ความสลับซับซ้อนของพลศาสตร์ไฟฟ้าเชิงควอนตัมแบบสมมาตรยวดยิ่ง

นายศุภปิยะ สิระนันท์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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INTRICACIES OF SUPERSYMMETRIC QUANTUM ELECTRODYNAMICS

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INTRICACIES OF SUPERSYMMETRIC

QUANTUM ELECTRODYNAMICS

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ศุภปิยะ สิระนันท์ : ความสลับซับซ้อนของพลศาสตร์ไฟฟ้าเชิงควอนตัมแบบสมมาตรยวดยิ่ง (INTRICACIES OF SUPERSYMMETRIC QUANTUM ELECTRODYNAMICS) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร. เอ็ดเวิร์ด มานูเกียน, 195 หน้า.

วิทยานิพนธ์นี้เกี่ยวข้องกับการวิเคราะห์ระเบียบวิธีแบบสมมาตรยวดยิ่ง ในพลศาสตร์ไฟฟ้าควอนตัมของระบบหลายอนุภาคในฟิสิกส์ควอนตัม และของพลศาสตร์ ไฟฟ้าควอนตัมแบบ สมมาตร ยวดยิ่ง ใน รูป นัย นิยม เชิง อนุพันธ์ ฟังก์ชัน นัล ของ ทฤษฎี สนาม ควอนตัม ในการปรากฏของสนามไม่อิสระ การเบี่ยงเบนที่สลับซับซ้อนในรายละเอียดของหลักเชิงพลวัตควอนตัมในการปรากฏของสนามไม่อิสระได้ถูกเขียนขึ้นเพื่อใช้จัดการกับพลศาสตร์ ไฟฟ้าควอนตัมแบบสมมาตรยวดยิ่งซึ่งสนามไม่อิสระจำเป็นต้องเกิดขึ้นอันเป็นผลมาจากเงื่อนไขบังคับเกจ ผลการประยุกต์ใช้ระเบียบวิธีแบบสมมาตรยวดยิ่ง ขอบเขตล่างอย่างเคร่งครัดของพลังงานสถานะพื้นของพลศาสตร์ไฟฟ้าเชิงควอนตัม ของระบบหลายอนุภาคโบซอนที่มีประจุในฟิสิกส์ควอนตัมได้ถูกเขียนขึ้น และพบว่า เป็นฟังก์ชันของจำนวนอนุภาคที่มีประจุลบ หลักเชิงพลวัตควอนตัมได้ถูกใช้ ในการวิเคราะห์ปัญหาเกจอย่างเป็นระบบในพลศาสตร์ไฟฟ้าเชิงควอนตัมที่เกี่ยวข้องกับ เกจแบบคูลอมบ์ เกจแบบฟอกค์–ชวิงเงอร์ เกจแบบตามแกน และเกจแบบโควาเรียนต์อื่นๆ เริ่มต้นจากลากรานเจียนของพลศาสตร์ ไฟฟ้าควอนตัมแบบสมมาตรยวดยิ่งที่เสนอโดยเวสส์และซูมิโน หลักเชิงพลวัตควอนตัมได้ถูกใช้เขียนนิพจน์ชัดแจ้งสำหรับแอมพลิจูดการเปลี่ยนสถานะ สุญญากาศไปสู่สุญญากาศของทฤษฎี เป็นการดำเนินการหาอนุพันธ์ฟังก์ชันนัลต่อ ฟังก์ชันนัลชัดแจ้งที่บรรยายการแผ่ของอนุภาคและคู่หูยวดยิ่งของมันระหว่าง ตัวปล่อยและตัวตรวจหาซึ่งแทนด้วยแหล่งต้นทางภายนอก ผลการประยุกต์ แอมพลิจูดการกระเจิงของอิเล็กตรอนกับโพสิตรอนไปสู่โฟติโนและแอนติโฟติโน และแอมพลิจูดของพลังงานตัวเองของอิเล็กตรอนได้ถูกเขียนขึ้นถึงอันดับนำ แอมพลิจูดของพลังงานตัวเองของอิเล็กตรอนเกี่ยวข้องกับแผนภาพเพิ่มเติมจาก กรณีพลศาสตร์ไฟฟ้าเชิงควอนตัมเดี่ยวๆ ที่สุดแล้วเราได้แสดงให้เห็นว่า ค่าคงตัวรีนอร์มัลไลเซชันของฟังก์ชันคลื่น Z_2 มีค่าจำกัดถึงอันดับนำ ในเกจแบบแลนเดา เหมือนกับกรณีพลศาสตร์ไฟฟ้าเชิงควอนตัมเดี่ยวๆ นัยยะของผลลัพธ์ในทฤษฎีสมมาตรยวดยิ่งแบบเต็มๆ กรณีไร้มวลที่พลังงานสูง และความสอดคล้องภายในของมันได้ถูกเน้นย้ำ เหตุผลคือมันได้เปิดประตูสู่การศึกษา โดยเฉพาะอย่างยิ่ง ปริมาณที่ไม่แปรเปลี่ยนเกจอันดับต่ออันดับ อันนำมาซึ่งผลลัพธ์ ใน

ทฤษฎีสมมาตรยวดยิ่งเต็มรูปแบบกรณีไร้มวลที่พลังงานสูง และความสอดคล้องภายใน ของมันได้ถูกเน้นย้ำ มันได้เปิดวิถีในการศึกษา ปริมาณ*ไม่แปรเปลี่ยน*เกจ อันดับต่ออันดับในทฤษฎีเต็มรูปแบบในย่านพลังงานสูงไร้มวล เมื่อมวลหรือมาตราพลังงาน ไม่มีความสำคัญโดยการเลือกเกจที่เหมาะสมอันดับต่ออันดับซึ่ง นิพจน์ที่สมนัยมีค่าจำกัดอันดับต่ออันดับและไม่มีความกำกวมในการตัดออกของพจน์ อนันต์ที่มีเครื่องหมายตรงข้ามกัน คู่หูสมมาตรยวดยิ่งของอนุภาคที่สังเกตได้ ให้ค่าตรวจแก้แบบปล่อยรังสีไม่ชัดเป็นเส้นภายในของกระบวนการ โดยที่เส้นภายนอก สมนัยกับอนุภาคที่สังเกตได้ และการดัดแปลงผลลัพธ์ในพลศาสตร์อาจไม่จำเป็นที่จะละเลย และอาจทดสอบได้โดยการทดลองทางอ้อม

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา ____________________

ปีการศึกษา 2551 ลายมือชื่ออาจารย์ที่ปรึกษา ____________________

SUPPIYA SIRANAN : INTRICACIES OF SUPERSYMMETRIC QUANTUM ELECTRODYNAMICS. THESIS ADVISOR : PROF. EDOUARD B. MANOUKIAN, Ph.D. 195 PP.

QUANTUM ELECTRODYNAMICS / QUANTUM ELECTRODYNAMICS OF MANY-PARTICLE SYSTEMS / SUPERSYMMETRIC METHODS / GAUGE THEORIES / GAUGE TRANSFORMATIONS / RENORMALIZATION CONSTANTS / FUNCTIONAL DIFFERENTIAL FORMALISM OF QUANTUM FIELD THEORY.

This thesis is involved with extensive analyses of supersymmetric methods in the quantum electrodynamics of many-particle systems in quantum physics and of supersymmetric quantum electrodynamics working in the functional *differential* formalism of quantum field theory in the presence of dependent fields. A detail intricate deviation of the Quantum Dynamical Principle is derived in the presence of dependent fields tailored made to handle supersymmetric quantum electrodynamics in which dependent fields necessarily arise as a consequence of a gauge constraint. As an application of supersymmetric methods, a rigorous lower bound is derived for the ground-state energy of the quantum electrodynamics of charged many-particle systems of bosonic type in quantum physics as a function of the number of the negatively charged particles. The Quantum Dynamical Principle is then used to carry out a systematic analysis of the gauge problem in quantum electrodynamics dealing with the Coulomb gauge, the Fock–Schwinger gauge, the axial gauge and *all* covariant gauges. Starting from the Lagrangian proposed by Wess and Zumino for supersymmetric quantum electrodynamics, the Quantum Dynamical Principle thus derived is used to obtain the explicit expression for the vacuum-to-vacuum transition amplitude of the theory as a functional derivative operation applied to an exact functional describing the propagation of the particles and their superpartners between emitters and detectors represented by external sources.

As applications, the scattering amplitude of the process electron–positron to photino–photino is derived to the leading order, as well as of the self-energy of the electron. The latter involves additional diagrams to the one in pure quantum electrodynamics. We finally show that the wave-function renormalization constant Z_2 is finite in the Landau gauge, to the leading order, as it is in pure quantum electrodynamics. The implication of the result in the high-energy massless full supersymmetric theory and of its internal consistency is to be stressed. The reason is that it opens the way to study, in particular, gauge *invariant* quantities order by order in the full theory in the massless high-energy regime where the mass, providing an energy scale, becomes unimportant by working appropriately in a gauge chosen consistently order by order so that their corresponding expressions are finite order by order and no ambiguous cancellation of infinite terms of opposite sign arise. Finally, it is worth emphasizing that the supersymmetric partners of observed particles provide, in particular, non-trivial radiative corrections as internal lines of processes with external lines corresponding to observed particles and the resulting modifications in the dynamics may not necessarily be omitted on physical grounds and may, hopefully, be thus tested in experiments perhaps indirectly.

School of Physics	Student's Signature ____________
Academic Year 2008	Advisor's Signature ____________

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CHAPTER I

INTRODUCTION

Supersymmetry (SUSY) was introduced to treat bosons and fermions on the same footing and construct theories with built in transformations between fermions and bosons. Quantum field theories, being relativistic, then necessitate the introduction of generators Q^j_{A} to carry out such transformation, [cf. Wess and Zumino (1974a,b,c); Salam and Strathdee (1974a,b,c, 1975a)], in addition to the generators P^μ, $J^{\mu\nu}$ associated with the Lorentz transformations, where the P^μ operators generate space-time translations, and the $J^{\mu\nu}$ ones generate the Lorentz transformations themselves. Due to the spin and statistics theorem [cf. Streater and Wightman (1964)], the fermions and bosons, obeying different statistics, implies that the so-called supersymmetric generators Q^j_{A} behave as spinors under Lorentz transformations and satisfy anti-commutation relations as opposed to P^μ, $J^{\mu\nu}$ which satisfy commutation relations. The so-called Lie algebra associated with P^μ, $J^{\mu\nu}$ then necessarily change to some new algebra referred to as a graded Lie algebra which accommodates these anti-commutation relations [cf. Haag *et al.* (1975)]. In Q^j_{A}, A is a spinor index ($\mathsf{A} = 1, \ldots, 4$) and $j = 1, \ldots, N$ denotes the number of supersymmetric generators needed. The theory with $N = 1$ is referred as a simple SUSY model [Parkes and West (1983b); Piguet and Sibold (1984); Veneziano and Yankielowicz (1982)] and for $N \geqslant 2$, these models are referred to as extended SUSY models [Parkes and West (1983a); Fayet (1979)]. With such transformations between bosons and fermions and vice versa a new rule arises according to these theories as far as the type of particles predicted to exist in nature. The simplest such rule is that for every known boson there exists a Fermi partner and for every known fermion there exists a Bose particle. A Bose partner of a known fermion is named by adding “s” to the beginning of the fermion, e.g., selectron as a Bose partner of an electron. A Fermi partner of a known Bose particle is named by replacing "on" at the end of the boson's name by "ino", e.g., photino, gluino.

One of the characteristic of a SUSY theory is that it has the vacuum invariant under SUSY transformations and that the ground state energy is exactly zero. This fact alone has independent applications of supersymmetric methods in setting up supersymmetric Hamiltonians which are non-negative and hence are *bounded below* by zero. We have used this important property of supersymmetric methods to investigate rigorously the quantum electrodynamics (QED) ground-state energy of many-particle systems in quantum physics and will be elaborated upon later in this introductory chapter. Quite generally, as mentioned above, an invariance property of the vacuum in quantum field theory (QFT) under SUSY transformations predicts that superpartners have the same mass (and, in general, are produced in pairs). The fact that SUSY partners (e.g., selectron, photino) have not been observed seems to indicate that the vacuum is not invariant under SUSY transformations and a large lower bound on the masses of SUSY partners exist which are high enough to be accessible experimentally, and a corresponding theory is then referred to as spontaneously broken [Affleck *et al.* (1984); Campbell-Smith *et al.* (1999); Cecotti and Girardello (1982); de Wit and Freedman (1975a); Fayet (1975a,b); Fayet and Iliopoulos (1974); Ferrara *et al.* (1979); Girardello and Grirasu (1982); Iliopoulos and Zumino (1974); O'Raifeartaigh (1975); Parkes and West (1983a,b); Witten (1981, 1982); Zumino (1975)]. Also another advantage is that SUSY theories are expected to have less divergences in carrying out the renormalization program than in non-symmetric ones [Avdeev *et al.* (1980); Caswell and Zanon (1981); Ferrara *et al.* (1974a); Grisaru *et al.* (1980); Hollik *et al.* (1999); Iliopoulos and Zumino (1974); Jones (1975, 1977); Kraus and Stöckinger (2002); Mandelstam (1983); Piguet and Rouet (1975); Piguet and Sibold (1984); Poggio and Pendleton (1977); Rupp *et al.* (2000); Slavnov and Stepanyantz (2003)]. When the transformations between fermions and bosons (and vice versa) depend on the space-time point, i.e., are locally defined, gravitational theory naturally arises and is referred to as Supergravity (SUGRA) [cf. Jacob (1986)]. On the other hand, the extension of point particle descriptions in field theory to strings starting from a SUSY theory has led to Superstrings Theory [cf. Green *et al.* (1988)]. A wide class of SUSY models have been introduced over the years [Gol'fand and Likhtman (1971); Wess and Zumino (1974a,b); West (1976); Veneziano and Yankielowicz (1982); Slavnov and Stepanyantz (2003); Rupp *et al.* (2000); Piguet and Sibold (1984); Meisler (1996); Kraus and Stöckinger (2002); Hollik *et al.* (1999); Grisaru *et al.* (1979); Ferrara and Zumino (1974, 1975); Fayet (1976); de Wit and Freedman (1975b); Clark *et al.* (1980a,b); Bagger and Witten (1982)] with a lot of success, but still much work has to be done. Our interest are the technical aspects of the SUSY extension of quantum electrodynamics (SQED) [Wess and Zumino (1974b); Zumino (1975); Slavnov and Stepanyantz (2003); Rupp *et al.* (2000); Poggio and Pendleton (1977); Piguet and Rouet (1975); Hollik *et al.* (1999); Ferrara *et al.* (1974a); Ferrara and Zumino (1975); Clark *et al.* (1980a); Campbell-Smith *et al.* (1999); Avdeev *et al.* (1980)] as well as in such supersymmetric methods as applied to the quantum electrodynamics of many-particle systems in potential theory.

The development of the dynamics of QED *and* SQED requires extensive applications of the Quantum Dynamical Principle in the *presence* of so-called dependent fields. This is a highly non-trivial problem and is intimately related to the gauge problem and of resulting constraints. Accordingly, as an important contribution to the analysis of the intricacies of SQED, in particular, a fairly detailed investigation of the Quantum Dynamical Principle in the presence of dependent fields and resulting rigorous derivations were recently carried out [Manoukian *et al.* (2007)]. This is reported in equal details in Chapter II. The latter chapter also involves rigorously in the analysis of the gauge problem in QED as many of the results are significant in SQED. This chapter establishes many of the intricate details used throughout the thesis and is necessarily quite long. Here many of the specific gauges used are discussed. The following chapter (Chapter III), deals with supersymmetric methods as applied in quantum physics in potential theory and, in particular, to the quantum electrodynamics of charged many-particle systems, in which a rigorous lower bound is derived for the corresponding ground-state energy, as a function of the number of negatively charged particles involved, as in so-called bosonic matter. Chapter IV, introduces generators of supersymmetric transformation together with their anti-commutation relations with the four-momentum. The transformation law of the vectors in the underlying Hilbert space of physical states is also introduced. The Lagrangian proposed by Wess and Zumino [Wess and Zumino (1974a,b,c)] for SQED is spelled out paying special attention to the rôles of the supersymmetric partners and their corresponding propagators. External sources are introduced coupled to the basic fields appearing in the Wess–Zumino Lagrangian in Chapter V and the explicit expression for the vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle$ is derived as a functional *differential* operation applied to the vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle_0$ involving the propagation of free particles and their superpartners between various sources represent emitters and detectors of such particles. This provides the entire dynamics of the underlying theory and gives rise to all the fundamental processes governed by it giving rules of computations appropriately referred to as Schwinger–Feynman rules using functional derivatives and in contrast to the old fashioned rules based on Feynman diagrams avoids altogether "guessing" multiplicative weight factors and signs of basic integrals involving propagators, vertex functions and their various convolutions, thus emphasizing the power and elegance of the present formalism. Chapter VI deals with applications, to derive, in particular, the leading contribution to the scattering amplitude of electron–positron scattering to photino–photino, where the photino is the supersymmetric partner of the photon. Another significant application is also given to the so-called self-energy of the electron so that a direct comparison of this result with the one occurring in pure quantum electrodynamics is made with emphasis on the renormalization constant Z_2. To our surprise it is found to the leading order, Z_2 is finite only in the so-called Landau gauge. The significance of this result will be discussed in our concluding chapter (Chapter VII). It is important to emphasize that even if the supersymmetric partners of the present known particles are not detected in experiment, such particles, as virtual particles, would have important contributions to the scattering of the fundamental particles in nature as internal lines in the diagrams describing their interactions. [This is reminiscent of the non-observability of quarks in describing physical properties of fundamental problems, and even of the so-called Higgs particle and its rôle in unified field theories to ensure renormalizability.] Also they seem to have an utmost importance in the divergence problem of quantum field theories and hence of their internal consistencies. Chapter VII, also summarizes our main results. Some notations and conventions are finally given in an appendix.

CHAPTER II

QUANTUM ELECTRODYNAMICS AND GAUGE TRANSFORMATIONS

About two decades ago, it was shown [Manoukian (1986, 1987a)] that the very elegant action principle [Schwinger (1951a,b, 1953a,b, 1954a)] may be used to quantize gauge theories in constructing the vacuum-to-vacuum transition amplitude and the Faddeev–Popov factor [Faddeev and Popov (1967)], encountered in non-abelian gauge theories, was obtained *directly* from the action principle without much effort. No appeal was made to path integrals, no commutation rules were used, and there was not even the need to go into the well known complicated structure of the Hamiltonian [Fradkin and Tyutin (1970)] in non-abelian gauge theories. Of course path integrals are extremely useful in many respects and may be formally derived from the action principle [cf. Symanzik (1954); Lam (1965); Manoukian (1985)]. We have worked in the Coulomb gauge, where the physical components are clear at the outset, to derive the expression for the vacuum-to-vacuum transition amplitude (generating functional) including the Faddeev–Popov factor in non-abelian gauge theories. It is interesting to note also that the Coulomb gauge naturally arises [Faddeev and Jackiw (1988); Ogawa *et al.* (1996)], see also [Joglekar and Mandal (2002)], in gauge field theories as constrained dynamics [cf. Henneaux and Teitelboim (1992); Garcia *et al.* (1996); Su (2001)]. To make transitions of the generating functional to arbitrary covariant gauges, we have made use [Manoukian (1986, 1987a)], in the process, of so-called δ functionals [Schwinger (1965)]. The δ functionals, however, are defined as infinite dimensional continual integrals corresponding to the different points of spacetime and hence the gauge transformations were carried out in the spirit of path integrals.

The purpose of the present chapter is, in particular, to remedy the above situation involved with delta functionals, and we here derive the gauge transformations, providing explicit expressions, for the full vacuum-to-vacuum transition amplitude to the generating functionals of arbitrary covariant gauges and, in turn, to the celebrated Fock–Schwinger (FS) gauge

$$x^{\mu} A_{\mu} = 0 \,, \tag{2.1}$$

[Fock (1937); Schwinger (1951c)], as well as the axial gauge

$$n^{\mu} A_{\mu} = 0 \,, \tag{2.2}$$

for a fixed vector n^{μ}, for the abelian (QED) gauge theory by an entirely *algebraic* approach dealing only with commuting (or anti-commuting) external sources. The interest in the FS gauge, in gauge theories, in general, is that it leads to Faddeev–Popov ghost-free theories, [cf. Kummer and Weiser (1986)], the gauge field may be expressed quite simply in terms of the field strength [Kummer and Weiser (1986); Durand and Mendel (1982)] and it turns out to be useful in non-perturbative studies, [cf. Shifman textitet al. (1979a,b)]. Needless to say, the complete expressions of such generating functionals allow one to obtain gauge transformations of *all* the Green functions in a theory simply by functional differentiations with respect to the external sources coupled to the quantum fields in question and avoids the rather tedious treatment, but provides information on, the gauge transformation of diagram by diagram [Handy (1979); Feng and Lam (1996)] occurring in a theory. A key point, whose importance cannot be overemphasized, in our analysis [Manoukian (1986, 1987a)] is that, a priori, *no* restrictions are set on the external source(s) J^{μ} coupled to the gauge field(s), such as a $\partial_{\mu} J^{\mu} = 0$—restriction, so that *variations of the components of J^{μ} may be carried out independently*, until the entire analysis is completed. The present method is expected to be applicable to non-abelian gauge theories including supersymmetric ones and the latter will be attempted in a forthcoming report. Some classic references which have set the stage of the investigation of the gauge problem in field theory are given in Landau *et al.* (1954); Landau and Khalatnikov (1955); Johnson and Zumino (1959); Zumino (1960); Białynicki-Birula (1968); Mills (1971); Slavnov (1972); Taylor (1971); Abers and Lee (1973); Wess and Zumino (1974a); Salam and Strathdee (1974a); Becchi *et al.* (1975); Utiyama and Sakamoto (1977). For more recent studies which are, however, more involved with field operator techniques and their gauge transformations may be found in Sardanashvily (1984); Kobe and Gray (1985); Oh *et al.* (1987); Sugano and Kimura (1990); Gastmans *et al.* (1996); Pons *et al.* (1997); Gastmans and Wu (1998); Banerjee and Mandal (2000). To generate the so-called vacuum-to-vacuum transition amplitude in the functional *differential* formalism of quantum field theory the so-called quantum dynamical principle is applied in the process. In the presence of dependent fields this application becomes difficult as the rules of applications are to be modified to remedy this program. Accordingly, in our next section, we first study and develop the quantum dynamical rigorously in the presence of dependent fields [Manoukian *et al.* (2007)], in view of applications to QED and SQED where dependent fields are necessarily present due to the gauge properties of such theories. The photon field in the interaction with an external current in the Coulomb gauge is then treated carefully in Section 2.2. The field equations of the latter theory are developed in §2.2.1, again in the celebrated Coulomb gauge, while the photon propagator is worked out in §2.2.2. The canonical conjugate momentum components of the (independent) fields and the canonical commutation relations are developed, respectively, in §2.2.3, §2.2.4, while in §2.2.5 some applications of the quantum dynamical principle are carried out. Section 2.3 deals with the central problem of this chapter, that is, of gauge transformations in QED involving the Coulomb gauge, the covariant ones, the Fock–Schwinger and the axial gauge in various subsections.

2.1 Quantum Dynamical Principle in View of Applications to QED and SQED

Consider a Hamiltonian of the general form

$$H(t,\lambda) = H_1(t) + H_2(t,\lambda)\,, \tag{2.1.1}$$

where $H_1(t)$, $H_2(t,\lambda)$ may be time-dependent but $H_2(t,\lambda)$ may, in addition, depend on some parameters denoted by λ. Typically, in quantum field theory, $H_1(t)$ may stand for the free Hamiltonian written in terms of the physically observed masses referred to renormalized masses and $H_1(t)$ will be time-independent. In this latter case, $H_2(t,\lambda)$ will denote the remaining part of the Hamiltonian which, in particular, depends on renormalization constants, coupling constants and so-called external sources coupled to the quantum fields. The coupling constants and the external sources will be then collectively denoted by λ. A derivative of a transformation function with respect to λ with the latter denoting an external source will then represent a functional derivative [see e.g. Manoukian (1987c)].

The time evolution operator $U(t,\lambda)$, corresponding to the Hamiltonian $H(t,\lambda)$, satisfies the equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}U(t,\lambda) = H(t,\lambda)\,U(t,\lambda)\,. \tag{2.1.2}$$

For the theory given in a specific description, we have

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,\langle at| = \langle at|\,H(t,\lambda)\,, \tag{2.1.3}$$

where the states $\langle at|$ will depend on the parameters λ. Typically, the states $\langle a|$, assumed independent of λ, may represent multi-particle states of free particles associated with a given self-adjoint operator such as the momentum operator, with the single particle energies written in terms of the observed masses, or may represent the vacuum-state. One may also introduce the time evolution operator $U_1(t)$, corresponding to $H_1(t)$, satisfying the equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_1(t) = H_1(t)\,U_1(t)\,, \tag{2.1.4}$$

and the states ${}_1\langle at|$ which are independent of the parameters λ, satisfy

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,{}_1\langle at| = {}_1\langle at|\,H_1(t)\,. \tag{2.1.5}$$

The states $\langle at|$ of interest are related to the states ${}_1\langle at|$ by the equation

$$\langle at| = {}_1\langle at|\,V(t,\lambda)\,, \tag{2.1.6}$$

where

$$V(t,\lambda) = U_1^\dagger(t)\,U(t,\lambda)\,, \tag{2.1.7}$$

with the latter satisfying

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}V(t,\lambda) = U_1^\dagger(t)\,H_2(t,\lambda)\,U(t,\lambda)\,. \tag{2.1.8}$$

The quantum dynamical principle is involved with the study of the variation of a transformation function $\langle at_2\,|\,bt_1\rangle$, with respect to the parameters λ.

For $\tau \neq t_2,\ \tau \neq t_1$ and $\lambda \neq \lambda'$, we have the following useful *key* identity in the entire analysis:

$$\begin{aligned}&\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}\Big[V(t_2,\lambda)\,V^\dagger(\tau,\lambda)\,V(\tau,\lambda')\,V^\dagger(t_1,\lambda')\Big]\\ &\quad = V(t_2,\lambda)\,\Big[U^\dagger(\tau,\lambda)\,\big(H(\tau,\lambda') - H(\tau,\lambda)\big)\,U(\tau,\lambda')\Big]\,V^\dagger(t_1,\lambda')\,,\end{aligned} \tag{2.1.9}$$

which will be subsequently used.

The independent quantum fields of the theory will be denoted by $\chi(x)$ and their canonical conjugate momenta by $\pi(x)$, suppressing all obvious indices. The dependent fields will be denoted by $\eta(x)$ whose canonical conjugate momenta vanish, by definition. Here $x = \big(t\,,\ \mathbf{x}\big)$. The Hamiltonian $H(t, \lambda)$ may be then written as

$$H(t, \lambda) = H(\chi, \pi, \lambda, t)\,, \tag{2.1.10}$$

which, in particular, is a functional of $\chi(\mathbf{x}),\ \pi(\mathbf{x})$ with the latter defined in the so-called Schrödinger representation at $t = 0$, which are independent of λ. In the Heisenberg representation we have

$$\chi(x) = U^{\dagger}(t, \lambda)\, \chi(\mathbf{x})\, U(t, \lambda)\,, \tag{2.1.11}$$

$$\pi(x) = U^{\dagger}(t, \lambda)\, \pi(\mathbf{x})\, U(t, \lambda)\,, \tag{2.1.12}$$

having non-trivial dependence on the parameters λ.

Now we integrate the relation in (2.1.9) over τ from t_1 to t_2 to obtain

$$\begin{aligned}\Big[V(t_2, \lambda')\, V^{\dagger}(t_1, \lambda') - V(t_2, \lambda)\, V^{\dagger}(t_1, \lambda)\Big] \\ = -\frac{\mathrm{i}}{\hbar} V(t_2, \lambda) \Bigg[\int_{t_1}^{t_2} \mathrm{d}\tau\, U^{\dagger}(\tau, \lambda)\, \big(H(\tau, \lambda') - H(\tau, \lambda)\big) \\ \times\, U(\tau, \lambda')\Bigg]\, V(t_1, \lambda')\,.\end{aligned} \tag{2.1.13}$$

By setting $\lambda' = \lambda + \delta\lambda$, one obtains the variational form of the above equation

$$\begin{aligned}\delta \Big[V(t_2, \lambda)\, V^{\dagger}(t_1, \lambda)\Big] \\ = -\frac{\mathrm{i}}{\hbar} V(t_2, \lambda) \Bigg[\int_{t_1}^{t_2} \mathrm{d}\tau\, U^{\dagger}(\tau, \lambda)\, \delta H(\tau, \lambda)\, U(\tau, \lambda)\Bigg]\, V^{\dagger}(t_1, \lambda)\,.\end{aligned} \tag{2.1.14}$$

Upon defining the Heisenberg representation of $H(\tau,\lambda)$ at time τ, by

$$\mathbb{H}(\tau,\lambda) = U^{\dagger}(\tau,\lambda)\, H(\chi,\pi,\tau,\lambda)\, U(\tau,\lambda)\,, \tag{2.1.15}$$

we may rewrite (2.1.14), as

$$\delta\left[V(t_2,\lambda)\, V^{\dagger}(t_1,\lambda)\right] = -\frac{\mathrm{i}}{\hbar} V(t_2,\lambda) \left[\int_{t_1}^{t_2} \mathrm{d}\tau\; \delta\mathbb{H}(\tau,\lambda)\right] V^{\dagger}(t_1,\lambda)\,, \tag{2.1.16}$$

provided the variations of $\mathbb{H}$ with respect to λ in (2.1.16) are carried out *by keeping* $\chi(x)$, $\pi(x)$, given in (2.1.11), (2.1.12), fixed.

We take the matrix elements of (2.1.16) with respect to ${}_1\langle at_2|,\ |bt_1\rangle_1$ [see (2.1.5)], use (2.1.6), *and* note the λ independence of ${}_1\langle at_2|,\ |bt_1\rangle_1$, to obtain

$$\delta\langle at_2\,|\,bt_1\rangle = -\frac{\mathrm{i}}{\hbar}\left\langle at_2 \left| \int_{t_1}^{t_2} \mathrm{d}\tau\; \delta\mathbb{H}(\tau,\lambda) \right| bt_1 \right\rangle\,, \tag{2.1.17}$$

with the variation in $\mathbb{H}$, with respect to λ, carried out with the independent fields $\chi(x)$ and their canonical conjugate momenta $\pi(x)$ kept *fixed*.

The Hamiltonian $\mathbb{H}$ in the Heisenberg representation in (2.1.15) may be rewritten as

$$\mathbb{H}(t,\lambda) = H\big(\chi(t),\pi(t),\lambda,t\big)\,, \tag{2.1.18}$$

as obtained from the Hamiltonian $H(t,\lambda)$ in (2.1.10) at t, by carrying out the explicit operation given in (2.1.15). Eq. (2.1.18) is, in particular, written in terms of the independent (Heisenberg) fields at time t and their canonical conjugate momenta. The effective Lagrangian L_* of the system is related to $\mathbb{H}$ by the equation

$$L_*\big(\chi(t),\dot{\chi}(t),\lambda,t\big) = \int \mathrm{d}^3\mathbf{x}\; \pi(x)\,\dot{\chi}(x) - H\big(\chi(t),\pi(t),\lambda,t\big)\,, \tag{2.1.19}$$

with a summation over the fields understood.

The canonical conjugate momenta $\pi(x)$ of the fields are defined through the equation

$$L_*\big(\chi(t),\dot{\chi}(t)+\delta\dot{\chi}(t),\lambda,t\big)-L_*\big(\chi(t),\dot{\chi}(t),\lambda,t\big)=\int \mathrm{d}^3\mathbf{x}\,\pi(x)\,\delta\dot{\chi}(x)\,. \qquad (2.1.20)$$

Eqs. (2.1.19), (2.1.20) allow us to consider the variation of $H\big(\chi(\tau),\pi(\tau),\lambda,\tau\big)$, with respect to λ, by keeping χ,π fixed as required in (2.1.17), in relationship to the variation of L_*. From (2.1.19), (2.1.20), we then obtain, with χ, π kept fixed, that

$$\delta L_*\big(\chi(\tau),\dot{\chi}(\tau),\lambda,\tau\big)=-\delta H\big(\chi(\tau),\pi(\tau),\lambda,\tau\big)\,, \qquad (2.1.21)$$

upon *cancelation* of the term on the right-hand side of (2.1.20), where, now the variation of L_* in (2.1.21) is carried out with respect to λ by keeping $\chi(\tau)$ *and* $\dot{\chi}(\tau)$ fixed.

The dependent fields will be denoted by $\eta(x)$ and their canonical conjugate momenta vanish, by definition. The Lagrangian of the underlying field theory may be written as $L\big(\chi(t),\dot{\chi}(t),\eta(t),\lambda,t\big)$, which upon the elimination of $\eta(t)$ in favor of $\chi(t)$, $\dot{\chi}(t)$ and λ generating the Hamiltonian under study as well as the effective Lagrangian L_*. We consider the variation of L, with respect to λ, by keeping $\chi(t)$, $\dot{\chi}(t)$ fixed. Now since $\eta(t)$ will, in general, *depend* on λ, we have

$$\delta L=E_\eta\,\frac{\partial\eta}{\partial\lambda}\,\delta\lambda+\delta L\Big|_{\chi,\dot{\chi},\eta}\,, \qquad (2.1.22)$$

where we note that the Lagrangian does not contain terms depending on $\dot{\eta}$, by definition. The first term on the right-hand side defined as an integral in abbreviated form, E_η in it corresponds to the Euler–Lagrange equation of η, which vanishes, and the second term on the right-hand denotes the variation of L, with respect to λ, by keeping χ, $\dot{\chi}$ *and* η fixed. The latter property was first noted in Lam (1965). The Lagrangian density

$\mathscr{L} = \mathscr{L}(x) = \mathscr{L}(x, \lambda)$ of the system is related to the Lagrangian L through

$$L\big(\chi(t), \dot{\chi}(t), \eta(t), \lambda, t\big) = \int \mathrm{d}^3\mathbf{x}\, \mathscr{L}(x, \lambda)\,. \tag{2.1.23}$$

From (2.1.21), (2.1.22) and (2.1.23), we obtain the celebrated quantum dynamical principle or the Schwinger dynamical (action) principle

$$\delta\langle at_2 \,|\, bt_1\rangle = \frac{\mathrm{i}}{\hbar}\left\langle at_2 \left| \int_{t_1}^{t_2} (\mathrm{d}x)\; \delta\mathscr{L}(x, \lambda) \right| bt_1 \right\rangle\,, \tag{2.1.24}$$

where $(\mathrm{d}x) = \mathrm{d}t\, \mathrm{d}^3\mathbf{x}$, and the variation $\delta\mathscr{L}(x, \lambda)$, with respect to λ, is carried out with the fields, independent *and* dependent, and their derivatives $\partial_\mu \chi$, $\boldsymbol{\nabla}\eta$, all kept *fixed*. The interesting thing to note is that although the states $|at_2\rangle$, $|bt_1\rangle$ depend on λ, in the variation of the transformation function $\langle at_2 \,|\, bt_1\rangle$, the same (non-varied) states appear on the right-hand side of (2.1.24) with the entire variation being applied to the Lagrangian density $\mathscr{L}(x, \lambda)$ with the fields and their canonical conjugate momenta kept fixed. This is thanks to the U and V operators elaborated upon in (2.1.2)–(2.1.8), the independence of the states ${}_1\langle at_2|$, ${}_1\langle bt_1|$ of λ, and the key identify given in (2.1.9). In practice the limits $t_2 \to +\infty$, $t_1 \to -\infty$ are taken in (2.1.24) in scattering processes.

Now consider an arbitrary function

$$B\big(\chi(x), \pi(x), \lambda, t\big) \equiv \mathbb{B}(t, \lambda)\,, \tag{2.1.25}$$

of the variables indicated, with $\chi(x)$, $\pi(x)$ in the Heisenberg representation in (2.1.11), (2.1.12). We may write

$$\mathbb{B}(t, \lambda) = U^\dagger(t, \lambda)\, B\big(\chi(\mathbf{x}), \pi(\mathbf{x}), \lambda, t\big)\, U(t, \lambda)\,, \tag{2.1.26}$$

with $\chi(\mathbf{x})$, $\pi(\mathbf{x})$ on the right-hand side in the Schrödinger representation at time $t = 0$.

We note the identity

$$V(t_2,\lambda)\,\mathbb{B}(\tau,\lambda)\,V^{\dagger}(t_1,\lambda) = V(t_2,\lambda)\,V^{\dagger}(\tau,\lambda)\,U_1^{\dagger}(\tau)\,B\big(\chi(\mathbf{x}),\pi(\mathbf{x}),\lambda,\tau\big)$$

$$\times\, U_1(\tau)\,V(\tau,\lambda)\,V^{\dagger}(t_1,\lambda)\,. \tag{2.1.27}$$

Hence (2.1.14), (2.1.27) give for the following variation with respect to λ $(t_1 < \tau < \tau_2)$:

$$\begin{aligned}
&\delta\Big[V(t_2,\lambda)\,\mathbb{B}(\tau,\lambda)\,V^{\dagger}(t_1,\lambda)\Big] \\
&\quad= -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_{\tau}^{t_2}\mathrm{d}\tau'\;\delta H(\tau',\lambda)\,\mathbb{B}(\tau,\lambda)\,V^{\dagger}(t_1,\lambda) \\
&\qquad+ V(t_2,\lambda)\,\delta\mathbb{B}(\tau,\lambda)\,V^{\dagger}(t_1,\lambda) \\
&\qquad- \frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_{t_1}^{\tau}\mathrm{d}\tau'\;\mathbb{B}(\tau,\lambda)\,\delta H(\tau',\lambda)\,V^{\dagger}(t_1,\lambda)\,,
\end{aligned} \tag{2.1.28}$$

where according to (2.1.28), the variation in $\delta\mathbb{B}(\tau,\lambda) = \delta B\big(\chi(\mathbf{x},\tau),\pi(\mathbf{x},\tau),\lambda,\tau\big)$, with respect to λ, is carried out by keeping the (Heisenberg) fields $\chi(\mathbf{x},\tau)$, $\pi(\mathbf{x},\tau)$ *fixed*.

We may use the definition of the chronological time ordering product to rewrite (2.1.28) in the more compact form

$$\begin{aligned}
&\delta\left[V(t_2,\lambda)\,\mathbb{B}(\tau,\lambda)\,V^{\dagger}(t_1,\lambda)\right] \\
&\quad= -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_{t_1}^{t_2}\mathrm{d}\tau'\;\big(\mathbb{B}(\tau,\lambda)\,\delta H(\tau',\lambda)\big)_+\,V^{\dagger}(t_1,\lambda) \\
&\qquad+ V(t_2,\lambda)\,\delta\mathbb{B}(\tau,\lambda)\,V^{\dagger}(t_1,\lambda)\,.
\end{aligned} \tag{2.1.29}$$

Upon taking the matrix element of (2.1.29) with respect to ${}_1\langle at_2|,\ \ |bt_1\rangle_1$, and using (2.1.6), (2.1.15) and (2.1.24) we have for $t_1 < \tau < t_2$

$$\delta\langle at_2\,|\mathbb{B}(\tau,\lambda)|bt_1\rangle = \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2}(\mathrm{d}x')\ \Big\langle at_2\Big|\big(\mathbb{B}(\tau,\lambda)\,\delta\mathscr{L}(x',\lambda)\big)_+\Big|bt_1\Big\rangle$$

$$+\ \langle at_2\,|\delta\mathbb{B}(\tau,\lambda)|bt_1\rangle\ , \tag{2.1.30}$$

where in the variation $\delta\mathscr{L}(x',\lambda)$, with respect to λ, all the fields and their derivatives $\partial_\mu\chi,\ \ \boldsymbol{\nabla}\eta$ are kept fixed, while in $\delta\mathbb{B}(\tau,\lambda)$, *expressed* in terms of $\chi(\mathbf{x},\tau),\ \ \pi(\mathbf{x},\tau)$, the latter are kept fixed, and an extra λ-dependence may arise from the elimination of η in favor of $\chi,\ \ \pi$. To our knowledge Eq. (2.1.30) appears first in Lam (1965). The second term in (2.1.30) is *responsible* for the generation of the Faddeev–Popov factor and its generalizations in gauge theories [see, Manoukian (1986, 1987a); Limboonsong and Manoukian (2006)].

2.2 Pure Photon Field in the Coulomb Gauge

The Lagrangian density for the photon field A^μ in the presence of an external source J^μ is

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\ , \tag{2.2.1}$$

and the action for photon field is

$$\mathcal{W} = \int(\mathrm{d}x)\,\mathscr{L} = \int(\mathrm{d}x)\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right]\ , \tag{2.2.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mu,\nu = 0,1,2,3.$

We note that (for $\alpha = 0,1,2,3$)

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = \partial_\alpha\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \partial_\mu\left(\partial_\nu A_\alpha - \partial_\alpha A_\nu\right)$$

$$+ \partial_\nu (\partial_\alpha A_\mu - \partial_\mu A_\alpha)$$

$$= (\partial_\alpha \partial_\mu - \partial_\mu \partial_\alpha) A_\nu + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) A_\alpha$$

$$+ (\partial_\nu \partial_\alpha - \partial_\alpha \partial_\nu) A_\mu \,.$$

On smooth spacetime manifold, we have

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0 \,. \tag{2.2.3}$$

This identity (2.2.3) has an analogue in general relativity, known as the Bianchi identity. We may similarly refer to (2.2.3) as the Bianchi identity.

In the Coulomb gauge, we have the constraint

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \partial_i A^i = 0, \qquad i = 1, 2, 3 \,. \tag{2.2.4}$$

Equation (2.2.4) allows us to solve, for example, A^3 in terms of A^1, A^2:

$$A^3 = -\frac{1}{\partial_3}\left(\partial_1 A^1 + \partial_2 A^2\right) = -(\partial_3)^{-1}\partial_{\mathsf{a}} A^{\mathsf{a}}, \qquad \mathsf{a} = 1, 2 \,. \tag{2.2.5}$$

We treat A^0, A^1, A^2 (not A^3) as dynamical variables, we obtain $\delta A^{\mathsf{a}} = \delta A^{\mathsf{a}}$ and $\delta A^3 = -(\partial_3)^{-1}\partial_{\mathsf{a}}\delta A^{\mathsf{a}}$ or combining into δA^i by

$$\delta A^i = \delta^{ij}\delta A^j = \delta^{i\mathsf{a}}\delta A^{\mathsf{a}} + \delta^{i3}\delta A^3 = \delta^{i\mathsf{a}}\delta A^{\mathsf{a}} - \delta^{i3}\frac{\partial_{\mathsf{a}}}{\partial_3}\delta A^{\mathsf{a}} \,,$$

$$\delta A^i = \left(\delta^{i\mathsf{a}} - \delta^{i3}\frac{\partial_{\mathsf{a}}}{\partial_3}\right)\delta A^{\mathsf{a}}, \qquad i = 1, 2, 3 \,. \tag{2.2.6}$$

2.2.1 Field Equations in the Coulomb Gauge

$$
\begin{aligned}
\delta\mathcal{W} &= \int(\mathrm{d}x)\left[-\frac{1}{4}\delta(F^{\mu\nu}F_{\mu\nu}) + J^{\mu}\delta A_{\mu}\right] \\
&= \int(\mathrm{d}x)\left[-\frac{1}{2}F^{\mu\nu}\delta F_{\mu\nu} + J^{\mu}\delta A_{\mu}\right] \\
&= \int(\mathrm{d}x)\left[-\frac{1}{2}F^{\mu\nu}\delta(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) + J^{\mu}\delta A_{\mu}\right] \\
&= \int(\mathrm{d}x)\left[-F^{\mu\nu}\partial_{\mu}\delta A_{\nu} + J^{\mu}\delta A_{\mu}\right] \\
&= \int(\mathrm{d}x)\left[-\partial_{\mu}(F^{\mu\nu}\delta A_{\nu}) + (\partial_{\mu}F^{\mu\nu})\delta A_{\nu} + J^{\mu}\delta A_{\mu}\right] .
\end{aligned}
$$

The surface term ST.I $\equiv -\int(\mathrm{d}x)\,\partial_{\mu}(F^{\mu\nu}\delta A_{\nu})$ does not contribute to the field equations.

$$
\begin{aligned}
\delta\mathcal{W} &= \int(\mathrm{d}x)\left[\left(\partial_{\mu}F^{\mu\nu} + J^{\nu}\right)\delta A_{\nu}\right] + \text{ST.I} \\
&= \int(\mathrm{d}x)\left[\left(\partial_{\mu}F^{\mu 0} + J^{0}\right)\delta A_{0} + \left(\partial_{\mu}F^{\mu i} + J^{i}\right)\delta A_{i}\right] + \text{ST.I}
\end{aligned}
$$

or

$$
\delta\mathcal{W} = \int(\mathrm{d}x)\left[-\left(\partial_{\mu}F^{\mu 0} + J^{0}\right)\delta A^{0} + \left(\partial_{\mu}F^{\mu i} + J^{i}\right)\delta A^{i}\right] + \text{ST.I}\,. \tag{2.2.7}
$$

Using (2.2.6):

$$
\begin{aligned}
\delta\mathcal{W} = \int(\mathrm{d}x)\Bigg[&-\left(\partial_{\mu}F^{\mu 0} + J^{0}\right)\delta A^{0} \\
&+ \left(\partial_{\mu}F^{\mu i} + J^{i}\right)\left(\delta^{i\mathsf{a}} - \delta^{i3}\frac{\partial_{\mathsf{a}}}{\partial_{3}}\right)\delta A^{\mathsf{a}}\Bigg] + \text{ST.I}
\end{aligned}
$$

$$= \int (\mathrm{d}x) \Big[-\big(\partial_\mu F^{\mu 0} + J^0\big)\, \delta A^0 + \big(\partial_\mu F^{\mu \mathsf{a}} + J^{\mathsf{a}}\big)\, \delta A^{\mathsf{a}}$$

$$- \big(\partial_\mu F^{\mu 3} + J^3\big)\, \frac{\partial_{\mathsf{a}}}{\partial_3} \delta A^{\mathsf{a}} \Big] + \text{ST.I}$$

$$= - \int (\mathrm{d}x)\, \big(\partial_\mu F^{\mu 0} + J^0\big)\, \delta A^0 + \int (\mathrm{d}x)\, \big(\partial_\mu F^{\mu \mathsf{a}} + J^{\mathsf{a}}\big)\, \delta A^{\mathsf{a}}$$

$$- \int (\mathrm{d}x)\, \big(\partial_\mu F^{\mu 3} + J^3\big)\, \frac{\partial_{\mathsf{a}}}{\partial_3} \delta A^{\mathsf{a}} + \text{ST.I}\,. \tag{2.2.8}$$

Consider the third term:

$$\boxed{\text{III}} \equiv - \int (\mathrm{d}x)\, \big(\partial_\mu F^{\mu 3} + J^3\big)\, \frac{\partial_{\mathsf{a}}}{\partial_3} \delta A^{\mathsf{a}}$$

$$= - \int (\mathrm{d}x)\, \partial_{\mathsf{a}} \Big[\big(\partial_\mu F^{\mu 3} + J^3\big)\, \frac{1}{\partial_3} \delta A^{\mathsf{a}} \Big]$$

$$+ \int (\mathrm{d}x)\, \Big[\partial_{\mathsf{a}} \big(\partial_\mu F^{\mu 3} + J^3\big) \Big]\, \frac{1}{\partial_3} \delta A^{\mathsf{a}}\,.$$

The surface term $\text{ST.II} \equiv - \int (\mathrm{d}x)\, \partial_{\mathsf{a}} \Big[\big(\partial_\mu F^{\mu 3} + J^3\big)\, \frac{1}{\partial_3} \delta A^{\mathsf{a}} \Big]$ also does not contribute to the field equations.

$$\boxed{\text{III}} = \int (\mathrm{d}x)\, \Big[\partial_{\mathsf{a}} \big(\partial_\mu F^{\mu 3} + J^3\big) \Big]\, \frac{1}{\partial_3} \delta A^{\mathsf{a}} + \text{ST.II}$$

$$= \int (\mathrm{d}x)\, \Big[\partial_3 \frac{1}{\partial_3} \partial_{\mathsf{a}} \big(\partial_\mu F^{\mu 3} + J^3\big) \Big]\, \frac{1}{\partial_3} \delta A^{\mathsf{a}} + \text{ST.II}$$

$$= \int (\mathrm{d}x)\, \partial_3 \Big\{ \Big[\frac{1}{\partial_3} \partial_{\mathsf{a}} \big(\partial_\mu F^{\mu 3} + J^3\big) \Big]\, \frac{1}{\partial_3} \delta A^{\mathsf{a}} \Big\}$$

$$- \int (\mathrm{d}x)\, \Big[\frac{1}{\partial_3} \partial_{\mathsf{a}} \big(\partial_\mu F^{\mu 3} + J^3\big) \Big]\, \partial_3 \frac{1}{\partial_3} \delta A^{\mathsf{a}} + \text{ST.II}\,.$$

The surface term $\text{ST.III} \equiv - \int (\mathrm{d}x)\, \partial_3 \Big\{ \Big[\frac{1}{\partial_3} \partial_{\mathsf{a}} \big(\partial_\mu F^{\mu 3} + J^3\big) \Big]\, \frac{1}{\partial_3} \delta A^{\mathsf{a}} \Big\}$ again does not contribute to the field equations.

$$\boxed{\text{III}} = -\int(\mathrm{d}x)\left[\frac{\partial_{\mathsf{a}}}{\partial_3}\left(\partial_\mu F^{\mu 3} + J^3\right)\right]\delta A^{\mathsf{a}} + \textbf{ST.II} + \textbf{ST.III}\,,$$

or

$$-\int(\mathrm{d}x)\left(\partial_\mu F^{\mu 3} + J^3\right)\frac{\partial_{\mathsf{a}}}{\partial_3}\delta A^{\mathsf{a}} = -\int(\mathrm{d}x)\left[\frac{\partial_{\mathsf{a}}}{\partial_3}\left(\partial_\mu F^{\mu 3} + J^3\right)\right]\delta A^{\mathsf{a}} + \textbf{ST.II} + \textbf{ST.III}\,. \tag{2.2.9}$$

Substituting (2.2.9) into (2.2.8):

$$\begin{aligned}\delta\mathcal{W} = &-\int(\mathrm{d}x)\left(\partial_\mu F^{\mu 0} + J^0\right)\delta A^0 \\ &+\int(\mathrm{d}x)\left[\left(\partial_\mu F^{\mu\mathsf{a}} + J^{\mathsf{a}}\right) - \frac{\partial_{\mathsf{a}}}{\partial_3}\left(\partial_\mu F^{\mu 3} + J^3\right)\right]\delta A^{\mathsf{a}} \\ &+ \textbf{ST.I} + \textbf{ST.II} + \textbf{ST.III}\,,\end{aligned} \tag{2.2.10}$$

or

$$\frac{\delta\mathcal{W}}{\delta A^0} = -\left(\partial_\mu F^{\mu 0} + J^0\right) \tag{2.2.11}$$

$$\frac{\delta\mathcal{W}}{\delta A^{\mathsf{a}}} = \left(\partial_\mu F^{\mu\mathsf{a}} + J^{\mathsf{a}}\right) - \frac{\partial_{\mathsf{a}}}{\partial_3}\left(\partial_\mu F^{\mu 3} + J^3\right)\ . \tag{2.2.12}$$

The field equations are

$$-\partial_\mu F^{\mu 0} = J^0 \tag{2.2.13}$$

$$-\partial_\mu F^{\mu\mathsf{a}} = J^{\mathsf{a}} - \frac{\partial_{\mathsf{a}}}{\partial_3}\left(\partial_\mu F^{\mu 3} + J^3\right)\ , \tag{2.2.14}$$

for $\mathtt{a} = 1, 2$. But (2.2.14) also true for $\mathtt{a} = 3$:

$$-\partial_\mu F^{\mu i} = J^i - \frac{\partial_i}{\partial_3}\big(\partial_\mu F^{\mu 3} + J^3\big)\,, \qquad i = 1, 2, 3\,. \tag{2.2.15}$$

Using (2.2.13) and (2.2.15):

$$\begin{aligned}
-\partial_\nu \partial_\mu F^{\mu\nu} &= -\partial_0 \partial_\mu F^{\mu 0} - \partial_i \partial_\mu F^{\mu i} \\
&= \partial_0 J^0 + \partial_i \left[J^i - \frac{\partial_i}{\partial_3}\big(\partial_\mu F^{\mu 3} + J^3\big)\right] \\
&= \partial_0 J^0 + \partial_i J^i - \frac{\partial_i \partial_i}{\partial_3}\big(\partial_\mu F^{\mu 3} + J^3\big) \\
&= \partial_\mu J^\mu - \frac{\nabla^2}{\partial_3}\big(\partial_\mu F^{\mu 3} + J^3\big)\ ,
\end{aligned}$$

and using $\partial_\nu \partial_\mu F^{\mu\nu} = 0$ we obtain the non-conservation of current (charge)

$$\partial_\mu J^\mu = \frac{\nabla^2}{\partial_3}\big(\partial_\mu F^{\mu 3} + J^3\big) \neq 0\,, \tag{2.2.16}$$

or

$$\frac{1}{\partial_3}\big(\partial_\mu F^{\mu 3} + J^3\big) = \frac{1}{\nabla^2}\partial_\mu J^\mu\,. \tag{2.2.17}$$

Substituting (2.2.17) into (2.2.15):

$$-\partial_\mu F^{\mu i} = J^i - \frac{\partial^i \partial_\mu}{\nabla^2} J^\mu = \left(g^{i\alpha} - \frac{\partial^i \partial^\alpha}{\nabla^2}\right) J_\alpha, \qquad i = 1, 2, 3\,. \tag{2.2.18}$$

Combining (2.2.13) and (2.2.18) into one equation:

$$\begin{aligned}
-\partial_\mu F^{\mu\nu} &= -\delta^\nu{}_\alpha\, \partial_\mu F^{\mu\alpha} \\
&= -\delta^\nu{}_0\, \partial_\mu F^{\mu 0} - \delta^\nu{}_i\, \partial_\mu F^{\mu i}
\end{aligned}$$

$$= -g^{\nu 0}J^0 + g^{\nu i}\left(g^{i\alpha} - \frac{\partial^i \partial^\alpha}{\nabla^2}\right) J_\alpha$$

$$= g^{\nu 0}J_0 + g^{\nu i}J_i - g^{\nu i}\frac{\partial_i \partial^\alpha}{\nabla^2} J_\alpha \,,$$

$$-\partial_\mu F^{\mu\nu} = \left(g^{\nu\alpha} - g^{\nu i}\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha \,, \qquad (2.2.19)$$

or

$$-\partial_\mu F^{\mu\nu} = \left(g^{\nu\alpha} - b^\nu \partial^\alpha\right) J_\alpha \,, \qquad (2.2.20)$$

where

$$b^\mu \equiv \left(0 \,;\, \frac{\boldsymbol{\nabla}}{\nabla^2}\right) = g^{\mu k}\frac{\partial_k}{\nabla^2} \,. \qquad (2.2.21)$$

For $\partial_\alpha J^\alpha = 0$, we obtain the Maxwell's equation

$$-\partial_\mu F^{\mu\nu}\Big|_{\partial_\alpha J^\alpha = 0} = J^\nu \,. \qquad (2.2.22)$$

In the Coulomb gauge, $-\partial_\mu F^{\mu\nu} = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\Box A^\nu + \partial^\nu \partial_0 A^0$

$$-\Box A^\nu + \partial^\nu \partial_0 A^0 = \left(g^{\nu\alpha} - g^{\nu i}\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha \,. \qquad (2.2.23)$$

For $\nu = 0$:

$$-\Box A^0 + \partial^0 \partial_0 A^0 = \left(g^{0\alpha} - g^{0i}\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha \,,$$

or

$$-\nabla^2 A^0 = J^0 \qquad \text{and} \qquad A^0 = \frac{1}{-\nabla^2} J^0 \,. \qquad (2.2.24)$$

Actually, we do not treat (2.2.24) as a field equation. We will see, in the next section, that A^0 is a dependent field since its canonical momentum π^0 vanishes and is given by (2.2.24).

