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ที่อุณหภูมิอันตะ

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QUANTUM ELECTRODYNAMICS
OF ČERENKOV RADIATION
AT FINITE TEMPERATURE

Miss Doojdao Charuchittipan

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การศึกษาการปลดปล่อยของการแผ่สีเซเรนโกฟที่อุณหภูมิอันตะในตัวกลางไอโซทรอปิกเอกพันธุ์ถึงอันดับค่าคงตัวโครงสร้างละเอียดในพลศาสตร์ไฟฟ้าควอนตัมได้มีขึ้นเป็นครั้งแรก ด้วยวิธีการหาปริพันธ์เชิงซ้อนเพื่อหลีกเลี่ยงการรวมตัวหารของตัวแผ่ไฟยน์แมนในรูปแบบอิงตัวแปรเสริม ซึ่งจะนำไปสู่การประมาณอันเนื่องมาจากผลการหาปริพันธ์ที่ซับซ้อนเช่นในอดีต วิธีการคำนวณแบบนี้จะให้ส่วนเสริมควอนตัม $\hbar^2\omega^2/E^2$ ในสเปกตรัมกำลังโดยอัตโนมัติ และยังทำให้ความยุ่งยากที่เกี่ยวข้องกับพจน์นี้ที่ศูนย์องศาสัมบูรณ์หมดไป ในวิทยานิพนธ์ฉบับนี้ยังแสดงวิธีการปริพันธ์เชิงซ้อนที่เกี่ยงข้องกับ ภาวะเอกฐานอย่างระมัดระวัง ซึ่งทำให้แก้ปัญหาได้ง่ายขึ้นมากกว่าการรวมตัวหารของตัวแผ่ โดยเฉพาะค่าจินตภาพของพลังงานในตัวของอิเล็กตรอนซึ่งสอดคล้องกับเงื่อนไขขอบที่ถูกต้องเป็นผลให้ไม่ต้องการพจน์สัมผัส (contact term) ในการคำนวณนี้ และแง่มุมที่น่าพอใจที่เกิดขึ้นในพลศาสตร์ไฟฟ้าควอนตัมนี้โดยไม่ปรากฏในแบบแผนเดิมคือ ที่ความถี่สูงๆ กำลังของการปลดปล่อยรังสีจะคัท-ออฟ (cut-off) โดยอัตโนมัติซึ่งเป็นสิ่งที่สำคัญมากในการแก้ปัญหาเชิงควอนตัม

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DOOJDAO CHARUCHITTIPAN : QUANTUM ELECTRODYNAMICS OF ČERENKOV RADIATION AT FINITE TEMPERTURE : THESIS ADVISOR : PROF. EDOUARD B. MANOUKIAN, Ph.D. 82PP.

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An exact, to order α (the fine-structure constant), study of Čerenkov radiation emission in Quantum Electrodynamics is carried out at finite temperature ($T \neq 0$) in isotropic homogeneous media for the first time. The method of complex integration is used to avoid the method of combining denominators of Feyman propagators in parametric form; which has led to approximations in the past due to the complexity of the resulting integrals. The $\hbar^2\omega^2/E^2$ -quantum contribution to the power spectrum is automatically evaluated by our method and settles the ambiguity associated with this term known to exist at $T = 0$. In this work we also show that complex integration, by careful analysis of the singularities involved, actually simplifies the problem tremendously over the usual method of combining the denominators of the propagators. In particular, the imaginary part of the electron self-energy satisfies the correct underlying boundary condition and no contact term is needed in its evaluation. One of the most pleasing aspects of Quantum Electrodynamics, unlike its classical counterpart, is that it introduces automatically a cut-off for higher frequencies of radiation emission emphasizing the importance of the quantum treatment.

สาขาวิชา ฟิสิกส์	ลายมือชื่อนักศึกษา..........................
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Doojdao Charuchittipan

Contents

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List of Symbols and some Physical Constants

The following symbols are used throughout this thesis unless otherwise stated. The numerical value of the physical constants given are based on the *Review of Particle Physics* by the Particle Data Group (1998).

c	=	Speed of light in vacuum, which is 299792458 m s^{-1}
h	=	Planck constant, which is $6.6260755 \times 10^{-34}$ J s
$\hbar \equiv h/2\pi$	=	Reduced Planck constant, which is $1.05457267 \times 10^{-34}$ J s or 6.582122×10^{-22} MeV s
e	=	Electron charged magnitude, which is $1.60217733 \times 10^{-19}$ C or $4.8032068 \times 10^{-10}$ esu
m	=	Electron mass, which is 0.51099906 MeV/c^2 or $9.1093898 \times 10^{-31}$ kg
$\alpha = e^2/4\pi\hbar c$	=	Fine-structure constant, which is 1/137.0353896
k	=	Boltzmann constant, which is 1.380658×10^{-23} J K^{-1} or 8.617386×10^{-5} eV K^{-1}
T	=	Absolute temperature
Γ	=	The decay rate
u	=	Dirac's spinor
Σ	=	Electron's self-energy
ω	=	Photon's energy (frequency)
$p = (x^0, \vec{\mathbf{x}})$	=	Space-time four-vector
$p = (E, \vec{\mathbf{p}})$	=	Electron's energy-momentum four-vector
E	=	Electron's energy
$q = (q^0, \vec{\mathbf{q}})$	=	Photon's energy-momentum four-vector
$\delta(x)$	=	Dirac delta function
$P(\omega)$	=	The power spectrum

$D_{\mu\nu}(q)$	=	Photon propagator
$S(p)$	=	Electron propagator
$g_{\mu\nu}$	=	Metric tensor
γ^{μ}	=	Dirac matrices
$\vec{\mathbf{D}}$	=	Electric displacement vector
$\vec{\mathbf{E}}$	=	Electric field intensity
$\vec{\mathbf{B}}$	=	Magnetic induction vector
$\vec{\mathbf{H}}$	=	Magnetic field intensity
$A^{\mu} = (\varphi, \vec{\mathbf{A}})$	=	Potential four-vector
ξ	=	Permittivity
μ	=	Permeability
n	=	Medium's refractive index
$J^{\mu} = (\rho, \vec{\mathbf{J}})$	=	Current density four vector
$\eta^{\mu} = (1, \vec{\mathbf{0}})$	=	Time-like unit vector

The three-vector notation is denoted by a bold-arrowhead character.

Chapter I

Introduction

In this chapter the basic ideas about Èerenkov radiation are introduced. This includes its history, earlier explanation of the classical and quantum theory. Finally the purpose of the thesis investigation followed by a detailed description of the contents of the coming chapters are spelled out.

1.1 The Very Basics of Èerenkov Radiation

Èerenkov radiation is the radiation emitted by a uniformly moving charged particle in a medium when its speed exceeds the speed of light in the same medium. That is when

$$v > \frac{c}{n}, \qquad (1.1)$$

where v is the velocity of a charged particle, c is the speed of light in vacuum and n is the refractive index of the medium.

According to the Huygen principle, each point on the path of a moving charged particle is a source of a spherical wave (Fig. 1.1). Every spherical wave along the path has a common envelope, which is a cone, the Èerenkov cone (similar to Mach's cone in acoustics), whose apex coincides with the instantaneous position of the charged particle. From this figure one readily obtains the Èerenkov relation

$$\cos\theta = \frac{1}{n\beta}, \tag{1.2}$$

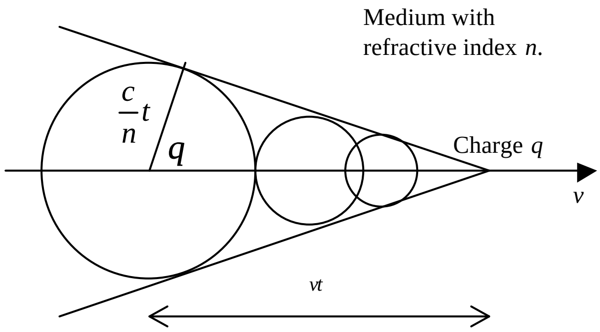

Figure 1.1 Formation of the Èerenkov cone

where $\beta = v/c$. Due to the constraint on the possible values of $\cos\theta$, one obtains the threshold condition for the radiation emission:

$$n\beta > 1. \tag{1.3}$$

1.2 Brief Historical Overview

Èerenkov radiation has been observed as a bluish-white light from transparency substances near strong radioactive sources by many physicists. These observations were not at first interpreted as a new phenomenon. In 1934, the Russian physicist, S I Vavilov and his post-graduate student, P A Èerenkov discovered that this was a new phenomenon by investigating the bluish-white light from uranyl salts under the influence of gamma rays from radium. Thus, in the Russian literature this radiation is also known as Vavilov-Èerenkov radiation. This discovery was first published in Russian in 1934 (Èerenkov, 1934, 1936, quoted in Ginzburg, 1996) and the first English version was published in 1937 (Èerenkov, 1937, quoted in Smith, 1993). A series of experiments carried out by Èerenkov have been recently summarized by Mukhin (1987).

In 1937, I M Frank and I E Tamm (Frank, and Tamm, 1937, quoted in Smith, 1993) introduced the theory of an electron moving uniformly (constant velocities) in a dielectric medium, which was based on classical electrodynamics. They have provided a theoretical description of Èerenkov's experimental observations. This new phenomenon, in its simplest terms, was described by stating that a charged particle in a medium may emit radiation even if it is not accelerating as long as its speed exceeds the speed of light in the medium.

In 1958, Èerenkov, Frank and Tamm won the Nobel Prize in physics for their important work.

There has been much interest in recent years in Èerenkov radiation emphasizing several aspects by different investigators (e.g., Schwinger, Tsai, and Erber, 1976; Bazylev, V., et al., 1981; Fülöp, 1993; Manoukian, 1993; Manoukian, and Bantitadawit, 1999) and most recently in Èerenkov radiation by neutrinos (Ioannisian, and Raffelt, Lanl. preprints, 1999) and in string theory (Manoukian, 1991). An outstanding application of the theory of Èerenkov radiation is in the so-called Èerenkov detector, which is widely used in accelerators in high-energy physics. This detector is used to detect and count high-energy charged particles and determine indirectly their velocities.

1.3 Classical and Quantum Theory of Èerenkov Radiation

Èerenkov radiation has been treated both classically and by the quantum theory. Early work on Èerenkov radiation is reported in the book by Jelly (1958).

For classical theory, Tamm and Frank (1937) obtained the expression for radiation intensity per unit time by calculating the flux associated with the Poynting vector across a cylindrical surface surrounding the path of an electron and the former is given by (e.g., Ginzburg, 1996)

$$\frac{dW}{dt} = \frac{e^2 v}{c} \int_{\beta n > 1} d\omega\, \omega \left(1 - \frac{1}{n^2 \beta^2}\right) \quad . \tag{1.4}$$

The quantum theory of Èerenkov radiation has been investigated by Ginzburg (1940 quoted in Ginzburg ,1996), Sokolov (1940, quoted in Jelly, 1958), Neamtan (1953), Schwinger, Tsai, and Erber (1976), Fülöp (1993) and others. However, the clearest and most detailed quantum treatment of the problem which we have found in the literature for investigators done over the years is that of Schwinger, et al. (1976). This work deals with the full Quantum Electrodynamics, to first order in the fine-structure constant α, in an isotropic homogeneous medium at absolute zero temperature ($T=0$). Unfortunately, due to the familiar method of combining the denominators of the Feynman propagators in parametric form, the resulting integrals turned out to be exceedingly complicated and approximations were necessarily made. This left, in particular, the $^2\omega^2/E^2$-contribution to the quantum correction undetermined and ambiguous, where E is the total energy of the electron and ω is the energy of a photon. In this thesis, we use units such that = 1, c = 1 (see Appendix A). The expression for the power spectrum found in their work is

$$P(\omega) = \alpha\omega\mu\beta\left[1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{E}(n^2 - 1)\right)\right], \tag{1.5}$$

where μ is permeability of a medium and ω is the frequency of an emitted photon, with the threshold condition

$$n\boldsymbol{b} > 1 + \frac{\boldsymbol{w}}{2E}(n^2 - 1)\ . \tag{1.6}$$

to be satisfied.

The classical expression of the power spectrum was also evaluated in their work in the language of source theory, and is given by

$$P(\boldsymbol{w}) = \boldsymbol{awmb}\left(1 - \frac{1}{n^2\boldsymbol{b}^2}\right), \tag{1.7}$$

with the threshold condition as in (1.3). It is seen that in classical limit, $E >> ù$, the expression for the power spectrum in (1.5) and the threshold condition in (1.6) are equivalent to the classical expression in (1.7) and (1.3), respectively.

For the full quantum treatment it is illuminating to describe the kinematics involved in Èerenkov radiation using the photon concept. This description follows. Suppose an electron of rest mass m is moving through a medium of refractive index n, with constant velocity $\vec{\mathbf{v}}$. At some point along the track, a photon of energy $\boldsymbol{w}$ is emitted at an angle $\boldsymbol{q}$ with respect to the original direction of the electron as shown in figure 1.2.

The energy-momentum four-vector for the electron before the emission can be written as

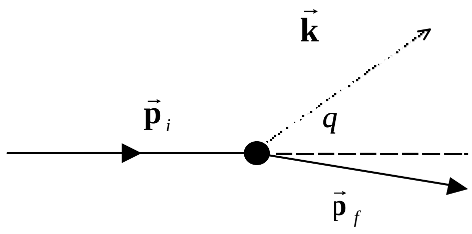

Figure 1.2 A photon emission by an electron at an angle $\boldsymbol{q}$, where $\vec{\mathbf{k}}$ is the momentum carried by the photon.

$$p_i = \left(\frac{m}{\sqrt{1-b^2}}, \frac{m\vec{\mathbf{v}}}{\sqrt{1-b^2}} \right), \quad (1.8)$$

After emission, the corresponding four-vector is given by

$$p_f = (\frac{m}{\sqrt{1-b^2}} - w, \frac{m\vec{\mathbf{v}}}{\sqrt{1-b^2}} - \vec{\mathbf{k}}) \quad , \quad (1.9)$$

where $b = \frac{v}{c}$ and $k = nw$. The scalar product of four-vectors is an invariant, This leads, in particular, to the mass-shell condition on energy and momentum:

$$p_i^2 = p_f^2 = -m^2 \quad (1.10)$$

or

$$\frac{m^2v^2}{1-b^2} - \frac{m^2}{1-b^2} = (\frac{m\vec{\mathbf{v}}}{\sqrt{1-b^2}} - \vec{\mathbf{k}})^2 - (\frac{m}{\sqrt{1-b^2}} - w)^2 \ . \quad (1.11)$$

This leads to

$$\cos q = \frac{1}{nb}[1 + \frac{w(n^2-1)}{2E}] \quad (1.12)$$

where E = ã m and ã = (1-â^2)$^{-1/2}$. The classical limits for cos è in (1.2) is obtained by taking the limit $\hbar \to 0$ of the expression in (1.12).

1.4 Purpose of the Thesis Investigation

The purpose of this thesis is to carry out exactly, to order a (the fine-structure constant), a study of the power for Èerenkov radiation emission in Quantum Electrodynamics at finite temperature (*T* ¹ *0*) (e.g., Manoukian, 1990; Kang, Kye, and Kim, 1993) for the first time, and investigate, in the process, the nature of the full quantum correction to the spectrum, including the $^{2}w^{2}/E^{2}$ correction, in an isotropic homogeneous medium, where E is the total energy of the electron and w is the energy of a photon. As far as the accuracy of Quantum Electrodynamics is concerned, the legendary R. P. Feynman states that: " If you were to measure the distance from Los Angeles to New York, this accuracy would be exact to the thickness of a human hair. "(Feynman, 1985).

The underlying theory used in this work is that of Quantum Electrodynamics. It is the most precise theory ever devised by man. The theoretical predictions of Quantum Electrodynamics with experiments are embarrassingly accurate. The analysis is done exactly, to the order a, in an isotropic homogeneous medium described by a given index of refraction n at finite temperature.

The basic departure from earlier work is that the denominators of the underlying Feynman propagators are not combined in parametric form, which had necessarily led to approximations for $T = 0$ in the past. Complex integration in the complex energy plane is used instead. This allows us for an exact evaluation of the power spectrum of Èerenkov emission (at any temperature). The nature of the singularities is studied rigorously by deriving lower bounds to them and check which singularities would contribute and which would not contribute to the imaginary part of the self-energy of the electron. The boundary condition of no radiation emission when the index of refraction goes to one is also investigated. Finally, the inclusion of temperature into the theory involving the Boltzmann factor will be incorporated rigorously.

1.5 Plan of the Work

In Chapter II, we establish rigorously, using the machinery of Quantum Electrodynamics, that the electron in vacuum is stable and does not radiate. In Chapter III, the photon propagator in a medium is evaluated through Maxwell's equations. In Chapter IV, the investigation in Chapter II is extended to a medium by using the result in Chapter III. This leads to a closed expression for the power spectrum of Èerenkov radiation. The analyses in Chapter II to Chapter IV are treated at the absolute zero temperature. In Chapter V, the earlier investigation is extended to finite temperatures and the expression of the power spectrum of Èerenkov radiation emission at any finite temperature is evaluated. In Appendix A, we spell out the units used in this work. We use units such that = 1, c = 1. Appendix B, deals with the four-vector notation, the Dirac equation and some pertinent properties involving gamma matrices. Some properties of the Dirac delta function are spell out in Appendix C. In the final Appendix (D), some properties of the so-called residue theorem in complex integration, which is so important in our work, are given.

