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INFINITE DIMENSIONAL PERIODIC SYSTEMS WITH IMPULSES

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วิทยานิพนธ์ฉบับนี้ศึกษาการมีผลเฉลยชนิดอ่อนอย่างเป็นคาบของระบบเป็นคาบแบบอิมพัลส์เชิงเส้นและกึ่งเชิงเส้นในมิติอนันต์ โดยมีตัวดำเนินการกณิกนันต์เป็นแบบ C_0- กึ่งกลุ่ม

ในตอนแรกจะเริ่มต้นด้วยการพิสูจน์เรื่องการมีจริงของผลเฉลยชนิดอ่อนอย่างเป็นคาบของระบบเป็นคาบแบบอิมพัลส์เชิงเส้น เราได้แก้ปัญหาด้วยการใช้ทฤษฎีแบบกึ่งกลุ่มและทฤษฎีบทจุดคงที่

โดยการใช้วิธีการเดียวกับข้างต้น เราได้พิสูจน์การมีจริงของผลเฉลยชนิดอ่อนอย่างเป็นคาบของระบบเป็นคาบแบบอิมพัลส์กึ่งเชิงเส้นด้วย

ในท้ายสุดได้มีการเสนอตัวอย่างซึ่งเกี่ยวข้องกับสมการเชิงอนุพันธ์ย่อยกึ่งกลุ่มชนิดพาราโบลิกแบบอิมพัลส์

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IMPULSIVE SYSTEMS / PERIODIC SOLUTION / SEMIGROUP / BANACH SPACE

This thesis systematically studies periodic mild solutions for linear and semilinear impulsive control systems on infinite dimensional space in those cases where the differential operator involved is the infinitesimal generator of C_0-semigroup.

First, a new existence result on periodic mild solutions for linear impulsive control systems is presented. An approach involving integrating the theory of semigroup and the fixed point theorem was used. This approach also yields an existence result of periodic mild solutions for the semilinear impulsive control systems.

Finally, the results are illustrated by example concerning semilinear partial differential equations of parabolic type with impulses.

School of Mathematics	Student's Signature ________________
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Sujitra Hinpang

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CHAPTER I

INTRODUCTION

Recently, differential equations with impulsive conditions have been studied quite extensively, where the traditional initial value problems

$$x(0) = x_0,$$

are replaced by the impulsive conditions

$$x(0) = x_0, \quad \Delta x(\tau_k) = B_k x(\tau_k), \quad k = 1, 2, \ldots$$

where $0 < \tau_1 < \tau_2 < \ldots, \quad \Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-), \quad k = 1, 2, \ldots,$ and B_k's are some operators.

That is, the impulsive conditions are the combinations of traditional initial value problems and short-term perturbations, the durations of which are negligible in comparison with the duration of the whole process. The Impulsive differential equations appear to represent a natural framework for mathematical modelings of several real world phenomena. For instance, systems with impulses effects have application in physics, in biotechnology, in industrial robotics, in pharmacokinetics, in population dynamics, in optimal control and so on. The qualitative investigation of impulsive differential equations began with the work of Mil'man and Myshkis (1960). The possibility of broad practical applications of impulsive differential equations in recent years and the publication of monographs about this subject is explored in Samoilenko and Perestyuk (1987), Lakshmikantham, Bainove and Simeonov (1989) and Bainove and Simeonove (1989). For the basic theory on impulsive differential equations, the reader is referred to Lakshmikantham (1989).

In the recent past, attention has been given to impulsive differential equations with results concerning the existence of periodic solutions for first-order impulsive differential equations appearing. See, for instance, the papers by Hristova and Bainov (1987), Nieto (1997) and Benchohra, Henderson and Ntouyas (2001). The fundamental tools used in the existence proofs of all the above mentioned works are essentially fixed point arguments, nonlinear alternative, topological transversality, degree theory or the monotone method combined with upper and lower solutions.

On the other hand, another important and interesting problem concerns the impulsive periodic systems arising naturally in the mathematical modeling of various physical processes. Since many processes are cyclic, for example, chemotherapeutic treatments in Lakmeche and Arino (2000), vaccinations against disease in Shulgin, Stone and Agur (1998) and inputs of substrates in Liu and Zhang (2005). An important trend in the investigation of impulsive differential equation is related to the periodic solution for the systems. Related basic theory on this aspect can be found in Bainove and Simeonov (1993), Yang (2001) and the references cited therein.

Although, there are some papers discussing the existence of periodic solutions on finite dimensional space, linear and semilinear periodic systems with impulses on infinite dimensional space have not been studied.

In this thesis, we systematically study the existence and uniqueness of periodic mild solutions for linear and semilinear periodic systems with impulses on infinite dimensional space, where the differential operator involved is the infinitesimal generator of C_0-semigroup by using semigroup theory and fixed point theorems.

The thesis is organized as follows. Chapter II presents some basic concepts and results from functional analysis, semigroup theory and evolution equations that are necessary for the presentation of the theory in later chapters. Chapter III deals with the existence of periodic mild solutions for linear periodic systems with impulses. In chapter IV, we study the existence of periodic mild solutions for semilinear periodic systems with impulses. In chapter V, we present two examples to demonstrate the applicability of our abstract results.

CHAPTER II

MATHEMATICAL PRELIMINARIES

In this chapter, we review the theoretical background from functional analysis, real analysis and semigroup theory which will be used throughout this thesis. Results are mostly without proof which can be found in standard textbooks (see Ahmed (1991), Erwin Kreyszig (1978) and Pazy (1983) for example).

2.1 Elements of Functional Analysis

2.1.1 Normed Linear Spaces

Definition 2.1.1. Let X be a vector space over field $\mathbb{F}$, (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{R}$ is said to be a *norm* on X if it satisfies :

(N1) $\|x\| \geq 0$,

(N2) $\|x\| = 0 \;\Leftrightarrow\; x = 0$,

(N3) $\|\alpha x\| = |\alpha|\|x\|$,

(N4) $\|x + y\| \leq \|x\| + \|y\|$,

for all $x, y \in X$ and $\alpha \in \mathbb{F}$.

Definition 2.1.2. Let X be a vector space over field $\mathbb{F}$, (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). A function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ is said to be an *inner product* on X if it satisfies :

(IP1) $\langle x, x \rangle \geq 0$,

(IP2) $\langle x, x \rangle = 0 \;\Leftrightarrow\; x = 0$,

(IP3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, (IP4) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,

for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$. In (IP3), the bar denotes the complex conjugate. Consequently, if X is a real vector space, we simply have $\langle x, y\rangle = \langle y, x\rangle$.

Hereafter, we denote a norm on X by $\|\cdot\|_X$. Similarly, we denote an inner product on X by $\langle\cdot,\cdot\rangle_X$. If X has a norm, then the pair $(X, \|\cdot\|_X)$ is called a normed linear space. The norm $\|\cdot\|_X$ induces a metric d on X by $d(x, y) = \|x - y\|_X$ and thus X become a topological space.

Definition 2.1.3. A sequence $\{x_n\}$ in a normed space $(X, \|\cdot\|)$ is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that

$$\|x_m - x_n\| < \varepsilon, \quad \text{for all } m, n > N.$$

Definition 2.1.4. A normed linear space X is said to be *complete* if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

Definition 2.1.5. A normed linear space X is said to be a *Banach space* if it is complete.

Definition 2.1.6. A Banach space X is *uniformly convex*, if whenever $\{x_n\}, \{y_n\} \in \overline{B_1(0)}$ and $\|x_n + y_n\| \to 2$, as $n \to \infty$, then $\|x_n - y_n\| \to 0$.

Definition 2.1.7. Let X be a linear space with inner product $\langle\cdot,\cdot\rangle_X$. The inner product induces a norm on X by $\|\cdot\|_X = \sqrt{\langle\cdot,\cdot\rangle_X}$. Then X is said to be a *Hilbert space* if it is complete under the norm $\|\cdot\|_X$.

2.1.2 Linear Operators

Definition 2.1.8. Let X and Y be vector spaces. A *linear operator* or a *linear map* T from X into Y is a function $T : X \to Y$ such that

(i) $T(x+y) = T(x) + T(y)$ for all $x, y \in X$,

(ii) $T(\alpha x) = \alpha T(x)$ for all $x \in X$ and $\alpha \in \mathbb{F}$.

Definition 2.1.9. Let X and Y be normed linear spaces and $T : X \to Y$ a linear operator. Then T is said to be *bounded* if there exists $M > 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Theorem 2.1.1. *Let X and Y be normed linear spaces and $T : X \to Y$ a linear operator. Then the following statement are equivalent :*

(i) *T is continuous at 0, the zero vector in X,*

(ii) *T is continuous on X,*

(iii) *T is bounded on X.*

Let X and Y be normed spaces. Consider the set $\mathcal{L}(X,Y)$ consisting of all bounded linear operator from X to Y. $\mathcal{L}(X,Y)$ becomes a normed linear space if we define vector operations in a natural way and define the operator norm $\|T\|_{\mathcal{L}(X,Y)} = \sup\limits_{\|x\|_X \leq 1} \|Tx\|_Y$. If $X = Y$, we simply write $\mathcal{L}(X)$. Moreover, we have the following theorem

Theorem 2.1.2. *If X is a normed linear space and Y is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space.*

Lemma 2.1.3. If $T : X \to Y$ and $S : Y \to Z$ are bounded linear operators, then $ST : X \to Z$ is also a bounded linear operator. Moreover,

$$\|ST\| \leq \|S\|\|T\|.$$

Theorem 2.1.4. *(Uniform Boundedness Principle). Let X and Y be Banach spaces and $\mathcal{T} \subset \mathcal{L}(X,Y)$. Then,*

$$\sup_{T\in\mathcal{T}} \|Tx\|_Y < \infty, \ \forall x \in X \quad \textit{implies that} \quad \sup_{T\in\mathcal{T}} \|T\|_{\mathcal{L}(X,Y)} < \infty.$$

2.1.3 Linear Functionals and Dual Spaces

Definition 2.1.10. A *linear functional* on a normed linear space X is a linear map from X into the scalar field $\mathbb{F}$.

We write X^* for the space of all bounded linear functionals on X and call it the *dual space* of X and $\langle x, y\rangle$ to denote the pairing of an element $x \in X^*$ with an element $y \in X$.

Definition 2.1.11. A Banach space is *reflexive* if $(X^*)^* = X$. More precisely, this means that for each $x^{**} \in (X^*)^*$, there exists $x \in X$ such that

$$\langle x^{**}, x^* \rangle = \langle x^*, x\rangle \quad \text{for all } x^* \in X^*.$$

2.1.4 Closed Operators

Definition 2.1.12. Let X and Y be normed linear spaces and $T : X \to Y$ a function. The *graph* of T, denote by $\mathcal{G}(T)$, is defined by

$$\mathcal{G}(T) = \{(x, Tx) \mid x \in X\} \subset X \times Y.$$

If T is linear, it is easy to verify that $\mathcal{G}(T)$ is a linear subspace of $X \times Y$. We say that the map $T : X \to Y$ has a *closed graph* or T is a *closed operator* if $\mathcal{G}(T)$ is a closed subspace of $X \times Y$.

The following lemma gives a characterization of the closedness of a linear operator in terms of sequences.

Lemma 2.1.5. Let X and Y be normed linear spaces and $T : X \to Y$ a linear operator. Then T has a closed graph if and only if for every sequence $\{x_n\}$ in X, if $x_n \to x$ and $Tx_n \to y$, then $y = Tx$.

Theorem 2.1.6. *(Closed Graph Theorem). Suppose that X and Y are Banach spaces and $T : X \to Y$ a linear operator. Then T is bounded if and only if T has a closed graph.*

Definition 2.1.13. Let X be a Banach space, Y a subspace(not necessarily closed) of X and let $A : D(A) \subset X \to X$ be a linear operator in X. The subspace Y of X is an *invariant subspace* of A if $A : D(A) \cap Y \to Y$.

2.1.5 Compact Linear Operators

First, we recall the following facts from topology.

Definition 2.1.14. A subset M of a topological space X is *compact* if every open cover of M contains a finite subcover.

Definition 2.1.15. Let X and Y be normed spaces. An operator $A : X \to Y$ is called a *compact linear operator* (or completely continuous linear operator) if A is linear and if for every bounded subset M of X, the image $A(M)$ is *relatively compact*, that is, the closure $\overline{A(M)}$ is compact.

Definition 2.1.16. (ε-net, total boundedness). Let B be a subset of a metric space X and $\varepsilon > 0$ be given. A set $M_\varepsilon \subset X$ is called an ε-net *for* B if for every point $z \in B$ there is a point of M_ε at a distance from z less than ε. The set B is said to be *totally bounded* if for every $\varepsilon > 0$ there is a *finite* $\varepsilon-$net $M_\varepsilon \subset X$ for B, where "finite" means that M_ε is a finite set (that is, consists of finitely many points).

Lemma 2.1.7. *Let B be a subset of a metric space X.*

1. *If B is relatively compact, then B is totally bounded.*

2. *If B is totally bounded and X is complete, then B is relatively compact.*

3. *If B is totally bounded, then for every $\varepsilon > 0$ it has a finite $\varepsilon-$net $M_\varepsilon \subset B$.*

Theorem 2.1.8. *Let $T : X \to X$ be a compact linear operator and $S : X \to X$ a bounded linear operator on a normed space X. Then TS and ST are compact.*

The following fixed point theorems are the main tools in the proof of the existence of periodic mild solutions for linear and semilinear periodic systems with impulses.

Definition 2.1.17. Let X be a Banach space and let $A : X \to X$ be an operator (not necessarily linear). A *fixed point* of A is a point $x \in X$ such that

$$Ax = x.$$

In other words, a fixed point of A is solution of the equation

$$Ax = x, \;\; x \in X.$$

Definition 2.1.18. Let X be a Banach space and let $A : X \to X$ be an operator. The operator A is called *Lipschitz continuous* (or, briefly, A is Lipschitz) if

$$\|Ax - Ay\| \leq L\|x - y\|$$

for some constant L and all $x, y \in X$. If $0 \leq L < 1$ is called a *contraction.*

Theorem 2.1.9. *(The Contraction Mapping Theorem). Let X be a Banach space and let $A : X \to X$ be a contraction. Then the equation*

$$Ax = x$$

has a unique solution in X, i.e., A has a unique fixed point x. Further, this fixed point may be obtained by the method of successive approximations as follow:

$$x_0 \in X \;\; arbitrary, \;\; x_n = Ax_{n-1}(n \geq 1)\,; \;\; x = \lim_{n\to\infty} x_n.$$

Corollary 2.1.10. *Let X_0 be a closed subset of the Banach space X and assume that A maps X_0 into itself and is a contraction on X_0. The equation $Ax = x$ has a unique solution $x \in X_0$.*

Theorem 2.1.11. *(Schauder Fixed Point Theorem). Let G be a compact convex set in a Banach space B and let T be a continuous mapping of G into itself. Then T has a fixed point.*

Corollary 2.1.12. *Let G be a compact convex set in a Banach space B and let T be a continuous mapping of G into itself such that the image TG is relatively compact. Then T has a fixed point.*

Theorem 2.1.13. *(Leray-Schauder Fixed Point Theorem). Let G be a compact mapping of a Banach space B into itself and suppose there exists a constant M such that*

$$\|x\|_B < M$$

for all $x \in B$ and $\lambda \in [0, 1]$ satisfying $x = \lambda Gx$. Then G has a fixed point.

The proof can be found in Gilbarg and Trudinger (1977).

Theorem 2.1.14. *(Arzela-Ascoli). Let X and Y be Banach spaces, $G \subset X$ be compact and $\mathcal{F} \subset C(G, Y)$. Suppose that*

1. *for each $x \in G$, the set $\{F(x) \mid F \in \mathcal{F}\}$ is relatively compact in Y.*

2. *$\mathcal{F}$ is uniformly bounded, i.e.,*

$$\sup_{F \in \mathcal{F}, x \in G} \|F(x)\|_Y < \infty.$$

3. *$\mathcal{F}$ is equicontinuous, i.e., for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\|F(x) - F(y)\|_Y < \varepsilon, \text{ when ever } \|x - y\|_X < \delta, \; F \in \mathcal{F}, \; x, y \in G.$$

Then there exists a sequence $\{F_k\} \subseteq \mathcal{F}$ *and* $F_0 \in C(G,Y)$ *such that*

$$\lim_{k\to\infty} \|F_k - F_0\|_{C(G,Y)} = 0$$

where $C(G,Y)$ *denotes the supremum norm*

The proof can be found in Xunjing Li and Jiongmin Yong (1995).

2.1.6 Spectral Properties of Compact Linear Operators

In this section, we consider spectral properties of a compact linear operator $T : X \to X$ on a normed space X. For this purpose we use the operator

$$T_\lambda = T - \lambda I \qquad (\lambda \in \mathbb{C}), \tag{2.1}$$

where I is the identity operator on X.

Definition 2.1.19. Let X be a complex Banach space and let $T : D(T) \subset X \to X$ be a linear, not necessarily bounded operator. The *resolvent set* $\rho(T)$ of T is the set of all complex numbers λ for which $T - \lambda I$ is invertible, i.e., $(T - \lambda I)^{-1}$ is a bounded linear operator in X, that is, the resolvent set $\rho(T)$ of T is given by

$$\rho(T) = \{ \lambda \in \mathbb{C} : (T - \lambda I)^{-1} \in \mathcal{L}(X)\},$$

I is the identity operator on X. When $\lambda \in \rho(T)$, $R(\lambda, T) = (T - \lambda I)^{-1}$ is called the *resolvent operator* of T at λ.

Definition 2.1.20. An *eigenvalue* of an operator T is a complex number λ such that

$$Tx = \lambda x$$

has a solution $x \neq 0$. This x is called an *eigenvector* of T corresponding to that eigenvalue λ.

Theorem 2.1.15. *The set of eigenvalues of a compact linear operator $T : X \to X$ on a normed space X is countable (perhaps finite or even empty) and the only possible point of accumulation is $\lambda = 0$.*

Theorem 2.1.16. *Let $T : X \to X$ be a compact linear operator on a normed space X. Then for every $\lambda \neq 0$ the null space $\mathcal{N}(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional.*

Theorem 2.1.17. *Let $T : X \to X$ be a compact linear operator on a normed space X. Then for every $\lambda \neq 0$ the range of $T_\lambda = T - \lambda I$ is closed.*

2.1.7 Operator Equations Involving Compact Linear Operators

I. Fredholm (1903) investigated linear integral equations and his famous work suggested a theory of solvability of certain equations involving a compact linear operator. We will consider a compact linear operator $T : X \to X$ on a normed space X, the adjoint operator $T^* : X^* \to X^*$, the equation

$$Tx - \lambda x = y \qquad (y \in X), \tag{2.2}$$

the corresponding homogeneous equation

$$Tx - \lambda x = 0, \tag{2.3}$$

and two similar equations involving the adjoint operator, namely

$$T^* f - \lambda f = g \qquad (g \in X^*), \tag{2.4}$$

the corresponding homogeneous equation

$$T^* f - \lambda f = 0, \tag{2.5}$$

for all nonzero $\lambda \in \mathbb{C}$. We will study the existence of solutions x and f, respectively.

