\vec{x}_0 \in P$, there exists an open neighborhood

$\mathcal{U}(\vec{x}_0)$ of $\vec{x}_0$ in $P$ so that

$$\begin{aligned} s(\vec{x}_0) - \delta_1 &< s(\vec{x}) < s(\vec{x}_0) + \delta_1 \\ t(\vec{x}_0) - \delta_1 &< t(\vec{x}) < t(\vec{x}_0) + \delta_1 \end{aligned} \tag{4.80}$$

for all $\vec{x} \in \mathcal{U}(\vec{x}_0)$, where we have picked $\delta_1$ so that $0 < \delta + \delta_1 < \frac{1}{2}$. Since $P$ is open in $\mathbb{R}^2$, $\mathcal{U}(\vec{x}_0)$ is also open in $\mathbb{R}^2$.

Now if $\vec{x} \in \mathcal{U}(\vec{x}_0)$ and $\vec{x}A^kB^l \in S_\delta$ then by (4.79) and (4.80),

$$s(\vec{x}_0) - \delta_1 - \delta < k < s(\vec{x}_0) + 1 + \delta + \delta_1$$

$$t(\vec{x}_0) - \delta_1 - \delta < l < t(\vec{x}_0) + 1 + \delta + \delta_1$$

so that

$$\begin{aligned} s(\vec{x}_0) - \frac{1}{2} < k < s(\vec{x}_0) + \frac{3}{2} \\ t(\vec{x}_0) - \frac{1}{2} < l < t(\vec{x}_0) + \frac{3}{2}. \end{aligned}$$

This shows that there exist at most 4 pairs $(k, l)$ so that $\vec{x}A^kB^l \in S_\delta$, independent of the choice of $\vec{x} \in \mathcal{U}(\vec{x}_0)$. □

The goal now is to construct an admissible function $\psi$ whose Fourier transform $\widehat{\psi}$ is smooth and has compact support. The following theorem shows the construction of $\widehat{\psi}$ explicitly.

**Theorem 4.16.** *Let $\Psi : \mathbb{Z}^2 \to GL_2(\mathbb{R})$ be a discrete admissible two-parameter group. Then there exists an admissible function $\psi$ such that $\widehat{\psi} \in C_c^\infty(\mathbb{R}^2)$.*

*Proof.* Let $D = \{A^kB^l : k, l \in \mathbb{Z}\}$. Fix $0 < \delta < \frac{1}{2}$, an let $S$ and $S_\delta$ be as above. First we construct a function $f \in C_c^\infty(\mathbb{R}^2)$ which is supported on $S_\delta$ and such that

$$0 \leq f \leq 1, \quad f(\vec{x}) = 1 \quad \text{for all } \vec{x} \in S.$$

**Case A1** : We split $S_\delta$ into the four parts which lie in the four quadrants.

In fact, if

$$S' = \{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$$

where $\vec{x}_1 = (1,1), \vec{x}_2 = (-1,1), \vec{x}_3 = (-1,-1)$ and $\vec{x}_4 = (1,-1)$, set

$$S^i = \{\vec{x}_i A^s B^t : 0 \leq s,t \leq 1\}$$

and

$$S^i_\delta = \{\vec{x}_i A^s B^t : -\delta \leq s,t \leq 1+\delta\}.$$

Next we construct diffeomorphisms from $\mathbb{R}^2$ onto each of the four quadrants $Q_i$. In fact, the maps

$$g_i : \mathbb{R}^2 \to Q_i$$

given by

$$g_i(s,t) = \vec{x}_i A^s B^t = (\pm e^{\alpha s + at}, \pm e^{\beta s + bt})$$

are injective since $\alpha b - a\beta \neq 0$, surjective and infinitely differentiable. Indeed, formulas (4.75) and (4.76) show that the inverse maps $g_i^{-1} : Q_i \to \mathbb{R}^2$ are given by

$$g_i^{-1}(u,v) = \Big(\frac{a \ln|v| - b \ln|u|}{\alpha b - a\beta}, \frac{\beta \ln|u| - \alpha \ln|v|}{\alpha b - a\beta}\Big)$$

which are clearly infinitely differentiable.