Chapter II

Stability of the Electron in Quantum Electrodynamics in Vacuum

In this chapter, we investigate the stability of the electron in vacuum. That is, we rigorously establish by using the machinery of Quantum Electrodynamics that an electron moving with uniform velocity does not radiate and continues to move with the same velocity. That is, our method of investigation, with complex analysis, provides automatically the physically expected result that the power of radiation is identically equal to zero when the index of refraction $n = 1$ (the vacuum). This is quite important as it shows that no contact term is needed in our analysis to satisfy the correct boundary condition of zero power for $n = 1$.

2.1 Expression for the Power Spectrum

The stability of an electron in vacuum may be expressed in terms of the power of radiation. In Classical Electrodynamics, the power can be calculated from the Poynting vector, but in Quantum Electrodynamics one uses a different approach. The latter may be obtained from the well-known relation of the decay rate of an electron (Schwinger, 1973; Schwinger, et. al., 1974, 1976, 1978; Tsai, 1973; Tsai, and Yildiz, 1973), which is given through the imaginary part of the electron self-energy by

$$\Gamma = -\frac{2m}{E}\,\mathrm{Im}(\bar{u}\Sigma u)\,, \tag{2.1}$$

where Σ is the electron self-energy and u is a Dirac spinor. That the decay rate is given by the electron self-energy is self-evident. Taking its imaginary part leads essentially to putting the particles in the intermediate states on their mass-shells. The decay rate is connected to the power spectrum by

$$\Gamma = \int d\omega \frac{P(\omega)}{\omega}\,. \tag{2.2}$$

The power spectrum for photon emission each with energy ω can be obtained by inserting the identity operation,

$$\mathbf{1} = \int_0^\infty d\omega\, \delta(\omega - \frac{|\vec{\mathbf{q}}|}{n})\,,$$

into (2.1), leading to

$$-\frac{2m}{E}\int d\omega \int (dq)\, I(p,q)\, \delta(\omega - \frac{|\vec{\mathbf{q}}|}{n}) = \int d\omega \frac{P(\omega)}{\omega}\,, \tag{2.3}$$

where we have set

$$\mathrm{Im}(\bar{u}\Sigma u) \equiv \int (dq)\, I(p,q)\,. \tag{2.4}$$

This finally leads to the expression for the power spectrum

$$P(\omega) = -\frac{2m\omega}{E}\int (dq)\, I(p,q)\, \delta(\omega - \frac{|\vec{\mathbf{q}}|}{n})\,. \tag{2.5}$$

By using the property (C. 5) of the delta function in Appendix C, (2.5) becomes

$$P(\omega) = -\frac{2mn\omega}{E}\int (dq)\, I(p,q)\, \delta(|\vec{\mathbf{q}}| - n\omega)\,. \tag{2.6}$$

In vacuum $n = 1$ and we may write

$$P(\omega) = -\frac{2m\omega}{E}\int (dq)\, I(p,q)\, \delta(|\vec{\mathbf{q}}| - \omega)\,. \tag{2.7}$$

2.2 The Electron Self-Energy

The electron self-energy corresponds to the diagram in Fig. 2.1. An electron with energy-momentum $p = (E, \vec{\mathbf{p}})$ emits a photon of energy-momentum $q = (q^0, \vec{\mathbf{q}})$ and then reabsorbs it back again. The expression of the electron self energy is well known (e.g., Jauch, and Rohrlich, 1980; Peskin, and Schoeder, 1995) to be given by

$$\Sigma(p) = ie^2 \int \frac{(dq)}{(2\boldsymbol{p})^4} Tr[\boldsymbol{g}^m S(p-q)\boldsymbol{g}^n] D_{mn}(q), \quad \boldsymbol{e} \to +0, \qquad (2.8)$$

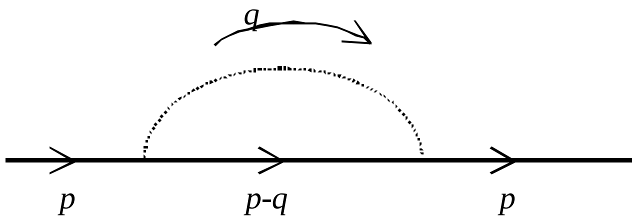

Figure 2.1 The electron self-energy diagram

where $S(p-q)$ is the electron propagator and $D_{mn}(q)$ is the photon propagator. The latter are given by

$$S(p-q) = \frac{-\boldsymbol{g}(p-q)+m}{(p-q)^2+m^2-i\boldsymbol{e}}, \qquad (2.9)$$

with the mass-shell condition $p^2 + m^2 = 0$,

and in the Feynman gauge

$$D_{\mu\nu}(q) = \frac{g_{\mu\nu}}{q^2 - i\varepsilon}. \qquad (2.10)$$

Substitute (2.10) and (2.9) into (2.8) to obtain for the expression of the electron self-energy:

$$\Sigma(p) = ie^2 \int \frac{(dq)}{(2\pi)^4} \frac{\gamma^\mu [-\gamma(p-q)+m]\gamma^\nu}{[(p-q)^2 + m^2 - i\varepsilon]} \frac{g_{\mu\nu}}{(q^2 - i\varepsilon)}, \tag{2.11}$$

or

$$\Sigma(p) = ie^2 \int \frac{(dq)}{(2\pi)^4} \frac{\gamma^\mu [-\gamma(p-q)+m]\gamma_\mu}{[(p-q)^2 + m^2 - i\varepsilon]} \frac{1}{(q^2 - i\varepsilon)}, \tag{2.12}$$

by using the relation $\gamma_\mu = g_{\mu\nu}\gamma^\nu$ as discussed in Appendix B.

2.3 Evaluation of the Power Spectrum in Vacuum

In Schwinger's work (Schwinger, et al., 1976), the denominators of the propagators were combined together by using the techniques of Schwinger-Feynman parameters. However, in this thesis, the power spectrum will be evaluated without combining the denominators of the propagators in parametric form. This simplifies the work tremendously and allows us later to obtain an exact result. After we substitute (2.12) into (2.7), one obtains

$$I(p,q) = \frac{e^2}{(2\pi)^4} \mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right],$$

then expression for the power spectrum becomes

$$P(\omega) = -\frac{2m\omega e^2}{E} \int \frac{(dq)}{(2\pi)^4} \delta(|\vec{\mathbf{q}}| - \omega)\, \mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right], \tag{2.13}$$

where

$$N = \gamma^\mu [-\gamma(p-q) + m]\gamma_\mu, \tag{2.14}$$

and

$$D = (q^2 - i\varepsilon)[(p-q)^2 + m^2 - i\varepsilon]. \tag{2.15}$$

The integral over (dq) is done in 4-dimensional space, that is

$$\int (dq) = \int d^3\vec{\mathbf{q}} \int dq^0 = \int_0^{\infty} d\left|\vec{\mathbf{q}}\right| \int_{-1}^{1} d(\cos q) \int_0^{2p} d f \int_{-\infty}^{\infty} dq^0 \left|\vec{\mathbf{q}}\right|^2 .$$

Thus

$$P(w) = -\frac{2mwe^2}{E(2p)^4} \int_0^{\infty} d\left|\vec{\mathbf{q}}\right| \int_{-1}^{1} d(\cos q) \int_0^{2p} d f \int_{-\infty}^{\infty} dq^0 \left|\vec{\mathbf{q}}\right|^2 \text{ä}(\left|\vec{\mathbf{q}}\right| - w)\, \mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right] \qquad (2.16)$$

Carrying out the $\left|\vec{\mathbf{q}}\right|$ integration by using the property (C. 4) of a delta function in Appendix C gives

$$P(w) = -\frac{2mwe^2}{E(2p)^4} \int_{-1}^{1} d(\cos q) \int_0^{2p} d f \int_{-\infty}^{\infty} dq^0 w^2\, \mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right], \qquad (2.17)$$

with $N = N(\left|\vec{\mathbf{q}}\right| = w)$. Then evaluate the angular f integration in (2.17). This gives

$$P(w) = -\frac{2m\, w^3 e^2}{E(2p)^4} (2p) \int_{-1}^{1} d(\cos q) \int_{-\infty}^{\infty} dq^0\ \mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right]. \qquad (2.18)$$

Next, we evaluate $\bar{u}Nu$, from (2.14)

$$\bar{u}Nu = \bar{u}\, g^m [-g(p-q) + m] g_m u . \qquad (2.19)$$

The algebra in Appendix B over the gamma matrices will be repeatedly used here. From (B. 16) and (B. 17) in Appendix B, (2.19) becomes

$$\begin{aligned} \bar{u}Nu &= \bar{u}[-2g^m (p-q)_m - 4m]u \\ &= -2\bar{u}g^m p_m + 2\bar{u}g^m q_m - 4m\bar{u}u. \end{aligned} \qquad (2.20)$$

By using (B. 8) in Appendix B, then

$$\bar{u}Nu = -2\bar{u}g^m p_m + 2\bar{u}g^m q_m - 4m\ . \qquad (2.21)$$

From the Dirac equation

$$\bar{u}(g^m p_m + m) = 0\ , \qquad (B. 6)$$

this gives

$$\bar{u}\mathbf{g}^{m} p_{m} = -m\bar{u}\ .$$

Hence,

$$\begin{aligned}\bar{u}Nu &= 2m\bar{u}u + 2\bar{u}\mathbf{g}^{m} q_{m} u - 4m \\ &= 2m + 2\bar{u}\mathbf{g}^{m} q_{m} u - 4m \\ &= -2m + 2\bar{u}\mathbf{g}^{m} q_{m} u.\end{aligned} \quad (2.22)$$

From (B. 18) in Appendix B, then

$$\bar{u}\mathbf{g}^{m} q_{m} u = \bar{u}\mathbf{g}^{m} u q_{m} = \frac{p^{m} q_{m}}{m}$$

Thus (2.22) becomes

$$\bar{u}Nu = -2m + 2\frac{p^{m} q_{m}}{m}\ . \quad (2.23)$$

Expand the product of four-vectors (see Appendix B) in (2.23). This gives

$$\begin{aligned}\bar{u}Nu &= -2m + \frac{2}{m}\left(\vec{\mathbf{p}} \bullet \vec{\mathbf{q}} - p^{0} q^{0}\right) \\ &= -2m + \frac{2}{m}\left(\left|\vec{\mathbf{p}}\right|\left|\vec{\mathbf{q}}\right| \cos \mathbf{q} - q^{0} E\right),\end{aligned} \quad (2.24)$$

with $p^{0} = E$ denoting the electron's energy. Factor out *2m* from (2.24) and replace $\left|\vec{\mathbf{q}}\right|$ by $\mathbf{w}$ from the delta function to obtain,

$$\bar{u}Nu = 2m\left[-1 + \frac{1}{m^{2}}\left(\left|\vec{\mathbf{p}}\right| \mathbf{w} \cos \mathbf{q} - q^{0} E\right)\right]. \quad (2.25)$$

From $\mathbf{b} = \frac{\left|\vec{\mathbf{p}}\right|}{E} = \frac{v}{c}$, then

$$\bar{u}Nu = 2m\left[-1 + \frac{1}{m^{2}}\left(\mathbf{b} E \mathbf{w} \cos \mathbf{q} - q^{0} E\right)\right]. \quad (2.26)$$

Rewrite (2.26) as

$$\bar{u}Nu = \frac{2\boldsymbol{b}^2 E^2}{m}\left[-\frac{m^2}{\boldsymbol{b}^2 E^2} + \frac{\boldsymbol{w}\cos\boldsymbol{q}}{\boldsymbol{b}E} + \frac{q^0 E}{\boldsymbol{b}^2 E^2}\right]. \qquad (2.27)$$

Consider the first term. By using $E^2 = (\boldsymbol{g}\, m)^2$ and the Lorentz factor $\boldsymbol{g}^2 = (1-\boldsymbol{b}^2)^{-1}$, then we may rewrite (2.27) as

$$\bar{u}Nu = \frac{2\boldsymbol{b}^2 E^2}{m}\left[-\frac{m^2}{\boldsymbol{b}^2\boldsymbol{g}^2 m^2} + \frac{\boldsymbol{w}\cos\boldsymbol{q}}{\boldsymbol{b}E} + \frac{q^0 E}{\boldsymbol{b}^2 E^2}\right]$$

$$= \frac{2\boldsymbol{b}^2 E^2}{m}\left[-\frac{1}{\boldsymbol{b}^2\boldsymbol{g}^2} + \frac{\boldsymbol{w}\cos\boldsymbol{q}}{\boldsymbol{b}E} + \frac{q^0 E}{\boldsymbol{b}^2 E^2}\right]$$

$$= \frac{2\boldsymbol{b}^2 E^2}{m}\left[-\frac{(1-\boldsymbol{b}^2)}{\boldsymbol{b}^2} + \frac{\boldsymbol{w}\cos\boldsymbol{q}}{\boldsymbol{b}E} + \frac{q^0 E}{\boldsymbol{b}^2 E^2}\right].$$

$$\bar{u}Nu = \frac{2\boldsymbol{b}^2 E^2}{m}\left[1-\frac{1}{\boldsymbol{b}^2} + \frac{\boldsymbol{w}\cos\boldsymbol{q}}{\boldsymbol{b}E} + \frac{q^0 E}{\boldsymbol{b}^2 E^2}\right]. \qquad (2.28)$$

Now, the possible singularities arising in (2.18) will be investigated. Expand the product of four-vectors in (2.15) and replace $|\vec{\mathbf{q}}|$ by $\boldsymbol{w}$ from the delta function to obtain

$$D = \left[\boldsymbol{w}^2 - (q^0)^2 - i\boldsymbol{e}\right]\left[p^2 + q^2 - 2p^m q_m + m^2 - i\boldsymbol{e}\right]$$

$$= \left[\boldsymbol{w}^2 - (q^0)^2 - i\boldsymbol{e}\right]\left[\boldsymbol{w}^2 - (q^0)^2 - 2(\boldsymbol{b}E\boldsymbol{w}\cos\boldsymbol{q} - q^0 E) - i\boldsymbol{e}\right]$$

$$= \left[(q^0)^2 - \boldsymbol{w}^2 + i\boldsymbol{e}\right]\left[(q^0)^2 - 2q^0 E - \boldsymbol{w}^2 + 2\boldsymbol{b}E\boldsymbol{w}\cos\boldsymbol{q} + i\boldsymbol{e}\right]. \qquad (2.29)$$

From this expression, the singularities occur at points (poles) which make *D=0*. Consider the first term,

$$(q^0)^2 - \boldsymbol{w}^2 + i\boldsymbol{e} = \left(q^0 + \boldsymbol{w} - i\boldsymbol{e}\right)\left(q^0 - \boldsymbol{w} + i\boldsymbol{e}\right) = 0\,.$$

Thus the poles from the first term are

$$q^0_{1\pm} = \pm\omega \mp i\varepsilon . \tag{2.30}$$

The second term is more complicated and leads to

$$(q^0)^2 - 2q^0 E - \omega^2 + 2\beta E\omega\cos\theta + i\varepsilon = 0 . \tag{2.31}$$

Upon using the roots of the quadratic equation,

$$ax^2 + bx + c = 0, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with

a=1, b= -2E and $c = -\omega^2 + 2\beta E\omega\cos\theta + i\varepsilon$,

the poles from the second term are obtained to be

$$\begin{aligned} q^0_{2\pm} &= \frac{2E \pm \sqrt{4E^2 - 4\left(2\beta E\omega\cos\theta - \omega^2 + i\varepsilon\right)}}{2} \\ &= E \pm E\sqrt{1 + \frac{\omega^2}{E^2} - \frac{2\beta\omega\cos\theta}{E} - i\varepsilon} \quad . \end{aligned} \tag{2.32}$$

Let

$$A^2(\theta) = 1 + \frac{\omega^2}{E^2} - \frac{2\beta\omega\cos\theta}{E} . \tag{2.33}$$

We prove that $A^2(\theta)$ is always positive. $A^2(\theta)$ is minimum at the maximum value of cosθ, which is cosθ = 1 at θ = 0. If $A^2(\theta)$ is positive at θ = 0, then it is always positive. That is

$$A^2(\theta) \geq A^2(\theta = 0) = 1 + \frac{\omega^2}{E^2} - \frac{2\beta\omega}{E} . \tag{2.34}$$

Let $\frac{\omega}{E} = x$, then the right-hand side of (2.34) is

$$f(x) = 1 + x^2 - 2\beta x . \tag{2.35}$$

The extremum of *f*(*x*) occurs at the point which gives $\frac{df}{dx} = 0$. That is

$$\frac{df}{dx} = 2x - 2\boldsymbol{b} = 0 \ .$$

Therefore *f(x)* is extremum at $x = \boldsymbol{b}$. Then differentiate $\frac{df}{dx}$ with respect to x again. That is

$$\frac{d^2 f}{dx^2} = 2 \ .$$

The second derivative of *f(x)* with respect to *x* is positive at $x = \boldsymbol{b}$, thus *f(x)* is minimum at this point. Hence,

$$f(x) \geq 1 + \boldsymbol{b}^2 - 2\boldsymbol{b}^2 = 1 - \boldsymbol{b}^2 \ .$$

This means

$$A^2(\boldsymbol{q}) \geq 1 - \boldsymbol{b}^2 \qquad \text{or} \qquad A(\boldsymbol{q}) \geq \sqrt{1 - \boldsymbol{b}^2} \ . \qquad (2.36)$$

Thus for $0 \leq \boldsymbol{b} < 1$

$$A(\boldsymbol{q}) \geq \sqrt{1 - \boldsymbol{b}^2} > 0 \qquad (2.37)$$

for all $\boldsymbol{q}$ in [0,$\boldsymbol{p}$].

