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Instability of "bosonic matter" in all dimensions ☆

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Abstract

An upper bound is derived for the exact ground state energy E_N of N negatively charged bosons and N motionless, i.e., fixed, positive charges with Coulombic interactions in arbitrary dimensions ν: $E_N < -N^{(2+\nu)/\nu}/16\pi^2\nu^3(2)^\nu$, in units of the Rydberg, for all $N \geqslant (2)^\nu$ thus establishing, in particular, that the instability of "bosonic matter" is not a characteristic of the dimensionality of space.

The instability of "bosonic matter" with N negatively charged bosons and N motionless, i.e., fixed, positive charges with Coulombic interactions, was investigated many years ago by Dyson and Lenard [1] giving rise to the famous $N^{5/3}$ law for the ground state energy E_N. Such a power law behaviour N^α, with $\alpha > 1$, implies the instability of such a system, since the formation of such matter consisting of $(2N + 2N)$ particles will be favourable over two separate systems brought into contact, each consisting of $(N + N)$ particles, and the energy released upon collapse, in the formation of the former system, being proportional to $((2N)^\alpha - 2(N)^\alpha)$, will be overwhelmingly large for realistically large N, e.g., $N \sim 10^{23}$. An elementary, but rigorous, upper bound for E_N was derived by Lieb [2] for $N = 8n^3$, $n = 1, 2, \ldots$, for the problem at hand, and extended for all $N \geqslant 8$ in [3] both in three dimensions. We were able to derive an explicit upper bound for the exact E_N in all dimensions ν

$$E_N < -\left(\frac{me^4}{2\hbar^2}\right)\frac{N^{(2+\nu)/\nu}}{16\pi^2\nu^3(2)^\nu} \tag{1}$$

for all $N \geqslant (2)^\nu$, where m is the smallest of the masses of the N negatively charged bosons. Thus we conclude, in particular, that the instability of such matter is not a characteristic of the dimensionality of space. For example, no stable planar configurations may be formed corresponding to $\nu = 2$. There has been much interest in recent years in the physics of arbitrary dimensions, e.g., [4–7] and the role of the spin and statistics theorem in such dimensions. It is well known that the latter is tied up, e.g., [4] to the dimensionality of space and we learn that such a system not being subjected to stringent constrained statistics is necessarily unstable in all dimensions. It is also an important theoretical question to investigate if the change of the dimensionality of space will change such matter from, e.g., an "implosive" to a "stable" or to an "explosive" phase. The present work shows that this does not hap-

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pen. (Some of present field theories speculate that at early stages of the universe the dimensionality of space was not necessarily three and, by a process, which may be referred to as compactification of space, the present three-dimensional character of space arose upon the evolution and the cooling of the universe.) The potentials considered are the $1/r$ type, and we do not dwell on the fate and the dynamics of the positive charges which, undoubtedly, involve complicated interactions, and instability is established down to and above the nuclear level. Also since both signs of the charges are involved in this work, the analysis becomes more involved than one dealing with one sign of the charges only, e.g., [8]. Needless to say, if for some ν, E_N goes to minus infinity, then (1) is automatically satisfied.

The Hamiltonian under study is given by

$$H = \sum_{i=1}^{N} p_i^2/2m_i + V_1 + V_2 + \sum_{i<j}^{N} e^2 |\vec{R}_i - \vec{R}_j|^{-1}, \tag{2}$$

where

$$V_1 = -\sum_{i=1}^{N}\sum_{j=1}^{N} e^2 |\vec{x}_i - \vec{R}_j|^{-1}, \tag{3}$$

$$V_2 = \sum_{i<j}^{N} e^2 |\vec{x}_i - \vec{x}_j|^{-1} \tag{4}$$

and the $\vec{x}_i$, $\vec{R}_j$ refer to the negative and positive charges, respectively. Quite generally, we may write for any $N \geqslant (2)^\nu$,

$$N = (2n)^\nu \left(1 + \frac{\varepsilon}{n}\right)^\nu < (2n)^\nu (2)^\nu, \quad (2n)^\nu \equiv k, \tag{5}$$

where $0 \leqslant \varepsilon < 1$ [3], $n = 1, 2, \ldots.$ We introduce an N-particle trial function

