N^2 LAW FOR BOSONS IN $2D$*

C. MUTHAPORN and E. B. MANOUKIAN†

School of Physics, Suranaree University of Technology,
111 University Avenue, Nakhon Ratchasima 30000, Thailand
(e-mail: edouard@ccs.sut.ac.th)

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A rigorous upper bound is derived for the exact ground-state energy of N negative charged bosons and N motionless, i.e. fixed, positive charges with Coulomb interactions in $2D$ for arbitrary $N \geqslant 4$ giving rise to an N^2-upper bound. The consistency of such an N^2 behaviour is also investigated by examining a lower bound to the ground-state energy.

Keywords: bosonic matter in the bulk, stability of matter, Lieb-Thirring bounds.

1. Introduction

There has been much interest in recent years in physics in $2D$ e.g. [1–4] and the role of the spin and statistics theorem which is tied up to the dimensionality of space [4]. It has thus become important to investigate the nature of matter in $2D$ in the simplest case when the system is not being subject to stringent constrained statistics. It is equally important to study the nature of such "bosonic matter" to see if the well-known instability (implosive character) of such matter in $3D$ [5–7] will persist in $2D$ or it will turn it to a stable or even to an explosive phase. To answer such questions, we derive a rigorous upper bound for the ground-state energy E_N of the system with N negatively charged bosons and N motionless, i.e., fixed, positive charges, with Coulombic interactions. By doing so, in particular, we do not dwell on the fate and dynamics of the positive background which undoubtedly involves complicated dynamics. We obtain an N^2 behaviour which is to be compared to the $N^{5/3}$ one of Dyson [5–7] in $3D$, implying even a more violent collapse of such a system in $2D$ since the system of $(2N+2N)$ particles will be favourable over two systems each with $(N+N)$ particles, brought into contact, and the energy release upon collapse will be proportional to $((2N)^2-2(N)^2)$ which will be overwhelmingly large for large N, e.g. $N \sim 10^{23}$. Thus the system becomes unstable and stable planar configurations, for example, do not even arise. The present paper deals with a mathematically rigorous treatment of such a system by deriving an explicit upper bound for the exact ground-state energy E_N.

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†Corresponding author.

For $N \geqslant 4$ denoting the number of the negative (or positive) charges, $(N/4)^{1/2}$ being a real number may be written as

$$\left(\frac{N}{4}\right)^{1/2} = n + \varepsilon, \qquad 0 \leqslant \varepsilon < 1, \tag{1.1}$$

where n is natural number. We then derive the following upper bound for ground-state energy E_N, which is the main result of our paper.

THEOREM 1.1.

$$E_N < -\left(\frac{me^4}{2\hbar^2}\right) \frac{N^2}{32\pi^2(1+\varepsilon/n)^4} \tag{1.2}$$

for all $N \geqslant 4$. Here m denotes the smallest mass of the negatively charged bosons.

This is to be compared with the corresponding $N^{5/3}$ law [5–7] in $3D$. Again (1.2) is consistent with the collapse of such matter. Intuitively, such an N^2 behaviour is easily seen as follows. Consider the system to be a square of sides L. The uncertainty principle provides a zero-point kinetic energy of the N bosons to be of the order $N\hbar^2/mL^2$. On the other hand, each particle feels the Coulomb potential of its nearest neighbour at some effective distance d, with negligible interactions with charges farther away due to screening, giving rise to an electrostatic energy of the order $-Ne^2/d$. To fit the N bosons in the square we then have $N \sim (L/d)^2$, giving $d \sim L/N^{1/2}$. Hence for the total energy E_N we have $N\hbar^2/mL^2 - N^{3/2}e^2/L$. Optimization over L gives $L \sim 2\hbar^2/me^2N^{1/2}$ leading to $E_N \sim -(me^4/4\hbar^2)N^2$.

In Appendix, the consistency of such an N^2 behaviour is also investigated by examining a lower bound to the ground-state energy.