The expression for $A(\boldsymbol{q})$ also leads to the inequality $E(1 + A(\boldsymbol{q})) - \boldsymbol{w} > 0$ for all $0 \leq \boldsymbol{q} \leq \boldsymbol{p}$. This relation can be proven by contradiction. Suppose the converse of this relation is true. That is, as an initial hypothesis suppose that

$$E(1 + A(\boldsymbol{q})) - \boldsymbol{w} \leq 0$$

or

$$\boldsymbol{w} \geq E(1 + A(\boldsymbol{q})) \qquad (2.38)$$

for some q in [$0,p$]. Let $x = \frac{w}{E}$ again. Then (2.38) implies that

$$x \geq 1 + A(q) \tag{2.39}$$

Due to the strict positivity of $A(q)$,

$$x \geq 1 + A(q) > 1 \tag{2.40}$$

This means

$$x - 1 \geq A(q) > 0 \tag{2.41}$$

Squaring (2.41) gives

$$x^2 - 2x + 1 \geq 1 + x^2 + 2bx\cos q \, ,$$

which leads to

$$2bx\cos q \geq 2x \tag{2.42}$$

or

$$\cos q \geq \frac{1}{b} > 1 \, . \tag{2.43}$$

The contradictory statement in (2.43) for a cosine function implies that the initial hypothesis (2.38) is false for all q in [0,p]. That is $E(1 + A(q))$ being some real number must satisfy

$$E(1 + A(q)) > w \tag{2.44}$$

Since

$$q_{2\pm}^{0} = E \pm E\sqrt{A^2 - ie}$$

and $A^2(q)$ is strictly positive for $0 \leq b < 1$ as indicated, thus it can be factored out from the square root leading to

$$q^0_{2\pm} = E\left(1 \pm A\sqrt{1-\frac{i\varepsilon}{A^2}}\right)$$

$$= E\left[1 \pm A\left(1-\frac{i\varepsilon}{2A^2}\right)\right].$$

Since $\varepsilon \to +0$ and $(1+\varepsilon)^n \approx 1+n\varepsilon$, this can be applied to the expression above to obtain

$$q^0_{2\pm} = E\left[1 \pm A\left(1-\frac{i\varepsilon}{2A^2}\right)\right]$$

$$q^0_{2\pm} = \left[E \pm (EA - i\varepsilon)\right]. \qquad (2.45)$$

Rewrite D as

$$D = (q^0 - q^0_{1+})(q^0 - q^0_{1-})(q^0 - q^0_{2+})(q^0 - q^0_{2-})$$

or

$$D = \left(q^0 - \omega + i\varepsilon\right)\left(q^0 + \omega - i\varepsilon\right)\left(q^0 - E - EA + i\varepsilon\right)\left(q^0 - E + EA - i\varepsilon\right) \qquad (2.46)$$

Next, we evaluate the integral q^0 over by using the residue theorem. Close the contour in the complex q^0-plane as in Fig. 2.2, by noting that D has enough powers in q^0 to make sure that the infinite semi-circle gives no contribution to the resulting integral. The enclosed poles in the lower complex plane are q^0_{1+} and q^0_{2+}.

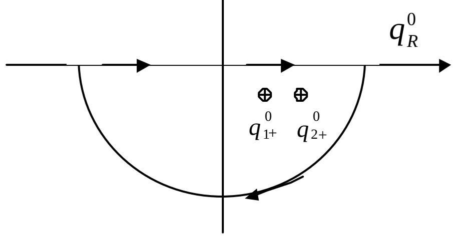

Figure 2.2 Closing the contour in a lower semi-circle of the complex q^0-plane.

From the residue theorem (see appendix D), the q^0 integration can be carried out by using the following integral,

$$\int_{-\infty}^{\infty} dq^0 \left[\frac{i\bar{u}Nu}{D}\right] = 2p\left[\text{Residue at } q^0_{1+} + \text{Residue at } q^0_{2+}\right]. \qquad (2.47)$$

First, compute the residue at q^0_{1+}:

$$\operatorname{Re}s(q^0_{1+}) = (q^0 - w + ie)\frac{\bar{u}Nu\,|_{q^0=q^0_{1+}}}{(q^0 + w - ie)(q^0 - w + ie)(q^0 - E - EA + ie)(q^0 - E + EA - ie)}$$

$$= \frac{\bar{u}Nu\,|_{q^0=q^0_{1+}}}{(w - ie + w - ie)(w - ie - E - EA + ie)(w - ie - E + EA - ie)}$$

$$= \frac{\bar{u}Nu\,|_{q^0=q^0_{1+}}}{2(w - ie)(w - E - EA)(w - E + EA - 2ie)}. \qquad (2.48)$$

The denominator of (2.48) is

$$d_1 = 2(w - ie)(w - E - EA)(w - E + EA - 2ie). \qquad (2.49)$$

Combine the second and the third parenthesis together by using the relation $(a-b)(a+b) = a^2 - b^2$. That is

$$(w - E - EA)(w - E + EA - 2ie) = (w - E)^2 - (EA)^2 + 2ie(E + EA - w). \qquad (2.50)$$

Due to (2.44) the imaginary part of (2.50) is positive. Hence for $e \to +0$,

$$(w - E - EA)(w - E + EA - 2ie) = (w - E)^2 - (EA)^2 + ie. \qquad (2.51)$$

Expand the terms squared and combine them together to obtain

$$(w - E)^2 - (EA)^2 + ie$$

$$= w^2 + E^2 - 2wE - E^2\left(1 + \frac{w^2}{E^2} - \frac{2bEw\cos q}{E^2}\right) + ie$$

$$= 2bEw\cos q - 2wE + ie = -2bEw\left(\frac{1}{b} - \cos q + ie\right). \qquad (2.52)$$

Substitute this into (2.50), then

$$d_{1} = -4bE w(w - ie)(\frac{1}{b} - \cos q + ie) \qquad (2.53)$$

Therefore

$$\mathrm{Res}(q_{1+}^{0}) = \frac{\bar{u}Nu\,|_{q^0=q_{1+}^0}}{-4bEw(w - ie)(\frac{1}{b} - \cos q + ie)} \qquad (2.54)$$

Now we use the well known relation (D. 5) given in Appendix D which leads to

$$\mathrm{Im}\left[\frac{1}{x + ie}\right] = -p\,\text{ä}(x) \qquad (2.55)$$

When $e \to +0$, the contribution of the imaginary part, which is related to the power spectrum, can be obtained from the denominator of (2.54). That is

$$\mathrm{Im}\left[\mathrm{Res}(q_{1+}^{0})\right] = \frac{\bar{u}Nu\,|_{q^0=w}}{4bEw^2}\,p\,\text{ä}\left(\frac{1}{b} - \cos q\right) \qquad (2.56)$$

Since $b < 1$, $\frac{1}{b} > 1$, also $1 \geq \cos q \geq -1$ (Fig. 2.3), thus the delta function in (2.56) is zero. Therefore, the integral from the first pole is zero, and gives no contribution to the power spectrum.

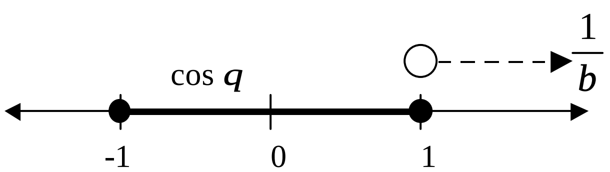

Figure 2.3 The range of values of $\cos q$ is represented by the bold-line and the range of the *1/b* values is represented by the dashed-line. The corresponding regions never overlap.

Next, compute the residue at q^0_{2+}:

$$\mathrm{Res}(q^0_{2+})$$
$$= \left(q^0 - E - EA + i\varepsilon\right)\frac{\bar{u}Nu\,|_{q^0=q^0_{2+}}}{\left(q^0 - \omega + i\varepsilon\right)\left(q^0 + \omega - i\varepsilon\right)\left(q^0 - E - EA + i\varepsilon\right)\left(q^0 - E + EA - i\varepsilon\right)}$$

$$= \frac{\bar{u}Nu\,|_{q^0=q^0_{2+}}}{(E + EA - i\varepsilon - \omega + i\varepsilon)(E + EA - i\varepsilon + \omega - i\varepsilon)(E + EA - i\varepsilon - E + EA - i\varepsilon)}$$

$$= \frac{\bar{u}Nu\,|_{q^0=q^0_{2+}}}{2(E + EA - \omega)(E + EA + \omega - 2i\varepsilon)(EA - i\varepsilon)}\,. \qquad (2.57)$$

The denominator of (2.57) is

$$d_2 = 2(E + EA - \omega)(E + EA + \omega - 2i\varepsilon)(EA - i\varepsilon)\,. \qquad (2.58)$$

Combine the first and the second parentheses together, thus

$$d_2 = 2(EA - i\varepsilon)\left[(E + EA)^2 - \omega^2 - 2i\varepsilon(E + EA - \omega)\right]. \qquad (2.59)$$

Due to the inequality in (2.44), (2.60) becomes

$$d_2 = 2(EA - i\varepsilon)\left[(E + EA)^2 - \omega^2 - i\varepsilon\right]. \qquad (2.60)$$

Therefore

$$\mathrm{Res}(q^0_{2+}) = \frac{\bar{u}Nu\,|_{q^0=q^0_{2+}}}{2(EA - i\varepsilon)\left[(E + EA)^2 - \omega^2 - i\varepsilon\right]} \qquad (2.61)$$

and

$$\lim_{\varepsilon\to+0}\mathrm{Res}(q^0_{2+}) = \frac{\bar{u}Nu\,|_{q^0=E+EA}}{2EA\left[(E + EA)^2 - \omega^2\right]}, \qquad (2.62)$$

which is real and gives no contribution to the power spectrum.

The integral in (2.47) gives no contribution to the power spectrum, thus

$$P(\omega) = 0\,. \qquad (2.63)$$

This means an electron does not radiate. That is, it is stable in vacuum. This analysis provides a rigorous justification that no contact term is needed in our method of investigation to properly normalize $P(ù) = 0$ for $n = 1$.

Chapter III

The Photon Propagator in a Medium

The electron self-energy, which was discussed in the previous chapter, required the photon propagator in vacuum. In order to extend the problem to a medium of refractive index *n*, this photon propagator must be modified, while the electron propagator is still the same. The method shown in this chapter will follow the Schwinger, et al. (1976) paper.

The photon propagator in the medium can be derived from Maxwell's equation through the following relation

$$A^{m}(x) = \int (dx')\, D^{mn}(x - x')\, J_{n}(x') \,, \qquad (3.1)$$

where $A^{m}(x)$ is the potential four-vector, $D^{mn}(x - x')$ is the photon propagator and $J_{n}(x')$ is an arbitrary current-density four-vectors.

The medium is assumed to be homogeneous and isotropic and the interaction with the electromagnetic field can be described by the permittivity x and the permeability m. The refractive index is given by $n = (xm)^{1/2}$, which is supposed to be a real quantity. Absorptive effects, anisotropy and spatial dispersion will be neglected.

We start from Maxwell's equations.

$$\vec{\nabla} \bullet \vec{\mathbf{D}} = r \quad , \qquad (3.2)$$

$$\vec{\nabla} \bullet \vec{\mathbf{B}} = 0\,, \qquad (3.3)$$

$$\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}\,, \qquad (3.4)$$

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}\,. \qquad (3.5)$$

For an isotropic, permeable medium,

$$\vec{\mathbf{D}} = x\vec{\mathbf{E}}\,, \quad \vec{\mathbf{B}} = m\vec{\mathbf{H}}\,, \quad n^2 = xm, \qquad (3.6)$$

From (3.3),

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}\,, \qquad (3.7)$$

where $\vec{\mathbf{A}}$ is a vector potential. Substitute (3.7) and (3.6) into (3.5), this gives

$$\begin{aligned} \vec{\nabla} \times \vec{\mathbf{E}} &= -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\mathbf{A}}) \\ &= -\vec{\nabla} \times \frac{\partial \vec{\mathbf{A}}}{\partial t} \end{aligned}$$

$$\vec{\nabla} \times (\vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t}) = 0\,, \qquad (3.8)$$

which implies that

$$\vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t} = -\vec{\nabla} f\,,$$

and leads to

$$\vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{A}}}{\partial t} - \vec{\nabla} f\,, \qquad (3.9)$$

where j is a scalar potential. (3.7) and (3.9) can be combined in tensor form as

$$F^{mn} = \partial^m A^n - \partial^n A^m\,, \qquad (3.10)$$

with

$$F^{0i} = E^i\,, \qquad F^{ij} = \in^{ijk} B_k\,, \qquad A^m = (f, \vec{\mathbf{A}}) \qquad (3.11)$$

Substitute (3.6) into (3.2), this leads to

$$\vec{\nabla} \bullet \vec{\mathbf{E}} = \frac{r}{x} . \tag{3.12}$$

Replace $\vec{\mathbf{E}}$ by (3.9), this gives

$$\vec{\nabla} \bullet (-\frac{\partial \vec{\mathbf{A}}}{\partial t} - \vec{\nabla} f) = \frac{r}{x} . \tag{3.13}$$

By using the following vector formula

$$\vec{\nabla} \bullet \left(\vec{\nabla} f \right) = \vec{\nabla}^2 f , \tag{3.14}$$

(3.13) becomes

$$-\nabla^2 f - \frac{\partial}{\partial t} (\vec{\nabla} \bullet \vec{\mathbf{A}}) = \frac{r}{x} . \tag{3.15}$$

Substitute (3.6) and (3.7) into (3.4), this gives

$$\frac{1}{m} (\vec{\nabla} \times \vec{\mathbf{B}}) = \vec{\mathbf{J}} + x \frac{\partial \vec{\mathbf{E}}}{\partial t} ,$$

thus

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{A}}) = m\vec{\mathbf{J}} + mx \frac{\partial}{\partial t} (-\frac{\partial \vec{\mathbf{A}}}{\partial t} - \vec{\nabla} f) . \tag{3.16}$$

By using the vector identity

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\mathbf{A}} \right) = \vec{\nabla} \left(\vec{\nabla} \bullet \vec{\mathbf{A}} \right) - \vec{\nabla}^2 \vec{\mathbf{A}} , \tag{3.17}$$

(3.16) becomes

$$-\nabla^2 \vec{\mathbf{A}} + \vec{\nabla} (\vec{\nabla} \bullet \vec{\mathbf{A}}) = m\vec{\mathbf{J}} - mx \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} - mx \vec{\nabla} \left(\frac{\partial f}{\partial t} \right) . \tag{3.18}$$

In the Lorentz gauge

$$\vec{\nabla} \bullet \vec{\mathbf{A}} - mx \frac{\partial f}{\partial t} = 0 , \tag{3.19}$$

or in relativistic notation form

$$\partial_m A^m - (mx - 1)(h\partial)(hA) = 0\,, \tag{3.20}$$

where $h^m = (1,0,0,0)$ is a time-like unit vector.

Therefore, (3.15) becomes

$$-\nabla^2 f - \frac{\partial}{\partial t}\left(-mx\frac{\partial f}{\partial t}\right) = \frac{r}{x}$$

or

$$\nabla^2 f - mx\frac{\partial^2 f}{\partial t^2} = -\frac{r}{x}\,, \tag{3.21}$$

and (3.18) becomes

$$\nabla^2 \vec{\mathbf{A}} - mx\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -m\vec{\mathbf{J}} - \vec{\nabla}\left(\vec{\nabla} \bullet \vec{\mathbf{A}} - mx\frac{\partial f}{\partial t}\right)$$

or

$$\nabla^2 \vec{\mathbf{A}} - mx\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -m\vec{\mathbf{J}}\,. \tag{3.22}$$

Equations (3.21) and (3.22) can be combined together as

$$\vec{\nabla}^2 A^m - mx\frac{\partial^2}{\partial t^2} A^m = -m\left[g^{mn} + \left(1 - \frac{1}{n^2}\right)h^m h^n\right] J_n\,, \tag{3.23}$$

where $J^m = (r, \vec{\mathbf{J}})$ is the current-density four-vector,

$x^m = (t, \vec{\mathbf{r}})$ is the space-time four-vector

and $k^m = (k^0, \vec{\mathbf{k}})$ is the energy-momentum four-vector of an electron.