Theorem 2.1.18. *Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then (2.2) has a solution x if and only if y is such that*

$$f(y) = 0 \tag{2.6}$$

for all $f \in X^*$ satisfying (2.5).

Hence if (2.5) has only the trivial solution $f = 0$, then (2.2) with any $y \in X$ is solvable.

Theorem 2.1.19. *Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then (2.4) has a solution f if and only if g is such that*

$$g(x) = 0 \tag{2.7}$$

for all $x \in X$ satisfying (A.2).

Hence if (A.2) has only the trivial solution $x = 0$, then (2.4) with any given $y \in X$ is solvable.

Theorem 2.1.20. *Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then :*

(a) *Equation (2.2) has a solution x for every $y \in X$ if and only if the homogeneous equation (A.2) has only the trivial solution $x = 0$. In this case the solution of (2.2) is unique and T_λ has a bounded inverse.*

(b) *Equation (2.4) has a solution f for every $g \in X^*$ if and only if (2.5) has only the trivial solution $f = 0$. In this case the solution of (2.4) is unique.*

Theorem 2.1.21. *Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then (A.2) and (2.5) have the same number of linearly independent solutions.*

2.1.8 Fredholm Alternative

Definition 2.1.21. A bounded linear operator $A : X \to X$ on a normed space X is said to satisfy the *Fredholm alternative* if A is such either one of the following holds :

(I) The nonhomogeneous equations

$$Ax = y, \qquad A^* f = g$$

($A^* : X^* \to X^*$ being the adjoint operator of A) have solutions x and f, respectively, for every given $y \in X$ and $g \in X^*$, the solutions being unique. The corresponding homogeneous equations

$$Ax = 0, \qquad A^* f = 0$$

have only the trivial solution $x = 0$ and $f = 0$, respectively.

(II) The homogeneous equations

$$Ax = 0, \qquad A^* f = 0$$

have the same number of linearly independent solutions

$$x_1, \ldots, x_n \qquad \text{and} \qquad f_1, \ldots, f_n$$

respectively. The nonhomogeneous equations

$$Ax = y, \qquad A^* f = g$$

are not solvable for all y and g, respectively ; they have a solution if and only if y and g are such that

$$f_k(y) = 0, \qquad g(x_k) = 0$$

$(k = 1, \ldots, n)$, respectively.

Theorem 2.1.22. *Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then $T_\lambda = T - \lambda I$ satisfies the Fredholm alternative.*

2.2 Integration Theory

In this section, we review some basic concept of measurable functions and Bochner integral for Banach space valued functions. We then state some standard convergence theorems for integrals and introduce the definition of Fréchet derivative. For details and proofs we refer to Zeidler (1990), unless we state otherwise.

2.2.1 Measurable Functions

Let $M \subset \mathbb{R}^n$ be a measurable set and X a Banach space.

Definition 2.2.1. 1. A function $f : M \to X$ is called a *step function* if there exist finitely many pairwise disjoint measurable subsets M_i of M such that $|M_i| < \infty$ for all i and element a_i of X such that

$$f(x) = \begin{cases} a_i, & \text{if} \quad x \in M_i, \\ 0, & \text{otherwise.} \end{cases}$$

That is, f is constant on each set M_i.

2. The integral of a step function is defined to be

$$\int_M f dx = \sum_i |M_i| a_i.$$

3. A function $f : M \to X$ is called (strongly) *measurable* if there exists a sequence $\{f_n\}$ of step functions $f_n : M \to X$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for almost all} \quad x \in M.$$

4. (Measurable functions via substitution). Let X, U be real and separable Banach spaces, $M \subseteq \mathbb{R}^n$ be measurable, $f : M \times U \to X$ and $u : M \to U$. Set

$$F(x) = f(x, u(x)).$$

If the function $u : M \to U$ is measurable, then the function $F : M \to X$ is also measurable provided that f satisfies the *Caratheodory condition* :

(i) $x \mapsto f(x,u)$ is measurable on M for all $u \in U$.

(ii) $u \mapsto f(x,u)$ is continuous on U for almost all $x \in M$.

2.2.2 Bochner Integral

Definition 2.2.2. A function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n \in X$ and $E_1, E_2, \ldots, E_n \in \mathcal{M}$ such that

$$f(x) = \sum_{i=1}^{n} x_i \chi_{E_i}(x),$$

where χ_{E_i} is the characteristic function of a measurable set E_i and the set E_i are pairwise disjoint with union Ω.

Definition 2.2.3. A function $f : \Omega \to X$ is called *Bochner integrable* if Ω is measurable and there exists a sequence $\{f_n\}$ of simple functions $f_n : \Omega \to X$ such that

1. $f(x) = \lim_{n\to\infty} f_n(x)$ for almost all $x \in \Omega$,

2. given $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\int_{\Omega} \|f_m(x) - f_n(x)\|_X dx < \varepsilon \quad \text{for all} \quad m, n \geq n_0(\varepsilon).$$

The second condition implies that the sequence$\left\{ \int_M f_n(x)dx \right\}$ is *Cauchy* in X, so that we can define the *Lebesgue integral* of f by

$$\int_M f(x)dx = \lim_{n\to\infty} \int_M f_n(x)dx \tag{2.8}$$

One can show that this integral is well defined, i.e., the limit in (2.8) does not depend on the choice of the step functions f_n. Furthermore if $B \in \mathcal{L}(X)$ and the integral of f exists, then the integral of Bf exists and

$$\int_M Bf(x)dx = B\int_M f(x)dx.$$

Theorem 2.2.1. *A strongly measurable function* $f : \Omega \to X$ *is Bochner integrable if and only if* $\int_\Omega \|f(x)\|dx < \infty$.

Theorem 2.2.2. *(Majorant criterion). Let* $f : \Omega \to X$ *be measurable. If there exists* $g : \Omega \to \mathbb{R}$ *such that* $\|f(x)\|_X \le g(x)$ *for almost all* $x \in \Omega$ *and* $\int_\Omega g(x)dx$ *exists, then* f *is integrable and*

$$\left\|\int_\Omega f(x)dx\right\|_X \le \int_\Omega \|f(x)\|_X dx \le \int_\Omega g(x)dx.$$

2.2.3 Fréchet Derivative

Definition 2.2.4. A function f defined on an open subset D of a normed space X with values in a normed space Y is *Fréchet differentiable* at $x \in D$ if there exists a bounded linear operator $\partial f(x) \in \mathcal{L}(X,Y)$ such that if

$$\rho(x,h) := f(x+h) - f(x) - \partial f(x)h, \quad (x, x+h \in D),$$

then

$$\lim_{h\to 0} \frac{\|\rho(x,h)\|_X}{\|h\|_Y} = 0.$$

The operator $\partial f(x)$ is called the *Fréchet differential* or *Fréchet derivative* of f at x. Obviously, Fréchet differentiability implies continuity. The mean value theorem holds for Fréchet differentiable maps : we need it in the form

$$\|f(x) - f(y)\| \le \|x-y\|_X \sup_{z\in I} \|\partial f(z)\|_{(X,Y)}$$

(I the segment joining x and y) valid for D convex. The Fréchet differentiable is of course the calculus differential if $X = \mathbb{R}^m$.

2.3 Theory of C_0-semigroup

In this section, we recall some basic concepts and results on C_0-semigroups. For more details and proofs, we refer to Ahmed (1991) and Pazy (1983).

2.3.1 C_0-semigroups

Definition 2.3.1. The family of operators $\{T(t), t \geq 0\}$ is said to be a *semigroup of bounded linear operators* on X if

(i) $T(0) = I$, (I is the identity operator on X).

(ii) $T(t+s) = T(t)T(s) = T(s)T(t)$ for all $t, s \geq 0$.

The semigroup $\{T(t), t \geq 0\}$ is said to be *uniformly continuous* if $t \mapsto T(t)$ is continuous on $[0, \infty)$ in the uniform operator topology, that is,

$$\lim_{t \to 0} \|T(t) - I\|_{\mathcal{L}(X)} = 0.$$

Equivalently, from the definition it is clear that if $\{T(t), t \geq 0\}$ is a uniformly continuous semigroup of bounded linear operators then

$$\lim_{t \to t_0} \|T(t) - T(t_0)\|_{\mathcal{L}(X)} = 0,$$

for all $t_0 \in [0, \infty)$.

Definition 2.3.2. The operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \left\{ x \in X \ : \ \lim_{t \to 0^+} A_t x \quad \text{exists in } X \right\}$$

$$Ax = \lim_{t \to 0^+} A_t x \ \text{ for } \ x \in D(A),$$

where for $t > 0$, $A_t x = \dfrac{T(t)x - x}{t}$, $x \in X$, is called the *infinitesimal generator* of the semigroup $\{T(t), t \geq 0\}$ on X.

Theorem 2.3.1. *A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of uniformly continuous semigroup of operator $\{T(t), t \geq 0\}$ in X if and only if A is a bounded linear operator.*

Definition 2.3.3. (C_0-semigroup). The semigroup $\{T(t), t \geq 0\}$ is said to be *strongly continuous* at the origin if for each $x \in X$,

$$\lim_{t \to 0^+} \|T(t)x - x\|_X = 0.$$

That is, $t \to T(t)x$ is continuous from the right at $t = 0$ for each $x \in X$.

A strongly continuous semigroup of bounded linear operator on X is called a C_0-*semigroup.*

It readily follows from the semigroup property that strong right continuity at origin implies strong continuity for every $t > 0$, we only have to note that $T(t+h)x - T(t)x = T(t)(T(h)x - x)$ for $h > 0$. To obtain left continuity, we have to invoke the uniform boundedness principle.

Theorem 2.3.2. *(Properties of C_0-semigroups). Let X be a Banach space and $\{T(t), t \geq 0\}$ a C_0-semigroup on X with A as its infinitesimal generator. Then,*

(1) *There exist constants $M \geq 1$ and $\omega \geq 0$ such that*

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \text{ for all } t \geq 0.$$

(2) *For each $x \in X, t \mapsto T(t)x$ is continuous X-valued function on $[0, \infty)$.*

(3) *For $x \in X$ and $t > 0$,*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(\tau)x d\tau = T(t)x.$$

(4) *For $x \in X, t \in [0, \infty)$, $\int_0^t T(\tau)x d\tau \in D(A)$ and*

$$A\left(\int_0^t T(\tau)x d\tau\right) = T(t)x - x.$$

(5) *For* $x \in D(A)$, $T(t)x \in D(A)$ *and* $\dfrac{d}{dt}T(t)x = AT(t)x = T(t)Ax$.

(6) *For* $x \in D(A)$, $0 \le s \le t$,

$$\int_s^t AT(\tau)x d\tau = \int_s^t T(\tau)Ax d\tau = T(t)x - T(s)x.$$

(7) $\overline{D(A)} = X$ *and* A *is closed operator or equivalently to its graph* $\Gamma(A) = \{(x, y) \in X \times X : y = Ax\}$ *is closed subset of* $X \times X$.

(8) *Let* B *be the infinitesimal generator of* C_0*-semigroup* $\{S(t), t \ge 0\}$. *If* $A = B$, *then* $T(t) = S(t)$ *for all* $t \ge 0$, *that is, each* C_0*-semigroup generator generates a unique semigroup.*

2.3.2 Semigroup of Compact Operators

Definition 2.3.4. A C_0-semigroup $\{T(t), t \ge 0\}$ in a Banach space X is called a *compact semigroup* for $t > t_0$ if $T(t)$ is a compact operator for every $t > t_0$. It is simply called *compact* if it is compact for all $t > 0$.

Note that if $T(0)$ is compact, then X must be a finite dimensional Banach space, since the identity operator is compact if and only if X is finite dimensional. Hence for a general Banach space, one can expect $T(t)$ to be compact only for $t > 0$. Note also that if $T(t_0)$ is compact for some $t_0 > 0$, then $T(t)$ is compact for all $t > t_0$. This follows from the fact that $T(t) = T(t - t_0)T(t_0)$, $t_0 > 0$ and that the composition of a compact operator with a bounded operator is always compact.

Definition 2.3.5. Let $\{T(t), t \ge 0\}$ be a C_0-semigroup. If $T(t)$ is compact for $t > t_0$, then $T(t)$ is continuous in the uniform operator topology for $t > t_0$.

2.3.3 Differentiable and Analytic Semigroups

Definition 2.3.6. A C_0-semigroup $\{T(t), t \geq 0\}$ in a Banach space X is said to be *differentiable* if, for each $x \in X, T(t)x$ is differentiable for all $t > 0$.

We have seen in Theorem 2.3.2(5) that if $T(t)$ is a C_0-semigroup with infinitesimal generator A and $x \in D(A)$ then $t \mapsto T(t)x$ is differentiable for $t > 0$.

Theorem 2.3.3. *If $\{T(t), t \geq 0\}$ is differentiable semigroup with A as its infinitesimal generator then it is differentiable infinitely many times and for each $n = 1, 2, 3, \ldots$*

(i) $\dfrac{d^n}{dt^n}T(t) = T^{(n)}(t) = A^nT(t) \in \mathcal{L}(X)$ *for all* $t > 0$.

(ii) $T^{(n)}(t) = \big(AT(\frac{t}{n})\big)^n$ *for all* $t > 0$.

(iii) $T^{(n)}(t)$ *is uniformly continuous for all* $t > 0$.

Definition 2.3.7. Let $\triangle = \{z \in \mathbb{C} : \theta_1 < \arg z < \theta_2; \theta_1 < 0 < \theta_2\}$ and suppose $T(z) \in \mathcal{L}(X)$ for all $z \in \triangle$. The family $\{T(z), z \in \triangle\}$ is called an *analytic semigroup* in $\triangle$ if it satisfies the following properties :

(i) $z \mapsto T(z)$ is analytic in $\triangle$, that is for each $x^* \in X^*$ and $x \in X$, the scalar valued function $z \mapsto x^*(T(z)x)$ is analytic in the usual sense uniformly with respect to $x^* \in B_1(X^*) = \{x^* : \|x^*\|_{X^*} \leq 1\}$ and $x \in B_1(X) = \{x : \|x\|_X \leq 1\}$,

(ii) $T(0) = I$ and $\displaystyle\lim_{z \to 0,\, z \in \triangle} T(z)x = x$ for each $x \in X$,

(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \triangle$.

A complete characterization of analytic semigroups is given in the following theorem.

Theorem 2.3.4. *Let A be the infinitesimal generator of a uniformly bounded C_0-semigroup $\{T(t), t \geq 0\}$ with $0 \in \rho(A)$. The following statements are equivalent:*

(i) $T(t)$ *can be extended to an analytic semigroup from* $[0,\infty)$ *to a sector around it given by*

$$\triangle_\delta = \{z : |\arg z| < \delta\} \text{ for some } \delta > 0,$$

and $\|T(z)\|_{\mathcal{L}(X)}$ *is uniformly bounded in every closed subsector* $\triangle_{\delta'} \subset \triangle_\delta,\ \delta' < \delta$.

(ii) *There exists a constant* $C > 0$ *such that for every* $\sigma > 0$ *and* $\tau \neq 0$

$$\|R(\sigma + i\tau, A)\|_{\mathcal{L}(X)} \leq \frac{C}{|\tau|}.$$

(iii) *There exists* $0 < \delta < \dfrac{\pi}{2}$ *and* $M \geq 1$ *such that*

$$\rho(A) \supset \Sigma = \left\{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\}$$

and

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|} \text{ for all } \lambda \in \Sigma,\ \lambda \neq 0.$$

(iv) $T(t)$ *is differentiable for* $t > 0$ *and there exists a constant* $C > 0$ *such that*

$$\|AT(t)\|_X \leq \frac{C}{t} \text{ for } t > 0.$$

2.4 Differential Equations on Banach Spaces

In this section, we introduce the concept and results on semigroups of operators via differential equations on Banach spaces which are abstract formulation of initial value problem for partial differential equations. For more details and proofs, we refer to Fattorini (1999).

2.4.1 The Homogeneous Initial Value Problem

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a given operator. Consider the differential equation on X given by

$$\begin{cases} \dot{x}(t) = Ax(t), & t > 0 \\ x(0) = x_0. \end{cases} \tag{2.9}$$

Definition 2.4.1. The Cauchy problem (2.9) is said to have a *classical solution* if for each given $x_0 \in D(A)$ there exists a function $x \in C([0,\infty), X)$ satisfying the following properties :

(i) $x \in C([0,\infty), X) \cap C^1((0,\infty), X)$,

(ii) $x(t) \in D(A)$ for all $t > 0$,

(iii) (2.9) is satisfied, i.e., $\begin{cases} \dot{x}(t) = Ax(t), t > 0 \\ x(0) = x_0. \end{cases}$

Theorem 2.4.1. *Let $\overline{D(A)} = X, \rho(A) \neq \emptyset$. Then (2.9) has a unique classical solution $x(t)$ which is continuously differentiable on $[0,\infty)$, for every initial value $x_0 \in D(A)$ if and only if A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X.*

Theorem 2.4.2.

(i) *If A is the infinitesimal generator of a differentiable semigroup $\{T(t), t \geq 0\}$ in X then for every $x_0 \in X$, (2.9) has a unique (classical) solution $x(t) = T(t)x_0, t > 0$.*

(ii) *If A is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ then for every $x_0 \in X$, (2.9) has a unique (classical) solution $x(t) = T(t)x_0, t > 0$.*

Proof. (i) Since $\{T(t), t \geq 0\}$ is a differentiable semigroup for $t > 0$, the X-valued function $t \mapsto T(t)x_0$ is differentiable for every $x_0 \in X$ and

$$\frac{d}{dt}T(t)x_0 = AT(t)x_0 \text{ for } t > 0.$$

Further, by Theorem 2.3.3(iii), $AT(t)x_0$ is Lipschitz continuous for $t > 0$ and hence we conclude that $x(t) = T(t)x_0, t > 0$, is the unique (classical) solution of (2.9).