2. Derivation of the upper bound

The Hamiltonian under study is given by

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} - \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{e^2}{\left|\vec{x}_i - \vec{R}_j\right|} + \sum_{i<j}^{N} \frac{e^2}{\left|\vec{x}_i - \vec{x}_j\right|} + \sum_{i<j}^{N} \frac{e^2}{\left|\vec{R}_i - \vec{R}_j\right|}, \tag{2.1}$$

where $\vec{x}_i$ and $\vec{R}_j$ refer, respectively, to the negative and positive charges, and m is taken to be the smallest of the masses of the negative charges.

We introduce an N-particle trial function

$$\Psi(\vec{x}_1, \ldots, \vec{x}_N) = \frac{1}{\sqrt{N!k!}} \sum_{\pi} \phi(\vec{x}(\pi_1)) \cdots \phi(\vec{x}(\pi_k))\, \psi_1(\vec{x}(\pi_{k+1})) \cdots \psi_{N-k}(\vec{x}(\pi_N)), \tag{2.2}$$

where $k = 4n^2$ (see (1.1)). The sum is over all permutations $\{\pi_1, \ldots, \pi_N\}$ of $\{1, \ldots, N\}$ such that

$$\int \mathrm{d}^2\vec{x}\; \psi_i(\vec{x})\, \psi_j(\vec{x}) = \delta_{ij}, \qquad \int \mathrm{d}^2\vec{x}\; \phi(\vec{x})\, \psi_i(\vec{x}) = 0, \qquad \int \mathrm{d}^2\vec{x}\; |\phi(\vec{x})|^2 = 1. \tag{2.3}$$

For the single-particle trial functions we take

$$\phi(\vec{x}) = \prod_i \left(\frac{1}{\sqrt{L}} \cos\left(\frac{\pi x_i}{2L}\right)\right) \equiv \phi_L(\vec{x}), \qquad |x_i| \leqslant L, \tag{2.4}$$

and is zero otherwise, and for $j = 1, \ldots, N-k$

$$\psi_j(\vec{x}) = \prod_i \left(\frac{1}{\sqrt{L_0}} \cos\left(\frac{\pi(x_i - L_i^j)}{2L_0}\right)\right) \equiv \phi_{L_0}(\vec{x} - \vec{L}^j), \qquad \left|x_i - L_i^j\right| \leqslant L_0, \tag{2.5}$$

and are zero otherwise, also

$$\vec{L}^j = jD_0(1,1), \tag{2.6}$$

where D_0 is a constant, constrained such that $L \leqslant L_0 \leqslant D_0/2$, and will be optimally chosen later on below Lemma 2.2. The intervals $\{-L \leqslant x_i \leqslant L\}$, $\{jD_0 - L_0 \leqslant x_i \leqslant jD_0 + L_0\}$ for $j = 1, \ldots, N-k$ $(i = 1, 2)$ are then all disjoint and the functions $\phi(\vec{x})$, $\psi_j(\vec{x})$ are nonoverlapping and automatically satisfy (2.3). Physically, they correspond, respectively, to particles localized in boxes of sides $2L$ and $2L_0$ with the center of the first at the origin of the coordinate system, while the others, of sides $2L_0$, are translated by the vector $\vec{L}^j$ from the origin. Such localizations of the particles in these $(N-k+1)$ boxes make the analysis manageable. The Coulomb interaction being of long range, there are nontrivial interactions between particles in different boxes as well. The key point is that we localize $k = 4n^2$ of the negative particles in the first box. Below we will also set k of the positive charges in the first box too.

Since Ψ does not necessarily coincide with the ground-state wave function, we have for the ground-state energy the upper bound

$$E_N \leqslant \langle\Psi| H |\Psi\rangle . \tag{2.7}$$

The single-particle average kinetic energies are given by

$$T = \frac{\hbar^2}{2m} \int \mathrm{d}^2\vec{x}\, \left|\vec{\nabla}\phi(\vec{x})\right|^2 = \frac{\pi^2\hbar^2}{4mL^2}, \tag{2.8}$$

$$T_j = \frac{\hbar^2}{2m} \int \mathrm{d}^2\vec{x}\, \left|\vec{\nabla}\psi_j(\vec{x})\right|^2 = \frac{\pi^2\hbar^2}{4mL_0^2} \equiv T^0, \tag{2.9}$$

and for the multi-particle state

$$\sum_{j=1}^{N} \frac{\hbar^2}{2m} \int \mathrm{d}^2\vec{x}_1 \ldots \mathrm{d}^2\vec{x}_N \left|\vec{\nabla}_j \Psi(\vec{x}_1, \ldots, \vec{x}_N)\right|^2 = \left[kT + (N-k)\,T^0\right]. \tag{2.10}$$