This can also be expressed in momentum representation by using the following Fourier transforms:

$$A^m(x) = \int \frac{(dk)}{(2p)^4} e^{ikx} A^m(k) \tag{3.24}$$

and

$$J_n(x) = \int \frac{(dk)}{(2p)^4} e^{ikx} J_n(k) \, . \tag{3.25}$$

Substitute (3.24) and (3.25) into (3.23), then

$$\vec{\nabla}^2 \int \frac{(dk)}{(2p)^4} e^{ikx} A^m(k) - mx \frac{\partial^2}{\partial t^2} \int \frac{(dk)}{(2p)^4} e^{ikx} A^m(k)$$

$$= -m\left[g^{mn} + \left(1 - \frac{1}{n^2}\right) h^m h^n \right] \int \frac{(dk)}{(2p)^4} e^{ikx} J_n(k) \, . \tag{3.26}$$

From $\vec{\nabla}^2 e^{ikx} = -\vec{\mathbf{k}}^2 e^{ikx}$ and $\frac{\partial^2}{\partial t^2} e^{ikx} = -(k^0)^2 e^{ikx}$, (3.25) becomes

$$\int \frac{(dk)}{(2p)^4} e^{ikx} A^m(k)(-\vec{\mathbf{k}}^2) + \int \frac{(dk)}{(2p)^4} e^{ikx} A^m(k) mx (k^0)^2$$

$$= \int \frac{(dk)}{(2p)^4} e^{ikx} J_n(k)(-m)\left[g^{mn} + \left(1 - \frac{1}{n^2}\right) h^m h^n \right] . \tag{3.27}$$

The integrands in (3.27) can be combined together as

$$\int \frac{(dk)}{(2p)^4} e^{ikx} \left[-\vec{\mathbf{k}}^2 A^m(k) + mx(k^0)^2 A^m(k) + m\left[g^{mn} + \left(1 - \frac{1}{n^2}\right) h^m h^n \right] J_n(k) \right] = 0 \, . \tag{3.28}$$

This implies from the inverse of the Fourier transform that

$$-\vec{\mathbf{k}}^2 A^m(k) + mx(k^0)^2 A^m(k) = -m\left[g^{mn} + \left(1 - \frac{1}{n^2}\right) h^m h^n \right] J_n(k) ,$$

or

$$\left(\vec{\mathbf{k}}^2 - n^2 (k^0)^2\right) A^m(k) = m\left[g^{mn} + \left(1 - \frac{1}{n^2}\right) h^m h^n \right] J_n(k) , \tag{3.29}$$

which is the Fourier transform of (3.1). That is

$$A^{\mu}(k) = D^{\mu\nu}(k)\, J_{\nu}(k)\,. \tag{3.30}$$

From (3.29)

$$A^{\mu}(k) = \mu\left[g^{\mu\nu} + \left(1 - \frac{1}{n^2}\right)\eta^{\mu}\eta^{\nu}\right]\frac{1}{\left(\vec{\mathbf{k}}^2 - n^2 (k^0)^2\right)} J_{\nu}(k)\,. \tag{3.31}$$

Thus the photon propagator in a medium is

$$D^{\mu\nu}(k) = \mu\left[g^{\mu\nu} + \left(1 - \frac{1}{n^2}\right)\eta^{\mu}\eta^{\nu}\right]\frac{1}{\left|\vec{\mathbf{k}}\right|^2 - n^2 (k^0)^2}\,, \tag{3.32}$$

with $\vec{\mathbf{k}}^2 = \vec{\mathbf{k}} \bullet \vec{\mathbf{k}} = \left|\vec{\mathbf{k}}\right|^2$.

In covariant indices form (see Appendix B)

$$D_{\mu\nu}(k) = \mu\left[g_{\mu\nu} + \left(1 - \frac{1}{n^2}\right)\eta_{\mu}\eta_{\nu}\right]\frac{1}{\left|\vec{\mathbf{k}}\right|^2 - n^2 (k^0)^2}\,, \tag{3.33}$$

where $\eta_{\mu} = (-1,0,0,0)$.

In vacuum, n = 1 and μ = 1 (3.33) becomes

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{\left|\vec{\mathbf{k}}\right|^2 - (k^0)^2}\,, \tag{3.34}$$

which is the photon propagator in vacuum as used in Chapter II.

There are some useful remarks that are worth emphasizing and which will be used later. When the vacuum is replaced by a medium, the photon propagator can be obtained by the following replacements

$$g_{\mu\nu} \rightarrow \mu\left[g_{\mu\nu} + \left(1 - \frac{1}{n^2}\right)\eta_{\mu}\eta_{\nu}\right] \tag{3.35}$$

and

$$k^2 = \left|\vec{\mathbf{k}}\right|^2 - (k^0)^2 \rightarrow \left|\vec{\mathbf{k}}\right|^2 - n^2 (k^0)^2\,. \tag{3.36}$$

Chapter IV

Quantum Electrodynamics of Èerenkov Radiation

for *T* = 0

The corresponding problem in vacuum was studied in detailed in Chapter II. In this chapter the electron is considered to move in a medium of refractive index n. The photon propagator, which has been derived in chapter III, will be used here. A closed form expression for the power spectrum will be derived by using complex analysis without combining denominators of the Feynman propagators in parametric form.

4.1 Expression for the Power Spectrum

The expression for the power spectrum is similar to (2.6). However, the photon propagator in a medium will be changed, as discussed in Chapter III. The photon propagator in (3.33) will be used in the self-energy expression. From Chapter III, the photon propagator in medium is

$$D_{mn}(q) = m\frac{\left[g_{mn} + \left(1 - \frac{1}{n^2}\right)h_m h_n\right]}{\vec{\mathbf{q}}^2 - n^2 (q^0)^2 - i e}, \tag{4.1}$$

where $h_m = (-1,0,0,0)$ is a time-like unit vector, with m and $n = \sqrt{xm}$ denoting, respectively, the permeability and the index of refraction of the medium. Substitute

(4.1) into the self-energy expression (2.8), using, in the process the same expression for the electron propagator, to obtain

$$\Sigma(p) = ie^2 \int \frac{(dq)}{(2\pi)^4} \frac{\gamma^\mu\left[-\gamma(p-q)+m\right]\gamma^\nu}{\left[(p-q)^2+m^2-i\varepsilon\right]} \mu \frac{\left[g_{\mu\nu} + \left(1-\frac{1}{n^2}\right)\eta_\mu\eta_\nu\right]}{\vec{\mathbf{q}}^2 - n^2(q^0)^2 - i\varepsilon}, \qquad (4.2)$$

and

$$I(p,q) = \frac{\mu e^2}{(2\pi)^4}\,\mathrm{Im}\left[\frac{i\bar{u}\Sigma u}{D}\right],$$

where now

$$N = \gamma^\mu\left[-\gamma(p-q)+m\right]\gamma^\nu\left[g_{\mu\nu} + \left(1-\frac{1}{n^2}\right)\eta_\mu\eta_\nu\right] \qquad (4.3)$$

and

$$D = \left[(p-q)^2 + m^2 - i\varepsilon\right]\left[\vec{\mathbf{q}}^2 - n^2(q^0)^2 - i\varepsilon\right]. \qquad (4.4)$$

The expression for the power spectrum is then

$$P(\omega) = -\frac{2\mu n\,\omega\mu e^2}{E}\int\frac{(dq)}{(2\pi)^4}\ \delta(|\vec{\mathbf{q}}| - n\omega)\,\mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right], \qquad (4.5)$$

As in Chapter II

$$\int(dq) = \int_{-0}^{1} d(\cos\theta)\int_{-\infty}^{\infty} dq^0 \int_0^{2\pi} d\phi \int_0^{\infty} d|\vec{\mathbf{q}}|\,|\vec{\mathbf{q}}|^2 .$$

Carrying out the $|\vec{\mathbf{q}}|$ integration by using the property (C. 4) of a delta function in Appendix C, then

$$P(\omega) = -\frac{2\mu n\omega\mu e^2}{E(2\pi)^4}\int_{-1}^{1} d(\cos\theta)\int_{-\infty}^{\infty} dq^0 \int_0^{2\pi} d\phi \int d|\mathbf{q}|\,|\vec{\mathbf{q}}|^2\ \delta(|\vec{\mathbf{q}}| - n\omega)\,\mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right]$$

$$= -\frac{2mnwme^2}{E(2p)^4}\int_{-1}^{1} d(\cos q)\int_{-\infty}^{\infty} dq^0 \int_{0}^{2p} df\, n^2 w^2\ \mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right] \quad , \tag{4.6}$$

with $N = N(|\vec{\mathbf{q}}| = n\,w)$, and the angular f integration in (4.6) gives

$$P(w) = -\frac{2mn^3 w^3 me^2}{E(2p)^4}(2p)\int_{-1}^{1} d(\cos q)\int_{-\infty}^{\infty} dq^0\ \mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right]. \tag{4.7}$$

Next, we evaluate $\bar{u}Nu$ in (4.4) to obtain

$$\bar{u}Nu = \bar{u}g^m\left[-g(p-q)+m\right]g^n\left[g_{mn} + \left(1-\frac{1}{n^2}\right)h_m h_n\right]u$$

$$= \bar{u}n_1 u + \bar{u}n_2 u\ , \tag{4.8}$$

where

$$\bar{u}n_1 u = \bar{u}\,g^m\left[-g(p-q)+m\right]g_m$$

$$= \frac{2b^2E^2}{m}\left[1-\frac{1}{b^2}-\frac{q^0}{b^2E}+\frac{nw\cos q}{bE}\right], \tag{4.9}$$

as done in Chapter II, and

$$\bar{u}n_2 u = \bar{u}g^m\left[-g(p-q)+m\right]g^n\left(1-\frac{1}{n^2}\right)h_m h_n\ u\ . \tag{4.10}$$

From $h_m = (-1,0,0,0)$, thus only the time-component contributes, and (4.10) becomes

$$\bar{u}n_2 u = \bar{u}g^0\left[-g(p-q)+m\right]g^0 u\left(1-\frac{1}{n^2}\right). \tag{4.11}$$

Expand the expressions in the first parenthesis as

$$-g(p-q)+m = -g^m(p-q)_m + m$$

$$= \vec{\mathbf{a}}\bullet(\vec{\mathbf{p}}-\vec{\mathbf{q}}) + g^0(p-q)_0 + m. \tag{4.12}$$

For the electron,

$$p^0 = E\ ,\ \ b = \frac{|\vec{\mathbf{p}}|}{E} = \frac{v}{c}\,, \qquad (4.13)$$

where E is the electron's energy. Apply (4.13) to (4.12) and substitute it into (4.11), to obtain

$$\bar{u}n_2u = \bar{u}g^0\left[-\tilde{\mathbf{a}}\bullet(\vec{\mathbf{p}}-\vec{\mathbf{q}}) + g^0(p-q)^0 + m\right]g^0u\left(1-\frac{1}{n^2}\right)$$

$$= \bar{u}g^0\left[-\vec{\tilde{\mathbf{a}}}\bullet\vec{\mathbf{p}} + \vec{\tilde{\mathbf{a}}}\bullet\vec{\mathbf{q}} + g^0E - g^0q^0 + m\right]g^0u\left(1-\frac{1}{n^2}\right). \qquad (4.14)$$

By using the Dirac gamma matrices in (B. 11)-(B. 12) in an appendix B one obtain

$$g^0g^ig^0 = -g^i\,,$$

$$g^0g^0g^0 = g^0\,,$$

(4.14) becomes

$$\bar{u}n_2u = \bar{u}\left[\tilde{\mathbf{a}}\bullet\vec{\mathbf{p}} - \tilde{\mathbf{a}}\bullet\vec{\mathbf{q}} + g^0E - g^0q^0 + m\right]u\left(1-\frac{1}{n^2}\right) \qquad (4.15)$$

From the Dirac equation (B. 6) and (B. 18), one obtains

$$\bar{u}\,(g^m p_m + m) = 0$$

$$\bar{u}\,(\vec{\tilde{\mathbf{a}}}\bullet\vec{\mathbf{p}} - g^0E + m) = 0$$

$$\bar{u}\,\vec{\tilde{\mathbf{a}}}\bullet\vec{\mathbf{p}} = \bar{u}g^0E - m\bar{u}$$

and

$$\bar{u}\,\tilde{\mathbf{a}}\bullet\vec{\mathbf{q}}\,u = \frac{\vec{\mathbf{p}}\bullet\vec{\mathbf{q}}}{m}\,.$$

Upon replacing these relations in (4.15), give

$$\bar{u}n_2u = [\bar{u}g^0Eu - m\bar{u}u - \frac{\vec{\mathbf{p}}\bullet\vec{\mathbf{q}}}{m} + \bar{u}g^0E\,u - \bar{u}g^0q^0u + m\bar{u}u]\left(1-\frac{1}{n^2}\right). \qquad (4.16)$$

Apply (4.13) to (4.16) to obtain

$$\bar{u}n_2u = [2\bar{u}\gamma^0 Eu - \frac{\beta En\omega\cos\theta}{m} - \bar{u}\gamma^0 q^0 u]\left(1-\frac{1}{n^2}\right). \tag{4.17}$$

From (B. 7) in Appendix B, (4.17) becomes

$$\bar{u}n_2u = [2u^+\gamma^0\gamma^0 Eu - \frac{\beta En\omega\cos\theta}{m} - u^+\gamma^0\gamma^0 q^0 u]\left(1-\frac{1}{n^2}\right). \tag{4.18}$$

From the anti-commutation relation (B. 10) in the appendix B and using the property $\gamma^0\gamma^0 = 1$ we obtain

$$\bar{u}n_2u = [2Eu^+u - \frac{\beta En\omega\cos\theta}{m} - q^0u^+u]\left(1-\frac{1}{n^2}\right). \tag{4.19}$$

By using the relation (B. 9) in Appendix B, the latter becomes

$$\bar{u}n_2u = [\frac{2E^2}{m} - \frac{\beta En\omega\cos\theta}{m} - \frac{q^0E}{m}]\left(1-\frac{1}{n^2}\right). \tag{4.20}$$

Finally, factoring out $\frac{2\beta^2E^2}{m}$ from (4.20) gives

$$\bar{u}n_2u = \frac{2\beta^2E^2}{m}\left[\frac{1}{\beta^2} - \frac{n\omega\cos\theta}{2\beta E} - \frac{q^0E}{2\beta^2E^2}\right]\left(1-\frac{1}{n^2}\right). \tag{4.21}$$

Combine (4.9) and (4.21) together, to get

$$\bar{u}Nu = \bar{u}n_1u + \bar{u}n_2u$$

$$= \frac{2\beta^2E^2}{m}\left[1-\frac{1}{\beta^2}-\frac{q^0}{\beta^2E}+\frac{n\omega\cos\theta}{\beta E}\right] + \frac{2\beta^2E^2}{m}\left[\frac{1}{\beta^2}-\frac{n\omega\cos\theta}{2\beta E}-\frac{q^0}{2\beta^2E}\right]\left(1-\frac{1}{n^2}\right)$$

$$= \frac{2\beta^2E^2}{m}\left[1-\frac{1}{\beta^2}-\frac{q^0}{\beta^2E}+\frac{n\omega\cos\theta}{\beta E}+\frac{1}{\beta^2}-\frac{n\omega\cos\theta}{2\beta E}-\frac{q^0}{2\beta^2E}\right.$$
$$\left.-\frac{1}{n^2\beta^2}+\frac{n\omega\cos\theta}{2n^2\beta E}+\frac{q^0}{2n^2\beta^2E}\right]$$

$$=\frac{2b^2E^2}{m}\left[1+\frac{1}{b^2}(-1+1-\frac{1}{n^2})-\frac{q^0}{2b^2E}(2+1-\frac{1}{n^2})+\frac{nw\cos q}{2bE}(2-1+\frac{1}{n^2})\right].$$

Therefore,

$$\bar{u}Nu=\frac{2b^2E^2}{m}\left[1-\frac{1}{n^2b^2}-\frac{q^0}{2b^2E}(3-\frac{1}{n^2})+\frac{nw\cos q}{2bE}(1+\frac{1}{n^2})\right]. \qquad (4.22)$$

Rewrite (4.4) by factoring out n^2 from the second term as

$$D=\left[p^2+q^2-2p^mq_m+m^2-ie\right]\left[\vec{\mathbf{q}}^2-n^2(q^0)^2-ie\right]$$

$$=n^2\left[p^2+q^2-2p^mq_m+m^2-ie\right]\left[\frac{\vec{\mathbf{q}}^2}{n^2}-(q^0)^2-\frac{ie}{n^2}\right]. \qquad (4.23)$$

Then replace $|\vec{\mathbf{q}}|=nw$ from the delta function, this gives

$$D=n^2\left[p^2+q^2-2p^mq_m+m^2-ie\right]\left[w^2-(q^0)^2-ie\right]. \qquad (4.24)$$

Let $D=n^2D_e$, where

$$D_e=\left[p^2+q^2-2p^mq_m+m^2-ie\right]\left[w^2-(q^0)^2-ie\right]. \qquad (4.25)$$

Therefore, the expression for the power spectrum in (4.6) becomes

$$P(w)=-\frac{2mnw^3me^2}{E(2p)^3}\int_{-1}^{1}d(\cos q)\int_{-\infty}^{\infty}dq^0\ \mathrm{Im}\left[\frac{i\bar{u}Nu}{D_e}\right]. \qquad (4.26)$$

4.2 Rigorous Estimates of the Singularities

The singularities occur at points in q^0 -plane, which make $D_e=0$ will be now investigated. Expand the expressions in (4.25) and use the mass-shell relation $p^2+m^2=0$, then

$$D_e=\left[\vec{\mathbf{q}}^2-(q^0)^2-2(\vec{\mathbf{p}}\bullet\vec{\mathbf{q}}-q^0E)-ie\right]\left[w^2-(q^0)^2-ie\right]. \qquad (4.27)$$

Apply (4.13) and substitute $|\vec{\mathbf{q}}|=nw$ from the delta function, then

$$D_\varepsilon = \left[-(q^0)^2 + 2q^0E + n^2\omega^2 - 2\beta En\omega\cos\theta - i\varepsilon\right]\left[-(q^0)^2 + \omega^2 - i\varepsilon\right]. \qquad (4.28)$$

Factor out the minus sign from each term of (4.28), then (4.28) becomes

$$D_\varepsilon = \left[(q^0)^2 - 2q^0E - n^2\omega^2 + 2\beta En\omega\cos\theta + i\varepsilon\right]\left[(q^0)^2 - \omega^2 + i\varepsilon\right]. \qquad (4.29)$$

The singularities from the first term in (4.29) can be obtained by using the quadratic formulae as done in Chapter II below (2.31), with

$$a = 1, \quad b = -2E,$$

and $$c = -n^2\omega^2 + 2\beta En\omega\cos\theta + i\varepsilon\,.$$

This gives two poles at

$$q^0_{1\pm} = \frac{2E \pm \sqrt{4E^2 - 4(-n^2\omega^2 + 2\beta En\omega\cos\theta + i\varepsilon)}}{2}$$

$$= E \pm \sqrt{E^2 + n^2\omega^2 - 2\beta En\omega\cos\theta - i\varepsilon}$$

$$= E\left(1 \pm \sqrt{1 + \frac{n^2\omega^2}{E^2} - \frac{2\beta n\omega\cos\theta}{E} - i\varepsilon}\right). \qquad (4.30)$$

Let

$$A^2(\theta) = 1 + \frac{n^2\omega^2}{E^2} - \frac{2\beta n\omega\cos\theta}{E}. \qquad (4.31)$$

Thus

$$q^0_{1\pm} = E\left(1 \pm \sqrt{A^2(\theta) - i\varepsilon}\right). \qquad (4.32)$$

Suppose for the moment that $A^2(\theta) > 0$ for $0 \le \beta < 1$, which will be <u>proved</u> later. Therefore, $A^2(\theta)$ can be factored out from the square root sign as

$$q^0_{1\pm} = E\left(1 \pm A(\theta)\sqrt{1 - \frac{i\varepsilon}{A^2(\theta)}}\right). \qquad (4.33)$$

As ε will be taken to go to zero, we may use the expansion

$$(1+e)^n \approx 1+ne, \tag{4.34}$$

which when applied to (4.34) gives

$$q^{0}_{1\pm} = E\left[1 \pm A(q)\left(1 - \frac{ie}{2A^{2}(q}\right)\right]$$

or

$$q^{0}_{1\pm} = E\left[1 \pm A(q) \mp ie\right]. \tag{4.35}$$

Consider the other singularities arising from the second term of (4.29). That is

$$(q^{0})^{2} - w^{2} + ie = 0. \tag{4.36}$$

By also using the quadratic formulae with $a = 1$, $b = 0$ and $c = -w^{2} + ie$, the solution of (4.36) is

$$q^{0}_{2\pm} = \pm\frac{\sqrt{4w^{2} - ie}}{2} = \pm w^{2} \mp ie. \tag{4.37}$$

Then

$$D_e = \left(q^0 - q^0_{1+}\right)\left(q^0 - q^0_{1-}\right)\left(q^0 - q^0_{2+}\right)\left(q^0 - q^0_{2-}\right)$$

$$= \left[q^0 - (E + EA - ie)\right]\left[q^0 - (E - EA + ie)\right]\left[q^0 - (w - ie)\right]\left[q^0 - (-w + ie)\right] \tag{4.38}$$

The proof that $A^{2}(q)$ >0 for $0 \le b < 1$, which was needed in (4.33) now follows.