(ii) This follows from the simple fact that for analytic semigroup, $T(t)x \in D(A)$ for every $x \in X$ and $t > 0$ and consequently every analytic semigroup is also a differentiable semigroup. □

If A is the infinitesimal generator of a C_0-semigroup which is not differentiable then, in general, if $x \notin D(A)$, the initial value problem (2.9) dose not have a solution. The function $t \mapsto T(t)x_0$ is then a *generalized solution* of (2.9) which we will call a *mild solution.*

2.4.2 The Inhomogeneous Initial Value Problem

Consider the inhomogeneous initial value problem

$$\begin{cases} \dot{u}(t) = Au(t) + f(t), & t > 0 \\ u(0) = x_0, \quad x_0 \in X \end{cases} \tag{2.10}$$

where A is the infinitesimal generator of C_0-semigroup $\{T(t), t \geq 0\}$ in X and $f \in L^1_{loc}([0, \infty), X)$.

Definition 2.4.2. A function $u : [0, T) \to X$ is a (classical) solution of (2.10) on $[0, T)$ if u is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $u(t) \in D(A)$ for $0 < t < T$ and (2.10) is satisfied on $[0, T)$.

Theorem 2.4.3 (Existence and Uniqueness)**.** *Let A be the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$. If $f \in L^1([0, T], X)$ then for every $x \in X$ the* *initial value problem (2.10) has at most one solution. If it has a solution, this solution is given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \quad 0 \le t \le T. \tag{2.11}$$

Definition 2.4.3. A function $u \in C([0,T],X)$ is said to be a *mild solution* of (2.10) corresponding to the initial state $x_0 \in X$ and the input $f \in L^1([0,T],X)$ if u is given by (2.11).

The definition of the mild solution of (2.10) coincides when $f \equiv 0$ with the definition of $T(t)x_0$ as the mild solution of the corresponding homogeneous equation. It is therefore clear that not every mild solution of (2.10) is a (classical) solution even in the case $f \equiv 0$.

Theorem 2.4.4. *Let A be the infinitesimal generator of a C_0-semigroup $\{T(t), t \ge 0\}$, let $f \in L^1([0,T],X)$ be continuous on $(0,T)$ and let*

$$v(t) = \int_0^t T(t-s)f(s)ds, \quad 0 \le t \le T.$$

The initial value problem (2.10) has a solution u on $[0,T)$ for every $x \in D(A)$ if one of the following conditions is satisfied;

(i) *$v(t)$ is continuously differentiable on $(0,T)$.*

(ii) *$v(t) \in D(A)$ for $0 < t < T$ and $Av(t)$ is continuous on $(0,T)$.*

Corollary 2.4.5. *Let A be the infinitesimal generator of a C_0-semigroup $\{T(t), t \ge 0\}$, $f(s)$ is continuously differentiable on $[0,T]$ then the initial value problem (2.10) has a solution u on $[0,T)$ for every $x \in D(A)$.*

Corollary 2.4.6. *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ and $f \in L^1([0,T],X)$ be continuous on $(0,T)$. If $f(s) \in D(A)$, then the initial value problem (2.10) has a solution on $[0,T)$.*

2.4.3 Semilinear Initial Value Problem and Perturbations Theory.

Consider the semilinear initial value problem

$$\dot{u}(t) = Au(t) + f(t, u(t)), \qquad u(s) = \zeta, \tag{2.12}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X and $f : [0, \infty) \times X \to X$. The assumption on A is that the initial value problem for the linear equation

$$\dot{u}(t) = Au(t) \tag{2.13}$$

is well posed in $0 \leq t \leq T$, as defined in Fattorini (1999), pp 207. Below, $S(t, s)$ denotes the solution operator of (2.13) is defined and strongly continuous in the triangle $0 \leq s \leq t \leq T$.

Define a solution of (2.12) as a solution of the integral equation

$$u(t) = S(t, s)\zeta(t) + \int_s^t S(t, \tau) f(\tau, u(\tau)) d\tau. \tag{2.14}$$

We summarize in this section the necessary existence-uniqueness theory of (2.12). Result will be proved under two hypotheses on $f(t, u)$. The second hypothesis is stronger than the first.

Hypothesis I. $f(t, u)$ is strongly measurable in t for fixed u. For every $c > 0$ there exists $K(\cdot, c) \in L^1(0, T)$ such that

$$\|f(t, u)\| \leq K(t, c) \qquad (0 \leq t \leq T, \|u\| \leq c). \tag{2.15}$$

Hypothesis II. $f(t, u)$ is strongly measurable in t for fixed u. For every $c > 0$ there exists $K(\cdot, c),\ L(\cdot, c) \in L^1(0, T)$ such that (2.17) holds and

$$\|f(t, u') - f(t, u)\| \leq L(t, c)\|u' - u\| \qquad (0 \leq t \leq T, \|u\|, \|u'\| \leq c). \tag{2.16}$$

Theorem 2.4.7. *Assume Hypothesis II holds in* $0 \leq t \leq T$. *Then the integral equation*

$$u(t) = \zeta(t) + \int_s^t S(t,\tau) f(\tau, u(\tau)) d\tau \tag{2.17}$$

has a unique solution in some interval $s \leq t \leq T'$, *where* $s \leq T' \leq T$.

Theorem 2.4.8. *Let* $u_1(\cdot)$ *(respectively,* $u_2(\cdot)$*) be solution of (2.17) in* $s \leq t \leq T'$ *with* $\zeta(t) = \zeta_1(t)$. *(respectively, with* $\zeta(t) = \zeta_2(t)$*). Let* c *be a bound for* $\|u_1(t)\|$, $\|u_2(t)\|$ *in* $s \leq t \leq T'$. *Then*

$$\|u_1(t) - u_2(t)\| \leq \sup_{s \leq t \leq T'} \|\zeta_1(t) - \zeta_2(t)\| \exp\left(M \int_s^t L(\tau, c) d\tau\right) \ (s \leq t \leq T). \tag{2.18}$$

In particular, if $u_1(\cdot)$ (respectively, $u_2(\cdot)$) is solution of (2.14) with $\zeta = \zeta_1$ (respectively, with $\zeta = \zeta_2$), then

$$\|u_1(t) - u_2(t)\| \leq M \|\zeta_1 - \zeta_2\| \exp\left(M \int_s^t L(\tau, c) d\tau\right) \quad (s \leq t \leq T). \tag{2.19}$$

Lemma 2.4.9. Let $u(t)$ be a solution of (2.17) in an interval $[s, T')$. Assume that

$$\|u(t)\| \leq c \quad (s \leq t \leq T'). \tag{2.20}$$

Then $u(\cdot)$ can be extended to an interval $[s, T'')$ with $T'' > T'$ (that is, a solution of (2.17) coinciding with $u(\cdot)$ in $s \leq t \leq T'$ exists in $[0, T'']$).

Corollary 2.4.10. *The solution* $u(\cdot)$ *of (2.17) exists in* $s \leq t \leq T$ *or in an interval* $[s, T_m), T_m \leq T$ *and*

$$\sup_{t \to T_m^-} \|u(t)\| \leq \infty. \tag{2.21}$$

Corollary 2.4.11. *Assume that there exists* $K(\cdot) \in L^1(0,T)$ *such that*

$$\|f(t,u)\| \leq K(t)(1 + \|u\|) \quad (0 \leq t \leq T, u \in X). \tag{2.22}$$

Then (2.20) holds in every interval where the solution $u(t)$ *of (2.17) exists accordingly,* $u(t)$ *exists in* $s \leq t \leq T$.

The following theorem is one of the main tools in the proof of the existence of periodic mild solutions for the semilinear impulsive peridic control systems with parameter perturbations discussed in this thesis. Its proof can be found in Fattorini (1999), pp.213.

Theorem 2.4.12. *Let the Cauchy problem for (2.13) be well posed in* $s \leq t \leq T$ *and let* $\{B(t), 0 \leq t \leq T\}$ *be a family of bounded linear operators in* X *such that (a) for each* $u \in X, t \to B(t)u$ *is strongly measurable, (b) there exists* $\alpha(\cdot) \in L^1(0,T)$ *such that*

$$\|B(t)\| \leq \alpha(t) \quad (0 \leq t \leq T). \tag{2.23}$$

Then the Cauchy problem for

$$\dot{u}(t) = (A(t) + B(t))u(t) \tag{2.24}$$

is well posed in $0 \leq t \leq T$*, solution of (2.24) with* $u(s) = \zeta$ *understood as solutions of the integral equation*

$$u(t) = S(t,s)\zeta + \int_s^t S(t,\tau)B(\tau)u(\tau)d\tau \tag{2.25}$$

If $U(t,s)$ *be the solution operator of (2.24), solutions of the inhomogeneous equation*

$$\dot{u}(t) = (A(t) + B(t))u(t) + f(t), \quad u(s) = \zeta \tag{2.26}$$

with $f(\cdot) \in L^1(0,T)$*, understood as solutions of the integral equation*

$$u(t) = S(t,s)\zeta + \int_s^t S(t,\tau)B(\tau)(u(\tau) + f(\tau))d\tau, \tag{2.27}$$

can be expressed by the variation of constants formula

$$u(t) = U(t,s)\zeta + \int_s^t U(t,\tau)f(\tau)d\tau. \tag{2.28}$$

2.5 Gronwall's Lemma

Theorem 2.5.1. *For $t > t_0$ let a nonnegative piecewise continuous function $u(t)$ satisfy*

$$u(t) \leq c + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0 \leq \tau_n < t} b_n u(\tau_n)$$

where $c \geq 0, b_n \geq 0, v(s) > 0, u(t)$ has discontinuous points of the first kind at τ_n. Then we have

$$u(t) \leq c \prod_{t_0 \leq \tau_n < t} (1 + b_n) exp\Big(\int_{t_0}^{t} v(s)ds \Big).$$

CHAPTER III

LINEAR PERIODIC SYSTEMS WITH IMPULSES

In this chapter, we study the existence and uniqueness of periodic mild solution for linear periodic systems with impulses. The first section contains some notations and basic assumptions. In the second section, regularity of mild solution, existence and uniqueness results of periodic mild solutions for homogenous linear impulsive periodic systems will be presented. In the third section, we will discuss the existence of periodic mild solutions for the nonhomogenous linear impulsive periodic control systems. Finally, in Section 4, we will discuss the existence of periodic mild solutions for the linear impulsive periodic systems with parameter perturbations.

3.1 Notations

Let $I := [0, T_0]$ be a closed bounded interval of the real line and define the sets $D := \{0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \ldots\} \subset [0, \infty)$.

Definition 3.1.1. A sequence (τ_k) is said to be an *impulsive moment* if $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \ldots$ and $\tau_k \to \infty$ as $k \to \infty$.

We now introduce the piecewise continuous function spaces. Let X be a Banach space and $0 < T_0 < \infty$.

(1) $PC([0, \infty), X)$ is the set of all functions $x : [0, \infty) \to X$ which are continuous at $t \neq \tau_k$, x is left continuous at $t = \tau_k$ and $x(\tau_k^+)$ exists for all $k \in \mathbb{N}$.

(2) $PC_{T_0}([0,\infty),X)$ is the set of all functions $x \in PC([0,\infty),X)$ such that $x(t) = x(t+T_0)$ for all $t \geq 0$.

3.2 Homogenous Linear Impulsive Periodic Systems

We consider the following homogenous linear impulsive periodic systems

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{3.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Suppose that system (A.5) satisfies the following assumption (A1).

Assumption (A1) ;

(A1.1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots,$ $\tau_k \to \infty$ as $k \to \infty$ and there exists a positive integer σ such that $\tau_{k+\sigma} = \tau_k + T_0$ for all $k \in \mathbb{N}$.

(A1.2) A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X.

(A1.3) $B_k \in \mathcal{L}(X)$ such that $B_{k+\sigma} = B_k$.

3.2.1 Impulsive Evolution Operator

Definition 3.2.1. Let assumption (A1) hold. An operator value function $U(t,s)$ with values in $\mathcal{L}(X)$, defined on the triangle $\Delta \equiv \{0 \leq s \leq t \leq a\}$ with $t, s \in (\tau_{k-1}, \tau_k]$ for all $k \in \mathbb{N}$, given by

$$U(t,s) = \begin{cases} T(t-s), & \tau_{k-1} \leq s \leq t \leq \tau_k, \\ T(t-\tau_k)(I+B_k)T(\tau_k - s), & \tau_{k-1} < s \leq \tau_k < t \leq \tau_{k+1}, \\ T(t-\tau_k)\left[\displaystyle\prod_{j=i+1}^{k}(I+B_j)T(\tau_j - \tau_{j-1})\right](I+B_i)T(\tau_i - s), & \\ \qquad \text{for } i < k,\ \tau_{i-1} < s \leq \tau_i < \ldots < \tau_k < t \leq \tau_{k+1} & \end{cases} \tag{3.2}$$

is called an *impulsive evolution operator*.

Proposition 3.2.1. *Let assumption (A1) hold and* $\{U(t,s), 0 \le s \le t \le a\}$ *be a family of impulsive evolution operators. For each fixed* $T_0 = \tau_\sigma > 0$, *then the following conditions are satisfied :*

(1) $U(t,t) = I$, *(*I *is the identity operator on* X*).*

(2) $U(t,s) = U(t,r)U(r,s)$ *for all* $0 \le s \le r \le t \le a$.

(3) $U(t+KT_0, s+KT_0) = U(t,s)$ *for all* $K \in \mathbb{N}$ *and* $0 \le s \le t \le T_0$ *with* $T_0 \le a$.

Proof. (1) Since $T(0) = I$, $U(t,t) = T(0) = I$. (2) By substitution in equation (A.3) and using semigroup property that $T(t+s) = T(t)T(s)$, the relation (2) follows. By assumption (A1.1), it is easy to prove (3). □

Proposition 3.2.2. *Let assumption (A1) hold. If* $\{U(t,s), 0 \le s \le t \le a\}$ *is a family of impulsive evolution operators, then there exist* $M > 1$ *and* $\omega > 0$ *such that*

$$\|U(t,s)\| \le M \exp\Big(\omega(t-s) + \sum_{s<\tau_n<t} \ln\left(M\|I+B_n\|\right)\Big).$$

for all $0 \le s \le t \le a$.

Proof. Suppose $\{U(t,s), 0 \le s \le t \le a\}$ is a family of impulsive evolution operators. Let $t,s \in (\tau_{k-1}, \tau_k]$ for $k \in \mathbb{N}$. By assumption (A1) and Theorem 2.3.2(1), there exist $M > 1$ and $\omega > 0$ such that

$$\|U(t,s)\| = \|T(t-s)\| \le Me^{\omega(t-s)} \qquad \text{for } \tau_{k-1} \le s < t \le \tau_k.$$

For $\tau_{k-1} \le s \le \tau_k < t \le \tau_{k+1}$, we have

$$\begin{aligned}
\|U(t,s)\| &= \|T(t-\tau_k)(I+B_k)T(\tau_k - s)\| \\
&\le \|T(t-\tau_k)\|\|I+B_k\|\|T(\tau_k - s)\| \\
&\le Me^{\omega(t-\tau_k)}\|I+B_k\|Me^{\omega(\tau_k - s)} \\
&= M\exp\Big(\omega(t-s) + \ln\left(M\|I+B_k\|\right)\Big).
\end{aligned}$$

For $\tau_{i-1} \leq s \leq \tau_i < \ldots < \tau_k < t \leq \tau_{k+1}$, we have

$$\begin{aligned}
\|U(t,s)\| &= \|T(t-\tau_k)(I+B_k)T(\tau_k-\tau_{k-1})\ldots(I+B_i)T(\tau_i-s)\| \\
&\leq \|T(t-\tau_k)\|\Big(\prod_{j=i+1}^{k}\|I+B_j\|\|T(\tau_j-\tau_{j-1})\|\Big)\|I+B_i\|\|T(\tau_i-s)\| \\
&\leq Me^{\omega(t-\tau_k)}\Big(\prod_{j=i+1}^{k}M\|I+B_j\|e^{\omega(\tau_j-\tau_{j-1})}\Big)\|I+B_i\|Me^{\omega(\tau_i-s)} \\
&= M\exp\Big(\omega(t-s)+\sum_{s<\tau_n<t}\ln\left(M\|I+B_n\|\right)\Big).
\end{aligned}$$

This completes the proof. □

Corollary 3.2.3. *Let assumption (A1) hold and* $\{U(t,s), 0 \leq s \leq t \leq a\}$ *be a family of impulsive evolution operators, then*

$$\sup_{0\leq s\leq t\leq a}\|U(t,s)\| < \infty$$

for all $a > 0$.

Proof. Let $a > 0$ be fixed. By Proposition 3.2.2 , there exist constants $M > 1$ and $\omega > 0$ such that

$$\|U(t,s)\| \leq M\exp\Big(\omega(t-s)+\sum_{s<\tau_n<t}\ln\left(M\|I+B_n\|\right)\Big)$$

for all $0 \leq s \leq t \leq a$. Since

$$\omega(t-s)+\sum_{s<\tau_n<t}\ln\left(M\|I+B_n\|\right) \leq \omega a+\sum_{0<\tau_n\leq a}\ln\left(M\|I+B_n\|\right)$$

for all $0 \leq s \leq t \leq a$, this implies that

$$\sup_{0\leq s<t\leq a}\|U(t,s)\| \leq M\exp\Big(\omega a+\sum_{0<\tau_n\leq a}\ln\left(M\|I+B_n\|\right)\Big).$$

This completes the proof. □

Corollary 3.2.4. *Let assumption (A1) hold and* $\{U(t,s), 0 \le s \le t \le a\}$ *be a family of impulsive evolution operators. For each* $a > 0$ *and* $\nu \in \mathbb{R}$*, there exists* $K > 0$ *such that*

$$\|U(t,s)\| < Ke^{\nu(t-s)}$$

for all $0 \le s \le t \le a$*.*

Proof. Let $a > 0$ and $\nu \in \mathbb{R}$ be given. By Corollary A.2, there exists a constant $K_1 > 0$ such that

$$\|U(t,s)\| \le K_1 = K_1 e^{-\nu(t-s)} e^{\nu(t-s)} \le K_1 e^{|\nu|a} e^{\nu(t-s)}$$

for all $0 \le s \le t \le a$. We let $K := K_1 e^{|\nu|a}$, this implies that

$$\|U(t,s)\| < Ke^{\nu(t-s)}$$

for all $0 \le s \le t \le a$. □

3.2.2 Definitions of Solutions

Definition 3.2.2. Let assumption (A1) hold. The impulsive system (A.5) is said to have a *classical solution* if for every $x_0 \in D(A)$ there exists a function $x \in PC([0,\infty), X)$ satisfies the following properties :

1. $x \in PC([0,\infty), X) \cap C^1((0,\infty) \setminus D, X)$,
2. For $t > 0$, $x(t) \in D(A)$ and $\dot{x}(t) = Ax(t)$ where $t \neq \tau_k$,
3. $x(0) = x_0$ and $\triangle x(t) = B_k x(t)$ where $t = \tau_k$.