In particular, we observe that $g_i$ maps the unit square $[0,1] \times [0,1]$ onto $S^i$, and $[-\delta, 1+\delta] \times [-\delta, 1+\delta]$ onto $S^i_\delta$. Now by Jones (1993) pp. 175-180, there exists a function $f_0 \in C_c^\infty(\mathbb{R}^2)$ such that $0 \leq f_0 \leq 1$, $\text{supp}\,(f_0) \subset [-\delta, 1+\delta] \times [-\delta, 1+\delta]$ and $f_0(\vec{x}) = 1 \;\; \forall \vec{x} \in [0,1] \times [0,1]$. Since $[-\delta, 1+\delta] \times [-\delta, 1+\delta]$ is compact, and each $g_i$ is continuous, each $S^i_\delta$ is compact in $Q_i$. Since $Q_i$ is open on $\mathbb{R}^2$, $S^i_\delta$ is also compact in $\mathbb{R}^2$. Set

$$f_i = f_0 \circ g_i^{-1} \qquad : i = 1, ..., 4.$$

Then $f_i : Q_i \to [0,1]$ is infinitely differentiable and has support contained in $S^i_\delta = g_i([-\delta, 1+\delta] \times [-\delta, 1+\delta])$, $0 \leq f_i \leq 1$ and $f_i(\vec{x}) = 1 \;\; \forall \vec{x} \in S^i = g_i([0,1] \times [0,1])$.

Next we extend each $f_i$ to $\mathbb{R}^2$ by setting $f_i(\vec{x}) = 0 \ \forall \vec{x} \notin Q_i$.

We need to verify that $f$ is in $C^\infty(\mathbb{R}^2)$. Suppose, $\vec{y}_0 \in \mathbb{R}^2 \setminus Q_i$. Then since $S^i_\delta$ is closed in $\mathbb{R}^2$ and $\vec{y}_0 \notin S^i_\delta$, we can find an open neighborhood $\mathcal{U}(\vec{y}_0)$ such that $\mathcal{U}(\vec{y}_0) \cap S^i_\delta = \emptyset$. Then $f_i(\vec{y}) = 0 \quad \forall \vec{y} \in \mathcal{U}(\vec{y}_0)$, so that obviously, $f_i$ is infinitely differentiable at $\vec{y}_0$. This shows that $f \in C^\infty(\mathbb{R}^2)$, $\text{supp}\,(f_i) \subset S^i_\delta$, $0 \le f_i \le 1$ and $f_i(\vec{x}) = 1 \ \forall \vec{x} \in S^i$.

Then $f = \sum_{i=1}^{4} f_i$ is the desired function.

**Case A2**: We proceed similarly:

If $S' = \{\vec{x}_1, \vec{x}_2\}$ with $\vec{x}_1 = (1,1)$ and $\vec{x}_2 = (-1,1)$, then we set

$$S^i = \{\vec{x}_i A^s B^t : 0 \le s, t \le 1\}$$

and

$$S^i_\delta = \{\vec{x}_i A^s B^t : -\delta \le s, t \le 1 + \delta\} \qquad : i = 1, 2.$$

Let $Q_1$ and $Q_2$ denote the open half planes

$$Q_1 = \{\vec{x} \in \mathbb{R}^2 : x_1 > 0\}$$

$$Q_2 = \{\vec{x} \in \mathbb{R}^2 : x_1 < 0\}$$

respectively. Then the maps

$$h_i : \mathbb{R}^2 \to Q_i$$

given by

$$h_i(s,t) = \vec{x}_i A^s B^t = \big(\pm e^{\alpha s + at}, \big(1 \pm (s + bt)\big) e^{\alpha s + at}\big)$$

are injective since $\alpha b - a \neq 0$, surjective and infinitely differentiable. Indeed, formulas (4.77) and (4.78) show that the inverse maps $h_i^{-1} : Q_i \to \mathbb{R}^2$ are given by

$$h_i^{-1}(u,v) = \Big(\frac{-\big(b \ln|u| + \frac{a}{u}(|u| - v)\big)}{\alpha b - a}, \frac{\ln|u| + \frac{\alpha}{u}(|u| - v)}{\alpha b - a}\Big)$$

which are clearly infinitely differentiable.