From (2.2.24), we obtain its time derivative

$$\partial_0 A^0 = -\frac{\partial_0}{\nabla^2} J^0 \,, \tag{2.2.25}$$

and substituting into (2.2.23):

$$\begin{aligned}
-\Box A^\nu &= \left(g^{\nu\alpha} - g^{\nu i} \frac{\partial_i \partial^\alpha}{\nabla^2} \right) J_\alpha + \frac{\partial^\nu \partial_0}{\nabla^2} J^0 \\
&= J^\nu - g^{\nu i} \frac{\partial_i \partial^\alpha}{\nabla^2} J_\alpha + \frac{\partial^\nu \partial^0}{\nabla^2} J_0 \\
&= J^\nu + \frac{\partial^\nu \partial^\alpha}{\nabla^2} J_\alpha - g^{\nu i} \frac{\partial_i \partial^\alpha}{\nabla^2} J_\alpha - \frac{\partial^\nu \partial^\alpha}{\nabla^2} J_\alpha + \frac{\partial^\nu \partial^0}{\nabla^2} J_0 \\
&= J^\nu + \frac{\partial^\nu \partial^\alpha}{\nabla^2} J_\alpha - g^{\nu i} \frac{\partial_i \partial^\alpha}{\nabla^2} J_\alpha - \frac{\partial^\nu \partial^i}{\nabla^2} J_i \\
&= J^\nu + \frac{\partial^\nu \partial^\alpha}{\nabla^2} J_\alpha - g^{\nu i} \frac{\partial_i \partial^\alpha}{\nabla^2} J_\alpha - g^{\alpha i} \frac{\partial^\nu \partial_i}{\nabla^2} J_\alpha \,,
\end{aligned}$$

$$\begin{aligned}
-\Box A^\nu &= \left[g^{\nu\alpha} + \frac{\partial^\nu \partial^\alpha}{\nabla^2} - g^{\nu i} \frac{\partial_i \partial^\alpha}{\nabla^2} - g^{\alpha i} \frac{\partial_i \partial^\nu}{\nabla^2} \right] J_\alpha \\
&= \left(g^{\nu\lambda} - g^{\lambda i} \frac{\partial_i \partial^\nu}{\nabla^2} \right) g_{\lambda\sigma} \left(g^{\sigma\alpha} - g^{\sigma k} \frac{\partial_k \partial^\alpha}{\nabla^2} \right) J_\alpha \,,
\end{aligned} \tag{2.2.26}$$

or

$$\begin{aligned}
-\Box A^\nu &= \left[g^{\nu\alpha} + \frac{\partial^\nu \partial^\alpha}{\nabla^2} - b^\nu \partial^\alpha - b^\alpha \partial^\nu \right] J_\alpha \\
&= \left(g^{\nu\lambda} - b^\lambda \partial^\nu \right) g_{\lambda\sigma} \left(g^{\sigma\alpha} - b^\sigma \partial^\alpha \right) J_\alpha \,.
\end{aligned} \tag{2.2.27}$$

For $\partial_\alpha J^\alpha = 0$, we obtain

$$-\Box A^\nu \Big|_{\partial_\alpha J^\alpha = 0} = \left(g^{\nu\alpha} - g^{\alpha i} \frac{\partial_i \partial^\nu}{\nabla^2} \right) J_\alpha \,. \tag{2.2.28}$$

For $\nu = k = 1, 2, 3$:

$$-\Box A^k = \left[g^{k\alpha} + \frac{\partial^k \partial^\alpha}{\nabla^2} - g^{ki} \frac{\partial_i \partial^\alpha}{\nabla^2} - g^{\alpha i} \frac{\partial^k \partial_i}{\nabla^2} \right] J_\alpha$$

$$= J^k + \frac{\partial^k \partial^\alpha}{\nabla^2} J_\alpha - \delta^k{}_i \frac{\partial^i \partial^\alpha}{\nabla^2} J_\alpha - \frac{\partial^k \partial_i}{\nabla^2} J^i$$

$$= J^k - \frac{\partial^k \partial_i}{\nabla^2} J^i ,$$

or

$$-\Box A^k = \left(\delta^{ki} - \frac{\partial_k \partial_i}{\nabla^2} \right) J^i . \tag{2.2.29}$$

From (2.2.26):

$$-\Box \partial^\mu A^\nu = \left[g^{\nu\alpha} \partial^\mu + \frac{\partial^\mu \partial^\nu \partial^\alpha}{\nabla^2} - g^{\nu i} \frac{\partial_i \partial^\mu \partial^\alpha}{\nabla^2} - g^{\alpha i} \frac{\partial_i \partial^\mu \partial^\nu}{\nabla^2} \right] J_\alpha ,$$

and

$$-\Box \partial^\nu A^\mu = \left[g^{\mu\alpha} \partial^\nu + \frac{\partial^\nu \partial^\mu \partial^\alpha}{\nabla^2} - g^{\mu i} \frac{\partial_i \partial^\nu \partial^\alpha}{\nabla^2} - g^{\alpha i} \frac{\partial_i \partial^\nu \partial^\mu}{\nabla^2} \right] J_\alpha ,$$

or

$$-\Box \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = \left[\left(g^{\nu\alpha} \partial^\mu - g^{\mu\alpha} \partial^\nu \right) - g^{\nu i} \frac{\partial_i \partial^\mu \partial^\alpha}{\nabla^2} + g^{\mu i} \frac{\partial_i \partial^\nu \partial^\alpha}{\nabla^2} \right] J_\alpha ,$$

$$-\Box F^{\mu\nu} = \left[\left(g^{\nu\alpha} \partial^\mu - g^{\mu\alpha} \partial^\nu \right) - \left(g^{\nu i} \partial^\mu - g^{\mu i} \partial^\nu \right) \frac{\partial_i \partial^\alpha}{\nabla^2} \right] J_\alpha$$

$$= \left(g^{\nu\lambda} \partial^\mu - g^{\mu\lambda} \partial^\nu \right) g_{\lambda\sigma} \left(g^{\sigma\alpha} - g^{\sigma i} \frac{\partial_i \partial^\alpha}{\nabla^2} \right) J_\alpha , \tag{2.2.30}$$

or

$$-\Box F^{\mu\nu} = \left[\left(g^{\nu\alpha} \partial^\mu - g^{\mu\alpha} \partial^\nu \right) - \left(b^\nu \partial^\mu - b^\mu \partial^\nu \right) \partial^\alpha \right] J_\alpha$$

$$= \left(g^{\nu\lambda}\partial^{\mu} - g^{\mu\lambda}\partial^{\nu}\right) g_{\lambda\sigma} \left(g^{\sigma\alpha} - b^{\sigma}\partial^{\alpha}\right) J_{\alpha} \,. \tag{2.2.31}$$

For $\partial_{\alpha} J^{\alpha} = 0$, we obtain

$$-\Box F^{\mu\nu}\Big|_{\partial_{\alpha}J^{\alpha}=0} = \partial^{\mu}J^{\nu} - \partial^{\nu}J^{\mu} \,. \tag{2.2.32}$$

2.2.2 Free Photon Propagator

From (2.2.26) and (2.2.27):

$$A^{\mu}(x) = \int (\mathrm{d}x')\, D^{\mu\nu}_{\mathrm{C}+}(x,x')\, J_{\nu}(x') \,, \tag{2.2.33}$$

where the free (retarded) photon propagator in the Coulomb gauge is

$$D^{\mu\nu}_{\mathrm{C}+}(x,x') = \left[g^{\mu\nu} + \frac{\partial^{\mu}\partial^{\nu}}{\nabla^2} - b^{\mu}\partial^{\nu} - b^{\nu}\partial^{\mu}\right] \frac{1}{-\Box - \mathrm{i}\epsilon}\delta^4(x-x') \,, \tag{2.2.34}$$

or

$$D^{\mu\nu}_{\mathrm{C}+}(x,x') = \left(g^{\mu\alpha} - b^{\alpha}\partial^{\mu}\right) g_{\alpha\beta} \left(g^{\beta\nu} - b^{\beta}\partial^{\nu}\right) \frac{1}{-\Box - \mathrm{i}\epsilon}\delta^4(x-x') \,, \tag{2.2.35}$$

$D^{\mu\nu}_{\mathrm{C}+}(x,x') = D^{\mu\nu}_{\mathrm{C}+}(x-x')$ is translational invariant.

We note that

$$\partial_{\mu} D^{\mu\nu}_{\mathrm{C}+}(x,x') = \left[\partial^{\nu} + \frac{\Box\partial^{\nu}}{\nabla^2} - \partial_{\mu} g^{\mu k}\frac{\partial_k}{\nabla^2}\partial^{\nu} - g^{\nu k}\frac{\partial_k}{\nabla^2}\Box\right] \frac{1}{-\Box - \mathrm{i}\epsilon}\delta^4(x-x')$$

$$= \left[\partial^{\nu} + \frac{\Box\partial^{\nu}}{\nabla^2} - \frac{\partial^k\partial_k}{\nabla^2}\partial^{\nu} - g^{\nu k}\frac{\partial_k}{\nabla^2}\Box\right] \frac{1}{-\Box - \mathrm{i}\epsilon}\delta^4(x-x')$$

$$= \frac{\Box}{\nabla^2}\left(\partial^{\nu} - g^{\nu k}\partial_k\right) \frac{1}{-\Box - \mathrm{i}\epsilon}\delta^4(x-x') \,,$$

or

$$\partial_\mu D^{\mu\nu}_{\mathrm{C}+}(x,x') = -g^{\nu 0}\frac{\partial_0}{\nabla^2}\delta^4(x-x') \tag{2.2.36}$$

$$\partial_\nu\partial_\mu D^{\mu\nu}_{\mathrm{C}+}(x,x') = \frac{(\partial_0)^2}{\nabla^2}\delta^4(x-x')\,. \tag{2.2.37}$$

Let

$$\begin{aligned}
\mathcal{A} \equiv & \left[\int (\mathrm{d}x)(\mathrm{d}x')\, J_\mu(x) D^{\mu\nu}_{\mathrm{C}+}(x,x') J_\nu(x')\right]_{\partial_\alpha J^\alpha = 0} \\
= & \left[\int (\mathrm{d}x)\, J_\mu(x)\left(g^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{\nabla^2} - b^\mu\partial^\nu - b^\nu\partial^\mu\right)\frac{1}{-\Box - \mathrm{i}\epsilon} J_\nu(x)\right]_{\partial_\alpha J^\alpha = 0} \\
= & \left[\int (\mathrm{d}x)\, J_\mu(x)\left(g^{\mu\nu} - b^\nu\partial^\mu\right)\frac{1}{-\Box - \mathrm{i}\epsilon} J_\nu(x)\right]_{\partial_\alpha J^\alpha = 0} \\
= & \int (\mathrm{d}x)\, J_\mu(x)\frac{g^{\mu\nu}}{-\Box - \mathrm{i}\epsilon} J_\nu(x) \\
= & \int (\mathrm{d}x)(\mathrm{d}x')\, J_\mu(x) D^{\mu\nu}_{\mathrm{F}+}(x,x') J_\nu(x')\,.
\end{aligned}$$

Therefore $\mathcal{A}$ is gauge invariant quantity,

$$\left[\int (\mathrm{d}x)(\mathrm{d}x')\, J_\mu(x) D^{\mu\nu}_{+}(x,x') J_\nu(x')\right]_{\partial_\alpha J^\alpha = 0} = \int (\mathrm{d}x)\, J_\mu(x)\frac{g^{\mu\nu}}{-\Box - \mathrm{i}\epsilon} J_\nu(x)\,. \tag{2.2.38}$$

In the momentum description

$$D^{\mu\nu}_{\mathrm{C}+}(k) = \int (\mathrm{d}x)\, \mathrm{e}^{-\mathrm{i}kx} D^{\mu\nu}_{\mathrm{C}+}(x) \quad \text{and} \quad D^{\mu\nu}_{\mathrm{C}+}(x) = \int \frac{(\mathrm{d}k)}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}kx} D^{\mu\nu}_{\mathrm{C}+}(k)\,, \tag{2.2.39}$$

$$D^{\mu\nu}_{\mathrm{C}+}(k) = \left[g^{\mu\nu} + \frac{k^\mu k^\nu}{\mathbf{k}^2} - a^\mu k^\nu - a^\nu k^\mu\right]\frac{1}{k^2 - \mathrm{i}\epsilon}\,, \tag{2.2.40}$$

or

$$D^{\mu\nu}_{\mathrm{C}+}(k) = \left(g^{\mu\alpha} - a^{\alpha}k^{\mu}\right) g_{\alpha\beta} \left(g^{\beta\nu} - a^{\beta}k^{\nu}\right) \frac{1}{k^2 - \mathrm{i}\epsilon}\,, \tag{2.2.41}$$

where

$$a^{\mu} \equiv \left(0\,;\,\frac{\mathbf{k}}{\mathbf{k}^2}\right) = g^{\mu i}\frac{k_i}{\mathbf{k}^2}\,. \tag{2.2.42}$$

$$\left.\begin{aligned} D^{00}_{\mathrm{C}+}(k) &= -\frac{1}{\mathbf{k}^2}\,, \\ D^{0i}_{\mathrm{C}+}(k) &= D^{i0}_{\mathrm{C}+}(k) = 0\,, \\ D^{ij}_{\mathrm{C}+}(k) &= \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\right)\frac{1}{k^2 - \mathrm{i}\epsilon}\,. \end{aligned}\right\} \tag{2.2.43}$$

By using the time-like vector $\xi^{\mu} = \left(-1\,;\,\mathbf{0}\right) = g^{\mu 0}$ and $\xi^{\mu}\xi_{\mu} = -1$, we have

$$D^{\mu\nu}_{\mathrm{C}+}(k) = \left[g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2 + (k{\cdot}\xi)^2} - \frac{(k{\cdot}\xi)\left(k^{\mu}\xi^{\nu} + k^{\nu}\xi^{\mu}\right)}{k^2 + (k{\cdot}\xi)^2}\right]\frac{1}{k^2 - \mathrm{i}\epsilon}\,, \tag{2.2.44}$$

where $k{\cdot}\xi = k_{\mu}\xi^{\mu} = k^0$ and $k^2 + (k{\cdot}\xi)^2 = \mathbf{k}^2$.

We note that

$$k_{\mu}D^{\mu\nu}_{\mathrm{C}+}(k) = g^{\nu 0}\frac{k_0}{\mathbf{k}^2} \qquad \text{and} \qquad k_{\mu}D^{\mu\nu}_{\mathrm{C}+}(k)\Big|_{k^2=0} = -\frac{g^{\nu 0}}{|\mathbf{k}|}\,, \tag{2.2.45}$$

or

$$k_{\mu}D^{\mu\nu}_{\mathrm{C}+}(k)k_{\nu} = -\frac{(k^0)^2}{\mathbf{k}^2} \qquad \text{and} \qquad k_{\mu}D^{\mu\nu}_{\mathrm{C}+}(k)k_{\nu}\Big|_{k^2=0} = -1\,. \tag{2.2.46}$$

For the real source $J^*_{\mu}(k) = J_{\mu}(-k)$, let

$$\begin{aligned} \mathcal{A}' &\equiv \left[\int\frac{(\mathrm{d}k)}{(2\pi)^4}\,J^*_{\mu}(k)D^{\mu\nu}_{\mathrm{C}+}(k)J_{\nu}(k)\right]_{k_{\alpha}J^{\alpha}=0} \\ &= \left[\int\frac{(\mathrm{d}k)}{(2\pi)^4}\,J_{\mu}(-k)\left(g^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{\mathbf{k}^2} - a^{\mu}k^{\nu} - a^{\nu}k^{\mu}\right)\frac{1}{k^2 - \mathrm{i}\epsilon}J_{\nu}(k)\right]_{k_{\alpha}J^{\alpha}=0} \end{aligned}$$

$$= \int \frac{(\mathrm{d}k)}{(2\pi)^4}\, J_\mu(-k) \frac{g^{\mu\nu}}{k^2 - \mathrm{i}\epsilon} J_\nu(k)$$

$$= \int \frac{(\mathrm{d}k)}{(2\pi)^4}\, J^*_\mu(k) D^{\mu\nu}_{\mathrm{F}+}(k) J_\nu(k)\,.$$

Therefore $\mathcal{A}'$ is gauge invariant quantity,

$$\left[\int \frac{(\mathrm{d}k)}{(2\pi)^4}\, J^*_\mu(k) D^{\mu\nu}_{+}(k) J_\nu(k) \right]_{k_\alpha J^\alpha = 0} = \int \frac{(\mathrm{d}k)}{(2\pi)^4}\, J_\mu(-k) \frac{g^{\mu\nu}}{k^2 - \mathrm{i}\epsilon} J_\nu(k)\,. \tag{2.2.47}$$

2.2.3 Canonical Conjugate Momenta of the Field Components

From (2.2.6)

$$\delta A^i = \left(\delta^{i\mathsf{a}} - \delta^{i3} \frac{\partial_\mathsf{a}}{\partial_3} \right) \delta A^\mathsf{a}\,,$$

$$\delta\big(\partial_\mu A^i\big) = \left(\delta^{i\mathsf{a}} - \delta^{i3} \frac{\partial_\mathsf{a}}{\partial_3} \right) \delta\big(\partial_\mu A^\mathsf{a}\big)\ , \tag{2.2.48}$$

and from (2.2.7):

$$\delta\mathcal{W} = \int (\mathrm{d}x) \left[-\left(\partial_\mu F^{\mu 0} + J^0\right) \delta A^0 + \left(\partial_\mu F^{\mu i} + J^i\right) \delta A^i \right] + \text{ST.I}$$

$$= \int (\mathrm{d}x) \left[\left(-\partial_k F^{k0}\right) \delta A^0 - J^0 \delta A^0 + J^i \delta A^i + \left(\partial_\mu F^{\mu i}\right) \delta A^i \right] + \text{ST.I}$$

$$= \int (\mathrm{d}x) \left[-\partial_k \left(F^{k0} \delta A^0\right) + F^{k0} \delta\big(\partial_k A^0\big) - J^0 \delta A^0 + J^i \delta A^i + \left(\partial_\mu F^{\mu i}\right) \delta A^i \right]$$

$$+ \text{ST.I}$$

$$= \int (\mathrm{d}x) \left[J^0 \delta A_0 + J^i \delta A^i - F^{k0} \delta(\partial_k A_0) + \left(\partial_\mu F^{\mu i}\right) \delta A^i \right]$$

$$+ \text{ST.I} + \text{ST.IV}\,. \tag{2.2.49}$$

Considering the second term in (2.2.49) and using (2.2.6)

$$\int (\mathrm{d}x)\, J^i \delta A^i = \int (\mathrm{d}x)\, J^i \left(\delta^{i\mathsf{a}} - \delta^{i3} \frac{\partial_\mathsf{a}}{\partial_3} \right) \delta A^\mathsf{a}$$

$$= \int (\mathrm{d}x)\, J^\mathsf{a} \delta A^\mathsf{a} - \int (\mathrm{d}x)\, J^3 \frac{\partial_\mathsf{a}}{\partial_3} \delta A^\mathsf{a}$$

$$= \int (\mathrm{d}x)\, J^\mathsf{a} \delta A^\mathsf{a} + \int (\mathrm{d}x)\, \left(\partial_\mathsf{a} J^3\right) \frac{1}{\partial_3} \delta A^\mathsf{a}$$

$$= \int (\mathrm{d}x)\, J^\mathsf{a} \delta A^\mathsf{a} - \int (\mathrm{d}x) \left(\frac{\partial_\mathsf{a}}{\partial_3} J^3 \right) \delta A^\mathsf{a} \,,$$

or

$$\int (\mathrm{d}x)\, J^i \delta A^i = \int (\mathrm{d}x) \left(J^\mathsf{a} - \frac{\partial_\mathsf{a}}{\partial_3} J^3 \right) \delta A_\mathsf{a} \,. \tag{2.2.50}$$

Considering the last term in (2.2.49) and using (2.2.48)

$$\int (\mathrm{d}x)\, \left(\partial_\mu F^{\mu i}\right) \delta A^i = - \int (\mathrm{d}x)\, F^{\mu i}\, \delta\big(\partial_\mu A^i\big)$$

$$= - \int (\mathrm{d}x)\, F^{\mu i} \left(\delta^{i\mathsf{a}} - \delta^{i3} \frac{\partial_\mathsf{a}}{\partial_3} \right) \delta(\partial_\mu A^\mathsf{a})$$

$$= - \int (\mathrm{d}x)\, F^{\mu \mathsf{a}}\, \delta(\partial_\mu A^\mathsf{a}) + \int (\mathrm{d}x)\, F^{\mu 3}\, \frac{\partial_\mathsf{a}}{\partial_3} \delta(\partial_\mu A^\mathsf{a})$$

$$= - \int (\mathrm{d}x)\, F^{\mu \mathsf{a}}\, \delta(\partial_\mu A^\mathsf{a}) - \int (\mathrm{d}x)\, \left(\partial_\mathsf{a} F^{\mu 3}\right) \frac{1}{\partial_3} \delta(\partial_\mu A^\mathsf{a})$$

$$= - \int (\mathrm{d}x)\, F^{\mu \mathsf{a}}\, \delta(\partial_\mu A^\mathsf{a}) + \int (\mathrm{d}x) \left(\frac{\partial_\mathsf{a}}{\partial_3} F^{\mu 3} \right) \delta(\partial_\mu A^\mathsf{a}) \,,$$

or

$$\int (\mathrm{d}x)\, \left(\partial_\mu F^{\mu i}\right) \delta A^i = \int (\mathrm{d}x) \left(-F^{\mu \mathsf{a}} + \frac{\partial_\mathsf{a}}{\partial_3} F^{\mu 3} \right) \delta(\partial_\mu A_\mathsf{a}) \,. \tag{2.2.51}$$

Substituting (2.2.50) and (2.2.51) into (2.2.49):

$$\delta\mathcal{W} = \int (\mathrm{d}x) \left[J^0 \delta A_0 + \left(J^{\mathsf{a}} - \frac{\partial_{\mathsf{a}}}{\partial_3} J^3 \right) \delta A_{\mathsf{a}} \right.$$
$$\left. - F^{k0} \delta(\partial_k A_0) + \left(-F^{\mu\mathsf{a}} + \frac{\partial_{\mathsf{a}}}{\partial_3} F^{\mu 3} \right) \delta(\partial_\mu A_{\mathsf{a}}) \right] . \tag{2.2.52}$$

The canonical momenta conjugate to A^μ is defined by

$$\pi^\mu \equiv \pi\left[A^\mu\right] = \frac{\delta\mathcal{W}}{\delta \dot{A}_\mu} = \frac{\delta\mathcal{W}}{\delta(\partial_0 A_\mu)} . \tag{2.2.53}$$

For photon field, A^0 is a dependent field since its canonical momentum vanishes ($\pi^0 = 0$) and is given by $A^0 = -(\nabla^2)^{-1} J^0$.

$$\pi^{\mathsf{a}} \equiv \pi\left[A^{\mathsf{a}}\right] = -F^{0\mathsf{a}} + \frac{\partial_{\mathsf{a}}}{\partial_3} F^{03}, \qquad \mathsf{a} = 1, 2 . \tag{2.2.54}$$

But $\pi^3 = 0$ because A^3 is not a dynamical variable.

Using (2.2.3), we have $\partial^3 F^{0\mathsf{a}} + \partial^0 F^{\mathsf{a}3} + \partial^{\mathsf{a}} F^{30} = 0$

$$-\partial^3 F^{0\mathsf{a}} - \partial^{\mathsf{a}} F^{30} = \partial^0 F^{\mathsf{a}3}$$

$$-\partial^3 F^{0\mathsf{a}} + \partial^{\mathsf{a}} F^{03} = \partial^0 F^{\mathsf{a}3}$$

$$-\partial_3 F^{0\mathsf{a}} + \partial_{\mathsf{a}} F^{03} = -\partial_0 F^{\mathsf{a}3} ,$$

or

$$-F^{0\mathsf{a}} + \frac{\partial_{\mathsf{a}}}{\partial_3} F^{03} = -\frac{\partial_0}{\partial_3} F^{\mathsf{a}3} .$$

Therefore

$$\pi^{\mathsf{a}} = -F^{0\mathsf{a}} + \frac{\partial_{\mathsf{a}}}{\partial_3} F^{03} = -\frac{\partial_0}{\partial_3} F^{\mathsf{a}3} ,$$

also true for $\mathtt{a} = k = 1, 2, 3$:

$$\pi^k = -F^{0k} + \frac{\partial_k}{\partial_3} F^{03} = -\frac{\partial_0}{\partial_3} F^{k3} \,. \qquad (2.2.55)$$

Because $\pi^0 = 0$:

$$\pi^\mu = g^{\mu\nu}\pi_\nu = g^{\mu 0}\pi_0 + g^{\mu k}\pi_k = -g^{\mu 0}\pi^0 + g^{\mu k}\pi^k = g^{\mu k}\pi^k \,,$$

$$\pi^\mu = -F^{0\mu} + g^{\mu k}\frac{\partial_k}{\partial_3} F^{03} = -g^{\mu k}\frac{\partial_0}{\partial_3} F^{k3} \,, \qquad (2.2.56)$$

for $\mu = 0, 1, 2, 3$ and solving for F^{03}:

$$\partial_\mu \pi^\mu = -\partial_\mu F^{0\mu} + g^{\mu k}\partial_\mu \frac{\partial_k}{\partial_3} F^{03}$$

$$= \partial_\mu F^{\mu 0} + \frac{\partial^k \partial_k}{\partial_3} F^{03}$$

$$= -J^0 + \frac{\nabla^2}{\partial_3} F^{03} \,,$$

or

$$F^{03} = \frac{\partial_3 \partial_\mu}{\nabla^2}\pi^\mu + \frac{\partial_3}{\nabla^2} J^0 \,. \qquad (2.2.57)$$

Substituting F^{03} from (2.2.57) to (2.2.56):

$$F^{0\mu} = -\pi^\mu + g^{\mu k}\frac{\partial_k}{\partial_3} F^{03}$$

$$= -\pi^\mu + g^{\mu k}\frac{\partial_k}{\partial_3}\left(\frac{\partial_3 \partial_\nu}{\nabla^2}\pi^\nu + \frac{\partial_3}{\nabla^2} J^0\right)$$

$$= -\pi^\mu + g^{\mu k}\frac{\partial_k \partial_\nu}{\nabla^2}\pi^\nu + g^{\mu k}\frac{\partial_k}{\nabla^2} J^0 \,,$$

or

$$F^{0\mu} = -\left(g^{\mu\nu} - g^{\mu k}\frac{\partial_k \partial^\nu}{\nabla^2}\right)\pi_\nu + g^{\mu k}\frac{\partial_k}{\nabla^2}J^0\,, \tag{2.2.58}$$

where $\mu, \nu = 0, 1, 2, 3$ but $k = 1, 2, 3$.

Because $\pi^0 = 0$ and $\pi^3 = 0$, we have $g^{\mu\nu}\pi_\nu = g^{\mu l}\pi^l = g^{\mu \mathtt{a}}\pi^{\mathtt{a}}$ and $\partial_\nu \pi^\nu = \partial_l \pi^l = \partial_{\mathtt{a}}\pi^{\mathtt{a}}$

$$F^{0k} = -\left(\delta^{k\mathtt{a}} - \frac{\partial_k \partial_{\mathtt{a}}}{\nabla^2}\right)\pi^{\mathtt{a}} + \frac{\partial_k}{\nabla^2}J^0\,, \tag{2.2.59}$$

or

$$\left(g^{k\mu} - \frac{\partial_k \partial^\mu}{\nabla^2}\right)\pi_\mu = \left(\delta^{k\mathtt{a}} - \frac{\partial_k \partial_{\mathtt{a}}}{\nabla^2}\right)\pi^{\mathtt{a}} = -F^{0k} + \frac{\partial_k}{\nabla^2}J^0\,. \tag{2.2.60}$$

We may generalize (2.2.58) into $F^{\mu\nu} = -F^{\nu\mu}$ by

$$\begin{aligned}
F^{\mu\nu} &= \frac{1}{2}\left(\delta^\mu{}_\alpha\,\delta^\nu{}_\beta - \delta^\nu{}_\alpha\,\delta^\mu{}_\beta\right)F^{\alpha\beta} \\
&= \frac{1}{2}\left(\delta^\mu{}_0\,\delta^\nu{}_\beta - \delta^\nu{}_0\,\delta^\mu{}_\beta\right)F^{0\beta} + \frac{1}{2}\left(\delta^\mu{}_k\,\delta^\nu{}_\beta - \delta^\nu{}_k\,\delta^\mu{}_\beta\right)F^{k\beta} \\
&= \frac{1}{2}\left(\delta^\mu{}_0\,\delta^\nu{}_k - \delta^\nu{}_0\,\delta^\mu{}_k\right)F^{0k} + \frac{1}{2}\left(\delta^\mu{}_k\,\delta^\nu{}_0 - \delta^\nu{}_k\,\delta^\mu{}_0\right)F^{k0} \\
&\quad + \frac{1}{2}\left(\delta^\mu{}_k\,\delta^\nu{}_l - \delta^\nu{}_k\,\delta^\mu{}_l\right)F^{kl} \\
&= -\frac{1}{2}\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)F^{0k} + \frac{1}{2}\left(g^{\mu k}g^{\nu 0} - g^{\nu k}g^{\mu 0}\right)F^{0k} \\
&\quad + \frac{1}{2}\left(\delta^\mu{}_k\,\delta^\nu{}_l - \delta^\nu{}_k\,\delta^\mu{}_l\right)F^{kl} \\
&= -\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)F^{0k} + \frac{1}{2}\left(\delta^\mu{}_k\,\delta^\nu{}_l - \delta^\nu{}_k\,\delta^\mu{}_l\right)F^{kl}\,,
\end{aligned}$$

therefore

$$F^{\mu\nu} = -g^{\mu 0}F^{0\nu} + g^{\nu 0}F^{0\mu} + \frac{1}{2}\left(\delta^\mu{}_k\,\delta^\nu{}_l - \delta^\nu{}_k\,\delta^\mu{}_l\right)F^{kl}\,,$$

$$F^{\mu\nu} = \left[g^{\mu 0} \left(g^{\nu\alpha} - g^{\nu k} \frac{\partial_k \partial^\alpha}{\nabla^2} \right) - g^{\nu 0} \left(g^{\mu\alpha} - g^{\mu k} \frac{\partial_k \partial^\alpha}{\nabla^2} \right) \right] \pi_\alpha$$

$$- \left(g^{\mu 0} g^{\nu k} - g^{\nu 0} g^{\mu k} \right) \frac{\partial_k}{\nabla^2} J^0 + \frac{1}{2} \left(\delta^\mu{}_k \, \delta^\nu{}_l - \delta^\nu{}_k \, \delta^\mu{}_l \right) F^{kl} \,, \tag{2.2.61}$$

by using a notation $b^\mu \equiv g^{\mu k} \dfrac{\partial_k}{\nabla^2}$

$$F^{\mu\nu} = \left[\left(g^{\mu 0} g^{\nu\alpha} - g^{\nu 0} g^{\mu\alpha} \right) - \left(g^{\mu 0} b^\nu - g^{\nu 0} b^\mu \right) \partial^\alpha \right] \pi_\alpha$$

$$- \left(g^{\mu 0} b^\nu - g^{\nu 0} b^\mu \right) J^0 + \frac{1}{2} \left(\delta^\mu{}_k \, \delta^\nu{}_l - \delta^\nu{}_k \, \delta^\mu{}_l \right) F^{kl} \,, \tag{2.2.62}$$

or

$$F^{\mu\nu} = \left(g^{\mu 0} g^{\nu\sigma} - g^{\nu 0} g^{\mu\sigma} \right) g_{\sigma\lambda} \left(g^{\lambda\alpha} - b^\lambda \partial^\alpha \right) \pi_\alpha$$

$$- \left(g^{\mu 0} b^\nu - g^{\nu 0} b^\mu \right) J^0 + \frac{1}{2} \left(\delta^\mu{}_k \, \delta^\nu{}_l - \delta^\nu{}_k \, \delta^\mu{}_l \right) F^{kl} \,, \tag{2.2.63}$$

where $\mu, \nu, \alpha = 0, 1, 2, 3$ but $k = 1, 2, 3.$

In the Coulomb gauge $A^3 = -(\partial_3)^{-1} \partial_{\mathsf{b}} A^{\mathsf{b}}$ with $\mathsf{b} = 1, 2$

$$\pi^\mu = -g^{\mu k} \frac{\partial_0}{\partial_3} F^{k3}$$

$$= -g^{\mu k} \frac{\partial_0}{\partial_3} \left(\partial^k A^3 - \partial^3 A^k \right)$$

$$= g^{\mu k} \partial_0 A^k - g^{\mu k} \frac{\partial_0 \partial^k}{\partial_3} A^3$$

$$= g^{\mu k} \dot{A}^k + g^{\mu k} \frac{\partial_0 \partial^k}{\partial_3} \left(\frac{\partial_{\mathsf{b}}}{\partial_3} A^{\mathsf{b}} \right)$$

$$= g^{\mu k} \dot{A}^k + g^{\mu k} \frac{\partial_k \partial_{\mathsf{b}}}{(\partial_3)^2} \dot{A}^{\mathsf{b}} \,,$$

$$\pi^{\mu} = g^{\mu k}\left[\dot{A}^{k} + \frac{\partial_k \partial_{\mathtt{b}}}{(\partial_3)^2}\dot{A}^{\mathtt{b}}\right] , \tag{2.2.64}$$

or by using the gauge constraint $A^3 = -(\partial_3)^{-1}\partial_{\mathtt{b}}A^{\mathtt{b}}$

$$\pi^{\mu} = g^{\mu \mathtt{a}}\left[\delta^{\mathtt{ab}} + \frac{\partial_{\mathtt{a}} \partial_{\mathtt{b}}}{(\partial_3)^2}\right]\dot{A}^{\mathtt{b}} , \tag{2.2.65}$$

with $\mu = 0, 1, 2, 3$, $k = 1, 2, 3$ and $\mathtt{a}, \mathtt{b} = 1, 2$.

From (2.2.64), solving for $\partial_{\mathtt{b}}\dot{A}^{\mathtt{b}}$

$$\begin{aligned}
\partial_\mu \pi^{\mu} &= g^{\mu k}\partial_\mu\left[\dot{A}^{k} + \frac{\partial_k \partial_{\mathtt{b}}}{(\partial_3)^2}\dot{A}^{\mathtt{b}}\right] \\
&= \partial_k \dot{A}^{k} + \frac{\partial^k \partial_k \partial_{\mathtt{b}}}{(\partial_3)^2}\dot{A}^{\mathtt{b}} \\
&= \frac{\nabla^2}{(\partial_3)^2}\partial_{\mathtt{b}}\dot{A}^{\mathtt{b}} ,
\end{aligned}$$

or

$$\partial_{\mathtt{b}}\dot{A}^{\mathtt{b}} = \frac{(\partial_3)^2}{\nabla^2}\partial_\mu \pi^\mu = \frac{(\partial_3)^2}{\nabla^2}\partial_{\mathtt{a}}\pi^{\mathtt{a}} , \tag{2.2.66}$$

and substituting to (2.2.64)

$$\begin{aligned}
\pi^{\mu} &= g^{\mu k}\dot{A}^{k} + g^{\mu k}\frac{\partial_k \partial_{\mathtt{b}}}{(\partial_3)^2}\dot{A}^{\mathtt{b}} \\
&= g^{\mu k}\dot{A}^{k} + g^{\mu k}\frac{\partial_k \partial_{\nu}}{\nabla^2}\pi^{\nu} ,
\end{aligned}$$

or

$$g^{\mu k}\dot{A}^{k} = \pi^{\mu} - g^{\mu k}\frac{\partial_k \partial_\nu}{\nabla^2}\pi^{\nu} = \left(g^{\mu\nu} - g^{\mu k}\frac{\partial_k \partial^\nu}{\nabla^2}\right)\pi_\nu ,$$

and

$$g_{i\mu}g^{\mu k}\dot{A}^{k} = g_{i\mu}\left(g^{\mu\nu} - g^{\mu k}\frac{\partial_k \partial^\nu}{\nabla^2}\right)\pi_\nu .$$

By using (2.2.60)

$$\dot{A}^i = \left(g^{i\nu} - \frac{\partial_i \partial^\nu}{\nabla^2} \right) \pi_\nu = -F^{0i} + \frac{\partial_i}{\nabla^2} J^0 \,, \tag{2.2.67}$$

or

$$\dot{A}^i = \left(\delta^{ik} - \frac{\partial_i \partial_k}{\nabla^2} \right) \pi^k = -F^{0\mathtt{a}} + \frac{\partial_\mathtt{a}}{\nabla^2} J^0 \,, \tag{2.2.68}$$

and

$$\dot{A}^\mathtt{a} = \left(\delta^{\mathtt{ab}} - \frac{\partial_\mathtt{a} \partial_\mathtt{b}}{\nabla^2} \right) \pi^\mathtt{b} = -F^{0\mathtt{a}} + \frac{\partial_\mathtt{a}}{\nabla^2} J^0 \,. \tag{2.2.69}$$

Combining (2.2.67) with $A^0 = -(\nabla^2)^{-1} J^0$

$$\dot{A}^\mu = g^{\mu\nu} \dot{A}_\nu = g^{\mu 0} \dot{A}_0 + g^{\mu i} \dot{A}_i = -g^{\mu 0} \dot{A}^0 + g^{\mu i} \dot{A}^i \,,$$

$$\dot{A}^\mu = g^{\mu i} \left(g_{i\nu} - \frac{\partial_i \partial_\nu}{\nabla^2} \right) \pi^\nu + g^{\mu 0} \frac{\partial_0}{\nabla^2} J^0 = -F^{0\mu} + \frac{\partial^\mu}{\nabla^2} J^0 \,. \tag{2.2.70}$$

2.2.4 Canonical Commutation Relations

For $\mathtt{a}, \mathtt{b} = 1, 2$

$$\left[A^\mathtt{a}(x) \,, \pi^\mathtt{b}(x') \right] \Big|_{x^0 = x'^0} = \mathrm{i} \delta^{\mathtt{ab}} \delta^3(\mathbf{x} - \mathbf{x}') \,, \tag{2.2.71}$$

or

$$\delta(x^0 - x'^0) \left[A^\mathtt{a}(x) \,, \pi^\mathtt{b}(x') \right] = \mathrm{i} \delta^{\mathtt{ab}} \delta^4(x - x') \,, \tag{2.2.72}$$

is the (equal-time) quantization rule for the physical degree of freedom A^1 and A^2.

Because $\pi^0 = 0$ and $\pi^3 = 0$, we generalize to

$$\delta(x^0 - x'^0) \left[A^\mathtt{a}(x) \,, \pi^\nu(x') \right] = \mathrm{i} g^{\nu \mathtt{a}} \delta^4(x - x') \,, \tag{2.2.73}$$

and using $A^3 = -(\partial_3)^{-1}\partial_{\mathtt{a}} A^{\mathtt{a}}$

$$\left[A^{\mathtt{a}}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0} = \mathrm{i}g^{\nu\mathtt{a}}\delta^3(\mathbf{x}-\mathbf{x}')$$

$$\left[-(\partial_3)^{-1}\partial_{\mathtt{a}} A^{\mathtt{a}}(x), \pi^{\nu}(x')\right]\Big|_{x^0=x'^0} = -\mathrm{i}g^{\nu\mathtt{a}}\frac{\partial_{\mathtt{a}}}{\partial_3}\delta^3(\mathbf{x}-\mathbf{x}')$$

$$\left[A^{3}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0} = -\mathrm{i}g^{\nu\mathtt{a}}\frac{\partial_{\mathtt{a}}}{\partial_3}\delta^3(\mathbf{x}-\mathbf{x}')\,,$$

or

$$\delta(x^0 - x'^0)\left[A^{3}(x)\,,\pi^{\nu}(x')\right] = -\mathrm{i}g^{\nu\mathtt{a}}\frac{\partial_{\mathtt{a}}}{\partial_3}\delta^4(x-x')\,. \tag{2.2.74}$$

Combining (2.2.73) and (2.2.74):

$$\begin{aligned}\left[A^{i}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0} &= \delta^{ij}\,\left[A^{j}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0}\\ &= \delta^{i\mathtt{a}}\,\left[A^{\mathtt{a}}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0}\\ &\quad + \delta^{i3}\,\left[A^{3}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0}\\ &= \mathrm{i}g^{\nu\mathtt{a}}\delta^{i\mathtt{a}}\delta^4(x-x') - \mathrm{i}g^{\nu\mathtt{a}}\delta^{i3}\frac{\partial_{\mathtt{a}}}{\partial_3}\delta^4(x-x')\,,\end{aligned}$$

or

$$\delta(x^0 - x'^0)\left[A^{i}(x)\,,\pi^{\nu}(x')\right] = \mathrm{i}g^{\nu\mathtt{a}}\left(\delta^{i\mathtt{a}} - \delta^{i3}\frac{\partial_{\mathtt{a}}}{\partial_3}\right)\delta^4(x-x')\,, \tag{2.2.75}$$

with $i, k = 1, 2, 3$ and $\mathtt{a} = 1, 2$.

Because $A^0 = -(\nabla^2)^{-1}J^0$ is just c-number,

$$\begin{aligned}\left[A^{\mu}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0} &= \delta^{\mu}{}_{\alpha}\,\left[A^{\alpha}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0}\\ &= \delta^{\mu}{}_{0}\,\left[A^{0}(x)\,,\pi^{\nu}(x')\right]\Big|_{x^0=x'^0}\end{aligned}$$

$$+ \delta^{\mu}{}_{i} \left[A^i(x)\,, \pi^\nu(x')\right] \Big|_{x^0=x'^0}$$

$$= g^{\mu i}\delta(x^0 - x'^0)\left[A^i(x)\,, \pi^\nu(x')\right]\,,$$

$$\delta(x^0 - x'^0)\left[A^\mu(x)\,, \pi^\nu(x')\right] = \mathrm{i}g^{\mu i}g^{\nu k}\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right)\delta^4(x - x')\,, \qquad (2.2.76)$$

we note that

$$g^{\nu k}\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right) = g^{\nu \mathsf{a}}\left(\delta^{i\mathsf{a}} - \delta^{i3}\frac{\partial_{\mathsf{a}}}{\partial_3}\right)\,.$$

Using (2.2.70) for $j = 1, 2, 3$

$$\left[A^\mu(x)\,, \dot{A}^\nu(x')\right]\Big|_{x^0=x'^0} = g^{\nu j}\left(g_{j\alpha} - \frac{\partial'_j\partial'_\alpha}{\nabla'^2}\right)\left[A^\mu(x)\,, \pi^\alpha(x')\right]\Big|_{x^0=x'^0}$$

$$= \mathrm{i}g^{\mu i}g^{\alpha k}g^{\nu j}\left(g_{j\alpha} - \frac{\partial'_j\partial'_\alpha}{\nabla'^2}\right)\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right)\delta^3(\mathbf{x} - \mathbf{x}')$$

$$= \mathrm{i}g^{\mu i}g^{\alpha k}g^{\nu j}\left(g_{j\alpha} - \frac{\partial_j\partial_\alpha}{\nabla^2}\right)\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right)\delta^3(\mathbf{x} - \mathbf{x}')$$

$$= \mathrm{i}g^{\mu i}g^{\nu j}\left(\delta^{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right)\delta^3(\mathbf{x} - \mathbf{x}')\,,$$

where

$$\partial'_\alpha \equiv \frac{\partial}{\partial x'^\alpha} \qquad \text{and} \qquad \nabla'^2 \equiv \delta^{ij}\frac{\partial}{\partial x'^i}\frac{\partial}{\partial x'^j}\,.$$

Considering

$$\left(\delta^{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right)$$

$$= \delta^{ik}\delta^{jk} - \delta^{i3}\delta^{jk}\frac{\partial_k}{\partial_3} - \delta^{ik}\frac{\partial_j\partial_k}{\nabla^2} + \delta^{i3}\frac{\partial_j}{\partial_3}\frac{\partial_k\partial_k}{\nabla^2}$$

$$= \delta^{ik}\delta^{jk} - \delta^{i3}\frac{\partial_j}{\partial_3} - \delta^{ik}\frac{\partial_j\partial_k}{\nabla^2} + \delta^{i3}\frac{\partial_j}{\partial_3}$$

$$= \delta^{ik}\delta^{jk} - \delta^{ik}\frac{\partial_j\partial_k}{\nabla^2}\,,$$

or

$$\left(\delta^{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right) = \delta^{ij} - \frac{\partial_i\partial_j}{\nabla^2}\,. \tag{2.2.77}$$

In terms of the transverse delta function

$$\delta_\perp^{ij}(\mathbf{x}-\mathbf{x}') \equiv \left(\delta^{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^3(\mathbf{x}-\mathbf{x}')$$

$$= \left(\delta^{ij} - b_i\partial_j\right)\delta^3(\mathbf{x}-\mathbf{x}')\,, \tag{2.2.78}$$

or

$$\delta_\perp^{ij}(\mathbf{x}-\mathbf{x}') = \delta^{ij}\delta^3(\mathbf{x}-\mathbf{x}') + \frac{1}{4\pi}\partial_i\partial_j\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{2.2.79}$$

and

$$\delta_\perp^{ij}(x-x') \equiv \delta\big(x^0 - x'^0\big)\delta_\perp^{ij}(\mathbf{x}-\mathbf{x}')\,, \tag{2.2.80}$$

we obtain

$$\big[A^\mu(x), \dot{A}^\nu(x')\big]\Big|_{x^0=x'^0} = \mathrm{i}g^{\mu i}g^{\nu j}\delta_\perp^{ij}(\mathbf{x}-\mathbf{x}')\,, \tag{2.2.81}$$

or

$$\delta\big(x^0 - x'^0\big)\big[A^\mu(x), \dot{A}^\nu(x')\big] = \mathrm{i}g^{\mu i}g^{\nu k}\delta_\perp^{ik}(x-x')\,. \tag{2.2.82}$$

From (2.2.70)

$$\dot{A}^\nu = -F^{0\nu} + \frac{\partial^\nu}{\nabla^2}J^0\,,$$

we have

$$\delta\big(x^0 - x'^0\big)\,\big[A^\mu(x)\,, F^{0\nu}(x')\big] = -\mathrm{i}g^{\mu i}g^{\nu k}\delta_\perp^{ik}(x-x')\,, \tag{2.2.83}$$

and

$$\delta\big(x^0 - x'^0\big)\,\big[A^\mu(x)\,, F^{0k}(x')\big] = -\mathrm{i}g^{\mu i}\delta_\perp^{ik}(x-x')\,.$$

Generalizing (2.2.83) for $F^{\alpha\beta}$ with $F^{\alpha\beta} = -F^{\beta\alpha}$:

$$F^{\alpha\beta} = -\left(g^{\alpha 0} g^{\beta k} - g^{\beta 0} g^{\alpha k}\right) F^{0k} + \frac{1}{2}\left(\delta^{\alpha}{}_{m}\, \delta^{\beta}{}_{n} - \delta^{\beta}{}_{m}\, \delta^{\alpha}{}_{n}\right) F^{mn} ,$$

and using $\delta\big(x^0 - x'^0\big)\, \big[A^{\mu}(x)\, , F^{mn}(x')\big] = 0$

$$\delta\big(x^0 - x'^0\big)\, \big[A^{\mu}(x)\, , F^{\alpha\beta}(x')\big] = \mathrm{i} g^{\mu i}\left(g^{\alpha 0} g^{\beta k} - g^{\beta 0} g^{\alpha k}\right) \delta_{\perp}^{ik}(x - x')\, , \qquad (2.2.84)$$

and

$$\begin{aligned} &\delta\big(x^0 - x'^0\big)\, \big[F^{\mu\nu}(x)\, , F^{\alpha\beta}(x')\big] \\ &\qquad = \mathrm{i}\Big[\left(g^{\alpha 0} g^{\beta k} - g^{\beta 0} g^{\alpha k}\right)\left(g^{\nu i} \partial^{\mu} - g^{\mu i} \partial^{\nu}\right) \\ &\qquad\qquad - \left(g^{\mu 0} g^{\nu k} - g^{\nu 0} g^{\mu k}\right)\left(g^{\beta i} \partial^{\alpha} - g^{\alpha i} \partial^{\beta}\right)\Big]\, \delta_{\perp}^{ik}(x - x')\, . \end{aligned} \qquad (2.2.85)$$

We note that

$$\begin{aligned} \delta\big(x^0 - x'^0\big)\, \big[F^{0i}(x)\, , F^{0j}(x')\big] &= 0 \\ \delta\big(x^0 - x'^0\big)\, \big[F^{ij}(x)\, , F^{mn}(x')\big] &= 0 \\ \delta\big(x^0 - x'^0\big)\, \big[F^{0i}(x)\, , F^{mn}(x')\big] &= +\mathrm{i}\big(\delta^{kn} \partial^{m} - \delta^{km} \partial^{n}\big)\delta_{\perp}^{ik}(x - x') \\ &= +\mathrm{i}\big(\delta^{in} \partial^{m} - \delta^{im} \partial^{n}\big)\delta^{4}(x - x') \\ \delta\big(x^0 - x'^0\big)\, \big[F^{mn}(x)\, , F^{0i}(x')\big] &= -\mathrm{i}\big(\delta^{kn} \partial^{m} - \delta^{km} \partial^{n}\big)\delta_{\perp}^{ik}(x - x') \\ &= -\mathrm{i}\big(\delta^{in} \partial^{m} - \delta^{im} \partial^{n}\big)\delta^{4}(x - x')\, , \end{aligned}$$

or in terms of the electric field $E^i = F^{0i}$ and the magnetic field $B^i = \frac{1}{2}\varepsilon^{ijk}F^{jk}$, we have

$$
\begin{aligned}
\delta\big(x^0 - x'^0\big)\,\big[E^i(x)\,, E^j(x')\big] &= 0 \\
\delta\big(x^0 - x'^0\big)\,\big[B^i(x)\,, B^j(x')\big] &= 0 \\
\delta\big(x^0 - x'^0\big)\,\big[E^i(x)\,, B^j(x')\big] &= +\mathrm{i}\varepsilon^{jmn}\partial^m\delta_\perp^{in}(x - x') \\
&= +\mathrm{i}\varepsilon^{ijk}\partial^k\delta^4(x - x') \\
\delta\big(x^0 - x'^0\big)\,\big[B^j(x)\,, E^i(x')\big] &= -\mathrm{i}\varepsilon^{jmn}\partial^m\delta_\perp^{in}(x - x') \\
&= -\mathrm{i}\varepsilon^{ijk}\partial^k\delta^4(x - x')\,.
\end{aligned}
$$

2.2.5 Applications of the Quantum Dynamical Principle

The Schwinger's quantum dynamical (action) principle [see (2.1.24)] imply the functional derivative of the vacuum-to-vacuum transition amplitude with respect to an external source J^μ is

$$
\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\,\langle 0_+\,|\,0_-\rangle = \big\langle 0_+\,\big|\,A^\mu(x)\big|0_-\big\rangle\ , \tag{2.2.86}
$$

and for any operator-valued (q-number) function $\mathcal{F}(x)$ [see (2.1.30)]

$$
\begin{aligned}
\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\,\big\langle 0_+\,\big|\mathcal{F}(x)\big|0_-\big\rangle &= \Big\langle 0_+\Big|\big(\mathcal{F}(x)A^\nu(x')\big)_+\Big|0_-\Big\rangle \\
&\quad + \left\langle 0_+\left|\frac{\delta\mathcal{F}(x)}{\mathrm{i}\delta J_\nu(x')}\right|0_-\right\rangle\ ,
\end{aligned} \tag{2.2.87}
$$

with no constraint on a c-number function $J^\mu(x)$. (That is J^μ need not be conserved.)

In terms of the expectation value of the photon field $A^\mu(x)$:

$$\big\langle A^\mu(x)\big\rangle \equiv \frac{\big\langle 0_+\big|A^\mu(x)\big|0_-\big\rangle}{\langle 0_+\,|\,0_-\rangle}\,, \tag{2.2.88}$$

we have

$$\big\langle A^\mu(x)\big\rangle = \frac{\delta}{\mathrm{i}\delta J_\mu(x)}\ln\langle 0_+\,|\,0_-\rangle\;. \tag{2.2.89}$$

We treat the equation (2.2.33):

$$A^\mu(x) = \int(\mathrm{d}x')\, D^{\mu\nu}_{\mathrm{C}+}(x,x')\, J_\nu(x')\,,$$

as a classical equation, replacing by the q-number field with its expectation value

$$\big\langle A^\mu(x)\big\rangle = \int(\mathrm{d}x')\, D^{\mu\nu}_{\mathrm{C}}(x,x')\, J_\nu(x')\,, \tag{2.2.90}$$

where $D^{\mu\nu}_{\mathrm{C}}(x,x')$ is the exact photon propagator. Obviously, the expectation value of the photon field $A^\mu(x)$ in the absence of an external source J^μ is

$$\big\langle A^\mu(x)\big\rangle\Big|_{J=0} = 0\,. \tag{2.2.91}$$

From (2.2.86) and (2.2.90), we obtain the functional differential equation for the vacuum-to-vacuum transition amplitude

$$\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\langle 0_+\,|\,0_-\rangle = \int(\mathrm{d}x')\, D^{\mu\nu}_{\mathrm{C}}(x,x')\, J_\nu(x')\,\langle 0_+\,|\,0_-\rangle\;, \tag{2.2.92}$$

and the solution is

$$\langle 0_+\,|\,0_-\rangle = \exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')\, J_\mu(x)\, D^{\mu\nu}_{\mathrm{C}}(x,x')\, J_\nu(x')\right]\,, \tag{2.2.93}$$

with normalizing by $\langle 0_+\,|\,0_-\rangle\big|_{J=0} = 1$.

For the functional derivative, the photon field $A^i(x)$ does not depend on an external source J^i but the dependent field $A^0(x)$ depend on J^0 by (2.2.24). We obtain

$$\frac{\delta A^0(x)}{\mathrm{i}\delta J^0(x')} = \frac{\mathrm{i}}{\nabla^2}\delta^4(x-x')\,, \tag{2.2.94}$$

and generalizing to

$$\begin{aligned}
\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')} &= \delta^\mu{}_\alpha\, g^{\nu\beta}\frac{\delta A^\alpha(x)}{\mathrm{i}\delta J^\beta(x')} \\
&= \delta^\mu{}_0\, g^{\nu 0}\frac{\delta A^0(x)}{\mathrm{i}\delta J^0(x')} + \delta^\mu{}_i\, g^{\nu 0}\frac{\delta A^i(x)}{\mathrm{i}\delta J^0(x')} \\
&\quad + \delta^\mu{}_0\, g^{\nu k}\frac{\delta A^0(x)}{\mathrm{i}\delta J^k(x')} + \delta^\mu{}_i\, g^{\nu k}\frac{\delta A^i(x)}{\mathrm{i}\delta J^k(x')} \\
&= -g^{\mu 0}g^{\nu 0}\frac{\delta A^0(x)}{\mathrm{i}\delta J^0(x')}\,,
\end{aligned}$$

$$\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')} = -\mathrm{i}g^{\mu 0}g^{\nu 0}\frac{1}{\nabla^2}\delta^4(x-x')\,, \tag{2.2.95}$$

are both c-numbers.