Definition 3.2.3. A function $x \in PC([0,\infty), X)$ is said to be a *mild solution* of system (A.5) with initial condition $x(0) = x_0$ if x is given by

$$x(t) = U(t,0)x_0 \tag{3.3}$$

where

$$U(t,0)=\begin{cases} T(t), & 0\le t\le \tau_1,\\ T(t-\tau_k)\left[\prod_{j=1}^{k}(I+B_j)T(\tau_j-\tau_{j-1})\right], & \tau_k<t\le\tau_{k+1}, \end{cases} \tag{3.4}$$

for all $k\in\mathbb{N}$.

Definition 3.2.4. A function $x\in PC([0,\infty),X)$ is said to be a *periodic mild solution* of system (A.5) if it is a mild solution and there exists $T_0>0$ such that $x(t+T_0)=x(t)$ for all $t\ge 0$.

Proposition 3.2.5. *Let assumption (A1) hold and $\{U(t,0),0\le t\le a\}$ be a family of impulsive evolution operators. For each fixed $T_0=\tau_\sigma>0$, then the following conditions are satisfied :*

(1) $U(0,0)=I$,

(2) $U(t,0)=U(\bar t,0)[U(T_0,0)]^M$ *where* $t=\bar t+MT_0$ *for* $\bar t\in[0,T_0]$ *and* $M\in\mathbb{N}\cup\{0\}$.

Proof. (1) Since $T(0)=I$ with equation (A.6), $U(0,0)=I$.

(2) We consider the following impulsive system for $t\in[MT_0,(M+1)T_0]$;

$$\begin{cases} \dot x(t) &= Ax(t), \qquad t\ne\tau_{M\sigma+m},\\ \Delta x(t) &= B_{M\sigma+m}x(t), \qquad t=\tau_{M\sigma+m},\ m=1,2,\dots,\sigma,\\ x(MT_0) &= [U(T_0,0)]^M x_0,\ x(0)=x_0, \end{cases} \tag{3.5}$$

where $MT_0=\tau_{M\sigma}<\tau_{M\sigma+1}<\ldots<\tau_{(M+1)\sigma}=(M+1)T_0$

and $\Delta x(\tau_{M\sigma+m})=x(\tau^+_{M\sigma+m})-x(\tau^-_{M\sigma+m})$ for all $M\in\mathbb{N}\cup\{0\}$.

First, we consider the impulsive system for $M=0$, that is for $t\in[0,T_0]$;

$$\begin{cases} \dot x(t) &= Ax(t), \qquad t\ne\tau_m,\\ \Delta x(t) &= B_m x(t), \qquad t=\tau_m,\ m=1,2,\dots,\sigma,\\ x(0) &= x_0, \end{cases} \tag{3.6}$$

where $\quad 0 < \tau_1 < \tau_2 < \ldots < \tau_\sigma = T_0 \quad$ and $\quad \Delta x(\tau_m) = x(\tau_m^+) - x(\tau_m^-)$.

For $t \in [0, \tau_1]$;

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq 0, \\ x(0) = x_0, & t = 0. \end{cases}$$

The mild solution is $\quad x(t) = T(t)x_0, \quad 0 \leq t \leq \tau_1$.

For $t \in (\tau_1, \tau_2]$;

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq \tau_1, \\ x(\tau_1^+) = (I + B_1)x(\tau_1), & t = \tau_1. \end{cases}$$

The mild solution is

$$\begin{aligned} x(t) &= T(t - \tau_1)x(\tau_1^+) \\ &= T(t - \tau_1)(I + B_1)x(\tau_1) \\ &= T(t - \tau_1)(I + B_1)T(\tau_1)x_0, \end{aligned}$$

for all $\quad \tau_1 < t \leq \tau_2$.

Proceeding in this way, we can be repeated on $(\tau_m, \tau_{m+1}], m = 2, 3, \ldots, \sigma - 1$ to get the solution of system (3.6) on $[0, T_0]$ with initial condition $x(0) = x_0$ is given by

$$x(t) = U(t, 0)x_0$$

where $U(t, 0)$ is defined in (A.6).

Next, we consider the impulsive system for $M = 1$, that is for $t \in [T_0, 2T_0]$;

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq \tau_{\sigma+m}, \\ \Delta x(t) = B_{\sigma+m}x(t), & t = \tau_{\sigma+m}, \ m = 1, 2, \ldots, \sigma, \\ x(T_0) = U(T_0, 0)x_0, & x(0) = x_0, \end{cases} \tag{3.7}$$

where $\quad T_0 = \tau_\sigma < \tau_{\sigma+1} < \ldots < \tau_{2\sigma} = 2T_0$ and $\Delta x(\tau_{\sigma+m}) = x(\tau_{\sigma+m}^+) - x(\tau_{\sigma+m}^-)$.

For $t \in [\tau_\sigma, \tau_{\sigma+1}]$;

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq \tau_\sigma, \\ x(T_0) = U(T_0, 0)x_0, & t = \tau_\sigma. \end{cases}$$

The mild solution is $\quad x(t) = T(t-\tau_\sigma)U(T_0,0)x_0 \quad$ for all $\quad \tau_\sigma \le t \le \tau_{\sigma+1}$.

For $t \in (\tau_{\sigma+1}, \tau_{\sigma+2}]$;

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq \tau_{\sigma+1}, \\ x(\tau_{\sigma+1}^+) = (I + B_{\sigma+1})x(\tau_{\sigma+1}), & t = \tau_{\sigma+1}. \end{cases}$$

The mild solution is

$$\begin{aligned} x(t) &= T(t-\tau_{\sigma+1})x(\tau_{\sigma+1}^+) \\ &= T(t-\tau_{\sigma+1})(I+B_{\sigma+1})x(\tau_{\sigma+1}) \\ &= T(t-\tau_{\sigma+1})(I+B_{\sigma+1})T(\tau_{\sigma+1}-\tau_\sigma)U(T_0,0)x_0, \end{aligned} \tag{3.8}$$

for all $\quad \tau_{\sigma+1} < t \le \tau_{\sigma+2}$.

By similarity procedure, we can be repeated on $(\tau_{\sigma+m}, \tau_{\sigma+m+1}]$, $m = 2, 3, \ldots, \sigma - 1$ to get the mild solution of system (3.7) on $[T_0, 2T_0]$ with initial condition $x(T_0) = U(T_0, 0)x_0$,

$$x(t) = T(t-\tau_{\sigma+m}) \left[\prod_{j=1}^{m} (I + B_{\sigma+j})T(\tau_{\sigma+j} - \tau_{\sigma+j-1}) \right] U(T_0,0)x_0 \tag{3.9}$$

Since $B_{\sigma+m} = B_m$, $\tau_{\sigma+m} = \tau_m + T_0$ and $t = \bar{t} + T_0$ for all $0 \le \bar{t} \le T_0$ and $m \in \mathbb{N}$, then the solution (3.9) can be rewritten in the form

$$\begin{aligned} x(t) = {} & T(t-\tau_{\sigma+m}) \left[\prod_{j=1}^{m} (I + B_{\sigma+j})T(\tau_{\sigma+j} - \tau_{\sigma+j-1}) \right] U(T_0,0)x_0 \\ = {} & T(\bar{t} + T_0 - (\tau_m + T_0)) \left[\prod_{j=1}^{m} (I + B_j)T(\tau_j + T_0 - (\tau_{j-1} + T_0)) \right] U(T_0,0)x_0 \\ = {} & T(\bar{t} - \tau_m) \left[\prod_{j=1}^{m} (I + B_j)T(\tau_j - \tau_{j-1}) \right] U(T_0,0)x_0 \end{aligned}$$

Then by Definition 3.2.3, we have

$$x(t) = U(t,0)x_0 = U(\bar{t},0)U(T_0,0)x_0, \quad \text{for all} \quad 0 \le \bar{t} \le T_0.$$

That is for $M = 1$ with $t = \bar{t} + T_0$, we obtain

$$U(t,0) = U(\bar{t},0)U(T_0,0), \quad \text{for all} \quad 0 \le \bar{t} \le T_0.$$

By mathematical induction, we proceeding in similar way. Then the mild solution of system (3.5) on $[MT_0, (M+1)T_0]$ with initial condition $x(MT_0) = [U(T_0, 0)]^M x_0$ is given by

$$x(t) = T(\bar{t} - \tau_m)\left[\prod_{j=1}^{m}(I + B_j)T(\tau_j - \tau_{j-1})\right][U(T_0, 0)]^M x_0.$$

Again by Definition 3.2.3, we have

$$x(t) = U(t, 0)x_0 = U(\bar{t}, 0)[U(T_0, 0)]^M x_0, \quad \text{for all} \quad 0 \leq \bar{t} \leq T_0.$$

That is for $M \in \mathbb{N} \cup \{0\}$ with $t = \bar{t} + MT_0$, we obtain

$$U(t, 0) = U(\bar{t}, 0)[U(T_0, 0)]^M, \quad \text{for all} \quad 0 \leq \bar{t} \leq T_0.$$

This completes the proof. □

Remark 3.1. If $\{T(t), t > 0\}$ is a compact semigroup in X, then $U(t, 0)$ is a compact operator. Particularly, $U(T_0, 0)$ is also a compact operator.

Since $\{T(t), t > 0\}$ is a compact semigroup, then $T(\tau_k - \tau_{k-1})$ is compact for all $\tau_k > \tau_{k-1}$ and $k \in \mathbb{N}$. From equation (A.6) and the fact that $B_k \in \mathcal{L}(x)$, then $U(t, 0)$ is a compact operator for $t > 0$. Particularly, $U(T_0, 0)$ is also a compact operator.

Lemma 3.2.6. If $\{T(t), t \geq 0\}$ is a C_0-semigroup, then for every $x \in X$, $t \mapsto T(t - m)x$ is a continuous X-valued function on $[m, \infty)$ and its right limit exists at m for all $m \geq 0$.

Proof. Suppose that $\{T(t), t \geq 0\}$ is a C_0-semigroup generated by A. Let $x \in X$ and $t - m \geq h \geq 0$, then by the semigroup property

$$T(t - m + h)x - T(t - m)x = T(t - m)\left[T(h)x - x\right].$$

By Theorem 2.3.2(1), there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$. Then we have

$$\begin{aligned}\|T(t-m+h)x - T(t-m)x\|_X &\leq \|T(t-m)\|_{\mathcal{L}(X)} \|T(h)x - x\|_X \\ &\leq Me^{\omega(t-m)} \|T(h)x - x\|_X,\end{aligned}$$

and by the C_0-property, we obtain

$$\lim_{h \to 0} \|T(t-m+h)x - T(t-m)x\|_X = 0.$$

Similarly for $t - m \geq h \geq 0$, we have

$$\begin{aligned}\|T(t-m-h)x - T(t-m)x\|_X &\leq \|T(t-m-h)\|_{\mathcal{L}(X)} \|T(h)x - x\|_X \\ &\leq Me^{\omega(t-m-h)} \|T(h)x - x\|_X.\end{aligned}$$

Hence

$$\lim_{h \to 0} \|T(t-m-h)x - T(t-m)x\|_X = 0$$

and

$$\lim_{t \to m^+} T(t-m)x = T(0)x = Ix = x.$$

This implies that $t \mapsto T(t-m)x$ is a continuous X-valued function on $[m, \infty)$ and its right limit exists at m. This completes the proof. □

Lemma 3.2.7. If A is an infinitesimal generator of C_0-semigroup $\{T(t), t \geq 0\}$, then for every $x_0 \in D(A)$, $t \mapsto T(t-m)x_0$ is differentiable for $t > m$ with the derivatives given by

$$\frac{d}{dt} T(t-m)x_0 = AT(t-m)x_0 = T(t-m)Ax_0, \quad \text{for all } t > m. \tag{3.10}$$

Proof. Suppose that $\{T(t), t \geq 0\}$ is a C_0-semigroup generated by A. For $t \geq 0$ and $x_0 \in D(A)$ we show that $t \mapsto T(t-m)x_0$ is differentiable for $t > m$, that is we must show that the right and left derivatives of $T(t-m)x_0$ exist and are equal to equation (3.10). By definition of the right derivative is given by

$$\begin{aligned}\frac{d^+}{dt}T(t-m)x_0 &:= \lim_{h\to 0}\left(\frac{T(t-m+h)x_0 - T(t-m)x_0}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{T(h)-I}{h}\right)T(t-m)x_0\\ &= AT(t-m)x_0\end{aligned}$$

and

$$\begin{aligned}\lim_{h\to 0}\left(\frac{T(t-m+h)x_0 - T(t-m)x_0}{h}\right) &= \lim_{h\to 0} T(t-m)\left(\frac{T(h)-I}{h}\right)x_0\\ &= T(t-m)Ax_0,\end{aligned}$$

thus, we have

$$\frac{d^+}{dt}T(t-m)x_0 = AT(t-m)x_0 = T(t-m)Ax_0.$$

This proves the existence of the right derivative and that $T(t-m)x_0 \in D(A)$. For the left derivative it suffices to verify that for $t-m \geq h$,

$$\lim_{h\to 0}\left\|\left(\frac{T(t-m)x_0 - T(t-m-h)x_0}{h}\right) - T(t-m)Ax_0\right\|_X = 0.$$

By using the semigroup property and Theorem 2.3.2(1), there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\begin{aligned}&\left\|\left(\frac{T(t-m)x_0 - T(t-m-h)x_0}{h}\right) - T(t-m)Ax_0\right\|_X\\ &= \left\|T(t-m-h)\left[\left(\frac{T(h)x_0 - x_0}{h}\right) - T(h)Ax_0\right]\right\|_X\\ &\leq \|T(t-m-h)\|_{\mathcal{L}(X)}\left\|\left(\frac{T(h)x_0 - x_0}{h} - Ax_0\right) + (I-T(h))Ax_0\right\|_X\\ &\leq M\exp^{\omega(t-m-h)}\left(\left\|\frac{T(h)x_0 - x_0}{h} - Ax_0\right\|_X + \|(I-T(h))Ax_0\|_X\right).\end{aligned}$$

Since A is the infinitesimal generator of the semigroup $\{T(t), t \geq 0\}$ and $T(t-m)$ is continuous, the expressions within the bracket converge to zero as $h \to 0$. Thus

$$\frac{d^-}{dt}T(t-m)x_0 = T(t-m)Ax_0.$$

Since $T(t-m)Ax_0 = AT(t-m)x_0$ and the continuity of $T(t-m)$, then

$$\frac{d^-}{dt}T(t-m)x_0 = AT(t-m)x_0 = T(t-m)Ax_0 \quad \text{for all } \ t > m.$$

Thus for $x_0 \in D(A)$, both the right and left derivative exist and are identical. Hence $t \mapsto T(t-m)x_0$ is differentiable for $t > m$. □

Theorem 3.2.8. *Let assumption (A1) hold and* $x_0 \in X$*. Then a function* $x : [0,\infty) \to X$ *defined by*

$$x(t) = U(t,0)x_0$$

or

$$x(t) = \begin{cases} T(t)x_0, & 0 \le t \le \tau_1, \\ T(t-\tau_k)\left[\prod_{j=1}^{k}(I+B_j)T(\tau_j-\tau_{j-1})\right]x_0, & \tau_k < t \le \tau_{k+1}, \end{cases} \tag{3.11}$$

belongs to $PC([0,\infty),X)$ *and* $\triangle x(\tau_k) = B_k x(\tau_k)$ *for all* $k \in \mathbb{N}$.

Proof. Let $x_0 \in X$. For $t \in [0,\tau_1]$, we obtain

$$x(t) = U(t,0)x_0 = T(t)x_0.$$

By Lemma 3.2.6, we have $T(t)x_0$ is continuous on $[0,\tau_1]$ and

$$x(0^+) = \lim_{t\to 0^+} T(t)x_0 = x_0.$$

For $t \in (\tau_k,\tau_{k+1}]$, we obtain

$$x(t) = U(t,0)x_0 = T(t-\tau_k)\left[\prod_{j=1}^{k}(I+B_j)T(\tau_j-\tau_{j-1})\right]x_0,$$

for all $k \in \mathbb{N}$.

Define $\quad x_k := \left[\prod_{j=1}^{k}(I+B_j)T(\tau_j-\tau_{j-1})\right]x_0$, then $x(t) = T(t-\tau_k)x_k$.

Again by Lemma 3.2.6, we have $T(t-\tau_k)x_k$ is continuous on $(\tau_k,\tau_{k+1}]$ and

$$x(\tau_k^+) = \lim_{t\to\tau_k^+} T(t-\tau_k)x_k = x_k.$$

That is, $x \in PC([0,\infty), X)$ and

$$\triangle x(\tau_k) = x(\tau_k^+) - x(\tau_k) = (I + B_k)x(\tau_k) - x(\tau_k) = B_k x(\tau_k),$$

for all $k \in \mathbb{N}$. This completes the proof. □

Corollary 3.2.9. *Let assumption (A1) hold and* $\{U(t,s), 0 \le s \le t < \infty\}$ *be a family of impulsive evolution operators. For each* $s \ge 0$ *and* $x \in X$*, a function* $U(\cdot,s)x : [s,\infty) \to X$ *belongs to* $PC([s,\infty), X)$.