Keep $f_0$ as in case A1, set

$$f_i = f_0 \circ h_i^{-1} \qquad : i = 1, 2.$$

Then $f_i \in C^\infty(Q_i)$, $0 \le f_i \le 1$, $\operatorname{supp}(f_i) \subset S^i_\delta$, and $f_i(\vec{x}) = 1 \ \forall \vec{x} \in S^i$.

Arguing as in case A1, one can extend the functions $f_i$ to $C^\infty$ functions defined on $\mathbb{R}^2$, by setting

$$f_i(\vec{x}) = 0 \ \ \forall \vec{x} \notin Q_i,$$

then $f = f_1 + f_2$ is the desired function.

Now set

$$\sigma(\vec{x}) = \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left(f(\vec{x}A^k B^l)\right)^2 : \quad \vec{x} \in P. \tag{4.81}$$

By lemma 4.15, $\forall \vec{x_0} \in P$, we can find a neighborhood $\mathcal{U}$ of $\vec{x_0}$ so that the terms in this summation (4.81) are zero for all except at most 4 pairs $(k, l)$, which are independent of $\vec{x}$, say $(k_i, l_i) \ : i = 1, ..., 4$. That is, for $\vec{x} \in \mathcal{U}(\vec{x}_0)$, (4.81) is a finite sum

$$\sigma(\vec{x}) = \left(f(\vec{x}A^{k_1}B^{l_2})\right)^2 + ... + \left(f(\vec{x}A^{k_4}B^{l_4})\right)^2.$$

Since each term on the right hand side is infinitely differentiable at $\vec{x}_0$, it follows that $\sigma$ is infinitely differentiable at $\vec{x}_0$, that is, $\sigma \in C^\infty(P)$. Since $0 \le f \le 1$, $f(\vec{x}) = 1$ for $\vec{x} \in S$, and $S$ is a cross-section, at least one term in this sum is equal to one, so that

$$1 \le \sigma(\vec{x}) \le 4$$

for all $\vec{x} \in \mathcal{U}(\vec{x_0})$. Set

$$\widehat{\psi}(\vec{x}) = \begin{cases} \frac{f(\vec{x})}{\sqrt{\sigma(\vec{x})}} & : \vec{x} \in P \\ 0 & : \vec{x} \notin P. \end{cases}$$

Since $f \in C_c^\infty(\mathbb{R}^2)$ is supported on $S_\delta$, and since as shown above, $S_\delta$ is a compact subset of $P$, there exists an open set $\mathcal{O}$ containing $P^c$, so that $\mathcal{O} \cap S_\delta = \emptyset$. Since $f(\vec{x}) = 0$ for $\vec{x} \in \mathcal{O}$, we conclude that $\widehat{\psi}$ is infinitely differentiable on $\mathcal{O}$, and hence $\widehat{\psi} \in C_c^\infty(\mathbb{R}^2)$. By construction, for all $\vec{x} \in P$,

$$\sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\widehat{\psi}(\vec{x}A^k B^l)|^2 = \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \frac{\left(f(\vec{x}A^k B^l)\right)^2}{\sigma(\vec{x}A^k B^l)} = 1. \tag{4.82}$$

Since $P^c$ has measure zero, the inverse Fourier transform of $\widehat{\psi}$ is the desired admissible function. □

# Chapter V

# Conclusion

The objective of this thesis was to investigate what matrix groups which depend on two parameters are admissible for wavelet analysis, and to construct explicit admissible functions. The tools used are mainly from the works of Weiss et al. (2002) and the preprint by Schulz and Taylor. We have obtained the following results

1. In proposition 4.4 we have given a complete classification of commuting matrix pairs $A, B \in GL_2(\mathbb{R})$ so that the continuous two-parameter group $\Phi : (s,t) \in \mathbb{R}^2 \to A^s B^t$ is defined and is admissible.