<u>*Lemma1*</u>*:* *For* $0 \le b < 1$,

$$A^{2}(q) \ge \sqrt{1 - b^{2}} > 0, \tag{4.39}$$

for all q *in* $[0,p]$.

As in Chapter II, $A^{2}(q)$ is minimum at $\cos q = 1$, then

$$A^2(\theta) = 1 + \frac{n^2\omega^2}{E^2} - \frac{2\beta n\omega\cos\theta}{E} \geq 1 + \frac{n^2\omega^2}{E^2} - \frac{2\beta n\omega}{E}. \qquad (4.40)$$

Let $x = \frac{n\omega}{E}$, then the right-hand side of the above inequality is

$$f(x) = 1 + x^2 - 2x\beta. \qquad (4.41)$$

As done in Chapter II, it is easily verified that the minimum of *f(x)* occurs for $x = \beta$, thus

$$A^2(\theta) \geq f(x = \beta) = 1 - \beta^2. \qquad (4.42)$$

Then (4.39) is true.

The following result is also needed later.

Lemma 2 : For n>1 , $0 \leq \beta < 1$,

$$E[1 + A(\theta)] > \omega, \qquad (4.43)$$

for all $0 \leq \theta \leq \pi$ *as a strict inequality.*

The proof of Lemma 2 is done in the same way as in Chapter II. Suppose the converse of (4.43) is true. That is, as an initial hypothesis suppose that

$$\omega \geq E[1 + A(\theta)], \qquad (4.44)$$

for some θ in [0,π]. Let $x = \frac{\omega}{E}$. Then (4.44) implies that

$$x \geq [1 + A(\theta)]. \qquad (4.45)$$

From Lemma 1, $A(\theta)$ is positive, thus

$$x \geq [1 + A(\theta)] > 1 \qquad (4.46)$$

or $\quad x-1 \geq A(\boldsymbol{q}) > 0\,.$ (4.47)

Squaring (4.47)

$$(x-1)^2 \geq 1+\frac{n^2\boldsymbol{w}^2}{E^2}-\frac{2\boldsymbol{b}n\boldsymbol{w}\cos\boldsymbol{q}}{E}=1+nx^2-2\boldsymbol{b}nx\cos\boldsymbol{q}\,. \tag{4.48}$$

This leads to the inequality

$$x^2-2x+1 \geq 1+nx^2-2n\boldsymbol{b}x\cos\boldsymbol{q}$$

or

$$2n\boldsymbol{b}x\cos\boldsymbol{q} \geq x^2(n^2-1)+2x \tag{4.49}$$

or $$\cos\boldsymbol{q} \geq \frac{1}{n\boldsymbol{b}}\left[x\frac{(n^2-1)}{2}+1\right]. \tag{4.50}$$

For $n > 1$ and from (4.46) $x > 1$ then

$$\cos\boldsymbol{q} \geq \frac{1}{n\boldsymbol{b}}\left[x\frac{(n^2-1)}{2}+1\right] > \frac{1}{n\boldsymbol{b}}\left[\frac{(n^2-1)}{2}+1\right]. \tag{4.51}$$

Consider the last expression in (4.51), that is

$$\frac{1}{n\boldsymbol{b}}\left[\frac{(n^2-1)}{2}+1\right]=\frac{(n^2+1)}{2n\boldsymbol{b}}. \tag{4.52}$$

From $n > 1$,

$$n-1>0$$
$$(n-1)^2=n^2-2n+1>0,$$

thus $$n^2+1>2n\,. \tag{4.53}$$

Apply (4.53) to (4.52), this gives

$$\frac{(n^2+1)}{2n\boldsymbol{b}}>\frac{1}{\boldsymbol{b}}\,. \tag{4.54}$$

(4.54) is always greater than 1 for $0 \le \boldsymbol{b} < 1$. Therefore we run into the false result that

$$\cos \boldsymbol{q} > 1, \tag{4.55}$$

which is, of course, not true. The contradictory statement in (4.55)) for a cosine function implies that the initial hypothesis in (4.44) is false for all $\boldsymbol{q}$ in [0,$\boldsymbol{p}$]. That is $E[1 + A(\boldsymbol{q})]$ being some real number must satisfy (4.43).

4.3 Closed Expression for the Power Spectrum and Complex Analysis

The residue theorem will be used again in the integration of the power spectrum. Close the q^0-contour in the complex q^0--plane from below (clockwise), as in Fig. 4.1, by noting that D_e has enough powers in q^0- to make sure that the infinite semi-circle gives no contribution to the resulting integral. The enclosed poles in the lower complex plane are q^0_{1+} and q^0_{2+}.

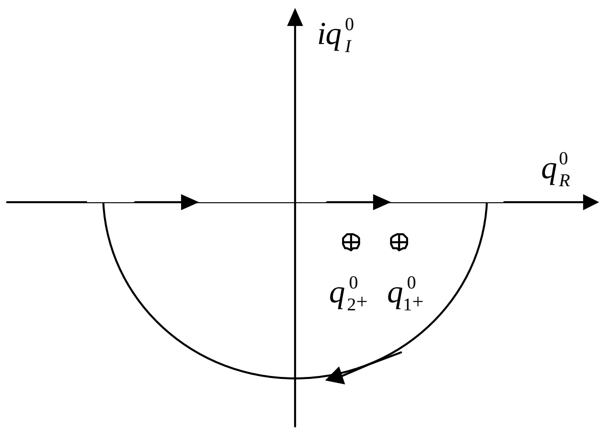

Figure 4.1 The q^0-contour in the complex q^0-plane.

From the residue theorem, the q^0-integral in (4.26) becomes

$$\int_{-\infty}^{\infty} dq^0 \left[\frac{i\bar{u}Nu}{D_e} \right] = 2p\left[\text{Resdue at } q^0_{1+} + \text{Resdue at } q^0_{2+}\right]. \tag{4.56}$$

First, compute the residue at q^0_{1+}:

$$\mathrm{Re}\, s(q^0_{1+})$$

$$= \left(q^0 - q^0_{1+}\right) \frac{\bar{u}Nu\,|_{q^0=q^0_{1+}}}{\left(q^0 - q^0_{1+}\right)\left(q^0 - q^0_{1-}\right)\left(q^0 - q^0_{2+}\right)\left(q^0 - q^0_{2-}\right)}$$

$$= \frac{\bar{u}Nu\,|_{q^0=q^0_{1+}}}{\left[q^0 - (E - EA + ie)\right]\left[q^0 - (w - ie)\right]\left[q^0 - (-w + ie)\right]}$$

$$= \frac{\bar{u}Nu\,|_{q^0=q^0_{1+}}}{[E + EA - ie - (E - EA + ie)][E + EA - ie - (w - ie)][E + EA - ie - (-w + ie)]}$$

(4.57)

When $e \to +0$, the imaginary part only comes from the denominator. The denominator of (4.57) is

$$d_1 = 2(EA - ie)\,(E + EA - w)\,(E + EA + w - 2ie)$$

$$= 2(EA - ie)\left[(E + EA)^2 - w^2 - 2ie(E + EA - w)\right]. \tag{4.58}$$

By using Lemma 2, (4.58) becomes

$$d_1 = 2(EA - ie)\left[(E + EA)^2 - w^2 - ie\right]. \tag{4.59}$$

In the limit $e \to +0$,

$$\lim_{e \to +0} d_1 = 2EA\left[(E + EA)^2 - w^2\right]. \tag{4.60}$$

By using Lemma 1,

$$\lim_{e \to +0} d_1 = 2EA(q)\left[E(1 + A(q))^2 - w^2\right] > 0 \quad (\neq 0)\,. \tag{4.61}$$

Therefore, the integral from the q^0_{1+} poles is real and gives no contribution to the power spectrum in (4.26).

Next, compute the residue at q^0_{2+}:

$$\operatorname{Re} s(q^0_{2+})$$
$$= \left(q^0 - q^0_{2+}\right)\frac{\bar{u}Nu\,|_{q^0=q^0_{2+}}}{\left(q^0 - q^0_{1+}\right)\left(q^0 - q^0_{1-}\right)\left(q^0 - q^0_{2+}\right)\left(q^0 - q^0_{2-}\right)}$$

$$= \frac{\bar{u}Nu\,|_{q^0=q^0_{2+}}}{\left[q^0 - (E + EA - ie)\right]\left[q^0 - (E - EA + ie)\right]\left[q^0 - (-w + ie)\right]}$$

$$= \frac{\bar{u}Nu\,|_{q^0=q^0_{2+}}}{[w - ie - (E + EA - ie)][w - ie - (E - EA + ie)][w - ie - (-w + ie)]}\,. \qquad (4.62)$$

The imaginary part of (4.62) also comes from the denominator. The denominator of (4.62) is

$$d_2 = 2(w - ie)[w - E - EA][(w - E + EA) - 2ie]$$
$$= 2(w - ie)\left[(w - E)^2 - EA^2 - 2ie(w - E - EA)\right]. \qquad (4.63)$$

From Lemma 2, the imaginary part of the second term has a plus sign. Substitute $A^2(q)$ in (4.31), then

$$d_2 = 2(w - ie)\left[w^2 + E^2 - 2Ew - E\left(1 + \frac{n^2w^2}{E^2} - \frac{2bnw\cos q}{E}\right) + ie\right]$$
$$= 2(w - ie)\left[w^2 - 2Ew - n^2w^2 + 2bnw\cos q + ie\right]$$
$$= 2(w - ie)\left[w^2(1 - n^2) + 2Ew(bn\cos q - 1) + ie\right]. \qquad (4.64)$$

As $e \to +0$, the second factor can be equal to zero at some value of *n,* thus the q^0_{2+} can give a contribution to the imaginary part or the power spectrum. The imaginary part of this integral can be obtained from the relation (2.55), which is

$$\mathrm{Im}\left[\frac{1}{x+i\varepsilon}\right]=-\pi\,\delta(x)\,. \tag{2.55}$$

Apply this to (4.64) with $x=\omega^2(1-n^2)+2E\omega(\beta n\cos\theta-1)$, that is

$$\mathrm{Im}\left(\frac{1}{d_2}\right)=\frac{-\pi}{2\omega}\,\delta\left(\omega^2(1-n^2)+2E\omega(\beta n\cos\theta-1)\right),\ \varepsilon\to+0\,. \tag{4.65}$$

Consider the delta function

$$\delta\left(\omega^2(1-n^2)+2E\omega(\beta n\cos\theta-1)\right)$$

$$=\delta\left(2\beta n\omega E\left(\cos\theta-\frac{\omega^2(n^2-1)}{2\beta n\omega E}-\frac{2E\omega}{2\beta n\omega E}\right)\right). \tag{4.66}$$

By using the property (C. 5) in Appendix C, (4.66) becomes

$$\delta\left(\omega^2(1-n^2)+2E\omega(\beta n\cos\theta-1)\right)$$

$$=\frac{1}{2\beta n\omega E}\,\delta\left(\cos\theta-\frac{\omega(n^2-1)}{2\beta nE}-\frac{1}{\beta n}\right)$$

$$=\frac{1}{2\beta n\omega E}\,\delta\left(\cos\theta-\frac{1}{\beta n}\left(1+\frac{\omega(n^2-1)}{2E}\right)\right). \tag{4.67}$$

Substitute (4.67) into (4.65), this gives

$$\mathrm{Im}\left(\frac{1}{d_2}\right)=\frac{-\pi}{2\omega}\frac{1}{2\beta n\omega E}\,\delta\left(\cos\theta-\frac{1}{\beta n}\left(1+\frac{\omega(n^2-1)}{2E}\right)\right),\ \varepsilon\to+0$$

$$=\frac{-\pi}{4\omega^2\beta nE}\,\delta\left(\cos\theta-\frac{1}{\beta n}\left(1+\frac{\omega(n^2-1)}{2E}\right)\right). \tag{4.68}$$

Therefore the imaginary part of (4.56) is

$$\mathrm{Im}\left[\int_{-\infty}^{\infty} dq^0 \left(\frac{i\bar{u}Nu}{D_e}\right)\right]$$

$$= 2\pi\,\bar{u}Nu\,|_{q^0=\omega} \left(\frac{-\pi}{4\omega^2 \beta n E}\right) \delta\left(\cos\theta - \frac{1}{\beta n}\left(1+\frac{\omega(n^2-1)}{2E}\right)\right)$$

$$= -\frac{\pi^2 \bar{u}Nu\,|_{q^0=\omega}}{2\omega^2 \beta n E}\, \delta\left(\cos\theta - \frac{1}{\beta n}\left(1+\frac{\omega(n^2-1)}{2E}\right)\right), \qquad (4.69)$$

where

$$\bar{u}Nu\,|_{q^0=\omega} = \frac{2\beta^2 E^2}{m}\left[1 - \frac{1}{n^2\beta^2} - \frac{\omega}{2\beta^2 E}(3-\frac{1}{n^2}) + \frac{n\,\omega\cos\theta}{2\beta E}(1+\frac{1}{n^2})\right]. \qquad (4.70)$$

Thus (4.26) becomes

$$P(\omega) = -\frac{2mn\omega^3 \mu e^2}{E(2\pi)^3}\int_{-1}^{1} d(\cos\theta) \int_{-\infty}^{\infty} dq^0 \;\mathrm{Im}\left[\frac{i\bar{u}Nu}{D_e}\right]$$

$$= -\frac{2mn\omega^3 \mu e^2}{E(2\pi)^3}\int_{-1}^{1} d(\cos\theta)\left(-\frac{\pi^2 \bar{u}Nu\,|_{q^0=\omega}}{2\omega^2 \beta n E}\,\delta\left(\cos\theta - \frac{1}{\beta n}\left(1+\frac{\omega(n^2-1)}{2E}\right)\right)\right)$$

$$P(\omega) = \frac{\mu m\,\omega\alpha}{2\beta E^2}\int_{-1}^{1} d\,(\cos\theta)\left(\bar{u}Nu\,|_{q^0=\omega}\right)\delta\left(\cos\theta - \frac{1}{\beta n}\left(1+\frac{\omega(n^2-1)}{2E}\right)\right), \qquad (4.71)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant.

The delta function in (4.71) gives a contribution only at

$$\cos\theta - \frac{1}{n\beta}\left(1+\frac{\omega(n^2-1)}{2E}\right) = 0 \qquad (4.72)$$

or

$$\cos\theta = \frac{1}{n\beta}\left(1+\frac{\omega(n^2-1)}{2E}\right). \qquad (4.73)$$

Since for the values of $\cos\theta$, we must have $-1 \le \cos\theta \le 1$, for all θ, this implies that

$$\frac{1}{n\beta}\left(1+\frac{\omega(n^2-1)}{2E}\right)<1, \tag{4.74}$$

which gives the following constraint for the non-vanishing of the power spectrum in (4.71),

$$n\beta > \left(1+\frac{\omega(n^2-1)}{2E}\right). \tag{4.75}$$

This threshold condition coincides with the one obtained by Schwinger et al. in (1.6).