From (2.2.87), we have

$$\begin{aligned}
\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\left\langle 0_+\middle|A^\mu(x)\middle|0_-\right\rangle &= \left\langle 0_+\middle|\left(A^\mu(x)A^\nu(x')\right)_+\middle|0_-\right\rangle \\
&\quad + \left\langle 0_+\middle|\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')}\middle|0_-\right\rangle\,,
\end{aligned} \tag{2.2.96}$$

imply

$$\left\langle 0_+\middle|\left(A^0(x)A^0(x')\right)_+\middle|0_-\right\rangle = 0\,. \tag{2.2.97}$$

The expectation value of A^μ depend on J^μ:

$$\left[\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\left\langle A^\mu(x)\right\rangle\right]_{J=0} = -\mathrm{i}D_{\mathrm{C}}^{\mu\nu}(x,x')\,. \tag{2.2.98}$$

We note that

$$\begin{aligned}\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\left\langle A^\mu(x)\right\rangle &= \frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\left\langle 0_+\middle|A^\mu(x)\middle|0_-\right\rangle \\ &\quad - \left\langle A^\mu(x)\right\rangle\frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\langle 0_+|0_-\rangle \\ &= \frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\left\langle 0_+\middle|A^\mu(x)\middle|0_-\right\rangle \\ &\quad - \left\langle A^\mu(x)\right\rangle\left\langle A^\nu(x')\right\rangle\,,\end{aligned}$$

or

$$\begin{aligned}\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\left\langle A^\mu(x)\right\rangle &= \left\langle\left(A^\mu(x)A^\nu(x')\right)_+\right\rangle + \left\langle\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')}\right\rangle \\ &\quad - \left\langle A^\mu(x)\right\rangle\left\langle A^\nu(x')\right\rangle\,,\end{aligned} \tag{2.2.99}$$

and

$$\left[\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\left\langle A^\mu(x)\right\rangle\right]_{J=0} = \left\langle\left(A^\mu(x)A^\nu(x')\right)_+\right\rangle_{J=0} + \left\langle\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')}\right\rangle_{J=0}\,. \tag{2.2.100}$$

The exact photon propagator is

$$D_{\mathrm{C}}^{\mu\nu}(x,x') = \mathrm{i}\left\langle\left(A^\mu(x)A^\nu(x')\right)_+\right\rangle_{J=0} + g^{\mu 0}g^{\nu 0}\frac{1}{\nabla^2}\delta^4(x-x')\,. \tag{2.2.101}$$

From the classical equation (2.2.30):

$$-\Box F^{\mu\nu} = \left[\left(g^{\nu\alpha}\partial^{\mu} - g^{\mu\alpha}\partial^{\nu}\right) - \left(g^{\nu i}\partial^{\mu} - g^{\mu i}\partial^{\nu}\right)\frac{\partial_i\partial^{\alpha}}{\nabla^2}\right] J_{\alpha}\,,$$

we have

$$\left\langle 0_+\middle| F^{\mu\nu}(x)\middle| 0_-\right\rangle = \mathcal{D}^{\mu\nu\alpha} J_{\alpha}(x)\left\langle 0_+\middle| 0_-\right\rangle\,, \tag{2.2.102}$$

or

$$\left\langle F^{\mu\nu}(x)\right\rangle = \mathcal{D}^{\mu\nu\alpha} J_{\alpha}(x)\,, \tag{2.2.103}$$

and

$$\frac{\delta}{\mathrm{i}\delta J_{\alpha}(x')}\left\langle F^{\mu\nu}(x)\right\rangle = -\mathrm{i}\mathcal{D}^{\mu\nu\alpha}\delta^4(x-x')\,, \tag{2.2.104}$$

where

$$\mathcal{D}^{\mu\nu\alpha} \equiv \frac{1}{-\Box - \mathrm{i}\epsilon}\left[\left(g^{\nu\alpha}\partial^{\mu} - g^{\mu\alpha}\partial^{\nu}\right) - \left(g^{\nu i}\partial^{\mu} - g^{\mu i}\partial^{\nu}\right)\frac{\partial_i\partial^{\alpha}}{\nabla^2}\right]\,. \tag{2.2.105}$$

From (2.2.87), we have

$$\frac{\delta}{\mathrm{i}\delta J_{\alpha}(x')}\left\langle 0_+\middle| F^{\mu\nu}(x)\middle| 0_-\right\rangle = \left\langle 0_+\middle|\frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_{\alpha}(x')}\middle| 0_-\right\rangle + \left\langle 0_+\middle|\left(F^{\mu\nu}(x)A^{\alpha}(x')\right)_+\middle| 0_-\right\rangle\,, \tag{2.2.106}$$

or

$$\frac{1}{\left\langle 0_+\middle| 0_-\right\rangle}\frac{\delta}{\mathrm{i}\delta J_{\alpha}(x')}\left\langle 0_+\middle| F^{\mu\nu}(x)\middle| 0_-\right\rangle = \left\langle\frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_{\alpha}(x')}\right\rangle + \left\langle\left(F^{\mu\nu}(x)A^{\alpha}(x')\right)_+\right\rangle\,. \tag{2.2.107}$$

But in terms of the expectation value of $F^{\mu\nu}(x)$, we have

$$\frac{\delta}{\mathrm{i}\delta J_\alpha(x')}\left\langle F^{\mu\nu}(x)\right\rangle = \frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\alpha(x')}\left\langle 0_+\left|F^{\mu\nu}(x)\right|0_-\right\rangle$$

$$-\left\langle F^{\mu\nu}(x)\right\rangle\frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\alpha(x')}\langle 0_+|0_-\rangle$$

$$= \frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\alpha(x')}\left\langle 0_+\left|F^{\mu\nu}(x)\right|0_-\right\rangle$$

$$-\left\langle F^{\mu\nu}(x)\right\rangle\left\langle A^\alpha(x')\right\rangle ,$$

or

$$\frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\alpha(x')}\left\langle 0_+\left|F^{\mu\nu}(x)\right|0_-\right\rangle = \frac{\delta}{\mathrm{i}\delta J_\alpha(x')}\left\langle F^{\mu\nu}(x)\right\rangle$$

$$+\left\langle F^{\mu\nu}(x)\right\rangle\left\langle A^\alpha(x')\right\rangle . \qquad (2.2.108)$$

We obtain

$$\left\langle\left(F^{\mu\nu}(x)A^\alpha(x')\right)_+\right\rangle = \frac{\delta}{\mathrm{i}\delta J_\alpha(x')}\left\langle F^{\mu\nu}(x)\right\rangle - \left\langle\frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\alpha(x')}\right\rangle$$

$$+\left\langle F^{\mu\nu}(x)\right\rangle\left\langle A^\alpha(x')\right\rangle , \qquad (2.2.109)$$

and

$$\left\langle\left(F^{\mu\nu}(x)A^\alpha(x')\right)_+\right\rangle_{J=0} = \left[\frac{\delta}{\mathrm{i}\delta J_\alpha(x')}\left\langle F^{\mu\nu}(x)\right\rangle\right]_{J=0} - \left\langle\frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\alpha(x')}\right\rangle_{J=0} . \qquad (2.2.110)$$

For the bosonic q-number functions $\mathcal{A}(x)$ and $\mathcal{B}(x')$, the (chronological) time-order product is defined by

$$\left(\mathcal{A}(x)\mathcal{B}(x')\right)_+ \overset{\text{def}}{=} \theta(x^0 - x'^0)\mathcal{A}(x)\mathcal{B}(x') + \theta(x'^0 - x^0)\mathcal{B}(x')\mathcal{A}(x) ,$$

or

$$\left\langle \left(\mathcal{A}(x)\mathcal{B}(x')\right)_+ \right\rangle = \theta(x^0 - x'^0) \left\langle \mathcal{A}(x)\mathcal{B}(x') \right\rangle$$

$$+ \theta(x'^0 - x^0) \left\langle \mathcal{B}(x')\mathcal{A}(x) \right\rangle \ . \tag{2.2.111}$$

Because

$$\partial'_0 \left\langle \left(\mathcal{A}(x)\mathcal{B}(x')\right)_+ \right\rangle = -\delta(x^0 - x'^0) \left\langle \mathcal{A}(x)\mathcal{B}(x') \right\rangle$$

$$+ \theta(x^0 - x'^0) \left\langle \mathcal{A}(x)\partial'_0\mathcal{B}(x') \right\rangle$$

$$+ \delta(x'^0 - x^0) \left\langle \mathcal{B}(x')\mathcal{A}(x) \right\rangle$$

$$+ \theta(x'^0 - x^0) \left\langle \partial'_0\mathcal{B}(x')\mathcal{A}(x) \right\rangle \ ,$$

or

$$\partial'_0 \left\langle \left(\mathcal{A}(x)\mathcal{B}(x')\right)_+ \right\rangle = \left\langle \left(\mathcal{A}(x)\partial'_0\mathcal{B}(x')\right)_+ \right\rangle$$

$$- \delta(x^0 - x'^0) \left\langle \left[\mathcal{A}(x) \, , \mathcal{B}(x')\right] \right\rangle \ , \tag{2.2.112}$$

and

$$\partial'_k \left\langle \left(\mathcal{A}(x)\mathcal{B}(x')\right)_+ \right\rangle = \left\langle \left(\mathcal{A}(x)\partial'_k\mathcal{B}(x')\right)_+ \right\rangle \ . \tag{2.2.113}$$

Combining (2.2.112) and (2.2.113) into

$$\partial'_\mu \left\langle \left(\mathcal{A}(x)\mathcal{B}(x')\right)_+ \right\rangle = \left\langle \left(\mathcal{A}(x)\partial'_\mu\mathcal{B}(x')\right)_+ \right\rangle$$

$$+ g_{\mu 0}\delta(x^0 - x'^0) \left\langle \left[\mathcal{A}(x) \, , \mathcal{B}(x')\right] \right\rangle \ , \tag{2.2.114}$$

or

$$\left\langle \left(\mathcal{A}(x)\partial'_\mu \mathcal{B}(x')\right)_+ \right\rangle = \partial'_\mu \left\langle \left(\mathcal{A}(x)\mathcal{B}(x')\right)_+ \right\rangle$$

$$- g_{\mu 0}\delta(x^0 - x'^0) \left\langle \left[\mathcal{A}(x)\, ,\, \mathcal{B}(x')\right] \right\rangle \, . \qquad (2.2.115)$$

From (2.2.109):

$$\left\langle \left(F^{\mu\nu}(x)A^\alpha(x')\right)_+ \right\rangle = \frac{\delta}{\mathrm{i}\delta J_\alpha(x')} \left\langle F^{\mu\nu}(x) \right\rangle - \left\langle \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\alpha(x')} \right\rangle$$

$$+ \left\langle F^{\mu\nu}(x) \right\rangle \left\langle A^\alpha(x') \right\rangle \, ,$$

we have

$$\left\langle \left(F^{\mu\nu}(x)\partial'^\beta A^\alpha(x')\right)_+ \right\rangle = \partial'^\beta \left\langle \left(F^{\mu\nu}(x)A^\alpha(x')\right)_+ \right\rangle$$

$$+ g^{\beta 0}\delta(x^0 - x'^0) \left\langle \left[F^{\mu\nu}(x)\, ,\, A^\alpha(x')\right] \right\rangle$$

$$= \partial'^\beta \frac{\delta}{\mathrm{i}\delta J_\alpha(x')} \left\langle F^{\mu\nu}(x) \right\rangle - \partial'^\beta \left\langle \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\alpha(x')} \right\rangle$$

$$+ \left\langle F^{\mu\nu}(x) \right\rangle \partial'^\beta \left\langle A^\alpha(x') \right\rangle$$

$$+ g^{\beta 0}\delta(x^0 - x'^0) \left\langle \left[F^{\mu\nu}(x)\, ,\, A^\alpha(x')\right] \right\rangle \, ,$$

or

$$\left\langle \left(F^{\mu\nu}(x)\partial'^\beta A^\alpha(x')\right)_+ \right\rangle = \partial'^\beta \frac{\delta}{\mathrm{i}\delta J_\alpha(x')} \left\langle F^{\mu\nu}(x) \right\rangle - \left\langle \partial'^\beta \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\alpha(x')} \right\rangle$$

$$+ \left\langle F^{\mu\nu}(x) \right\rangle \left\langle \partial'^\beta A^\alpha(x') \right\rangle$$

$$+ g^{\beta 0}\delta(x^0 - x'^0)\left\langle \left[F^{\mu\nu}(x), A^\alpha(x')\right]\right\rangle . \tag{2.2.116}$$

By defining the operator

$$\widehat{A}^\mu(x) \equiv \frac{\delta}{\mathrm{i}\delta J_\mu(x)} , \tag{2.2.117}$$

and the operator

$$\widehat{F}^{\mu\nu}(x) \equiv \partial^\mu \widehat{A}^\nu(x) - \partial^\nu \widehat{A}^\mu(x)$$

$$= \partial^\mu \frac{\delta}{\mathrm{i}\delta J_\nu(x)} - \partial^\nu \frac{\delta}{\mathrm{i}\delta J_\mu(x)} , \tag{2.2.118}$$

we write the quantum dynamical principle (2.2.86) as

$$\widehat{A}^\mu(x)\,\langle 0_+ | 0_-\rangle = \langle 0_+ \big| A^\mu(x) \big| 0_-\rangle , \tag{2.2.119}$$

and write the equation (2.2.99) as

$$\widehat{A}^\nu(x')\,\big\langle A^\mu(x)\big\rangle = \Big\langle \big(A^\mu(x)A^\nu(x')\big)_+\Big\rangle - \big\langle A^\mu(x)\big\rangle\,\big\langle A^\nu(x')\big\rangle , \tag{2.2.120}$$

where

$$\big\langle A^\mu(x)\big\rangle = \frac{1}{\langle 0_+ | 0_-\rangle}\widehat{A}^\mu(x)\,\langle 0_+ | 0_-\rangle . \tag{2.2.121}$$

Writing the equation (2.2.109) as

$$\Big\langle \big(F^{\mu\nu}(x)A^\alpha(x')\big)_+\Big\rangle = \widehat{A}^\alpha(x')\,\big\langle F^{\mu\nu}(x)\big\rangle - \Big\langle \widehat{A}^\alpha(x')F^{\mu\nu}(x)\Big\rangle$$

$$+ \big\langle F^{\mu\nu}(x)\big\rangle\,\big\langle A^\alpha(x')\big\rangle , \tag{2.2.122}$$

and from (2.2.116), we obtain

$$\Big\langle \big(F^{\mu\nu}(x)F^{\alpha\beta}(x')\big)_{+}\Big\rangle = \widehat{F}^{\alpha\beta}(x')\left\langle F^{\mu\nu}(x)\right\rangle - \Big\langle \widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x)\Big\rangle$$

$$+ \left\langle F^{\mu\nu}(x)\right\rangle\left\langle F^{\alpha\beta}(x')\right\rangle$$

$$+ g^{\alpha 0}\delta(x^0 - x'^0)\Big\langle \big[F^{\mu\nu}(x), A^{\beta}(x')\big]\Big\rangle$$

$$- g^{\beta 0}\delta(x^0 - x'^0)\Big\langle \big[F^{\mu\nu}(x), A^{\alpha}(x')\big]\Big\rangle , \qquad (2.2.123)$$

where

$$\left\langle F^{\mu\nu}(x)\right\rangle = \frac{1}{\langle 0_+ | 0_- \rangle}\widehat{F}^{\mu\nu}(x)\,\langle 0_+ | 0_- \rangle \ . \qquad (2.2.124)$$

We note that

$$\frac{\delta}{\mathrm{i}\delta J_\beta(x')}\left\langle F^{\mu\nu}(x)\right\rangle = \frac{1}{\langle 0_+ | 0_- \rangle}\frac{\delta}{\mathrm{i}\delta J_\beta(x')}\widehat{F}^{\mu\nu}(x)\,\langle 0_+ | 0_- \rangle$$

$$- \frac{1}{\langle 0_+ | 0_- \rangle}\left[\frac{1}{\langle 0_+ | 0_- \rangle}\frac{\delta}{\mathrm{i}\delta J_\beta(x')}\langle 0_+ | 0_- \rangle\right]\widehat{F}^{\mu\nu}(x)\,\langle 0_+ | 0_- \rangle$$

$$= \frac{1}{\langle 0_+ | 0_- \rangle}\frac{\delta}{\mathrm{i}\delta J_\beta(x')}\widehat{F}^{\mu\nu}(x)\,\langle 0_+ | 0_- \rangle$$

$$- \left\langle A^{\beta}(x')\right\rangle\left\langle F^{\mu\nu}(x)\right\rangle ,$$

or

$$\widehat{A}^{\beta}(x')\left\langle F^{\mu\nu}(x)\right\rangle = \frac{1}{\langle 0_+ | 0_- \rangle}\widehat{A}^{\beta}(x')\widehat{F}^{\mu\nu}(x)\,\langle 0_+ | 0_- \rangle$$

$$- \left\langle A^{\beta}(x')\right\rangle\left\langle F^{\mu\nu}(x)\right\rangle , \qquad (2.2.125)$$

$$\widehat{F}^{\alpha\beta}(x')\left\langle F^{\mu\nu}(x)\right\rangle = \frac{1}{\langle 0_+\,|\,0_-\rangle}\widehat{F}^{\alpha\beta}(x')\widehat{F}^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle$$

$$-\left\langle F^{\mu\nu}(x)\right\rangle\left\langle F^{\alpha\beta}(x')\right\rangle\,, \tag{2.2.126}$$

and rewriting (2.2.123) as

$$\left\langle \left(F^{\mu\nu}(x)F^{\alpha\beta}(x')\right)_+\right\rangle = \frac{1}{\langle 0_+\,|\,0_-\rangle}\widehat{F}^{\alpha\beta}(x')\widehat{F}^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle$$

$$-\left\langle \widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x)\right\rangle$$

$$+\,g^{\alpha 0}\delta(x^0-x'^0)\left[F^{\mu\nu}(x)\,,A^{\beta}(x')\right]$$

$$-\,g^{\beta 0}\delta(x^0-x'^0)\left[F^{\mu\nu}(x)\,,A^{\alpha}(x')\right]\,, \tag{2.2.127}$$

since $\left[F^{\mu\nu}(x)\,,A^{\alpha}(x')\right]$ is a c-numbers, $\left\langle\left[F^{\mu\nu}(x)\,,A^{\alpha}(x')\right]\right\rangle = \left[F^{\mu\nu}(x)\,,A^{\alpha}(x')\right]$.

From (2.2.62):

$$F^{\mu\nu} = \left[\left(g^{\mu 0}g^{\nu\alpha}-g^{\nu 0}g^{\mu\alpha}\right)-\left(g^{\mu 0}b^{\nu}-g^{\nu 0}b^{\mu}\right)\partial^{\alpha}\right]\pi_{\alpha}$$

$$-\left(g^{\mu 0}b^{\nu}-g^{\nu 0}b^{\mu}\right)J^0+\frac{1}{2}\left(\delta^{\mu}{}_{k}\,\delta^{\nu}{}_{l}-\delta^{\nu}{}_{k}\,\delta^{\mu}{}_{l}\right)F^{kl}\,,$$

we have

$$\frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_{\alpha}(x')} = \mathrm{i}g^{\alpha 0}\left(g^{\mu 0}b^{\nu}-g^{\nu 0}b^{\mu}\right)\delta^4(x-x')\,, \tag{2.2.128}$$

or

$$\widehat{A}^{\alpha}(x')\,F^{\mu\nu}(x) = \mathrm{i}g^{\alpha 0}\left(g^{\mu 0}b^{\nu}-g^{\nu 0}b^{\mu}\right)\delta^4(x-x')\,. \tag{2.2.129}$$

$$\partial'^{\beta}\widehat{A}^{\alpha}(x')\,F^{\mu\nu}(x) = \mathrm{i}g^{\alpha 0}\partial'^{\beta}\left(g^{\mu 0}b^{\nu}-g^{\nu 0}b^{\mu}\right)\delta^4(x-x')$$

$$= -\mathrm{i}g^{\alpha 0}\partial^{\beta}\left(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu}\right)\delta^4(x - x')\,, \tag{2.2.130}$$

and

$$\partial'^{\alpha}\widehat{A}^{\beta}(x')\,F^{\mu\nu}(x) = -\mathrm{i}g^{\beta 0}\partial^{\alpha}\left(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu}\right)\delta^4(x - x')\,. \tag{2.2.131}$$

Therefore

$$\widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x) = -\mathrm{i}\left(g^{\beta 0}\partial^{\alpha} - g^{\alpha 0}\partial^{\beta}\right)\left(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu}\right)\delta^4(x - x')\,, \tag{2.2.132}$$

is just a c-number, $\left\langle \widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x)\right\rangle = \widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x)$.

From the commutation relation (2.2.84):

$$\delta\big(x^0 - x'^0\big)\left[A^{\mu}(x)\,, F^{\alpha\beta}(x')\right] = \mathrm{i}g^{\mu i}\left(g^{\alpha 0}g^{\beta k} - g^{\beta 0}g^{\alpha k}\right)\delta_{\perp}^{ik}(x - x')\,,$$

where $\delta_{\perp}^{ik}(x - x') = \left(\delta^{ik} - b^i\partial^k\right)\delta^4(x - x')$ and $b^{\mu} = g^{\mu i}\,\dfrac{\partial_i}{\nabla^2}$, we have

$$\delta\big(x'^0 - x^0\big)\left[A^{\mu}(x')\,, F^{\alpha\beta}(x)\right] = \mathrm{i}g^{\mu i}\left(g^{\alpha 0}g^{\beta k} - g^{\beta 0}g^{\alpha k}\right)\delta_{\perp}^{ik}(x' - x)$$

$$\delta\big(x^0 - x'^0\big)\left[F^{\alpha\beta}(x)\,, A^{\mu}(x')\right] = -\mathrm{i}g^{\mu i}\left(g^{\alpha 0}g^{\beta k} - g^{\beta 0}g^{\alpha k}\right)\delta_{\perp}^{ik}(x - x')\,,$$

because $\delta_{\perp}^{ik}(x' - x) = \delta_{\perp}^{ik}(x - x')$. Therefore

$$\delta\big(x^0 - x'^0\big)\left[F^{\mu\nu}(x)\,, A^{\alpha}(x')\right] = -\mathrm{i}g^{\alpha i}\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)\delta_{\perp}^{ik}(x - x')\,. \tag{2.2.133}$$

Let

$$\begin{aligned}\varPhi \equiv\; & g^{\alpha 0}\delta(x^0 - x'^0)\left[F^{\mu\nu}(x)\,, A^{\beta}(x')\right] \\ & - g^{\beta 0}\delta(x^0 - x'^0)\left[F^{\mu\nu}(x)\,, A^{\alpha}(x')\right]\,,\end{aligned} \tag{2.2.134}$$

we obtain

$$\begin{aligned}
\Phi &= -\mathrm{i}\left(g^{\alpha 0}g^{\beta i} - g^{\beta 0}g^{\alpha i}\right)\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)\delta_{\perp}^{ik}(x-x') \\
&= -\mathrm{i}\left(g^{\alpha 0}g^{\beta i} - g^{\beta 0}g^{\alpha i}\right)\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)\left(\delta^{ik} - b^i\partial^k\right)\delta^4(x-x') \\
&= -\mathrm{i}\left(g^{\alpha 0}g^{\beta i} - g^{\beta 0}g^{\alpha i}\right)\delta^{ik}\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)\delta^4(x-x') \\
&\quad + \mathrm{i}\left(g^{\alpha 0}g^{\beta i} - g^{\beta 0}g^{\alpha i}\right)\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)b^i\partial^k\delta^4(x-x') \\
&= -\mathrm{i}\left(g^{\alpha 0}g^{\beta i} - g^{\beta 0}g^{\alpha i}\right)g_{ik}\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)\delta^4(x-x') \\
&\quad + \mathrm{i}\left(g^{\alpha 0}g^{\beta i} - g^{\beta 0}g^{\alpha i}\right)\partial_i b_k\left(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k}\right)\delta^4(x-x') \\
&= -\mathrm{i}\left(g^{\alpha 0}g^{\beta \sigma} - g^{\beta 0}g^{\alpha \sigma}\right)g_{\sigma\lambda}\left(g^{\mu 0}g^{\nu \lambda} - g^{\nu 0}g^{\mu \lambda}\right)\delta^4(x-x') \\
&\quad + \mathrm{i}\left(g^{\alpha 0}g^{\beta \sigma} - g^{\beta 0}g^{\alpha \sigma}\right)\partial_\sigma b_\lambda\left(g^{\mu 0}g^{\nu \lambda} - g^{\nu 0}g^{\mu \lambda}\right)\delta^4(x-x')\,,
\end{aligned}$$

or

$$\begin{aligned}
\Phi &= -\mathrm{i}\left(g^{\alpha 0}g^{\beta \sigma} - g^{\beta 0}g^{\alpha \sigma}\right)g_{\sigma\lambda}\left(g^{\mu 0}g^{\nu \lambda} - g^{\nu 0}g^{\mu \lambda}\right)\delta^4(x-x') \\
&\quad + \mathrm{i}\left(g^{\alpha 0}\partial^\beta - g^{\beta 0}\partial^\alpha\right)\left(g^{\mu 0}b^\nu - g^{\nu 0}b^\mu\right)\delta^4(x-x')\,. \qquad (2.2.135)
\end{aligned}$$

But from (2.2.132):

$$\left\langle \widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x)\right\rangle = -\mathrm{i}\left(g^{\beta 0}\partial^\alpha - g^{\alpha 0}\partial^\beta\right)\left(g^{\mu 0}b^\nu - g^{\nu 0}b^\mu\right)\delta^4(x-x')\,,$$

we have

$$\Phi = -\mathrm{i}\left(g^{\alpha 0}g^{\beta \sigma} - g^{\beta 0}g^{\alpha \sigma}\right)g_{\sigma\lambda}\left(g^{\mu 0}g^{\nu \lambda} - g^{\nu 0}g^{\mu \lambda}\right)\delta^4(x-x')$$

$$+\left\langle \widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x)\right\rangle ,$$

or

$$-\left\langle \widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x)\right\rangle + g^{\alpha 0}\delta(x^0 - x'^0)\left[F^{\mu\nu}(x), A^{\beta}(x')\right]$$

$$- g^{\beta 0}\delta(x^0 - x'^0)\left[F^{\mu\nu}(x), A^{\alpha}(x')\right]$$

$$= -\mathrm{i}\left(g^{\alpha 0}g^{\beta\sigma} - g^{\beta 0}g^{\alpha\sigma}\right) g_{\sigma\lambda}\left(g^{\mu 0}g^{\nu\lambda} - g^{\nu 0}g^{\mu\lambda}\right)\delta^4(x - x') . \qquad (2.2.136)$$

From (2.2.127)

$$\left\langle \left(F^{\mu\nu}(x)F^{\alpha\beta}(x')\right)_+\right\rangle = \frac{1}{\langle 0_+ | 0_-\rangle}\widehat{F}^{\alpha\beta}(x')\widehat{F}^{\mu\nu}(x)\langle 0_+ | 0_-\rangle$$

$$- \left\langle \widehat{F}^{\alpha\beta}(x')F^{\mu\nu}(x)\right\rangle$$

$$+ g^{\alpha 0}\delta(x^0 - x'^0)\left[F^{\mu\nu}(x), A^{\beta}(x')\right]$$

$$- g^{\beta 0}\delta(x^0 - x'^0)\left[F^{\mu\nu}(x), A^{\alpha}(x')\right] ,$$

we have

$$\left\langle \left(F^{\mu\nu}(x)F^{\alpha\beta}(x')\right)_+\right\rangle$$

$$= \frac{1}{\langle 0_+ | 0_-\rangle}\widehat{F}^{\alpha\beta}(x')\widehat{F}^{\mu\nu}(x)\langle 0_+ | 0_-\rangle$$

$$-\mathrm{i}\left(g^{\alpha 0}g^{\beta\sigma} - g^{\beta 0}g^{\alpha\sigma}\right) g_{\sigma\lambda}\left(g^{\mu 0}g^{\nu\lambda} - g^{\nu 0}g^{\mu\lambda}\right)\delta^4(x - x') . \qquad (2.2.137)$$

From (2.2.31), we obtain

$$\left\langle F^{\mu\nu}(x)\right\rangle = \frac{1}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\gamma} - b^{\lambda}\partial^{\gamma}\right) J_{\gamma}(x)\,, \tag{2.2.138}$$

or

$$\begin{aligned}&\left\langle 0_+\middle|F^{\mu\nu}(x)\middle|0_-\right\rangle \\ &\quad = \frac{1}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\gamma} - b^{\lambda}\partial^{\gamma}\right) J_{\gamma}(x)\left\langle 0_+\middle|0_-\right\rangle\,.\end{aligned} \tag{2.2.139}$$

Because of

$$\widehat{F}^{\mu\nu}(x)\left\langle 0_+\middle|0_-\right\rangle = \left\langle 0_+\middle|F^{\mu\nu}(x)\middle|0_-\right\rangle\,, \tag{2.2.140}$$

therefore

$$\begin{aligned}&\widehat{F}^{\mu\nu}(x)\left\langle 0_+\middle|0_-\right\rangle \\ &\quad = \frac{1}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\gamma} - b^{\lambda}\partial^{\gamma}\right) J_{\gamma}(x)\left\langle 0_+\middle|0_-\right\rangle\,,\end{aligned} \tag{2.2.141}$$

and

$$\begin{aligned}&\frac{\delta}{\mathrm{i}\delta J_{\beta}(x')}\,\widehat{F}^{\mu\nu}(x)\left\langle 0_+\middle|0_-\right\rangle \\ &\quad = \frac{-\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\gamma} - b^{\lambda}\partial^{\gamma}\right) \delta^{\beta}{}_{\gamma}\,\delta^{4}(x - x')\left\langle 0_+\middle|0_-\right\rangle \\ &\qquad + \frac{1}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\gamma} - b^{\lambda}\partial^{\gamma}\right) J_{\gamma}(x)\,\frac{\delta}{\mathrm{i}\delta J_{\beta}(x')}\left\langle 0_+\middle|0_-\right\rangle \\ &\quad = \frac{-\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\beta} - b^{\lambda}\partial^{\beta}\right) \delta^{4}(x - x')\left\langle 0_+\middle|0_-\right\rangle \\ &\qquad + \left\langle F^{\mu\nu}(x)\right\rangle \frac{\delta}{\mathrm{i}\delta J_{\beta}(x')}\left\langle 0_+\middle|0_-\right\rangle\,,\end{aligned} \tag{2.2.142}$$

or

$$\widehat{A}^{\beta}(x')\,\widehat{F}^{\mu\nu}(x)\,\langle 0_+|0_-\rangle$$

$$= \frac{-\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\beta} - b^{\lambda}\partial^{\beta}\right)\delta^4(x-x')\,\langle 0_+|0_-\rangle$$

$$+ \left\langle F^{\mu\nu}(x)\right\rangle \left\langle 0_+\middle|A^{\beta}(x')\middle|0_-\right\rangle\,, \tag{2.2.143}$$

$$\partial'^{\alpha}\widehat{A}^{\beta}(x')\,\widehat{F}^{\mu\nu}(x)\,\langle 0_+|0_-\rangle$$

$$= \frac{-\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\beta}\partial'^{\alpha} - b^{\lambda}\partial'^{\alpha}\partial^{\beta}\right)\delta^4(x-x')\,\langle 0_+|0_-\rangle$$

$$+ \left\langle F^{\mu\nu}(x)\right\rangle \left\langle 0_+\middle|\partial'^{\alpha}A^{\beta}(x')\middle|0_-\right\rangle\,. \tag{2.2.144}$$

Because of

$$\partial'^{\alpha}\delta^4(x-x') = \frac{\partial}{\partial x'_{\alpha}}\,\delta^4(x-x') = -\frac{\partial}{\partial x_{\alpha}}\,\delta^4(x-x') = -\partial^{\alpha}\delta^4(x-x')\,,$$

therefore

$$\partial'^{\alpha}\widehat{A}^{\beta}(x')\,\widehat{F}^{\mu\nu}(x)\,\langle 0_+|0_-\rangle$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda}\left(g^{\lambda\beta}\partial^{\alpha} - b^{\lambda}\partial^{\alpha}\partial^{\beta}\right)\delta^4(x-x')\,\langle 0_+|0_-\rangle$$

$$+ \left\langle F^{\mu\nu}(x)\right\rangle \left\langle 0_+\middle|\partial'^{\alpha}A^{\beta}(x')\middle|0_-\right\rangle\,, \tag{2.2.145}$$

or

$$\partial'^{\beta}\widehat{A}^{\alpha}(x')\,\widehat{F}^{\mu\nu}(x)\,\langle 0_+|0_-\rangle$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda} \left(g^{\lambda\alpha}\partial^{\beta} - b^{\lambda}\partial^{\beta}\partial^{\alpha}\right) \delta^4(x - x') \left\langle 0_+ \middle| 0_- \right\rangle$$

$$+ \left\langle F^{\mu\nu}(x) \right\rangle \left\langle 0_+ \middle| \partial'^{\beta} A^{\alpha}(x') \middle| 0_- \right\rangle \,, \qquad (2.2.146)$$

and hence

$$\widehat{F}^{\alpha\beta}(x')\, \widehat{F}^{\mu\nu}(x) \left\langle 0_+ \middle| 0_- \right\rangle$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \left(g^{\nu\sigma}\partial^{\mu} - g^{\mu\sigma}\partial^{\nu}\right) g_{\sigma\lambda} \left(g^{\lambda\beta}\partial^{\alpha} - g^{\lambda\alpha}\partial^{\beta}\right) \delta^4(x - x') \left\langle 0_+ \middle| 0_- \right\rangle$$

$$+ \left\langle F^{\mu\nu}(x) \right\rangle \left\langle 0_+ \middle| F^{\alpha\beta}(x') \middle| 0_- \right\rangle \,,$$

or

$$\frac{1}{\left\langle 0_+ \middle| 0_- \right\rangle} \widehat{F}^{\alpha\beta}(x')\, \widehat{F}^{\mu\nu}(x) \left\langle 0_+ \middle| 0_- \right\rangle$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \left(g^{\alpha\sigma}\partial^{\beta} - g^{\beta\sigma}\partial^{\alpha}\right) g_{\sigma\lambda} \left(g^{\mu\lambda}\partial^{\nu} - g^{\nu\lambda}\partial^{\mu}\right) \delta^4(x - x')$$

$$+ \left\langle F^{\mu\nu}(x) \right\rangle \left\langle F^{\alpha\beta}(x') \right\rangle \,, \qquad (2.2.147)$$

$$\frac{1}{\left\langle 0_+ \middle| 0_- \right\rangle} \widehat{F}^{\alpha\beta}(x')\, \widehat{F}^{\mu\nu}(x) \left\langle 0_+ \middle| 0_- \right\rangle = \frac{1}{\left\langle 0_+ \middle| 0_- \right\rangle} \widehat{F}^{\mu\nu}(x')\, \widehat{F}^{\alpha\beta}(x) \left\langle 0_+ \middle| 0_- \right\rangle$$

symmetric in $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \nu$

$$\left[\frac{1}{\left\langle 0_+ \middle| 0_- \right\rangle} \widehat{F}^{\alpha\beta}(x')\, \widehat{F}^{\mu\nu}(x) \left\langle 0_+ \middle| 0_- \right\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \left(g^{\alpha\sigma}\partial^{\beta} - g^{\beta\sigma}\partial^{\alpha}\right) g_{\sigma\lambda} \left(g^{\mu\lambda}\partial^{\nu} - g^{\nu\lambda}\partial^{\mu}\right) \delta^4(x - x') \,,$$

and also

$$\left[\widehat{F}^{\alpha\beta}(x')\,\widehat{F}^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{\alpha\sigma}\partial^\beta - g^{\beta\sigma}\partial^\alpha\right) g_{\sigma\lambda}\left(g^{\mu\lambda}\partial^\nu - g^{\nu\lambda}\partial^\mu\right)\delta^4(x-x')\,. \tag{2.2.148}$$

- From (2.2.148), we have

$$\left[\widehat{F}^{0i}(x')\,\widehat{F}^{0j}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{0\sigma}\partial^i - g^{i\sigma}\partial^0\right) g_{\sigma\lambda}\left(g^{0\lambda}\partial^j - g^{j\lambda}\partial^0\right)\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[g^{00}\partial^i\partial^j - g^{0j}\partial^i\partial^0 - g^{i0}\partial^0\partial^j + g^{ij}\partial^0\partial^0\right]\delta^4(x-x')\,,$$

or

$$\left[\widehat{F}^{0i}(x')\,\widehat{F}^{0j}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[\delta^{ij}\left(\partial_0\right)^2 - \partial^i\partial^j\right]\delta^4(x-x')\,. \tag{2.2.149}$$

Therefore

$$\left[\widehat{E}^{i}(x')\,\widehat{E}^{j}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[\delta^{ij}\left(\partial_0\right)^2 - \partial^i\partial^j\right]\delta^4(x-x')\,. \tag{2.2.150}$$

Note that

$$\left[\widehat{\mathbf{E}}(x')\cdot\widehat{\mathbf{E}}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[3\left(\partial_0\right)^2 - \nabla^2\right]\delta^4(x-x')\,, \tag{2.2.151}$$

$$\left[\widehat{\mathbf{E}}(x')\times\widehat{\mathbf{E}}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{-\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\delta^4(x-x') = \mathbf{0}\,. \tag{2.2.152}$$

- From (2.2.148), we have

$$\left[\widehat{F}^{ij}(x')\,\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{i\sigma}\partial^j - g^{j\sigma}\partial^i\right) g_{\sigma\lambda}\left(g^{m\lambda}\partial^n - g^{n\lambda}\partial^m\right)\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[g^{im}\partial^j\partial^n - g^{in}\partial^j\partial^m - g^{jm}\partial^i\partial^n + g^{jn}\partial^i\partial^m\right]\delta^4(x-x')\,,$$

or

$$\left[\widehat{F}^{ij}(x')\,\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[\delta^{im}\partial^j\partial^n - \delta^{in}\partial^j\partial^m - \delta^{jm}\partial^i\partial^n + \delta^{jn}\partial^i\partial^m\right]\delta^4(x-x')\,, \tag{2.2.153}$$

and

$$\left[\frac{1}{2}\,\varepsilon^{kij}\,\widehat{F}^{ij}(x')\,\frac{1}{2}\,\varepsilon^{lmn}\,\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\frac{1}{4}\left[\varepsilon^{kij}\varepsilon^{lmn}\delta^{im}\partial^j\partial^n - \varepsilon^{kij}\varepsilon^{lmn}\delta^{in}\partial^j\partial^m\right.$$

$$\left. -\varepsilon^{kij}\varepsilon^{lmn}\delta^{jm}\partial^i\partial^n + \varepsilon^{kij}\varepsilon^{lmn}\delta^{jn}\partial^i\partial^m\right]\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\frac{1}{4}\left[\varepsilon^{kij}\varepsilon^{lin}\partial^j\partial^n - \varepsilon^{kij}\varepsilon^{lmi}\partial^j\partial^m\right.$$

$$\left. -\varepsilon^{kij}\varepsilon^{ljn}\partial^i\partial^n + \varepsilon^{kij}\varepsilon^{lmj}\partial^i\partial^m\right]\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\frac{1}{4}\left[\varepsilon^{ikj}\varepsilon^{iln}\partial^j\partial^n + \varepsilon^{ikj}\varepsilon^{ilm}\partial^j\partial^m\right.$$

$$\left. +\varepsilon^{jki}\varepsilon^{jln}\partial^i\partial^n + \varepsilon^{jki}\varepsilon^{jlm}\partial^i\partial^m\right]\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\, \varepsilon^{ikj}\varepsilon^{iln}\partial^j\partial^n\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\, \big(\delta^{kl}\delta^{jn} - \delta^{kn}\delta^{jl}\big)\partial^j\partial^n\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\, \big(\delta^{kl}\nabla^2 - \partial^k\partial^l\big)\delta^4(x-x')\,.$$

Therefore

$$\Big[\widehat{B}^i(x')\,\widehat{B}^j(x)\,\langle 0_+\,|\,0_-\rangle\Big]_{J=0} = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\, \big(\delta^{ij}\nabla^2 - \partial^i\partial^j\big)\delta^4(x-x')\,. \quad (2.2.154)$$

From (2.2.150) and (2.2.154)

$$\Big[\Big(\widehat{E}^i(x')\,\widehat{E}^j(x) - \widehat{B}^i(x')\,\widehat{B}^j(x)\Big)\,\langle 0_+\,|\,0_-\rangle\Big]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\Big[\delta^{ij}\big(\partial_0\big)^2 - \delta^{ij}\nabla^2\Big]\,\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\big[-\delta^{ij}\Box\big]\,\delta^4(x-x')\,,$$

or

$$\Big[\Big(\widehat{E}^i(x')\,\widehat{E}^j(x) - \widehat{B}^i(x')\,\widehat{B}^j(x)\Big)\,\langle 0_+\,|\,0_-\rangle\Big]_{J=0} = \mathrm{i}\delta^{ij}\delta^4(x-x')\,, \quad (2.2.155)$$

or

$$\Big[\Big(\widehat{\mathbf{E}}(x')\cdot\widehat{\mathbf{E}}(x) - \widehat{\mathbf{B}}(x')\cdot\widehat{\mathbf{B}}(x)\Big)\,\langle 0_+\,|\,0_-\rangle\Big]_{J=0} = 3\mathrm{i}\delta^4(x-x')\,. \quad (2.2.156)$$

Note that

$$\Big[\widehat{\mathbf{B}}(x')\cdot\widehat{\mathbf{B}}(x)\,\langle 0_+\,|\,0_-\rangle\Big]_{J=0} = \frac{2\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\nabla^2\delta^4(x-x')\,, \quad (2.2.157)$$

$$\left[\widehat{\mathbf{B}}(x') \times \widehat{\mathbf{B}}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{-\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\boldsymbol{\nabla} \times \boldsymbol{\nabla}\delta^4(x-x') = \mathbf{0}\,. \quad (2.2.158)$$

- From (2.2.148), we have

$$\left[\widehat{F}^{0i}(x')\,\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{0\sigma}\partial^i - g^{i\sigma}\partial^0\right) g_{\sigma\lambda}\left(g^{m\lambda}\partial^n - g^{n\lambda}\partial^m\right)\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[g^{0m}\partial^i\partial^n - g^{0n}\partial^i\partial^m - g^{im}\partial^0\partial^n + g^{in}\partial^0\partial^m\right]\delta^4(x-x')\,,$$

or

$$\left[\widehat{F}^{0i}(x')\,\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[\delta^{im}\partial^n - \delta^{in}\partial^m\right]\partial_0\delta^4(x-x')\,, \quad (2.2.159)$$

and

$$\left[\widehat{F}^{0i}(x')\,\frac{1}{2}\,\varepsilon^{jmn}\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\frac{1}{2}\left[\varepsilon^{jmn}\delta^{im}\partial^n - \varepsilon^{jmn}\delta^{in}\partial^m\right]\partial_0\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\frac{1}{2}\left[\varepsilon^{jin}\partial^n - \varepsilon^{jmi}\partial^m\right]\partial_0\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\frac{1}{2}\left[-\varepsilon^{ijk}\partial^k - \varepsilon^{ijk}\partial^k\right]\partial_0\delta^4(x-x')\,.$$

Therefore

$$\left[\widehat{E}^i(x')\,\widehat{B}^j(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{-\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\varepsilon^{ijk}\partial^k\partial_0\delta^4(x-x')\,, \quad (2.2.160)$$

and

$$\left[\varepsilon^{lij}\,\widehat{E}^{i}(x')\,\widehat{B}^{j}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{-\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\varepsilon^{lij}\varepsilon^{ijk}\partial^k\partial_0\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\varepsilon^{ilj}\varepsilon^{ijk}\partial^k\partial_0\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\big(\delta^{lj}\delta^{jk}-\delta^{lk}\delta^{jj}\big)\partial^k\partial_0\delta^4(x-x')$$

$$= \frac{-2\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\partial^l\partial_0\delta^4(x-x')\,.$$

Note that

$$\left[\widehat{\mathbf{E}}(x')\cdot\widehat{\mathbf{B}}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \mathbf{0}\,, \tag{2.2.161}$$

$$\left[\widehat{\mathbf{E}}(x')\times\widehat{\mathbf{B}}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{-2\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\partial_0\boldsymbol{\nabla}\delta^4(x-x')\,. \tag{2.2.162}$$

- From (2.2.148), we have

$$\left[\widehat{F}^{mn}(x')\,\widehat{F}^{0i}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\left(g^{m\sigma}\partial^n-g^{n\sigma}\partial^m\right)g_{\sigma\lambda}\left(g^{0\lambda}\partial^i-g^{i\lambda}\partial^0\right)\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\left[g^{m0}\partial^n\partial^i-g^{mi}\partial^n\partial^0-g^{n0}\partial^m\partial^i+g^{ni}\partial^m\partial^0\right]\delta^4(x-x')\,,$$

or

$$\left[\widehat{F}^{mn}(x')\,\widehat{F}^{0i}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\left[\delta^{im}\partial^n-\delta^{in}\partial^m\right]\partial_0\delta^4(x-x')\,, \tag{2.2.163}$$

and

$$\left[\frac{1}{2}\,\varepsilon^{jmn}\widehat{F}^{mn}(x')\,\widehat{F}^{0i}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\frac{1}{2}\left[\varepsilon^{jmn}\delta^{im}\partial^n-\varepsilon^{jmn}\delta^{in}\partial^m\right]\partial_0\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\frac{1}{2}\left[\varepsilon^{jin}\partial^n-\varepsilon^{jmi}\partial^m\right]\partial_0\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\frac{1}{2}\left[-\varepsilon^{ijk}\partial^k-\varepsilon^{ijk}\partial^k\right]\partial_0\delta^4(x-x')\,.$$

Therefore

$$\left[\widehat{B}^j(x')\,\widehat{E}^i(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{-\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\varepsilon^{ijk}\partial^k\partial_0\delta^4(x-x')\,, \qquad (2.2.164)$$

and

$$\left[\varepsilon^{lji}\widehat{B}^j(x')\,\widehat{E}^i(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{-\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\varepsilon^{lji}\varepsilon^{ijk}\partial^k\partial_0\delta^4(x-x')$$

$$= \frac{-\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\varepsilon^{jli}\varepsilon^{jik}\partial^k\partial_0\delta^4(x-x')$$

$$= \frac{-\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\big(\delta^{li}\delta^{ik}-\delta^{lk}\delta^{ii}\big)\partial^k\partial_0\delta^4(x-x')$$

$$= \frac{2\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\partial^l\partial_0\delta^4(x-x')\,.$$

Note that

$$\left[\widehat{\mathbf{B}}(x')\cdot\widehat{\mathbf{E}}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \mathbf{0}\,, \qquad (2.2.165)$$

$$\left[\widehat{\mathbf{B}}(x')\times\widehat{\mathbf{E}}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} = \frac{2\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\,\partial_0\boldsymbol{\nabla}\delta^4(x-x')\,. \qquad (2.2.166)$$

Let

$$\Psi^{\alpha\beta,\mu\nu} \equiv -\mathrm{i}\big(g^{\alpha 0}g^{\beta\sigma} - g^{\beta 0}g^{\alpha\sigma}\big)g_{\sigma\lambda}\big(g^{\mu 0}g^{\nu\lambda} - g^{\nu 0}g^{\mu\lambda}\big)\delta^4(x-x')\,, \tag{2.2.167}$$

symmetric in $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \nu$, we obtain

$$\left.\begin{aligned} \Psi^{0j,0i} &= -\mathrm{i}\delta^{ij}\delta^4(x-x') \\ \Psi^{mn,0i} &= \Psi^{0i,mn} = \Psi^{mn,ij} = 0\,, \end{aligned}\right\} \tag{2.2.168}$$

and from (2.2.137), we have

$$\begin{aligned} \Big\langle \big(F^{\mu\nu}(x)F^{\alpha\beta}(x')\big)_+ \Big\rangle &= \frac{1}{\langle 0_+\,|\,0_-\rangle}\widehat{F}^{\alpha\beta}(x')\widehat{F}^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle + \Psi^{\alpha\beta,\mu\nu} \\ &= \frac{1}{\langle 0_+\,|\,0_-\rangle}\widehat{F}^{\mu\nu}(x')\widehat{F}^{\alpha\beta}(x)\,\langle 0_+\,|\,0_-\rangle + \Psi^{\mu\nu,\alpha\beta}\,, \end{aligned} \tag{2.2.169}$$

$$\Big\langle \big(F^{\mu\nu}(x)F^{\alpha\beta}(x')\big)_+ \Big\rangle = \Big\langle \big(F^{\alpha\beta}(x)F^{\mu\nu}(x')\big)_+ \Big\rangle\,,$$

symmetric in $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \nu$. Together with (2.2.147):

$$\begin{aligned} &\frac{1}{\langle 0_+\,|\,0_-\rangle}\,\widehat{F}^{\alpha\beta}(x')\,\widehat{F}^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle \\ &\qquad = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{\alpha\sigma}\partial^\beta - g^{\beta\sigma}\partial^\alpha\right)g_{\sigma\lambda}\left(g^{\mu\lambda}\partial^\nu - g^{\nu\lambda}\partial^\mu\right)\delta^4(x-x') \\ &\qquad\quad + \big\langle F^{\mu\nu}(x)\big\rangle\big\langle F^{\alpha\beta}(x')\big\rangle\,, \end{aligned}$$

we obtain

$$\begin{aligned} &\Big\langle \big(F^{\alpha\beta}(x)F^{\mu\nu}(x')\big)_+ \Big\rangle \\ &\qquad = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(g^{\alpha\sigma}\partial^\beta - g^{\beta\sigma}\partial^\alpha\right)g_{\sigma\lambda}\left(g^{\mu\lambda}\partial^\nu - g^{\nu\lambda}\partial^\mu\right)\delta^4(x-x') \end{aligned}$$

$$+\left\langle F^{\alpha\beta}(x)\right\rangle\left\langle F^{\mu\nu}(x')\right\rangle+\Psi^{\alpha\beta,\mu\nu}\,, \tag{2.2.170}$$

or

$$\begin{aligned}
&\left\langle \left(F^{\alpha\beta}(x)F^{\mu\nu}(x')\right)_+\right\rangle\\
&\qquad=\frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\left(g^{\alpha\sigma}\partial^\beta-g^{\beta\sigma}\partial^\alpha\right)g_{\sigma\lambda}\left(g^{\mu\lambda}\partial^\nu-g^{\nu\lambda}\partial^\mu\right)\delta^4(x-x')\\
&\qquad-\mathrm{i}\big(g^{\alpha0}g^{\beta\sigma}-g^{\beta0}g^{\alpha\sigma}\big)g_{\sigma\lambda}\big(g^{\mu0}g^{\nu\lambda}-g^{\nu0}g^{\mu\lambda}\big)\delta^4(x-x')\\
&\qquad+\left\langle F^{\alpha\beta}(x)\right\rangle\left\langle F^{\mu\nu}(x')\right\rangle\,,
\end{aligned} \tag{2.2.171}$$

and

$$\begin{aligned}
&\left\langle \left(F^{\alpha\beta}(x)F^{\mu\nu}(x')\right)_+\right\rangle_{J=0}\\
&\qquad=\frac{\mathrm{i}}{-\Box-\mathrm{i}\epsilon}\left(g^{\alpha\sigma}\partial^\beta-g^{\beta\sigma}\partial^\alpha\right)g_{\sigma\lambda}\left(g^{\mu\lambda}\partial^\nu-g^{\nu\lambda}\partial^\mu\right)\delta^4(x-x')\\
&\qquad-\mathrm{i}\big(g^{\alpha0}g^{\beta\sigma}-g^{\beta0}g^{\alpha\sigma}\big)g_{\sigma\lambda}\big(g^{\mu0}g^{\nu\lambda}-g^{\nu0}g^{\mu\lambda}\big)\delta^4(x-x')\,.
\end{aligned} \tag{2.2.172}$$

- From (2.2.169) and (2.2.168), we have

$$\begin{aligned}
\left\langle \left(F^{0i}(x)F^{0j}(x')\right)_+\right\rangle&=\frac{1}{\langle 0_+\,|\,0_-\rangle}\widehat{F}^{0i}(x')\widehat{F}^{0j}(x)\,\langle 0_+\,|\,0_-\rangle+\Psi^{0i,0j}\\
&=\frac{1}{\langle 0_+\,|\,0_-\rangle}\widehat{F}^{0i}(x')\widehat{F}^{0j}(x)\,\langle 0_+\,|\,0_-\rangle\\
&\qquad-\mathrm{i}\delta^{ij}\delta^4(x-x')\,,
\end{aligned} \tag{2.2.173}$$

and from (2.2.149), we have

$$\left\langle \left(F^{0i}(x)F^{0j}(x')\right)_+ \right\rangle_{J=0} = \left[\widehat{F}^{0i}(x')\widehat{F}^{0j}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0} - \mathrm{i}\delta^{ij}\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left[\delta^{ij}\left(\partial_0\right)^2 - \partial^i\partial^j\right]\delta^4(x-x')$$

$$+ \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\,\delta^{ij}\,\Box\,\delta^4(x-x')\,,$$

$$\left\langle \left(F^{0i}(x)F^{0j}(x')\right)_+ \right\rangle_{J=0} = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(\delta^{ij}\nabla^2 - \partial^i\partial^j\right)\delta^4(x-x')\,, \qquad (2.2.174)$$

or

$$\left\langle \left(E^{i}(x)E^{j}(x')\right)_+ \right\rangle_{J=0} = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\left(\delta^{ij}\nabla^2 - \partial^i\partial^j\right)\delta^4(x-x')\,. \qquad (2.2.175)$$

Note that

$$\left\langle \left(\mathbf{E}(x)\cdot\mathbf{E}(x')\right)_+ \right\rangle_{J=0} = \frac{2\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\nabla^2\delta^4(x-x')\,, \qquad (2.2.176)$$

$$\left\langle \left(\mathbf{E}(x)\times\mathbf{E}(x')\right)_+ \right\rangle_{J=0} = \mathbf{0}\,. \qquad (2.2.177)$$

- From (2.2.169) and (2.2.168), we have

$$\left\langle \left(F^{ij}(x)F^{mn}(x')\right)_+ \right\rangle = \frac{1}{\langle 0_+\,|\,0_-\rangle}\widehat{F}^{ij}(x')\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle + \Psi^{ij,mn}$$

$$= \frac{1}{\langle 0_+\,|\,0_-\rangle}\widehat{F}^{ij}(x')\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle\;, \qquad (2.2.178)$$

and from (2.2.153):

$$\left[\widehat{F}^{ij}(x')\,\widehat{F}^{mn}(x)\,\langle 0_+\,|\,0_-\rangle\right]_{J=0}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \left[\delta^{im}\partial^j\partial^n - \delta^{in}\partial^j\partial^m - \delta^{jm}\partial^i\partial^n + \delta^{jn}\partial^i\partial^m \right] \delta^4(x - x') \, ,$$

we have

$$\left\langle \left(F^{ij}(x) F^{mn}(x') \right)_+ \right\rangle_{J=0} = \left[\widehat{F}^{ij}(x') \widehat{F}^{mn}(x) \, \langle 0_+ \, | \, 0_- \rangle \right]_{J=0} \, ,$$

$$\begin{aligned} &\left\langle \left(F^{ij}(x) F^{mn}(x') \right)_+ \right\rangle_{J=0} \\ &\quad = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \left[\delta^{im}\partial^j\partial^n - \delta^{in}\partial^j\partial^m \right. \\ &\qquad\qquad \left. -\delta^{jm}\partial^i\partial^n + \delta^{jn}\partial^i\partial^m \right] \delta^4(x - x') \, , \end{aligned} \tag{2.2.179}$$

$$\begin{aligned} &\frac{1}{4} \varepsilon^{kij} \varepsilon^{lmn} \left\langle \left(F^{ij}(x) F^{mn}(x') \right)_+ \right\rangle_{J=0} \\ &\quad = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \frac{1}{4} \left[\varepsilon^{kij}\varepsilon^{lmn}\delta^{im}\partial^j\partial^n - \varepsilon^{kij}\varepsilon^{lmn}\delta^{in}\partial^j\partial^m \right. \\ &\qquad\qquad \left. -\varepsilon^{kij}\varepsilon^{lmn}\delta^{jm}\partial^i\partial^n + \varepsilon^{kij}\varepsilon^{lmn}\delta^{jn}\partial^i\partial^m \right] \delta^4(x - x') \\ &\quad = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \frac{1}{4} \left[\varepsilon^{kij}\varepsilon^{lin}\partial^j\partial^n - \varepsilon^{kij}\varepsilon^{lmi}\partial^j\partial^m \right. \\ &\qquad\qquad \left. -\varepsilon^{kij}\varepsilon^{ljn}\partial^i\partial^n + \varepsilon^{kij}\varepsilon^{lmj}\partial^i\partial^m \right] \delta^4(x - x') \\ &\quad = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \frac{1}{4} \left[\varepsilon^{ikj}\varepsilon^{iln}\partial^j\partial^n + \varepsilon^{ikj}\varepsilon^{ilm}\partial^j\partial^m \right. \\ &\qquad\qquad \left. +\varepsilon^{jki}\varepsilon^{jln}\partial^i\partial^n + \varepsilon^{jki}\varepsilon^{jlm}\partial^i\partial^m \right] \delta^4(x - x') \\ &\quad = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon} \varepsilon^{ikj}\varepsilon^{iln}\partial^j\partial^n\delta^4(x - x') \end{aligned}$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\big(\delta^{kl}\delta^{jn} - \delta^{kn}\delta^{jl}\big)\partial^j\partial^n\delta^4(x-x')$$

$$= \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\big(\delta^{kl}\nabla^2 - \partial^k\partial^l\big)\delta^4(x-x')\,,$$

or

$$\Big\langle \big(B^i(x)B^j(x')\big)_+\Big\rangle_{J=0} = \frac{\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\Big(\delta^{ij}\nabla^2 - \partial^i\partial^j\Big)\,\delta^4(x-x')\,. \tag{2.2.180}$$

Note that

$$\Big\langle \big(\mathbf{B}(x)\cdot\mathbf{B}(x')\big)_+\Big\rangle_{J=0} = \frac{2\mathrm{i}}{-\Box - \mathrm{i}\epsilon}\nabla^2\delta^4(x-x')\,, \tag{2.2.181}$$

$$\Big\langle \big(\mathbf{B}(x)\times\mathbf{B}(x')\big)_+\Big\rangle_{J=0} = \mathbf{0}\,. \tag{2.2.182}$$

2.3 Gauge Transformations in QED

The Lagrangian density for QED is given by a well known expression [Manoukian (1986, 1987a)]

$$\begin{aligned}\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[\left(\frac{\partial_\mu\overline{\psi}}{\mathrm{i}}\right)\gamma^\mu\psi - \overline{\psi}\gamma^\mu\frac{\partial_\mu\psi}{\mathrm{i}}\right] - m_0\overline{\psi}\psi \\ + e_0\overline{\psi}\gamma_\mu\psi A^\mu + \overline{\eta}\psi + \overline{\psi}\eta + A_\mu J^\mu\,,\end{aligned} \tag{2.3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{2.3.2}$$

$\overline{\eta}$, η, J^μ are external (c-number) sources with $\overline{\eta}$, η anticommuting, and no restriction is set on J^μ (such as $\partial_\mu J^\mu = 0$) in order to carry out functional differentiations with respect to all of its components *independently*.