For the expectation value on the right-hand side of (2.7) we have

$$\langle\Psi| H |\Psi\rangle = \left[kT + (N-k)\,T^0\right] + \langle V_1\rangle + \langle V_2\rangle + \sum_{i<j}^{N} \frac{e^2}{\left|\vec{R}_i - \vec{R}_j\right|}, \tag{2.11}$$

where

$$\langle V_1 \rangle = -e^2 \sum_{j=1}^{N} \int \mathrm{d}^2\vec{x} \left[\frac{k}{|\vec{x} - \vec{R}_j|} \phi_L^2(\vec{x}) + \left(\sum_{i=1}^{N-k} \frac{1}{|\vec{x} + \vec{L}^i - \vec{R}_j|} \right) \phi_{L_0}^2(\vec{x}) \right], \tag{2.12}$$

$$\begin{aligned}\langle V_2 \rangle = {} & \frac{e^2}{2} k(k-1) \int \mathrm{d}^2\vec{x}\, \mathrm{d}^2\vec{x}'\, \phi_L^2(\vec{x}) \frac{1}{|\vec{x} - \vec{x}'|} \phi_L^2(\vec{x}') \\ & + e^2 \sum_{j=1}^{N-k} \int \mathrm{d}^2\vec{x}\, \mathrm{d}^2\vec{x}'\, \phi_L^2(\vec{x}) \frac{1}{|\vec{x} - \vec{x}' - \vec{L}^j|} \phi_{L_0}^2(\vec{x}') \\ & + e^2 \sum_{i<j}^{N-k} \int \mathrm{d}^2\vec{x}\, \mathrm{d}^2\vec{x}'\, \phi_{L_0}^2(\vec{x}) \frac{1}{|\vec{x} - \vec{x}' + \vec{L}^i - \vec{L}^j|} \phi_{L_0}^2(\vec{x}').\end{aligned} \tag{2.13}$$

We set

$$\vec{R}_{k+1} = \vec{L}^1, \quad \ldots, \quad \vec{R}_N = \vec{L}^{N-k} \tag{2.14}$$

and choose the vectors $\vec{R}_1, \ldots, \vec{R}_k$ to lie within the first box, with center at the origin, thus placing k positive charges in this box. We then establish the following key inequality embodied in the following lemma.

LEMMA 2.1.

$$\langle \Psi |\, H\, | \Psi \rangle \leqslant kT + \langle H_1 \rangle + (N-k) \left[T^0 + \frac{e^2(N+k-1)}{D_0} - \frac{e^2}{\sqrt{2} L_0} \right], \tag{2.15}$$

where

$$\begin{aligned}\langle H_1 \rangle = {} & -ke^2 \sum_{j=1}^{k} \int \frac{\mathrm{d}^2\vec{x}}{|\vec{x} - \vec{R}_j|} \phi_L^2(\vec{x}) + \frac{e^2}{2} k(k-1) \int \mathrm{d}^2\vec{x}\, \mathrm{d}^2\vec{x}'\, \phi_L^2(\vec{x}) \frac{1}{|\vec{x} - \vec{x}'|} \phi_{L_0}^2(\vec{x}') \\ & + e^2 \sum_{i<j}^{k} \frac{1}{|\vec{R}_i - \vec{R}_j|}.\end{aligned} \tag{2.16}$$

To derive the above inequality, we note that $\langle V_1 \rangle$, defined in (2.12), may be bounded as follows

$$\langle V_1 \rangle \leqslant -e^2 k \sum_{j=1}^{k} \int \mathrm{d}^2\vec{x}\, \frac{1}{|\vec{x} - \vec{R}_j|} \phi_L^2(\vec{x}) - e^2(N-k) \int \frac{\mathrm{d}^2\vec{x}}{|\vec{x}|}\, \phi_{L_0}^2(\vec{x}), \tag{2.17}$$

where we have noted the overall negative sign of $\langle V_1 \rangle$, and we have conveniently chosen an upper bound with the summation going up to k instead of up to N.