$$\begin{aligned} \Psi(\vec{x}_1, \ldots, \vec{x}_N) &= (N!k!)^{-1/2} \sum_\pi \phi\big(\vec{x}(\pi_1)\big) \cdots \phi\big(\vec{x}(\pi_k)\big) \\ &\quad \times \psi_1\big(\vec{x}(\pi_{k+1})\big) \cdots \psi_{N-k}\big(\vec{x}(\pi_N)\big), \end{aligned} \tag{6}$$

with the sum over all permutations $\{\pi_1, \ldots, \pi_N\}$ of $\{1, \ldots, N\}$, such that $\phi(\vec{x})$, $\psi_j(\vec{x})$ are pairwise orthonormal,

$$\phi(\vec{x}) = \prod_{i=1}^{\nu}\left(\frac{1}{\sqrt{L}}\cos\left(\frac{\pi x_i}{2L}\right)\right) \equiv \phi_L(\vec{x}), \quad |x_i| \leqslant L \tag{7}$$

and is zero otherwise, and for $j = 1, \ldots, N-k$,

$$\begin{aligned} \psi_j(\vec{x}) &= \prod_{i=1}^{\nu}\left(\frac{1}{\sqrt{L_0}}\cos\left(\frac{\pi(x_i - L_i^{(j)})}{2L_0}\right)\right) \\ &\equiv \phi_{L_0}\big(\vec{x} - \vec{L}^{(j)}\big), \quad \big|x_i - L_i^{(j)}\big| \leqslant L_0 \end{aligned} \tag{8}$$

and are zero otherwise, $\vec{L}^{(j)} = jD(1, 1, \ldots, 1)$. With $L \leqslant L_0 < D/2$, the intervals $\{-L \leqslant x_i \leqslant L\}$, $\{jD - L_0 \leqslant x_i \leqslant jD + L_0\}$, $j = 1, \ldots, N-k$ are pairwise disjoint. $\{-L \leqslant x_i \leqslant L; i = 1, \ldots, \nu\}$ defines a box centered at the origin, while $\{jD - L_0 \leqslant x_i \leqslant jD + L_0; i = 1, \ldots, \nu\}$ defines boxes with centers at the tip of the vectors $\vec{L}^{(j)}$, respectively. We choose the vectors $\vec{R}_1, \ldots, \vec{R}_k$ to lie in the box centered at the origin, while we choose $\vec{R}_{k+1} = \vec{L}^{(1)}, \ldots, \vec{R}_N = \vec{L}^{(N-k)}$.

By using the explicit bound for the expectation value of the kinetic energy

$$\langle\Psi|\sum_{i=1}^{N} p_i^2/2m_i|\Psi\rangle \leqslant \frac{\nu\pi^2\hbar^2}{8m}\left(\frac{k}{L^2} + \frac{(N-k)}{L_0^2}\right), \tag{9}$$

where m is the smallest of the masses of the negatively charged bosons, the following bound for the expectation value of H is readily obtained

$$\begin{aligned} &\langle\Psi|H|\Psi\rangle \\ &\quad \leqslant \frac{\nu\pi^2\hbar^2}{8m}\frac{k}{L^2} + \langle H_1\rangle \\ &\qquad + (N-k)\left[\frac{\nu\pi^2\hbar^2}{8mL_0^2} + e^2\frac{(N+k-1)}{D} - \frac{e^2}{\sqrt{\nu}L_0}\right] \end{aligned} \tag{10}$$

with

$$\begin{aligned} \langle H_1\rangle &= -e^2 k \sum_{j=1}^{k}\int \mathrm{d}^\nu\vec{x}\,\phi_L^2(\vec{x})|\vec{x} - \vec{R}_j|^{-1} \\ &\quad + \frac{e^2 k(k-1)}{2} \\ &\quad \times \int \mathrm{d}^\nu\vec{x}\,\mathrm{d}^\nu\vec{x}'\,\phi_L^2(\vec{x})|\vec{x} - \vec{x}'|^{-1}\phi_L^2(\vec{x}') \\ &\quad + \sum_{i<j}^{k} e^2|\vec{R}_i - \vec{R}_j|^{-1}. \end{aligned} \tag{11}$$