Proof. Let $x_s \in X$. For $0 \le s \le t \le \tau_1$, we obtain

$$x(t) = U(t,s)x_s = T(t-s)x_s.$$

By Lemma 3.2.6, we have $T(t-s)x_s$ is continuous on $[s,\tau_1]$ and

$$x(s^+) = \lim_{t \to s^+} T(t-s)x_s = x_s.$$

For $\tau_{k-1} \le s \le \tau_k < t \le \tau_{k+1}$, we obtain

$$x(t) = U(t,s)x_s = T(t-\tau_k)(I + B_k)T(\tau_k - s)x_s \quad \text{for all} \quad k \in \mathbb{N}.$$

Define $x_k = (I + B_k)T(\tau_k - s)x_s$, then $x(t) = T(t - \tau_k)x_k$.
Again by Lemma 3.2.6, we have $T(t-\tau_k)x_k$ is continuous on $(\tau_k, \tau_{k+1}]$ and

$$x(s^+) = \lim_{t \to s^+} T(t-\tau_k)x_k = x_k.$$

For $\tau_{i-1} < s \le \tau_i < \ldots < \tau_k < t \le \tau_{k+1}$, we obtain

$$\begin{aligned} x(t) = & \; U(t,s)x_s \\ = & \; T(t-\tau_k)\left[\prod_{j=i+1}^{k}(I + B_j)T(\tau_j - \tau_{j-1})\right](I + B_i)T(\tau_i - s)x_s, \end{aligned}$$

for all $k \in \mathbb{N}$.
Define $x_k := \left[\prod_{j=1}^{k}(I + B_j)T(\tau_j - \tau_{j-1})\right](I + B_i)T(\tau_i - s)x_s$, then

$$x(t) = T(t-\tau_k)x_k.$$

Also by Lemma 3.2.6, we have $T(t-\tau_k)x_k$ is continuous on $t \in (\tau_k, \tau_{k+1}]$ and

$$x(\tau_k^+) = \lim_{t \to \tau_k^+} T(t-\tau_k)x_k = x_k.$$

That is, $x \in PC([s,\infty), X)$ and

$$\triangle x(\tau_k) = x(\tau_k^+) - x(\tau_k) = (I+B_k)x(\tau_k) - x(\tau_k) = B_k x(\tau_k),$$

for all $k \in \mathbb{N}$. This completes the proof. □

3.2.3 Regularity of Mild Solutions

Theorem 3.2.10. *Let assumption (A1) hold. If $D(A)$ is an invariant subspace of $T(t)$ and $(I+B_k)$ for all $k \in \mathbb{N}$ and $t \in [0,\infty)$, then for every $x_0 \in D(A)$ the mild solution of impulsive system (A.5) is a classical solution.*

Proof. Assume that x is a mild solution of impulsive system (A.5) with initial condition $x_0 \in D(A)$. Next, we want to show that the mild solution is differentiable on (τ_k, τ_{k+1}) for all $k \in \mathbb{N} \cup \{0\}$. By Lemma 3.2.7, we have x is differentiable on $(0, \tau_1)$,

$$\dot{x}(t) = \frac{d}{dt}U(t,0)x_0 = \frac{d}{dt}T(t)x_0 = AT(t)x_0 = Ax(t)$$

and $x(t) = T(t)x_0 \in D(A)$. Since $D(A)$ is an invariant subspace of $T(t)$ and $(I+B_k)$,

$$x_k := \left[\prod_{j=1}^{k}(I+B_j)T(\tau_j - \tau_{j-1})\right] x_0 \in D(A),$$

for all $k \in \mathbb{N}$ and $t \in [0,\infty)$. By Lemma 3.2.7, we have x is differentiable on (τ_k, τ_{k+1}),

$$\begin{aligned} \dot{x}(t) &= \frac{d}{dt}U(t,0)x_0 = \frac{d}{dt}T(t-\tau_k)\left[\prod_{j=1}^{k}(I+B_j)T(\tau_j-\tau_{j-1})\right] x_0 \\ &= \frac{d}{dt}T(t-\tau_k)x_k = AT(t-\tau_k)x_k = Ax(t) \end{aligned}$$

and $x(t) = T(t-\tau_k)x_k \in D(A)$.

That is, $x \in PC([0,\infty),X) \cap C^1((0,\infty) \setminus D, X)$ and $\dot{x}(t) = Ax(t)$, where $t \neq \tau_k$. Therefore it follows from Theorem 3.2.8 that $x \in PC([0,\infty),X)$ and $\triangle x(\tau_k) = B_k x(\tau_k)$ for all $k \in \mathbb{N}$. Hence, x is a classical solution. □

Corollary 3.2.11. *Let assumption (A1) hold and $\{T(t), t \geq 0\}$ be a differential semigroup generated by A. If $D(A)$ is an invariant subspace of $T(t)$ and $(I + B_k)$ for all $k \in \mathbb{N}$, then for every $x_0 \in X$ the mild solution of impulsive system (A.5) is a classical solution.*

Proof. Since $\{T(t), t \geq 0\}$ is a differential semigroup, $\{T(t), t \geq 0\}$ is C_0-semigroup. From Theorem 3.2.10, the proof follows immediately. □

Corollary 3.2.12. *Let assumption (A1) hold and $\{T(t), t \geq 0\}$ be an analytic semigroup generated by A. If $D(A)$ is an invariant subspace of $T(t)$ and $(I+B_k)$ for all $k \in \mathbb{N}$, then for every $x_0 \in X$ the mild solution of impulsive system (A.5) is a classical solution.*

Proof. This follows from the simple fact that every analytic semigroup is also a differentiable semigroup and C_0-semigroup. From Theorem 3.2.10, the proof follows immediately. □

3.2.4 Existence and Uniqueness of periodic mild solutions

Theorem 3.2.13. *Let assumption (A1) hold. System (A.5) has a periodic mild solution if and only if the operator $U(T_0, 0)$ has a fixed point $x_0 \in X$.*

Proof. Let $x(t)$ be a periodic mild solution of system (A.5). Suppose $x(0) = x_0$ be the initial condition of system (A.5), then $x(T_0) = x(0) = x_0$. Since $x(T_0) = U(T_0, 0)x_0$, $x_0 = U(T_0, 0)x_0$. That is, the operator $U(T_0, 0)$ has a fixed point $x_0 \in X$. Conversely, assume that x_0 be a fixed point of $U(T_0, 0)$. Use x_0 as the initial condition of system (A.5), then the solution is $x(t) = U(t,0)x_0$, where $t = \bar{t} + MT_0$ for all $\bar{t} \in [0, T_0]$ and $M \in \mathbb{N} \cup \{0\}$. By assumption and Proposition A.1 (2), we have $x(t) = x(\bar{t} + MT_0) = U(\bar{t}, 0)[U(T_0, 0)]^M x_0 = U(\bar{t}, 0)x_0 = x(\bar{t})$. Hence x is a periodic mild solution of system (A.5). □

Theorem 3.2.14. *Let assumption (A1) hold. Furthermore, assume that A is the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$ in X. Then system (A.5) either has a unique trivial solution or have finitely many linearly independent nontrivial periodic mild solutions in $PC([0,\infty), X)$.*

Proof. Since $U(T_0, 0) : X \to X$ is a compact linear operator, applying Fredholm alternative theorem, we obtain $U(T_0, 0)$ satisfy Fredholm alternative that either (a) or (b) holds : (a) The homogenous equations $[I - U(T_0, 0)]x = 0$ have only the trivial solution $x = 0$. That is $U(T_0, 0)$ has only a unique fixed point $x = 0$ (i.e., by Theorem A.3, this means that system (A.5) has a unique trivial solution). (b) The homogenous equations $[I - U(T_0, 0)]x = 0$ have nontrivial solutions, then all of linearly independent nontrivial solutions are finite. Suppose all of nontrivial solutions $x_0^1, x_0^2, \ldots, x_0^m$ be such that $[I - U(T_0, 0)]x_0^i = 0$, $i = 1, 2, \ldots, m$. So $x_0^1, x_0^2, \ldots, x_0^m$ are fixed points of $U(T_0, 0)$. Again by Theorem A.3, this means that system (A.5) has periodic mild solutions, say $x^1, x^2, \ldots, x^m$ where x^i are the solutions of system (A.5) corresponding to initial conditions $x^i(0) = x_0^i$, $i = 1, 2, \ldots, m$. Hence the number of linearly independent nontrivial periodic mild solutions of system (A.5) are finite. □

3.3 Nonhomogeneous Linear Impulsive Periodic Control Systems

We consider the following nonhomogeneous linear impulsive periodic control systems

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k, & t = \tau_k, \end{cases} \tag{3.12}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Suppose that system (A.7) satisfies assumption (A1). We impose the following assumption for the remaining.

Assumption (A2) ;

(A2.1) A is the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$ in X.

(A2.2) $u \in PC([0, \infty), X)$ such that $u(t + T_0) = u(t)$.

(A2.3) $c_k \in X$ and $c_{k+\sigma} = c_k$ for all $k \in \mathbb{N}$.

3.3.1 Definitions of Solutions

Definition 3.3.1. A function $x \in PC([0, \infty), X)$ is said to be a *mild solution* of system (A.7) with initial condition $x(0) = x_0 \in X$ and the input $u \in L^1_{loc}([0, \infty), X)$ if x is given by

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)u(s)ds + \sum_{0 \leq \tau_k < t} U(t, \tau_k)c_k, \tag{3.13}$$

for all $k \in \mathbb{N}$.

Definition 3.3.2. A function $x \in PC([0, \infty), X)$ is said to be a *periodic mild solution* of system (A.7) if it is a mild solution and there exists $T_0 > 0$ such that $x(t + T_0) = x(t)$ for all $t \geq 0$.

Definition 3.3.3. A function $x \in PC([0,\infty), X)$ is said to be a *T_0-periodic mild solution* of system (A.7) if it is a mild solution and $x(t+T_0) = x(t)$ for all $t \geq 0$.

3.3.2 Existence and Uniqueness of Periodic Mild Solutions

Consider the nonhomogeneous system without impulses

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t), \quad t > 0 \\ x(0) = x_0. \end{cases} \tag{3.14}$$

where A is the infinitesimal generator of C_0-semigroup $\{T(t), t \geq 0\}$ in X.

Lemma 3.3.1. If $u \in L^1_{loc}([0,\infty), X)$, then for every $x_0 \in X$ the initial value problem (A.9) has a unique solution which satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-s)u(s)ds, \quad 0 \leq t \leq T_0. \tag{3.15}$$

Proof. See Pazy (1983), pp.106. □

Theorem 3.3.2. *Let assumptions (A1) and (A2) hold. If $u \in L^1_{loc}([0,\infty), X)$, then system (A.7) has a unique mild solution $x \in PC([0,T_0], X)$.*

Proof. For $t \in [0,\tau_1]$, Lemma A.5 implies that system

$$\dot{x}(t) = Ax(t) + u(t), \quad 0 \leq t \leq \tau_1, \quad x(0) = x_0, \tag{3.16}$$

has a unique mild solution on $I_1 = [0,\tau_1]$ which satisfies

$$x_1(t) = \ T(t)x_0 + \int_0^t T(t-s)u(s)ds, \quad t \in [0,\tau_1]. \tag{3.17}$$

Now, define

$$x_1(\tau_1) = T(\tau_1)x_0 + \int_0^{\tau_1} T(\tau_1 - s)u(s)ds, \tag{3.18}$$

so that $x_1(\cdot)$ is left continuous at τ_1.

Next, on $I_2 = (\tau_1, \tau_2]$, consider system

$$\dot{x}(t) = Ax(t) + u(t), \quad \tau_1 < t \leq \tau_2, \quad x_1(\tau_1^+) = (I + B_1)x_1(\tau_1) + c_1, \tag{3.19}$$

Since $x_1 \in X$, we can use Lemma A.5 again to get a unique mild solution on $(\tau_1, \tau_2]$ which satisfy

$$x_2(t) = T(t - \tau_1)\left[(I + B_1)x_1(\tau_1) + c_1\right] + \int_{\tau_1}^{t} T(t - s)u(s)ds. \tag{3.20}$$

Now, define $x_2(\tau_2)$ accordingly so that $x_2(\cdot)$ is left continuous at τ_2.

It is easy to see that Lemma A.5 can be applied to interval $(\tau_1, \tau_2]$ to verify that $x_2(\tau_2) \in X$. Proceeding in this way, we can be repeated on $I_k = (\tau_{k-1}, \tau_k]$, $k = 3, 4, \ldots, \sigma$ $(\tau_\sigma = T_0)$ to get a mild solution

$$x_k(t) = \; T(t - \tau_{k-1})\left[(I + B_{k-1})x_{k-1}(\tau_{k-1}) + c_{k-1}\right] + \int_{\tau_{k-1}}^{t} T(t - s)u(s)ds.$$

for $t \in (\tau_{k-}, \tau_k]$ and define $x_k(\tau_k)$ accordingly with $x_k(\cdot)$ left continuous at τ_k and $x_k(\tau_k) \in X$, $k = 1, 2, \ldots, \sigma$.

Thus we obtain $x \in PC([0, T_0], X)$ is a unique mild solution of system (A.7) and given by

$$x(t) = \begin{cases} x_1(t), & 0 \leq t \leq \tau_1, \\ x_k(t), & \tau_{k-1} < t \leq \tau_k, \; k = 2, 3, \ldots, \sigma. \end{cases}$$

Next, we use mathematical induction to show that (A.8) is satisfied on $[0, T_0]$. First, (A.8) is satisfied on $[0, \tau_1]$. If (A.8) is satisfied on $(\tau_{k-1}, \tau_k]$, then for $t \in (\tau_k, \tau_{k+1}]$,

$$\begin{aligned} x(t) = \; & x_{k+1}(t) = T(t - \tau_k)\left[(I + B_k)x_k(\tau_k) + c_k\right] + \int_{\tau_k}^{t} T(t - s)u(s)ds \\ = \; & T(t - \tau_k)(I + B_k)x(\tau_k) + T(t - \tau_k)c_k + \int_{\tau_k}^{t} T(t - s)u(s)ds \end{aligned}$$

$$
\begin{aligned}
&= T(t-\tau_k)(I+B_k)\Big[U(\tau_k,0)x_0+\int_0^{\tau_k}U(\tau_k,s)u(s)ds+\sum_{0\le\tau_i<\tau_k}U(\tau_k,\tau_i)c_i\Big]\\
&\quad +T(t-\tau_k)c_k+\int_{\tau_k}^{t}T(t-s)u(s)ds\\
&= U(t,0)x_0+\int_0^{\tau_k}U(t,s)u(s)ds+\int_{\tau_k}^{t}U(t,s)u(s)ds+\sum_{0\le\tau_i<\tau_k}U(t,\tau_i)c_i+U(t,\tau_k)c_k\\
&= U(t,0)x_0+\int_0^{t}U(t,s)u(s)ds+\sum_{0\le\tau_i<t}U(t,\tau_i)c_i.
\end{aligned}
$$

Thus (A.8) is also true on $(\tau_k,\tau_{k+1}]$. Therefore (A.8) is true on $[0,T_0]$. □

According to Definition 3.3.1, the mild solution of system (A.7) is given by

$$
x(t)=U(t,0)x_0+\int_0^{t}U(t,s)u(s)ds+\sum_{0\le\tau_k<t}U(t,\tau_k)c_k,
$$

where $x(0)=x_0$ for all $k\in\mathbb{N}$.

If $x(t)$ is T_0-periodic mild solution of system (A.7), then we have $x(T_0)=x(0)$; namely,

$$
[I-U(T_0,0)]x(0)=\int_0^{T_0}U(T_0,s)u(s)ds+\sum_{0\le\tau_k<T_0}U(T_0,\tau_k)c_k. \tag{3.21}
$$

We consider in 2 cases.

Case 1 : $[I-U(T_0,0)]^{-1}$ exists.

Theorem 3.3.3. *If system (A.5) has only trivial solution , then system (A.7) has a unique T_0-periodic mild solution*

$$
\begin{aligned}
x_{T_0}(t)= {}& U(t,0)[I-U(T_0,0)]^{-1}\left(\int_0^{T_0}U(T_0,s)u(s)\,ds\right.\\
&\left.+\sum_{0\le\tau_k<T_0}U(T_0,\tau_k)c_k\right)+\int_0^{t}U(t,s)u(s)ds \\
&+\sum_{0\le\tau_k<t}U(t,\tau_k)c_k.
\end{aligned}\tag{3.22}
$$

Proof. Suppose that system (A.5) has only trivial solution, then $[I-U(T_0,0)]^{-1}$ exists. This implies that (A.16) gives

$$
x(0)=[I-U(T_0,0)]^{-1}\left(\int_0^{T_0}U(T_0,s)u(s)ds+\sum_{0\le\tau_k<T_0}U(T_0,\tau_k)c_k\right).
$$

Substituting $x(0) = x_0$ into equation (A.10), we get

$$\begin{aligned} x(t) = \;& U(t,0)[I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)u(s)\,ds \right.\\ &\left. + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\right) + \int_0^t U(t,s)u(s)ds \\ & + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k. \end{aligned} \tag{3.23}$$

which is a mild solution of system (A.7).

Next, we want to show that a mild solution is unique and is T_0-periodic. Suppose that $y(t) = x(t+T_0)$ is a mild solution of system (A.7). By Proposition (3.2.1), we have

$$\begin{aligned} y(t) = \;& x(t+T_0) = U(t+T_0,0)x_0 + \int_0^{t+T_0} U(t+T_0,s)u(s)ds \\ & + \sum_{0\le\tau_k<t+T_0} U(t+T_0,\tau_k)c_k \\ = \;& U(t+T_0,T_0)U(T_0,0)x_0 + \int_0^{T_0} U(t+T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(t+T_0,\tau_k)c_k \\ & + \int_{T_0}^{t+T_0} U(t+T_0,s)u(s)ds + \sum_{T_0\le\tau_k<t+T_0} U(t+T_0,\tau_k)c_k \\ = \;& U(t,0)U(T_0,0)x_0 + \int_0^{T_0} U(t+T_0,T_0)U(T_0,s)u(s)ds \\ & + \sum_{0\le\tau_k<T_0} U(t+T_0,T_0)U(T_0,\tau_k)c_k + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k \\ = \;& U(t,0)U(T_0,0)x_0 + U(t,0)\int_0^{T_0} U(T_0,s)u(s)ds + U(t,0)\sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k \\ & + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k \\ = \;& U(t,0)\left[U(T_0,0)x_0 + \int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\right] \\ & + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k \end{aligned}$$

$$= U(t,0)x(T_0) + \int_0^t U(t,s)u(s)ds + \sum_{0\le \tau_k < t} U(t,\tau_k)c_k$$
$$= U(t,0)y(0) + \int_0^t U(t,s)u(s)ds + \sum_{0\le \tau_k < t} U(t,\tau_k)c_k$$

This implies that $y(t)$ is also a solution. It follows from Lemma A.5 that $y(t) = x(t+T_0) = x(t)$ for all $t \ge 0$. So $x(t)$ is a T_0-periodic mild solution of system (A.7), which is exactly (3.22). This completes the proof. □

Case 2 : $[I - U(T_0, 0)]^{-1}$ does not exists.

In this case, system (A.5) has nontrivial periodic mild solutions. Let us construct the following adjoint equation of system (A.5),

$$\begin{cases} \dot{y}(t) = -A^* y, & t \ne \tau_k, \\ -\Delta y(t) = B_k^* y(t), & t = \tau_k, \end{cases} \tag{3.24}$$

where A^* is the adjoint operator of A, $0 = \tau_0 < \tau_1 < \ldots < \tau_k < \ldots < \tau_\sigma = T_0$ and $\Delta y(\tau_k) = y(\tau_k^+) - y(\tau_k^-)$ for all $k = 1, 2, \ldots, \sigma$. Suppose that system (A.19) satisfies the following assumption (A3).