Now we use the following bounds

$$|\vec{L}^j| \geqslant \sqrt{2} D_0, \qquad |\vec{L}^i - \vec{L}^j| \geqslant \sqrt{2} D_0 \tag{2.18}$$

for $i \neq j$,

$$\left|\vec{x} - \vec{x}' - \vec{L}^j\right| \geqslant D_0, \qquad \left|\vec{x} - \vec{x}' + \vec{L}^i - \vec{L}^j\right| \geqslant D_0. \tag{2.19}$$

Also

$$\left|\vec{R}_i - \vec{R}_j\right| \geqslant D_0 \tag{2.20}$$

for $j = k+1, \ldots, N$ (see (2.14)), and all i such that $1 \leqslant i < j$, and the decomposition

$$\sum_{i<j}^{N} \frac{1}{\left|\vec{R}_i - \vec{R}_j\right|} = \sum_{i<j}^{k} \frac{1}{\left|\vec{R}_i - \vec{R}_j\right|} + \sum_{j=k+1}^{N} \sum_{i=1}^{j-1} \frac{1}{\left|\vec{R}_i - \vec{R}_j\right|}, \tag{2.21}$$

and the strict negativity of (2.12), to obtain the inequality given in (2.15).

We finally use the following lemma, proved below, which gives an upper bound for $\langle H_1 \rangle$ defined in (2.16).

LEMMA 2.2.

$$\langle H_1 \rangle \leqslant -\frac{e^2 k^{3/2}}{8L}. \tag{2.22}$$

This inequality then leads to

$$E_N \leqslant \frac{\pi^2 \hbar^2}{4mL^2} k - \frac{e^2 k^{3/2}}{8L} + (N-k)\left[\frac{\pi^2\hbar^2}{4mL_0^2} + \frac{e^2(N+k-1)}{xL_0} - \frac{e^2}{\sqrt{2}L_0}\right], \tag{2.23}$$

where we have set $D_0 = xL_0$, $x \geqslant 2$, which will be conveniently and consistently chosen.

Optimizations over L and L_0 give

$$L = \frac{4\pi^2\hbar^2}{me^2k^{1/2}}, \qquad L_0 = \frac{2\sqrt{2}\pi^2\hbar^2}{me^2 4} \frac{1}{\left[1 - \frac{\sqrt{2}(N+k-1)}{x}\right]} \tag{2.24}$$

with

$$0 < 4\left[1 - \frac{\sqrt{2}(N+k-1)}{x}\right] \leqslant \frac{\sqrt{2}}{2} k^{1/2}, \qquad x \geqslant 2. \tag{2.25}$$

With $k^{1/2} \geqslant 2$, we may choose

$$x = \frac{4\sqrt{2}}{3}(N+k-1) \tag{2.26}$$

which is obviously greater than 2, giving

$$L_0 = \frac{2\sqrt{2}\pi^2\hbar^2}{me^2} > L. \tag{2.27}$$

Since the last term on the right-hand side of (2.23), involving the $(N-k)$ factor, now leads to a strictly negative contribution proportional to N, we may further bound (2.23) from above by the sum of the first two terms only which is proportional to N^2.

Accordingly, we obtain the strict bound

$$E_N < -\left(\frac{me^4}{2\hbar^2}\right)\frac{N^2}{32\pi^2}\frac{1}{[1+\varepsilon/n]^4} \tag{2.28}$$

for *all* $N \geqslant 4$, where we have used the fact that $k = N(1+\varepsilon/n)^{-2} = 4n^2$.

Since $(1+\varepsilon/n) < 2$, we also have the conservative bound

$$E_N < -\left(\frac{me^4}{2\hbar^2}\right)\frac{N^2}{512\pi^2} \tag{2.29}$$

for all $N \geqslant 4$.