2.3.1 The Coulomb Gauge and Arbitrary Covariant Gauges

Our starting point is the vacuum-to-vacuum transition amplitude in the Coulomb gauge given by Manoukian (1986, 1987a)

$$\langle 0_+ \,|\, 0_- \rangle = \exp\left[\mathrm{i} \int (\mathrm{d}x)\, \mathscr{L}_{\mathrm{I}}' \right] \langle 0_+ \,|\, 0_- \rangle_0 \equiv Z_{\mathrm{C}}\big[\eta, \overline{\eta}, J\big] \,, \tag{2.3.3}$$

$$\mathscr{L}_{\mathrm{I}}'(\eta, \overline{\eta}, J) = e_0 \, \frac{\delta}{\mathrm{i}\delta\eta(x)} \, \gamma^\mu \frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)} \, \frac{\delta}{\mathrm{i}\delta J^\mu(x)} \,, \tag{2.3.4}$$

where

$$\langle 0_+ \,|\, 0_- \rangle_0 = \exp\left[\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')\, \overline{\eta}(x) S_+(x, x') \eta(x') \right]$$
$$\times \exp\left[\frac{\mathrm{i}}{2} \int (\mathrm{d}x)(\mathrm{d}x')\; J^\mu(x) D^{\mathrm{C}+}_{\mu\nu}(x, x') J^\nu(x') \right] \,, \tag{2.3.5}$$

with $S_+(x, x')$ denoting the free electron propagator,

$$S_+(x, x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4} \, S_+(p) \, \mathrm{e}^{\mathrm{i}p(x - x')} \,, \tag{2.3.6}$$

where

$$S_+(p) = \frac{-\gamma p + m_0}{p^2 + m_0^2 - \mathrm{i}\epsilon} \,, \tag{2.3.7}$$

and $D^{\mu\nu}_{\mathrm{C}+}(x, x')$ denoting the free photon propagator in the Coulomb gauge,

$$D^{\mu\nu}_{\mathrm{C}+}(x, x') = \int \frac{(\mathrm{d}q)}{(2\pi)^4} \, D^{\mu\nu}_{\mathrm{C}+}(q) \, \mathrm{e}^{\mathrm{i}q(x - x')} \,, \tag{2.3.8}$$

where

$$\left.\begin{aligned} D^{ik}_{\mathrm{C}+}(q) &= \left(\delta^{ik} - \frac{q^i q^k}{\mathbf{q}^2}\right)\frac{1}{q^2 - \mathrm{i}\epsilon}\,, \\ D^{i0}_{\mathrm{C}+}(q) &= 0 = D^{0i}_{\mathrm{C}+}(q)\,, \\ D^{00}_{\mathrm{C}+}(q) &= -\frac{1}{\mathbf{q}^2}\,, \end{aligned}\right\} \tag{2.3.9}$$

or

$$D^{\mu\nu}_{\mathrm{C}+}(q) = \left(g^{\mu\alpha} - g^{\alpha i}\frac{q_i q^\mu}{\mathbf{q}^2}\right) g_{\alpha\beta} \left(g^{\beta\nu} - g^{\beta k}\frac{q_k q^\nu}{\mathbf{q}^2}\right)\frac{1}{q^2 - \mathrm{i}\epsilon}\,. \tag{2.3.10}$$

We introduce the generating functional

$$\begin{aligned} Z\big[\rho, \overline{\rho}, K; G\big] = \exp&\left[\mathrm{i}\int \mathscr{L}'_{\mathrm{I}}(\rho, \overline{\rho}, K)\right] \\ &\times \exp\left[\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')\,\overline{\rho}(x) S_+(x - x')\rho(x')\right] \\ &\times \exp\left[\frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}x')\,K_\mu(x) D^{\mu\nu}_{G+}(x, x') K_\nu(x')\right]\,, \end{aligned} \tag{2.3.11}$$

where in the momentum description

$$D^{\mu\nu}_{G+}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{1}{q^2 - \mathrm{i}\epsilon} + q^\mu q^\nu G(q^2)\,, \tag{2.3.12}$$

and $G(q^2)$ is arbitrary.

We show that

$$Z_{\mathrm{C}}\big[\eta, \overline{\eta}, J\big] = \mathrm{e}^{\mathrm{i}W'} Z\big[\rho, \overline{\rho}, K; G\big]\Big|_{\rho=0,\,\overline{\rho}=0,\,K=0}\,, \tag{2.3.13}$$

where

$$W' = \int (\mathrm{d}x)\,\overline{\eta}(x)\exp\left[-\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu(x)}\right]\frac{\delta}{\mathrm{i}\delta\overline{\rho}(x)}$$

$$+\int(\mathrm{d}x)\,\frac{\delta}{\mathrm{i}\delta\rho(x)}\exp\left[\mathrm{i}e_0 a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu(x)}\right]\eta(x)$$

$$+\int(\mathrm{d}x)\left(\left(g^{\mu\sigma}-a^\mu\partial^\sigma\right)J_\sigma(x)\right)\frac{\delta}{\mathrm{i}\delta K^\mu(x)}\,, \tag{2.3.14}$$

and

$$a^\mu=\left(0\,,\ \frac{\boldsymbol{\nabla}}{\nabla^2}\right)=g^{\mu k}\frac{\partial^k}{\nabla^2}\,, \tag{2.3.15}$$

relating the Coulomb gauge to arbitrary covariant gauges.

To establish (2.3.13), we start from its right-hand side. We note, in a matrix notation, that

$$\mathrm{e}^{\mathrm{i}W'}\exp\left[\mathrm{i}\overline{\rho}S_+\rho\right]\exp\left[\frac{\mathrm{i}}{2}K_\mu D^{\mu\nu}_{G+}K_\nu\right]$$

$$=\exp\left[\mathrm{i}\left(\overline{\rho}+\overline{\eta}\exp\left[-\mathrm{i}e_0a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu}\right]\right)S_+\left(\rho+\exp\left[\mathrm{i}e_0a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu}\right]\eta\right)\right]$$

$$\times\exp\left[\frac{\mathrm{i}}{2}\Big(K_\mu+\left(g_{\mu\sigma}-a_\mu\partial_\sigma\right)J^\sigma\Big)D^{\mu\nu}_{G+}\Big(K_\nu+\left(g_{\nu\lambda}-a_\nu\partial_\lambda\right)J^\lambda\Big)\right]\,, \tag{2.3.16}$$

and since $\mathscr{L}'_{\mathrm{I}}(\rho,\overline{\rho},K)$, is classical, is invariant under transformations

$$\rho(x)\to\rho(x)\exp\big(\mathrm{i}\Lambda(x)\big)\quad\text{and}\quad\overline{\rho}(x)\to\exp\big(-\mathrm{i}\Lambda(x)\big)\overline{\rho}(x)\,, \tag{2.3.17}$$

for an arbitrary numerical function $\Lambda(x)$, and we eventually set $\rho=0$, $\overline{\rho}=0$, the right-hand side of (2.3.13) becomes

$$\exp\left[\mathrm{i}\int\mathscr{L}'_{\mathrm{I}}(\eta,\overline{\eta},J)\right]$$

$$\times\exp\left[\mathrm{i}\left(\overline{\eta}\exp\left[-\mathrm{i}e_0a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu}\right]\right)S_+\left(\exp\left[\mathrm{i}e_0a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu}\right]\eta\right)\right]$$

$$\times \exp\left[\frac{\mathrm{i}}{2}\Big(K_\mu + \big(g_{\mu\sigma} - a_\mu\partial_\sigma\big)J^\sigma\Big)D^{\mu\nu}_{G+}\Big(K_\nu + \big(g_{\nu\lambda} - a_\nu\partial_\lambda\big)J^\lambda\Big)\right] , \qquad (2.3.18)$$

with $K_\mu \to 0$. Now we use the identity

$$\exp\left[\mathrm{i}e_0\int(\mathrm{d}x)\left(\frac{\delta}{\mathrm{i}\delta\eta(x)}\gamma^\mu\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\partial_\mu\Lambda(x)\right)\right]\exp\left[\mathrm{i}\overline{\eta}S_+\eta\right]$$

$$= \exp\left[\mathrm{i}\left(\overline{\eta}\,\mathrm{e}^{\mathrm{i}e_0\Lambda}\right)S_+\left(\mathrm{e}^{-\mathrm{i}e_0\Lambda}\eta\right)\right], \qquad (2.3.19)$$

to rewrite the above expression as

$$\exp\left[\mathrm{i}e_0\int(\mathrm{d}x)\left(\frac{\delta}{\mathrm{i}\delta\eta(x)}\gamma_\mu\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\big(g^{\mu\sigma} - a^\mu\partial^\sigma\big)\frac{\delta}{\mathrm{i}\delta K^\sigma(x)}\right)\right]\exp\left[\mathrm{i}\overline{\eta}S_+\eta\right]$$

$$\times \exp\left[\frac{\mathrm{i}}{2}\Big(K_\mu + \big(g_{\mu\sigma} - a_\mu\partial_\sigma\big)J^\sigma\Big)D^{\mu\nu}_{G+}\Big(K_\nu + \big(g_{\nu\lambda} - a_\nu\partial_\lambda\big)J^\lambda\Big)\right] , \qquad (2.3.20)$$

which for $K_\mu \to 0$ reduces to the left-hand side of (2.3.13) *since*

$$\big(g_{\mu\sigma} - a_\mu\partial_\sigma\big)D^{\mu\nu}_{G+}\big(g_{\nu\lambda} - a_\nu\partial_\lambda\big) = D^{\mathrm{C}+}_{\sigma\lambda}\,. \qquad (2.3.21)$$

Almost an identical analysis as above shows, by noting in the process,

$$\big(g_{\mu\sigma} - \widetilde{a}_\mu\partial_\sigma\big)D^{\mu\nu}_{G+}\big(g_{\nu\lambda} - \widetilde{a}_\nu\partial_\lambda\big) = \big(D_0\big)_{\sigma\lambda} \equiv D^{\mathrm{L}+}_{\sigma\lambda}\,, \qquad (2.3.22)$$

with

$$\widetilde{a}_\mu = \frac{\partial_\mu}{\Box}, \qquad \Box \equiv \partial_\mu\partial^\mu\,, \qquad (2.3.23)$$

where the right-hand side of (2.3.22) defines the photon propagator in the Landau gauge, with G in (2.3.23) set equal to zero, that

$$Z\big[\eta,\overline{\eta},J;G=0\big] = \mathrm{e}^{\mathrm{i}\widetilde{W}'}Z\big[\rho,\overline{\rho},K;G\big]\Big|_{\rho=0,\overline{\rho}=0,K=0}\,, \qquad (2.3.24)$$

where $\widetilde{W}'$ is given by the expression defined in (2.3.14) with a^μ in it simply replaced by $\widetilde{a}^\mu$, thus relating the Landau gauge to arbitrary covariant gauges.

2.3.2 The Fock–Schwinger Gauge

The Fock–Schwinger gauge $x^\mu A_\mu = 0$, allows one to write

$$A^0 = \frac{x^k A_k}{x^0}\,, \tag{2.3.25}$$

which upon substitution in (2.3.1), and varying $\mathscr{L}$ with respect to A^k yields

$$\partial_\mu F^{\mu k} - \frac{x^k}{x^0}\partial_\mu F^{\mu 0} = -j^k + j^0 \frac{x^k}{x^0}\,, \tag{2.3.26}$$

where

$$j^\mu = e_0 \overline{\psi}\gamma^\mu \psi + J^\mu\,. \tag{2.3.27}$$

We note that (2.3.26) holds true with k replaced by 0 in it giving $0 = 0$, i.e., we may rewrite (2.3.26) as

$$\partial_\mu F^{\mu\nu} - \frac{x^\nu}{x^0}\partial_\mu F^{\mu 0} = -j^\nu + j^0 \frac{x^\nu}{x^0} \equiv S^\nu\,. \tag{2.3.28}$$

By taking the derivative ∂_ν of (2.3.28), we may solve for $(\partial_\mu F^{\mu 0})/x^0$,

$$-\frac{\partial_\mu F^{\mu 0}}{x^0} = (\partial\, x)^{-1}\partial_\sigma \left(-j^\sigma + j^0 \frac{x^\sigma}{x^0}\right)\,, \tag{2.3.29}$$

which upon substituting in (2.3.28) gives

$$\partial_\mu F^{\mu\nu} = -\Big[g^{\nu\sigma} - x^\nu (\partial\, x)^{-1}\partial^\sigma\Big] j_\sigma\,. \tag{2.3.30}$$

By taking $\nu = k$, and taking the derivative ∂_k of (2.3.30), we may write

$$-\partial_0 A^0 = \frac{1}{\nabla^2}\left(\partial_0^2\,\partial_k A^k + \partial_k S^k\right), \tag{2.3.31}$$

which when substituted in (2.3.30) gives

$$A^\nu = \Box^{-1} S^\nu + \frac{\partial^\nu}{\nabla^2}\left(\partial_k A^k - \frac{1}{\Box}\partial_k S^k\right). \tag{2.3.32}$$

That is, A^ν is of the form

$$A^\nu = \Box^{-1} S^\nu + \partial^\nu a\,. \tag{2.3.33}$$

For $\nu = k$, and multiplying (2.3.33) by x^k/x^0, we have from (2.3.25)

$$A^0 = \frac{x^k}{x^0}\Box^{-1} S^k + \frac{x^k}{x^0}\partial^k a\,. \tag{2.3.34}$$

On the other hand, directly from (2.3.33) with $\nu = 0$ in it,

$$A^0 = \Box^{-1} S^0 + \partial^0 a\,, \tag{2.3.35}$$

which upon comparison with (2.3.34) leads to

$$x\,\partial\,a = -x^\mu \Box^{-1} S_\mu\,. \tag{2.3.36}$$

From (2.3.33), (2.3.36) and the definition of S^ν in (2.3.28), we obtain

$$A^\nu = -\frac{1}{\Box}\left(g^{\nu\mu} - \partial^\nu \frac{1}{x\,\partial + 2} x^\mu\right)\left(g_{\mu\sigma} - x_\mu \frac{1}{\partial\,x}\partial_\sigma\right) j^\sigma\,, \tag{2.3.37}$$

where we have noted that $\partial\,x = 4 + x\,\partial$. It is straightforward to check from (2.3.37) that $x_\nu A^\nu = 0$ is indeed satisfied.

To establish the transformation from covariant gauges to the FS gauge, we have to pull $\Box^{-1}$ in (2.3.37) between the two round brackets. To this end we note that

$$\Box\, x\,\partial = \big(x\,\partial + 2\big)\,\Box\,, \tag{2.3.38}$$

and hence

$$\big(\Box\, x\,\partial\big)^{-1} = \big(x\,\partial\big)^{-1}\Box^{-1} = \Box^{-1}\big(x\,\partial + 2\big)^{-1}\,, \tag{2.3.39}$$

i.e.,

$$\frac{1}{\Box}\,\frac{1}{x\,\partial + 2} = \frac{1}{x\,\partial}\,\frac{1}{\Box}\,. \tag{2.3.40}$$

We may also use the identity

$$\frac{1}{\Box}x^\mu = x^\mu\frac{1}{\Box} - 2\frac{\partial^\mu}{\Box}\,, \tag{2.3.41}$$

and since ∂^μ when applied to the second factor in (2.3.37) gives

$$\partial^\mu\left(g_{\mu\sigma} - x_\mu\frac{1}{\partial\,x}\partial_\sigma\right) = 0\,. \tag{2.3.42}$$

We obtain from (2.3.40)–(2.3.42), (2.3.37)

$$A^\nu = \left(g^{\nu\mu} - \partial^\nu\frac{1}{x\,\partial}x^\mu\right)\frac{1}{(-\Box)}\left(g_{\mu\sigma} - x_\mu\frac{1}{\partial\,x}\partial_\sigma\right)j^\sigma\,. \tag{2.3.43}$$

Now we invoke the transversality property in (2.3.42) to *rewrite* (2.3.43) as

$$A^\nu = \left(g^{\nu\mu} - \partial^\nu\frac{1}{x\,\partial}x^\mu\right)\frac{1}{(-\Box)}\Big[g_{\mu\rho} - H(\Box)\partial_\mu\partial_\rho\Big]\left(g^{\rho\sigma} - x^\rho\frac{1}{\partial\,x}\partial^\sigma\right)j_\sigma\,, \tag{2.3.44}$$

where $H(\Box)$ is *arbitrary* on account of (2.3.42).

It remains to set

$$g^{\rho\sigma} - x^\rho\frac{1}{\partial\,x}\partial^\sigma = O^{\rho\sigma}\,, \tag{2.3.45}$$

and note that for the factor multiplying j_σ on the right-hand side of (2.3.44),

$$\langle x|(\bullet)|x'\rangle = \int (\mathrm{d}x'')(\mathrm{d}x''') \langle x''|O^{\mu\nu}|x\rangle \left\langle x''\middle|(D_H)_{\mu\rho}\middle|x'''\right\rangle \langle x'''|O^{\rho\sigma}|x'\rangle \ , \quad (2.3.46)$$

where, as shown in detail in the next subsection (§2.3.4), we have noted that

$$\left\langle x\middle|\partial^\nu (x\,\partial)^{-1} x^\mu\middle|x'\right\rangle = \left\langle x'\middle|x^\mu (\partial\,x)^{-1}\partial^\nu\middle|x\right\rangle \ , \quad (2.3.47)$$

and we recognize $\left\langle x''\middle|(D_H)_{\mu\rho}\middle|x'''\right\rangle$ to have the very general structure in (2.3.12). Hence we may write, as in (2.3.13),

$$Z_{\mathrm{FS}}[\eta,\overline{\eta},J] = \mathrm{e}^{\mathrm{i}W''} Z[\rho,\overline{\rho},K;G]\Big|_{\rho=0,\overline{\rho}=0,K=0} \ , \quad (2.3.48)$$

where W'' is given by (2.3.14) with a^μ in the latter replaced by $x^\mu(\partial\,x)^{-1}$. [For interpretation of $x^\mu(\partial\,x)^{-1}\partial^\nu$ see the subsection (§2.3.4) that follows and also Kummer and Weiser (1986).]

2.3.3 The Axial Gauge

The axial gauge $n^\mu A_\mu = 0$, with n^ν a fixed vector, is handled similarly, with A^ν in (2.3.43) now replaced by

$$A^\nu = \left(g^{\nu\mu} - \partial^\nu \frac{1}{n\,\partial} n^\mu\right) \frac{1}{(-\Box)} \left(g_{\mu\sigma} - n_\mu \frac{1}{n\,\partial}\partial_\sigma\right) j^\sigma \ , \quad (2.3.49)$$

and a similar expression as in (2.3.48) holds with a^μ in (2.3.13) replaced by $n^\mu(n\,\partial)^{-1}$ in it.

2.3.4 Explicit Derivation of the Identity (2.3.44)

For an explicit derivation of (2.3.47), we multiply ∂^ν by $-\mathrm{i}$ and write

$$\partial^\nu \big(x\,\partial\big)^{-1} x^\mu = (xp+1)^{-1} p^\nu x^\mu = \sum_{n=0}^{\infty} (-1)^n (xp)^n p^\nu x^\mu \,, \tag{2.3.50}$$

upon moving, in the process, p^ν to the right. Using the identity

$$(x^\mu p_\mu)_{\mathrm{op}} = \int (\mathrm{d}x) \frac{(\mathrm{d}p)}{(2\pi)^4} \, |x\,\rangle\langle p| \, xp \, \mathrm{e}^{\mathrm{i}xp} \,, \tag{2.3.51}$$

we note that

$$\big(xp\big)^n = \int \left[\prod_{i=1}^{n} (\mathrm{d}x_i) \frac{(\mathrm{d}p_i)}{(2\pi)^4} x_i p_i \right]$$

$$\times \, \mathrm{e}^{\mathrm{i}x_n(p_n - p_{n-1})} \mathrm{e}^{\mathrm{i}x_{n-1}(p_{n-1} - p_{n-2})} \ldots \mathrm{e}^{\mathrm{i}x_1 p_1} \, |x_1\,\rangle\langle p_n| \,, \tag{2.3.52}$$

and hence

$$\Big\langle x \Big| \partial^\nu \big(x\,\partial\big)^{-1} x^\mu \Big| x' \Big\rangle = \sum_{n=0}^{\infty} (-1)^n \int \left[\prod_{i=1}^{n} (\mathrm{d}x_i) \frac{(\mathrm{d}p_i)}{(2\pi)^4} x_i p_i \right] p_n^\nu x'^\mu \delta(x - x_1)$$

$$\times \, \mathrm{e}^{\mathrm{i}x_n(p_n - p_{n-1})} \mathrm{e}^{\mathrm{i}x_{n-1}(p_{n-1} - p_{n-2})} \ldots \mathrm{e}^{\mathrm{i}x_1 p_1} \mathrm{e}^{-\mathrm{i}p_n x} \,. \tag{2.3.53}$$

This may be rewritten in an equivalent form by making the change of variables

$$x_1 = y_n, \ldots, x_n = y_1 \,; \quad p_1 = -q_n, \ldots, p_n = -q_1 \,, \tag{2.3.54}$$

leading to

$$\Big\langle x \Big| \partial^\nu \big(x\,\partial\big)^{-1} x^\mu \Big| x' \Big\rangle = -\sum_{n=0}^{\infty} \int \left[\prod_{i=1}^{n} (\mathrm{d}y_i) \frac{(\mathrm{d}q_i)}{(2\pi)^4} y_i q_i \right] x'^\mu q_1^\nu \delta(y_n - x)$$

$$\times\, \mathrm{e}^{\mathrm{i}xq_1}\mathrm{e}^{\mathrm{i}y_1(q_2-q_1)}\mathrm{e}^{\mathrm{i}y_2(q_3-q_2)}\dots\mathrm{e}^{-\mathrm{i}y_nq_n}\,. \tag{2.3.55}$$

On the other hand,

$$\begin{aligned}\left\langle x\left|x^\mu(\partial\,x)^{-1}\partial^\nu\right|x'\right\rangle &= \left\langle x\left|x^\mu p^\nu(p\,x-1)^{-1}\right|x'\right\rangle\\ &= -\sum_{n=0}^{\infty}\left\langle x\left|x^\mu p^\nu(p\,x)^n\right|x'\right\rangle\,,\end{aligned} \tag{2.3.56}$$

and

$$(p^\mu x_\mu)_{\mathrm{op}} = \int(\mathrm{d}x)\frac{(\mathrm{d}p)}{(2\pi)^4}\,|p\rangle\langle x|\,p\,x\,\mathrm{e}^{-\mathrm{i}px}\,, \tag{2.3.57}$$

$$\begin{aligned}(p\,x)^n = \int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}p_ix_i\right]&\\ \times\,\mathrm{e}^{\mathrm{i}x_1(p_2-p_1)}\dots\mathrm{e}^{\mathrm{i}x_{n-1}(p_n-p_{n-1})}\mathrm{e}^{-\mathrm{i}x_np_n}\,|p_1\rangle\langle x_n|\,,&\end{aligned} \tag{2.3.58}$$

leading to

$$\begin{aligned}\left\langle x\left|x^\mu(\partial\,x)^{-1}\partial^\nu\right|x'\right\rangle = -\sum_{n=0}^{\infty}\int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}p_ix_i\right]x^\mu p_1^\nu\delta(x_n-x')&\\ \times\,\mathrm{e}^{\mathrm{i}xp_1}\mathrm{e}^{\mathrm{i}x_1(p_2-p_1)}\dots\mathrm{e}^{\mathrm{i}x_{n-1}(p_n-p_{n-1})}\mathrm{e}^{-\mathrm{i}x_np_n}\,,&\end{aligned} \tag{2.3.59}$$

which upon comparison with (2.3.55) establishes (2.3.47).

CHAPTER III

SUPERSYMMETRIC METHODS IN THE INVESTIGATION OF THE QUANTUM ELECTRODYNAMICS OF MANY-PARTICLE SYSTEMS

Supersymmetric methods have had interesting applications in potential theory in quantum physics as well. Such supersymmetric methods are applied, in the present chapter, in potential theory in the quantum physics of many-particle systems with very general Hamiltonians of the form given below in (3.1.2) with potential energy defined in (3.1.1), and, in particular, a rigorous application is given to develop a lower bound for the ground-state energy of the quantum electrodynamics of charged many-particle systems of bosonic types. The study of the nature of the ground-state energy of Hamiltonians of interacting many-particle systems is of central importance for the investigation of the stability of such complex systems. Over the years much work has been done in deriving rigorous bounds [cf. Dyson and Lenard (1967); Dyson (1967); Hall (2000); Lenard and Dyson (1968); Lieb and Thirring (1975); Lieb (1979); Manoukian and Muthaporn (2002, 2003a,b); Manoukian and Siririnlakul (2004); Muthaporn and Manoukian (2004)] on the exact ground-state energy of such Hamiltonians and, in turn, establish stability or instability of the underlying systems with main emphasis on systems pertaining to matter in bulk. The instability of so-called "bosonic matter", i.e., for matter obtained by relaxing the Pauli exclusion constraint [cf. Dyson (1967); Lieb (1979); Manoukian and Muthaporn (2002, 2003a,b); Manoukian and Siririnlakul (2004); Muthaporn and Manoukian (2004)] is a result of a power law behaviour N^{γ} of the ground-state energy, where N is the number of negatively charged particles, *with* the exponent γ such that $\gamma > 1$. Such a power law behaviour, with $\gamma > 1$, implies instability of the underlying system, since the formation of such matter consisting of $(2N + 2N)$ particles will be favourable over two separate systems brought into contact, each consisting of $(N + N)$ particles, and the energy released upon collapse, in the formation of the former system, being proportional to $\left[(2N)^{\gamma} - 2N^{\gamma}\right]$, will be overwhelmingly large for realistic large N, e.g., $N \sim 10^{23}$. It is interesting to point out that if collapse occurs, then the radial extension of such a system does not decrease faster than $N^{-1/3}$ [Manoukian *et al.* (2006)] upon collapse, as N increases for large N. On the other hand, for ordinary matter, i.e., for which the Pauli exclusion constraint is invoked, the ground-state energy has the single power law behaviour $\sim N$ [Lieb and Thirring (1975); Thirring (2005)] consistent with stability. In this respect, as the number N is made to increase such matter inflates and its radial extension increases not any slower than $N^{1/3}$ [Manoukian and Sirininlakul (2005)]. In recent years there has been also much interest in physics of arbitrary dimensions [cf. Forte (1992); Hatfield (1992); Manoukian and Muthaporn (2003a,b); Muthaporn and Manoukian (2004); Semenoff and Wijewardhana (1987)]. In this respect it is also quite important to investigate if the change of the dimensionality of space will change the properties of many-particle systems and if a given property, such as instability, is a characteristic of the three-dimensional property of space. [Some present field theories speculate that at early stages of the universe, the dimensionality of space was not necessarily three and, by a process which may be referred to as compactification, the present three-dimensional character of space arose upon the evolution and the cooling of the universe.] The purpose of this communication is to use supersymmetry methods to derive *rigorously* lower bounds to a class of Hamiltonians, to be defined in the next section, with particular emphasis on "bosonic matter" in *arbitrary dimensions of space*. The basic idea of supersymmetry methods [cf. Manoukian (2006) for a pedagogical treatment] is to introduce generators $\mathbf{Q}$ and write the Hamiltonian H under consideration, or more precisely a part H' of the Hamiltonian, as $\mathbf{Q}^\dagger \cdot \mathbf{Q}$, where $\mathbf{Q}^\dagger$ is the adjoint of $\mathbf{Q}$, and then use positivity constraints to derive a lower bound for H.

3.1 Supersymmetry Methods and the Ground-State Energy: Application to "Bosonic Matter"

For an N-particle system, we introduce N real vector fields $G_j(x_1, \ldots, x_N; \varrho)$, $j = 1, \ldots, N$, as functions of N dynamical variables $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^\nu$, which may also depend on some parameters which we denote collectively by ϱ. The space dimension is denoted by ν. We consider a class of potential energies $V(\mathbf{x}_1, \ldots, \mathbf{x}_N; \varrho)$ defined by

$$V(\mathbf{x}_1, \ldots, \mathbf{x}_N; \varrho) = -\sum_{j=1}^{N} \boldsymbol{\nabla}_j \cdot \mathbf{G}_j(x_1, \ldots, x_N; \varrho)\,, \tag{3.1.1}$$

where $\boldsymbol{\nabla}_j = \dfrac{\partial}{\partial \mathbf{x}_j}$, and define the multi-particle Hamiltonian by

$$H = \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m_j} + V(\mathbf{x}_1, \ldots, \mathbf{x}_N; \varrho)\,, \tag{3.1.2}$$

with $\mathbf{p}_j = -\mathrm{i}\hbar\boldsymbol{\nabla}_j$, and the m_j denoting the masses of the underlying particles.

Introduce the N operators

$$Q_j = \frac{\hbar\boldsymbol{\nabla}_j}{\sqrt{2m_j}} + \frac{\sqrt{2m_j}}{\hbar}\mathbf{G}_j\,, \tag{3.1.3}$$

and their adjoints

$$Q_j^\dagger = -\frac{\hbar\boldsymbol{\nabla}_j}{\sqrt{2m_j}} + \frac{\sqrt{2m_j}}{\hbar}\mathbf{G}_j\,, \tag{3.1.4}$$

$j = 1, \ldots, N$, and use the property $\boldsymbol{\nabla}_j \cdot \mathbf{G}_j = \big(\boldsymbol{\nabla}_j \cdot \mathbf{G}_j\big) + \mathbf{G}_j \cdot \boldsymbol{\nabla}_j$ to obtain for any normalized state $|\Psi\rangle$

$$0 \leqslant \sum_{j=1}^{N} \left\|Q_j\Psi\right\|^2 = \sum_{j=1}^{N} \left\langle \Psi \middle| Q_j^\dagger Q_j \middle| \Psi \right\rangle$$

$$= \sum_{j=1}^{N} \left\langle \Psi \middle| \left[-\frac{\hbar^2}{2m_j} \boldsymbol{\nabla}_j^2 - \boldsymbol{\nabla}_j \cdot \mathbf{G}_j + \frac{2m_j}{\hbar^2}\, \mathbf{G}_j^2 \right] \middle| \Psi \right\rangle , \qquad (3.1.5)$$

an idea often used in supersymmetry methods, from which we obtain the basic lower bound

$$\left\langle \Psi \middle| H \middle| \Psi \right\rangle \geqslant - \sum_{j=1}^{N} \frac{2m_j}{\hbar^2} \left\langle \Psi \middle| \mathbf{G}_j^2 \middle| \Psi \right\rangle , \qquad (3.1.6)$$

for any Hamiltonian defined by (3.1.2), (3.1.1), giving a lower bound for the expectation value of the Hamiltonian in the state $|\Psi\rangle$.

A classic application of the above is to the Hamiltonian of matter given by

$$H = \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m_j} + \sum_{i<j}^{N} \frac{e^2}{\left|\mathbf{x}_i - \mathbf{x}_j\right|}$$

$$- \sum_{i=1}^{N} \sum_{j=1}^{k} \frac{Z_j\, e^2}{\left|\mathbf{x}_i - \mathbf{R}_j\right|} + \sum_{i<j}^{k} \frac{Z_i\, Z_j\, e^2}{\left|\mathbf{R}_i - \mathbf{R}_j\right|} , \qquad (3.1.7)$$

where k denotes the number of nuclei situated at $\mathbf{R}_1, \dots, \mathbf{R}_k$ with total charges $Z_1|e|, \dots, Z_k|e|$ such that $\sum_{j=1}^{k} Z_j = N$ for neutral matter.

The potential energy in (3.1.7) may be generated *exactly* from the vector fields $\mathbf{G}_j(\mathbf{x}_1, \dots, \mathbf{x}_N; \mathbf{R}_1, \dots, \mathbf{R}_k)$ defined by

$$\mathbf{G}_j(\mathbf{x}_1, \dots, \mathbf{x}_N; \mathbf{R}_1, \dots, \mathbf{R}_k) = -\frac{e^2}{(\nu - 1)} \sum_{\ell=1}^{j-1} \mathbf{n}_{j\ell} + \frac{e^2}{(\nu - 1)} \sum_{\ell=1}^{k} Z_\ell\, \mathbf{k}_{j\ell}$$

$$- \frac{\mathbf{x}_j\, e^2}{\nu N} \sum_{i<\ell}^{k} \frac{Z_i\, Z_\ell}{\left|\mathbf{R}_i - \mathbf{R}_\ell\right|} , \qquad (3.1.8)$$

with $\nu \geqslant 2$ the dimensionality of the space considered, and $\mathbf{n}_{j\ell}$, $\mathbf{k}_{j\ell}$ are *unit* vector fields defined by

$$\mathbf{n}_{j\ell} = \frac{\mathbf{x}_j - \mathbf{x}_\ell}{|\mathbf{x}_j - \mathbf{x}_\ell|}\,, \qquad \mathbf{k}_{j\ell} = \frac{\mathbf{x}_j - \mathbf{R}_\ell}{|\mathbf{x}_j - \mathbf{R}_\ell|}\,, \tag{3.1.9}$$

by using, in the process, the facts that

$$\sum_{j=1}^{N} \boldsymbol{\nabla}_j \cdot \mathbf{x}_j = \nu N \tag{3.1.10}$$

$$\sum_{j=2}^{N} \sum_{\ell=1}^{j-1} \boldsymbol{\nabla}_j \cdot \mathbf{n}_{j\ell} = (\nu - 1) \sum_{\ell<j}^{N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_\ell|} \tag{3.1.11}$$

$$\sum_{j=1}^{N} \sum_{\ell=1}^{k} \boldsymbol{\nabla}_j \cdot \mathbf{k}_{j\ell} = (\nu - 1) \sum_{j=1}^{N} \sum_{\ell=1}^{k} \frac{1}{|\mathbf{x}_j - \mathbf{R}_\ell|}\,, \tag{3.1.12}$$

giving

$$\begin{aligned} -\sum_{j=1}^{N} \mathbf{G}_j(\mathbf{x}_1, \ldots, \mathbf{x}_N; \mathbf{R}_1, \ldots, \mathbf{R}_k) = \sum_{i<j}^{N} \frac{e^2}{|\mathbf{x}_i - \mathbf{x}_j|} \\ - \sum_{i=1}^{N} \sum_{j=1}^{k} \frac{Z_j\, e^2}{|\mathbf{x}_i - \mathbf{R}_j|} \\ + \sum_{i<j}^{k} \frac{Z_i\, Z_j\, e^2}{|\mathbf{R}_i - \mathbf{R}_j|}\,, \end{aligned} \tag{3.1.13}$$

which is the potential energy for matter in (3.1.7).

Due to the presence of the $\mathbf{x}_j$ factor in the last term on the right-hand side of (3.1.8), the lower bound in (3.1.6) for the Hamiltonian H *in* (3.1.7) will involve unmanageable terms such as $-\|\mathbf{x}_j \Psi\|^2$ for which no further lower bounds may be directly obtained. Accordingly, the definition in (3.1.8) suggests to introduce instead the vector fields $\mathbf{G}'_j(\mathbf{x}_1,\ldots,\mathbf{x}_N;\mathbf{R}_1,\ldots,\mathbf{R}_k)$

$$\mathbf{G}'_j(\mathbf{x}_1,\ldots,\mathbf{x}_N;\mathbf{R}_1,\ldots,\mathbf{R}_N) = -\frac{e^2}{(\nu-1)}\sum_{\ell=1}^{j-1}\mathbf{n}_{j\ell} + \frac{e^2}{(\nu-1)}\sum_{\ell=1}^{k} Z_\ell\,\mathbf{k}_{j\ell}\,, \quad (3.1.14)$$

with the unit vector fields $\mathbf{n}_{j\ell},\ \mathbf{k}_{j\ell}$ defined as before, yielding

$$-\sum_{j=1}^{N}\boldsymbol{\nabla}_j\cdot\mathbf{G}'_j(\mathbf{x}_1,\ldots,\mathbf{x}_N;\mathbf{R}_1,\ldots,\mathbf{R}_N) = \sum_{i<j}^{N}\frac{e^2}{|\mathbf{x}_i-\mathbf{x}_j|} - \sum_{i=1}^{N}\sum_{j=1}^{k}\frac{Z_j\,e^2}{|\mathbf{x}_i-\mathbf{R}_j|}\,. \quad (3.1.15)$$

From (3.1.6), (3.1.15), we then obtain the following lower bound for the expectation value of the Hamiltonian in (3.1.7) in a state $|\Psi\rangle$

$$\langle\Psi|H|\Psi\rangle \geqslant -\frac{2m}{\hbar^2}\sum_{j=1}^{N}\Big\langle\Psi\Big|\mathbf{G}'^{\,2}_j(\mathbf{x}_1,\ldots,\mathbf{x}_N;\mathbf{R}_1,\ldots,\mathbf{R}_N)\Big|\Psi\Big\rangle + \sum_{i<j}^{k}\frac{Z_i\,Z_j\,e^2}{|\mathbf{R}_i-\mathbf{R}_j|}$$

$$\geqslant -\frac{2m}{\hbar^2}\sum_{j=1}^{N}\Big\langle\Psi\Big|\mathbf{G}'^{\,2}_j(\mathbf{x}_1,\ldots,\mathbf{x}_N;\mathbf{R}_1,\ldots,\mathbf{R}_N)\Big|\Psi\Big\rangle\,, \quad (3.1.16)$$

with $\mathbf{G}'_j(\mathbf{x}_1,\ldots,\mathbf{x}_N;\mathbf{R}_1,\ldots,\mathbf{R}_N)$ defined in (3.1.14).

Upon using the facts that $\mathbf{n}_{j\ell},\ \mathbf{k}_{j\ell}$, defined in (3.1.9), are unit vector fields, i.e., $\mathbf{n}_{j\ell}\cdot\mathbf{n}_{j\ell'}\leqslant 1,\ \mathbf{k}_{j\ell}\cdot\mathbf{k}_{j\ell'}\leqslant 1,\ -\mathbf{n}_{j\ell}\cdot\mathbf{k}_{j\ell}\leqslant 1,$ we obtain from (3.1.14)

$$\Big\langle\Psi\Big|\mathbf{G}'^{\,2}_j(\mathbf{x}_1,\ldots,\mathbf{x}_N;\mathbf{R}_1,\ldots,\mathbf{R}_N)\Big|\Psi\Big\rangle \leqslant \frac{e^4}{(\nu-1)^2}\,(j-1+N)^2 \quad (3.1.17)$$

where we have used, in the process, the property $\sum_{j=1}^{k} Z_j = N$ for neutral matter.

Summing over j from 1 to N, (3.1.17), (3.1.16) give the following lower bound for the ground-state energy E_N for the Hamiltonian in (3.1.7)

$$E_N \geqslant -\frac{2m}{\hbar^2} \frac{e^4}{(\nu-1)^2} \sum_{j=1}^{N} (j-1+N)^2 \,, \tag{3.1.18}$$

or

$$E_N > -\left(\frac{m\,e^4}{2\hbar^2}\right) \frac{16}{3} \frac{N^3}{(\nu-1)^2} \,. \tag{3.1.19}$$

Needless to say for $\nu \to 1$, we do not obtain any contradiction with $-$infinity as the lower limit of the set of real numbers—which is, however, not interesting.

3.2 Basic Remarks

We may combine the above result with an earlier one [Muthaporn and Manoukian (2004)] which derives instead an *upper* bound for E_N valid also for all space dimensions ν and for $N \geqslant 2^\nu$. The combined results now state that for the Hamiltonian H in (3.1.7) for so-called "bosonic matter"

$$-\left(\frac{m\,e^4}{2\hbar^2}\right) \frac{N^{(2+\nu)/\nu}}{16\pi^2\nu^3 2^\nu} > E_N > -\left(\frac{m\,e^4}{2\hbar^2}\right) \frac{16N^3}{3(\nu-1)^2} \,, \tag{3.2.1}$$

valid for *all* ν and for $N \geqslant 2^\nu$. It is easy to check the consistency relation $\frac{16N^3}{3(\nu-1)^2} > \frac{N^{(2+\nu)/\nu}}{16\pi^2\nu^3 2^\nu}$ in relation to the above double inequalities. It is well known that for $\nu = 3$, the power 3 of N in the inequality on the right-hand side of (3.2.1) may be reduced to $\frac{5}{3}$. Also for $\nu = 3$, for Fermionic, i.e., standard matter with the negatively charged particles obeying the Pauli exclusion principle, the power 3 of N is reduced to one, as mentioned in the introductory section, consistent with the stability criterion of matter. Our result obtained for *arbitrary* dimensions is obviously far from trivial. In

(3.1.7), the so-called positively charged particles (nuclei) are treated non-dynamically being much heavier than the negatively charged particles which is the common practice. Our lower bound for the ground-state energy E_N given in (3.1.19) *is still valid in all dimensions* for an overall neutral system of bosonic charged particles with the positively charged particles treated dynamically as well with the simplification that all the charges are equal in absolute values, provided m on the right-hand side of (3.1.19) denotes the *largest* mass in the set of masses of all the positively as well as negatively charged particles and N, being now even, denotes the total number of particles. The inequalities in (3.2.1) are consistent with a famous remark made by Dyson and Lenard [Dyson and Lenard (1967)] concerning bosonic matter and the release of an overwhelmingly large amount of energy, as also discussed in the introductory section, when two such systems are brought into contact: *"[Bosonic] matter in bulk would collapse into a condensed high density phase. The assembly of any two macroscopic objects would release energy comparable to that of an atomic bomb…"*. Such a property will be also shared in higher dimensional spaces than three, as well as in two dimensions. We will not speculate on the physical significance of higher dimensional spaces [cf. Forte (1992); Hatfield (1992); Semenoff and Wijewardhana (1987)] except to re-iterate that it is important to investigate if the change of the dimensionality of space will change the properties of many-particle systems and if a given property, such as instability, is a characteristic of the three-dimensional property of space. Needless to say, two dimensional space, however, seems to be physically relevant at least in condensed matter physics.

CHAPTER IV

SETTING UP SUPERSYMMETRIC QUANTUM ELECTRODYNAMICS

In this chapter, we introduce the anti-commutation rules of the generators of supersymmetry as well as their commutation properties with the four-momentum vector, as well as the rule of transformation of the vectors in the underlying Hilbert space of physical states. The Lagrangian proposed for SQED by Wess and Zumino [Wess and Zumino (1974a,b,c)] is then spelled out in the so-called Wess–Zumino gauge in which the non-polynomial character of the Lagrangian is reduced to a non-polynomial one to meet the requirements of renormalizability as based on the Power Counting Theorem [cf. Manoukian (1983)]. Special emphasis is also given in defining the supersymmetric partners in the theory as well as providing the expressions for their corresponding propagators. The vacuum-to-vacuum transition amplitude will then be constructed in the celebrated Coulomb gauge in the next chapter.

4.1 Transformation of Vectors in the Hilbert Space

The supersymmetry algebra [cf. Bailin and Love (1994)] is given by

$$[Q_{\mathsf{a}}\,, P_\mu] = \left[\overline{Q}_{\dot{\mathsf{a}}}\,, P_\mu\right] = [P_\mu\,, P_\nu] = 0\,, \tag{4.1.1}$$

$$\{Q_{\mathsf{a}}\,, Q_{\mathsf{b}}\} = \left\{\overline{Q}_{\dot{\mathsf{a}}}\,, \overline{Q}_{\dot{\mathsf{b}}}\right\} = 0\,, \tag{4.1.2}$$

and

$$\left\{Q_{\mathsf{a}}\,, \overline{Q}_{\dot{\mathsf{b}}}\right\} = -2\left(\sigma^\mu\right)_{\mathsf{a}\dot{\mathsf{b}}} P_\mu\,, \tag{4.1.3}$$

defining Q_{a}, $\overline{Q}_{\dot{\mathsf{a}}}$ as generators of supersymmetry ($\mathsf{a}, \mathsf{b}, \dot{\mathsf{a}}, \dot{\mathsf{b}} = 1, 2$) and P^{μ} as the energy-momentum vector components ($\mu, \nu = 0, 1, 2, 3$).

The vectors in the underlying Hilbert space are then transformed via the operator

$$U(x^{\mu}, \theta, \overline{\theta}) = \exp\left[\mathrm{i}\left(\theta^{\mathsf{a}} Q_{\mathsf{a}} + \overline{\theta}^{\dot{\mathsf{a}}}\, \overline{Q}_{\dot{\mathsf{a}}} + x^{\mu} P_{\mu}\right)\right] . \tag{4.1.4}$$

4.2 On the Lagrangian of SQED Versus Supersymmetric Particles

Based on renormalizability requirements, a Lagrangian for chiral superfields may be taken as

$$\begin{aligned} \mathscr{L} = & -\partial_{\mu}\varphi_i^{\dagger}\, \partial^{\mu}\varphi_i - \mathrm{i}\overline{\psi}_i\, \overline{\sigma}^{\mu}\partial_{\mu}\psi_i - F_i^{\dagger} F_i \\ & - \left(\frac{1}{2} m_{ij}\, \psi_i\psi_j + \lambda_{ijk}\, \psi_i\psi_j\psi_k + \text{H.C.}\right) , \end{aligned} \tag{4.2.1}$$

with m_{ij} and λ_{ijk} real and symmetric in their indices, where the terms depending on the field components F_i are so adjusted to ensure the supersymmetric invariance of the action in question.

$$\begin{aligned} \mathscr{L} = & -\partial_{\mu}\varphi_i^{\dagger}\, \partial^{\mu}\varphi_i - \mathrm{i}\overline{\psi}_i\, \overline{\sigma}^{\mu}\partial_{\mu}\psi_i - \left|m_{ij}\, \varphi_j + \lambda_{ijk}\, \varphi_j\varphi_k\right|^2 \\ & - \left(\frac{1}{2} m_{ij}\, \psi_i\psi_j + \lambda_{ijk}\, \psi_i\psi_j\psi_k + \text{H.C.}\right) . \end{aligned} \tag{4.2.2}$$

Similarly, for the vector superfield components may be equivalently constructed by imposing, in the process, the well known gauge transformation of the electromagnetic vector potential $\delta A^{\mu} = \partial^{\mu}\Lambda$. Based on these constructions the SQED Lagrangian suggested by Wess and Zumino [Wess and Zumino (1974a,b,c)] in a specific gauge, appropriately referred to as the Wess–Zumino gauge is

$$
\begin{aligned}
\mathscr{L} = \frac{1}{2}\bigg[\bigg(\frac{\mathrm{D}^{\dagger}_{\mu}}{\mathrm{i}}\overline{\psi}\bigg)\gamma^{\mu}\psi - \overline{\psi}\gamma^{\mu}\bigg(\frac{\mathrm{D}_{\mu}}{\mathrm{i}}\psi\bigg)\bigg] - m_0\overline{\psi}\psi \\
- (\mathrm{D}_{\mu}\phi_1)^{\dagger}(\mathrm{D}^{\mu}\phi_1) - m_0^2\phi_1^{\dagger}\phi_1 - (\mathrm{D}_{\mu}\phi_2^{\dagger})^{\dagger}(\mathrm{D}^{\mu}\phi_2^{\dagger}) - m_0^2\phi_2^{\dagger}\phi_2 \\
- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\bigg[\bigg(\frac{\partial_{\mu}}{\mathrm{i}}\overline{\lambda}\bigg)\gamma^{\mu}\lambda - \overline{\lambda}\gamma^{\mu}\bigg(\frac{\partial_{\mu}}{\mathrm{i}}\lambda\bigg)\bigg] \\
+ \frac{q_0}{\sqrt{2}}\big[\,\overline{\lambda}\psi(\phi_1^{\dagger} + \phi_2) + \overline{\lambda}\mathrm{i}\gamma_5\psi(\phi_2 - \phi_1^{\dagger}) \\
- \overline{\psi}\lambda(\phi_1 + \phi_2^{\dagger}) + \overline{\psi}\mathrm{i}\gamma_5\lambda(\phi_2^{\dagger} - \phi_1)\big] \\
- \frac{q_0^2}{2}\big(\phi_1^{\dagger}\phi_1 - \phi_2^{\dagger}\phi_2\big)^2 , \qquad (4.2.3)
\end{aligned}
$$

where

$$\mathrm{D}_{\mu} \equiv \partial_{\mu} - \mathrm{i}q_0 A_{\mu} \,, \qquad (4.2.4)$$

is a covariant derivative.

The supersymmetric partners defined through supersymmetric links are as follows:

ψ : electron field, $\big(\phi_1, \phi_2\big)$: SUSY partner (selectron field),

A^{μ} : photon field, λ : SUSY partner (photino field),

with the corresponding free propagators defined in the momentum representations as follows:

- the free electron propagator is

$$S_+(p) = \frac{-\gamma p + m_0}{p^2 + m_0^2 - \mathrm{i}\epsilon}\,, \tag{4.2.5}$$

- the free photon propagator in the Coulomb gauge is

$$\left.\begin{aligned} D^{ij}_{\mathrm{C}+}(q) &= \left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}\right)\frac{1}{q^2 - \mathrm{i}\epsilon}\,, \\ D^{i0}_{\mathrm{C}+}(q) &= 0 = D^{0i}_{\mathrm{C}+}(q)\,, \\ D^{00}_{\mathrm{C}+}(q) &= -\frac{1}{\mathbf{q}^2}\,, \end{aligned}\right\} \tag{4.2.6}$$

or

$$D^{\mu\nu}_{\mathrm{C}+}(q) = \left(g^{\mu\alpha} - g^{\alpha i}\frac{q_i q^\mu}{\mathbf{q}^2}\right) g_{\alpha\beta} \left(g^{\beta\nu} - g^{\beta k}\frac{q_k q^\nu}{\mathbf{q}^2}\right)\frac{1}{q^2 - \mathrm{i}\epsilon}\,, \tag{4.2.7}$$

- the free photino propagator is

$$R_+(p) = \frac{-\gamma p}{p^2 - \mathrm{i}\epsilon}\,, \tag{4.2.8}$$

- the free selectron propagators are

$$\left.\begin{aligned} \Delta_{1+}(k_1) &= \frac{1}{k_1^2 + m_0^2 - \mathrm{i}\epsilon}\,, \\ \Delta_{2+}(k_2) &= \frac{1}{k_2^2 + m_0^2 - \mathrm{i}\epsilon}\,, \end{aligned}\right\} \tag{4.2.9}$$

respectively, for the fields introduced above.

CHAPTER V

VACUUM-TO-VACUUM TRANSITION AMPLITUDE, EXTERNAL SOURCES AND THE COULOMB GAUGE

This chapter is involved in constructing the vacuum-to-vacuum amplitude $\langle 0_+ \,|\, 0_- \rangle$ of the underlying SQED theory given the Wess–Zumino Lagrangian (4.2.3). We work throughout in the celebrated Coulomb gauge. In Section 5.1, we couple the fields to external sources to generate $\langle 0_+ \,|\, 0_- \rangle$ thus introduce a new Lagrangian in the presence of such sources. Using *functional derivative techniques*, we derive the explicit expression for $\langle 0_+ \,|\, 0_- \rangle$ as a functional derivative operation acting on $\langle 0_+ \,|\, 0_- \rangle_0$ involving the propagation of free particles between these sources as emitters and detectors. Our final expression is given in (5.1.8) in the Coulomb gauge. To the leading up to second order the latter simplifies to the one given in (5.2.1) which allows us to develop rules for computations of fundamental process. Applications of these useful formulae are given in the next chapter.