The integrals in (11) involve integrations over the box $\{-L \leqslant x_i \leqslant L; i = 1, \ldots, \nu\}$ centered at the origin, and we recall that $\vec{R}_1, \ldots, \vec{R}_k$ were chosen to lie in this box. By partitioning the unit interval $[0, 1]$: $0 =$

$a_0 < a_1 < \cdots < a_n = 1$, such that [2]

$$\int\limits_{a_{j-1}}^{a_j} \mathrm{d}x \cos^2\left(\frac{\pi x}{L}\right) = \frac{1}{2n} = \left(\frac{1}{k}\right)^{1/\nu} \tag{12}$$

letting $\alpha_j = (a_j - a_{j-1})$, $j = 1, \ldots, n$, $\sum_{j=1}^n \alpha_j = 1$, we generate boxes $B(i_1, \ldots, i_\nu)$ of sides $\alpha_{i_1} \times \alpha_{i_2} \times \cdots \times \alpha_{i_\nu}$ to bound (11) as a sum of integrations over these boxes [2]

$$\begin{aligned}\langle H_1\rangle \leqslant{}& -\frac{e^2k^2}{2}\frac{(2)^\nu}{L} \\ &\times \sum_{B(i_1,\ldots,i_\nu)} \int \mathrm{d}^\nu\vec{x}\,\mathrm{d}^\nu\vec{x}\,' \\ &\times \phi_1^2(\vec{x})|\vec{x}-\vec{x}\,'|^{-1}\phi_1^2(\vec{x}\,'),\end{aligned} \tag{13}$$

where $(2)^\nu$ is obtained by symmetry. Since $|\vec{x} - \vec{x}\,'| \leqslant \sqrt{\alpha_{i_1}^2 + \cdots + \alpha_{i_\nu}^2}$ in each of the integrals in the summand in (13), we obtain from (12)

$$\langle H_1\rangle \leqslant -\frac{e^2}{2L}(2)^\nu \sum_{i_1,\ldots,i_\nu}^n 1 \Bigg/ \sqrt{\alpha_{i_1}^2 + \cdots + \alpha_{i_\nu}^2}. \tag{14}$$

From the Cauchy–Schwarz inequality this leads to

$$\begin{aligned}\langle H_1\rangle &\leqslant -\frac{e^2}{2L}(2)^\nu n^{2\nu} \Bigg/ \sum_{i_1,\ldots,i_\nu}^n \sqrt{\alpha_{i_1}^2 + \cdots + \alpha_{i_\nu}^2} \\ &\leqslant -\frac{e^2}{4\nu L}k^{(\nu+1)/\nu},\end{aligned} \tag{15}$$

where we have finally used that

$$\begin{aligned}&\sum_{i_1,\ldots,i_\nu}^n \sqrt{\alpha_{i_1}^2 + \cdots + \alpha_{i_\nu}^2} \\ &\quad\leqslant \sum_{i_1,\ldots,i_\nu}^n (\alpha_{i_1} + \cdots + \alpha_{i_\nu}) = \nu(n)^{\nu-1},\end{aligned}$$

and the definition of k. All told we obtain from (10)

$$\begin{aligned}E_N &\leqslant \langle\Psi|H|\Psi\rangle \\ &\leqslant \left(\frac{\nu\pi^2\hbar^2}{8mL^2}k - \frac{e^2}{4\nu}\frac{k^{(\nu+1)/\nu}}{L}\right) \\ &\quad + (N-k) \\ &\quad \times \left(\frac{\nu\pi^2\hbar^2}{8mL_0^2} + e^2\frac{(N+k-1)}{xL_0} - \frac{e^2}{\sqrt{\nu}\,L_0}\right),\end{aligned} \tag{16}$$

where we have set $D = xL_0$. Optimization over L, gives

$$L = \nu^2\pi^2\hbar^2/me^2k^{1/\nu}. \tag{17}$$

Optimization over L_0, and choosing x sufficiently large, the second term on the right-hand side of (16), involving the $(N-k)$ factor, leads to a strict negative contribution with a power of N less than that of the first one. That is, we may further bound the right-hand side of the inequality in (16) by the first term only from above, which when the value of L in (17) is substituted leads to the strict inequality in (1) by finally using (5).

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