The integration in (4.71) can be evaluated by using the property (C. 4) of the delta function in Appendix C. Using the expression for $\cos\theta$ in (4.73) in (4.72), the integrand becomes

$$\bar{u}Nu = \frac{2\beta^2 E^2}{m}\left[1-\frac{1}{n^2\beta^2}-\frac{\omega}{2\beta^2 E}(3-\frac{1}{n^2})+\frac{n\omega}{2\beta E}\frac{1}{\beta n}\left(1+\frac{\omega(n^2-1)}{2E}\right)(1+\frac{1}{n^2})\right]. \tag{4.76}$$

Consider the last term of (4.76)

$$\frac{\omega}{2\beta^2 E}\left(1+\frac{\omega(n^2-1)}{2E}\right)(1+\frac{1}{n^2})$$

$$=\frac{\omega}{2\beta^2 E}\left[1+\frac{1}{n^2}+\frac{\omega(n^2-1)}{2E}+\frac{\omega(n^2-1)}{2En^2}\right]$$

$$=\frac{\omega}{2\beta^2 E}\left[1+\frac{1}{n^2}+\frac{n^2\omega}{2E}-\frac{\omega}{2En^2}\right]. \tag{4.77}$$

Substitute this into (4.76), thus

$$\bar{u}Nu = \frac{2\beta^2 E^2}{m}\left[1-\frac{1}{n^2\beta^2}-\frac{\omega}{2\beta^2 E}(3-\frac{1}{n^2})+\frac{\omega}{2\beta^2 E}\left(1+\frac{1}{n^2}+\frac{n^2\omega}{2E}-\frac{\omega}{2En^2}\right)\right]$$

$$= \frac{2b^2E^2}{m}\left[1 - \frac{1}{n^2b^2} - \frac{w}{2b^2E}(3 - \frac{1}{n^2}) + \frac{w}{2b^2E}\left(1 + \frac{1}{n^2} + \frac{n^2w}{2E} - \frac{w}{2En^2}\right)\right]$$

$$= \frac{2b^2E^2}{m}\left[1 - \frac{1}{n^2b^2} - \frac{w}{2b^2E}(3 - \frac{1}{n^2} - 1 - \frac{1}{n^2}) + \frac{w^2}{4b^2E^2}\left(n^2 - \frac{1}{n^2}\right)\right]$$

$$= \frac{2b^2E^2}{m}\left[1 - \frac{1}{n^2b^2} - \frac{w}{2b^2E}(2 - \frac{2}{n^2}) + \frac{w^2}{4b^2E^2}\left(n^2 - \frac{1}{n^2}\right)\right]$$

$$= \frac{2b^2E^2}{m}\left[1 - \frac{1}{n^2b^2} - \frac{w}{n^2b^2E}(n^2 - 1) + \frac{w^2}{4n^2b^2E^2}\left(n^4 - 1\right)\right]$$

$$\bar{u}Nu = \frac{2b^2E^2}{m}\left[1 - \frac{1}{n^2b^2}\left(1 + \frac{w}{E}(n^2 - 1)\right) + \frac{w^2}{4n^2b^2E^2}\left(n^4 - 1\right)\right]. \qquad (4.78)$$

Thus, the expression for the power spectrum of Èerenkov radiation in medium of the refractive index *n* at absolute zero temperature is

$$P(w) = \frac{mm\,wa}{2bE^2}\frac{2b^2E^2}{m}\left[1 - \frac{1}{n^2b^2}\left(1 + \frac{w}{E}(n^2 - 1)\right) + \frac{w^2}{4n^2b^2E^2}\left(n^4 - 1\right)\right]$$

or

$$P(w) = awbm\left[1 - \frac{1}{n^2b^2}\left(1 + \frac{w}{E}(n^2 - 1)\right) + \frac{w^2}{4n^2b^2E^2}\left(n^4 - 1\right)\right]. \qquad (4.79)$$

This expression is valid with the threshold condition given through (4.75) for the emission of radiation. Compare this with the result obtained by Schwinger et al. in (1.5). This expression differs from the one given by the Schwinger group in the last term only. Hence this last term is the $^2w^2/E^2$-contribution to the quantum correction. Finally we note that no contact term is necessary in our investigation since for $n = 1$, (4.73) and (4.75) do not hold and *P(ù)* ≡ 0 – a result which we have already established rigorously in Chapter II.

It is interesting to dwell further on the fact as to why the pole at $q^0_{2+} = w - ie$ contributes and the pole at $q^0_{1+} = E(1 + A(q)) - ie$ dose not? We note that the condition $E(1 - A(q)) = w$ is in the domain of integration over cosθ. Accordingly, at this point, the upper pole q^0_{1-} is just above the lower pole q^0_{2+}. For $e \to +0$ they pinch the q^0-contour and no deformation of the latter is possible at this point to avoid the q^0_{2+} pole in the lower complex plane. The poles q^0_{1-} and q^0_{2-}, however, never coincide with q^0_{1+} for $e \to +0$ according to Lemmas 1 and 2.

4.4 Positivity of the Power Spectrum

Although the expression for the power spectrum has been obtained, the task is not complete. It remains to verify that the power spectrum is indeed strictly positive under the constraint in (4.75) and that no further restrictions are necessary.

Combine the second and the third term in the square brackets of (4.79) by factoring out $\left(-\frac{1}{n^2 b^2}\right)$, that is

$$P(w) = awbm\left[1 - \frac{1}{n^2 b^2}\left(1 + \frac{w}{E}(n^2 - 1) - \frac{w^2}{4E^2}(n^4 - 1)\right)\right] \quad (4.80)$$

An extra term, $\frac{w^2}{4E^2}(n^2 - 1)^2$, will be now added and then subtracted to complete the square. Thus

$$P(\omega) = \alpha\omega\beta\mu$$
$$\times\left[1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{E}(n^2-1) + \frac{\omega^2}{4E^2}(n^2-1)^2 - \frac{\omega^2}{4E^2}(n^2-1)^2 - \frac{\omega^2}{4E^2}(n^4-1)\right)\right]$$

or

$$P(\omega) = \alpha\omega\beta\mu\left[1 - \frac{1}{n^2\beta^2}\left(\left(1 + \frac{\omega}{2E}(n^2-1)\right)^2 - \frac{\omega^2}{4E^2}\left((n^2-1)^2 + (n^4-1)\right)\right)\right] \quad (4.81)$$

Simplifying the last term, this gives

$$P(\omega) = \alpha\omega\beta\mu\left[1 - \frac{1}{n^2\beta^2}\left(\left(1 + \frac{\omega}{2E}(n^2-1)\right)^2 - \frac{\omega^2}{4E^2}\left(n^4 - 2n^2 + 1 + n^4 - 1\right)\right)\right]$$

$$= \alpha\omega\beta\mu\left[1 - \frac{1}{n^2\beta^2}\left(\left(1 + \frac{\omega}{2E}(n^2-1)\right)^2 - \frac{\omega^2}{4E^2}\left(2n^4 - 2n^2\right)\right)\right].$$

Therefore, the expression for the power spectrum becomes

$$P(\omega) = \alpha\omega\beta\mu\left[1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{2E}(n^2-1)\right)^2 + \frac{\omega^2}{2E^2\beta^2}\left(n^2-1\right)\right]. \quad (4.82)$$

For $\omega > 0$, by using the constraint in (4.75), the second term is less than 1. The last term is always positive for $n > 1$, thus the power spectrum is strictly positive with the constraint in (4.75).

4.5 Quantum Mechanical Induced High Energy Cut-Off

For given $0 < \beta < 1$ and $n > 1$, (4.75) provides a cut-off for higher frequencies

$$\omega < 2E\frac{(n\beta - 1)}{(n^2-1)} \quad \text{or} \quad \omega < \omega_c \quad (4.83)$$

with

$$\omega_c \equiv 2E\frac{(n\beta - 1)}{(n^2-1)} \quad (4.84)$$

Beyond this limit, the power of radiation emission is zero. This cut-off upper limit $\boldsymbol{w}_c$ is still bounded above by the electron energy E. That is, $\boldsymbol{w} < E$ - a result which is expected on physical grounds. The proof of the latter bound follows from the following inequalities:

$$\boldsymbol{w}_c = 2E\frac{(n\boldsymbol{b}-1)}{(n^2-1)} < 2E\frac{(n-1)}{(n^2-1)} \qquad (4.85)$$

for $0 < \boldsymbol{b} < 1$ and $n > 1$. Consider the right-hand side of the above inequalities,

$$2E\frac{(n-1)}{(n^2-1)} = \frac{2E}{(n+1)} < E \qquad (4.86)$$

since $(n+1) > 2$. That is, necessarily, $\boldsymbol{w} < E$.

It is one of the most pleasing aspects of the quantum treatment that Quantum Electrodynamics, unlike its classical counterpart, produces automatically a high-energy cut-off beyond which the power of radiation is zero. This point cannot be overemphasized.

Chapter V

Quantum Electrodynamics of Èerenkov Radiation Emission at Finite Temperature

In this chapter, we extend our earlier investigation for radiation emission to finite temperature $T \neq 0$. At finite temperature the photon and electron propagators will be modified. This causes the power spectrum to change accordingly.

5.1 Electron and Photon Propagators for T ¹ 0

At finite temperature, the denominators of the Feynman propagators become replaced by the well-known expressions (Dolan, and Jackiw, 1974; Bechler, 1981; Niemi, and Semenoff, 1984; Donoghue, Holstein, and Robinett, 1985, Manoukian, 1990; Kang, Kye, and Kim, 1993)

$$\frac{1}{\left[(p-q)^2+m^2-i\varepsilon\right]} \rightarrow \frac{1}{\left[(p-q)^2+m^2-i\varepsilon\right]} - \frac{2\pi i\,\delta\left((p-q)^2+m^2\right)}{1+\exp\left(\beta\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)} \qquad (5.1)$$

in the denominator of the electron propagator and

$$\frac{1}{q^2-i\varepsilon} \rightarrow \frac{1}{q^2-i\varepsilon} - \frac{2\pi i\,\delta\left(q^2\right)}{1-\exp(\beta\left|q^0\right|)} \qquad (5.2)$$

in the denominator of the photon propagator, where $\beta = 1/kT$ and k is the Boltzmann constant. However, the photon propagator is also modified in a medium as we have already shown in Chapter III. That is

$$g_{\mu\nu} \to \mu\left[g_{\mu\nu} + \left(1 - \frac{1}{\nu^2} \right) \eta_\mu \eta_\nu \right] \tag{3.35}$$

and

$$q^2 = |\vec{\mathbf{q}}|^2 - (q^0)^2 \to |\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2 \tag{3.36}$$

Accordingly the photon propagator is obtained by making the following replacement

$$\frac{1}{q^2 - i\varepsilon} \to \frac{1}{|\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2 - i\varepsilon} - 2\pi i \frac{\delta\left(|\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2 \right)}{1 - \exp(\beta |q^0|)} . \tag{5.3}$$

5.2 The Electron Self-Energy for $T \neq 0$

By using the above modifications, the electron self-energy becomes

$$\Sigma(p) = ie^2 \mu \int \frac{(dq)}{(2\pi)^4} \gamma^\mu [-\gamma(p-q) + m] \gamma^\nu \left[g_{\mu\nu} + \left(1 - \frac{1}{\nu^2} \right) \eta_\mu \eta_\nu \right] \times$$
$$\left[\frac{1}{\left[(p-q)^2 + m^2 - i\varepsilon \right]} - 2\pi i \frac{\delta\left((p-q)^2 + m^2 \right)}{1 + \exp\left(\beta \sqrt{(\vec{\mathbf{p}} - \vec{\mathbf{q}})^2 + m^2} \right)} \right] \times$$
$$\left[\frac{1}{|\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2 - i\varepsilon} - 2\pi i \frac{\delta\left(|\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2 \right)}{1 - \exp(\beta |q^0|)} \right] \tag{5.4}$$

or

$$\Sigma(p) = \Sigma_0(p) + \Sigma_1(p) + \Sigma_2(p) + \Sigma_3(p) , \tag{5.5}$$

where

$$\Sigma_0(p) = ie^2 \mu \int \frac{(dq)}{(2\pi)^4} \frac{N}{\left[(p-q)^2 + m^2 - i\varepsilon \right] \left[|\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2 - i\varepsilon \right]} , \tag{5.6}$$

$$\Sigma_1(p) = 2\pi e^2 \mu \int \frac{(dq)}{(2\pi)^4} \frac{N}{\left[(p-q)^2 + m^2 - i\varepsilon\right]} \frac{\delta\left(|\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2\right)}{\left(1 - \exp(\beta |q^0|)\right)}\ , \qquad (5.7)$$

$$\Sigma_2(p) = 2\pi e^2 \mu \int \frac{(dq)}{(2\pi)^4} \frac{N}{|\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2 - i\varepsilon} \frac{\delta\left((p-q)^2 + m^2\right)}{\left(1 + \exp\left(\beta\sqrt{(\vec{\mathbf{p}} - \vec{\mathbf{q}})^2 + m^2}\right)\right)} \qquad (5.8)$$

and

$$\Sigma_3(p) = -i4\pi^2 e^2 \mu \int \frac{(dq)}{(2\pi)^4} \frac{N\,\delta\left((p-q)^2 + m^2\right)}{\left(1 + \exp\left(\beta\sqrt{(\vec{\mathbf{p}} - \vec{\mathbf{q}})^2 + m^2}\right)\right)} \frac{\delta\left(|\vec{\mathbf{q}}|^2 - \nu^2 (q^0)^2\right)}{\left(1 - \exp(\beta |q^0|)\right)}\ , \qquad (5.9)$$

with

$$N = \gamma^\mu [-\gamma(p-q) + m]\gamma^\nu \left[g_{\mu\nu} + \left(1 - \frac{1}{\nu^2}\right)\eta_\mu \eta_\nu \right]. \qquad (5.10)$$

5.3 Expression for the Power Spectrum for $T \neq 0$

Upon inserting (5.4)-(5.10) into the expression for the power spectrum in (2.6), one obtains

$$P(\omega) = -\frac{2m\nu\omega}{E} \int (dq)\,\delta(|\vec{\mathbf{q}}| - \nu\omega)\left[I_0(p,q) + I_1(p,q) + I_2(p,q) + I_3(p,q)\right], \qquad (5.11)$$

where

$$\mathrm{Im}(\bar{u}\Sigma_s u) = \int (dq)\, I_s(p,q) \qquad \text{for } s = 0,1,2,3.$$

Or

$$P(\omega) = P_0(\omega) + P_1(\omega) + P_2(\omega) + P_3(\omega)\ , \qquad (5.12)$$

where

$$P_0(\omega) = -\frac{2mn\omega}{E}\int(dq)\,\delta(|\vec{\mathbf{q}}| - n\omega)I_0(p,q)\,, \tag{5.13}$$

$$P_1(\omega) = -\frac{2mn\omega}{E}\int(dq)\,\delta(|\vec{\mathbf{q}}| - n\omega)I_1(p,q)\,, \tag{5.14}$$

$$P_2(\omega) = -\frac{2mn\omega}{E}\int(dq)\,\delta(|\vec{\mathbf{q}}| - n\omega)I_2(p,q)\,, \tag{5.15}$$

and

$$P_3(\omega) = -\frac{2mn\omega}{E}\int dq\,\delta(|\vec{\mathbf{q}}| - n\omega)I_3(p,q) \tag{5.16}$$

Each term will be evaluated separately by using the same method as in the previous chapter.

For the first term, we have

$$P_0(\omega) = -\frac{2mn\,\omega m e^2}{E}\int\frac{(dq)}{(2\pi)^4}\,\mathrm{Im}\left[i\,\frac{\bar{u}Nu}{D_0}\right]\delta(|\vec{\mathbf{q}}| - n\omega) \tag{5.17}$$

where

$$D_0 = \left[(p-q)^2 + m^2 - i\varepsilon\right]\left[|\vec{\mathbf{q}}|^2 - n^2(q^0)^2 - i\varepsilon\right]. \tag{5.18}$$

As shown in chapter IV

$$\bar{u}Nu = \frac{2\beta^2E^2}{m}\left[1 - \frac{1}{n^2\beta^2} - \frac{q^0}{2\beta^2E}(3 - \frac{1}{n^2}) + \frac{n\omega\cos\theta}{2\beta E}(1 + \frac{1}{n^2})\right]. \tag{5.19}$$

Equation (5.17) is identical to the expression of the temperature independent power spectrum studied in chapter IV, hence

$$P_0(\omega) = \alpha\omega\beta m\left[1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{2E}(n^2 - 1)\right)^2 + \frac{\omega^2}{2E^2\beta^2}\left(n^2 - 1\right)\right]. \tag{5.20}$$

The other three terms are just the temperature correction terms.

5.4 Temperature Correction for Radiation Emission

In the previous section, we have shown that the expression for the power spectrum for $T \neq 0$ gives three correction terms. These terms will be evaluated in this section.