2. In corollary 4.7 we have given a complete classification of commuting matrix pairs $A, B \in GL_2(\mathbb{R})$ so that the discrete two-parameter group $\Psi : (k,l) \in \mathbb{Z}^2 \to A^k B^l$ is defined and is admissible.

3. In examples 1-4 we have investigated classes of non-abelian subgroup of $GL_2(\mathbb{R})$ depending on two parameters, and have classified which of these group are admissible.

4. In theorem 4.16 we have constructed admissible functions for the admissible two-parameter groups in 1 and 2 which lie in the Schwartz class.

5. In propositions 4.5 and 4.6 we have clarified the relationship between generators $A, B$ of discrete and of continuous two-parameter groups. In particular, we have shown that every admissible function for the discrete two-parameter

group is admissible for the continuous two-parameter group, and every admissible function for the continuous two-parameter group can be modified to be admissible for the discrete two-parameter group. Furthermore, in the discrete case, we may assume that $A$ and $B$ are exponential matrices.

6. In theorem 4.8 we have given a complete classification of commuting matrix pairs $A, B \in GL_3(\mathbb{R})$ so that the continuous two-parameter group $\Phi : (s,t) \in \mathbb{R}^2 \to A^s B^t$ is defined and is admissible.

7. In theorem 4.12 we have presented a large classes of commuting matrices $M, N \in GL_n(\mathbb{R})$ so that the continuous two-parameter group $\Phi : (s,t) \in \mathbb{R}^2 \to A^s B^t$ is defined and is admissible.

The goal of classifying all two-parameter matrix groups which are admissible is far from complete, because it is not easy to classify all commuting $n \times n$ matrix pairs $M$ and $N$ for $n > 3$. It is also interesting to study what matrices give rise to admissible $n$-parameter groups.

In the examples of non-commuting $2 \times 2$ matrix group, we omitted the case where both parameters are discrete because the tools provided by theorem 3.5 can no longer be applied. It would be an interesting question to investigate whether such groups can be admissible at all.

Finally, for practical computations, one requires admissible functions which are in the Schwartz class. We have been able to find these in the case of $2 \times 2$ matrix groups only. In higher dimensions such admissible functions do not always exist as shown in the preprint by Schulz and Taylor, so one needs to classify for what groups such functions do exist.

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# Appendix

# Appendix A

# Classification of Commuting $2 \times 2$ Matrix Pairs

Let $A, B \in M_2(\mathbb{R})$ with $AB = BA$. We bring matrix $A$ into real Jordan normal form. There are 4 possibilities for $A$.

1. $A$ is diagonalizable and has only one real eigenvalue,

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

   Because $A$ is a multiple of the identity, every $2 \times 2$ matrix $B$ commutes with $A$ and by a change of basis leaving $A$ unchanged, we can bring $B$ also into Jordan normal form as follows:

   (1a) $B$ is diagonalizable, $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

   (1b) $B$ has only one real eigenvalue but is not diagonalizable, $B = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$

   (1c) $B$ has complex eigenvalues, $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where $b \neq 0$.

2. $A$ is diagonalizable and has distinct real eigenvalues, $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\alpha \neq \beta$.