Assumption (A3) ;

(A3.1) A^* is the infinitesimal generator of the adjoint semigroup $\{T^*(t), t \ge 0\}$ in X^*.

(A3.2) $B_k^* \in \mathcal{L}(X^*)$ such that $B_{k+\sigma}^* = B_k^*$ for all $k \in \mathbb{N}$.

Definition 3.3.4. A function $y \in PC([0, T_0], X)$ is said to be a *periodic mild solution* of system (A.19) with initial condition $y(T_0) = y(0)$ if y is given by

$$y(t) = U^*(T_0, t)y(0), \quad 0 \le t \le T_0, \tag{3.25}$$

where

$$U^*(T_0,t)=\begin{cases} T^*(T_0-t), & \tau_{\sigma-1}<t\le\tau_\sigma=T_0,\\ T^*(\tau_i-t)(I+B_i^*)\left[\prod_{j=i+1}^{k}(I+B_j)T(\tau_j-\tau_{j-1})\right]^* T^*(T_0-\tau_k), & \\ & 0\le\tau_{i-1}<t\le\tau_i\le\tau_\sigma=T_0,\end{cases} \quad (3.26)$$

for all $i=1,2,\dots,\sigma$.

Theorem 3.3.4. *Assume that (A1) and (A2) hold. Furthermore, assume that* X *is a Hilbert space and* $u\in L^1_{loc}([0,\infty),X)$. *If system (A.5) have* m *linearly independent periodic mild solutions* $x^1, x^2,\dots, x^m$ *with* $1\le m\le n$ *where* x^i *are periodic mild solutions of system (A.5) corresponding to initial conditions* $x^i(0)=x_0^i,\ i=1,2,\dots,m,$ *then*

1. *the adjoint system (A.19) also have* m *linearly independent periodic mild solutions* $y^1,y^2,\dots,y^m$.

2. *system (A.7) has a* T_0*-periodic mild solution if and only if*

$$\langle\, y\,,\,z\,\rangle=0, \quad (3.27)$$

where $y\in X^*$ *satisfying*

$$[I-U^*(T_0,0)]y=0 \quad (3.28)$$

and $z:=\int_0^{T_0}U(T_0,s)u(s)ds+\sum_{0\le\tau_k<T_0}U(T_0,\tau_k)c_k,$

or if and only if

$$\int_0^{T_0}\langle\, y(s),\,u(s)\,\rangle ds+\sum_{0\le\tau_k<T_0}\langle\, y(\tau_k),\,c_k\,\rangle=0. \quad (3.29)$$

Furthermore, let $x_a(t)$ be a particular T_0-periodic mild solution of system (A.7), each T_0-periodic mild solution of system (A.7) has the form

$$x(t)=x_a(t)+\sum_{i=1}^{m}\alpha_i x^i(t),$$

where $\alpha_i,\ i=1,2,\dots,m,$ are constants.

Proof. 1. Suppose system (A.5) have m linearly independent periodic mild solutions $x^1, x^2, \ldots, x^m$ with $1 \leq m \leq n$ where x^i are periodic mild solutions of system (A.5) corresponding to initial conditions $x^i(0) = x_0^i,\ i = 1, 2, \ldots, m.$ By Theorem A.3, this means that the equations

$$[I - U(T_0, 0)]x_0^i = 0 \tag{3.30}$$

have fixed points $x_0^1, x_0^2, \ldots, x_0^m$. Then from Theorem 2.1.21, we know that the following adjoint equations of (A.25)

$$[I - U^*(T_0, 0)]y_0^i = 0, \quad \text{where} \quad y_0^i = y^i(0) \tag{3.31}$$

also have m linearly independent solutions $y_0^1, y_0^2, \ldots, y_0^m$. So $y_0^1, y_0^2, \ldots, y_0^m$ are fixed points of $U^*(T_0, 0)$. Again by Theorem A.3, this means that system (A.19) have periodic mild solutions, say $y^1, y^2, \ldots, y^m$ where y^i are periodic mild solutions of system (A.5) corresponding to initial conditions $y^i(0) = y_0^i$, $i = 1, 2, \ldots, m$.

2. System (A.7) has a T_0-periodic mild solution $x(t)$ if and only if the equation

$$[I - U(T_0, 0)]x(0) = \int_0^{T_0} U(T_0, s)u(s)ds + \sum_{0 \leq \tau_k < T_0} U(T_0, \tau_k)c_k := z \tag{3.32}$$

has a solution $x(0)$. It follows from Theorem 2.1.18 that the above condition is equivalent to

$$\langle y\,,\, z \rangle = 0, \tag{3.33}$$

for any $y \in X^*$ satisfying

$$[I - U^*(T_0, 0)]y = 0 \tag{3.34}$$

From equation (A.28), we obtain

$$
\begin{aligned}
\langle\, y\,,\, z\,\rangle = 0 \quad \Leftrightarrow \quad & \langle\, y,\, \int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\,\rangle &= 0\\
\Leftrightarrow \quad & \langle\, y,\, \int_0^{T_0} U(T_0,s)u(s)ds\,\rangle + \langle\, y,\, \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\,\rangle &= 0\\
\Leftrightarrow \quad & \int_0^{T_0} \langle\, y,\, U(T_0,s)u(s)\,\rangle ds + \sum_{0\le\tau_k<T_0} \langle\, y,\, U(T_0,\tau_k)c_k\,\rangle &= 0\\
\Leftrightarrow \quad & \int_0^{T_0} \langle\, U^*(T_0,s)y,\, u(s)\,\rangle ds + \sum_{0\le\tau_k<T_0} \langle\, U^*(T_0,\tau_k)y,\, c_k\,\rangle &= 0\\
\Leftrightarrow \quad & \int_0^{T_0} \langle\, y(s),\, u(s)\,\rangle ds + \sum_{0\le\tau_k<T_0} \langle\, y(\tau_k),\, c_k\,\rangle &= 0
\end{aligned}
$$

from which we immediately get (A.24). This completes the proof. □

The following theorem guarantee the existence of periodic mild solution. The proof is based on boundedness property.

Theorem 3.3.5. *If system (A.7) has a bounded mild solution, then it has at least one T_0-periodic mild solution.*

Proof. Assume that $x(t)$ is a bounded mild solution of system (A.7). For any $t \geq 0,$ we have

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k,$$

where $x(0) = x_0$ and

$$x(T_0) = U(T_0,0)x_0 + \int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k.$$

Define $\quad z := \displaystyle\int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k$, then

$$x(T_0) = U(T_0,0)x_0 + z.$$

We know that the function $x(t+T_0)$ is also a solution of system (A.7) on $[T_0, 2T_0]$ for $t \in [0, T_0]$ and its value at $t = 0$ is $x(T_0)$. So

$$x(t+T_0) = U(T_0,0)x(T_0) + \int_0^t U(t,s)u(s)ds + \sum_{0 \leq \tau_k < t} U(t,\tau_k)c_k$$

and

$$x(2T_0) = U(T_0,0)x(T_0) + z = U^2(T_0,0)x_0 + [U(T_0,0)+I]z.$$

Proceeding in this way, we get

$$x(mT_0) = U^m(T_0,0)x_0 + \sum_{i=0}^{m-1} U^i(T_0,0)z \quad \text{for all } m \in \mathbb{N}. \tag{3.35}$$

By contradiction, we assume that (A.7) has no T_0-periodic mild solution. This means that the periodicity condition

$$x(T_0) = U(T_0,0)x_0 + z = x_0 \tag{3.36}$$

has no solution, i.e., the equation

$$[I - U(T_0,0)]x_0 = z \tag{3.37}$$

has no solution. By Theorem 2.1.18, this implies that there exists $y \in X^*$ such that

$$[I - U^*(T_0,0)]y = 0 \quad \text{and} \quad \langle y\,,\,z\rangle \neq 0. \tag{3.38}$$

The first condition means that $U^*(T_0,0)y = y$, so

$$U^{*^m}(T_0,0)y = y \quad \text{for all } m \in \mathbb{N}. \tag{3.39}$$

Assume that $\langle y\,,\,z\rangle = \gamma \neq 0$. Then from equation (A.30), we have

$$\begin{aligned}
\langle y\,,\,x(mT_0)\,\rangle &= \langle y\,,\,U^m(T_0,0)x_0\,\rangle + \sum_{i=0}^{m-1}\langle y\,,\,U^i(T_0,0)z\rangle \\
&= \langle U^{*^m}(T_0,0)y\,,\,x_0\,\rangle + \sum_{i=0}^{m-1}\langle U^{*^i}(T_0,0)y\,,\,z\rangle \\
&= \langle y\,,\,x_0\,\rangle + \sum_{i=0}^{m-1}\langle y\,,\,z\rangle \\
&= \langle y\,,\,x_0\,\rangle + m\gamma.
\end{aligned}$$

Letting $m \to \infty$, we obtain

$$\lim_{m\to\infty} \langle\, y\,,\, x(mT_0)\,\rangle = \infty. \tag{3.40}$$

Since $x(t)$ is bounded mild solution and $y \in X^*$, then

$$|\langle\, y\,,\, x(mT_0)\,\rangle| \le \|y\|_{X^*}\|x(mT_0)\|_X \le M\|y\|_{X^*} < \infty.$$

But $\lim_{m\to\infty} \langle\, y\,,\, x(mT_0)\,\rangle < \infty$. It contradicts (A.35) and the theorem is proved. □

Corollary 3.3.6.

1. *Assume that system (A.7) has no T_0-periodic mild solution, then all of its solutions are unbounded for* $t \ge 0$.

2. *Assume that system (A.7) has a unique bounded mild solution for* $t \ge 0$, *then this solution is T_0-periodic.*

3.4 Linear Impulsive Periodic Control Systems with Parameter Perturbations

We consider the following linear impulsive periodic control systems with parameter perturbations

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t) + p(t, x(t), \xi), & t \ne \tau_k, \\ \Delta x(t) = B_k x(t) + c_k + q_k(x(t), \xi), & t = \tau_k, \end{cases} \tag{3.41}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Suppose that system (3.41) satisfy assumptions (A1) and (A2).

We impose the following assumption for the remaining.

Assumption (A4) ;

(A4.1) $p(\cdot, x, \xi) \in PC([0, \infty)\,, X)$ such that $p(t + T_0, x, \xi) = p(t, x, \xi)$ for all

$(t, x, \xi) \in [0, \infty) \times \mathcal{B}_\rho \times [0, \xi_0]$.

(A4.2) $q_k \in C(\mathcal{B}_\rho \times [0, \xi_0], X)$ such that $q_{k+\sigma}(x, \xi) = q_k(x, \xi)$ for all $k \in \mathbb{N}$

(A4.3) For each $(t, x, \xi) \in [0, \infty) \times \mathcal{B}_\rho \times [0, \xi_0]$, there exists a nonnegative function $\chi(\xi)$ such that

$$\lim_{\xi \to 0} \chi(\xi) = \chi(0) = 0$$

and
$$\|p(t, x, \xi)\|_X \leq \chi(\xi), \quad \|q_k(x, \xi)\|_X \leq \chi(\xi) \tag{3.42}$$

for all $k \in \mathbb{N}$.

3.4.1 Definitions of Solutions

Definition 3.4.1. A function $x \in PC([0, \infty), X)$ is said to be a *mild solution* of system (3.41) with initial condition $x(0) = x_0 \in X$ and the input $u \in L^1_{loc}([0, \infty), X)$ if x is given by

$$\begin{aligned} x(t) = \ & U(t, 0)x_0 + \int_0^t U(t, s)[u(s) + p(s, x(s), \xi)]ds \\ & + \sum_{0 \leq \tau_k < t} U(t, \tau_k)[c_k + q_k(x(\tau_k), \xi)], \end{aligned} \tag{3.43}$$

for all $k \in \mathbb{N}$.

Definition 3.4.2. A function $x \in PC([0, \infty), X)$ is said to be a *periodic mild solution* of system (3.41) if it is a mild solution and there exists $T_0 > 0$ such that $x(t + T_0) = x(t)$ for all $t \geq 0$.

Definition 3.4.3. A function $x \in PC([0, \infty), X)$ is said to be a T_0*-periodic mild solution* of system (3.41) if it is a mild solution and $x(t + T_0) = x(t)$ for all $t \geq 0$.

From now on, we will find sufficient conditions for the existence of T_0-periodic mild solutions of system (3.41). We assume that system (A.5) has only trivial solution. Let $\xi = 0$, then system (3.41) has the same form as system (A.7) because of the fact from (3.42) that $p(t, x, 0) = 0$ and $q_k(x, 0) = 0$. It follows from Theorem A.7 that system (3.41) has the following T_0-periodic mild solution

$$\begin{aligned} x_{T_0}(t) = \; & U(t,0)[I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)u(s)\,ds \right. \\ & \left. + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\right) + \int_0^t U(t,s)u(s)ds \\ & + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k, \end{aligned} \tag{3.44}$$

where $U(t, s)$ is defined in (A.3). Then we have the following theorem to show that for small ξ system (3.41) has a T_0-periodic mild solution which is closed to $x_{T_0}(t)$.

3.4.2 Existence and Uniqueness of Periodic Mild Solutions

Theorem 3.4.1. *Let assumptions (A1), (A2) and (A4) hold. Assume that*

1. *system (A.5) has only trivial solution,*

2. *the following inequality is valid*

$$\rho_0 = \sup_{t\in[0,T_0]} \|x_{T_0}(t)\|_X < \rho \tag{3.45}$$

   where ρ be any positive real number,

3. *$p(t, x, \xi)$ and $q_k(x, \xi)$ satisfy Lipschitz conditions, i.e. for any (t, x, ξ), $(t, y, \xi) \in [0, \infty) \times \mathcal{B}_\rho \times [0, \xi_0]$, there exists a constant $N(\xi) > 0$ such that*

$$\|p(t, x, \xi) - p(t, y, \xi)\|_X \le N(\xi)\|x - y\|_X$$

   and
$$\|q_k(x, \xi) - q_k(y, \xi)\|_X \le N(\xi)\|x - y\|_X.$$

Then for any constant $\rho > \rho_0 > 0$, there exists a sufficiently small $\xi_0 > 0$ such that for every fixed $\xi \in [0, \xi_0]$ system (3.41) has a unique T_0-periodic mild *solution* $x^{\xi}_{T_0}(t)$ *satisfying*

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\|_X < \rho - \rho_0 \tag{3.46}$$

and

$$\lim_{\xi\to 0} x^{\xi}_{T_0}(t) = x_{T_0}(t) \tag{3.47}$$

uniformly on t.

Proof. Let $PC_{T_0}([0,\infty),X) := \{x \in PC([0,\infty),X) \mid x(t+T_0) = x(t), \forall t \geq 0\,\}$. Moreover, $PC_{T_0}([0,T_0],X)$ is a Banach space with the norm

$$\|x\|_{PC_{T_0}} = \sup_{t\in[0,T_0]} \|x(t)\|_X.$$

Let us define

$$\begin{aligned}
\mathcal{B} :=\ & \mathcal{B}(x_{T_0},\rho_1) = \{x \in PC_{T_0}([0,T_0],X) \mid \|x - x_{T_0}\|_{PC_{T_0}} \leq \rho_1 := \rho - \rho_0\} \\
L_1 =\ & \sup_{0\leq s\leq t\leq T_0} \|U(t,s)\|_{\mathcal{L}(X)} \\
L_2 =\ & \|[I - U(T_0,0)]^{-1}\|_{\mathcal{L}(X)}
\end{aligned} \tag{3.48}$$

and an operator $\Omega : \mathcal{B} \to PC_{T_0}([0,T_0]\,,\,X)$ such that

$$\begin{aligned}
\Omega(x)(t) :=\ & U(t,0)[I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)[u(s) + p(s,x(s),\xi)]\,ds\right. \\
& \left. + \sum_{0\leq\tau_k<T_0} U(T_0,\tau_k)[c_k + q_k(x(\tau_k),\xi)]\right) + \int_0^t U(t,s)[u(s) \\
& + p(s,x(s),\xi)]ds + \sum_{0\leq\tau_k<t} U(t,\tau_k)[c_k + q_k(x(\tau_k),\xi)].
\end{aligned} \tag{3.49}$$

We note that $\Omega(x) \in PC_{T_0}([0,\infty)\,,\,X)$. Since

$$\begin{aligned}
\Omega(x)(T_0) =\ & U(T_0,0)[I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)[u(s) + p(s,x(s),\xi)]\,ds\right. \\
& \left. + \sum_{0\leq\tau_k<T_0} U(T_0,\tau_k)[c_k + q_k(x(\tau_k),\xi)]\right) \\
& + \int_0^{T_0} U(T_0,s)[u(s),p(s,x(s),\xi)]ds \\
& + \sum_{0\leq\tau_k<T_0} U(T_0,\tau_k)[c_k + q_k(x(\tau_k),\xi)]
\end{aligned}$$

$$= \Big(U(T_0,0)[I-U(T_0,0)]^{-1}-I\Big)\left(\int_0^{T_0} U(T_0,s)[u(s)+p(s,x(s),\xi)]\,ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[c_k+q_k(x(\tau_k),\xi)]\right)$$

$$= [I-U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)[u(s)+p(s,x(s),\xi)]\,ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[c_k+q_k(x(\tau_k),\xi)]\right) = \Omega(x)(0).$$

From (A.37) and (A.40), we know that if $x\in\mathcal{B}$, then

$$\|x\|_{PC_{T_0}} \le \|x-x_{T_0}\|_{PC_{T_0}} + \|x_{T_0}\|_{PC_{T_0}} = \rho_1+\rho_0=\rho. \tag{3.50}$$