5.1 Lagrangian Density for SQED

The lagrangian (density) for SQED with the sources is

$$\mathscr{L} = \frac{1}{2}\left[\left(\frac{\mathrm{D}^{\dagger}_{\mu}}{\mathrm{i}}\overline{\psi}\right)\gamma^{\mu}\psi - \overline{\psi}\gamma^{\mu}\left(\frac{\mathrm{D}_{\mu}}{\mathrm{i}}\psi\right)\right] - m_0\overline{\psi}\psi$$

$$- (\mathrm{D}_{\mu}\phi_1)^{\dagger}(\mathrm{D}^{\mu}\phi_1) - m_0^2\phi_1^{\dagger}\phi_1 - (\mathrm{D}_{\mu}\phi_2^{\dagger})^{\dagger}(\mathrm{D}^{\mu}\phi_2^{\dagger}) - m_0^2\phi_2^{\dagger}\phi_2$$

$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\Big[\Big(\frac{\partial_\mu}{\mathrm{i}}\overline{\lambda}\Big)\gamma^\mu\lambda - \overline{\lambda}\gamma^\mu\Big(\frac{\partial_\mu}{\mathrm{i}}\lambda\Big)\Big]$$

$$+ \frac{q_0}{\sqrt{2}}\big[\,\overline{\lambda}\psi(\phi_1^\dagger + \phi_2) + \overline{\lambda}\mathrm{i}\gamma_5\psi(\phi_2 - \phi_1^\dagger)$$

$$- \overline{\psi}\lambda(\phi_1 + \phi_2^\dagger) + \overline{\psi}\mathrm{i}\gamma_5\lambda(\phi_2^\dagger - \phi_1)\big]$$

$$- \frac{q_0^2}{2}\big(\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2\big)^2$$

$$+ \overline{\psi}\eta + \overline{\eta}\psi + J^\mu A_\mu + K_1^\dagger\phi_1 + \phi_1^\dagger K_1$$

$$+ K_2^\dagger\phi_2 + \phi_2^\dagger K_2 + \overline{\lambda}\xi + \overline{\xi}\lambda\,, \tag{5.1.1}$$

where

$$\mathrm{D}_\mu \equiv \partial_\mu - \mathrm{i}q_0 A_\mu\,, \tag{5.1.2}$$

is a covariant derivative and η, $\overline{\eta}$, J^μ, K_1, $K_1^\dagger$, K_2, $K_2^\dagger$, ξ, $\overline{\xi}$ are external (c-number) sources with $\eta, \overline{\eta}, \xi, \overline{\xi}$ anticommuting.

One may rewrite the SQED lagrangian (5.1.1) as

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_\mathrm{I} + \mathscr{L}_\mathrm{S}\,, \tag{5.1.3}$$

where

$$\mathscr{L}_0 \equiv \frac{1}{2}\Big[\Big(\frac{\partial_\mu}{\mathrm{i}}\overline{\psi}\Big)\gamma^\mu\psi - \overline{\psi}\gamma^\mu\Big(\frac{\partial_\mu}{\mathrm{i}}\psi\Big)\Big] - m_0\overline{\psi}\psi$$

$$- (\partial_\mu\phi_1^\dagger)(\partial^\mu\phi_1) - m_0^2\phi_1^\dagger\phi_1 - (\partial_\mu\phi_2^\dagger)(\partial^\mu\phi_2) - m_0^2\phi_2^\dagger\phi_2$$

$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\Big[\Big(\frac{\partial_\mu}{\mathrm{i}}\overline{\lambda}\Big)\gamma^\mu\lambda - \overline{\lambda}\gamma^\mu\Big(\frac{\partial_\mu}{\mathrm{i}}\lambda\Big)\Big]\,, \tag{5.1.4}$$

is the free lagrangian,

$$\mathscr{L}_{\mathrm{I}} \equiv q_0\overline{\psi}\gamma^\mu\psi A_\mu - \mathrm{i}q_0 A^\mu\big[\phi_1^\dagger(\partial_\mu\phi_1) - (\partial_\mu\phi_1^\dagger)\phi_1\big]$$

$$+ \mathrm{i}q_0 A^\mu\big[\phi_2^\dagger(\partial_\mu\phi_2) - (\partial_\mu\phi_2^\dagger)\phi_2\big]$$

$$+ \frac{q_0}{\sqrt{2}}\big[\,\overline{\lambda}\psi(\phi_1^\dagger + \phi_2) + \overline{\lambda}\mathrm{i}\gamma_5\psi(\phi_2 - \phi_1^\dagger)$$

$$- \overline{\psi}\lambda(\phi_1 + \phi_2^\dagger) + \overline{\psi}\mathrm{i}\gamma_5\lambda(\phi_2^\dagger - \phi_1)\big]$$

$$- \frac{q_0^2}{2}\big(\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2\big)^2 - q_0^2 A_\mu A^\mu\big(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2\big)\,, \qquad (5.1.5)$$

or

$$\mathscr{L}_{\mathrm{I}} = q_0\bigg\{\overline{\psi}\gamma^\mu\psi A_\mu - \mathrm{i}A^\mu\big[\phi_1^\dagger(\partial_\mu\phi_1) - (\partial_\mu\phi_1^\dagger)\phi_1\big]$$

$$+ \mathrm{i}A^\mu\big[\phi_2^\dagger(\partial_\mu\phi_2) - (\partial_\mu\phi_2^\dagger)\phi_2\big]$$

$$+ \frac{1}{\sqrt{2}}\Big[\overline{\lambda}\big(\mathbb{1} - \mathrm{i}\gamma_5\big)\psi\phi_1^\dagger + \overline{\lambda}\big(\mathbb{1} + \mathrm{i}\gamma_5\big)\psi\phi_2$$

$$- \overline{\psi}\big(\mathbb{1} + \mathrm{i}\gamma_5\big)\lambda\phi_1 - \overline{\psi}\big(\mathbb{1} - \mathrm{i}\gamma_5\big)\lambda\phi_2^\dagger\Big]\bigg\}$$

$$- q_0^2\left[\frac{1}{2}\big(\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2\big)^2 + A_\mu A^\mu\big(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2\big)\right]\,, \qquad (5.1.6)$$

is the interaction lagrangian and

$$\mathscr{L}_{\mathrm{S}} \equiv \overline{\psi}\eta + \overline{\eta}\psi + J^\mu A_\mu + K_1^\dagger\phi_1 + \phi_1^\dagger K_1 + K_2^\dagger\phi_2 + \phi_2^\dagger K_2 + \overline{\lambda}\xi + \overline{\xi}\lambda\,, \qquad (5.1.7)$$

is the external source terms. Let $\mathscr{L}_{\mathrm{I}}'$ stands for the interaction lagrangian (5.1.5), (5.1.6) with $\overline{\psi}$ replaced by $\frac{\delta}{\mathrm{i}\delta\eta}$, ψ replaced by $\frac{\delta}{\mathrm{i}\delta\overline{\eta}}$, A_μ replaced by $\frac{\delta}{\mathrm{i}\delta J^\mu}$, ϕ_1 replaced by $\frac{\delta}{\mathrm{i}\delta K_1^\dagger}$, $\phi_1^\dagger$ replaced by $\frac{\delta}{\mathrm{i}\delta K_1}$, ϕ_2 replaced by $\frac{\delta}{\mathrm{i}\delta K_2^\dagger}$, $\phi_2^\dagger$ replaced by $\frac{\delta}{\mathrm{i}\delta K_2}$, $\overline{\lambda}$ replaced by $\dfrac{\delta}{\mathrm{i}\delta\xi}$ and λ replaced by $\dfrac{\delta}{\mathrm{i}\delta\overline{\xi}}$, i.e.,

$$\begin{aligned}
\mathscr{L}_{\mathrm{I}}' = {} & q_0 \frac{\delta}{\mathrm{i}\delta\eta}\gamma^\mu \frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta J^\mu} \\
& - \mathrm{i}q_0 \frac{\delta}{\mathrm{i}\delta J^\mu}\left[\frac{\delta}{\mathrm{i}\delta K_1}\left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_1^\dagger}\right) - \left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_1}\right)\frac{\delta}{\mathrm{i}\delta K_1^\dagger}\right] \\
& + \mathrm{i}q_0 \frac{\delta}{\mathrm{i}\delta J^\mu}\left[\frac{\delta}{\mathrm{i}\delta K_2}\left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right) - \left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_2}\right)\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right] \\
& + \frac{q_0}{\sqrt{2}}\left[\frac{\delta}{\mathrm{i}\delta\xi}\left(\mathbb{1} - \mathrm{i}\gamma_5\right)\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta K_1} + \frac{\delta}{\mathrm{i}\delta\xi}\left(\mathbb{1} + \mathrm{i}\gamma_5\right)\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right. \\
& \qquad \left. - \frac{\delta}{\mathrm{i}\delta\eta}\left(\mathbb{1} + \mathrm{i}\gamma_5\right)\frac{\delta}{\mathrm{i}\delta\overline{\xi}}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} - \frac{\delta}{\mathrm{i}\delta\eta}\left(\mathbb{1} - \mathrm{i}\gamma_5\right)\frac{\delta}{\mathrm{i}\delta\overline{\xi}}\frac{\delta}{\mathrm{i}\delta K_2}\right] \\
& - \frac{q_0^2}{2}\left(\frac{\delta}{\mathrm{i}\delta K_1}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} - \frac{\delta}{\mathrm{i}\delta K_2}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right)^2 \\
& - q_0^2 \frac{\delta}{\mathrm{i}\delta J^\mu}\frac{\delta}{\mathrm{i}\delta J_\mu}\left(\frac{\delta}{\mathrm{i}\delta K_1}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} + \frac{\delta}{\mathrm{i}\delta K_2}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right) . \qquad (5.1.8)
\end{aligned}$$

Hence the vacuum-to-vacuum transition amplitude in the Coulomb gauge, in the presence of the external sources, is

$$\langle 0_+ \,|\, 0_- \rangle = \exp\left[\mathrm{i}\int (\mathrm{d}x)\; \mathscr{L}_{\mathrm{I}}'\right] \langle 0_+ \,|\, 0_- \rangle_0 \,, \qquad (5.1.9)$$

where $\langle 0_+ \,|\, 0_- \rangle_0 \equiv \langle 0_+ \,|\, 0_- \rangle \big|_{q_0=0}$ is given by

$$\begin{aligned}
\langle 0_+ \,|\, 0_- \rangle_0 = {} & \exp\left[\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')\; \overline{\eta}(x) S_+(x,x')\eta(x')\right] \\
& \times \exp\left[\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')\; K_1^\dagger(x)\mathit{\Delta}_{1+}(x,x')K_1(x')\right] \\
& \times \exp\left[\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')\; K_2^\dagger(x)\mathit{\Delta}_{2+}(x,x')K_2(x')\right]
\end{aligned}$$

$$\times \exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')\; J_\mu(x) D_{\mathrm{C}+}^{\mu\nu}(x,x') J_\nu(x')\right]$$

$$\times \exp\left[\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')\,\bar{\xi}(x) R_+(x,x')\xi(x')\right]. \tag{5.1.10}$$

with the free electron propagator is

$$S_+(x,x') = \int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\frac{-\gamma p + m_0}{p^2 + m_0^2 - \mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}p(x-x')}\,, \tag{5.1.11}$$

the free photon propagator in the Coulomb gauge is

$$D_{\mathrm{C}+}^{\mu\nu}(x,x') = \int\frac{(\mathrm{d}q)}{(2\pi)^4}\,D_{\mathrm{C}+}^{\mu\nu}(q)\,\mathrm{e}^{\mathrm{i}q(x-x')}\,, \tag{5.1.12}$$

and

$$\left.\begin{aligned} D_{\mathrm{C}+}^{ij}(q) &= \left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}\right)\frac{1}{q^2 - \mathrm{i}\epsilon}\,, \\ D_{\mathrm{C}+}^{i0}(q) &= 0 = D_{\mathrm{C}+}^{0i}(q)\,, \\ D_{\mathrm{C}+}^{00}(q) &= -\frac{1}{\mathbf{q}^2}\,, \end{aligned}\right\} \tag{5.1.13}$$

or

$$D_{\mathrm{C}+}^{\mu\nu}(q) = \left(g^{\mu\alpha} - g^{\alpha i}\frac{q_i q^\mu}{\mathbf{q}^2}\right) g_{\alpha\beta}\left(g^{\beta\nu} - g^{\beta k}\frac{q_k q^\nu}{\mathbf{q}^2}\right)\frac{1}{q^2 - \mathrm{i}\epsilon}\,. \tag{5.1.14}$$

The free photino propagator is

$$R_+(x,x') = \int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\frac{-\gamma p}{p^2 - \mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}p(x-x')}\,, \tag{5.1.15}$$

and the free selectron propagators are

$$\left.\begin{aligned} \Delta_{1+}(x,x') &= \int\frac{(\mathrm{d}k_1)}{(2\pi)^4}\,\frac{1}{k_1^2 + m_0^2 - \mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}k_1(x-x')}\,, \\ \Delta_{2+}(x,x') &= \int\frac{(\mathrm{d}k_2)}{(2\pi)^4}\,\frac{1}{k_2^2 + m_0^2 - \mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}k_2(x-x')}\,. \end{aligned}\right\} \tag{5.1.16}$$

5.2 Schwinger–Feynman Rules Using Functional Derivatives

The vacuum-to-vacuum transition amplitude in the Coulomb gauge, in the presence of the external sources, is

$$\langle 0_+ | 0_- \rangle = \exp\left(\mathrm{i}q_0\mathcal{A} - \mathrm{i}q_0^2\mathcal{B}\right) \langle 0_+ | 0_- \rangle_0 \,, \tag{5.2.1}$$

where

$$\begin{aligned}\mathcal{A} \equiv \int(\mathrm{d}x)\Bigg\{ & \frac{\delta}{\mathrm{i}\delta\eta}\gamma^\mu \frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta J^\mu} \\ & - \mathrm{i}\frac{\delta}{\mathrm{i}\delta J^\mu}\left[\frac{\delta}{\mathrm{i}\delta K_1}\left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_1^\dagger}\right) - \left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_1}\right)\frac{\delta}{\mathrm{i}\delta K_1^\dagger}\right] \\ & + \mathrm{i}\frac{\delta}{\mathrm{i}\delta J^\mu}\left[\frac{\delta}{\mathrm{i}\delta K_2}\left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right) - \left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_2}\right)\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right] \\ & + \frac{1}{\sqrt{2}}\left[\frac{\delta}{\mathrm{i}\delta\xi}\left(\mathbb{1} - \mathrm{i}\gamma_5\right)\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta K_1} + \frac{\delta}{\mathrm{i}\delta\xi}\left(\mathbb{1} + \mathrm{i}\gamma_5\right)\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right. \\ & \left.\left. - \frac{\delta}{\mathrm{i}\delta\eta}\left(\mathbb{1} + \mathrm{i}\gamma_5\right)\frac{\delta}{\mathrm{i}\delta\overline{\xi}}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} - \frac{\delta}{\mathrm{i}\delta\eta}\left(\mathbb{1} - \mathrm{i}\gamma_5\right)\frac{\delta}{\mathrm{i}\delta\overline{\xi}}\frac{\delta}{\mathrm{i}\delta K_2}\right]\right\}, \end{aligned} \tag{5.2.2}$$

$$\begin{aligned}\mathcal{B} \equiv \int(\mathrm{d}x)\Bigg[& \frac{1}{2}\left(\frac{\delta}{\mathrm{i}\delta K_1}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} - \frac{\delta}{\mathrm{i}\delta K_2}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right)^2 \\ & + \frac{\delta}{\mathrm{i}\delta J^\mu}\frac{\delta}{\mathrm{i}\delta J_\mu}\left(\frac{\delta}{\mathrm{i}\delta K_1}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} + \frac{\delta}{\mathrm{i}\delta K_2}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right)\Bigg]. \end{aligned} \tag{5.2.3}$$

By differentiation (5.2.1) with respect to bare charge q_0,

$$\frac{\partial}{\partial q_0}\langle 0_+ | 0_- \rangle = \mathrm{i}\left(\mathcal{A} - 2q_0\mathcal{B}\right)\langle 0_+ | 0_- \rangle \,, \tag{5.2.4}$$

$$\frac{\partial^2}{\partial q_0^2}\langle 0_+ | 0_- \rangle = -2\mathrm{i}\mathcal{B}\langle 0_+ | 0_- \rangle - \left(\mathcal{A} - 2q_0\mathcal{B}\right)^2\langle 0_+ | 0_- \rangle \,, \tag{5.2.5}$$

$$\vdots$$

and

$$\left[\frac{\partial}{\partial q_0}\langle 0_+|0_-\rangle\right]_{q_0=0} = \mathrm{i}\mathcal{A}\langle 0_+|0_-\rangle_0 \ , \tag{5.2.6}$$

$$\left[\frac{\partial^2}{\partial q_0^2}\langle 0_+|0_-\rangle\right]_{q_0=0} = -2\mathrm{i}\mathcal{B}\langle 0_+|0_-\rangle_0 - \mathcal{A}^2\langle 0_+|0_-\rangle_0 \ , \tag{5.2.7}$$

$$\vdots$$

The vacuum-to-vacuum transition amplitude (5.2.1) may be written as

$$\langle 0_+|0_-\rangle = \exp\left(a_0 + a_1 q_0 + a_2 q_0^2 + a_3 q_0^3 + \dots\right) \ , \tag{5.2.8}$$

where

$$a_0 = \ln\langle 0_+|0_-\rangle_0 \ , \tag{5.2.9}$$

$$a_1 = \frac{1}{\langle 0_+|0_-\rangle_0}\left[\frac{\partial}{\partial q_0}\langle 0_+|0_-\rangle\right]_{q_0=0}$$

$$= \frac{\mathrm{i}}{\langle 0_+|0_-\rangle_0}\mathcal{A}\langle 0_+|0_-\rangle_0 \ , \tag{5.2.10}$$

$$a_2 = \frac{1}{2\langle 0_+|0_-\rangle_0}\left[\frac{\partial^2}{\partial q_0^2}\langle 0_+|0_-\rangle\right]_{q_0=0} - \frac{a_1^2}{2}$$

$$= \frac{-\mathrm{i}}{\langle 0_+|0_-\rangle_0}\mathcal{B}\langle 0_+|0_-\rangle_0 - \frac{1}{2\langle 0_+|0_-\rangle_0}\mathcal{A}^2\langle 0_+|0_-\rangle_0 - \frac{a_1^2}{2} \ , \tag{5.2.11}$$

$$\vdots$$

CHAPTER VI

APPLICATIONS

In this chapter we carry out applications of the vacuum-to-vacuum transition amplitude derived and obtained in the previous chapter, and we use notably our expression given in (5.2.1). We carry out an application to the scattering of electron–positron to photino–photino ($\mathrm{e}^-\mathrm{e}^+ \to \tilde{\gamma}\tilde{\gamma}$) and obtain the explicit expression for the corresponding amplitude and is given in (6.1.4). As another application, of significance importance, we carry out a study of the self-energy $\Sigma(p)$ of the electron which necessarily involves the photino to make a comparison with the one occurring in pure QED. In contrast to the latter theory, the present one involves three diagrams, not just one, to the leading order, and our explicit expression for $\Sigma(p)$ is given in (6.2.59). To our surprise it is shown that the wave-function renormalization constant Z_2 is finite only in the Landau gauge, to the leading order, as it is in pure QED, i.e., for the photon propagator given in the covariant form

$$D_+^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu\, q^\nu}{q^2} \right) \frac{1}{q^2 - \mathrm{i}\epsilon}\,. \tag{6.1}$$

The significance of this result will be discussed in detail in our concluding chapter (Chapter VII).

The vacuum-to-vacuum transition amplitude in the Coulomb gauge, in the presence of the external sources, is

$$\langle 0_+ \,|\, 0_- \rangle = \exp\left(\mathrm{i}q_0\mathcal{A} - \mathrm{i}q_0^2\mathcal{B}\right) \langle 0_+ \,|\, 0_- \rangle_0\ , \tag{6.2}$$

where

$$\mathcal{A} \equiv \int(\mathrm{d}x)\left\{\frac{\delta}{\mathrm{i}\delta\eta}\gamma^\mu\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta J^\mu}\right.$$

$$-\mathrm{i}\frac{\delta}{\mathrm{i}\delta J^\mu}\left[\frac{\delta}{\mathrm{i}\delta K_1}\left(\partial^\mu\frac{\delta}{\mathrm{i}\delta K_1^\dagger}\right)-\left(\partial^\mu\frac{\delta}{\mathrm{i}\delta K_1}\right)\frac{\delta}{\mathrm{i}\delta K_1^\dagger}\right]$$

$$+\mathrm{i}\frac{\delta}{\mathrm{i}\delta J^\mu}\left[\frac{\delta}{\mathrm{i}\delta K_2}\left(\partial^\mu\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right)-\left(\partial^\mu\frac{\delta}{\mathrm{i}\delta K_2}\right)\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right]$$

$$+\frac{1}{\sqrt{2}}\left[\frac{\delta}{\mathrm{i}\delta\xi}(\mathbb{1}-\mathrm{i}\gamma_5)\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta K_1}+\frac{\delta}{\mathrm{i}\delta\xi}(\mathbb{1}+\mathrm{i}\gamma_5)\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right.$$

$$\left.\left.-\frac{\delta}{\mathrm{i}\delta\eta}(\mathbb{1}+\mathrm{i}\gamma_5)\frac{\delta}{\mathrm{i}\delta\overline{\xi}}\frac{\delta}{\mathrm{i}\delta K_1^\dagger}-\frac{\delta}{\mathrm{i}\delta\eta}(\mathbb{1}-\mathrm{i}\gamma_5)\frac{\delta}{\mathrm{i}\delta\overline{\xi}}\frac{\delta}{\mathrm{i}\delta K_2}\right]\right\}, \tag{6.3}$$

$$\mathcal{B} \equiv \int(\mathrm{d}x)\left[\frac{1}{2}\left(\frac{\delta}{\mathrm{i}\delta K_1}\frac{\delta}{\mathrm{i}\delta K_1^\dagger}-\frac{\delta}{\mathrm{i}\delta K_2}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right)^2\right.$$

$$\left.+\frac{\delta}{\mathrm{i}\delta J^\mu}\frac{\delta}{\mathrm{i}\delta J_\mu}\left(\frac{\delta}{\mathrm{i}\delta K_1}\frac{\delta}{\mathrm{i}\delta K_1^\dagger}+\frac{\delta}{\mathrm{i}\delta K_2}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right)\right]. \tag{6.4}$$

6.1 Electron–Positron-to-Photino–Photino Scattering

We are looking for the amplitude of $\mathrm{e}^-\mathrm{e}^+ \to \tilde{\gamma}\tilde{\gamma}$ scattering:

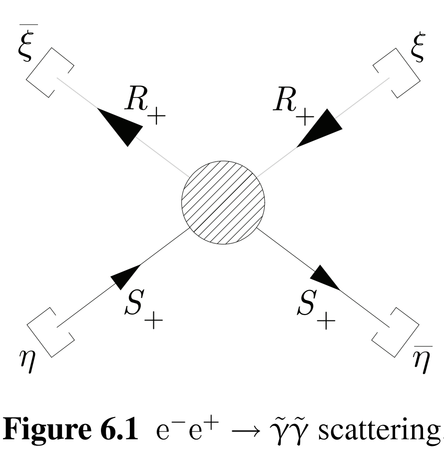


Figure 6.1 $\mathrm{e}^-\mathrm{e}^+ \to \tilde{\gamma}\tilde{\gamma}$ scattering.

For the process just discussed, the vacuum-to-vacuum transition amplitude becomes effectively replaced through the following steps:

$$
\begin{aligned}
&\langle 0_+ \,|\, 0_- \rangle \\
&\to \exp\Bigg\{ \mathrm{i} q_0 \left[\frac{1}{\sqrt{2}} \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x)} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \frac{\delta}{\mathrm{i}\delta\bar{\eta}_{\mathsf{D}}(x)} \frac{\delta}{\mathrm{i}\delta K_1(x)} \right. \\
&\qquad \left. - \frac{1}{\sqrt{2}} \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)} \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)} \frac{\delta}{\mathrm{i}\delta K_1^{\dagger}(x)} \right] \\
&\qquad + \mathrm{i} q_0 \left[\frac{1}{\sqrt{2}} \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{A}}(x)} \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \frac{\delta}{\mathrm{i}\delta\bar{\eta}_{\mathsf{B}}(x)} \frac{\delta}{\mathrm{i}\delta K_2^{\dagger}(x)} \right. \\
&\qquad \left. - \frac{1}{\sqrt{2}} \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x)} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{D}}(x)} \frac{\delta}{\mathrm{i}\delta K_2(x)} \right] \Bigg\} \langle 0_+ \,|\, 0_- \rangle_0 \,, \qquad (6.1.1)
\end{aligned}
$$

and set $K_1(x) = 0$, $K_2(x) = 0$ and $J^{\mu}(x) = 0$ after doing the differentiations.

$$
\begin{aligned}
&\langle 0_+ \,|\, 0_- \rangle \\
&\to \exp\Bigg\{ \mathrm{i} q_0^2 \left(-\frac{1}{2}\right) \left[\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)} \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)} \Delta_{1+}(x, x') \right. \\
&\qquad \left. \times \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \frac{\delta}{\mathrm{i}\delta\bar{\eta}_{\mathsf{D}}(x')} \right] \\
&\qquad + \mathrm{i} q_0^2 \left(-\frac{1}{2}\right) \left[\frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{A}}(x)} \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \frac{\delta}{\mathrm{i}\delta\bar{\eta}_{\mathsf{B}}(x)} \Delta_{2+}(x, x') \right. \\
&\qquad \left. \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{D}}(x')} \right] \Bigg\} \langle 0_+ \,|\, 0_- \rangle_0 \\
&\to \exp\Bigg\{ -\frac{\mathrm{i}}{2} q_0^2 \Bigg[\left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \Delta_{1+}(x, x') \\
&\qquad \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)} \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')} \\
&\qquad + \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AD}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CB}} \Delta_{2+}(x', x)
\end{aligned}
$$

$$\times \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{A}}(x')} \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x)} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}\Bigg] \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{D}}(x')}\Bigg\} \langle 0_+ | 0_- \rangle_0$$

$$\to \exp\Bigg\{ -\frac{\mathrm{i}}{2} q_0^2 \Bigg[\left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \mathit{\Delta}_{1+}(x, x')$$

$$\times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)} \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')}$$

$$+ \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AD}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CB}} \mathit{\Delta}_{2+}(x', x)$$

$$\times \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{A}}(x')} \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x)} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}\Bigg]$$

$$\times S_+^{\mathsf{DE}}(x', z)\, \eta_{\mathsf{E}}(z) \Bigg\} \langle 0_+ | 0_- \rangle_0$$

$$\to \exp\Bigg\{ -\frac{\mathrm{i}}{2} q_0^2 \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)}$$

$$\times \Bigg[\left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \mathit{\Delta}_{1+}(x, x') \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)} \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')}$$

$$- \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \mathit{\Delta}_{2+}(x', x) \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}\Bigg]$$

$$\times S_+^{\mathsf{DE}}(x', z)\, \eta_{\mathsf{E}}(z) \Bigg\} \langle 0_+ | 0_- \rangle_0$$

$$\to \exp\Bigg\{ -\frac{\mathrm{i}}{2} q_0^2\, \overline{\eta}_{\mathsf{F}}(y)\, S_+^{\mathsf{FA}}(y, x)$$

$$\times \Bigg[\left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \mathit{\Delta}_{1+}(x, x') \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)} \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')}$$

$$- \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \mathit{\Delta}_{2+}(x', x) \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}\Bigg]$$

$$\times S_+^{\mathsf{DE}}(x', z)\, \eta_{\mathsf{E}}(z) \Bigg\} \langle 0_+ | 0_- \rangle_0$$

$$\to \exp\left\{-\frac{\mathrm{i}}{2}q_0^2\,\overline{\eta}_{\mathsf{F}}(y)\,S_+^{\mathsf{FA}}(y,x)\right.$$

$$\times\left[\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\,\Delta_{1+}(x,x')\,\frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}\right.$$

$$\left.+\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\,\Delta_{2+}(x',x)\,\frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}\right]\frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')}$$

$$\left.\times\,S_+^{\mathsf{DE}}(x',z)\,\eta_{\mathsf{E}}(z)\right\}\langle 0_+\,|\,0_-\rangle_0$$

$$\to \exp\left\{+\frac{\mathrm{i}}{2}q_0^2\,\overline{\eta}_{\mathsf{F}}(y)\,S_+^{\mathsf{FA}}(y,x)\right.$$

$$\times\left[\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\,\Delta_{1+}(x,x')\,\frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}\right.$$

$$\left.+\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\,\Delta_{2+}(x',x)\,\frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}\right]$$

$$\left.\times\,\bar{\xi}_{\mathsf{G}}(z')\,R_+^{\mathsf{GC}}(z',x')\,S_+^{\mathsf{DE}}(x',z)\,\eta_{\mathsf{E}}(z)\right\}\langle 0_+\,|\,0_-\rangle_0$$

$$\to \exp\left\{+\frac{\mathrm{i}}{2}q_0^2\,\overline{\eta}_{\mathsf{F}}(y)\,S_+^{\mathsf{FA}}(y,x)\right.$$

$$\times\left[\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\,\Delta_{1+}(x,x')\right.$$

$$\left.+\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\,\Delta_{2+}(x',x)\right]\frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)}$$

$$\left.\times\,\bar{\xi}_{\mathsf{G}}(z')\,R_+^{\mathsf{GC}}(z',x')\,S_+^{\mathsf{DE}}(x',z)\,\eta_{\mathsf{E}}(z)\right\}\langle 0_+\,|\,0_-\rangle_0$$

$$\to \exp\left\{-\frac{\mathrm{i}}{2}q_0^2\,\overline{\eta}_{\mathsf{F}}(y)\,S_+^{\mathsf{FA}}(y,x)\right.$$

$$\times\left[\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\,\Delta_{1+}(x,x')\right.$$

$$+\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\,\varDelta_{2+}(x',x)\Big]$$

$$\times\,\overline{\xi}_{\mathsf{G}}(z')\,R_+^{\mathsf{GC}}(z',x')\,R_+^{\mathsf{BH}}(x,z'')\,\xi_{\mathsf{H}}(z'')\,S_+^{\mathsf{DE}}(x',z)\,\eta_{\mathsf{E}}(z)\Big\}\,\langle 0_+\,|\,0_-\rangle_0$$

$$\to\exp\Big\{+\frac{\mathrm{i}}{2}q_0^2\Big[\overline{\eta}_{\mathsf{F}}(y)\,S_+^{\mathsf{FA}}(y,x)\,R_+^{\mathsf{BH}}(x,z'')\,\xi_{\mathsf{H}}(z'')\Big]$$

$$\times\,\Big[\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\,\varDelta_{1+}(x,x')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{CD}}$$

$$+\big(\mathbb{1}-\mathrm{i}\gamma_5\big)_{\mathsf{AB}}\,\varDelta_{2+}(x',x)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)_{\mathsf{CD}}\Big]$$

$$\times\,\Big[\overline{\xi}_{\mathsf{G}}(z')\,R_+^{\mathsf{GC}}(z',x')\,S_+^{\mathsf{DE}}(x',z)\,\eta_{\mathsf{E}}(z)\Big]\,\Big\}\,\langle 0_+\,|\,0_-\rangle_0 \qquad (6.1.2)$$

The amplitude of $\mathrm{e}^-\mathrm{e}^+\to\tilde{\gamma}\tilde{\gamma}$ scattering is

$$\mathcal{M}=\frac{\mathrm{i}}{2}q_0^2\Big[\overline{\eta}(y)\,S_+(y,x)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,R_+(x,z'')\,\xi(z'')\Big]\,\varDelta_{1+}(x,x')$$

$$\times\,\Big[\overline{\xi}(z')\,R_+(z',x')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,S_+(x',z)\,\eta(z)\Big]$$

$$+\,\frac{\mathrm{i}}{2}q_0^2\Big[\overline{\eta}(y)\,S_+(y,x)\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(x,z'')\,\xi(z'')\Big]\,\varDelta_{2+}(x',x)$$

$$\times\,\Big[\overline{\xi}(z')\,R_+(z',x')\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,S_+(x',z)\,\eta(z)\Big]\,, \qquad (6.1.3)$$

or

$$\mathcal{M}=\frac{\mathrm{i}}{2}q_0^2\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}z'')$$

$$\times\,\Big[\overline{\eta}(y)\,S_+(y,x)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,R_+(x,z'')\,\xi(z'')\Big]\,\varDelta_{1+}(x,x')$$

$$\times\,\Big[\overline{\xi}(z')\,R_+(z',x')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,S_+(x',z)\,\eta(z)\Big]$$

$$+\,\frac{\mathrm{i}}{2}q_0^2\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}z'')$$

$$\times \left[\overline{\eta}(y)\, S_+(y,x)\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, R_+(x,z'')\, \xi(z'') \right] \Delta_{2+}(x',x)$$

$$\times \left[\overline{\xi}(z')\, R_+(z',x')\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, S_+(x',z)\, \eta(z) \right] . \tag{6.1.4}$$

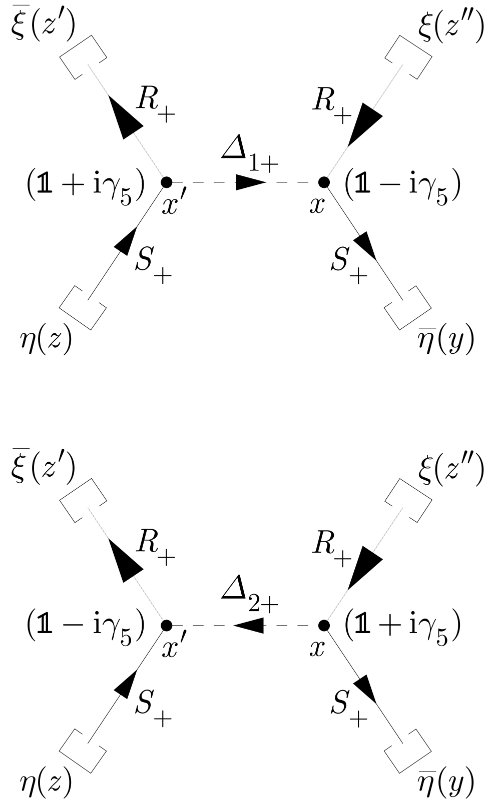


Figure 6.2 Two diagrams contributing to the $\mathrm{e}^-\mathrm{e}^+ \to \tilde{\gamma}\tilde{\gamma}$ scattering (in spacetime variables).

The free electron propagator is

$$S_+(x',z) = \int \frac{(\mathrm{d}p_1')}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p_1'(x'-z)}\, S_+(p_1')\,, \tag{6.1.5}$$

and

$$S_+(y,x) = \int \frac{(\mathrm{d}p_2')}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p_2'(y-x)}\, S_+(p_2')\,. \tag{6.1.6}$$

The free photino propagator is

$$R_+(z',x') = \int \frac{(\mathrm{d}p_1)}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p_1(z'-x')}\, R_+(p_1)\,, \tag{6.1.7}$$

and

$$R_+(x,z'') = \int \frac{(\mathrm{d}p_2)}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p_2(x-z'')}\, R_+(p_2)\,. \tag{6.1.8}$$

The free selectron propagators are

$$\varDelta_{1+}(x,x') = \int \frac{(\mathrm{d}k)}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}k(x-x')}\, \varDelta_{1+}(k)\,, \tag{6.1.9}$$

and

$$\varDelta_{2+}(x',x) = \int \frac{(\mathrm{d}k')}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}k'(x'-x)}\, \varDelta_{2+}(k')\,. \tag{6.1.10}$$

The amplitude of $\mathrm{e}^-\mathrm{e}^+ \to \tilde{\gamma}\tilde{\gamma}$ scattering (6.1.4) is

$$\begin{aligned}
\mathcal{M} = {}& \frac{\mathrm{i}}{2} q_0^2 \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}z'') \\
&\qquad \times \Big[\overline{\eta}(y)\, S_+(y,x)\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, R_+(x,z'')\, \xi(z'')\Big]\, \varDelta_{1+}(x,x') \\
&\qquad \times \Big[\overline{\xi}(z')\, R_+(z',x')\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, S_+(x',z)\, \eta(z)\Big] \\
&+ \frac{\mathrm{i}}{2} q_0^2 \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}z'') \\
&\qquad \times \Big[\overline{\eta}(y)\, S_+(y,x)\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, R_+(x,z'')\, \xi(z'')\Big]\, \varDelta_{2+}(x',x) \\
&\qquad \times \Big[\overline{\xi}(z')\, R_+(z',x')\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, S_+(x',z)\, \eta(z)\Big] \\
= {}& \frac{\mathrm{i}}{2} q_0^2 \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}z'') \frac{(\mathrm{d}p'_1)}{(2\pi)^4}\frac{(\mathrm{d}p'_2)}{(2\pi)^4}\frac{(\mathrm{d}p_1)}{(2\pi)^4}\frac{(\mathrm{d}p_2)}{(2\pi)^4}\frac{(\mathrm{d}k)}{(2\pi)^4} \\
&\qquad \times \mathrm{e}^{\mathrm{i}p'_1(x'-z)}\, \mathrm{e}^{\mathrm{i}p'_2(y-x)}\, \mathrm{e}^{\mathrm{i}p_1(z'-x')}\, \mathrm{e}^{\mathrm{i}p_2(x-z'')}\, \mathrm{e}^{\mathrm{i}k(x-x')}
\end{aligned}$$

$$\times\, \overline{\eta}(y)\, S_+(p_2')\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, R_+(p_2)\, \xi(z'')\, \Delta_{1+}(k)$$

$$\times\, \overline{\xi}(z')\, R_+(p_1)\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, S_+(p_1')\, \eta(z)$$

$$+\, \frac{\mathrm{i}}{2} q_0^2 \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}z'') \frac{(\mathrm{d}p_1')}{(2\pi)^4} \frac{(\mathrm{d}p_2')}{(2\pi)^4} \frac{(\mathrm{d}p_1)}{(2\pi)^4} \frac{(\mathrm{d}p_2)}{(2\pi)^4} \frac{(\mathrm{d}k')}{(2\pi)^4}$$

$$\times\, \mathrm{e}^{\mathrm{i}p_1'(x'-z)}\, \mathrm{e}^{\mathrm{i}p_2'(y-x)}\, \mathrm{e}^{\mathrm{i}p_1(z'-x')}\, \mathrm{e}^{\mathrm{i}p_2(x-z'')}\, \mathrm{e}^{\mathrm{i}k'(x'-x)}$$

$$\times\, \overline{\eta}(y)\, S_+(p_2')\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, R_+(p_2)\, \xi(z'')\, \Delta_{2+}(k')$$

$$\times\, \overline{\xi}(z')\, R_+(p_1)\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, S_+(p_1')\, \eta(z)$$

$$= \frac{\mathrm{i}}{2} q_0^2 \int \frac{(\mathrm{d}p_1')}{(2\pi)^4} \frac{(\mathrm{d}p_2')}{(2\pi)^4} \frac{(\mathrm{d}p_1)}{(2\pi)^4} \frac{(\mathrm{d}p_2)}{(2\pi)^4} \frac{(\mathrm{d}k)}{(2\pi)^4} (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}z'')$$

$$\times\, \mathrm{e}^{\mathrm{i}(p_1'-p_1-k)x'}\, \mathrm{e}^{\mathrm{i}(p_2-p_2'+k)x}$$

$$\times\, \Big[\mathrm{e}^{\mathrm{i}p_2'y}\, \overline{\eta}(y)\Big] S_+(p_2')\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, R_+(p_2) \Big[\mathrm{e}^{-\mathrm{i}p_2z''}\, \xi(z'')\Big] \Delta_{1+}(k)$$

$$\times\, \Big[\mathrm{e}^{\mathrm{i}p_1z'}\, \overline{\xi}(z')\Big] R_+(p_1)\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, S_+(p_1') \Big[\mathrm{e}^{-\mathrm{i}p_1'z}\, \eta(z)\Big]$$

$$+\, \frac{\mathrm{i}}{2} q_0^2 \int \frac{(\mathrm{d}p_1')}{(2\pi)^4} \frac{(\mathrm{d}p_2')}{(2\pi)^4} \frac{(\mathrm{d}p_1)}{(2\pi)^4} \frac{(\mathrm{d}p_2)}{(2\pi)^4} \frac{(\mathrm{d}k')}{(2\pi)^4} (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}z'')$$

$$\times\, \mathrm{e}^{\mathrm{i}(p_1'-p_1+k')x'}\, \mathrm{e}^{\mathrm{i}(p_2-p_2'-k')x}$$

$$\times\, \Big[\mathrm{e}^{\mathrm{i}p_2'y}\, \overline{\eta}(y)\Big] S_+(p_2')\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, R_+(p_2) \Big[\mathrm{e}^{-\mathrm{i}p_2z''}\, \xi(z'')\Big] \Delta_{2+}(k')$$

$$\times\, \Big[\mathrm{e}^{\mathrm{i}p_1z'}\, \overline{\xi}(z')\Big] R_+(p_1)\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, S_+(p_1') \Big[\mathrm{e}^{-\mathrm{i}p_1'z}\, \eta(z)\Big]$$

$$= \frac{\mathrm{i}}{2} q_0^2 \int \frac{(\mathrm{d}p_1')}{(2\pi)^4} \frac{(\mathrm{d}p_2')}{(2\pi)^4} \frac{(\mathrm{d}p_1)}{(2\pi)^4} \frac{(\mathrm{d}p_2)}{(2\pi)^4} \frac{(\mathrm{d}k)}{(2\pi)^4}$$

$$\times\, (2\pi)^4\, \delta^4(p_1' - p_1 - k)\, (2\pi)^4\, \delta^4(p_2 - p_2' + k)$$

$$\times\, \overline{\eta}(p_2')\, S_+(p_2')\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, R_+(p_2)\, \xi(p_2)\, \Delta_{1+}(k)$$

$$\times\,\overline{\xi}(p_1)\,R_+(p_1)\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,S_+(p_1')\,\eta(p_1')$$

$$+\,\frac{\mathrm{i}}{2}q_0^2\int\frac{(\mathrm{d}p_1')}{(2\pi)^4}\frac{(\mathrm{d}p_2')}{(2\pi)^4}\frac{(\mathrm{d}p_1)}{(2\pi)^4}\frac{(\mathrm{d}p_2)}{(2\pi)^4}\frac{(\mathrm{d}k')}{(2\pi)^4}$$

$$\times\,(2\pi)^4\,\delta^4(p_1'-p_1+k')\,(2\pi)^4\,\delta^4(p_2-p_2'-k')$$

$$\times\,\overline{\eta}(p_2')\,S_+(p_2')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(p_2)\,\xi(p_2)\,\Delta_{2+}(k')$$

$$\times\,\overline{\xi}(p_1)\,R_+(p_1)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,S_+(p_1')\,\eta(p_1')$$

$$=\frac{\mathrm{i}}{2}q_0^2\int\frac{(\mathrm{d}p_1')}{(2\pi)^4}\frac{(\mathrm{d}p_2')}{(2\pi)^4}\frac{(\mathrm{d}p_1)}{(2\pi)^4}\frac{(\mathrm{d}p_2)}{(2\pi)^4}\,(2\pi)^4\,\delta^4(p_2-p_2'+p_1'-p_1)$$

$$\times\,\overline{\eta}(p_2')\,S_+(p_2')\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,R_+(p_2)\,\xi(p_2)\,\Delta_{1+}(p_1'-p_1)$$

$$\times\,\overline{\xi}(p_1)\,R_+(p_1)\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,S_+(p_1')\,\eta(p_1')$$

$$+\,\frac{\mathrm{i}}{2}q_0^2\int\frac{(\mathrm{d}p_1')}{(2\pi)^4}\frac{(\mathrm{d}p_2')}{(2\pi)^4}\frac{(\mathrm{d}p_1)}{(2\pi)^4}\frac{(\mathrm{d}p_2)}{(2\pi)^4}\,(2\pi)^4\,\delta^4(p_1'-p_1+p_2-p_2')$$

$$\times\,\overline{\eta}(p_2')\,S_+(p_2')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(p_2)\,\xi(p_2)\,\Delta_{2+}(p_2-p_2')$$

$$\times\,\overline{\xi}(p_1)\,R_+(p_1)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,S_+(p_1')\,\eta(p_1')\,. \tag{6.1.11}$$

Therefore

$$\mathcal{M}=\frac{\mathrm{i}}{2}q_0^2\int\frac{(\mathrm{d}p_1')}{(2\pi)^4}\frac{(\mathrm{d}p_2')}{(2\pi)^4}\frac{(\mathrm{d}p_1)}{(2\pi)^4}\frac{(\mathrm{d}p_2)}{(2\pi)^4}\,(2\pi)^4\,\delta^4(p_2-p_2'+p_1'-p_1)$$

$$\times\,\Big[\overline{\eta}(p_2')\,S_+(p_2')\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,R_+(p_2)\,\xi(p_2)\Big]\Delta_{1+}(p_1'-p_1)$$

$$\times\,\Big[\overline{\xi}(p_1)\,R_+(p_1)\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,S_+(p_1')\,\eta(p_1')\Big]$$

$$+\,\frac{\mathrm{i}}{2}q_0^2\int\frac{(\mathrm{d}p_1')}{(2\pi)^4}\frac{(\mathrm{d}p_2')}{(2\pi)^4}\frac{(\mathrm{d}p_1)}{(2\pi)^4}\frac{(\mathrm{d}p_2)}{(2\pi)^4}\,(2\pi)^4\,\delta^4(p_1'-p_1+p_2-p_2')$$

$$\times\,\Big[\overline{\eta}(p_2')\,S_+(p_2')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(p_2)\,\xi(p_2)\Big]\Delta_{2+}(p_2-p_2')$$

$$\times \left[\bar{\xi}(p_1)\, R_+(p_1)\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, S_+(p_1')\, \eta(p_1') \right] . \tag{6.1.12}$$

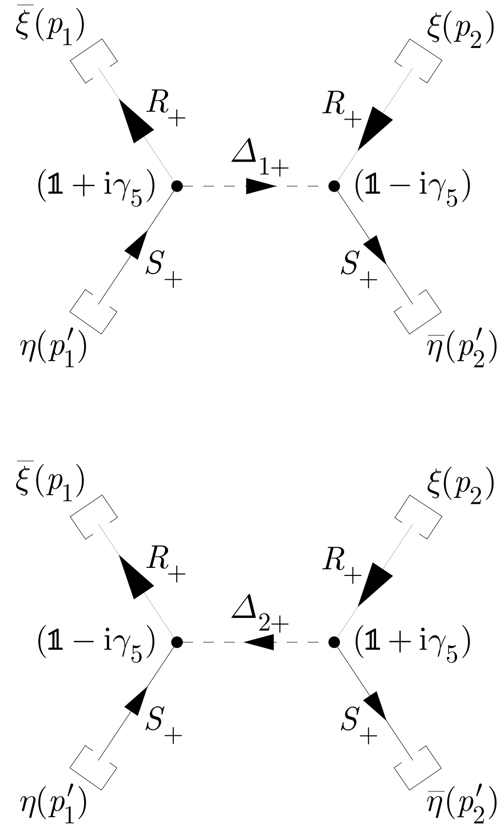
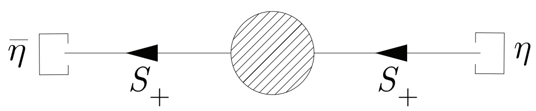


Figure 6.3 Two diagrams contributing to the $\mathrm{e}^-\mathrm{e}^+ \to \tilde{\gamma}\tilde{\gamma}$ scattering (in momentum description).

6.2 Self-Energy of the Electron

We are looking for the amplitude of electron self-energy:



We may then effectively consider the following replacement for $\langle 0_+ \,|\, 0_- \rangle$:

$\langle 0_+ \,|\, 0_- \rangle$

$$\to \exp\left\{ \mathrm{i}q_0 \left[\frac{\delta}{\mathrm{i}\delta\eta(x)}\,\gamma^\mu\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,\frac{\delta}{\mathrm{i}\delta J^\mu(x)} \right.\right.$$

$$+\frac{1}{\sqrt{2}}\,\frac{\delta}{\mathrm{i}\delta\xi(x)}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,\frac{\delta}{\mathrm{i}\delta K_1(x)}$$

$$-\frac{1}{\sqrt{2}}\,\frac{\delta}{\mathrm{i}\delta\eta(x)}\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\xi}(x)}\,\frac{\delta}{\mathrm{i}\delta K_1^\dagger(x)}$$

$$+\frac{1}{\sqrt{2}}\,\frac{\delta}{\mathrm{i}\delta\xi(x)}\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,\frac{\delta}{\mathrm{i}\delta K_2^\dagger(x)}$$

$$\left.\left.-\frac{1}{\sqrt{2}}\,\frac{\delta}{\mathrm{i}\delta\eta(x)}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\xi}(x)}\,\frac{\delta}{\mathrm{i}\delta K_2(x)}\right]\right\}\langle 0_+\,|\,0_-\rangle_0\,, \qquad (6.2.1)$$

and set $K_1(x) = 0$, $K_2(x) = 0$, $\xi(x) = 0$, $\overline{\xi}(x) = 0$ and $J^\mu(x) = 0$ after doing the differentiations.