   Every matrix $B$ commuting with $A$ must be of the form $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

3. $A$ has one real eigenvalue and is not diagonalizable, $A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$. Then every matrix $B$ commuting with $A$ is of the form $B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

4. $A$ has complex eigenvalues, $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ with $\beta \neq 0$. Every matrix $B$ commuting with $A$ is of the form $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

We can summerize the above as follows: After exchanging $A$ and $B$, if necessary, and an appropriate change of basis, exactly one of the following holds:

**Case 1** Both matrices are simultaneously diagonalizable,

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

**Case 2** Both matrices have only one real eigenvalue and at least one, say $A$, is not diagonalizable. Then in the Jordan basis of $A$,

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

**Case 3** At least one matrix, say $A$, has complex eigenvalues,

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{where } \beta \neq 0.$$

# Appendix B

# Classification of Commuting $3 \times 3$ Matrix Pairs

Let $A, B \in M_3(\mathbb{R})$ with $AB = BA$. Bringing matrix $A$ into real Jordan normal form we distinguish between 4 possibilities for $A$ as follows:

1. $A$ has exactly one eigenvalue $\alpha$ which is real.

(1a) $A$ has 3 Jordan blocks, $A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$.

Since $A$ is a multiple of the identity, every matrix $B \in M_3(\mathbb{R})$ commutes with $A$, and by a change of basis leaving $A$ unchanged, we can also bring $B$ into real Jordan normal form. There are 5 possibilities for $B$:

(1a.1) $B$ has 3 real Jordan blocks, $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$.

(1a.2) $B$ has 2 real Jordan blocks, $B = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$.

(1a.3) $B$ has one real Jordan block, $B = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$.

(1a.4) $B$ has one complex and one real Jordan block,

$$B = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix} \qquad b \neq 0.$$

(1b) $A$ has 2 real Jordan blocks, $A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$. Then

(1b.1) $B = \begin{pmatrix} a & b & d_2 \\ 0 & a & 0 \\ 0 & d_1 & c \end{pmatrix}$ where $a \neq c$.

Note that by a change of basis which leaves $A$ unchanged, we can bring $B$ into the form $B = \begin{pmatrix} a & b' & d \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$ where $d = d_1 d_2, b' = b + \frac{d}{a-c}$.

(1b.2) $B = \begin{pmatrix} a & b & d_2 \\ 0 & a & 0 \\ 0 & d_1 & a \end{pmatrix}$.

(1c) $A$ has 1 real Jordan block,

$$A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}, \text{ then } B = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}.$$

2. $A$ has two distinct eigenvalues $\alpha \neq \beta$ which are real.

(2a) $A$ has 3 real Jordan blocks,

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}, \text{ then } B = \begin{pmatrix} B_1 & 0 \\ 0 & c \end{pmatrix} \text{ where } B_1 = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

We can bring $B_1$ into real Jordan normal form leaving $A$ unchanged.

Then there are 3 possibilities,

(2a.1) $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where $B_1$ has 2 real Jordan blocks.

(2a.2) $B = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$ where $B_1$ has 1 real Jordan block.

(2a.3) $B = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}$ where $B_1$ has a complex pairs of eigenvalues and $b \neq 0$.

(2b) $A$ has 2 real Jordan blocks, $A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$. Every matrix $B$ commuting with $A$ must be of the form $B = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$.

3. $A$ has 3 distinct real eigenvalues $\alpha, \beta, \gamma$.
$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$. Every matrix $B$ commuting with $A$ must be of the form $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$.

4. $A$ has one real eigenvalue and a complex pairs of eigenvalues,

$A = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ $\beta \neq 0$. Every matrix $B$ commuting with $A$ must be of the form $B = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}$.

We can summarize this into 6 distinct cases as follows: After exchanging $A$ and $B$ if necessary, and choosing an appropriate basis, exactly one of the following holds:

(1) $A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$, $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$.

(2) $A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$, $B = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$.

(3) $A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$, $B = \begin{pmatrix} a & b & d \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$

where $a \neq c, d \neq 0$ ( if $d = 0$, this can be reduced to case (5) by exchanging $A$ and $B$ and changing basis.

(4) $A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$, $B = \begin{pmatrix} a & b & d_2 \\ 0 & a & 0 \\ 0 & d_1 & a \end{pmatrix}$.

(5) $A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$, $B = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$ where $\alpha \neq \beta$.

(6) $A = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$, $B = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}$ where $\beta \neq 0$.

# Curriculum Vitae

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