For large bosonic systems, e.g. for $n \geqslant 50$, i.e. for $N \geqslant 10^4$,

$$E_N < -\left(\frac{me^4}{2\hbar^2}\right)\frac{N^2}{36\pi^2}. \tag{2.30}$$

Therefore, it remains to prove the bound in (2.22) of Lemma 2.2. To this end, we follow the construction given in [6] and partition the unit interval $[0, 1]$: $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ such that

$$\int_{a_{j-1}}^{a_j} \mathrm{d}x \cos^2\left(\frac{\pi x}{2}\right) = \frac{1}{2n} = \left(\frac{1}{k}\right)^{1/2} \tag{2.31}$$

for $j = 1, \ldots, n$.

Let

$$\alpha_j = a_j - a_{j-1} \tag{2.32}$$

and note that

$$\sum_{j=1}^{n} \alpha_j = 1. \tag{2.33}$$

Let $B(i, j)$ denote a box of sides $\alpha_i \times \alpha_j$, then [6]

$$\langle H_1\rangle \leqslant -\frac{k^2}{2}\frac{4}{L}\sum_{B(i,j)}\int_{B(i,j)} \mathrm{d}^2\vec{x}\,\mathrm{d}^2\vec{x}'\,\phi_1^2(\vec{x})\,\frac{1}{|\vec{x}-\vec{x}'|}\,\phi_1^2(\vec{x}'), \tag{2.34}$$

where $\phi_1(\vec{x})$ denotes $\phi_L(\vec{x})$ with L formally replaced by one. The factor 4 in (2.34) arises as a consequence of the fact that for the box at the origin of sides $2L \times 2L$, $-L \leqslant x_i \leqslant L$, thus defining 4 regions of smaller boxes $B(i, j)$. Since $|\vec{x}-\vec{x}'| \leqslant \sqrt{\alpha_i^2+\alpha_j^2}$ in the integrals in (2.34), we obtain

$$\langle H_1\rangle \leqslant -\frac{2}{L}\sum_{i,j}^{n}\frac{1}{\sqrt{\alpha_i^2+\alpha_j^2}} \leqslant -\frac{2}{L}\,\frac{n^4}{\sum_{i,j}^{n}\sqrt{\alpha_i^2+\alpha_j^2}}, \tag{2.35}$$

where in writing the last inequality we have used the Cauchy–Schwarz one. On the other hand,

$$\sum_{i,j}^{n} \sqrt{\alpha_i^2 + \alpha_j^2} \leqslant \sum_{i,j}^{n} (\alpha_i + \alpha_j) = 2n, \tag{2.36}$$

which when substituted in the last inequality in (2.35) gives (2.22), since $k = 4n^2$.

The single-particle trial functions we have chosen in (2.4), (2.5), turned out to be relatively optimal in a large class of trial functions we have investigated. These functions defined on bounded domains, vanishing at their boundaries, were also suitable for the problem at hand as these domains are nonoverlapping and made our analysis manageable. Their simplicity also allowed us to make sharp explicit estimates. A proliferation of the method given in (1.1) for N may be continued as follows by writing (if $\varepsilon_1 \neq 0$)

$$\left(\frac{N}{4}\right)^{1/2} = n_1 + \varepsilon_1, \tag{2.37}$$

where $n_1 \geqslant 1$, letting $k_1 = 4n_1^2$, we continue in this manner to define $n_2 \geqslant 1$, by

$$\left(\frac{N - k_1}{4}\right)^{1/2} = n_2 + \varepsilon_2 \tag{2.38}$$

if $\varepsilon_2 \neq 0$, and so on, by defining in turn $k_2 = 4n_2^2, \dots, k_b = 4n_b^2$, until we reach a natural number b such that

$$\left(\frac{N - (k_1 + \cdots + k_b)}{4}\right)^{1/2} = \varepsilon_{b+1}. \tag{2.39}$$

For example, for $N = 23$, $k_1 = 16$, $k_2 = 4$ and $b = 2$. Unfortunately, b is not always greater than one, and our method of grouping the particles, as done above, with the proliferation just spelled out through the steps (2.37)–(2.39) turns out to be not useful.