For any $x,y\in\mathcal{B}$, we have

$$\begin{aligned}
\|\Omega(x)-\Omega(y)\|_{PC_{T_0}} = {} & \sup_{t\in[0,T_0]} \Big\| U(t,0)[I-U(T_0,0)]^{-1} \\
& \left(\int_0^{T_0} U(T_0,s)[p(s,x(s),\xi)-p(s,y(s),\xi)]\,ds \right. \\
& \left. + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[q_k(x(\tau_k),\xi)-q_k(y(\tau_k),\xi)]\right) \\
& + \int_0^t U(t,s)[p(s,x(s),\xi)-p(s,y(s),\xi)]ds \\
& + \sum_{0\le\tau_k<t} U(t,\tau_k)[q_k(x(\tau_k),\xi)-q_k(y(\tau_k),\xi)]\Big\|_X \\
\le {} & \sup_{t\in[0,T_0]} \Big(\|U(t,0)\|_{\mathcal{L}(X)} \|[I-U(T_0,0)]^{-1}\|_{\mathcal{L}(X)} \\
& \left(\int_0^{T_0} \|U(T_0,s)\|_{\mathcal{L}(X)} \|p(s,x(s),\xi)-p(s,y(s),\xi)\|_X\,ds \right. \\
& \left. + \sum_{0\le\tau_k<T_0} \|U(T_0,\tau_k)\|_{\mathcal{L}(X)} \|q_k(x(\tau_k),\xi)-q_k(y(\tau_k),\xi)\|_X\right) \\
& + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|p(s,x(s),\xi)-p(s,y(s),\xi)\|_X ds \\
& + \sum_{0\le\tau_k<t} \|U(T_0,\tau_k)\|_{\mathcal{L}(X)} \|q_k(x(\tau_k),\xi)-q_k(y(\tau_k),\xi)\|_X\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le L_1L_2\Big(L_1T_0N(\xi)\|x-y\|_{PC_{T_0}} + L_1\sigma N(\xi)\|x-y\|_{PC_{T_0}}\Big)\\
&\quad +L_1T_0N(\xi)\|x-y\|_{PC_{T_0}} + L_1\sigma N(\xi)\|x-y\|_{PC_{T_0}}\\
&= \Big(L_1^2L_2T_0 + L_1^2L_2\sigma + L_1T_0 + L_1\sigma\Big)N(\xi)\|x-y\|_{PC_{T_0}}.
\end{aligned}
$$

So
$$\|\Omega(x)-\Omega(y)\|_{PC_{T_0}} \le LN(\xi)\|x-y\|_{PC_{T_0}}, \tag{3.51}$$

where $\quad L = L_1^2L_2T_0 + L_1^2L_2\sigma + L_1T_0 + L_1\sigma \quad$ and

$$
\begin{aligned}
\|\Omega(x_{T_0}) - x_{T_0}\|_{PC_{T_0}} &= \sup_{t\in[0,T_0]} \|U(t,0)[I-U(T_0,0)]^{-1}\\
&\quad \Big(\int_0^{T_0} U(T_0,s)p(s,x_{T_0}(s),\xi)\,ds\\
&\quad + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)q_k(x_{T_0}(\tau_k),\xi)\Big)\\
&\quad + \int_0^t U(t,s)p(s,x_{T_0}(s),\xi)ds\\
&\quad + \sum_{0\le\tau_k<t} U(t,\tau_k)q_k(x_{T_0}(\tau_k),\xi)\Big\|_X\\
&\le \sup_{t\in[0,T_0]} \|U(t,0)\|_{\mathcal{L}(X)}\|[I-U(T_0,0)]^{-1}\|_{\mathcal{L}(X)}\\
&\quad \Big(\int_0^{T_0} \|U(T_0,s)\|_{\mathcal{L}(X)}\|p(s,x_{T_0}(s),\xi)\|_X\,ds\\
&\quad + \sum_{0\le\tau_k<T_0} \|U(T_0,\tau_k)\|_{\mathcal{L}(X)}\|q_k(x_{T_0}(\tau_k),\xi)\|_X\Big)\\
&\quad + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)}\|p(s,x_{T_0}(s),\xi)\|_X ds\\
&\quad + \sum_{0\le\tau_k<t} \|U(t,\tau_k)\|_{\mathcal{L}(X)}\|q_k(x_{T_0}(\tau_k),\xi)\|_X\\
&\le L_1L_2\Big(L_1T_0\chi(\xi) + L_1\sigma\chi(\xi)\Big) + L_1T_0\chi(\xi) + L_1\sigma\chi(\xi)\\
&= \Big(L_1^2L_2T_0 + L_1^2L_2\sigma + L_1T_0 + L_1\sigma\Big)\chi(\xi)
\end{aligned}
$$

So
$$\|\Omega(x_{T_0}) - x_{T_0}\|_{PC_{T_0}} \le L\chi(\xi), \tag{3.52}$$

where $\quad L = L_1^2L_2T_0 + L_1^2L_2\sigma + L_1T_0 + L_1\sigma.$

Let us choose $\xi_0 > 0$ such that

$$\begin{aligned} \eta = L \sup_{\xi\in[0,\xi_0]} N(\xi) &< \quad 1, \\ L \sup_{\xi\in[0,\xi_0]} \chi(\xi) &\leq \quad \rho_1(1-\eta). \end{aligned} \tag{3.53}$$

Assume that $\xi \in [0, \xi_0]$, then it follows from (A.43), (A.44) and (A.45) that

$$\begin{aligned} \|\Omega(x) - \Omega(y)\|_{PC_{T_0}} &\leq \quad \eta\|x-y\|_{PC_{T_0}}, \\ \|\Omega(x_{T_0}) - x_{T_0}\|_{PC_{T_0}} &\leq \quad \rho_1(1-\eta). \end{aligned} \tag{3.54}$$

This implies that $\Omega(x^{\xi}_{T_0}) \in \mathcal{B}$ and $\Omega : \mathcal{B} \to \mathcal{B}$ is a contraction mapping. So Ω has a unique fixed point $x^{\xi}_{T_0} \in \mathcal{B}$ and satisfy

$$\begin{aligned} x^{\xi}_{T_0}(t) = \quad & U(t,0)[I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)[u(s) + p(s, x^{\xi}_{T_0}(s), \xi)]\, ds \right. \\ & \left. + \sum_{0\leq\tau_k<T_0} U(T_0,\tau_k)[c_k + q_k(x^{\xi}_{T_0}(\tau_k),\xi)]\right) \\ & + \int_0^t U(t,s)[u(s), p(s, x^{\xi}_{T_0}(s),\xi)]ds \\ & + \sum_{0\leq\tau_k<T_0} U(t,\tau_k)[c_k + q_k(x^{\xi}_{T_0}(\tau_k),\xi)]. \end{aligned} \tag{3.55}$$

Since

$$\begin{aligned} x^{\xi}_{T_0}(T_0) = \quad & U(T_0,0)[I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)[u(s) + p(s, x^{\xi}_{T_0}(s), \xi)]\, ds \right. \\ & \left. + \sum_{0\leq\tau_k<T_0} U(T_0,\tau_k)[c_k + q_k(x^{\xi}_{T_0}(\tau_k),\xi)]\right) \\ & + \int_0^{T_0} U(T_0,s)[u(s), p(s, x^{\xi}_{T_0}(s),\xi)]ds \\ & + \sum_{0\leq\tau_k<T_0} U(T_0,\tau_k)[c_k + q_k(x^{\xi}_{T_0}(\tau_k),\xi)] \\ = \quad & \Big(U(T_0,0)[I - U(T_0,0)]^{-1} - I\Big)\left(\int_0^{T_0} U(T_0,s)[u(s) + p(s, x^{\xi}_{T_0}(s), \xi)]\, ds \right. \\ & \left. + \sum_{0\leq\tau_k<T_0} U(T_0,\tau_k)[c_k + q_k(x^{\xi}_{T_0}(\tau_k),\xi)]\right) \end{aligned}$$

$$
\begin{aligned}
&= [I - U(T_0, 0)]^{-1} \left(\int_0^{T_0} U(T_0, s)[u(s) + p(s, x^{\xi}_{T_0}(s), \xi)]\, ds \right. \\
&\quad \left. + \sum_{0 \le \tau_k < T_0} U(T_0, \tau_k)[c_k + q_k(x^{\xi}_{T_0}(\tau_k), \xi)] \right) \\
&= x^{\xi}_{T_0}(0),
\end{aligned}
$$

$x^{\xi}_{T_0}(t)$ is a T_0-periodic mild solution of system (3.41) which satisfies the estimate (A.38) because

$$
\begin{aligned}
x^{\xi}_{T_0} \in \mathcal{B} \Rightarrow &\ \|x^{\xi}_{T_0} - x_{T_0}\|_{PC_{T_0}} \le \rho_1 \\
\Rightarrow &\ \sup_{t \in [0, T_0]} \|x^{\xi}_{T_0}(t) - x_{T_0}(t)\|_X \le \rho_1 \\
\Rightarrow &\ \|x^{\xi}_{T_0}(t) - x_{T_0}(t)\|_X \le \rho_1 = \rho - \rho_0.
\end{aligned}
$$

Because we know that $\Omega(x^{\xi}_{T_0})(t) = x^{\xi}_{T_0}(t)$ for all $t \in [0, T_0]$.

Then $\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\|_X = \|\Omega(x^{\xi}_{T_0})(t) - x_{T_0}(t)\|_X \le L\chi(\xi)$.

Letting $\xi \to 0$, we obtain (A.39). This completes the proof. □

The following definition and lemma will be used in the proof of Theorem A.13.

Definition 3.4.4. A set $\mathcal{S} \subset PC([0, T_0], X)$ is *quasiequicontinuous* in $[0, T_0]$ if for any $\delta > 0$ there exists $\varepsilon > 0$ such that if $x \in \mathcal{S}, t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap [0, T_0], k \in \mathbb{N}$ and $|t_1 - t_2| < \varepsilon$, then $\|x(t_1) - x(t_2)\|_X < \delta$.

Lemma 3.4.2. A set $\mathcal{S} \subset PC([0, T_0], X)$ is relatively compact if and only if

1. $\mathcal{S}$ is bounded for each $x \in \mathcal{S}$,

2. $\mathcal{S}$ is quasiequicontinuous in $[0, T_0]$.

Proof. The proof can be found in D.D. Bainov and P.S. Simeonov (1993). □

Theorem 3.4.3. *Let assumptions (A1), (A2) and (A4) hold. Assume that*

1. *system (A.5) has only trivial solution,*

2. *the following inequality is valid*

$$\rho_0 = \sup_{t\in[0,\infty]} \|x_{T_0}(t)\|_X < \rho \tag{3.56}$$

Then for any constant $\rho > \rho_0 > 0$, *there exists a sufficiently small* $\xi_0 > 0$ *such that for every fixed* $\xi \in [0, \xi_0]$ *system (3.41) has a unique* T_0*-periodic mild solution* $x^{\xi}_{T_0}(t)$ *satisfying*

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\|_X \leq \rho - \rho_0. \tag{3.57}$$

Proof. As in the proof of Theorem A.11, we determine the number $\rho_1 = \rho - \rho_0$, the Banach space $PC_{T_0}([0, T_0], X)$, the set $\mathcal{B} := \mathcal{B}(x_{T_0}; \rho_1)$ and the operator $\Omega : \mathcal{B} \to PC_{T_0}([0, T_0], X)$ is defined in (A.41). Obviously, $\mathcal{B}$ is a non-empty bounded closed and convex set. It follows from the condition in (A.42) that if $x \in \mathcal{B}$, then $\|x\|_{PC_{T_0}} \leq \rho$. For any $x \in \mathcal{B}$, we have

$$\begin{aligned}
\|\Omega(x) - x_{T_0}\|_{PC_{T_0}} = \ & \sup_{t\in[0,T_0]} \Big\| U(t,0)[I - U(T_0, 0)]^{-1} \\
& \left(\int_0^{T_0} U(T_0, s)p(s, x(s), \xi)\, ds \right. \\
& \left. + \sum_{0\leq\tau_k<T_0} U(T_0, \tau_k)q_k(x(\tau_k), \xi) \right) \\
& + \int_0^{t} U(t, s)p(s, x(s), \xi)ds \\
& + \sum_{0\leq\tau_k<T_0} U(t, \tau_k)q_k(x(\tau_k), \xi) \Big\|_X
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\in[0,T_0]} \|U(t,0)\|_{\mathcal{L}(X)} \|[I-U(T_0,0)]^{-1}\|_{\mathcal{L}(X)} \\
&\quad \Big(\int_0^{T_0} \|U(T_0,s)\|_{\mathcal{L}(X)} \|p(s,x(s),\xi)\|_X \, ds \\
&\quad + \sum_{0\leq\tau_k<T_0} \|U(T_0,\tau_k)\|_{\mathcal{L}(X)} \|q_k(x(\tau_k),\xi)\|_X \Big) \\
&\quad + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|p(s,x(s),\xi)\|_X ds \\
&\quad + \sum_{0\leq\tau_k<T_0} \|U(t,\tau_k)\|_{\mathcal{L}(X)} \|q_k(x(\tau_k),\xi)\|_X \\
&\leq \Big(L_1^2 L_2 T_0 + L_1^2 L_2 \sigma + L_1 T_0 + L_1 \sigma\Big)\chi(\xi).
\end{aligned}
$$

So
$$\|\Omega(x) - x_{T_0}\|_{PC_{T_0}} \leq L\chi(\xi). \tag{3.58}$$

where $\quad L = L_1^2 L_2 T_0 + L_1^2 L_2 \sigma + L_1 T_0 + L_1 \sigma.$

Let us choose $\xi \in [0,\xi_0]$ such that

$$L \sup_{\xi\in[0,\,\xi_0]} \chi(\xi) \leq \rho_1. \tag{3.59}$$

Then for $\xi \in [0,\xi_0]$, we have

$$\|\Omega(x) - x_{T_0}\|_{PC_{T_0}} \leq L\chi(\xi) \leq \rho_1, \tag{3.60}$$

From which we know that $\Omega(x) \in \mathcal{B}$ and therefore $\Omega : \mathcal{B} \to \mathcal{B}$.

Next, we want show that Ω satisfies the assumptions of Corollary 2.1.12.

1) The map $\Omega : \mathcal{B} \to \mathcal{B}$ is a continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $\mathcal{B}$. Then we have

$$
\begin{aligned}
\|\Omega(x_n) - \Omega(x)\|_{PC_{T_0}} = &\sup_{t\in[0,T_0]} \|U(t,0)[I-U(T_0,0)]^{-1} \\
&\Big(\int_0^{T_0} U(T_0,s)[p(s,x_n(s),\xi) - p(s,x(s),\xi)] \, ds \\
&+ \sum_{0\leq\tau_k<T_0} U(T_0,\tau_k)[q_k(x_n(\tau_k),\xi) - q_k(x(\tau_k),\xi)]\Big)
\end{aligned}
$$

$$
\begin{aligned}
& + \int_0^t U(t,s)[p(s,x_n(s),\xi) - p(s,x(s),\xi)]ds \\
& + \sum_{0 \le \tau_k < t} U(t,\tau_k)[q_k(x_n(\tau_k),\xi) - q_k(x(\tau_k),\xi)] \Bigg\|_X \\
\le \; & \sup_{t\in[0,T_0]} \|U(t,0)\|_{\mathcal{L}(X)} \|[I - U(T_0,0)]^{-1}\|_{\mathcal{L}(X)} \\
& \Bigg(\int_0^{T_0} \|U(T_0,s)\|_{\mathcal{L}(X)} \|p(s,x(s),\xi)\|_X \; ds \\
& + \sum_{0\le\tau_k<T_0} \|U(T_0,\tau_k)\|_{\mathcal{L}(X)} \|q_k(x(\tau_k),\xi)\|_X \Bigg) \\
& + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|p(s,x(s),\xi)\|_X ds \\
& + \sum_{0\le\tau_k<T_0} \|U(t,\tau_k)\|_{\mathcal{L}(X)} \|q_k(x(\tau_k),\xi)\|_X \\
\le \; & \sup_{t\in[0,T_0]} \Bigg(\|U(t,0)\|_{\mathcal{L}(X)} \|[I - U(T_0,0)]^{-1}\|_{\mathcal{L}(X)} \\
& \Bigg(\int_0^{T_0} \|U(T_0,s)\|_{\mathcal{L}(X)} \|p(s,x_n(s),\xi) - p(s,x(s),\xi)\|_X \; ds \\
& + \sum_{0\le\tau_k<T_0} \|U(T_0,\tau_k)\|_{\mathcal{L}(X)} \|q_k(x_n(\tau_k),\xi) - q_k(x(\tau_k),\xi)\|_X \Bigg) \\
& + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|p(s,x_n(s),\xi) - p(s,x(s),\xi)\|_X ds \\
& + \sum_{0\le\tau_k<t} \|U(T_0,\tau_k)\|_{\mathcal{L}(X)} \|q_k(x_n(\tau_k),\xi) - q_k(x(\tau_k),\xi)\|_X \Bigg) \\
\le \; & L_1 L_2 \Big(L_1 T_0 N(\xi) \|x_n - x\|_{PC_{T_0}} + L_1 \sigma N(\xi) \|x_n - x\|_{PC_{T_0}} \Big) \\
& + L_1 T_0 N(\xi) \|x_n - x\|_{PC_{T_0}} + L_1 \sigma N(\xi) \|x_n - x\|_{PC_{T_0}} \\
= \; & \Big(L_1^2 L_2 T_0 + L_1^2 L_2 \sigma + L_1 T_0 + L_1 \sigma \Big) N(\xi) \|x_n - x\|_{PC_{T_0}},
\end{aligned}
$$

which tends to zero as $n \to \infty$.

2) $\Omega(\mathcal{B})$ is uniformly bounded

By assumption and equation (A.52),

$$
\|\Omega(x)\|_{PC_{T_0}} \le \|\Omega(x) - x_{T_0}\|_{PC_{T_0}} + \|x_{T_0}\|_{PC_{T_0}} \le \rho_1 + \rho_0 = \rho, \tag{3.61}
$$

from which we know that $\Omega(\mathcal{B})$ is uniformly bounded.

3) $\Omega(\mathcal{B})$ is quasiequicontinuous in $[0, T_0]$.

Let $x \in \mathcal{B}_\rho$ and $t_1, t_2 \in (\tau_{i-1}, \tau_i] \cap [0, T_0]$, $i = 1, 2, \ldots, \sigma$, where $\tau_0 = 0$ and $\tau_\sigma = T_0$. For $0 < \varepsilon < t_1 < t_2 \leq T_0$, then we have

$$\begin{aligned}
\|(\Omega x)(t_1) - (\Omega x)(t_2)\|_X \leq\ & \|U(t_1, 0) - U(t_2, 0)\|_{\mathcal{L}(X)} \|[I - U(T_0, 0)]^{-1}\|_{\mathcal{L}(X)} \\
& \left(\int_0^{T_0} \|U(T_0, s)\|_{\mathcal{L}(X)} \|u(s) + p(s, x(s), \xi)\|_X \, ds \right. \\
& \left. + \sum_{0 \leq \tau_k < T_0} \|U(T_0, \tau_k)\|_{\mathcal{L}(X)} \|c_k + q_k(x(\tau_k), \xi)\|_X \right) \\
& + \int_0^{t_1 - \epsilon} \|U(t_1, s) - U(t_2, s)\|_{\mathcal{L}(X)} \|u(s) + p(s, x(s), \xi)\|_X ds \\
& + \int_{t_1 - \epsilon}^{t_1} \|U(t_1, s) - U(t_2, s)\|_{\mathcal{L}(X)} \|u(s) + p(s, x(s), \xi)\|_X ds \\
& + \int_{t_1}^{t_2} \|U(t_2, s)\|_{\mathcal{L}(X)} \|u(s) + p(s, x(s), \xi)\|_X ds \\
& + \sum_{0 \leq \tau_k < t} \|U(t_1, \tau_k) - U(t_2, \tau_k)\|_{\mathcal{L}(X)} \|c_k + q_k(x(\tau_k), \xi)\|_X,
\end{aligned}$$

from which we know that for any $\delta > 0$, there exists $\varepsilon > 0$ such that if $t_1 - t_2 < \varepsilon$, then $\|(\Omega x)(t_1) - (\Omega x)(t_2)\|_X < \delta$. Thus $\Omega(\mathcal{B})$ is quasiequicontinuous.