6.2.1 The Coulomb Gauge

For the process just discussed, the vacuum-to-vacuum transition amplitude becomes effectively replaced through the following steps:

$$\langle 0_+\,|\,0_-\rangle$$

$$\to \exp\left\{ \mathrm{i}q_0^2\left(+\frac{1}{2}\right)\left[\frac{\delta}{\mathrm{i}\delta\eta(x)}\,\gamma^\mu\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\frac{\delta}{\mathrm{i}\delta\eta(x')}\,\gamma^\nu\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x')}\right]\right.$$

$$+\,\mathrm{i}q_0^2\left(-\frac{1}{2}\right)\left[\frac{\delta}{\mathrm{i}\delta\eta(x)}\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\xi}(x)}\,\varDelta_{1+}(x,x')\right.$$

$$\left.\times\frac{\delta}{\mathrm{i}\delta\xi(x')}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x')}\right]$$

$$+\,\mathrm{i}q_0^2\left(-\frac{1}{2}\right)\left[\frac{\delta}{\mathrm{i}\delta\xi(x)}\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,\varDelta_{2+}(x,x')\right.$$

$$\left.\left.\times\frac{\delta}{\mathrm{i}\delta\eta(x')}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\xi}(x')}\right]\right\}\langle 0_+\,|\,0_-\rangle_0$$

$$= \exp\left\{ \mathrm{i}\,\frac{q_0^2}{2}\left[\frac{\delta}{\mathrm{i}\delta\eta(x)}\,\gamma^{\mu}\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\frac{\delta}{\mathrm{i}\delta\eta(x')}\,\gamma^{\nu}\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x')}\right]\right.$$

$$- \mathrm{i}\,\frac{q_0^2}{2}\left[\frac{\delta}{\mathrm{i}\delta\eta(x)}\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\bar{\xi}(x)}\,\Delta_{1+}(x,x')\right.$$

$$\left.\times\frac{\delta}{\mathrm{i}\delta\xi(x')}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x')}\right]$$

$$- \mathrm{i}\,\frac{q_0^2}{2}\left[\frac{\delta}{\mathrm{i}\delta\xi(x)}\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,\Delta_{2+}(x,x')\right.$$

$$\left.\left.\times\frac{\delta}{\mathrm{i}\delta\eta(x')}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\bar{\xi}(x')}\right]\right\}\langle 0_+\,|\,0_-\rangle_0$$

$$\equiv \boxed{\mathrm{I}} + \boxed{\mathrm{II}} + \boxed{\mathrm{III}}\,, \tag{6.2.2}$$

where

$$\boxed{\mathrm{I}} \equiv \exp\left\{ \mathrm{i}\,\frac{q_0^2}{2}\left[\frac{\delta}{\mathrm{i}\delta\eta(x)}\,\gamma^{\mu}\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\right.\right.$$

$$\left.\left.\times\frac{\delta}{\mathrm{i}\delta\eta(x')}\,\gamma^{\nu}\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x')}\right]\right\}\langle 0_+\,|\,0_-\rangle_0\ , \tag{6.2.3}$$

$$\boxed{\mathrm{II}} \equiv \exp\left\{ -\mathrm{i}\,\frac{q_0^2}{2}\left[\frac{\delta}{\mathrm{i}\delta\eta(x)}\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\bar{\xi}(x)}\,\Delta_{1+}(x,x')\right.\right.$$

$$\left.\left.\times\frac{\delta}{\mathrm{i}\delta\xi(x')}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x')}\right]\right\}\langle 0_+\,|\,0_-\rangle_0\ , \tag{6.2.4}$$

$$\boxed{\mathrm{III}} \equiv \exp\left\{ -\mathrm{i}\,\frac{q_0^2}{2}\left[\frac{\delta}{\mathrm{i}\delta\xi(x)}\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\,\Delta_{2+}(x,x')\right.\right.$$

$$\left.\left.\times\frac{\delta}{\mathrm{i}\delta\eta(x')}\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\frac{\delta}{\mathrm{i}\delta\bar{\xi}(x')}\right]\right\}\langle 0_+\,|\,0_-\rangle_0\ . \tag{6.2.5}$$

Consider the first term (6.2.3):

$$\boxed{\mathrm{I}} = \exp\left\{ \mathrm{i}\,\frac{q_0^2}{2}\,\left(\gamma^\mu\right)_{\mathsf{AB}}\left(\gamma^\nu\right)_{\mathsf{CD}}\,D^{\mathrm{C}+}_{\mu\nu}(x,x') \right.$$

$$\left. \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)}\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{D}}(x')} \right\} \langle 0_+ \,|\, 0_- \rangle_0$$

$$= \exp\left\{ \mathrm{i}\,\frac{q_0^2}{2}\,\left(\gamma^\mu\right)_{\mathsf{AB}}\left(\gamma^\nu\right)_{\mathsf{CD}}\,D^{\mathrm{C}+}_{\mu\nu}(x,x') \right.$$

$$\left. \times \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)}\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{D}}(x')} \right\} \langle 0_+ \,|\, 0_- \rangle_0$$

$$\rightarrow \mathrm{i}\,\frac{q_0^2}{2}\,\left(\gamma^\mu\right)_{\mathsf{AB}}\left(\gamma^\nu\right)_{\mathsf{CD}}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')$$

$$\times \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)}\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{D}}(x')}\,\langle 0_+ \,|\, 0_- \rangle_0$$

$$= \mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\left(\gamma^\mu\right)_{\mathsf{AB}}\left(\gamma^\nu\right)_{\mathsf{CD}}$$

$$\times \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)}\,\left(S_+(x',y)\right)_{\mathsf{DE}}\,\eta_{\mathsf{E}}(y)\,\langle 0_+ \,|\, 0_- \rangle_0$$

$$= \mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\left(\gamma^\mu\right)_{\mathsf{AB}}\left(\gamma^\nu\right)_{\mathsf{CD}}\left(S_+(x',y)\right)_{\mathsf{DE}}$$

$$\times \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)}\,\eta_{\mathsf{E}}(y)\,\langle 0_+ \,|\, 0_- \rangle_0$$

$$= \mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\left(\gamma^\mu\right)_{\mathsf{AB}}\left(\gamma^\nu S_+(x',y)\right)_{\mathsf{CE}}$$

$$\times \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\left[-\mathrm{i}\delta_{\mathsf{AE}}\,\delta^4(x-y) - \eta_{\mathsf{E}}(y)\,\frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)}\right]\langle 0_+ \,|\, 0_- \rangle_0$$

$$= \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(\gamma^\mu\big)_{\mathsf{AB}}\, \big(\gamma^\nu S_+(x',x)\big)_{\mathsf{CA}}\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\, \langle 0_+\,|\,0_-\rangle_0$$

$$- \mathrm{i}\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(\gamma^\mu\big)_{\mathsf{AB}}\, \big(\gamma^\nu S_+(x',y)\big)_{\mathsf{CE}}$$

$$\times\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\, \left[\eta_{\mathsf{E}}(y)\, \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)}\right] \langle 0_+\,|\,0_-\rangle_0$$

$$= -\frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(\gamma^\nu S_+(x',x)\gamma^\mu\big)_{\mathsf{CB}}$$

$$\times\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \overline{\eta}_{\mathsf{F}}(y)\big(S_+(y,x')\big)_{\mathsf{FC}}\, \langle 0_+\,|\,0_-\rangle_0$$

$$+ \mathrm{i}\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(\gamma^\mu\big)_{\mathsf{AB}}\, \big(\gamma^\nu S_+(x',y)\big)_{\mathsf{CE}}$$

$$\times\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\, \left[\eta_{\mathsf{E}}(y)\, \overline{\eta}_{\mathsf{F}}(y')\big(S_+(y',x)\big)_{\mathsf{FA}}\right] \langle 0_+\,|\,0_-\rangle_0$$

$$= -\frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(\gamma^\nu S_+(x',x)\gamma^\mu\big)_{\mathsf{CB}}\, \big(S_+(y,x')\big)_{\mathsf{FC}}$$

$$\times\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \overline{\eta}_{\mathsf{F}}(y)\, \langle 0_+\,|\,0_-\rangle_0$$

$$+ \mathrm{i}\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(\gamma^\mu\big)_{\mathsf{AB}}\, \big(\gamma^\nu S_+(x',y)\big)_{\mathsf{CE}}\, \big(S_+(y',x)\big)_{\mathsf{FA}}$$

$$\times\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\, \eta_{\mathsf{E}}(y)\, \overline{\eta}_{\mathsf{F}}(y')\, \langle 0_+\,|\,0_-\rangle_0$$

$$= -\frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(S_+(y,x')\gamma^\nu S_+(x',x)\gamma^\mu\big)_{\mathsf{FB}}$$

$$\times\, \left[-\mathrm{i}\delta_{\mathsf{BF}}\, \delta^4(x-y) - \overline{\eta}_{\mathsf{F}}(y)\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\right] \langle 0_+\,|\,0_-\rangle_0$$

$$+ \mathrm{i}\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(S_+(y',x)\,\gamma^\mu\big)_{\mathsf{FB}}\, \big(\gamma^\nu S_+(x',y)\big)_{\mathsf{CE}}$$

$$\times \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)} \left[-\mathrm{i}\delta_{\mathsf{CE}}\, \delta^4(x'-y) - \eta_{\mathsf{E}}(y)\, \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')} \right] \overline{\eta}_{\mathsf{F}}(y')\, \langle 0_+ \,|\, 0_- \rangle_0$$

$$= \mathrm{i}\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[S_+(x,x')\, \gamma^\nu S_+(x',x)\, \gamma^\mu \big]\, \langle 0_+ \,|\, 0_- \rangle_0$$

$$+ \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x') \big(S_+(y,x')\, \gamma^\nu S_+(x',x)\, \gamma^\mu \big)_{\mathsf{FB}}$$

$$\times \overline{\eta}_{\mathsf{F}}(y) \big(S_+(x,y') \big)_{\mathsf{BA}}\, \eta_{\mathsf{A}}(y')\, \langle 0_+ \,|\, 0_- \rangle_0$$

$$+ \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x') \big(S_+(y',x)\, \gamma^\mu \big)_{\mathsf{FB}}\ \mathrm{Tr}\big[\gamma^\nu S_+(x',x') \big]$$

$$\times \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \overline{\eta}_{\mathsf{F}}(y')\, \langle 0_+ \,|\, 0_- \rangle_0$$

$$+ \mathrm{i}\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x') \big(S_+(y',x)\, \gamma^\mu \big)_{\mathsf{FB}} \big(\gamma^\nu S_+(x',y) \big)_{\mathsf{CE}}$$

$$\times \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \eta_{\mathsf{E}}(y)\, \overline{\eta}_{\mathsf{F}}(y')\, \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')}\, \langle 0_+ \,|\, 0_- \rangle_0$$

$$= \mathrm{i}\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[S_+(x,x')\, \gamma^\nu S_+(x',x)\, \gamma^\mu \big]\, \langle 0_+ \,|\, 0_- \rangle_0$$

$$+ \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \overline{\eta}(y)\, S_+(y,x')\, \gamma^\nu S_+(x',x)$$

$$\times \gamma^\mu S_+(x,y')\, \eta(y')\, \langle 0_+ \,|\, 0_- \rangle_0$$

$$+ \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x') \big(S_+(y',x)\, \gamma^\mu \big)_{\mathsf{FB}}\ \mathrm{Tr}\big[\gamma^\nu S_+(x',x') \big]$$

$$\times \left[-\mathrm{i}\delta_{\mathsf{BF}}\, \delta^4(x-y') - \overline{\eta}_{\mathsf{F}}(y')\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)} \right] \langle 0_+ \,|\, 0_- \rangle_0$$

$$- \mathrm{i}\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x') \big(S_+(y',x)\, \gamma^\mu \big)_{\mathsf{FB}} \big(\gamma^\nu S_+(x',y) \big)_{\mathsf{CE}}$$

$$\times\ \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\eta_{\mathsf{E}}(y)\,\overline{\eta}_{\mathsf{F}}(y')\,\overline{\eta}_{\mathsf{D}}(z)\big(S_{+}(z,x')\big)_{\mathsf{DC}}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$= \mathrm{i}\,\frac{q_0^2}{2}\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\ \mathrm{Tr}\big[S_{+}(x,x')\,\gamma^{\nu}S_{+}(x',x)\,\gamma^{\mu}\big]\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$+\ \frac{q_0^2}{2}\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\,\overline{\eta}(y)\,S_{+}(y,x')\,\gamma^{\nu}S_{+}(x',x)$$

$$\times\ \gamma^{\mu}S_{+}(x,y')\,\eta(y')\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$-\ \mathrm{i}\,\frac{q_0^2}{2}\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\ \mathrm{Tr}\big[S_{+}(x,x)\,\gamma^{\mu}\big]\ \mathrm{Tr}\big[\gamma^{\nu}S_{+}(x',x')\big]\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$-\ \frac{q_0^2}{2}\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\,\big(S_{+}(y',x)\,\gamma^{\mu}\big)_{\mathsf{FB}}\ \mathrm{Tr}\big[\gamma^{\nu}S_{+}(x',x')\big]$$

$$\times\ \overline{\eta}_{\mathsf{F}}(y')\,\big(S_{+}(x,y)\big)_{\mathsf{BA}}\,\eta_{\mathsf{A}}(y)\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$+\ \mathrm{i}\,\frac{q_0^2}{2}\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\,\big(S_{+}(y',x)\,\gamma^{\mu}\big)_{\mathsf{FB}}\,\big(S_{+}(z,x')\,\gamma^{\nu}S_{+}(x',y)\big)_{\mathsf{DE}}$$

$$\times\ \eta_{\mathsf{E}}(y)\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\overline{\eta}_{\mathsf{F}}(y')\,\overline{\eta}_{\mathsf{D}}(z)\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$= \mathrm{i}\,\frac{q_0^2}{2}\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\ \mathrm{Tr}\big[S_{+}(x,x')\,\gamma^{\nu}S_{+}(x',x)\,\gamma^{\mu}\big]\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$+\ \frac{q_0^2}{2}\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\,\overline{\eta}(y)\,S_{+}(y,x')\,\gamma^{\nu}S_{+}(x',x)$$

$$\times\ \gamma^{\mu}S_{+}(x,y')\,\eta(y')\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$-\ \mathrm{i}\,\frac{q_0^2}{2}\ \mathrm{Tr}\big[S_{+}(x,x)\,\gamma^{\mu}\big]\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\ \mathrm{Tr}\big[\gamma^{\nu}S_{+}(x',x')\big]\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$-\ \frac{q_0^2}{2}\,D_{\mu\nu}^{\mathrm{C}+}(x,x')\ \mathrm{Tr}\big[\gamma^{\nu}S_{+}(x',x')\big]$$

$$\times\ \overline{\eta}(y')\,S_{+}(y',x)\,\gamma^{\mu}\,S_{+}(x,y)\,\eta(y)\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$+\,\mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\big(S_+(y',x)\,\gamma^\mu\big)_{\mathsf{FB}}\,\big(S_+(z,x')\,\gamma^\nu S_+(x',y)\big)_{\mathsf{DE}}$$

$$\times\,\eta_{\mathsf{E}}(y)\,\left[-\mathrm{i}\delta_{\mathsf{BF}}\,\delta^4(x-y')-\overline{\eta}_{\mathsf{F}}(y')\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\right]\overline{\eta}_{\mathsf{D}}(z)\,\langle 0_+\,|\,0_-\rangle_0$$

$$=\mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[S_+(x,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu\big]\,\langle 0_+\,|\,0_-\rangle_0$$

$$+\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\overline{\eta}(y)\,S_+(y,x')\,\gamma^\nu S_+(x',x)$$

$$\times\,\gamma^\mu S_+(x,y')\,\eta(y')\,\langle 0_+\,|\,0_-\rangle_0$$

$$-\,\mathrm{i}\,\frac{q_0^2}{2}\;\mathrm{Tr}\big[S_+(x,x)\,\gamma^\mu\big]\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]\,\langle 0_+\,|\,0_-\rangle_0$$

$$-\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]$$

$$\times\,\overline{\eta}(y')\,S_+(y',x)\,\gamma^\mu\,S_+(x,y)\,\eta(y)\,\langle 0_+\,|\,0_-\rangle_0$$

$$+\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[S_+(x,x)\,\gamma^\mu\big]\,\big(S_+(z,x')\,\gamma^\nu S_+(x',y)\big)_{\mathsf{DE}}$$

$$\times\,\eta_{\mathsf{E}}(y)\,\overline{\eta}_{\mathsf{D}}(z)\,\langle 0_+\,|\,0_-\rangle_0$$

$$-\,\mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\big(S_+(y',x)\,\gamma^\mu\big)_{\mathsf{FB}}\,\big(S_+(z,x')\,\gamma^\nu S_+(x',y)\big)_{\mathsf{DE}}$$

$$\times\,\eta_{\mathsf{E}}(y)\,\overline{\eta}_{\mathsf{F}}(y')\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\,\overline{\eta}_{\mathsf{D}}(z)\,\langle 0_+\,|\,0_-\rangle_0$$

$$=\mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[S_+(x,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu\big]\,\langle 0_+\,|\,0_-\rangle_0$$

$$+\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\overline{\eta}(y)\,S_+(y,x')\,\gamma^\nu S_+(x',x)$$

$$\times\,\gamma^\mu S_+(x,y')\,\eta(y')\,\langle 0_+\,|\,0_-\rangle_0$$

$$- \mathrm{i}\,\frac{q_0^2}{2}\;\mathrm{Tr}\big[S_+(x,x)\,\gamma^\mu\big]\;D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]\,\langle 0_+\,|\,0_-\rangle_0$$

$$- \frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]$$

$$\times\,\overline{\eta}(y')\,S_+(y',x)\,\gamma^\mu\,S_+(x,y)\,\eta(y)\,\langle 0_+\,|\,0_-\rangle_0$$

$$- \frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[S_+(x,x)\,\gamma^\mu\big]$$

$$\times\,\overline{\eta}_{\mathsf{D}}(z)\,\big(S_+(z,x')\,\gamma^\nu S_+(x',y)\big)_{\mathsf{DE}}\,\eta_{\mathsf{E}}(y)\,\langle 0_+\,|\,0_-\rangle_0$$

$$- \mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\big(S_+(y',x)\,\gamma^\mu\big)_{\mathsf{FB}}\,\big(S_+(z,x')\,\gamma^\nu S_+(x',y)\big)_{\mathsf{DE}}$$

$$\times\,\eta_{\mathsf{E}}(y)\,\overline{\eta}_{\mathsf{F}}(y')\left[-\mathrm{i}\delta_{\mathsf{BD}}\,\delta^4(x-z)-\overline{\eta}_{\mathsf{D}}(z)\,\frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\right]\langle 0_+\,|\,0_-\rangle_0$$

$$= \mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[S_+(x,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu\big]\,\langle 0_+\,|\,0_-\rangle_0$$

$$+ \frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\overline{\eta}(y)\,S_+(y,x')\,\gamma^\nu S_+(x',x)$$

$$\times\,\gamma^\mu S_+(x,y')\,\eta(y')\,\langle 0_+\,|\,0_-\rangle_0$$

$$- \mathrm{i}\,\frac{q_0^2}{2}\;\mathrm{Tr}\big[S_+(x,x)\,\gamma^\mu\big]\;D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]\,\langle 0_+\,|\,0_-\rangle_0$$

$$- \frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]$$

$$\times\,\overline{\eta}(y')\,S_+(y',x)\,\gamma^\mu\,S_+(x,y)\,\eta(y)\,\langle 0_+\,|\,0_-\rangle_0$$

$$- \frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\;\mathrm{Tr}\big[S_+(x,x)\,\gamma^\mu\big]$$

$$\times\,\overline{\eta}(z)\,S_+(z,x')\,\gamma^\nu S_+(x',y)\,\eta(y)\,\langle 0_+\,|\,0_-\rangle_0$$

$$- \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(S_+(y',x)\,\gamma^\mu\big)_{\mathsf{FD}}\, \big(S_+(x,x')\,\gamma^\nu S_+(x',y)\big)_{\mathsf{DE}}$$

$$\times\, \eta_{\mathsf{E}}(y)\, \overline{\eta}_{\mathsf{F}}(y')\, \langle 0_+\,|\,0_-\rangle_0$$

$$+\,\mathrm{i}\,\frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(S_+(y',x)\,\gamma^\mu\big)_{\mathsf{FB}}\, \big(S_+(z,x')\,\gamma^\nu S_+(x',y)\big)_{\mathsf{DE}}$$

$$\times\, \eta_{\mathsf{E}}(y)\, \overline{\eta}_{\mathsf{F}}(y')\, \overline{\eta}_{\mathsf{D}}(z)\, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)}\, \langle 0_+\,|\,0_-\rangle_0$$

$$= \mathrm{i}\,\frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[S_+(x,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu\big]\, \langle 0_+\,|\,0_-\rangle_0$$

$$+\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \overline{\eta}(y)\, S_+(y,x')\,\gamma^\nu S_+(x',x)$$

$$\times\, \gamma^\mu S_+(x,y')\, \eta(y')\, \langle 0_+\,|\,0_-\rangle_0$$

$$-\,\mathrm{i}\,\frac{q_0^2}{2}\ \mathrm{Tr}\big[S_+(x,x)\,\gamma^\mu\big]\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]\, \langle 0_+\,|\,0_-\rangle_0$$

$$-\, q_0^2\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]$$

$$\times\, \overline{\eta}(y')\, S_+(y',x)\,\gamma^\mu\, S_+(x,y)\,\eta(y)\, \langle 0_+\,|\,0_-\rangle_0$$

$$+\, \frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \overline{\eta}(y')\, S_+(y',x)\,\gamma^\mu S_+(x,x')$$

$$\times\, \gamma^\nu S_+(x',y)\, \eta(y)\, \langle 0_+\,|\,0_-\rangle_0$$

$$+\,\mathrm{i}\,\frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\, \big(S_+(y',x)\,\gamma^\mu\big)_{\mathsf{FB}}\, \big(S_+(z,x')\,\gamma^\nu S_+(x',y)\big)_{\mathsf{DE}}$$

$$\times\, \eta_{\mathsf{E}}(y)\, \overline{\eta}_{\mathsf{F}}(y')\, \overline{\eta}_{\mathsf{D}}(z)\, \big(S_+(x,z')\big)_{\mathsf{BA}}\, \eta_{\mathsf{A}}(z')\, \langle 0_+\,|\,0_-\rangle_0$$

$$= \mathrm{i}\,\frac{q_0^2}{2}\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[S_+(x,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu\big]\, \langle 0_+\,|\,0_-\rangle_0$$

$$+ \, q_0^2 \, D^{\rm C+}_{\mu\nu}(x,x') \, \overline{\eta}(y) \, S_+(y,x') \, \gamma^\nu S_+(x',x)$$

$$\times \, \gamma^\mu S_+(x,y') \, \eta(y') \, \langle 0_+ \, | \, 0_- \rangle_0$$

$$- \, {\rm i} \, \frac{q_0^2}{2} \, {\rm Tr}\big[S_+(x,x) \, \gamma^\mu\big] \, D^{\rm C+}_{\mu\nu}(x,x') \, {\rm Tr}\big[\gamma^\nu S_+(x',x')\big] \, \langle 0_+ \, | \, 0_- \rangle_0$$

$$- \, q_0^2 \, D^{\rm C+}_{\mu\nu}(x,x') \, {\rm Tr}\big[\gamma^\nu S_+(x',x')\big]$$

$$\times \, \overline{\eta}(y') \, S_+(y',x) \, \gamma^\mu \, S_+(x,y) \, \eta(y) \, \langle 0_+ \, | \, 0_- \rangle_0$$

$$+ \, {\rm i} \, \frac{q_0^2}{2} \Big[\overline{\eta}(y') \, S_+(y',x) \, \gamma^\mu S_+(x,z') \, \eta(z')\Big] \, D^{\rm C+}_{\mu\nu}(x,x')$$

$$\times \, \Big[\overline{\eta}(z) \, S_+(z,x') \, \gamma^\nu S_+(x',y) \, \eta(y)\Big] \, \langle 0_+ \, | \, 0_- \rangle_0 \; ,$$

we obtain

$$\boxed{\rm I} \rightarrow \bigg\{ {\rm i} \, \frac{q_0^2}{2} \, D^{\rm C+}_{\mu\nu}(x,x') \, {\rm Tr}\big[S_+(x,x') \, \gamma^\nu S_+(x',x) \, \gamma^\mu\big]$$

$$+ \, q_0^2 \, D^{\rm C+}_{\mu\nu}(x,x') \, \overline{\eta}(y) \, S_+(y,x') \, \gamma^\nu S_+(x',x) \, \gamma^\mu S_+(x,y') \, \eta(y')$$

$$- \, {\rm i} \, \frac{q_0^2}{2} \, {\rm Tr}\big[S_+(x,x) \, \gamma^\mu\big] \, D^{\rm C+}_{\mu\nu}(x,x') \, {\rm Tr}\big[\gamma^\nu S_+(x',x')\big]$$

$$- \, q_0^2 \, D^{\rm C+}_{\mu\nu}(x,x') \, {\rm Tr}\big[\gamma^\nu S_+(x',x')\big] \, \overline{\eta}(y') \, S_+(y',x) \, \gamma^\mu \, S_+(x,y) \, \eta(y)$$

$$+ \, {\rm i} \, \frac{q_0^2}{2} \Big[\overline{\eta}(y') \, S_+(y',x) \, \gamma^\mu S_+(x,z') \, \eta(z')\Big] \, D^{\rm C+}_{\mu\nu}(x,x')$$

$$\times \, \Big[\overline{\eta}(z) \, S_+(z,x') \, \gamma^\nu S_+(x',y) \, \eta(y)\Big] \bigg\} \, \langle 0_+ \, | \, 0_- \rangle_0 \; , \tag{6.2.6}$$

or

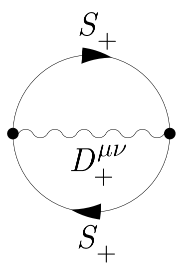
$$\boxed{\rm I} \rightarrow \Big[\mathcal{M}^{\rm I}_1 + \mathcal{M}^{\rm I}_2 + \mathcal{M}^{\rm I}_3 + \mathcal{M}^{\rm I}_4 + \mathcal{M}^{\rm I}_5\Big] \, \langle 0_+ \, | \, 0_- \rangle_0 \; . \tag{6.2.7}$$

The first term $\boxed{\mathrm{I}}$ generates amplitude of 5 processes:

- Process 1:

$$\mathcal{M}_1^{\mathrm{I}} = \mathrm{i}\,\frac{q_0^2}{2}\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[S_+(x,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu\big]\,, \tag{6.2.8}$$

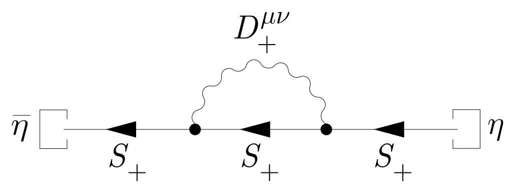
represented by the diagram



- Process 2:

$$\mathcal{M}_2^{\mathrm{I}} = q_0^2\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\overline{\eta}(y)\,S_+(y,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu S_+(x,y')\,\eta(y')\,, \tag{6.2.9}$$

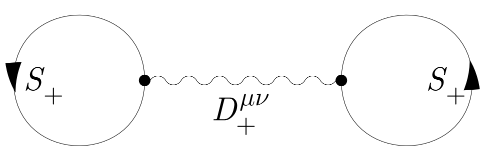
represented by the diagram



- Process 3:

$$\mathcal{M}_3^{\mathrm{I}} = -\mathrm{i}\,\frac{q_0^2}{2}\ \mathrm{Tr}\big[S_+(x,x)\,\gamma^\mu\big]\,D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]\,, \tag{6.2.10}$$

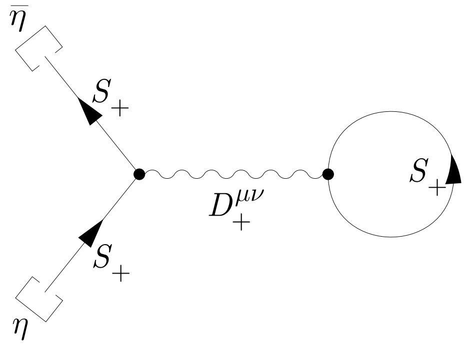
represented by the diagram



- Process 4:

$$\mathcal{M}^{\mathrm{I}}_4 = -q_0^2\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\ \mathrm{Tr}\big[\gamma^\nu S_+(x',x')\big]\,\overline{\eta}(y')\,S_+(y',x)\,\gamma^\mu\,S_+(x,y)\,\eta(y)\,, \quad (6.2.11)$$

represented by the diagram

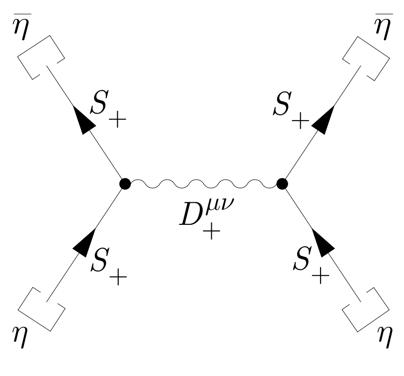


- Process 5:

$$\mathcal{M}^{\mathrm{I}}_5 = \mathrm{i}\,\frac{q_0^2}{2}\Big[\overline{\eta}(y')\,S_+(y',x)\,\gamma^\mu S_+(x,z')\,\eta(z')\Big]\,D^{\mathrm{C}+}_{\mu\nu}(x,x')$$

$$\times\Big[\overline{\eta}(z)\,S_+(z,x')\,\gamma^\nu S_+(x',y)\,\eta(y)\Big]\,, \quad (6.2.12)$$

represented by the diagram



Only the second process, eq. (6.2.9), contributes to the electron self-energy

$$\mathcal{M}^{\mathrm{I}} \equiv \mathcal{M}^{\mathrm{I}}_2 = q_0^2\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\overline{\eta}(y)\,S_+(y,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu S_+(x,y')\,\eta(y')\,,$$

or

$$\mathcal{M}^{\mathrm{I}} = q_0^2 \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}y')$$

$$\times\, D^{\mathrm{C}+}_{\mu\nu}(x,x')\,\overline{\eta}(y)\,S_+(y,x')\,\gamma^\nu S_+(x',x)\,\gamma^\mu S_+(x,y')\,\eta(y') \tag{6.2.13}$$

$$= q_0^2 \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}y')$$

$$\times \left[\int \frac{(\mathrm{d}q)}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}q(x-x')}\, D^{\mathrm{C}+}_{\mu\nu}(q)\right] \overline{\eta}(y) \left[\int \frac{(\mathrm{d}p)}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p(y-x')}\, S_+(p)\right] \gamma^\nu$$

$$\times \left[\int \frac{(\mathrm{d}p')}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p'(x'-x)}\, S_+(p')\right] \gamma^\mu \left[\int \frac{(\mathrm{d}p'')}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p''(x-y')}\, S_+(p'')\right] \eta(y')$$

$$= q_0^2 \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{(\mathrm{d}p)}{(2\pi)^4}\, \frac{(\mathrm{d}p')}{(2\pi)^4}\, \frac{(\mathrm{d}p'')}{(2\pi)^4}\, (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}y')$$

$$\times\, \mathrm{e}^{\mathrm{i}(q-p'+p'')x}\, \mathrm{e}^{\mathrm{i}(p'-p-q)x'}\, \mathrm{e}^{\mathrm{i}py}\, \mathrm{e}^{-\mathrm{i}p''y'}$$

$$\times\, D^{\mathrm{C}+}_{\mu\nu}(q)\,\overline{\eta}(y)\,S_+(p)\,\gamma^\nu S_+(p')\,\gamma^\mu S_+(p'')\,\eta(y')$$

$$= q_0^2 \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{(\mathrm{d}p)}{(2\pi)^4}\, \frac{(\mathrm{d}p')}{(2\pi)^4}\, \frac{(\mathrm{d}p'')}{(2\pi)^4}$$

$$\times\, (2\pi)^4\delta^4(q-p'+p'')\,(2\pi)^4\delta^4(p'-p-q)$$

$$\times\, D^{\mathrm{C}+}_{\mu\nu}(q)\,\overline{\eta}(p)\,S_+(p)\,\gamma^\nu S_+(p')\,\gamma^\mu S_+(p'')\,\eta(p'')\,,$$

therefore

$$\mathcal{M}^{\mathrm{I}} = q_0^2 \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{(\mathrm{d}p)}{(2\pi)^4}\, D^{\mathrm{C}+}_{\mu\nu}(q)\,\overline{\eta}(p)\,S_+(p)\,\gamma^\nu S_+(p+q)\,\gamma^\mu S_+(p)\,\eta(p)\,, \tag{6.2.14}$$

or

$$\mathcal{M}^{\mathrm{I}} = q_0^2 \int \frac{(\mathrm{d}p)}{(2\pi)^4}\, \overline{\eta}(p)\, S_+(p)\, \varSigma_{\mathrm{C}}^{\mathrm{I}}(p)\, S_+(p)\, \eta(p)\,, \tag{6.2.15}$$

where

$$\varSigma_{\mathrm{C}}^{\mathrm{I}}(p) \equiv \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, D_{\mu\nu}^{\mathrm{C}+}(q)\, \gamma^{\nu} S_+(p+q)\, \gamma^{\mu}\,, \tag{6.2.16}$$

and in the mass zero limit:

$$\varSigma_{\mathrm{C}}^{\mathrm{I}}(p)\Big|_{m_0=0} \equiv \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, D_{\mu\nu}^{\mathrm{C}+}(q)\, \gamma^{\nu}\, S_+(p+q)\Big|_{m_0=0}\, \gamma^{\mu}\,. \tag{6.2.17}$$

The free photon propagator in the Coulomb gauge is

$$D_{\mu\nu}^{\mathrm{C}+}(q) = \left(g_{\mu\alpha} - g_{\alpha i}\, \frac{q_i\, q_\mu}{\mathbf{q}^2}\right) g^{\alpha\beta} \left(g_{\beta\nu} - g_{\beta k}\, \frac{q_k\, q_\nu}{\mathbf{q}^2}\right) \frac{1}{q^2 - \mathrm{i}\epsilon}\,. \tag{6.2.18}$$

The free electron propagator is

$$S_+(p+q) = \frac{\left[-\gamma^{\alpha}(p+q)_{\alpha} + m_0\right]}{(p+q)^2 + m_0^2 - \mathrm{i}\epsilon}\,, \tag{6.2.19}$$

and in the mass zero limit

$$S_+(p+q)\Big|_{m_0=0} = -\gamma^{\alpha}\, \frac{(p+q)_{\alpha}}{(p+q)^2 - \mathrm{i}\epsilon}\,, \tag{6.2.20}$$

$$\begin{aligned}
\frac{\partial}{\partial p_\sigma}\, S_+(p+q)\Big|_{m_0=0} &= -\gamma^{\alpha}\, \frac{\partial}{\partial p_\sigma} \left[\frac{(p+q)_{\alpha}}{(p+q)^2 - \mathrm{i}\epsilon}\right] \\
&= -\gamma^{\alpha}\, \frac{\left[(p+q)^2 - \mathrm{i}\epsilon\right] \delta^{\sigma}{}_{\alpha} - 2(p+q)_{\alpha}\, (p+q)^{\sigma}}{\left[(p+q)^2 - \mathrm{i}\epsilon\right]^2} \\
&= -\frac{\left[(p+q)^2 - \mathrm{i}\epsilon\right] \gamma^{\sigma} - 2(p+q)^{\sigma}\, (p+q)_{\alpha}\, \gamma^{\alpha}}{\left[(p+q)^2 - \mathrm{i}\epsilon\right]^2}\,.
\end{aligned}$$

For very large photon momentum q:

$$\frac{\partial}{\partial p_\sigma}\, S_+(p+q)\Big|_{m_0=0} \xrightarrow[q\to\infty]{} -\frac{\big[q^2\,\gamma^\sigma - 2q^\sigma\, q_\alpha\,\gamma^\alpha\big]}{q^4}\,, \tag{6.2.21}$$

the contribution to the expression on the left-hand side of the following equation for $q^2 \gg p^2$ is effectively

$$\frac{\partial}{\partial p_\sigma}\, \Sigma^{\mathrm{I}}_{\mathrm{C}}(p)\Big|_{m_0=0} \longrightarrow -\int \frac{(\mathrm{d}q)}{(2\pi)^4}\, D^{\mathrm{C}+}_{\mu\nu}(q)\,\gamma^\nu\,\frac{\big[q^2\,\gamma^\sigma - 2q^\sigma\, q_\alpha\,\gamma^\alpha\big]}{q^4}\,\gamma^\mu\,, \tag{6.2.22}$$

and hence effectively

$$\gamma_\sigma\,\frac{\partial}{\partial p_\sigma}\, \Sigma^{\mathrm{I}}_{\mathrm{C}}(p)\Big|_{m_0=0}$$
$$\longrightarrow -\int\limits_{q^2\gg p^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\,\frac{1}{q^4}\, D^{\mathrm{C}+}_{\mu\nu}(q)\,\gamma_\sigma\,\gamma^\nu\big[q^2\,\gamma^\sigma - 2q^\sigma\, q_\alpha\,\gamma^\alpha\big]\,\gamma^\mu\,. \tag{6.2.23}$$

Using the identities

$$\gamma_\sigma\,\gamma^\nu\, q^2\,\gamma^\sigma\,\gamma^\mu = 2q^2\gamma^\nu\,\gamma^\mu\,,$$

$$-2\gamma^\sigma\,\gamma^\nu\, q_\sigma\, q_\alpha\,\gamma^\alpha\,\gamma^\mu = -2q^2\,\gamma^\nu\,\gamma^\mu + 4q^\nu\big(q_\alpha\,\gamma^\alpha\big)\gamma^\mu\,,$$

$$\gamma_\sigma\,\gamma^\nu\big[q^2\,\gamma^\sigma - 2q^\sigma\, q_\alpha\,\gamma^\alpha\big]\,\gamma^\mu = 4q^\nu\big(q_\alpha\,\gamma^\alpha\big)\gamma^\mu\,, \tag{6.2.24}$$

we effectively have for (6.2.23)

$$\gamma_\sigma\,\frac{\partial}{\partial p_\sigma}\, \Sigma^{\mathrm{I}}_{\mathrm{C}}(p)\Big|_{m_0=0} \longrightarrow -4\int\limits_{q^2\gg p^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\,\frac{1}{q^4}\, D^{\mathrm{C}+}_{\mu\nu}(q)\, q^\nu\big(q_\sigma\,\gamma^\sigma\big)\gamma^\mu\,. \tag{6.2.25}$$

From (6.2.18)

$$D^{\mathrm{C+}}_{\mu\nu}(q)\, q^{\nu} = \left(g_{\mu\alpha} - g_{\alpha i}\, \frac{q_i\, q_\mu}{\mathbf{q}^2} \right) g^{\alpha\beta} \left(q_\beta - g_{\beta k}\, \frac{q_k\, q^2}{\mathbf{q}^2} \right) \frac{1}{q^2 - \mathrm{i}\epsilon}\,,$$

$$\gamma^{\mu}\, D^{\mathrm{C+}}_{\mu\nu}(q)\, q^{\nu} = \left[\gamma_\alpha - g_{\alpha i}\, \frac{q_i \big(q_\mu\, \gamma^{\mu}\big)}{\mathbf{q}^2} \right] g^{\alpha\beta} \left[q_\beta - g_{\beta k}\, \frac{q_k\, q^2}{\mathbf{q}^2} \right] \frac{1}{q^2 - \mathrm{i}\epsilon}\,,$$

$$\begin{aligned}
\big(q_\sigma\, \gamma^\sigma\big) \gamma^{\mu}\, D^{\mathrm{C+}}_{\mu\nu}(q)\, q^{\nu} &= \left[\gamma_\alpha \big(q_\sigma\, \gamma^\sigma\big) + g_{\alpha i}\, \frac{q_i\, q^2}{\mathbf{q}^2} \right] g^{\alpha\beta} \left[q_\beta - g_{\beta k}\, \frac{q_k\, q^2}{\mathbf{q}^2} \right] \frac{1}{q^2 - \mathrm{i}\epsilon} \\
&= \left[\gamma_\alpha \big(q_\sigma\, \gamma^\sigma\big) + g_{\alpha i}\, \frac{q_i\, q^2}{\mathbf{q}^2} \right] \left[q^\alpha - \delta^\alpha{}_k\, \frac{q_k\, q^2}{\mathbf{q}^2} \right] \frac{1}{q^2 - \mathrm{i}\epsilon} \\
&= \left[-q^2 - \frac{q^2}{\mathbf{q}^2} \big(q_k\, \gamma^k\big) \big(q_\sigma\, \gamma^\sigma\big) + q^2 - \frac{q^4}{\mathbf{q}^2} \right] \frac{1}{q^2 - \mathrm{i}\epsilon} \\
&= -\frac{1}{\mathbf{q}^2} \left[\big(q_k\, \gamma^k\big) \big(q_\sigma\, \gamma^\sigma\big) + q^2 \right] \\
&= -\frac{1}{\mathbf{q}^2} \left[\big(q_k\, \gamma^k\big) - \big(q_\mu\, \gamma^\mu\big) \right] \big(q_\sigma\, \gamma^\sigma\big)\,,
\end{aligned}$$

or

$$\big(q_\sigma\, \gamma^\sigma\big) \gamma^{\mu}\, D^{\mathrm{C+}}_{\mu\nu}(q)\, q^{\nu} = \frac{1}{\mathbf{q}^2} \big(q_0\, \gamma^0\big) \big(q_\sigma\, \gamma^\sigma\big)\,. \tag{6.2.26}$$

Therefore for the contribution to $\gamma_\sigma\, \dfrac{\partial}{\partial p_\sigma}\, \Sigma^{\mathrm{I}}_{\mathrm{C}}(p)\Big|_{m_0=0}$ in the asymptotic region of the momentum of the photon,

$$\gamma_\sigma\, \frac{\partial}{\partial p_\sigma}\, \Sigma^{\mathrm{I}}_{\mathrm{C}}(p)\Big|_{m_0=0} \longrightarrow -4 \int\limits_{q^2 \gg p^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4}\, \frac{1}{\mathbf{q}^2} \big(q_0\, \gamma^0\big) \big(q_\sigma\, \gamma^\sigma\big)\,. \tag{6.2.27}$$

Now consider the second term (6.2.4):

$$\boxed{\mathbf{II}} = \exp\left\{ -\mathrm{i}\, \frac{q_0^2}{2} \left[\frac{\delta}{\mathrm{i}\delta \eta_{\mathsf{A}}(x)} \big(\mathbb{1} + \mathrm{i}\gamma_5\big)_{\mathsf{AB}}\, \frac{\delta}{\mathrm{i}\delta \bar{\xi}_{\mathsf{B}}(x)}\, \Delta_{1+}(x, x') \right.\right.$$

$$\times \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{D}}(x')}\Bigg]\Bigg\} \langle 0_+ | 0_- \rangle_0$$

$$= \exp\left\{ -\mathrm{i}\,\frac{q_0^2}{2}\,\mathit{\Delta}_{1+}(x,x') \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \right.$$

$$\left. \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)} \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')} \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{D}}(x')} \right\} \langle 0_+ | 0_- \rangle_0$$

$$= \exp\left\{ \mathrm{i}\,\frac{q_0^2}{2}\,\mathit{\Delta}_{1+}(x,x') \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \right.$$

$$\left. \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)} \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{D}}(x')} \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{C}}(x')} \frac{\delta}{\mathrm{i}\delta\bar{\xi}_{\mathsf{B}}(x)} \right\} \langle 0_+ | 0_- \rangle_0$$

$$\to \exp\left\{ \mathrm{i}\,\frac{q_0^2}{2}\,\mathit{\Delta}_{1+}(x,x') \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \right.$$

$$\left. \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{A}}(x)} \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{D}}(x')} (-\mathrm{i}) \big(R_+(x,x')\big)_{\mathsf{BC}} \right\} \langle 0_+ | 0_- \rangle_0$$

$$\to \exp\left\{ -\frac{q_0^2}{2}\,\mathit{\Delta}_{1+}(x,x') \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}}\, \overline{\eta}_{\mathsf{F}}(z) \left(S_+(z,x)\right)_{\mathsf{FA}} \right.$$

$$\left. \times \big(S_+(x',y)\big)_{\mathsf{DE}}\, \eta_{\mathsf{E}}(y) \left(R_+(x,x')\right)_{\mathsf{BC}} \right\} \langle 0_+ | 0_- \rangle_0$$

$$\to \exp\left\{ -\frac{q_0^2}{2}\, \overline{\eta}(z)\, S_+(z,x) \left(\mathbb{1} + \mathrm{i}\gamma_5\right) R_+(x,x')\, \mathit{\Delta}_{1+}(x,x') \right.$$

$$\left. \times \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, S_+(x',y)\, \eta(y) \right\} \langle 0_+ | 0_- \rangle_0 \ .$$

The second term $\boxed{\text{II}}$ generates amplitude of process:

$$\mathcal{M}^{\text{II}} = -\frac{q_0^2}{2}\,\overline{\eta}(z)\,S_+(z,x)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,R_+(x,x')\,\mathit{\Delta}_{1+}(x,x')$$

$$\times\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,S_+(x',y)\,\eta(y)\,, \tag{6.2.28}$$

or

$$\mathcal{M}^{\text{II}} = -\frac{q_0^2}{2}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)\,\overline{\eta}(z)\,S_+(z,x)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)$$

$$\times\,R_+(x,x')\,\mathit{\Delta}_{1+}(x,x')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,S_+(x',y)\,\eta(y) \tag{6.2.29}$$

$$= -\frac{q_0^2}{2}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)$$

$$\times\,\overline{\eta}(z)\left[\int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p(z-x)}\,S_+(p)\right]\big(\mathbb{1}+\mathrm{i}\gamma_5\big)$$

$$\times\left[\int\frac{(\mathrm{d}p'')}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p''(x-x')}\,R_+(p'')\right]\left[\int\frac{(\mathrm{d}q)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}q(x-x')}\,\mathit{\Delta}_{1+}(q)\right]$$

$$\times\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\left[\int\frac{(\mathrm{d}p')}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p'(x'-y)}\,S_+(p')\right]\eta(y)$$

$$= -\frac{q_0^2}{2}\int\frac{(\mathrm{d}q)}{(2\pi)^4}\,\frac{(\mathrm{d}p)}{(2\pi)^4}\,\frac{(\mathrm{d}p')}{(2\pi)^4}\,\frac{(\mathrm{d}p'')}{(2\pi)^4}$$

$$\times\,(2\pi)^4\,\delta^4(q-p+p'')\,(2\pi)^4\,\delta^4(p'-q-p'')$$

$$\times\,\overline{\eta}(p)\,S_+(p)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,R_+(p'')\,\mathit{\Delta}_{1+}(q)\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,S_+(p')\,\eta(p')\,,$$

therefore

$$\mathcal{M}^{\rm II} = -\frac{q_0^2}{2}\int \frac{({\rm d}q)}{(2\pi)^4}\,\frac{({\rm d}p)}{(2\pi)^4}\,\overline{\eta}(p)\,S_+(p)\,\big(\mathbb{1}+{\rm i}\gamma_5\big)$$

$$\times\, R_+(p-q)\,\varDelta_{1+}(q)\,\big(\mathbb{1}-{\rm i}\gamma_5\big)\,S_+(p)\,\eta(p)\,, \qquad (6.2.30)$$

or

$$\mathcal{M}^{\rm II} = q_0^2\int \frac{({\rm d}p)}{(2\pi)^4}\,\overline{\eta}(p)\,S_+(p)\,\varSigma^{\rm II}(p)\,S_+(p)\,\eta(p)\,, \qquad (6.2.31)$$

where

$$\varSigma^{\rm II}(p) \equiv -\frac{1}{2}\int \frac{({\rm d}q)}{(2\pi)^4}\,\big(\mathbb{1}+{\rm i}\gamma_5\big)\,R_+(p-q)\,\big(\mathbb{1}-{\rm i}\gamma_5\big)\,\varDelta_{1+}(q)\,, \qquad (6.2.32)$$

or in the mass zero limit

$$\varSigma^{\rm II}(p)\Big|_{m_0=0} \equiv -\frac{1}{2}\int \frac{({\rm d}q)}{(2\pi)^4}\,\big(\mathbb{1}+{\rm i}\gamma_5\big)\,R_+(p-q)\,\big(\mathbb{1}-{\rm i}\gamma_5\big)\,\varDelta_{1+}(q)\Big|_{m_0=0}\,. \qquad (6.2.33)$$

The first free selectron propagators is

$$\varDelta_{1+}(q) = \frac{1}{q^2+m_0^2-{\rm i}\epsilon}\,, \qquad (6.2.34)$$

and

$$\varDelta_{1+}(q)\Big|_{m_0=0} = \frac{1}{q^2-{\rm i}\epsilon}\,. \qquad (6.2.35)$$

The free photino propagator is

$$R_+(p-q) = \frac{-\gamma^\alpha\,(p-q)_\alpha}{(p-q)^2-{\rm i}\epsilon}\,. \qquad (6.2.36)$$

$$\frac{\partial}{\partial p_\sigma}R_+(p-q) = -\gamma^\alpha\,\frac{\partial}{\partial p_\sigma}\left[\frac{(p-q)_\alpha}{(p-q)^2-{\rm i}\epsilon}\right]$$

$$= -\gamma^{\alpha}\, \frac{\big[(p-q)^2 - \mathrm{i}\epsilon\big]\, \delta^{\sigma}{}_{\alpha} - 2(p-q)_{\alpha}\, (p-q)^{\sigma}}{\big[(p-q)^2 - \mathrm{i}\epsilon\big]^2}$$

$$= -\frac{\big[(p-q)^2 - \mathrm{i}\epsilon\big]\, \gamma^{\sigma} - 2(p-q)^{\sigma}\, (p-q)_{\alpha}\, \gamma^{\alpha}}{\big[(p-q)^2 - \mathrm{i}\epsilon\big]^2}\,.$$

For very large selectron momentum q

$$\frac{\partial}{\partial p_{\sigma}} R_{+}(p-q) \xrightarrow[q\to\infty]{} -\frac{\big[q^2\, \gamma^{\sigma} - 2q^{\sigma}\, q_{\alpha}\, \gamma^{\alpha}\big]}{q^4}\,. \tag{6.2.37}$$

As before for the contribution to the left-hand side of the equation below for $q^2 \gg p^2$, we effectively have

$$\frac{\partial}{\partial p_{\sigma}}\, \Sigma^{\mathrm{II}}(p)\Big|_{m_0=0} \longrightarrow$$

$$\frac{1}{2} \int\limits_{q^2 \gg p^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, \frac{\big[q^2\, \gamma^{\sigma} - 2q^{\sigma}\, q_{\alpha}\, \gamma^{\alpha}\big]}{q^4}\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\, \frac{1}{q^2}\,, \tag{6.2.38}$$

and hence effectively,

$$\gamma_{\sigma}\, \frac{\partial}{\partial p_{\sigma}}\, \Sigma^{\mathrm{II}}(p)\Big|_{m_0=0} \longrightarrow$$

$$\frac{1}{2} \int\limits_{q^2 \gg p^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^6}\, \gamma_{\sigma}\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, \big[q^2\, \gamma^{\sigma} - 2q^{\sigma}\, q_{\alpha}\, \gamma^{\alpha}\big]\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big)\,. \tag{6.2.39}$$

Using the identities

$$\gamma_{\sigma}\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, q^2\, \gamma^{\sigma}\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big) = 8\mathrm{i}\, q^2\, \gamma_5\,,$$

$$2\gamma_{\sigma}\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, q^{\sigma}\, q_{\alpha}\, \gamma^{\alpha}\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big) = 4\mathrm{i}\, q^2\, \gamma_5\,,$$

$$\gamma_{\sigma}\, \big(\mathbb{1} + \mathrm{i}\gamma_5\big)\, \big[q^2\, \gamma^{\sigma} - 2q^{\sigma}\, q_{\alpha}\, \gamma^{\alpha}\big]\, \big(\mathbb{1} - \mathrm{i}\gamma_5\big) = 4\mathrm{i}\, q^2\, \gamma_5\,, \tag{6.2.40}$$

we effectively have

$$\gamma_\sigma \frac{\partial}{\partial p_\sigma} \, \Sigma^{\mathrm{II}}(p)\Big|_{m_0=0} \longrightarrow 2\mathrm{i} \int \frac{(\mathrm{d}q)}{(2\pi)^4} \, \frac{\gamma_5}{q^4} \, . \tag{6.2.41}$$

Finally consider the third term (6.2.5):

$$\boxed{\mathrm{III}} = \exp\left\{ -\mathrm{i} \, \frac{q_0^2}{2} \left[\frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{A}}(x)} \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)} \, \varDelta_{2+}(x,x') \right.\right.$$

$$\left.\left. \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \frac{\delta}{\mathrm{i}\delta\overline{\xi}_{\mathsf{D}}(x')} \right] \right\} \langle 0_+ \,|\, 0_- \rangle_0$$

$$= \exp\left\{ -\mathrm{i} \, \frac{q_0^2}{2} \, \varDelta_{2+}(x,x') \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \right.$$

$$\left. \times \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{A}}(x)} \, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)} \, \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')} \, \frac{\delta}{\mathrm{i}\delta\overline{\xi}_{\mathsf{D}}(x')} \right\} \langle 0_+ \,|\, 0_- \rangle_0$$

$$= \exp\left\{ \mathrm{i} \, \frac{q_0^2}{2} \, \varDelta_{2+}(x,x') \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \right.$$

$$\left. \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')} \, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)} \, \frac{\delta}{\mathrm{i}\delta\xi_{\mathsf{A}}(x)} \, \frac{\delta}{\mathrm{i}\delta\overline{\xi}_{\mathsf{D}}(x')} \right\} \langle 0_+ \,|\, 0_- \rangle_0$$

$$\rightarrow \exp\left\{ \mathrm{i} \, \frac{q_0^2}{2} \, \varDelta_{2+}(x,x') \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \right.$$

$$\left. \times \frac{\delta}{\mathrm{i}\delta\eta_{\mathsf{C}}(x')} \, \frac{\delta}{\mathrm{i}\delta\overline{\eta}_{\mathsf{B}}(x)} \, (-\mathrm{i}) \big(R_+(x',x)\big)_{\mathsf{DA}} \right\} \langle 0_+ \,|\, 0_- \rangle_0$$

$$\rightarrow \exp\left\{ -\frac{q_0^2}{2} \, \varDelta_{2+}(x,x') \left(\mathbb{1} + \mathrm{i}\gamma_5\right)_{\mathsf{AB}} \left(\mathbb{1} - \mathrm{i}\gamma_5\right)_{\mathsf{CD}} \, \overline{\eta}_{\mathsf{F}}(z) \, \big(S_+(z,x')\big)_{\mathsf{FC}} \right.$$

$$\left. \times \big(S_+(x,y)\big)_{\mathsf{BE}} \, \eta_{\mathsf{E}}(y) \, \big(R_+(x',x)\big)_{\mathsf{DA}} \right\} \langle 0_+ \,|\, 0_- \rangle_0$$

$$\to \exp\Big\{-\frac{q_0^2}{2}\,\overline{\eta}(z)\,S_+(z,x')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(x',x)\,\Delta_{2+}(x,x')$$

$$\times\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,S_+(x,y)\,\eta(y)\Big\}\,\langle 0_+\,|\,0_-\rangle_0\ .$$

The third term $\boxed{\text{III}}$ generates amplitude of process:

$$\mathcal{M}^{\text{III}} = -\frac{q_0^2}{2}\,\overline{\eta}(z)\,S_+(z,x')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(x',x)\,\Delta_{2+}(x,x')$$

$$\times\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,S_+(x,y)\,\eta(y)\,, \tag{6.2.42}$$

or

$$\mathcal{M}^{\text{III}} = -\frac{q_0^2}{2}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)\,\overline{\eta}(z)\,S_+(z,x')\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)$$

$$\times\,R_+(x',x)\,\Delta_{2+}(x,x')\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,S_+(x,y)\,\eta(y) \tag{6.2.43}$$

$$= -\frac{q_0^2}{2}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)$$

$$\times\,\overline{\eta}(z)\left[\int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p(z-x')}\,S_+(p)\right]\big(\mathbb{1}-\mathrm{i}\gamma_5\big)$$

$$\times\,\left[\int\frac{(\mathrm{d}p'')}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p''(x'-x)}\,R_+(p'')\right]\left[\int\frac{(\mathrm{d}q)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}q(x-x')}\,\Delta_{2+}(q)\right]$$

$$\times\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\left[\int\frac{(\mathrm{d}p')}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p'(x-y)}\,S_+(p')\right]\eta(y)$$

$$= -\frac{q_0^2}{2}\int\frac{(\mathrm{d}q)}{(2\pi)^4}\,\frac{(\mathrm{d}p)}{(2\pi)^4}\,\frac{(\mathrm{d}p')}{(2\pi)^4}\,\frac{(\mathrm{d}p'')}{(2\pi)^4}$$

$$\times\,(2\pi)^4\,\delta^4(p'+q-p'')\,(2\pi)^4\,\delta^4(p''-q-p)$$

$$\times\,\overline{\eta}(p)\,S_+(p)\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(p'')\,\Delta_{2+}(q)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,S_+(p')\,\eta(p')\,,$$

therefore

$$\mathcal{M}^{\mathrm{III}}=-\frac{q_0^2}{2}\int\frac{(\mathrm{d}q)}{(2\pi)^4}\,\frac{(\mathrm{d}p)}{(2\pi)^4}\,\overline{\eta}(p)\,S_+(p)\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)$$

$$\times\,R_+(p+q)\,\Delta_{2+}(q)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,S_+(p)\,\eta(p)\,, \tag{6.2.44}$$

or

$$\mathcal{M}^{\mathrm{III}}=q_0^2\int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\overline{\eta}(p)\,S_+(p)\,\Sigma^{\mathrm{III}}(p)\,S_+(p)\,\eta(p)\,, \tag{6.2.45}$$

where

$$\Sigma^{\mathrm{III}}(p)\equiv-\frac{1}{2}\int\frac{(\mathrm{d}q)}{(2\pi)^4}\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(p+q)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,\Delta_{2+}(q)\,, \tag{6.2.46}$$

or in the mass zero limit

$$\Sigma^{\mathrm{III}}(p)\Big|_{m_0=0}\equiv-\frac{1}{2}\int\frac{(\mathrm{d}q)}{(2\pi)^4}\,\big(\mathbb{1}-\mathrm{i}\gamma_5\big)\,R_+(p+q)\,\big(\mathbb{1}+\mathrm{i}\gamma_5\big)\,\Delta_{2+}(q)\Big|_{m_0=0}\,. \tag{6.2.47}$$

The second free selectron propagator is

$$\Delta_{2+}(q)=\frac{1}{q^2+m_0^2-\mathrm{i}\epsilon}\,, \tag{6.2.48}$$

and

$$\Delta_{2+}(q)\Big|_{m_0=0}=\frac{1}{q^2-\mathrm{i}\epsilon}\,. \tag{6.2.49}$$

The free photino propagator is

$$R_+(p+q) = \frac{-\gamma^\alpha\,(p+q)_\alpha}{(p+q)^2 - \mathrm{i}\epsilon}\,, \tag{6.2.50}$$

$$\begin{aligned}\frac{\partial}{\partial p_\sigma} R_+(p+q) &= -\gamma^\alpha\,\frac{\partial}{\partial p_\sigma}\left[\frac{(p+q)_\alpha}{(p+q)^2 - \mathrm{i}\epsilon}\right] \\ &= -\gamma^\alpha\,\frac{\big[(p+q)^2 - \mathrm{i}\epsilon\big]\delta^\sigma{}_\alpha - 2(p+q)_\alpha\,(p+q)^\sigma}{\big[(p+q)^2 - \mathrm{i}\epsilon\big]^2} \\ &= -\frac{\big[(p+q)^2 - \mathrm{i}\epsilon\big]\gamma^\sigma - 2(p+q)^\sigma\,(p+q)_\alpha\,\gamma^\alpha}{\big[(p+q)^2 - \mathrm{i}\epsilon\big]^2}\,.\end{aligned}$$

For very large selectron momentum q

$$\frac{\partial}{\partial p_\sigma} R_+(p+q) \xrightarrow[q\to\infty]{} -\frac{\big[q^2\,\gamma^\sigma - 2q^\sigma\,q_\alpha\,\gamma^\alpha\big]}{q^4}\,. \tag{6.2.51}$$

Hence for the contribution to the left-hand side of the expression below for $q^2 \gg p^2$, we effectively have

$$\frac{\partial}{\partial p_\sigma}\,\Sigma^{\mathrm{III}}(p)\Big|_{m_0=0} \longrightarrow$$

$$\frac{1}{2}\int\limits_{q^2\gg p^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\,\big(\mathbb{1} - \mathrm{i}\gamma_5\big)\,\frac{\big[q^2\,\gamma^\sigma - 2q^\sigma\,q_\alpha\,\gamma^\alpha\big]}{q^4}\,\big(\mathbb{1} + \mathrm{i}\gamma_5\big)\,\frac{1}{q^2}\,, \tag{6.2.52}$$

and effectively

$$\gamma_\sigma\,\frac{\partial}{\partial p_\sigma}\,\Sigma^{\mathrm{III}}(p)\Big|_{m_0=0} \longrightarrow$$

$$\frac{1}{2}\int \frac{(\mathrm{d}q)}{(2\pi)^4}\,\frac{1}{q^6}\,\gamma_\sigma\,\big(\mathbb{1} - \mathrm{i}\gamma_5\big)\,\big[q^2\,\gamma^\sigma - 2q^\sigma\,q_\alpha\,\gamma^\alpha\big]\,\big(\mathbb{1} + \mathrm{i}\gamma_5\big)\,. \tag{6.2.53}$$

Using the identities

$$\gamma_\sigma \left(\mathbb{1} - \mathrm{i}\gamma_5\right) q^2 \gamma^\sigma \left(\mathbb{1} + \mathrm{i}\gamma_5\right) = -8\mathrm{i}\, q^2 \gamma_5 \,,$$

$$2\gamma_\sigma \left(\mathbb{1} - \mathrm{i}\gamma_5\right) q^\sigma q_\alpha \gamma^\alpha \left(\mathbb{1} + \mathrm{i}\gamma_5\right) = -4\mathrm{i}\, q^2 \gamma_5 \,,$$

$$\gamma_\sigma \left(\mathbb{1} - \mathrm{i}\gamma_5\right) \left[q^2 \gamma^\sigma - 2q^\sigma q_\alpha \gamma^\alpha\right] \left(\mathbb{1} + \mathrm{i}\gamma_5\right) = -4\mathrm{i}\, q^2 \gamma_5 \,, \tag{6.2.54}$$

we have for the contribution to $\gamma_\sigma \dfrac{\partial}{\partial p_\sigma} \varSigma^{\mathrm{III}}(p)\Big|_{m_0=0}$ in the asymptotic region of the integration variable, effectively

$$\gamma_\sigma \frac{\partial}{\partial p_\sigma} \varSigma^{\mathrm{III}}(p)\Big|_{m_0=0} \longrightarrow -2\mathrm{i} \int\limits_{q^2 \gg p^2} \frac{(\mathrm{d}q)}{(2\pi)^4} \frac{\gamma_5}{q^4} \,, \tag{6.2.55}$$

leading effectively

$$\gamma_\sigma \frac{\partial}{\partial p_\sigma} \left[\varSigma^{\mathrm{II}}(p) + \varSigma^{\mathrm{III}}(p)\right]\Big|_{m_0=0} \longrightarrow 0 \,, \tag{6.2.56}$$

for the contribution of the integral of defining $\dfrac{\partial}{\partial p_\sigma} \left[\varSigma^{\mathrm{II}}(p) + \varSigma^{\mathrm{III}}(p)\right]\Big|_{m_0=0}$ in the asymptotic region of the integration variable q. That is, the latter integral gives a *finite* contribution to $\dfrac{\partial}{\partial p_\sigma} \left[\varSigma^{\mathrm{II}}(p) + \varSigma^{\mathrm{III}}(p)\right]\Big|_{m_0=0}$.