The first correction term is given by

$$P_1(\omega) = -\frac{4\pi m n\, \omega \mu e^2}{E} \int \frac{(dq)}{(2\pi)^4} \,\mathrm{Im}\left[\frac{\bar{u} N u}{D_1}\right] \delta(|\vec{\mathbf{q}}| - n\omega) \frac{\delta\left(|\vec{\mathbf{q}}|^2 - n^2 (q^0)^2\right)}{\left(1 - \exp(\rho|q^0|)\right)}, \qquad (5.21)$$

where

$$D_1 = \left[(p-q)^2 + m^2 - i\varepsilon\right]. \qquad (5.22)$$

Carrying out the $|\vec{\mathbf{q}}|$ and the angular ϕ integration in (5.22), this gives

$$P_1(\omega) = -\frac{2mn^3\omega^3\mu e^2}{E(2\pi)^2} \int_{-1}^{1} d(\cos\theta) \int_{-\infty}^{\infty} dq^0 \,\mathrm{Im}\left[\frac{\bar{u}Nu}{D_1}\right]$$

$$\times \frac{\delta\left(n^2\omega^2 - n^2 (q^0)^2\right)}{\left(1 - \exp(\rho|q^0|)\right)}, \quad |\vec{\mathbf{q}}| = n\omega. \qquad (5.23)$$

Replace $|\vec{\mathbf{q}}|$ in (5.22) by $n\omega$, to obtain

$$D_1 = \left[n^2\omega^2 - (q^0)^2 - 2\beta E n\omega\cos\theta + 2q^0 E - i\varepsilon\right]. \qquad (5.24)$$

As done in Chapter II and Chapter IV, we also have

$$\mathrm{Im}\left[\frac{1}{D_1}\right] = \pi\, \delta\left(n^2\omega^2 - (q^0)^2 - 2\beta E n\omega\cos\theta + 2q^0 E\right). \qquad (5.25)$$

Hence,

$$P_1(\omega) = -\frac{mn^3\omega^3\mu e^2}{E(2\pi)} \int_{-1}^{1} d(\cos\theta) \int_{-\infty}^{\infty} dq^0 \bar{u}Nu \times$$

$$\times \frac{\text{ä}\left(n^2\boldsymbol{w}^2 - n^2(q^0)^2\right)\text{ä}\left(n^2\boldsymbol{w}^2 - (q^0)^2 - 2\boldsymbol{b}En\boldsymbol{w}\cos\boldsymbol{q} + 2q^0E\right)}{\left(1-\exp(\boldsymbol{r}|q^0|)\right)}, \ |\vec{\mathbf{q}}| = n\boldsymbol{w}.$$

(5.26)

The second correction term is given by

$$P_2(\boldsymbol{w}) = -\frac{4\boldsymbol{p}mn\boldsymbol{w}\boldsymbol{m}\boldsymbol{e}^2}{E}\int\frac{(dq)}{(2\boldsymbol{p})^4}\text{ä}(|\vec{\mathbf{q}}| - n\boldsymbol{w})\ \mathrm{Im}\left[\frac{\bar{u}Nu}{D_2}\right]\frac{\text{ä}\left((p-q)^2+m^2\right)}{\left(1+\exp\ \boldsymbol{r}\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)},$$

(5.27)

where

$$D_2 = |\vec{\mathbf{q}}|^2 - n^2(q^0)^2 - i\boldsymbol{e}. \qquad (5.28)$$

Upon integration over $|\vec{\mathbf{q}}|$ and $\boldsymbol{f}$ we obtain

$$P_2(\boldsymbol{w}) = -\frac{2mn^3\boldsymbol{w}^3\boldsymbol{m}\boldsymbol{e}^2}{(2\boldsymbol{p})^2E}\int_{-1}^{1}d(\cos\boldsymbol{q})\int_{-\infty}^{\infty}dq^0\ \mathrm{Im}\left[\frac{\bar{u}Nu}{D_2}\right]$$

$$\times\frac{\text{ä}\left((p-q)^2+m^2\right)}{\left(1+\exp\ \boldsymbol{r}\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)}, \quad |\vec{\mathbf{q}}| = n\boldsymbol{w}. \qquad (5.29)$$

Replace $|\vec{\mathbf{q}}|$ in (5.28) by $n\text{ù}$ and pick up only its imaginary part, as done before to obtain

$$\mathrm{Im}\left(\frac{1}{D_2}\right) = \mathrm{Im}\left(\frac{1}{n^2\boldsymbol{w}^2 - n^2(q^0)^2 - i\boldsymbol{e}}\right)$$

$$= \boldsymbol{p}\,\text{ä}\left(n^2\boldsymbol{w}^2 - n^2(q^0)^2\right). \qquad (5.30)$$

Hence,

$$P_2(\boldsymbol{w}) = -\frac{mn^3\boldsymbol{w}^3\boldsymbol{m}\boldsymbol{e}^2}{(2\boldsymbol{p})E}\int_{-1}^{1}d(\cos\boldsymbol{q})\int_{-\infty}^{\infty}dq^0\bar{u}Nu$$

$$\times \frac{\delta\left(n^2\omega^2 - n^2(q^0)^2\right)\delta\left((p-q)^2+m^2\right)}{\left(1+\exp\ \rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)}, \quad |\vec{\mathbf{q}}| = n\omega. \tag{5.31}$$

The second delta function is given in detail to be

$$\delta\left((p-q)^2+m^2\right) = \delta\left(n^2\omega^2 - (q^0)^2 - 2\beta E n\omega\cos\theta + 2q^0E\right). \tag{5.32}$$

Therefore,

$$P_2(\omega) = -\frac{mn^3\omega^3\mu e^2}{(2\pi)E}\int_{-1}^{1} d(\cos\theta)\int_{-\infty}^{\infty} dq^0\,\bar{u}Nu$$

$$\times \frac{\delta\left(n^2\omega^2 - n^2(q^0)^2\right)\delta\left(n^2\omega^2 - (q^0)^2 - 2\beta E n\omega\cos\theta + 2q^0E\right)}{\left(1+\exp\ \rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)}, \quad |\vec{\mathbf{q}}| = n\omega. \tag{5.33}$$

The third correction term is given by

$$P_3(\omega) = \frac{8\pi^2 mn\omega\mu e^2}{E}$$

$$\times \int \frac{(dq)}{(2\pi)^4}\frac{\delta\left((p-q)^2+m^2\right)\delta(|\vec{\mathbf{q}}|-n\omega)\,\delta\left(|\vec{\mathbf{q}}|^2 - n^2(q^0)^2\right)}{\left(1+\exp\left(\rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)\right)\left(1-\exp(\rho|q^0|)\right)}\mathrm{Im}\left(i\bar{u}Nu\right). \tag{5.34}$$

Upon carrying out the $|\vec{\mathbf{q}}|$ and φ integrations, one obtains

$$P_3(\omega) = \frac{mn^3\omega^3\mu e^2}{E\pi}\int_{-1}^{1} d(\cos\theta)\int dq^0\ \mathrm{Im}\left(i\bar{u}Nu\right)$$

$$\times \frac{\delta\left(n^2\omega^2 - n^2(q^0)^2\right)\delta\left((p-q)^2+m^2\right)}{\left(1+\exp\left(\rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)\right)\left(1-\exp(\rho|q^0|)\right)}, \quad |\vec{\mathbf{q}}| = n\omega. \tag{5.35}$$

From (5.19), $\bar{u}Nu$ is real. Thus

$$\mathrm{Im}\left(i\bar{u}Nu\right) = \bar{u}Nu\ .$$

Replace the second delta function by (5.32), that is

$$P_3(\omega) = \frac{mn^3\omega^3\mu e^2}{E\pi}\int_{-1}^{1} d(\cos\theta)\int dq^0 \bar{u}Nu$$

$$\times\frac{\delta\left(n^2\omega^2 - n^2(q^0)^2\right)\delta\left(n^2\omega^2-(q^0)^2-2\beta En\omega\cos\theta+2q^0E\right)}{\left(1+\exp\left(\rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)\right)\left(1-\exp(\rho|q^0|)\right)}\ ,\quad |\vec{\mathbf{q}}| = n\omega.$$

(5.36)

Adding up the three correction terms as given, respectively, in (5.26), (5.33) and (5.36) we obtain for the temperature correction for the power spectrum the general result

$$\Delta_T P(\omega) = P_1(\omega)+P_2(\omega)+P_3(\omega)$$

$$= -\frac{mn^3\omega^3\mu e^2}{2\pi E}\int_{-1}^{1} d(\cos\theta)\int dq^0\bar{u}Nu$$

$$\times\,\delta\left(n^2\omega^2-n^2(q^0)^2\right)\delta\left(n^2\omega^2-(q^0)^2-2\beta En\omega\cos\theta+2q^0E\right)$$

$$\times\left[\frac{1}{\left(1-\exp(\rho|q^0|)\right)}+\frac{1}{\left(1+\exp\left(\rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)\right)}\right.$$

$$\left.-\frac{2}{\left(1+\exp\left(\rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)\right)\left(1-\exp(\rho|q^0|)\right)}\right],\ |\vec{\mathbf{q}}| = n\omega. \qquad (5.37)$$

The last factor in the square brackets is a function of q^0, which suggests to define the function

$$F_T(q^0) = \left[\frac{1}{\left(1-\exp(\rho|q^0|)\right)}+\frac{1}{\left(1+\exp\left(\rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)\right)}\right.$$

$$-\frac{2}{\left(1+\exp\left(\rho\sqrt{(\vec{\mathbf{p}}-\vec{\mathbf{q}})^2+m^2}\right)\right)\left(1-\exp(\rho|q^0|)\right)}\Bigg]. \tag{5.38}$$

Hence,

$$\Delta_T P(\omega) = -\frac{m n^3 \omega^3 \mu \varepsilon^2}{2\pi E}\int_{-1}^{1} d(\cos\theta)\int dq^0\ \bar{u}Nu\ \delta\left(n^2\omega^2 - n^2(q^0)^2\right)$$

$$\times\,\delta\left(n^2\omega^2 - (q^0)^2 - 2\beta E n\omega\cos\theta + 2q^0 E\right)F_T(q^0),\quad |\vec{\mathbf{q}}| = n\omega. \tag{5.39}$$

Factor out $(-n^2)$ from the first delta function to write

$$\delta\left(n^2\omega^2 - n^2(q^0)^2\right) = \delta\left(-n^2\left[(q^0)^2-\omega^2\right]\right)$$

$$= \frac{1}{n^2}\delta\left((q^0)^2-\omega^2\right). \tag{5.40}$$

Using the property (C. 6) of the delta function in the appendix. Then (5.40) becomes

$$\delta\left(n^2\omega^2 - n^2(q^0)^2\right) = \frac{1}{2\omega n^2}\left[\delta(q^0-\omega)+\delta(q^0+\omega)\right]. \tag{5.41}$$

However, for radiation emission of frequency $\omega > 0$, then only the positive of q^0 will contribute. Upon inserting this into (5.39) gives

$$\Delta_T P(\omega) = -\frac{m n \omega^2 \mu \alpha}{E}\int_{-1}^{1} d(\cos\theta)\int dq^0\ \bar{u}Nu\ \delta\left(q^0-\omega\right)$$

$$\times\,\delta\left(n^2\omega^2 - (q^0)^2 - 2\beta E n\omega\cos\theta + 2q^0 E\right)F_T(q^0),\ |\vec{\mathbf{q}}| = n\omega. \tag{5.42}$$

Doing the q^0 integration by using, in the process, the property (C. 4) of the delta function in Appendix C, we obtain

$$\Delta_T P(\omega) = -\frac{m n \omega^2 \mu \alpha}{E}\int_{-1}^{1} d(\cos\theta)\ \bar{u}Nu$$

$$\times\,\delta\left(n^2\omega^2 - \omega^2 - 2\beta E n\omega\cos\theta + 2\omega E\right)F_T(q^0),\ |\vec{\mathbf{q}}| = n\omega, q^0 = \omega. \tag{5.43}$$

Consider the delta function in (5.43) rewritten as

$$\delta\left(n^2\omega^2 - \omega^2 - 2\beta E n\omega\cos\theta + 2\omega E\right) = \delta\left(-2\beta E n\omega\left[\cos\theta - \frac{\omega(n^2-1)}{2\beta E n} - \frac{1}{\beta n}\right]\right)$$

$$= \frac{1}{2\beta E n\omega}\,\delta\left(\cos\theta - \frac{1}{\beta n}\left(1 + \frac{\omega}{2E}(n^2-1)\right)\right).$$

(5.44)

This delta function is non-zero only for

$$\cos\theta = \frac{1}{\beta n}\left(1 + \frac{\omega}{2E}\left(n^2-1\right)\right). \tag{5.45}$$

Due to the possible values of $\cos\theta$, the right-hand side of this equation must be less than 1. This gives the threshold condition, which is

$$n\beta > 1 + \frac{\omega}{2E}\left(n^2-1\right). \tag{5.46}$$

This threshold condition coincides with the one obtained for the temperature independent case. Substitute the expression of the delta function into (5.43) to obtain

$$\Delta_T P(\omega) = -\frac{m\omega\mu\alpha}{2\beta E^2}\int_{-1}^{1} d(\cos\theta)\,\bar{u}Nu\,\delta\left(\cos\theta - \frac{1}{\beta n}\left(1 + \frac{\omega}{2E}(n^2-1)\right)\right)F_T(q^0),$$
$$|\vec{\mathbf{q}}| = n\omega,\ q^0 = \omega.$$

(5.47)

Doing the $\cos\theta$-integration by using, in the process, the property (C. 4) of the delta function in Appendix C, this gives

$$\Delta_T P(\omega) = -\frac{m\omega\mu\alpha}{2\beta E^2}\bar{u}Nu\ F_T(q^0),\ \ |\vec{\mathbf{q}}| = n\omega,\ q^0 = \omega,\ \cos\theta = \frac{1}{\beta n}\left(1 + \frac{\omega}{2E}\left(n^2-1\right)\right).$$

(5.48)

As shown in (4.78) in Chapter IV,

$$\bar{u}Nu = \frac{2b^2E^2}{m}\left[1 - \frac{1}{n^2b^2}\left(1 + \frac{w}{E}(n^2 - 1)\right) + \frac{w^2}{4n^2b^2E^2}\left(n^4 - 1\right)\right], \qquad (5.49)$$

since $|\vec{\mathbf{q}}| = nw, q^0 = w$ and $\cos q = \frac{1}{bn}\left(1 + \frac{w}{2E}\left(n^2 - 1\right)\right).$

Consider the exponential factor in the second term in $F_T(q^0)$ as given in (5.38) depending on:

$$(\vec{\mathbf{p}} - \vec{\mathbf{q}})^2 + m^2 = \vec{\mathbf{p}}^2 + \vec{\mathbf{q}}^2 - 2\vec{\mathbf{p}} \bullet \vec{\mathbf{q}} + m^2$$

$$= \vec{\mathbf{p}}^2 + n^2w^2 - 2bEnw\cos q + m^2. \qquad (5.50)$$

From the mass-shell condition $p^2 + m^2 = 0$, we may write

$$\begin{aligned} p^2 &= -m^2 = \vec{\mathbf{p}}^2 - E^2 \\ \vec{\mathbf{p}}^2 &= E^2 - m^2. \end{aligned} \qquad (5.51)$$

Thus (5.50) becomes

$$(\vec{\mathbf{p}} - \vec{\mathbf{q}})^2 + m^2 = E^2 - m^2 + n^2w^2 - 2bEnw\cos q + m^2$$

$$= E^2 + n^2w^2 - 2bEnw\cos q. \qquad (5.52)$$

Replace $\cos q$ by $\frac{1}{bn}\left(1 + \frac{w}{2E}(n^2 - 1)\right)$, to finally rewrite (5.53) as

$$(\vec{\mathbf{p}} - \vec{\mathbf{q}})^2 + m^2 = E^2 + n^2w^2 - 2bEnw\frac{1}{bn}\left(1 + \frac{w}{2E}(n^2 - 1)\right)$$

$$= E^2 + n^2w^2 - 2Ew - n^2w^2 + w^2$$

$$= E^2 - 2Ew + w^2 = (E - w)^2. \qquad (5.53)$$

Then (5.38) becomes

$$F_T(w) = \frac{1}{\left(1-\exp(\,rw)\right)} + \frac{1}{\left(1+\exp\left(\,r|E-w|\right)\right)} - \frac{2}{\left(1+\exp\left(\,r|E-w|\right)\right)\left(1-\exp(\,rw)\right)}$$

(5.54)

By using the definition *ñ ≡ 1/kT*, we rewrite the latter as

$$F_T(w) = \frac{1}{\left(1-\exp(\frac{w}{kT})\right)} + \frac{1}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)} - \frac{2}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)\left(1-\exp(\frac{w}{kT})\right)}$$

(5.55)

Therefore, the temperature correction for radiation emission of Èerenkov radiation at finite temperature is

$$\Delta_T P(w) = -\frac{mwma}{2bE^2}\frac{2b^2E^2}{m}\left[1-\frac{1}{n^2b^2}\left(1+\frac{w}{E}(n^2-1)\right)+\frac{w^2}{4n^2b^2E^2}\left(n^4-1\right)\right]$$

$$\times\left[\frac{1}{\left(1-\exp(\frac{w}{kT})\right)} + \frac{1}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)} - \frac{2}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)\left(1-\exp(\frac{w}{kT})\right)}\right].$$

(5.56)

or

$$\Delta_T P(w) = -awbm\left[1-\frac{1}{n^2b^2}\left(1+\frac{w}{E}(n^2-1)\right)+\frac{w^2}{4n^2b^2E^2}\left(n^4-1\right)\right]$$

$$\times\left[\frac{1}{\left(1-\exp(\frac{w}{kT})\right)} + \frac{1}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)} - \frac{2}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)\left(1-\exp(\frac{w}{kT})\right)}\right].$$

(5.57)

5.5 Closed Expression for the Power Spectrum of Radiation Emission at Finite Temperature and High Energy Cut-Off

We now combine the temperature correction term in (5.57) with the temperature independent one in (5.20). This gives

$$P_T(\omega) = \alpha\omega\beta\mu\left[1-\frac{1}{n^2\beta^2}\left(1+\frac{\omega}{2E}(n^2-1)\right)^2+\frac{\omega^2}{2E^2\beta^2}\left(n^2-1\right)\right]$$

$$\times\left[1-\frac{1}{\left(1-\exp(\frac{\omega}{kT})\right)}-\frac{1}{\left(1+\exp\left(\frac{|E-\omega|}{kT}\right)\right)}+\frac{2}{\left(1+\exp\left(\frac{|E-\omega|}{kT}\right)\right)\left(1-\exp(\frac{\omega}{kT})\right)}\right].$$