The consistency of the N^2 behaviour is also investigated by examining a lower bound to the ground-state energy in Appendix.

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Appendix

In this appendix, the consistency of an N^2 behaviour is investigated by examining the nature of a lower bound to E_N. To this end, we use a well-known lower bound

[8] to the repulsive Coulomb energy established originally in $3D$,

$$\sum_{i<j}^{N}\left|\vec{X}_i-\vec{X}_j\right|^{-1} \geqslant \sum_{i=1}^{N}\int \mathrm{d}^3\vec{X}\,\rho(\vec{X})\left|\vec{X}-\vec{X}_i\right|^{-1} -\frac{1}{2}\int \mathrm{d}^3\vec{X}\,\mathrm{d}^3\vec{X}'\,\rho(\vec{X})\left|\vec{X}-\vec{X}'\right|^{-1}\rho(\vec{X}') -\frac{2\pi}{\mu^2}\int \mathrm{d}^3\vec{X}\,\rho^2(\vec{X})-\frac{N\mu}{2} \tag{A.1}$$

valid for any real $\rho(\vec{X})$ and any constant $\mu > 0$.

Although the $\vec{X}_i$ in (A.1) are $3D$-vectors, the inequality should apply, as is, for the $\vec{X}_i$ having zero third components. To this end, we may take $\vec{X}_i = \left(x_i^{(1)}, x_i^{(2)}, 0\right)$, $\vec{x}_i = (x_i^{(1)}, x_i^{(2)})$.

Upon introducing the particle number density in $2D$,

$$\sigma(\vec{x}) = N\int \mathrm{d}^2\vec{x}_2 \ldots \mathrm{d}^2\vec{x}_N\ \left|\Psi\left(\vec{x}, \vec{x}_2, \ldots, \vec{x}_N\right)\right|^2, \tag{A.2}$$

$$\int \mathrm{d}^2\vec{x}\ \sigma(\vec{x}) = N, \tag{A.3}$$

and in view of investigating the nature of a lower bound to E_N, we may choose

$$\rho(\vec{X}) = \sigma(\vec{x})\frac{\Theta\left(a-\left|x^{(3)}\right|\right)}{2a}, \tag{A.4}$$

where $a > 0$ is so far arbitrary. In anticipation of an N^2-lower bound, we may also rewrite the arbitrary parameter $\mu = N\lambda$, where $\lambda > 0$ is also arbitrary.

From (A.4), (A.1), we then obtain the elementary bound

$$\sum_{i<j}^{N}\left|\vec{x}_i-\vec{x}_j\right|^{-1} \geqslant \sum_{i=1}^{N}\int \mathrm{d}^2\vec{x}\ \sigma(\vec{x})\frac{1}{\sqrt{(\vec{x}-\vec{x}_i)^2+a^2}} -\frac{1}{2}\int \mathrm{d}^2\vec{x}\,\mathrm{d}^2\vec{x}'\ \sigma(\vec{x})\left|\vec{x}-\vec{x}'\right|^{-1}\sigma(\vec{x}') -\frac{\pi}{\lambda^2N^2a}\int \mathrm{d}^2\vec{x}\ \sigma^2(\vec{x})-\frac{N^2\lambda}{2}, \tag{A.5}$$

where in obtaining a lower bound to the second term on the right-hand side of (A.1), we have noted that

$$\sqrt{(\vec{x}-\vec{x}')^2+\left(x^{(3)}-x'^{(3)}\right)^2} \geqslant \left|\vec{x}-\vec{x}'\right|$$

and used the normalizability of $\Theta\left(a-\left|x^{(3)}\right|\right)/2a$ to one over $x^{(3)}$.

An identical copy to (A.5) may be written down for the positively charged particles' coordinates $\vec{R}_i$.

Finally, in the spirit of the $3D$ analysis [9], we use a lower bound to the expectation value T of the kinetic energy operator in $2D$ [10].

LEMMA A.1 (A Lieb–Thirring bound)

$$T \geqslant \frac{\pi \hbar^2}{2MN} \int \mathrm{d}^2\vec{x}\, \sigma^2(\vec{x}), \tag{A.6}$$

where M may be taken to be the largest of the masses of the negative charges.