The total contribution to the self-energy of electron is obtained by combining the expressions in (6.2.15), (6.2.31) and (6.2.45):

$$\mathcal{M} = \mathcal{M}^{\mathrm{I}} + \mathcal{M}^{\mathrm{II}} + \mathcal{M}^{\mathrm{III}} \,, \tag{6.2.57}$$

or

$$\mathcal{M} = q_0^2 \int \frac{(\mathrm{d}p)}{(2\pi)^4}\, \overline{\eta}(p)\, S_+(p)\, \varSigma_{\mathrm{C}}(p)\, S_+(p)\, \eta(p) \,, \tag{6.2.58}$$

where

$$\Sigma_{\rm C}(p) = \Sigma^{\rm I}_{\rm C}(p) + \Sigma^{\rm II}(p) + \Sigma^{\rm III}(p)\,, \tag{6.2.59}$$

and

$$\Sigma^{\rm I}_{\rm C}(p) \equiv \int \frac{({\rm d}q)}{(2\pi)^4}\, D^{\rm C+}_{\mu\nu}(q)\,\gamma^{\nu} S_{+}(p+q)\,\gamma^{\mu}\,, \tag{6.2.60}$$

$$\Sigma^{\rm II}(p) \equiv -\frac{1}{2}\int \frac{({\rm d}q)}{(2\pi)^4}\,\big(\mathbb{1} + {\rm i}\gamma_5\big)\,R_{+}(p-q)\,\big(\mathbb{1} - {\rm i}\gamma_5\big)\,\Delta_{1+}(q)\,, \tag{6.2.61}$$

$$\Sigma^{\rm III}(p) \equiv -\frac{1}{2}\int \frac{({\rm d}q)}{(2\pi)^4}\,\big(\mathbb{1} - {\rm i}\gamma_5\big)\,R_{+}(p+q)\,\big(\mathbb{1} + {\rm i}\gamma_5\big)\,\Delta_{2+}(q)\,. \tag{6.2.62}$$

Combine (6.2.27) and (6.2.56), we obtain effectively,

$$\gamma_{\sigma}\,\frac{\partial}{\partial p_{\sigma}}\,\Sigma_{\rm C}(p)\Big|_{m_0=0} \longrightarrow \gamma_{\sigma}\,\frac{\partial}{\partial p_{\sigma}}\,\Sigma^{\rm I}_{\rm C}(p)\Big|_{m_0=0}\,, \tag{6.2.63}$$

or that for the contribution to $\gamma_{\sigma}\,\frac{\partial}{\partial p_{\sigma}}\,\Sigma_{\rm C}(p)\Big|_{m_0=0}$ in the asymptotic q-region

$$\gamma_{\sigma}\,\frac{\partial}{\partial p_{\sigma}}\,\Sigma_{\rm C}(p)\Big|_{m_0=0} \longrightarrow -4\int\limits_{q^2\gg p^2} \frac{({\rm d}q)}{(2\pi)^4}\,\frac{1}{q^4}\,\frac{1}{\mathbf{q}^2}\,\big(q_0\,\gamma^0\big)\big(q_{\sigma}\,\gamma^{\sigma}\big)\,. \tag{6.2.64}$$

We may write $\Sigma_{\rm C}(p)\Big|_{m_0=0}$ in the form

$$\Sigma_{\rm C}(p)\Big|_{m_0=0} \longrightarrow \big(\gamma\,p\big)\,A_{\rm C}\,, \tag{6.2.65}$$

where, in general, $A_{\rm C}$ is may be defined with are ultraviolet cut-off. Hence we formally have

$$\frac{\partial}{\partial p_{\sigma}}\,\Sigma_{\rm C}(p)\Big|_{m_0=0} \longrightarrow \gamma^{\sigma}\,A_{\rm C}\,, \tag{6.2.66}$$

$$\gamma_{\sigma}\,\frac{\partial}{\partial p_{\sigma}}\,\Sigma_{\rm C}(p)\Big|_{m_0=0} \longrightarrow -4\,A_{\rm C}\,, \tag{6.2.67}$$

$$\mathrm{Tr}\left[\gamma_\sigma \frac{\partial}{\partial p_\sigma}\, \Sigma_{\mathrm{C}}(p)\Big|_{m_0=0}\right] \longrightarrow -16\, A_{\mathrm{C}}\,, \tag{6.2.68}$$

form which we will determine A_{C}.

Using (6.2.64), the so-called expected divergent part of the expression on the left-hand below is given by

$$\mathrm{Tr}\left[\gamma_\sigma \frac{\partial}{\partial p_\sigma}\, \Sigma_{\mathrm{C}}(p)\Big|_{m_0=0}\right] \longrightarrow -4 \int\limits_{q^2\to\Lambda^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4}\, \frac{1}{\mathbf{q}^2}\, q_0\, q_\sigma\, \mathrm{Tr}\big[\gamma^0\gamma^\sigma\big]\,, \tag{6.2.69}$$

therefore

$$A_{\mathrm{C}} = \frac{1}{4} \int\limits_{q^2\to\Lambda^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4}\, \frac{1}{\mathbf{q}^2}\, q_0\, q_\sigma\, \mathrm{Tr}\big[\gamma^0\gamma^\sigma\big]\,, \tag{6.2.70}$$

defined rigorously with an ultraviolet cut-off Λ^2. Since $\mathrm{Tr}\big[\gamma^\mu\gamma^\nu\big] = -4g^{\mu\nu}$, we obtain

$$\begin{aligned} A_{\mathrm{C}} &= \frac{1}{4} \int\limits_{q^2\to\Lambda^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4}\, \frac{1}{\mathbf{q}^2} \times 4\big(q_0\big)^2 \\ &= \int\limits_{q^2\to\Lambda^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4} \left(1 - \frac{q^2}{\mathbf{q}^2}\right)\,. \end{aligned}$$

By writing the 3-vector $\mathbf{q}$ as

$$\mathbf{q} = Q\,\big(\cos\phi\sin\theta\,,\ \sin\phi\sin\theta\,,\ \cos\theta\big)\ , \tag{6.2.71}$$

and the 4-vector q as

$$q = Q\,\big(\cos\phi\sin\theta\sin\chi\,,\ \sin\phi\sin\theta\sin\chi\,,\ \cos\theta\sin\chi\,,\ \cos\chi\big)\ . \tag{6.2.72}$$

We can evaluate A_{C} in a standard manner by using the substitutions [cf. Jauch and Rohrlich (1980), page 456 and 457]:

$$\mathbf{q}^2 \longrightarrow Q^2 \sin^2\chi \quad \text{and} \quad q^2 \longrightarrow Q^2\,, \tag{6.2.73}$$

to obtain

$$\begin{aligned} A_{\mathrm{C}} &= \int \frac{(\mathrm{d}q)}{(2\pi)^4}\,\frac{1}{q^4}\left(1-\frac{q^2}{\mathbf{q}^2}\right) \\ &= \frac{1}{(2\pi)^4}\left[\int^{\Lambda}\frac{Q^3\,\mathrm{d}Q}{Q^4}\right]\left[\int_0^{2\pi}\mathrm{d}\phi\right]\left[\int_0^{\pi}\mathrm{d}\theta\right]\left[\int_0^{\pi}\mathrm{d}\chi\,\sin^2\chi\left(1-\frac{1}{\sin^2\chi}\right)\right] \\ &= \frac{1}{8\pi^2}\left[\int^{\Lambda}\frac{\mathrm{d}Q}{Q}\right]\left[-\int_0^{\pi}\mathrm{d}\chi\,\cos^2\chi\right], \end{aligned}$$

therefore in the Coulomb gauge we have

$$A_{\mathrm{C}} = -\frac{1}{16\pi}\int^{\Lambda}\frac{\mathrm{d}Q}{Q}\,, \tag{6.2.74}$$

and the divergent part of the self energy of the electron in pure QED in the Coulomb gauge is

$$\varSigma_{\mathrm{C}}(p)\Big|_{m_0=0} \longrightarrow -\frac{1}{16\pi}\,(\gamma p)\int^{\Lambda}\frac{\mathrm{d}Q}{Q}\,. \tag{6.2.75}$$

6.2.2 Arbitrary Covariant Gauges

We just replace $\varSigma_{\mathrm{C}}(p)$ by $\varSigma_G(p)$, and $D^{\mathrm{C}+}_{\mu\nu}$ by $D^{G+}_{\mu\nu}$, defined below in (6.2.83), so that the amplitude in (6.2.58) becomes replaced by

$$\mathcal{M} = q_0^2\int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\overline{\eta}(p)\,S_+(p)\,\varSigma_G(p)\,S_+(p)\,\eta(p)\,, \tag{6.2.76}$$

where

$$\varSigma_G(p) = \varSigma_G^{\mathrm{I}}(p) + \varSigma^{\mathrm{II}}(p) + \varSigma^{\mathrm{III}}(p)\,, \tag{6.2.77}$$

and

$$\varSigma_G^{\mathrm{I}}(p) \equiv \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, D^{G+}_{\mu\nu}(q)\, \gamma^\nu S_+(p+q)\, \gamma^\mu\,, \tag{6.2.78}$$

to obtain effectively

$$\gamma_\sigma \frac{\partial}{\partial p_\sigma}\, \varSigma_G(p)\Big|_{m_0=0} \longrightarrow \gamma_\sigma \frac{\partial}{\partial p_\sigma}\, \varSigma_G^{\mathrm{I}}(p)\Big|_{m_0=0}\,, \tag{6.2.79}$$

and from (6.2.25), we effectively have in the asymptotic q-region

$$\gamma_\sigma \frac{\partial}{\partial p_\sigma}\, \varSigma_G^{\mathrm{I}}(p)\Big|_{m_0=0} \longrightarrow -4 \int\limits_{q^2 \gg p^2} \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4}\, D^{G+}_{\mu\nu}(q)\, q^\nu \big(q_\sigma\, \gamma^\sigma\big) \gamma^\mu\,, \tag{6.2.80}$$

$$\begin{aligned} &\mathrm{Tr}\Big[\gamma_\sigma \frac{\partial}{\partial p_\sigma}\, \varSigma_G^{\mathrm{I}}(p)\Big|_{m_0=0}\Big] \\ &\qquad \longrightarrow -4 \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4}\, \mathrm{Tr}\Big[D^{G+}_{\mu\nu}(q)\, q^\nu \big(q_\sigma\, \gamma^\sigma\big) \gamma^\mu\Big]\,. \end{aligned} \tag{6.2.81}$$

Therefore

$$A_G = \frac{1}{4} \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4}\, \mathrm{Tr}\Big[D^{G+}_{\mu\nu}(q)\, q^\nu \big(q_\sigma\, \gamma^\sigma\big) \gamma^\mu\Big]\,. \tag{6.2.82}$$

The free photon propagator in the Coulomb gauge is

$$D^{G+}_{\mu\nu}(q) = \left(g_{\mu\nu} - \frac{q_\mu\, q_\nu}{q^2}\right) \frac{1}{q^2 - \mathrm{i}\epsilon} + q_\mu\, q_\nu\, G(q^2)\,, \tag{6.2.83}$$

where

$$G(q^2) = \frac{\xi_0}{q^2\big(q^2 - \mathrm{i}\epsilon\big)}\,, \tag{6.2.84}$$

with

$$\xi_0 = \begin{cases} 0 & \text{(the Landau gauge)}, \\ 1 & \text{(the Feynman gauge)}, \\ 3 & \text{(the Fried–Yennie gauge)}. \end{cases} \tag{6.2.85}$$

$$\begin{aligned} D^{G+}_{\mu\nu}(q)\, q^\nu &= \left(g_{\mu\nu} - \frac{q_\mu\, q_\nu}{q^2} \right) \frac{q^\nu}{q^2 - \mathrm{i}\epsilon} + \xi_0\, \frac{q_\mu\, q_\nu\, q^\nu}{q^2 \big(q^2 - \mathrm{i}\epsilon\big)} \\ &= \xi_0\, \frac{q_\mu}{\big(q^2 - \mathrm{i}\epsilon\big)}\,, \end{aligned}$$

$$\begin{aligned} D^{G+}_{\mu\nu}(q)\, q^\nu \big(q_\sigma\, \gamma^\sigma\big) \gamma^\mu &= \xi_0\, \frac{\big(q_\sigma\, \gamma^\sigma\big)\big(q_\mu\, \gamma^\mu\big)}{\big(q^2 - \mathrm{i}\epsilon\big)} \\ &= \xi_0\, \frac{-q^2}{\big(q^2 - \mathrm{i}\epsilon\big)}\,. \end{aligned}$$

Or

$$D^{G+}_{\mu\nu}(q)\, q^\nu \big(q_\sigma\, \gamma^\sigma\big) \gamma^\mu = -\xi_0\,, \tag{6.2.86}$$

and

$$\mathrm{Tr}\Big[D^{G+}_{\mu\nu}(q)\, q^\nu \big(q_\sigma\, \gamma^\sigma\big) \gamma^\mu \Big] = -4\xi_0\,. \tag{6.2.87}$$

From (6.2.82), we obtain

$$\begin{aligned} A_G &= -\xi_0 \int \frac{(\mathrm{d}q)}{(2\pi)^4}\, \frac{1}{q^4} \\ &= -\frac{\xi_0}{(2\pi)^4} \left[\int^\Lambda \frac{Q^3\, \mathrm{d}Q}{Q^4} \right] \left[\int_0^{2\pi} \mathrm{d}\phi \right] \left[\int_0^\pi \mathrm{d}\theta \right] \left[\int_0^\pi \mathrm{d}\chi\, \sin^2 \chi \right] \\ &= -\frac{\xi_0}{8\pi^2} \left[\int^\Lambda \frac{\mathrm{d}Q}{Q} \right] \left(\frac{\pi}{2} \right)\,, \end{aligned}$$

therefore

$$A_G = -\frac{\xi_0}{16\pi}\int^{\Lambda}\frac{\mathrm{d}Q}{Q}\,, \tag{6.2.88}$$

and for the contribution to $\varSigma_G(p)\Big|_{m_0=0}$ in the asymptotic Q-region,

$$\varSigma_G(p)\Big|_{m_0=0} \longrightarrow -\frac{\xi_0}{16\pi}\,(\gamma\,p)\int^{\Lambda}\frac{\mathrm{d}Q}{Q}\,. \tag{6.2.89}$$

Note that

- in the Landau gauge, $\xi_0 = 0$, $A_{\mathrm{L}} = 0$ and for the contribution in the asymptotic Q-region

$$\varSigma_{\mathrm{L}}(p)\Big|_{m_0=0} \longrightarrow 0\,. \tag{6.2.90}$$

  That is, $\varSigma_{\mathrm{L}}(p)\Big|_{m_0=0}$ is finite.

- In Feynman gauge, $\xi_0 = 1$, $A_{\mathrm{F}} = -\dfrac{1}{16\pi}\displaystyle\int^{\Lambda}\frac{\mathrm{d}Q}{Q} = A_{\mathrm{C}}$ (the Coulomb gauge).

6.3 Wave-Function Renormalization Constant Z_2

The leading contribution to the exact electron propagator S'_+ is

$$S'_+(x,x') = S_+(x,x') + \int(\mathrm{d}y)(\mathrm{d}y')\,S_+(x,y)\,\varSigma(y,y')\,S_+(y',x') + \ldots\,, \tag{6.3.1}$$

with

$$\varSigma(y,y') = \int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p(y-y')}\,\varSigma(p)\,, \tag{6.3.2}$$

where

$$\varSigma(p) = \varSigma^{\mathrm{I}}(p) + \varSigma^{\mathrm{II}}(p) + \varSigma^{\mathrm{III}}(p)\,. \tag{6.3.3}$$

Using the defining of the wave-function renormalization constant Z_2 given through the following equation:

$$\Sigma(p)\Big|_{m_0=0} = \frac{(\gamma\, p)}{Z_2}\,, \tag{6.3.4}$$

as defined in the mass zero limit, we obtain for the expected divergent part of $\frac{1}{Z_2}$, the following expression:

$$\left(\frac{1}{Z_2}\right)_{\text{Divergent part}} = A\,. \tag{6.3.5}$$

- In the Coulomb gauge or the Feynman gauge:

$$\left(\frac{1}{Z_2}\right)_{\text{Divergent part}} = -\frac{1}{16\pi}\int^{\Lambda}\frac{\mathrm{d}Q}{Q}\,. \tag{6.3.6}$$

- In the Landau gauge:

$$\left(\frac{1}{Z_2}\right)_{\text{Divergent part}} = 0\,. \tag{6.3.7}$$

- In arbitrary covariant gauge:

$$\left(\frac{1}{Z_2}\right)_{\text{Divergent part}} = -\frac{\xi_0}{16\pi}\int^{\Lambda}\frac{\mathrm{d}Q}{Q}\,. \tag{6.3.8}$$

Thus we conclude that as in quantum electrodynamics (QED), the wave-function renormalization constant Z_2 is finite in the Landau gauge in supersymmetric quantum electrodynamics (SQED) to the leading order. The significance and the importance of this result will be discussed in our concluding chapter that follows.

CHAPTER VII

CONCLUSION

In this final chapter, we summarize our findings and make several pertinent comments. Extensive analyses were carried out of supersymmetric methods dealing with the quantum electrodynamics of many-particle systems in potential theory in quantum physics as well as of supersymmetric quantum electrodynamics. As the study is carried out in the functional *differential* formalism of quantum field theory, extensive applications of the so-called Quantum Dynamical Principle in the *presence* of dependent fields which are inherently present not only in quantum electrodynamics but also in its supersymmetric version. Accordingly, in our studies of the intricacies of the latter theory, a complete, detailed and rigorous derivation was given of the Quantum Dynamical Principle in the presence of dependent fields. The basic equation involved here is provided by the one given by

$$\delta \left\langle at_2 \middle| \mathbb{B}(\tau,\lambda) \middle| bt_1 \right\rangle = \frac{\mathrm{i}}{\hbar} \int_{t_1}^{t_2} (\mathrm{d}x') \left\langle at_2 \middle| \left(\mathbb{B}(\tau,\lambda)\, \delta\mathscr{L}(x',\lambda) \right)_+ \middle| bt_1 \right\rangle + \left\langle at_2 \middle| \delta\mathbb{B}(\tau,\lambda) \middle| bt_1 \right\rangle \,, \tag{7.1}$$

where λ is a generic coupling parameter, and the variations are with respect to coupling parameters, masses and so on. $\mathbb{B}(\tau,\lambda)$ is a Heisenberg operator which may depend on λ evaluated at a time τ: $t_1 < \tau < t_2$. The variations are taken by keeping the independent fields and their canonical conjugate momenta fixed. Supersymmetric quantum electrodynamics turns out to be a very special case of the ones embodied in the general result in (7.1). Detailed analyses were also carried of the gauge problem in quantum electrodynamics which is also present in its supersymmetric version. Supersymmetric methods were applied, in particular, to derive a lower bound for the ground-state energy E_N of the quantum electrodynamic of charged many-particle systems in potential theory in quantum physics, for bosonic systems, as a function of the number of the negatively charged particle N given by

$$E_N > -\left(\frac{m\,e^4}{2\hbar^2}\right)\frac{16}{3}\,\frac{N^3}{(\nu-1)^2}\,, \tag{7.2}$$

where ν is the dimensionality of space, $-|e|$ is the charge of a negatively charged particle and m is its mass. The Wess–Zumino Lagrangian was spelled out in (4.2.3) putting emphasis on the fields describing the particles and their superpartners. In the presence of external sources, acting as emitters and detectors of the particles, the Lagrangian is given by

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_\mathrm{I} + \mathscr{L}_\mathrm{S}\,, \tag{7.3}$$

where

$$\begin{aligned}\mathscr{L}_0 \equiv{}& \frac{1}{2}\left[\left(\frac{\partial_\mu}{\mathrm{i}}\overline{\psi}\right)\gamma^\mu\psi - \overline{\psi}\gamma^\mu\left(\frac{\partial_\mu}{\mathrm{i}}\psi\right)\right] - m_0\overline{\psi}\psi \\ &- (\partial_\mu\phi_1^\dagger)(\partial^\mu\phi_1) - m_0^2\phi_1^\dagger\phi_1 - (\partial_\mu\phi_2^\dagger)(\partial^\mu\phi_2) - m_0^2\phi_2^\dagger\phi_2 \\ &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\left[\left(\frac{\partial_\mu}{\mathrm{i}}\overline{\lambda}\right)\gamma^\mu\lambda - \overline{\lambda}\gamma^\mu\left(\frac{\partial_\mu}{\mathrm{i}}\lambda\right)\right],\end{aligned} \tag{7.4}$$

is the free lagrangian,

$$\begin{aligned}\mathscr{L}_\mathrm{I} = q_0\Bigg\{&\overline{\psi}\gamma^\mu\psi A_\mu - \mathrm{i}A^\mu\big[\phi_1^\dagger(\partial_\mu\phi_1) - (\partial_\mu\phi_1^\dagger)\phi_1\big] \\ &+ \mathrm{i}A^\mu\big[\phi_2^\dagger(\partial_\mu\phi_2) - (\partial_\mu\phi_2^\dagger)\phi_2\big] \\ &+ \frac{1}{\sqrt{2}}\Big[\overline{\lambda}(\mathbb{1} - \mathrm{i}\gamma_5)\psi\phi_1^\dagger + \overline{\lambda}(\mathbb{1} + \mathrm{i}\gamma_5)\psi\phi_2\end{aligned}$$

$$- \overline{\psi}(\mathbb{1} + \mathrm{i}\gamma_5)\lambda\phi_1 - \overline{\psi}(\mathbb{1} - \mathrm{i}\gamma_5)\lambda\phi_2^\dagger\Big]\Big\}$$

$$- q_0^2 \left[\frac{1}{2}\left(\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2\right)^2 + A_\mu A^\mu\left(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2\right)\right] , \tag{7.5}$$

is the interaction lagrangian and

$$\mathscr{L}_\mathrm{S} \equiv \overline{\psi}\eta + \overline{\eta}\psi + J^\mu A_\mu + K_1^\dagger\phi_1 + \phi_1^\dagger K_1 + K_2^\dagger\phi_2 + \phi_2^\dagger K_2 + \overline{\lambda}\xi + \overline{\xi}\lambda\,, \tag{7.6}$$

is the external source terms. The *exact* vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle$ describing all the dynamical processes in the theory was derived to be

$$\langle 0_+ \,|\, 0_- \rangle = \exp\left[\mathrm{i}\int(\mathrm{d}x)\; \mathscr{L}_\mathrm{I}'\right] \langle 0_+ \,|\, 0_- \rangle_0 \;, \tag{7.7}$$

where

$$\begin{aligned}
\mathscr{L}_\mathrm{I}' = {} & q_0 \frac{\delta}{\mathrm{i}\delta\eta}\gamma^\mu \frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta J^\mu} \\
& - \mathrm{i}q_0 \frac{\delta}{\mathrm{i}\delta J^\mu}\left[\frac{\delta}{\mathrm{i}\delta K_1}\left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_1^\dagger}\right) - \left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_1}\right)\frac{\delta}{\mathrm{i}\delta K_1^\dagger}\right] \\
& + \mathrm{i}q_0 \frac{\delta}{\mathrm{i}\delta J^\mu}\left[\frac{\delta}{\mathrm{i}\delta K_2}\left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right) - \left(\partial^\mu \frac{\delta}{\mathrm{i}\delta K_2}\right)\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right] \\
& + \frac{q_0}{\sqrt{2}}\left[\frac{\delta}{\mathrm{i}\delta\xi}(\mathbb{1} - \mathrm{i}\gamma_5)\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta K_1} + \frac{\delta}{\mathrm{i}\delta\xi}(\mathbb{1} + \mathrm{i}\gamma_5)\frac{\delta}{\mathrm{i}\delta\overline{\eta}}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right. \\
& \qquad \left. - \frac{\delta}{\mathrm{i}\delta\eta}(\mathbb{1} + \mathrm{i}\gamma_5)\frac{\delta}{\mathrm{i}\delta\overline{\xi}}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} - \frac{\delta}{\mathrm{i}\delta\eta}(\mathbb{1} - \mathrm{i}\gamma_5)\frac{\delta}{\mathrm{i}\delta\overline{\xi}}\frac{\delta}{\mathrm{i}\delta K_2}\right] \\
& - \frac{q_0^2}{2}\left(\frac{\delta}{\mathrm{i}\delta K_1}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} - \frac{\delta}{\mathrm{i}\delta K_2}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right)^2 \\
& - q_0^2 \frac{\delta}{\mathrm{i}\delta J^\mu}\frac{\delta}{\mathrm{i}\delta J_\mu}\left(\frac{\delta}{\mathrm{i}\delta K_1}\frac{\delta}{\mathrm{i}\delta K_1^\dagger} + \frac{\delta}{\mathrm{i}\delta K_2}\frac{\delta}{\mathrm{i}\delta K_2^\dagger}\right) ,
\end{aligned} \tag{7.8}$$

and

$$\langle 0_+ \,|\, 0_- \rangle_0 = \exp\left[\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')\, \overline{\eta}(x) S_+(x,x') \eta(x') \right]$$

$$\times \exp\left[\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')\, K_1^\dagger(x) \varDelta_{1+}(x,x') K_1(x') \right]$$

$$\times \exp\left[\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')\, K_2^\dagger(x) \varDelta_{2+}(x,x') K_2(x') \right]$$

$$\times \exp\left[\frac{\mathrm{i}}{2} \int (\mathrm{d}x)(\mathrm{d}x')\, J_\mu(x) D^{\mu\nu}_{\mathrm{C}+}(x,x') J_\nu(x') \right]$$

$$\times \exp\left[\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')\, \overline{\xi}(x) R_+(x,x') \xi(x') \right], \tag{7.9}$$

with the free electron propagator is given by

$$S_+(x,x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4} \, \frac{-\gamma p + m_0}{p^2 + m_0^2 - \mathrm{i}\epsilon} \, \mathrm{e}^{\mathrm{i}p(x-x')}, \tag{7.10}$$

and the free photon propagator in the Coulomb gauge is

$$D^{\mu\nu}_{\mathrm{C}+}(x,x') = \int \frac{(\mathrm{d}q)}{(2\pi)^4} \, D^{\mu\nu}_{\mathrm{C}+}(q) \, \mathrm{e}^{\mathrm{i}q(x-x')}, \tag{7.11}$$

with

$$\left. \begin{aligned} D^{ij}_{\mathrm{C}+}(q) &= \left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2} \right) \frac{1}{q^2 - \mathrm{i}\epsilon}, \\ D^{i0}_{\mathrm{C}+}(q) &= 0 = D^{0i}_{\mathrm{C}+}(q), \\ D^{00}_{\mathrm{C}+}(q) &= -\frac{1}{\mathbf{q}^2}, \end{aligned} \right\} \tag{7.12}$$

or

$$D^{\mu\nu}_{\mathrm{C}+}(q) = \left(g^{\mu\alpha} - g^{\alpha i} \frac{q_i q^\mu}{\mathbf{q}^2} \right) g_{\alpha\beta} \left(g^{\beta\nu} - g^{\beta k} \frac{q_k q^\nu}{\mathbf{q}^2} \right) \frac{1}{q^2 - \mathrm{i}\epsilon}. \tag{7.13}$$

The free photino propagator is

$$R_+(x,x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4} \frac{-\gamma p}{p^2 - \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}p(x-x')} , \tag{7.14}$$

and the free selectron propagators are

$$\left.\begin{aligned} \Delta_{1+}(x,x') &= \int \frac{(\mathrm{d}k_1)}{(2\pi)^4} \frac{1}{k_1^2 + m_0^2 - \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}k_1(x-x')} , \\ \Delta_{2+}(x,x') &= \int \frac{(\mathrm{d}k_2)}{(2\pi)^4} \frac{1}{k_2^2 + m_0^2 - \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}k_2(x-x')} . \end{aligned}\right\} \tag{7.15}$$

Inherit with the explicit expression for $\langle 0_+ | 0_- \rangle$ in (7.7)–(7.15) specific applications were carried out. The scattering amplitude to the leading order for the process $\mathrm{e}^-\mathrm{e}^+ \to \tilde{\gamma}\tilde{\gamma}$ was explicitly derived, where $\tilde{\gamma}$, the photino, is the superpartner of the photon. Another application was carried out in the analysis of the self-energy of the electron which now includes three diagrams in comparison to the pure quantum electrodynamics case,

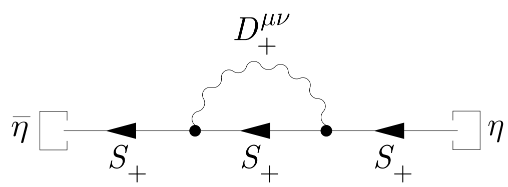


Figure 7.1 The self-energy of the electron in pure quantum electrodynamics with virtual photon line represented by $D^+_{\mu\nu}$.

where in the second diagram the dashed lines represent the virtual selectrons denoted by Δ_{1+} and Δ_{2+}, and R_+ represent photino lines, with the self-energy given in Figure **7.2** . Most importantly we have shown that the wave-function renormalization constant Z_2 is finite, to the leading order, only in the Landau gauge with the photon propagator in the explicit form

$$D_+^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu\, g^\nu}{q^2} \right) \frac{1}{q^2 - \mathrm{i}\epsilon} , \tag{7.16}$$

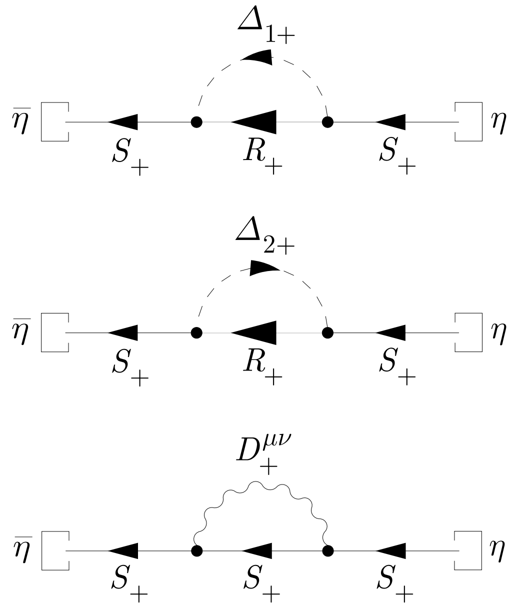


Figure 7.2 The self-energy of the electron in supersymmetric quantum electrodynamics with virtual selectron lines represented by Δ_{1+} and Δ_{2+}, and photino line represented by R_+, while the last diagram represents the pure QED contribution.

where $-\mathrm{i}\epsilon$ is the Schwinger–Feynman boundary condition, as it is in pure quantum electrodynamics. This result of finiteness of Z_2 in this specific gauge is quite of significance. This opens the way to study supersymmetric quantum electrodynamics at *high energies* in which the mass term m_0, providing an energy scale, may be neglected. Accordingly, one hopes that a systematic perturbation expansion may be carried out in the coupling parameter q_0 in (7.7) for the gauge parameter ξ_0 as appearing in the covariant form of the photon propagator

$$D_{\mu\nu}^{G+}(q) = \left(g_{\mu\nu} - \frac{q_\mu\, g_\nu}{q^2} \right) \frac{1}{q^2 - \mathrm{i}\epsilon} + \frac{q_\mu\, g_\nu}{q^2} \, \frac{\xi_0}{q^2 - \mathrm{i}\epsilon} \,, \tag{7.17}$$

which makes graph by graph *finite* in *each* order in q_0 *in the mass* m_0 *zero limit*. We hope that such a program may be carried out in the near future as such a result would imply automatically the finiteness of any *gauge invariant* amplitude order by order in perturbation theory without the need of cancelling infinite terms with other infinite terms of different signs. Finally, it is worth mentioning that even if the supersymmetric partners of the present known particles are not detected experimentally, such particles, as virtual particles, would have important non-trivial contributions to the scattering of the fundamental observed particles in nature as internal lines in the diagrams describing their interactions. This is somehow reminiscent of the so-called Higgs Boson in unified field theory and of quarks in describing physical properties of fundamental particles.

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APPENDIX

PUBLICATIONS

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Action Principle and Algebraic Approach to Gauge Transformations in Gauge Theories

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The action principle is used to derive, by an entirely *algebraic* approach, gauge transformations of the full vacuum-to-vacuum transition amplitude (generating functional) from the Coulomb gauge to arbitrary covariant gauges and in turn to the celebrated Fock–Schwinger (FS) gauge for the Abelian (QED) gauge theory without recourse to path integrals or to commutation rules and without making use of delta functionals. The interest in the FS gauge, in particular, is that it leads to Faddeev–Popov ghosts-free non-Abelian gauge theories. This method is expected to be applicable to non-Abelian gauge theories including supersymmetric ones.

KEY WORDS: action principle; gauge transformations; Coulomb gauge; Fock–Schwinger gauge.
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1. INTRODUCTION

About two decades ago, we have seen (Manoukian, 1986, 1987) that the very elegant action principle (Schwinger, 1951a,b, 1953a,b, 1954) may be used to quantize gauge theories in constructing the vacuum-to-vacuum transition amplitude and the Faddeev–Popov factor (Faddeev and Popov, 1967), encountered in non-Abelian gauge theories, was obtained *directly* from the action principle without much effort. No appeal was made to path integrals, no commutation rules were used, and there was not even the need to go into the well-known complicated structure of the Hamiltonian (Fradkin and Tyutin, 1970) in non-Abelian gauge theories. Of course path integrals are extremely useful in many respects and may be formally derived from the action principle cf. (Symanzik, 1954; Lam, 1965; Manoukian, 1985). We have worked in the Coulomb gauge, where the physical components are clear at the outset, to derive the expression for the vacuum-to-vacuum transition amplitude (generating functional) including the Faddeev–Popov

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factor in non-Abelian gauge theories. It is interesting to note also that the Coulomb gauge naturally arises (Faddeev and Jackiw, 1988; Ogawa *et al*., 1986), see also (Jogleker and Mandal, 2002), in gauge field theories as constrained dynamics cf. (Henneaux and Teitelboim, 1992; Garcia and Vergara, 1996; Su, 2001). To make transitions of the generating functional to arbitrary covariant gauges, we have made use (Manoukian, 1986, 1987), in the process, of so-called δ functionals (Schwinger, 1972, 1973). The δ functionals, however, are defined as infinite dimensional continual integrals corresponding to the different points of spacetime and hence the gauge transformations were carried out in the spirit of path integrals.

The purpose of the present investigation is, in particular, to remedy the above situation involved with delta functionals, and we here derive the gauge transformations, providing explicit expressions, for the full vacuum-to-vacuum transition amplitude to the generating functionals of arbitrary covariant gauges and, in turn, to the celebrated Fock–Schwinger (FS) gauge $x^\mu A_\mu = 0$ (Fock, 1937), as well as the axial gauge $n^\mu A_\mu = 0$ for a fixed vector n^μ, for the Abelian (QED) gauge theory by an entirely *algebraic* approach dealing only with commuting (or anti-commuting) external sources. The interest in the FS gauge, in gauge theories, in general, is that it leads to Faddeev–Popov ghost-free theories, cf. (Kummer and Weiser, 1986), the gauge field may be expressed quite simply in terms of the field strength (Kummer and Weiser, 1986; Durand and Mendel, 1982) and it turns out to be useful in non-perturbative studies, cf. (Shifman *et al*., 1979). Needless to say, the complete expressions of such generating functionals allow one to obtain gauge transformations of *all* the Green functions in a theory simply by functional differentiations with respect to the external sources coupled to the quantum fields in question and avoids the rather tedious treatment, but provides information on, the gauge transformation of diagram by diagram (Handy, 1979; Feng and Lam, 1996) occurring in a theory. A key point, whose importance cannot be overemphasized, in our analysis (Manoukian, 1986, 1987) is that, a priori, *no* restrictions are set on the external source(s) J^μ coupled to the gauge field(s), such as a $\partial_\mu J^\mu = 0$—restriction, so that *variations of the components of* J^μ *may be carried out independently*, until the entire analysis is completed. The present method is expected to be applicable to non-Abelian gauge theories including supersymmetric ones and the latter will be attempted in a forthcoming report. Some classic references which have set the stage of the investigation of the gauge problem in field theory are given in Landau and Khalatnikov (1954), Landau and Khalatnikov (1956), Johnson and Zumino (1959), Zumino (1960), Bialynicki-Birula (1968), Mills (1971), Slavnov (1972), Taylor (1971), Abers and Lee (1973), Wess and Zumino (1974), Salam and Strathdee (1974), Becchi *et al.* (1975), Utiyama and Sakamoto (1977). For more recent studies which are, however, more involved with field operator techniques and their gauge transformations may be found in Sardanashvily (1984), Kobe, (1985), Oh and Soo (1987), Sugano and Kimura (1990), Gastmans *et al.* (1996), Pons *et al.* (1997), Gastmans and Wu (1998), and Banerjee (2000).

2. GAUGE TRANSFORMATIONS

The Lagrangian density under consideration is given by a well-known expression (Manoukian, 1986, 1987)

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[\left(\frac{\partial_\mu\overline{\psi}}{\mathrm{i}}\right)\gamma^\mu\psi - \overline{\psi}\gamma^\mu\frac{\partial_\mu\psi}{\mathrm{i}}\right] - m_0\overline{\psi}\psi$$
$$+ e_0\overline{\psi}\gamma_\mu\psi A^\mu + \overline{\eta}\psi + \overline{\psi}\eta + A_\mu J^\mu \quad (1)$$

where $\overline{\eta}$, η, J^μ are external sources, and no restriction is set on J^μ (such as $\partial_\mu J^\mu = 0$) in order to carry out functional differentiations with respect to all of its components *independently*.

Our starting point is the vacuum-to-vacuum transition amplitude in the Coulomb gauge given by Manoukian (1986, 1987)

$$\langle 0_+|0_-\rangle = \exp\left[\mathrm{i}\int \mathscr{L}'_{\mathrm{I}}\right]\langle 0_+|0_-\rangle_0 \equiv F_{\mathrm{C}}[\eta, \bar{\eta}, J] \quad (2)$$

$$\int \mathscr{L}'_{\mathrm{I}}(\eta, \bar{\eta}, J) = \int (\mathrm{d}x)\left(e_0\frac{\delta}{\mathrm{i}\delta\eta(x)}\gamma^\mu\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\frac{\delta}{\mathrm{i}\delta J^\mu x}\right) \quad (3)$$

where

$$\langle 0_+\,|\,0_-\rangle_0 = \exp\left[\mathrm{i}\int(\mathrm{d}x)\,(\mathrm{d}x')\,\overline{\eta}(x)S_+(x-x')\eta(x')\right]$$
$$\times \exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)\,(\mathrm{d}x')\,J^\mu(x)D^{\mathrm{C}}_{\mu\nu}(x,x')J^\nu(x')\right] \quad (4)$$

with $S_+(x-x')$ denoting the free electron propagator, and, in the momentum description, $(k, m = 1, 2, 3)$,

$$D^{\mathrm{C}}_{km}(q) = \left(\delta_{km} - \frac{q_k q_m}{\vec{q}^{\,2}}\right)\frac{1}{q^2 - \mathrm{i}\varepsilon} \quad (5)$$

$$D^{\mathrm{C}}_{0k}(q) = 0 = D^{\mathrm{C}}_{k0}(q) \quad (6)$$

$$D^{\mathrm{C}}_{00}(q) = -\frac{1}{\vec{q}^{\,2}}. \quad (7)$$

We introduce the generating functional

$$F\big[\rho, \overline{\rho}, K; G\big] = \exp\left[\mathrm{i}\int\mathscr{L}'_{\mathrm{I}}(\rho, \overline{\rho}, K)\right]\exp\left[\mathrm{i}\int(\mathrm{d}x)\,(\mathrm{d}x')\,\overline{\rho}(x)S_+(x-x')\rho(x')\right]$$
$$\times \exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)\,(\mathrm{d}x')\,K_\mu(x)D^{\mu\nu}_G(x,x')K_\nu(x')\right] \quad (8)$$

where in the momentum description

$$D_G^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{1}{q^2 - \mathrm{i}\varepsilon} + q^\mu q^\nu G(q^2) \tag{9}$$

and $G(q^2)$ is arbitrary.

We show that

$$F_{\mathrm{C}}[\eta, \overline{\eta}, J] = \mathrm{e}^{\mathrm{i}W'} F\big[\rho, \overline{\rho}, K; G\big]\big|_{\rho=0, \overline{\rho}=0, K=0} \tag{10}$$

where

$$\begin{aligned} W' = &\int (\mathrm{d}x)\, \overline{\eta}(x) \exp\left[-\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu(x)}\right] \frac{\delta}{\mathrm{i}\delta\overline{\rho}(x)} \\ &+ \int (\mathrm{d}x)\, \frac{\delta}{\mathrm{i}\delta\rho(x)} \exp\left[\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu(x)}\right] \eta(x) \\ &+ \int (\mathrm{d}x)\, ((g^{\mu\sigma} - a^\mu \partial^\sigma) J_\sigma(x)) \frac{\delta}{\mathrm{i}\delta K^\mu(x)} \end{aligned} \tag{11}$$

and

$$a^\mu = \left(0, \frac{\vec{\nabla}}{\nabla^2}\right) = g^{\mu k} \frac{\partial^k}{\nabla^2} \tag{12}$$

relating the Coulomb gauge to arbitrary covariant gauges.

To establish (10), we start from its right-hand side. We note, in a matrix notation, that

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}W'} \exp\big[\mathrm{i}\overline{\rho} S_+ \rho\big] \exp\left[\frac{\mathrm{i}}{2} K_\mu D_G^{\mu\nu} K_\nu\right] \\ &\quad = \exp\left[\mathrm{i}\left(\overline{\rho} + \overline{\eta}\exp\left[-\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu}\right]\right) S_+ \left(\rho + \exp\left[\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu}\right]\eta\right)\right] \\ &\quad \times \exp\left[\frac{\mathrm{i}}{2}\Big(K_\mu + (g_{\mu\sigma} - a_\mu \partial_\sigma) J^\sigma\Big) D_G^{\mu\nu} \Big(K_\nu + (g_{\nu\lambda} - a_\nu \partial_\lambda) J^\lambda\Big)\right] \end{aligned} \tag{13}$$

and since $\mathscr{L}'_{\mathrm{I}}(\rho, \overline{\rho}, K)$, is classical, is invariant under transformations $\rho(x) \to \rho(x)\exp\big(\mathrm{i}\Lambda(x)\big)$, $\overline{\rho}(x) \to \exp\big(-\mathrm{i}\Lambda(x)\big)\,\overline{\rho}(x)$ for an arbitrary numerical function $\Lambda(x)$, and we eventually set $\rho = 0$, $\overline{\rho} = 0$, the right-hand side of (10) becomes

$$\begin{aligned} &\exp\left[\mathrm{i}\int \mathscr{L}'_{\mathrm{I}}(\eta, \overline{\eta}, J)\right] \exp\left[\mathrm{i}\left(\overline{\eta}\exp\left[-\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu}\right]\right) S_+ \left(\exp\left[\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu}\right]\eta\right)\right] \\ &\quad \times \exp\left[\frac{\mathrm{i}}{2}(K_\mu + (g_{\mu\sigma} - a_\mu \partial_\sigma) J^\sigma) D_G^{\mu\nu} (K_\nu + (g_{\nu\lambda} - a_\nu \partial_\lambda) J^\lambda)\right] \end{aligned} \tag{14}$$

with $K_\mu \to 0$. Now we use the identity

$$\exp\left[\mathrm{i}e_0 \int(\mathrm{d}x)\left(\frac{\delta}{\mathrm{i}\delta\eta(x)}\gamma^\mu \frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\partial_\mu\Lambda(x)\right)\right]\exp[\mathrm{i}\overline{\eta}S_+\eta]$$
$$= \exp[\mathrm{i}(\overline{\eta}\,\mathrm{e}^{\mathrm{i}e_0\Lambda})S_+(\mathrm{e}^{-\mathrm{i}e_0\Lambda}\eta)\Big] \tag{15}$$

to rewrite the above expression as

$$\exp\left[\mathrm{i}e_0 \int(\mathrm{d}x)\ \left(\frac{\delta}{\mathrm{i}\delta\eta(x)}\gamma_\mu \frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\big(g^{\mu\sigma} - a^\sigma\partial^\mu\big)\frac{\delta}{\mathrm{i}\delta K^\sigma(x)}\right)\right]\exp[\mathrm{i}\overline{\eta}S_+\eta]$$
$$\times\exp\left[\frac{\mathrm{i}}{2}\big(K_\mu + \big(g_{\mu\sigma} - a_\mu\partial_\sigma\big)J^\sigma\big)D_G^{\mu\nu}\big(K_\nu + \big(g_{\nu\lambda} - a_\nu\partial_\lambda\big)J^\lambda\big)\right] \tag{16}$$

which for $K_\mu \to 0$ reduces to the left-hand side of (10) *since*

$$\big(g_{\mu\sigma} - a_\mu\partial_\sigma\big)D_G^{\mu\nu}\big(g_{\nu\lambda} - a_\nu\partial_\lambda\big) = D^{\mathrm{C}}_{\sigma\lambda}. \tag{17}$$

Almost an identical analysis as above shows, by noting in the process,

$$\big(g_{\mu\sigma} - \widetilde{a}_\mu\partial_\sigma\big)D_G^{\mu\nu}\big(g_{\nu\lambda} - \widetilde{a}_\nu\partial_\lambda\big) = \big(D_0\big)_{\sigma\lambda} \equiv D^{\mathrm{L}}_{\sigma\lambda} \tag{18}$$

with

$$\widetilde{a}_\mu = \frac{\partial_\mu}{\Box}, \Box \equiv \partial_\mu\partial^\mu \tag{19}$$

where the right-hand side of (18) defines the photon propagator in the Landau gauge, with G in (9) set equal to zero, that

$$F[\eta, \overline{\eta}, J; G = 0] = \mathrm{e}^{\mathrm{i}\widetilde{W}'}F[\rho, \overline{\rho}, K; G]|_{\rho=0,\overline{\rho}=0,K=0} \tag{20}$$

where $\widetilde{W}'$ is given by the expression defined in (11) with a^μ in it simply replaced by $\widetilde{a}^\mu$, thus relating the Landau gauge to arbitrary covariant gauges.

The Fock–Schwinger gauge $x^\mu A_\mu = 0$, allows one to write

$$A^0 = \frac{x^k A_k}{x^0} \tag{21}$$

which upon substitution in (1), and varying $\mathscr{L}$ with respect to A^k yields

$$\partial_\mu F^{\mu k} - \frac{x^k}{x^0}\partial_\mu F^{\mu 0} = -j^k + j^0\frac{x^k}{x^0} \tag{22}$$

where

$$j^\mu = e_0\overline{\psi}\gamma^\mu\psi + J^\mu. \tag{23}$$

We note that (22) holds true with k replaced by 0 in it giving $0 = 0$, i.e., we may rewrite (22) as

$$\partial_\mu F^{\mu\nu} - \frac{x^\nu}{x^0}\partial_\mu F^{\mu 0} = -j^\nu + j^0\frac{x^\nu}{x^0} \equiv S^\nu. \tag{24}$$

By taking the derivative ∂_ν of (24), we may solve for $(\partial_\mu F^{\mu 0})/x^0$,

$$-\frac{\partial_\mu F^{\mu 0}}{x^0} = (\partial\, x)^{-1}\partial_\sigma\left(-j^\sigma + j^0\frac{x^\sigma}{x^0}\right) \tag{25}$$

which upon substituting in (24) gives

$$\partial_\mu F^{\mu\nu} = -\left[g^{\nu\sigma} - x^\nu(\partial\, x)^{-1}\partial^\sigma\right]j_\sigma. \tag{26}$$

By taking $\nu = k$, and taking the derivative ∂_k of (26), we may write

$$-\partial_0 A^0 = \frac{1}{\nabla^2}\left(\partial_0^2\,\partial_k A^k + \partial_k S^k\right) \tag{27}$$

which when substituted in (26) gives

$$A^\nu = \Box^{-1}S^\nu + \frac{\partial^\nu}{\nabla^2}\left(\partial_k A^k - \frac{1}{\Box}\partial_k S^k\right). \tag{28}$$

That is, A^ν is of the form

$$A^\nu = \Box^{-1}S^\nu + \partial^\nu a. \tag{29}$$

For $\nu = k$, and multiplying (29) by x^k/x^0, we have from (21)

$$A^0 = \frac{x^k}{x^0}\Box^{-1}S^k + \frac{x^k}{x^0}\partial^k a. \tag{30}$$

On the other hand, directly from (29) with $\nu = 0$ in it,

$$A^0 = \Box^{-1}S^0 + \partial^0 a \tag{31}$$

which upon comparison with (30) leads to

$$x\partial a = -x^\mu\Box^{-1}S_\mu. \tag{32}$$

From (29), (32) and the definition of S^ν in (24), we obtain

$$A^\nu = -\frac{1}{\Box}\left(g^{\nu\mu} - \partial^\nu\frac{1}{x\,\partial + 2}x^\mu\right)\left(g_{\mu\sigma} - x_\mu\frac{1}{\partial x}\partial_\sigma\right)j^\sigma \tag{33}$$

where we have noted that $\partial x = 4 + x\partial$. It is straightforward to check from (33) that $x_\nu A^\nu = 0$ is indeed satisfied.

To establish the transformation from covariant gauges to the FS gauge, we have to pull $\Box^{-1}$ in (33) between the two round brackets. To this end we note that

$$\Box x\partial = (x\partial + 2)\,\Box \tag{34}$$

and hence

$$(\Box\, x\partial)^{-1} = (x\partial)^{-1}\Box^{-1} = \Box^{-1}(x\partial + 2)^{-1} \tag{35}$$

i.e.,

$$\frac{1}{\Box}\frac{1}{x\partial+2} = \frac{1}{x\partial}\frac{1}{\Box}. \tag{36}$$

We may also use the identity

$$\frac{1}{\Box}x^{\mu} = x^{\mu}\frac{1}{\Box} - 2\frac{\partial^{\mu}}{\Box} \tag{37}$$

and since ∂^{μ} when applied to the second factor in (33) gives

$$\partial^{\mu}\left(g_{\mu\sigma} - x_{\mu}\frac{1}{\partial x}\partial_{\sigma}\right) = 0. \tag{38}$$

We obtain from (36)–(38), (33)

$$A^{\nu} = \left(g^{\nu\mu} - \partial^{\nu}\frac{1}{x\partial}x^{\mu}\right)\frac{1}{(-\Box)}\left(g_{\mu\sigma} - x_{\mu}\frac{1}{\partial x}\partial_{\sigma}\right)j^{\sigma}. \tag{39}$$

Now we invoke the transversality property in (38) to *rewrite* (39) as

$$A^{\nu} = \left(g^{\nu\mu} - \partial^{\nu}\frac{1}{x\partial}x^{\mu}\right)\frac{1}{(-\Box)}\Big[g_{\mu\rho} - H(\Box)\partial_{\mu}\partial_{\rho}\Big]\left(g^{\rho\sigma} - x^{\rho}\frac{1}{\partial x}\partial^{\sigma}\right)j_{\sigma} \tag{40}$$

where $H(\Box)$ is *arbitrary* on account of (38).

It remains to set

$$g^{\rho\sigma} - x^{\rho}\frac{1}{\partial x}\partial^{\sigma} = O^{\rho\sigma} \tag{41}$$

and note that for the factor multiplying j_{σ} on the right-hand side of (40),

$$\langle x|(\bullet)|x'\rangle = \int(\mathrm{d}x'')(\mathrm{d}x''')\langle x''|O^{\mu\nu}|x\rangle\langle x''|(D_{\mathrm{H}})_{\mu\rho}|x'''\rangle\langle x'''|O^{\rho\sigma}|x'\rangle \tag{42}$$

where, as shown in the appendix, we have noted that

$$\langle x|\partial^{\nu}(x\partial)^{-1}x^{\mu}|x'\rangle = \langle x'|x^{\mu}(\partial x)^{-1}\partial^{\nu}|x\rangle \tag{43}$$

and we recognize $\langle x''|(D_H)_{\mu\rho}|x'''\rangle$ to have the very general structure in (9). Hence we may write, as in (10),

$$F_{\mathrm{FS}}[\eta,\overline{\eta},J] = \mathrm{e}^{\mathrm{i}W''}F[\rho,\overline{\rho},K;G]|_{\rho=0,\overline{\rho}=0,K=0} \tag{44}$$

where W'' is given by (11) with a^{μ} in the latter replaced by $x^{\mu}(\partial x)^{-1}$. [For interpretation of $x^{\mu}(\partial x)^{-1}\partial^{\nu}$ see the appendix and also Kummer and Weiser (1986).]

The axial gauge $n^{\mu}A_{\mu} = 0$, with n^{ν} a fixed vector, is handled similarly, with A^{ν} in (39) now replaced by

$$A^{\nu} = \left(g^{\nu\mu} - \partial^{\nu}\frac{1}{n\,\partial}n^{\mu}\right)\frac{1}{(-\Box)}\left(g_{\mu\sigma} - n_{\mu}\frac{1}{n\,\partial}\partial_{\sigma}\right)j^{\sigma} \tag{45}$$

and a similar expression as in (44) holds with a^μ in (10) replaced by $n^\mu(n\,\partial)^{-1}$ in it.