(5.58)

Let

$$A_T(\omega) = \left[1-\frac{1}{\left(1-\exp(\frac{\omega}{kT})\right)}-\frac{1}{\left(1+\exp\left(\frac{|E-\omega|}{kT}\right)\right)}+\frac{2}{\left(1+\exp\left(\frac{|E-\omega|}{kT}\right)\right)\left(1-\exp(\frac{\omega}{kT})\right)}\right].$$

(5.59)

Rewrite $A_T(\omega)$ as

$$A_T(\omega) = \left[1+\frac{1}{\left(\exp(\frac{\omega}{kT})-1\right)}-\frac{1}{\left(1+\exp\left(\frac{|E-\omega|}{kT}\right)\right)}-\frac{2}{\left(1+\exp\left(\frac{|E-\omega|}{kT}\right)\right)\left(\exp(\frac{\omega}{kT})-1\right)}\right].$$

(5.60)

Factor out $\frac{1}{\left(\exp(\frac{\omega}{kT})-1\right)}$ from (5.60) to obtain

$$A_T(w) = \frac{1}{\left(\exp(\frac{w}{kT})-1\right)}\left[\left(\exp(\frac{w}{kT})-1\right)+1-\frac{\left(\exp(\frac{w}{kT})-1\right)}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)}-\frac{2}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)}\right]. \tag{5.61}$$

Factor out $\exp(\frac{w}{kT})$ from (5.61), this gives

$$A_T(w) = \frac{\exp(\frac{w}{kT})}{\left(\exp(\frac{w}{kT})-1\right)}\left[1-\frac{\left(1-\exp(-\frac{w}{kT})\right)}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)}-\frac{2\exp(-\frac{w}{kT})}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)}\right]. \tag{5.62}$$

Evaluate the second term separately as

$$1-\frac{\left(1-\exp(-\frac{w}{kT})\right)}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)}-\frac{2\exp(-\frac{w}{kT})}{\left(1+\exp\left(\frac{|E-w|}{kT}\right)\right)}$$

$$=\left[\frac{1+\exp\left(\frac{|E-w|}{kT}\right)-1+\exp(-\frac{w}{kT})-2\exp(-\frac{w}{kT})}{\exp\left(\frac{|E-w|}{kT}\right)+1}\right]$$

$$=\left[\frac{\exp\left(\frac{|E-w|}{kT}\right)-\exp(-\frac{w}{kT})}{\exp\left(\frac{|E-w|}{kT}\right)+1}\right]. \tag{5.63}$$

Thus

$$A_T(w) = \frac{\exp(\frac{w}{kT})}{\left(\exp(\frac{w}{kT}) - 1\right)} \left[\frac{\exp\left(\frac{|E-w|}{kT}\right) - \exp(-\frac{w}{kT})}{\left(\exp\left(\frac{|E-w|}{kT}\right) + 1\right)} \right]. \tag{5.64}$$

Therefore, the temperature dependent power spectrum of photon emission is

$$P_T(w) = awbm\left[1 - \frac{1}{n^2 b^2}\left(1 + \frac{w}{2E}(n^2-1)\right)^2 + \frac{w^2}{2E^2 b^2}\left(n^2-1\right)\right]$$

$$\times \frac{\exp(\frac{w}{kT})}{\left(\exp(\frac{w}{kT}) - 1\right)} \left[\frac{\exp\left(\frac{|E-w|}{kT}\right) - \exp(-\frac{w}{kT})}{\left(\exp\left(\frac{|E-w|}{kT}\right) + 1\right)} \right], \tag{5.65}$$

which is strictly positive and holds only with the threshold condition in (5.46)

$$n b > 1 + \frac{w}{2E}\left(n^2 - 1\right) \tag{5.46}$$

satisfied, otherwise $P_T(ù)$ is zero.

For given 0 < $\hat{a}$ <1, n > 1 (with necessarily $n\hat{a}$ > 1), (5.46) automatically provides a cut-off for higher frequencies, that is

$$w < 2E\frac{\left(n b - 1\right)}{(n^2 - 1)} \quad \text{or} \quad w < w_c \tag{5.66}$$

with

$$w_c \equiv 2E\frac{(nb - 1)}{(n^2) - 1} \tag{5.67}$$

Beyond this limit, the power of radiation emission is zero. This cut-off upper limit w_c is still bounded above by the electron energy E. That is, $w < E$ - a result which is expected on physical grounds. The proof of the latter bound follows from the following inequalities:

$$w_c = 2E\frac{(nb-1)}{(n^2-1)} < 2E\frac{(n-1)}{(n^2-1)} \tag{5.68}$$

for $0 < b < 1$ and $n > 1$. Consider the right-hand side of the above inequalities,

$$2E\frac{(n-1)}{(n^2-1)} = \frac{2E}{(n+1)} < E \tag{5.69}$$

since $(n+1) > 2$. That is, necessarily, $w < E$. This means that the absolute value sign in $|E\text{-}ù|$ appearing in (5.66) may be removed.

5.6 Low and High Temperature Limit of the Power Spectrum

Consider the asymptotic behavior of $A_T(ù)$ at low and high temperatures, respectively.

At low temperatures, $kT << ù$. $\exp(\frac{w}{kT})$ is very large, then

$$\frac{\exp(\frac{w}{kT})}{\left(\exp(\frac{w}{kT})-1\right)} \approx 1 \;, \quad \exp(-\frac{w}{kT}) \approx 0 \;. \tag{5.70}$$

and

$$\frac{\exp\left(\frac{|E-w|}{kT}\right)}{\left(\exp\left(\frac{|E-w|}{kT}\right)+1\right)} \approx 1 \tag{5.71}$$

Thus

$$A_T(w) \sim 1 \,, \quad kT << w. \tag{5.72}$$

This means at low temperature, the power spectrum tends to be the temperature independent one as expected.

At high temperatures, $kT >> E$. $\frac{w}{kT}$ and $\frac{E-w}{kT}$ are very small, then

$$\frac{\exp(\frac{w}{kT})}{\left(\exp(\frac{w}{kT})-1\right)} \approx \frac{1}{1+\frac{w}{kT}-1} = \frac{kT}{w}, \qquad (5.73)$$

and

$$\frac{\exp\left(\frac{|E-w|}{kT}\right)-\exp(-\frac{w}{kT})}{\left(\exp\left(\frac{|E-w|}{kT}\right)+1\right)} \approx \frac{1+\frac{E-w}{kT}-1+\frac{w}{kT}}{1+\frac{E-w}{kT}+1} = \frac{\frac{E}{kT}}{2+\frac{E-w}{kT}} \approx \frac{E}{2kT}. \qquad (5.74)$$

Thus

$$A_T(w) \sim \frac{kT}{w}\cdot\frac{E}{2kT} = \frac{E}{2w}, \quad kT >> E\ . \qquad (5.75)$$

Therefore, at high temperature, the power spectrum becomes

$$P_T(w) \underset{kT>>E}{\sim} \frac{abmE}{2}\left[1-\frac{1}{n^2 b^2}\left(1+\frac{w}{2E}(n^2-1)\right)^2+\frac{w^2}{2E^2 b^2}\left(n^2-1\right)\right]. \qquad (5.76)$$

Accordingly, the power of emission of radiation of energies *ù* < *E/2* is enhanced, and of radiation of energies *E/2* < *ù* <*E* is suppressed in the high temperature limit.

Chapter VI

Conclusion

In this thesis, we have carried out, to order α, an exact evaluation of the power spectrum $P(\omega)$ of Čerenkov radiation emission in Quantum Electrodynamics at finite temperature ($T \neq 0$) in isotropic homogeneous media of index of refraction n for the first time. Our strategy of attack was to use complex integration analysis instead of combining the denominators of Feynman propagators in parametric form, which has necessarily led to approximations in the past. We have first established that for n = 1, $P(\omega) \equiv 0$ (as it should be), thus showing that our method of study does not necessitate the introduction of a contact term in the definition of $P(\omega)$.

The expression for $P(\omega)$ at $T = 0$ derived in this work is given by

$$P(\omega) = \alpha\omega\beta\mu\left[1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{2E}(n^2 - 1)\right)^2 + \frac{\omega^2}{2E^2\beta^2}\left(n^2 - 1\right)\right], \qquad (4.82)$$

with the threshold condition as

$$n\beta > \left(1 + \frac{\omega(n^2 - 1)}{2E}\right) \qquad (4.75)$$

and the upper limit for the frequency of emitted photons is:

$$\omega < \omega_c = 2E\frac{(n\beta - 1)}{n^2 - 1}. \qquad (4.83)$$

Beyond ω_c, $P(\omega)$ = 0. With the constraint in (4.75), there is no question of the positivity of $P(\omega)$ in (4.82).

One of the most pleasing aspects of the quantum treatment, over its classical counter-part, that it provides naturally a cut-off for the frequency of photons emitted (as given in (4.83)).

Our derived expression for the power spectrum of Èerenkov radiation emission at arbitrary temperature T is given by

$$P_T(w) = awbm\left[1 - \frac{1}{n^2 b^2}\left(1 + \frac{w}{2E}(n^2 - 1)\right)^2 + \frac{w^2}{2E^2 b^2}\left(n^2 - 1\right)\right]$$

$$\times \frac{\exp(\frac{w}{kT})}{\left(\exp(\frac{w}{kT}) - 1\right)}\left[\frac{\exp\left(\frac{|E - w|}{kT}\right) - \exp(-\frac{w}{kT})}{\left(\exp\left(\frac{|E - w|}{kT}\right) + 1\right)}\right], \qquad (5.65)$$

with the same threshold condition and upper limit as in (4.75) and (4.83), respectively. This expression tends to (4.82) at low temperatures as expected and in the high temperature limit it tends to

$$P_T(w) \underset{kT >> E}{\sim} \frac{abmE}{2}\left[1 - \frac{1}{n^2 b^2}\left(1 + \frac{w}{2E}(n^2 - 1)\right)^2 + \frac{w^2}{2E^2 b^2}\left(n^2 - 1\right)\right]. \qquad (5.76)$$

Upon comparison of (5.76) with (4.82), we infer that in the high temperature limit, the power of radiation emission is enhanced for energies $ù < E/2$ and suppressed for $ù$ in $E/2 < ù < E$. The latter opens the intriguing possibility of Èerenkov radiation absorption, which may occur at finite temperatures. No attempt, however, will be made in this thesis to develop the underlying respective theory.

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Appendices

Appendix A

Units

Our units are such that

$$\hbar = c = 1 .$$

Thus the following have the same dimensions

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}.$$

The equations of electrodynamics are taken in rationalized c.g.s. units.

Appendix B

Four-Vectors, The Dirac Equation and Gamma Matrices

B.1 Four-Vectors

A *four-vector* (four-dimensional vector) is a set of four quantities A^0, A^1, A^2, A^3, which transform under the Lorentz transformations. The component A^0 is called the *time component*. The A^1, A^2, A^3 components are called the *space components* and can be combined as a three-dimensional vector $\vec{\mathbf{A}}$. Two types of four-vectors, *contravaraint* and *covariant*, are introduced. Each type can be transformed into the other by using the *metric tensor*. Our metric tensor is defined by

$$\left(g_{mn}\right)=\left(g^{mn}\right)=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \text{(B. 1)}$$

This convention is also used by Schwinger (1998) and Jauch and Rohrlich (1980). In contravariant form, the four-vector is denoted by

$$A^m = (A^0, \vec{\mathbf{A}}) . \qquad \text{(B. 2)}$$

The covariant form can be obtained by using the metric tensor through the operation

$$A_m = g_{mn} A^n = (-A^0, \vec{\mathbf{A}}) . \qquad \text{(B. 3)}$$

The contravariant form can also be obtained from the covariant through the operation

$$A^{m} = g^{mn} A_{n} \,. \qquad \text{(B. 4)}$$

The scalar product of 2 four-vectors is invariant under Lorentz transformations, that is

$$A \cdot B = A^{m} B_{m} = A_{m} B^{m} = \vec{\mathbf{A}} \bullet \vec{\mathbf{B}} - A^{0} B^{0} \,. \qquad \text{(B. 5)}$$

is invariant

B.2 The Dirac Equation and Gamma Matrices

The Dirac spinors u is the solution of the Dirac equation,

$$\begin{aligned} &\left(g^{m} p_{m} + m\right) u = 0 \\ &\bar{u}\left(g^{m} p_{m} + m\right) = 0 \end{aligned} , \qquad \text{(B. 6)}$$

where $\hat{u}$ is the adjoint of u defined as

$$\hat{u} = u^{\dagger} \tilde{a}^{0}. \qquad \text{(B. 7)}$$

Our normalization condition are

$$\bar{u} u = 1 \qquad \text{(B. 8)}$$

and

$$u^{\dagger} u = p^{0}/m. \qquad \text{(B. 9)}$$

The Dirac matrices $\tilde{a}^{\grave{\imath}}$ satisfy the anticommutation relations

$$\left\{g^{m}, g^{n}\right\} = -2 g^{mn} \,. \qquad \text{(B. 10)}$$

A typical representation of the gamma matrices is given by

$$g^{0} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \text{(B. 11)}$$

and

$$g^{i} = \begin{pmatrix} 0 & s^{i} \\ -s^{i} & 0 \end{pmatrix}, \qquad \text{(B. 12)}$$

where σ^i are the Pauli matrices defined by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{(B. 13)}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{(B. 14)}$$

and

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \text{(B. 15)}$$

The following important properties of the gamma matrices should be noted:

$$\gamma^\mu \gamma_\mu = -4I\ , \qquad \text{(B. 16)}$$

$$\gamma^\mu \gamma^\rho \gamma_\mu = 2\gamma^\rho\ , \qquad \text{(B. 17)}$$

and

$$\bar{u}\, \vec{\boldsymbol{\alpha}}\, u = \frac{\vec{\mathbf{p}}}{m}. \qquad \text{(B. 18)}$$

Appendix C

The Dirac Delta Function

The Dirac delta function is defined by

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ik(x - x_0)} \,. \tag{C. 1}$$

It takes formally the following values

$$\delta(x - x_0) = \begin{cases} 0, & \text{if} \;\; x \neq x_0 \\ \infty, & \text{if} \;\; x = x_0 \end{cases} , \tag{C. 2}$$

and satisfies

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1 \,. \tag{C. 3}$$

In a mathematical sense, it defines a “distribution” or a “generalized function” rather than a function.

The following properties of the delta function were used in this thesis:

1. If x lies between a and b, then

$$\int_a^b f(x)\,\delta(x - x_0)\,dx = f(x_0) \,. \tag{C. 4}$$

2. $$\delta(ax) = \frac{\delta(x)}{|a|} \,. \tag{C. 5}$$

3. $$\delta\left(x^2 - x_0^2\right) = \frac{\delta\left(x - x_0\right) + \delta\left(x + x_0\right)}{2|x_0|} \,. \tag{C. 6}$$

Appendix D

Residue Theorem

In the complex z-plane, if $f(z)$ is analytic in any closed (clockwise) contour except at some poles $z = a_i$, then

$$\oint dz\ f(z) = -2\pi i\ (\text{summation of the residues at each pole}) . \qquad \text{(D. 1)}$$

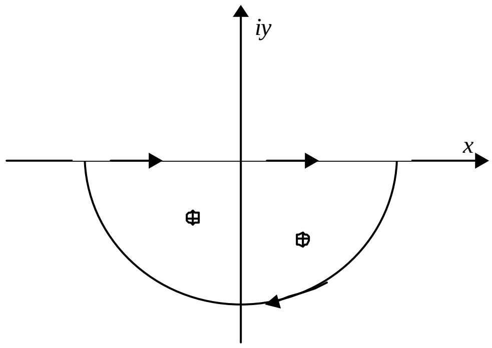

Fig D.1 Clockwise direction of the contour of integration in the residue theorem

If we close the contour as a lower semi-circle in clockwise direction, the residue theorem allows us to evaluate a real integral under the condition that if for some k greater than 1 the following is true

$$\lim_{|z| \to \infty} |z|^k \, |f(z)| \neq \infty , \qquad \text{(D. 2)}$$

that is the limit exists, then

$$\int_{-\infty}^{\infty} dx\ f(x) = \oint dz\ f(z) . \qquad \text{(D. 3)}$$

The residue at $z = a$ may be found by the following formula

$$\text{Residue} = \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}\left[(z-a)^n f(z)\right]\Big|_{z=a}, \qquad \text{(D. 4)}$$

where n is order of the pole.

However, if the poles lie on the real axis, the Cauchy principal value of integrals (e.g., Brown, and Churchill, 1996) leads to the well-known relation

$$\lim_{\epsilon \to +0} \frac{1}{x + i\epsilon} = \frac{P}{x} - i\pi\delta(x), \qquad \text{(D. 5)}$$

where P denotes the principal value.

Biography

Doojdao Charuchittipan was born on April 9th, 1977 in Koh Samui, Southern Thailand. She obtained her High School Diploma from Demonstration School of Srinakharinwirot University Prasarnmit in 1993. After that she went to study in the Department of Physics, Faculty of Science, Mahidol University and graduated with a B.Sc. Degree in 1998. In 1998, she decided to study for her Master's degree in the School of Physics, Institute of Science of the Suranaree University of Technology.