With m replaced by M in (2.1), we obtain from (A.5) and the corresponding expression for the $\vec{R}_i$ coordinates

$$\begin{aligned}\langle\Psi|\, H\,|\Psi\rangle \geqslant T &- e^2 N \int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2}\, |\tilde{\sigma}(\vec{k})| \left(\frac{1-e^{-ka}}{k}\right) \\ &- e^2 \int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2}\, |\tilde{\sigma}(\vec{k})|^2 \left(\frac{1-e^{-ka}}{k}\right) \\ &- \frac{2\pi e^2}{\lambda^2 N^2 a} \int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2}\, |\tilde{\sigma}(\vec{k})|^2 - N^2\lambda e^2,\end{aligned} \tag{A.7}$$

where $\tilde{\sigma}(\vec{k})$ is the Fourier transform of $\sigma(\vec{x})$, and in writing the second term on the right-hand side of (A.7), we have used the bound

$$\left|\sum_{j=1}^{N} e^{-i\vec{k}\cdot\vec{R}_j}\right| \leqslant N, \tag{A.8}$$

uniformly for all $\vec{R}_j$.

Since $(1-e^{-ka})/k \leqslant a$, (A.6) and (A.7) give

$$\begin{aligned}\langle\Psi|\, H\,|\Psi\rangle \geqslant &\int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2}\, |\tilde{\sigma}(\vec{k})| \left[\left(\frac{\pi\hbar^2}{2MN} - e^2 a - \frac{2\pi e^2}{\lambda^2 N^2 a}\right)\tilde{\sigma}(\vec{k}) - e^2 N a\right] \\ &- N^2\lambda e^2\end{aligned} \tag{A.9}$$

Optimization over a gives $a = \sqrt{2\pi}/N\lambda$, leading to

$$\begin{aligned}\langle\Psi|\, H\,|\Psi\rangle \geqslant &\int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2}\, |\tilde{\sigma}(\vec{k})| \left[\frac{1}{N}\left(\frac{\pi\hbar^2}{2M} - \frac{2\sqrt{2\pi}e^2}{\lambda}\right)|\tilde{\sigma}(\vec{k})| - \frac{e^2\sqrt{2\pi}}{\lambda}\right] \\ &- N^2\lambda e^2.\end{aligned} \tag{A.10}$$

Finally, we may choose $\lambda = 4Me^2\sqrt{2/\pi}/\hbar^2$, and introduce the one-particle density

$$\sigma_0(\vec{x}) = \sigma(\vec{x})/N, \qquad \int d^2\vec{x}\ \sigma_0(\vec{x}) = 1, \tag{A.11}$$

to obtain the lower bound

$$\langle\Psi|\, H\, |\Psi\rangle \geqslant -\left(\frac{Me^4}{2\hbar^2}\right)\left(8\sqrt{\frac{2}{\pi}}\right)N^2 - \frac{\hbar^2\pi}{4M}N\int\frac{d^2\vec{k}}{(2\pi)^2}\,\left|\widetilde{\sigma}_0(\vec{k})\right| \tag{A.12}$$

which for N sufficiently large gives rise to an N^2-power behaviour provided $\left|\widetilde{\sigma}_0(\vec{k})\right|$ is integrable, or more precisely provided

$$\frac{1}{N}\int\frac{d^2\vec{k}}{(2\pi)^2}\,\left|\widetilde{\sigma}_0(\vec{k})\right| \tag{A.13}$$

remains bounded for $N \to \infty$. Intuitively, an N^2 behaviour of the right-hand side of (A.12) for large N may be inferred. The reason is that for large N, σ_0 scales as $\sigma_0(\vec{x}N^{1/2})$, therefore $\widetilde{\sigma}_0(\vec{k}N^{-1/2})$ for the Fourier transform and $\int d^2\vec{k}\,\left|\widetilde{\sigma}_0(\vec{k}N^{-1/2})\right| = N\int d^2\vec{k}\,\left|\widetilde{\sigma}_0(\vec{k})\right|$. Thus altogether the right-hand side of (A.12) would grow as N^2.

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