3. CONCLUSION

We have seen that the algebraic method developed in this work solves the gauge transformation problem relating generating functionals in different gauges starting from the vacuum-to-vacuum transition amplitude in the Coulomb gauge. Needless to say, their transformation rules give the transformations of *all* the Green functions encountered in the theory and avoids unnecessary tedious steps otherwise involved. The simplicity and the power of the method is evident and it is expected to be applicable to non-Abelian gauge theories, with (Manoukian, 1986, 1987) or without Faddeev–Popov ghosts, as well as to supersymmetric theories. We have not, however, touched upon uniqueness problems such as the Gribov ambiguity (Gribov, 1978; Zwanziger, 1981). This and extensions to non-Abelian cases and supersymmetric theories will be attempted in a forthcoming report.

APPENDIX

For an explicit derivation of (43), we multiply ∂^ν by $-\mathrm{i}$ and write

$$\partial^\nu(x\partial)^{-1}x^\mu = (xp+1)^{-1}p^\nu x^\mu = \sum_{n=0}^{\infty}(-1)^n(xp)^n p^\nu x^\mu \tag{A.1}$$

upon moving, in the process, p^ν to the right. Using the identity

$$(x^\mu p_\mu)_{\mathrm{op}} = \int (\mathrm{d}x)\frac{(\mathrm{d}p)}{(2\pi)^4}\,|x\rangle\langle p|\,xp\,\mathrm{e}^{\mathrm{i}xp} \tag{A.2}$$

we note that

$$(xp)^n = \int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}x_i p_i\right]\mathrm{e}^{\mathrm{i}x_n(p_n-p_{n-1})}\mathrm{e}^{\mathrm{i}x_{n-1}(p_{n-1}-p_{n-2})}\ldots\mathrm{e}^{\mathrm{i}x_1p_1}\,|x_1\rangle\langle p_n| \tag{A.3}$$

and hence

$$\langle x|\partial^\nu(x\partial)^{-1}x^\mu|x'\rangle = \sum_{n=0}^{\infty}(-1)^n\int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}x_i p_i\right]p_n^\nu x'^\mu\delta(x-x_1)$$
$$\times\mathrm{e}^{\mathrm{i}x_n(p_n-p_{n-1})}\mathrm{e}^{\mathrm{i}x_{n-1}(p_{n-1}-p_{n-2})}\ldots\mathrm{e}^{\mathrm{i}x_1p_1}\mathrm{e}^{-\mathrm{i}p_nx}. \tag{A.4}$$

This may be rewritten in an equivalent form by making the change of variables

$$x_1 = y_n,\ldots,x_n = y_1;\quad p_1 = -q_n,\ldots,p_n = -q_1 \tag{A.5}$$

leading to

$$\langle x|\partial^{\nu}(x\partial)^{-1}x^{\mu}|x'\rangle = -\sum_{n=0}^{\infty}\int\left[\prod_{i=1}^{n}(\mathrm{d}y_i)\frac{(\mathrm{d}q_i)}{(2\pi)^4}y_i q_i\right]x'^{\mu}q_1^{\nu}\delta(y_n - x)$$
$$\times \mathrm{e}^{\mathrm{i}xq_1}\mathrm{e}^{\mathrm{i}y_1(q_2-q_1)}\mathrm{e}^{\mathrm{i}y_2(q_3-q_2)}\ldots\mathrm{e}^{-\mathrm{i}y_n q_n}. \tag{A.6}$$

On the other hand,

$$\langle x|x^{\mu}(\partial x)^{-1}\partial^{\nu}x'\rangle = \langle x|x^{\mu}p^{\nu}(p\,x-1)^{-1}|x'\rangle$$
$$= -\sum_{n=0}^{\infty}\langle x|x^{\mu}p^{\nu}(p\,x)^n|x'\rangle \tag{A.7}$$

and

$$(p^{\mu}x_{\mu})_{\mathrm{op}} = \int(\mathrm{d}x)\frac{(\mathrm{d}p)}{(2\pi)^4}\,|p\rangle\langle x|\,px\mathrm{e}^{-\mathrm{i}px} \tag{A.8}$$

$$(p\,x)^n = \int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}p_i x_i\right]\mathrm{e}^{\mathrm{i}x_1(p_2-p_1)}\ldots\mathrm{e}^{\mathrm{i}x_{n-1}(p_n-p_{n-1})}\mathrm{e}^{-\mathrm{i}x_n p_n}\,|p_1\rangle\langle x_n| \tag{A.9}$$

leading to

$$\langle x|x^{\mu}(\partial x)^{-1}\partial^{\nu}x'\rangle = -\sum_{n=0}^{\infty}\int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}p_i x_i\right]x^{\mu}p_1^{\nu}\delta(x_n - x')$$
$$\times\,\mathrm{e}^{\mathrm{i}xp_1}\mathrm{e}^{\mathrm{i}x_1(p_2-p_1)}\ldots\mathrm{e}^{\mathrm{i}x_{n-1}(p_n-p_{n-1})}\mathrm{e}^{-\mathrm{i}x_n p_n} \tag{A.10}$$

which upon comparison with (A.6) establishes (43).

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POLARIZATION CORRELATIONS IN MUON PAIR PRODUCTION IN THE ELECTROWEAK MODEL*

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Explicit field theory computations are carried out of the joint probabilities associated with spin correlations of $\mu^-\mu^+$ produced in e^-e^+ collision in the standard electroweak model to the leading order. The derived expressions are found to *depend not only on the speed of the e^-e^+ pair but also on the underlying couplings*. These expressions are unlike the ones obtained from simply combining the spins of the relevant particles which are of kinematical nature. It is remarkable that these explicit results obtained from quantum field theory show a clear violation of Bell's inequality.

Keywords: Polarization correlations; quantum field theory; high-energy computation; the Standard Electroweak Model; Bell's test.

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Several experiments have been performed over the years on particles' polarizations correlations[1–5] in the light of Bell's inequality and many Bell-like experiments have been proposed recently in high energy physics.[6–11] We have been particularly interested in actual quantum field theory computations of polarization correlation probabilities of particles produced in basic processes because of the novelties encountered in dynamical calculations as opposed to kinematical considerations to be discussed. Here it is worth recalling that quantum field theory originates from the combination of quantum physics *and* relativity and involve nontrivial dynamics. Many such computations have been done in QED[12,13] as well as in e^-e^+ pair production from some charged and neutral strings.[14] All of these polarization correlation probabilities based on dynamical analyses following from field theory share the interesting property that they depend on the energy (speed) of the colliding particles due to the mere fact that typically the latter carry speed in order to collide. Such analyses are unlikely based on formal arguments of simply combining

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spins,[17] as is usually done, and are of kinematical nature, void of dynamical considerations. Here it is worth recalling that the total spin of a two-particle system each with spin, such as of two spin-1/2's, is obtained not only from combining the spins of the latter but also from any orbital angular momentum residing in their center-of-mass system. For low speed, one expects that the argument based simply on combining the spins of the colliding particles should provide an accurate description of the polarization correlations sought and all of our QED computations[12,13] show the correctness of such an argument in the limit of low speed. Needless to say, we are interested in the *relativistic* regime as well, and the formal arguments just mentioned fail to provide the correct expressions for the correlations. As a by-product of the work, our computations of the joint polarizations correlations carried out in a full quantum field theory setting show a clear violation of Bell's inequality.

In the present communication we encounter additional *completely novel properties* not encountered in our earlier QED[12,13] calculations. We consider the process $e^-e^+ \to \mu^-\mu^+$ as described in the standard electroweak (EW) model. It is well known that this process[15] as computed in the EW model is in much better agreement with experiments than that of a QED computation. The reasons for considering such a process in the EW model are many, one of which is the high precision of the differential cross-section obtained as just discussed. Reasons which are, however, more directly relevant to our anylyses are the following. *Due to the theshold energy needed to create the $\mu^-\mu^+$ pair, the limit of the speed β of the colliding particles cannot be taken to go to zero.* This is different from the processes treated by the authors in QED such as in $e^-e^- \to e^-e^-$, $e^+e^- \to 2\gamma$, thus all arguments based simply on combining the spins of e^-, e^+, without dynamical considerations, *fail.* [As a matter of fact the latter argument would lead to the joint probability *in* (7) we are seeking, the incorrect result $\sin^2((\chi_1 - \chi_2)/2)$ — an expression which has been used for years.] Another novelty we encounter in the present investigation is that the polarization correlations *not only depend on speed but also have an explicit dependence on the underlying couplings.* Again this latter explicit dependence is different from the situation arising in QED.[12,13]

The relevant quantity of interest here in testing Bell's inequality[16,17] is, in a standard notation,

$$S = \frac{p_{12}(a_1, a_2)}{p_{12}(\infty,\infty)} - \frac{p_{12}(a_1, a_2')}{p_{12}(\infty,\infty)} + \frac{p_{12}(a_1', a_2)}{p_{12}(\infty,\infty)} + \frac{p_{12}(a_1', a_2')}{p_{12}(\infty,\infty)} - \frac{p_{12}(a_1', \infty)}{p_{12}(\infty,\infty)} - \frac{p_{12}(\infty, a_2)}{p_{12}(\infty,\infty)} \quad (1)$$

as is *computed from* the electroweak model. Here a_1, a_2 (a_1', a_2') specify directions along which the polarizations of two particles are measured, with $p_{12}(a_1, a_2)/p_{12}(\infty,\infty)$ denoting the joint probability, and $p_{12}(a_1,\infty)/p_{12}(\infty,\infty)$, $p_{12}(\infty, a_2)/p_{12}(\infty,\infty)$ denoting the probabilities when the polarization of only one of the particles is measured. [$p_{12}(\infty,\infty)$ is a normalization factor.] The corresponding probabilities as computed from the electroweak model will be denoted by $P(\chi_1, \chi_2)$, $P(\chi_1, -)$, $P(-, \chi_2)$ with χ_1, χ_2 denoting angles specifying directions along which spin measurements are carried out with respect to certain axes spelt out in the bulk of the paper. To show that the electroweak model is in violation with Bell's inequality of LHV, it is sufficient to find one set of angles χ_1, χ_2, χ_1', χ_2', such that S, as computed in the electroweak model, leads to a value of S outside the interval $[-1, 0]$. In this work, it is implicitly assumed that the polarization parameters in the particle states are directly observable and may be used for Bell-type measurements as discussed.

We consider the process $e^-e^+ \to \mu^-\mu^+$ in the center-of-mass frame (see Fig. 1) with the momentum of, say, e^- chosen to be $\mathbf{p} = \gamma\beta m_e(0, 1, 0) = -\mathbf{k}$, m_e denoting its mass and $\gamma = 1/\sqrt{1-\beta^2}$. The momentum of the emerging μ^- will be taken to be $\mathbf{p}' = \gamma'\beta' m_\mu(1, 0, 0) = -\mathbf{k}'$, $\gamma' = 1/\sqrt{1-\beta'^2}$, and m_μ is the mass of $\mu^-(\mu^+)$, the spinors of e^-, e^+ are chosen as

$$u(p) = \sqrt{\frac{\gamma+1}{2}} \begin{pmatrix} \uparrow \\ i\dfrac{\gamma\beta}{\gamma+1} \downarrow \end{pmatrix} \quad \text{and} \quad v(k) = \sqrt{\frac{\gamma+1}{2}} \begin{pmatrix} i\dfrac{\gamma\beta}{\gamma+1} \uparrow \\ \downarrow \end{pmatrix}. \tag{2}$$

Obviously, there is a nonzero probability of occurrence of the above process. Given that such a process has occurred, we compute the conditional joint probability of spins measurements of μ^-, μ^+ along directions specified by the angles χ_1, χ_2 as shown in Fig. 1. Here we have considered the so-called singlet state. The triplet

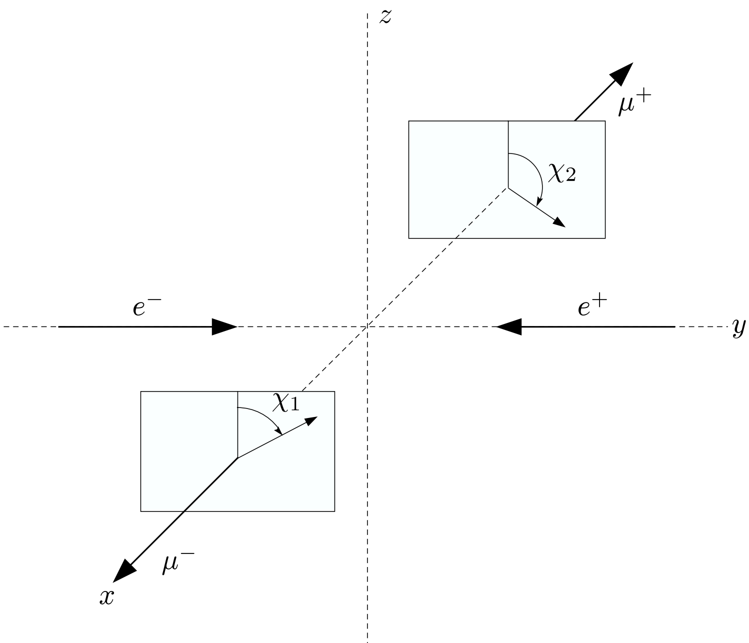

Fig. 1. The figure depicts the process $e^-e^+ \to \mu^-\mu^+$, with e^-, e^+ moving along the y-axis, and the emerging muons moving along the x-axis. χ_1 and χ_2 denote the angles with the z-axis specifying the directions of measurements of the spins of μ^- and μ^+, respectively.

state leads to an expression similar to the one in (7) for the probability in question with different coefficients $A(\mathcal{E}),\dots,E(\mathcal{E})$, $N(\mathcal{E})$ and leads again to a violation of Bell's inequality. The corresponding details may be obtained from the authors by the interested reader.

A fairly tedious computation for the invariant amplitude of the process[18–20] in Fig. 1 leads to

$$\mathcal{M} \propto \left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right) + B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right) + C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right] - i\left[D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right) + E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right], \tag{3}$$

where

$$A(\mathcal{E}) = \left(\frac{M_Z^2}{4\mathcal{E}^2} + ab^2 - 1\right), \tag{4a}$$

$$B(\mathcal{E}) = -\left(\frac{m_e}{m_\mu}\right)\left(\frac{M_Z^2}{4\mathcal{E}^2} + ab^2 - 1\right), \tag{4b}$$

$$C(\mathcal{E}) = \frac{abm_e}{\mathcal{E}m_\mu}\sqrt{\mathcal{E}^2 - m_\mu^2}, \tag{4c}$$

$$D(\mathcal{E}) = \frac{a}{m_\mu\mathcal{E}}\sqrt{\mathcal{E}^2 - m_\mu^2}\sqrt{\mathcal{E}^2 - m_e^2}, \tag{4d}$$

$$E(\mathcal{E}) = -\frac{ab}{m_\mu}\sqrt{\mathcal{E}^2 - m_e^2} \tag{4e}$$

and

$$a \equiv \frac{g^2}{16e^2\cos^2\theta_{\rm W}} \cong 0.353, \qquad b \equiv 1 - 4\sin^2\theta_{\rm W} \cong 0.08 \tag{4f}$$

g denotes the weak coupling constant, $\theta_{\rm W}$ is the Weinberg angle, and e denotes the electric charge. The contribution of the Higgs particles turns out to be too small and is negligible.[19]

Using the notation $F(\chi_1,\chi_2)$ for the absolute value squared of the right-hand side of (3), the conditional joint probability distribution of spin measurements along the directions specified by the angles χ_1, χ_2 is given by

$$P(\chi_1,\chi_2) = \frac{F(\chi_1,\chi_2)}{N(\mathcal{E})}, \tag{5}$$

where the normalization factor $N(\mathcal{E})$ is

$$N(\mathcal{E}) \equiv F(\chi_1,\chi_2) + F(\chi_1+\pi,\chi_2) + F(\chi_1,\chi_2+\pi) + F(\chi_1+\pi,\chi_2+\pi) = 2\{[A(\mathcal{E})]^2 + [B(\mathcal{E})]^2 + [C(\mathcal{E})]^2 + [D(\mathcal{E})]^2 + [E(\mathcal{E})]^2\} \tag{6}$$

giving

$$P(\chi_1,\chi_2) = \frac{1}{N(\mathcal{E})}\left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right) + B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)\right.$$
$$\left. + C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 + \frac{1}{N(\mathcal{E})}\left[D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)\right.$$
$$\left. + E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2. \tag{7}$$

The probabilities associated with the measurement of only one of the polarizations are given respectively, by

$$P(\chi_1,-) = \frac{1}{2} - \frac{2B(\mathcal{E})}{N(\mathcal{E})}[A(\mathcal{E})\cos\chi_1 + C(\mathcal{E})\sin\chi_1] \tag{8}$$

and similarly for χ_2

$$P(-,\chi_2) = \frac{1}{2} + \frac{2B(\mathcal{E})}{N(\mathcal{E})}[A(\mathcal{E})\cos\chi_2 + C(\mathcal{E})\sin\chi_2]\,. \tag{9}$$

It is important to note that $P(\chi_1,\chi_2) \neq P(\chi_1,-)P(-,\chi_2)$, in general, showing the obvious correlations occurring between the two spins.

The indicator S in (1) computed according to the probabilities $P(\chi_1,\chi_2)$, $P(\chi_1,-)$, $P(-,\chi_2)$ in (7)–(9) may be readily evaluated. To show violation of Bell's inequality, it is sufficient to find four angles χ_1, χ_2, χ_1', χ_2' at accessible energies, for which S falls outside the interval $[-1,0]$. For $\mathcal{E} = 105.656$ MeV, i.e. near threshold, an optimal value of S is obtained equal to -1.28203, for $\chi_1 = 0°$, $\chi_2 = 45°$, $\chi_1' = 90°$, $\chi_2' = 135°$, clearly violating Bell's inequality. For the energies originally carried out in the experiment on the differential cross-section at $\mathcal{E} \sim 34$ GeV, an optimal value of S is obtained equal to -1.22094 for $\chi_1 = 0°$, $\chi_2 = 45°$, $\chi_1' = 51.13°$, $\chi_2' = 170.85°$.

As mentioned in the introductory part of the paper, one of the reasons for this investigation arose from the fact that the limit of the speed β of e^-e^+ cannot be taken to go to zero due to the threshold energy needed to create the $\mu^-\mu^+$ pair and methods used for years by simply combining the spins of the particles in question completely fail. The present computations are expected to be relevant near the threshold energy for measuring the spins of the $\mu^-\mu^+$ pair. Near the threshold, the indicator S_{QED} computed within QED coincides with that of S given above in the electroweak model, and varies slightly at higher energies, thus confirming that the weak effects are negligible. Due to the persistence of the dependence of the indicator S on speed, as seen above, in a nontrivial way, it would be interesting if any experiments may be carried out to assess the accuracy of the indicator S as computed within (relativistic) quantum field theory. As there is ample support of the dependence of polarizations correlations, as we have shown by explicit computations in quantum field theory in the electroweak interaction as well as QED ones,[12,13] on speed, we hope that some new experiments will be carried out in the light of Bell-like tests which monitor speed as further practical tests of quantum physics in the relativistic regime.

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Variational derivatives of transformation functions in quantum field theory

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Abstract
A systematic explicit derivation is given for variational derivatives of transformation functions in field theory with respect to parameters variations, also known as the quantum dynamical principle (QDP), by introducing, in the process, two unitary time-dependent operators which in turn allow an otherwise non-trivial interchange of the orders of the parameters variations of transformation functions with specific time-dependent ones. Special emphasis is put on dependent fields, as appearing, particularly, in gauge theories, and on the Lagrangian formalism. The importance of the QDP and its practicality as a powerful tool in field theory are spelled out, which cannot be overemphasized, and a complete derivation of it is certainly lacking in the literature. The derivation applies to gauge theories as well.

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1. Introduction

The purpose of this study is to *derive* systematically variational derivatives of transformation functions, also referred to as the quantum dynamical principle (QDP), with respect to parameters occurring in the theory and with respect to external sources, coupled to the underlying fields, in quantum field theory. The very elegant QDP [1–14] is undisputably recognized as a very powerful tool for carrying out explicit computations in quantum field theory, and in the quantization problem, in general. How these applications and constructions are carried out using variational derivatives of transformation functions, derived below, and will be spelled out for the convenience of the reader in the concluding section. In particular, the QDP has been used to quantize gauge theories [10–13] in constructing the vacuum-to-vacuum transition amplitude and the direct generation and *derivation* [10–13] of Faddeev–Popov (FP) [15] factors, encountered in non-abelian gauge theories and their further generalizations [13] with not much effort and without making an appeal to path integrals or to commutation rules and without [10–13] even going into the well known complicated structure of the Hamiltonian [16]. In particular, it has been shown [13] that the so-called FP factor needs to be modified in more general cases of gauge theories and that a gauge invariant theory does not necessarily imply the familiar FP factor for proper quantization as may be otherwise naïvely expected based on symmetry arguments. As the QDP provides the variations of transformation functions with respect to external parameters, such as coupling constants and external sources coupled to the quantum fields, upon integrations of the amplitudes over these parameters yield the expression for the latter (see e.g. [8–14]). To derive variational derivatives of transformation functions, we introduce in the process, two unitary time-dependent operators which in turn allow an otherwise non-trivial interchange of the orders of parameters variations with specific time-dependent ones. This procedure answers the otherwise rather mysterious question as to why the variation of a transformation function, with respect to given parameters, is restricted solely to the variation of the Lagrangian in question with the states defining the transformation function, which may depend on these parameters, kept non-varied! The answer is based, mostly on equations (6) and (7) below and a key identity derived in (9) written in terms of the two unitary time-dependent operators mentioned above. The derivation is an extension of the corresponding one in quantum mechanics [9] to the more complicated case of quantum field theory, where now emphasis is also put on dependent fields, as occurring, particularly, in gauge theories and on the Lagrangian formalism. There has been renewed interest recently in Schwinger's action principle (see e.g. [17–20]) emphasizing generally operator aspects of a theory, as deriving, for example, various commutation relations, rather than dealing with the computational aspects directly related to transformation functions and transition amplitudes through their variational derivatives as done here, and most importantly, to be derived in this work. In the concluding section, we spell out how variational derivatives of transformation functions are used in various aspects of the theory, emphasizing the underlying method as a powerful tool in quantum field theory.

2. The QDP

Consider a Hamiltonian of the general form

$$H(t,\lambda) = H_1(t) + H_2(t,\lambda), \tag{1}$$

where $H_1(t)$, $H_2(t,\lambda)$ may be time-dependent but $H_2(t,\lambda)$ may, in addition, depend on some parameters denoted by λ. Typically, in quantum field theory, $H_1(t)$ may stand for the free Hamiltonian written in terms of the physically observed masses referred to renormalized masses and $H_1(t)$ will be time-independent. In this latter case, $H_2(t,\lambda)$ will denote the remaining part of the Hamiltonian which, in particular, depends on renormalization constants, coupling constants and so-called external sources coupled to the quantum fields. The coupling constants and the external sources will be then collectively denoted by λ. A derivative of a transformation function with respect to λ with the latter denoting an external source will then represent a functional derivative (see e.g. [10]).

The time evolution operator $U(t,\lambda)$, corresponding to the Hamiltonian $H(t,\lambda)$, satisfies the equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}U(t,\lambda) = H(t,\lambda)U(t,\lambda). \tag{2}$$

For the theory given in a specific description, we have

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle at| = \langle at|H(t,\lambda), \tag{3}$$

where the states $\langle at|$ will depend on the parameters λ. Typically, the states $\langle a|$, assumed independent of λ, may represent multi-particle states of free particles associated with a given self-adjoint operator such as the momentum operator, with the single particle energies written in terms of the observed masses, or may represent the vacuum-state. One may also introduce the time evolution operator $U_1(t)$, corresponding to $H_1(t)$, satisfying the equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_1(t) = H_1(t)U_1(t), \tag{4}$$

and the states ${}_1\langle at|$ which are independent of the parameters λ, satisfy

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,{}_1\langle at| = {}_1\langle at|H_1(t). \tag{5}$$

The states $\langle at|$ of interest are related to the states ${}_1\langle at|$ by the equation

$$\langle at| = {}_1\langle at|V(t,\lambda), \tag{6}$$

where

$$V(t,\lambda) = U_1^\dagger(t)U(t,\lambda), \tag{7}$$

with the latter satisfying

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}V(t,\lambda) = U_1^\dagger(t)H_2(t,\lambda)U(t,\lambda). \tag{8}$$

The QDP is involved with the study of the variation of a transformation function $\langle at_2|bt_1\rangle$, with respect to the parameters λ.

For $\tau \neq t_2$, $\tau \neq t_1$ and $\lambda \neq \lambda'$, we have the following useful *key* identity in the entire analysis

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}\left[V(t_2,\lambda)V^\dagger(\tau,\lambda)V(\tau,\lambda')V^\dagger(t_1,\lambda')\right]$$
$$= V(t_2,\lambda)\left[U^\dagger(\tau,\lambda)(H(\tau,\lambda') - H(\tau,\lambda))U(\tau,\lambda')\right]V^\dagger(t_1,\lambda'), \tag{9}$$

which will be subsequently used.

The independent quantum fields of the theory will be denoted by $\chi(x)$ and their canonical conjugate momenta by $\pi(x)$, suppressing all obvious indices. The dependent fields will be denoted by $\eta(x)$ whose canonical conjugate momenta vanish, by definition. Here $x = (t,\ \mathbf{x})$. The Hamiltonian $H(t,\lambda)$ may be then written as

$$H(t,\lambda) = H(\chi,\pi,\lambda,t), \tag{10}$$

which, in particular, is a function of $\chi(\mathbf{x})$, $\pi(\mathbf{x})$ with the latter defined in the so-called Schrödinger representation at $t = 0$, which are independent of λ. In the Heisenberg representation we have

$$\chi(x) = U^\dagger(t,\lambda)\chi(\mathbf{x})U(t,\lambda), \tag{11}$$

$$\pi(x) = U^\dagger(t,\lambda)\pi(\mathbf{x})U(t,\lambda) \tag{12}$$

having non-trivial dependence on the parameters λ.

Now we integrate the relation in (9) over τ from t_1 to t_2 to obtain

$$\left[V(t_2,\lambda')V^\dagger(t_1,\lambda') - V(t_2,\lambda)V^\dagger(t_1,\lambda)\right] = -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)$$
$$\times\left[\int_{t1}^{t_2}\mathrm{d}\tau\, U^\dagger(\tau,\lambda)(H(\tau,\lambda') - H(\tau,\lambda))U(\tau,\lambda')\right]V(t_1,\lambda'). \tag{13}$$

By setting $\lambda' = \lambda + \delta\lambda$, one obtains the variational form of the above equation

$$\delta[V(t_2,\lambda)V^\dagger(t_1,\lambda)]$$
$$= -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\left[\int_{t_1}^{t_2}\mathrm{d}\tau\, U^\dagger(\tau,\lambda)\delta H(\tau,\lambda)U(\tau,\lambda)\right]V^\dagger(t_1,\lambda). \tag{14}$$

Upon defining the Heisenberg representation of $H(\tau,\lambda)$ at time τ, by

$$I\!H(\tau,\lambda) = U^\dagger(\tau,\lambda)H(\chi,\pi,\tau,\lambda)U(\tau,\lambda), \tag{15}$$

we may rewrite (14), as

$$\delta[V(t_2,\lambda)V^\dagger(t_1,\lambda)] = -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\left[\int_{t_1}^{t_2}\mathrm{d}\tau\,\delta I\!H(\tau,\lambda)\right]V^\dagger(t_1,\lambda) \tag{16}$$

provided the variations of $I\!H$ with respect to λ in (16) are carried out *by keeping* $\chi(x)$, $\pi(x)$, given in (11) and (12), fixed.

We take the matrix elements of (16) with respect to ${}_1\langle at_2|$, $|bt_1\rangle_1$ (see (5)), use (6), *and* note the λ independence of ${}_1\langle at_2|$, $|bt_1\rangle_1$, to obtain

$$\delta\,\langle at_2|bt_1\rangle = -\frac{\mathrm{i}}{\hbar}\left\langle at_2\left|\int_{t_1}^{t_2}\mathrm{d}\tau\,\delta I\!H(\tau,\lambda)\right|bt_1\right\rangle, \tag{17}$$

with the variation in $I\!H$, with respect to λ, carried out with the independent fields $\chi(x)$ and their canonical conjugate momenta $\pi(x)$ kept *fixed*.

The Hamiltonian $I\!H$ in the Heisenberg representation in (15) may be rewritten as

$$I\!H(t,\lambda) = H(\chi(t),\pi(t),\lambda,t), \tag{18}$$

as obtained from the Hamiltonian $H(t,\lambda)$ in (10) at t, by carrying out the explicit operation given in (15). Equation (18) is, in particular, written in terms of the independent (Heisenberg) fields at time t and their canonical conjugate momenta. The effective Lagrangian L_* of the system is related to $I\!H$ by the equation

$$L_*(\chi(t),\dot\chi(t),\lambda,t) = \int \mathrm{d}^3\mathbf{x}\,\pi(x)\dot\chi(x) - H(\chi(t),\pi(t),\lambda,t), \tag{19}$$

with a summation over the fields understood.

The canonical conjugate momenta $\pi(x)$ of the fields are defined through the equation

$$L_*(\chi(t),\dot\chi(t)+\delta\dot\chi(t),\lambda,t) - L_*(\chi(t),\dot\chi(t),\lambda,t) = \int \mathrm{d}^3\mathbf{x}\,\pi(x)\delta\dot\chi(x). \tag{20}$$

Equations (19) and (20) allow us to consider the variation of $H(\chi(\tau),\pi(\tau),\lambda,\tau)$, with respect to λ, by keeping χ,π fixed as required in (17), in relationship to the variation of L_*. From (19) and (20), we then obtain, with χ,π kept fixed, that

$$\delta L_*(\chi(\tau),\dot\chi(\tau),\lambda,\tau) = -\delta H(\chi(\tau),\pi(\tau),\lambda,\tau), \tag{21}$$

upon *cancellation* of the term on the right-hand side of (20), where, now the variation of L_* in (21) is carried out with respect to λ by keeping $\chi(\tau)$ and $\dot\chi(\tau)$ fixed.

The dependent fields will be denoted by $\eta(x)$ and their canonical conjugate momenta vanish, by definition. The Lagrangian of the underlying field theory may be written as $L(\chi(t),\dot\chi(t),\eta(t),\lambda,t)$, which upon the elimination of $\eta(t)$ in favour of $\chi(t)$, $\dot\chi(t)$ and λ generating the Hamiltonian under study as well as the effective Lagrangian L_*. We consider the variation of L, with respect to λ, by keeping $\chi(t),\dot\chi(t)$ fixed. Now since $\eta(t)$ will, in general, *depend* on λ, we have

$$\delta L = E_\eta \frac{\partial\eta}{\partial\lambda}\delta\lambda + \delta L\Big|_{\chi,\dot\chi,\eta}, \tag{22}$$

where we note that the Lagrangian does not contain terms depending on $\dot\eta$, by definition. The first term on the right-hand side defined as an integral in abbreviated form, E_η in it corresponds to the Euler–Lagrange equation of η, which vanishes, and the second term on the right-hand denotes the variation of L, with respect to λ, by keeping $\chi,\dot\chi$ and η fixed. The latter property was first noted in [7]. The Lagrangian density $\mathscr{L} = \mathscr{L}(x) = \mathscr{L}(x,\lambda)$ of the system is related to the Lagrangian L through

$$L(\chi(t),\dot\chi(t),\eta(t),\lambda,t) = \int \mathrm{d}^3\mathbf{x}\,\mathscr{L}(x,\lambda). \tag{23}$$

From (21), (22) and (23), we obtain the celebrated QDP or the Schwinger dynamical (action) principle

$$\delta\langle at_2|bt_1\rangle = \frac{\mathrm{i}}{\hbar}\left\langle at_2\left|\int_{t_1}^{t_2}(\mathrm{d}x)\,\delta\mathscr{L}(x,\lambda)\right|bt_1\right\rangle, \tag{24}$$

where $(\mathrm{d}x) = \mathrm{d}t\,\mathrm{d}^3\mathbf{x}$, and the variation $\delta\mathscr{L}(x,\lambda)$, with respect to λ, is carried out with the fields, independent *and* dependent, and their derivatives $\partial_\mu\chi$, $\nabla\eta$, all kept *fixed*. The interesting thing to note is that although the states $|at_2\rangle$, $|bt_1\rangle$ depend on λ, in the variation of the transformation function $\langle at_2|bt_1\rangle$, the same (non-varied) states appear on the right-hand side of (24) with the entire variation being applied to the Lagrangian density $\mathscr{L}(x,\lambda)$ with the fields and their canonical conjugate momenta kept fixed. This is thanks to the U and V operators elaborated upon in (2)–(8), the independence of the states${}_1\langle at_2|$,${}_1\langle bt_1|$ of λ, and the key identify given in (9). In practice the limits $t_2\to+\infty$, $t_1\to-\infty$ are taken in (24) in scattering processes.

Now consider an arbitrary function

$$B(\chi(x),\pi(x),\lambda,t) \equiv I\!B(t,\lambda), \tag{25}$$

of the variables indicated, with $\chi(x)$, $\pi(x)$ in the Heisenberg representation in (11) and (12). We may write

$$I\!B(t,\lambda) = U^\dagger(t,\lambda)B(\chi(\mathbf{x}),\pi(\mathbf{x}),\lambda,t)U(t,\lambda), \tag{26}$$

with $\chi(\mathbf{x})$, $\pi(\mathbf{x})$ on the right-hand side in the Schrödinger representation at time $t=0$. We note the identity

$$\begin{aligned} &V(t_2,\lambda)I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)\\ &\quad = V(t_2,\lambda)V^\dagger(\tau,\lambda)U_1^\dagger(\tau)B(\chi(\mathbf{x}),\pi(\mathbf{x}),\lambda,\tau)\\ &\qquad \times U_1(\tau)V(\tau,\lambda)V^\dagger(t_1,\lambda). \end{aligned} \tag{27}$$

Hence (14) and (27) give for the following variation with respect to λ ($t_1<\tau<\tau_2$)

$$\begin{aligned} &\delta[V(t_2,\lambda)I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)]\\ &\quad = -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_\tau^{t_2}\mathrm{d}\tau'\,\delta H(\tau',\lambda)I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)\\ &\qquad + V(t_2,\lambda)\delta I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)\\ &\qquad - \frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_{t_1}^{\tau}\mathrm{d}\tau'\,I\!B(\tau,\lambda)\delta H(\tau',\lambda)V^\dagger(t_1,\lambda), \end{aligned} \tag{28}$$

where according to (28), the variation in $\delta I\!B(\tau,\lambda) = \delta B(\chi(\mathbf{x},\tau),\pi(\mathbf{x},\tau),\lambda,\tau)$, with respect to λ, is carried out by keeping the (Heisenberg) fields $\chi(\mathbf{x},\tau)$, $\pi(\mathbf{x},\tau)$ *fixed*.

We may use the definition of the chronological time ordering product to rewrite (28) in the more compact form

$$\begin{aligned} &\delta[V(t_2,\lambda)I\!B(\tau,\lambda)V^\dagger(t_1,\lambda)]\\ &\quad = -\frac{\mathrm{i}}{\hbar}V(t_2,\lambda)\int_{t_1}^{t_2}\mathrm{d}\tau'(I\!B(\tau,\lambda)\delta H(\tau',\lambda))_+V^\dagger(t_1,\lambda)\\ &\qquad + V(t_2,\lambda)\delta I\!B(\tau,\lambda)V^\dagger(t_1,\lambda). \end{aligned} \tag{29}$$

Upon taking the matrix element of (29) with respect to ${}_1\langle at_2|$, $|bt_1\rangle_1$, and using (6), (15) and (24) we have for $t_1 < \tau < t_2$

$$\delta\langle at_2|\mathbb{B}(\tau,\lambda)|bt_1\rangle = \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2}(\mathrm{d}x')\langle at_2\,|(\mathbb{B}(\tau,\lambda)\delta\mathscr{L}(x',\lambda))_+|\,bt_1\rangle + \langle at_2\,|\delta\mathbb{B}(\tau,\lambda)|\,bt_1\rangle\,, \quad (30)$$

where in the variation $\delta\mathscr{L}(x',\lambda)$, with respect to λ, all the fields and their derivatives $\partial_\mu\chi$, $\nabla\eta$ are kept fixed, while in $\delta\mathbb{B}(\tau,\lambda)$, *expressed* in terms of $\chi(\mathbf{x},\tau)$, $\pi(\mathbf{x},\tau)$, the latter are kept fixed, and an extra λ-dependence may arise from the elimination of η in favour of χ,π. To our knowledge Equation (30) appears first in [7]. The second term in (30) is *responsible* for the generation of the FP factor and its generalizations in gauge theories (see [10, 12, 13]).

3. Conclusion

The importance of the QDP as a powerful tool in field theory cannot be overemphasized and a detailed derivation of it was given by introducing, in the process, two unitary time-dependent operators. The latter in turn allowed the interchange of variations of transformation functions with respect to given parameters with specific time-dependent operations so crucial for the validity of the QDP. A key identity has been derived in (9) which was essential for the entire derivation. For the convenience of the reader we spell out how variational derivatives of transformation functions are used in some aspects of an underlying theory. (i) The integration of (24) for the QDP over λ is carried out by introducing, in the process, external sources coupled to the fields, where the external sources (currents) are necessarily taken initially to be non-conserved so that variations of all of their components may be varied independently (see [10, 13]). From the expression of the vacuum-to-vacuum transition amplitude, for example, thus obtained, transition amplitudes of *all* processes may be extracted by factoring out amplitudes for the emission and absorption of the underlying particles by the external sources. By functional differentiation of the vacuum-to-vacuum transition amplitude with respect to the external sources, integral equations, such as Schwinger–Dyson equations, relating various Green's functions may be derived. We also recall that the path integral expressions may be derived, for example, directly from the application of the QDP principle (see e.g. [8, 9]). It is also far simpler to carry out (functional) differentiations than to deal with infinite dimensional continual integrals. (ii) In the presence of dependent fields, with no time derivatives of them occurring in their respective field equations, these dependent fields will, in general, be functions of independent fields (and their conjugate momenta) *and* external sources. With the rules set up in (24) and in (30), additional terms will then occur coming from the second term on the right-hand side of (30) by taking functional derivatives of matrix elements of such dependent fields in (30) with respect to external sources by keeping the independent fields (and their conjugate momenta) fixed. Such terms lead precisely to FP factors and their generalizations, for example, in gauge theories, from the applications of (30), as just mentioned, in the present formalism (see [8, 10–14]). For such intricate and additional details, the reader may refer to the just given references as well as to some of the earlier ones such as [21, 22].

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Supersymmetry Methods in the Analysis of Ground-State Energy of Many-Particle Systems[1]

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Abstract

Supersymmetry methods are used to derive *rigorously* a lower bound for the exact ground-state energy of many-particle systems for a general class of interactions, in *arbitrary* dimensions, with main emphasis on the state of matter in bulk with Coulomb interactions. In particular, we derive a lower bound for the ground-state energy E_N of so-called "bosonic matter" as a *cubic* power of N - the number of negatively charged particles - valid for *all dimensions* $\nu \geq 2$ providing an upper as well as a lower bound for E_N for such matter in all such dimensions.

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1 Introduction

The study of the nature of the ground-state energy of Hamiltonians of interacting many-particle systems is of central importance for the investigation of the stability of such complex systems. Over the years much work has been done in deriving rigorous bounds (cf. [1, 2, 5, 7–9, 11–13, 15, 17]) on the exact ground-state energy of such Hamiltonians and, in turn, establish stability or instability of the underlying systems with main emphasis on systems pertaining to matter in bulk. The instability of so-called "bosonic matter", i.e., for matter obtained by relaxing the Pauli exclusion constraint (cf. [2, 9, 11–13, 15, 17]) is a result of a power law behaviour N^{γ} of the ground-state energy, where N is the number of negatively charged particles, *with* the exponent γ such that

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$\gamma > 1$. Such a power law behaviour, with $\gamma > 1$, implies instability of the underlying system, since the formation of such matter consisting of $(2N + 2N)$ particles will be favourable over two separate systems brought into contact, each consisting of $(N + N)$ particles, and the energy released upon collapse, in the formation of the former system, being proportional to $[(2N)^\gamma - 2N^\gamma]$, will be overwhelmingly large for realistic large N, e.g., $N \sim 10^{23}$. It is interesting to point out that if collapse occurs, then the radial extension of such a system does not decrease faster than $N^{-1/3}$ [14] upon collapse, as N increases for large N . On the other hand, for ordinary matter, i.e., for which the Pauli exclusion constraint is invoked, the ground-state energy has the single power law behaviour $\sim N$ [8, 19] consistent with stability. In this respect, as the number N is made to increase such matter inflates and its radial extension increases not any slower than $N^{1/3}$ [16]. In recent years there has been also much interest in physics of arbitrary dimensions (cf. [3, 6, 12, 13, 17, 18]). In this respect it is also quite important to investigate if the change of the dimensionality of space will change the properties of many-particle systems and if a given property, such as instability, is a characteristic of the three-dimensional property of space. [Some present field theories speculate that at early stages of the universe, the dimensionality of space was not necessarily three and, by a process which may be referred to as compactification, the present three-dimensional character of space arose upon the evolution and the cooling of the universe.] The purpose of this communication is to use supersymmetry methods to derive *rigorously* lower bounds to a class of Hamiltonians, to be defined in the next section, with particular emphasis on "bosonic matter" in *arbitrary dimensions of space.* The basic idea of supersymmetry methods (cf. [10] for a pedagogical treatment) is to introduce generators $\mathbf{Q}$ and write the Hamiltonian H under consideration, or more precisely a part H' of the Hamiltonian, as $\mathbf{Q}^\dagger . \mathbf{Q}$, where $\mathbf{Q}^\dagger$ is the adjoint of $\mathbf{Q}$, and then use *positivity* constraints to derive a lower bound for H. In the concluding section, further comments on our findings are made.

2 Supersymmetry Methods and the Ground-State Energy: Application to "Bosonic Matter"

For an N-particle system, we introduce N real vector fields $G_j(\mathbf{x}_1, ..., \mathbf{x}_N; \varrho)$, $j = 1, ..., N$, as functions of N dynamical variables $\mathbf{x}_1, ..., \mathbf{x}_N \ \epsilon\ \mathbb{R}^\nu$, which may also depend on some parameters which we denote collectively by ϱ. The space dimension is denoted by ν. We consider a class of potential energies $V(\mathbf{x}_1, ..., \mathbf{x}_N; \varrho)$ defined by

$$V(\mathbf{x}_1, ..., \mathbf{x}_N; \varrho) = -\sum_{j=1}^{N} \boldsymbol{\nabla}_j.\mathbf{G}_j(\mathbf{x}_1, ..., \mathbf{x}_N; \varrho) \tag{1}$$

where $\boldsymbol{\nabla}_j = \partial/\partial\mathbf{x}_j$, and define the multi-particle Hamiltonian by

$$H = \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m_j} \; + \; V(\mathbf{x}_1, ...\mathbf{x}_N; \varrho) \tag{2}$$

with $\mathbf{p}_j = -\mathrm{i}\hbar\boldsymbol{\nabla}_j$, and the m_j denoting the masses of the underlying particles.

Introduce the N operators

$$Q_j = \frac{\hbar\boldsymbol{\nabla}_j}{\sqrt{2m_j}} \; + \; \frac{\sqrt{2m_j}}{\hbar}\mathbf{G}_j \tag{3}$$

and their adjoints

$$Q_j^{\dagger} = -\frac{\hbar\boldsymbol{\nabla}_j}{\sqrt{2m_j}} \; + \; \frac{\sqrt{2m_j}}{\hbar}\mathbf{G}_j \tag{4}$$

$j = 1, ..., N$, and use the property $\boldsymbol{\nabla}_j.\mathbf{G}_j = (\boldsymbol{\nabla}_j.\mathbf{G}_j) + \mathbf{G}_j.\boldsymbol{\nabla}_j$ to obtain for any normalized state $|\Psi\rangle$

$$\begin{aligned} 0 \leq \sum_{j=1}^{N} \parallel Q_j\Psi \parallel^2 &= \sum_{j=1}^{N} \left\langle \Psi \middle| Q_j^{\dagger} Q_j \middle| \Psi \right\rangle \\ &= \sum_{j=1}^{N} \left\langle \Psi \middle| \left[\frac{-\hbar^2\boldsymbol{\nabla}_j^2}{2m_j} \; - \; \boldsymbol{\nabla}_j.\mathbf{G}_j + \frac{2m_j}{\hbar^2}\mathbf{G}_j^2 \right] \middle| \Psi \right\rangle \end{aligned} \tag{5}$$

an idea often used in supersymmetry methods, from which we obtain the basic lower bound

$$\langle \Psi | H | \Psi \rangle \geq -\sum_{j=1}^{N} \frac{2m_j}{\hbar^2} \left\langle \Psi \middle| \mathbf{G}_j^2 \middle| \Psi \right\rangle \tag{6}$$

for *any* Hamiltonian defined by (2), (1), giving a lower bound for the expectation value of the Hamiltonian in the state $|\Psi\rangle$.

A classic application of the above is to the Hamiltonian of matter given by

$$H = \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m} + \sum_{i<j}^{N} \frac{\mathrm{e}^2}{\mid \mathbf{x}_i - \mathbf{x}_j \mid} - \sum_{i=1}^{N}\sum_{j=1}^{k} \frac{Z_j \mathrm{e}^2}{\mid \mathbf{x}_i - \mathbf{R}_j \mid} + \sum_{i<j}^{k} \frac{Z_i Z_j \mathrm{e}^2}{\mid \mathbf{R}_i - \mathbf{R}_j \mid} \tag{7}$$

where k denotes the number of nuclei situated at $\mathbf{R}_1, ...\mathbf{R}_k$ with total charges $Z_1 \mid \mathrm{e} \mid, ..., Z_k \mid \mathrm{e} \mid$ such that $\Sigma_{j=1}^{k} Z_j = N$ for neutral matter.

The potential energy in (7) may be generated *exactly* from the vector fields $\mathbf{G}_j(\mathbf{x}_1, ..., \mathbf{x}_N; \mathbf{R}_1, ..., \mathbf{R}_k)$ defined by

$$\begin{aligned} \mathbf{G}_j(\mathbf{x}_1, ..., \mathbf{x}_N; \mathbf{R}_1, ..., \mathbf{R}_k) = &- \frac{\mathrm{e}^2}{(\nu - 1)} \sum_{\ell=1}^{j-1} \mathbf{n}_{j\ell} + \frac{\mathrm{e}^2}{(\nu - 1)} \sum_{\ell=1}^{k} Z_\ell \, \mathbf{k}_{j\ell} \\ &- \frac{\mathbf{x}_j \, \mathrm{e}^2}{\nu N} \sum_{i<\ell}^{k} \frac{Z_i Z_\ell}{\mid \mathbf{R}_i - \mathbf{R}_\ell \mid} \end{aligned} \tag{8}$$

with $\nu \geq 2$ the dimensionality of the space considered, and $\mathbf{n}_{j\ell}$, $\mathbf{k}_{j\ell}$ are *unit* vector fields defined by

$$\mathbf{n}_{j\ell} = \frac{\mathbf{x}_j - \mathbf{x}_\ell}{|\mathbf{x}_j - \mathbf{x}_\ell|}, \qquad \mathbf{k}_{j\ell} = \frac{\mathbf{x}_j - \mathbf{R}_\ell}{|\mathbf{x}_j - \mathbf{R}_\ell|} \tag{9}$$

by using, in the process, the facts that

$$\sum_{j=1}^{N} \boldsymbol{\nabla}_j . \mathbf{x}_j = \nu N \tag{10}$$

$$\sum_{j=2}^{N} \sum_{\ell=1}^{j-1} \boldsymbol{\nabla}_j . \mathbf{n}_{j\ell} = (\nu - 1) \sum_{\ell<j}^{N} \frac{1}{\mid \mathbf{x}_j - \mathbf{x}_\ell \mid} \tag{11}$$

$$\sum_{j=1}^{N} \sum_{\ell=1}^{k} \boldsymbol{\nabla}_j . \mathbf{k}_{j\ell} = (\nu - 1) \sum_{j=1}^{N} \sum_{\ell=1}^{k} \frac{1}{\mid \mathbf{x}_j - \mathbf{R}_\ell \mid} \tag{12}$$

giving

$$-\sum_{j=1}^{N}\boldsymbol{\nabla}_j.\mathbf{G}_j(\mathbf{x}_1,...,\mathbf{x}_N;\mathbf{R}_1,...,\mathbf{R}_k)=\sum_{i<j}^{N}\frac{\mathrm{e}^2}{\mid\mathbf{x}_i-\mathbf{x}_j\mid}-\sum_{i=1}^{N}\sum_{j=1}^{k}\frac{Z_j\mathrm{e}^2}{\mid\mathbf{x}_i-\mathbf{R}_j\mid}+\sum_{i<j}^{k}\frac{Z_iZ_j\mathrm{e}^2}{\mid\mathbf{R}_i-\mathbf{R}_j\mid}\tag{13}$$

which is the potential energy for matter in (7).

Due to the presence of the $\mathbf{x}_j$ factor in the last term on the right-hand side of (8), the lower bound in (6) for the Hamiltonian H *in* (7) will involve unmanageable terms such as $-\parallel\mathbf{x}_j\Psi\parallel^2$ for which no further lower bounds may be directly obtained. Accordingly, the definition in (8) suggests to introduce instead the vector fields $\mathbf{G}'_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...\mathbf{R}_N)$ given by

$$\mathbf{G}'_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N)=-\frac{\mathrm{e}^2}{(\nu-1)}\sum_{\ell=1}^{j-1}\mathbf{n}_{j\ell}+\frac{\mathrm{e}^2}{(\nu-1)}\sum_{\ell=1}^{k}Z_\ell\,\mathbf{k}_{j\ell}\tag{14}$$

with the unit vector fields $\mathbf{n}_{j\ell}$, $\mathbf{k}_{j\ell}$ defined as before, yielding

$$-\sum_{j=1}^{N}\boldsymbol{\nabla}_j.\mathbf{G}'_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N)=\sum_{i<j}^{N}\frac{\mathrm{e}^2}{\mid\mathbf{x}_i-\mathbf{x}_j\mid}-\sum_{i=1}^{N}\sum_{j=1}^{k}\frac{Z_j\mathrm{e}^2}{\mid\mathbf{x}_i-\mathbf{R}_j\mid}\tag{15}$$

From (6), (15), we then obtain the following lower bound for the expectation value of the Hamiltonian in (7) in a state $|\Psi\rangle$

$$\begin{aligned}\langle\Psi|H|\Psi\rangle&\geq-\frac{2m}{\hbar^2}\sum_{j=1}^{N}\left\langle\Psi\middle|\mathbf{G}'^2_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N)\middle|\Psi\right\rangle+\sum_{i<j}^{k}\frac{Z_iZ_j\mathrm{e}^2}{\mid\mathbf{R}_i-\mathbf{R}_j\mid}\\&\geq-\frac{2m}{\hbar^2}\sum_{j=1}^{N}\left\langle\Psi\middle|\mathbf{G}'^2_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N)\middle|\Psi\right\rangle\end{aligned}\tag{16}$$

with $\mathbf{G}'_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N)$ defined in (14).

Upon using the facts that $\mathbf{n}_{j\ell}$, $\mathbf{k}_{j\ell}$, defined in (9), are unit vector fields, i.e., $\mathbf{n}_{j\ell}.\mathbf{n}_{j\ell'}\leq 1$, $\mathbf{k}_{j\ell}.\mathbf{k}_{j\ell'}\leq 1$, $-\mathbf{n}_{j\ell}.\mathbf{k}_{j\ell}\leq 1$, we obtain from (14)

$$\left\langle\Psi\middle|\mathbf{G}'^2_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...\mathbf{R}_N)\middle|\Psi\right\rangle\leq\frac{\mathrm{e}^4}{(\nu-1)^2}(j-1+N)^2\tag{17}$$

where we have used, in the process, the property $\sum_{j=1}^{k} Z_j = N$ for neutral matter.

Summing over j from 1 to N, (17), (16) give the following lower bound for the ground-state energy E_N for the Hamiltonian in (7)

$$E_N \geq -\frac{2m}{\hbar^2}\frac{\mathrm{e}^4}{(\nu-1)^2}\sum_{j=1}^{N}(j-1+N)^2 \tag{18}$$

or

$$E_N > -\Big(\frac{m\mathrm{e}^4}{2\hbar^2}\Big)\frac{16}{3}\frac{N^3}{(\nu-1)^2} \tag{19}$$

Needless to say for $\nu \rightarrow 1$, we do not obtain any contradiction with $-$ infinity as the lower limit of the set of real numbers - which is, however, not interesting.

3 Conclusion

We may combine the above result with an earlier one [17] which derives instead an *upper* bound for E_N valid also for all space dimensions ν and for $N \geq (2)^\nu$. The combined results now state that for the Hamiltonian H in (7) for so-called "bosonic matter"

$$-\Big(\frac{m\mathrm{e}^4}{2\hbar^2}\Big)\frac{N^{(2+\nu)/\nu}}{16\pi^2\nu^3(2)^\nu} \;>\; E_N \;>\; -\Big(\frac{m\mathrm{e}^4}{2\hbar^2}\Big)\frac{16N^3}{3(\nu-1)^2} \tag{20}$$

valid for *all* ν and for $N \geq (2)^\nu$. It is easy to check the consistency relation $16N^3/3(\nu-1)^2 > N^{(2+\nu)/\nu}/16\pi^2\nu^3(2)^\nu$ in relation to the above double inequalities. It is well known that for $\nu = 3$, the power 3 of N in the inequality on the right-hand side of (20) may be reduced to 5/3. Also for $\nu = 3$, for Fermionic, i.e., standard matter with the negatively charged particles obeying the Pauli exclusion principle, the power 3 of N is reduced to one, as mentioned in the introductory section, consistent with the stability criterion of matter. Our result obtained for *arbitrary* dimensions is obviously far from trivial. In (7), the so-called positively charged particles (nuclei) are treated non-dynamically being much heavier than the negatively charged particles which is the common practice. Our lower bound for the ground-state energy E_N given in (19) *is still valid in all dimensions* for an overall neutral system of bosonic charged particles with the positively charged particles treated dynamically as well with the simplification that all the charges are equal in absolute values, provided m on the right-hand side of (19) denotes the *largest* mass in the set of masses of all the positively as well as negatively charged particles and N, being now even, denotes the total number of particles. The inequalities in (20) are consistent with a famous remark made by Dyson [1] concerning bosonic matter and the release of an overwhelmingly large amount of energy, as also discussed in the introductory section, when two such systems are brought into contact: "*[Bosonic] matter in bulk would collapse into a condensed high density phase. The assembly of any two macroscopic objects would release energy comparable to that of an atomic bomb ...*". Such a property will be also shared in higher dimensional spaces than three, as well as in two dimensions. We will not speculate on the physical significance of higher dimensional spaces (cf. [3, 6, 18]) except to re-iterate that it is important to investigate if the change of the dimensionality of space will change the properties of many-particle systems and if a given property, such as instability, is a characteristic of the three-dimensional property of space. Needless to say, two dimensional space, however, seems to be physically relevant at least in condensed matter physics.

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