From Lemma A.12, we know that the following set

$$\mathcal{S} = \{y \in \mathcal{B} \mid y = \Omega(x),\ x \in \mathcal{B}\}.$$

is relatively compact in $PC_{T_0}([0, T_0], X)$. Applying Corollary 2.1.12, it follows that the operator Ω has a fixed point $x^\xi_{T_0} \in \mathcal{B}$ and satisfy

$$\begin{aligned}
x^\xi_{T_0}(t) =\ & U(t, 0)[I - U(T_0, 0)]^{-1} \left(\int_0^{T_0} U(T_0, s)[u(s) + p(s, x^\xi_{T_0}(s), \xi)] \, ds \right. \\
& \left. + \sum_{0 \leq \tau_k < T_0} U(T_0, \tau_k)[c_k + q_k(x^\xi_{T_0}(\tau_k), \xi)] \right) \\
& + \int_0^t U(t, s)[u(s), p(s, x^\xi_{T_0}(s), \xi)] ds \\
& + \sum_{0 \leq \tau_k < T_0} U(t, \tau_k)[c_k + q_k(x^\xi_{T_0}(\tau_k), \xi)].
\end{aligned}$$

Since

$$
\begin{aligned}
x^{\xi}_{T_0}(T_0) = \; & U(T_0,0)[I-U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)[u(s)+p(s,x^{\xi}_{T_0}(s),\xi)]\,ds\right. \\
& \left. + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[c_k+q_k(x^{\xi}_{T_0}(\tau_k),\xi)]\right) \\
& + \int_0^{T_0} U(T_0,s)[u(s),p(s,x^{\xi}_{T_0}(s),\xi)]ds \\
& + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[c_k+q_k(x^{\xi}_{T_0}(\tau_k),\xi)] \\
= \; & \Big(U(T_0,0)[I-U(T_0,0)]^{-1}-I\Big)\left(\int_0^{T_0} U(T_0,s)[u(s)+p(s,x^{\xi}_{T_0}(s),\xi)]\,ds\right. \\
& \left. + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[c_k+q_k(x^{\xi}_{T_0}(\tau_k),\xi)]\right) \\
= \; & [I-U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)[u(s)+p(s,x^{\xi}_{T_0}(s),\xi)]\,ds\right. \\
& \left. + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[c_k+q_k(x^{\xi}_{T_0}(\tau_k),\xi)]\right) \\
= \; & x^{\xi}_{T_0}(0),
\end{aligned}
$$

$x^{\xi}_{T_0}(t)$ is a T_0-periodic mild solution of system (3.41). This completes the proof.

□

CHAPTER IV

SEMILINEAR PERIODIC SYSTEMS WITH IMPULSES

In this chapter, we study the existence and uniqueness of periodic mild solution of semilinear impulsive periodic systems, semilinear impulsive periodic control systems and semilinear impulsive periodic control systems with parameter perturbations.

4.1 Semiliner Impulsive Periodic Systems

We consider the following semilinear impulsive periodic systems

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{4.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Furthermore, we suppose that A is infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$.

In addition to assumption (A1), we introduce the following assumption.

Assumption (A5) ;

(A5) $f : [0, \infty) \times X \to X$ is an operator such that $f(t + T_0, x) = f(t, x)$ and $t \mapsto f(t, x)$ is strongly measurable. For every $\rho > 0$, there exist constants $K_1(\rho),\ K_2(\rho) > 0$ such that

$$\|f(t, x)\|_X \leq K_1(\rho)$$

and
$$\|f(t, x) - f(t, y)\|_X \leq K_2(\rho)\|x - y\|_X,$$

for all $t \geq 0$ and all $x, y \in X$ such that $\|x\|_X,\ \|y\|_X \leq \rho$.

4.1.1 Definitions of Solutions

Definition 4.1.1. A function $x \in PC([0,\infty), X)$ is said to be a *mild solution* of system (4.1) with initial condition $x(0) = x_0 \in X$ if x is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))ds \tag{4.2}$$

Definition 4.1.2. A function $x \in PC([0,\infty); X)$ is said to be a *periodic mild solution* of system (4.1) if it is a mild solution and there exists $T_0 > 0$ such that $x(t+T_0) = x(t)$ for all $t \geq 0$.

Definition 4.1.3. A function $x \in PC([0,\infty); X)$ is said to be a T_0*-periodic mild solution* of system (4.1) if it is a mild solution and $x(t+T_0) = x(t)$ for all $t \geq 0$.

4.1.2 Existence and Uniqueness of Periodic Mild Solutions

At first, we consider the following semilinear system without impulses,

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t,x(t)), \\ x(0) = x_0, \end{cases} \tag{4.3}$$

where A is the infinitesimal generator a compact semigroup $\{T(t), t > 0\}$ in X and $f : [0,\infty) \times X \to X$.

Definition 4.1.4. A function $x \in C([0,T_0], X)$ is said to be a mild solution of system (4.3) if x is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s,x(s))ds.$$

Theorem 4.1.1. *Suppose (a1) A be the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$, (a2) $f : [0,\infty) \times X \to X$ is continuous and map a bounded set of $[0,\infty) \times X$ into a bounded set of X. Then for every $x_0 \in X$ system*

(4.3) has a mild solution $x \in C([0,t_{\max}),X)$, *where* $[0,t_{\max})$ *is a maximal interval solution of existence. Further, if* $t_{\max} < \infty$, *then* $\lim_{t \to t_{\max}} \|x(t)\|_X = \infty$.

Proof. Let $T_0 \in (0,\infty)$. For any $\tau' > 0$, $\rho > 0$ be such that

$$\mathcal{B}_\rho(x_0) := \{x \in X \mid \|x - x_0\|_X \le \rho\}.$$

Then there exists a constant $N > 0$ such that $\|f(t,x)\|_X \le N$ for $0 \le t \le \tau'$ and $x \in \mathcal{B}_\rho(x_0)$. Clearly, due to the strong continuous of C_0-semigroup of $\{T(t), t \ge 0\}$, there exists $\tau'' > 0$ such that

$$\|T(t)x_0 - x_0\|_X < \frac{\rho}{2}, \quad \text{for} \quad t \in [0,\tau''].$$

Let $M = \sup_{0 \le s \le t \le T_0} \|T(t-s)\|_{\mathcal{L}(X)}$ and define

$$Y_\rho := \{x \in C([0,t_1],X) \mid x(t) \in \mathcal{B}_\rho(x_0),\ t \in [0,t_1]\},$$

where $t_1 = \min\left\{\tau',\ \tau'',\ T_0,\ \dfrac{\rho}{2MN}\right\}$. Then Y_ρ is a closed bounded convex subset of $Y = C([0,t_1],X)$. We define a mapping $F : Y \to Y_\rho$ as follow,

$$(Fx)(t) = T(t)x_0 + \int_0^t T(t-s)f(s,x(s))ds. \tag{4.4}$$

Since

$$\begin{aligned}\|(Fx)(t) - x_0\|_X \le \ & \|T(t)x_0 - x_0\|_X + \int_0^t \|T(t-s)\|_{\mathcal{L}(X)} \|f(s,x(s))\|_X ds \\ \le \ & \frac{\rho}{2} + tMN \ \le \ \rho,\end{aligned}$$

so $FY_\rho \subseteq Y_\rho$. Next, we want to show that F has a fixed point. According to Schauder's fixed point theorem, first we want to show that $F : Y_\rho \to Y_\rho$ is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in Y_ρ. Then, we have

$$\begin{aligned}\|(Fx_n)(t) - (Fx)(t)\|_X \le \ & \int_0^t \|T(t-s)\|_{\mathcal{L}(X)} \|f(s,x_n(s)) - f(s,x(s))\|_X ds \\ \le \ & M \int_0^t \|f(s,x_n(s)) - f(s,x(s))\|_X ds.\end{aligned} \tag{4.5}$$

By continuity of f, we obtain the right hand side of (4.5) tends to zero as $n \to \infty$. Thus, F is continuous.

Moreover, FY_ρ is a relatively compact subset of Y_ρ. By Ascoli-Arzela theorem states that a subset $W \subseteq C(I, X)$ is relatively compact if and only if

(a) its t-section $W(t) = \{(Fx)(t) \mid x \in W\}$ is relatively compact subset of X,

(b) the set W is equicontinuous.

Define $W = FY_\rho$ and $W(t) = \{(Fx)(t) \mid x \in Y_\rho\}$ for $t \in [0, t_1]$. Clearly, $W(0) = \{x_0\}$ is compact. Let $0 < t \leq t_1$ be fixed and let $0 < \varepsilon < t$. For $x \in Y_\rho$, we define

$$\begin{aligned}(F_\varepsilon x)(t) = &\; T(t)x_0 + \int_0^{t-\varepsilon} T(t-s)f(s, x(s))ds \\ = &\; T(t)x_0 + T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s, x(s))ds.\end{aligned}$$

Since $T(t)$ is a compact operator, the set $W_\varepsilon(t) = \{(F_\varepsilon x)(t) \mid x \in Y_\rho\}$ is relatively compact in X for every $\varepsilon,\ 0 < \varepsilon < t$. Furthermore, for $x \in Y_\rho$ we have

$$\begin{aligned}\sup_{x \in Y_\rho} \|(Fx)(t) - (F_\varepsilon x)(t)\|_X \leq &\; \sup_{x \in Y_\rho} \left\| \int_{t-\varepsilon}^t T(t-s)f(s, x(s))ds \right\|_X \\ \leq &\; \varepsilon MN.\end{aligned}$$

This shows that the set $W(t)$ can be approximated to an arbitrary degree of accuracy by a relatively compact set. Hence $W(t)$ itself is relatively compact. For equicontinuity, we note that for $0 < z_1 < z_2 \leq t$ and $x \in Y_\rho$,

$$\begin{aligned}\|(Fx)(z_1) - (Fx)(z_2)\|_X \leq &\; \|T(z_1) - T(z_2)\|_{\mathcal{L}(X)} \|x_0\|_X \\ &+ \int_0^{z_1} \|T(z_1 - s) - T(z_2 - s)\|_{\mathcal{L}(X)} \|f(s, x(s))\|_X ds \\ &+ \int_{z_1}^{z_2} \|T(z_2 - s)\|_{\mathcal{L}(X)} \|f(s, x(s))\|_X ds \\ \leq &\; \|T(z_1) - T(z_2)\|_{\mathcal{L}(X)} \|x_0\|_X \\ &+ N \int_0^{z_1} \|T(z_1 - s) - T(z_2 - s)\|_{\mathcal{L}(X)} ds \\ &+ MN(z_2 - z_1).\end{aligned}$$

from which we know that for any $\delta > 0$, there exists $\varepsilon > 0$ such that if $|z_1 - z_2| < \varepsilon$, then $\|(Fx)(z_1) - (Fx)(z_2)\|_X < \delta$. Therefore, W is equicontinuous. This implies that FY_ρ is a relatively compact subset of Y_ρ and hence F has a fixed point in Y_ρ which is a mild solution of system (4.3) on $[0, t_1]$.

We note that a mild solution x of system (4.3) defined on a closed interval $[0, t_1]$ can be extended to a larger interval $[0, t_1 + \delta]$, $\delta > 0$, by defining $y(t) = x(t_1 + t)$ where $y(t)$ is a mild solution of the following system

$$\begin{cases} \dot{y}(t) = Ay(t) + f(t, y(t)), \\ y(0) = x(t_1). \end{cases} \tag{4.6}$$

The existence of positive constant $\delta > 0$ is guarantee by the above assertion. Repeating this procedure, one continues the solution till time $t_{\max}$ where $[0, t_{\max})$ is the maximal interval of existence of mild solution. Thus system (4.3) has a unique solution $x \in C([0, t_{\max}), X)$. If $t_{\max} < \infty$, then $\lim\limits_{t \to t_{\max}} \|x(t)\|_X = \infty$. If not, there exists a sequence $\{t_n\}$ such that $t_n \to t_{\max}$ and $\|x(t_n)\|_X \leq \beta$ for all n. Taking n sufficiently large, so that $\{t_n\}$ near enough to $t_{\max}$, one can use the previous arguments to extend the solution beyond $t_{\max}$ which is a contradiction to the definition of $t_{\max}$. This proves the theorem. □

Remark 4.1. It is not difficult to see that Theorem 4.1.1 holds if $T_0 = \infty$.

Corollary 4.1.2. *Suppose assumptions of Theorem 4.1.1 hold. If there exists a constant $\beta > 0$ such that $\|x(t)\|_X \leq \beta$ hold for every mild solution x. Then system (4.3) has a global mild solution $x \in C([0, T_0], X)$.*

Theorem 4.1.3. *Suppose A be the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$. If assumptions (A1) and (A5) hold, then system (4.1) has a unique mild solution $x \in PC([0, T_0], X)$.*

Proof. For $t \in [0, \tau_1]$, Theorem 4.1.1 implies that system

$$\dot{x}(t) = Ax(t) + f(t, x(t)), \quad 0 < t \le \tau_1, \quad x(0) = x_0, \tag{4.7}$$

has a mild solution on $I_1 = [0, \tau_1]$ which satisfies

$$x_1(t) = \ T(t)x_0 + \int_0^t T(t-s) f(s, x_1(s)) ds, \quad t \in [0, \tau_1]. \tag{4.8}$$

Now, define

$$x_1(\tau_1) = T(\tau_1)x_0 + \int_0^{\tau_1} T(\tau_1 - s) f(s, x_1(s)) ds, \tag{4.9}$$

so that $x_1(\cdot)$ is left continuous at τ_1.

Next, on $I_2 = (\tau_1, \tau_2]$, consider system

$$\dot{x}(t) = Ax(t) + f(t, x(t)), \quad \tau_1 < t < \tau_2, \quad x_1(\tau_1^+) = (I + B_1)x_1(\tau_1), \tag{4.10}$$

Since $x_1 \in X$, we can use Theorem 4.1.1 again to get a mild solution on $(\tau_1, \tau_2]$ which satisfying

$$x_2(t) = T(t - \tau_1)x_1(\tau_1^+) + \int_{\tau_1}^t T(t-s) f(s, x_2(s)) ds. \tag{4.11}$$

Now, define $x_2(\tau_2)$ accordingly so that $x_2(\cdot)$ is left continuous at τ_2. It is easy to see that Theorem 4.1.1 can be applied to interval $(\tau_1, \tau_2]$ to verify that $x_2(\tau_2) \in X$. Repeat the procedure above, use step-by-step approach on intervals $I_k = (\tau_{k-1}, \tau_k],\ k = 3, 4, \ldots, \sigma$ $(\tau_\sigma = T_0)$ to get a mild solutions

$$x_k(t) = \ T(t - \tau_{k-1})x_{k-1}(\tau_{k-1}^+) + \int_{\tau_{k-1}}^t T(t-s) f(s, x_k(s)) ds.$$

for $t \in (\tau_{k-}, \tau_k]$ and define $x_k(\tau_k)$ accordingly with $x_k(\cdot)$ left continuous at τ_k and $x_k(\tau_k) \in X,\ k = 1, 2, \ldots, \sigma$.

Thus we obtain $x \in PC([0, T_0], X)$ is a mild solution of system (4.1) and given by

$$x(t) = \begin{cases} x_1(t), & 0 \le t \le \tau_1, \\ x_k(t), & \tau_{k-1} < t \le \tau_k,\ k = 2, 3, \ldots, \sigma. \end{cases}$$

Next, by mathematical induction we can show that (4.2) is satisfied on $[0,T_0]$. First, (4.2) is satisfied on $[0,\tau_1]$. If (4.2) is satisfied on $(\tau_{k-1},\tau_k]$, then for $t \in (\tau_k,\tau_{k+1}]$,

$$\begin{aligned}
x(t) = {} & x_{k+1}(t) = T(t-\tau_k)x_k(\tau_k^+) + \int_{\tau_k}^t T(t-s)f(s,x_{k+1}(s))ds \\
= {} & T(t-\tau_k)(I+B_k)x(\tau_k) + \int_{\tau_k}^t T(t-s)f(s,x_{k+1}(s))ds \\
= {} & T(t-\tau_k)(I+B_k)\Big[U(\tau_k,0)x_0 + \int_0^{\tau_k} U(\tau_k,s)f(s,x(s))ds\Big] \\
& + \int_{\tau_k}^t T(t-s)f(s,x_{k+1}(s))ds \\
= {} & U(t,0)x_0 + \int_0^{\tau_k} U(t,s)f(s,x(s))ds + \int_{\tau_k}^t U(t,s)f(s,x(s))ds \\
= {} & U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))ds.
\end{aligned}$$

Thus (4.2) is also true on $(\tau_k,\tau_{k+1}]$. Therefore (4.2) is true on $[0,T_0]$.

Next, we want to show that a mild solution is unique on $PC([0,T_0],X)$. Suppose that x, y are mild solutions of system (4.1) on $PC([0,T_0],X)$. Then by Corollary 3.2.4, we have

$$\begin{aligned}
\|x(t)-y(t)\|_X \leq {} & \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|f(s,x(s)) - f(s,y(s))\|_X ds \\
\leq {} & Ke^{\nu t} \int_0^t e^{-\nu s} \|x(s)-y(s)\|_X ds.
\end{aligned}$$

It follows from Gronwall Lemma, we obtain $\|x(t)-y(t)\| = 0$ for all $t \in [0,T_0]$. That is, $x = y$. Therefore, system (4.1) has a unique mild solution. This completes the proof. □

We consider the following system,

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t,x(t)), \quad t \geq 0 \\ x(0) = x_0, \end{cases} \tag{4.12}$$

and we suppose that it has a global mild solution $x(t)$.

We also consider the following system,

$$\begin{cases} \dot{y}(t) &= Ay(t) + f(t, x(t)), \quad t \geq 0 \\ y(0) &= x(0). \end{cases} \tag{4.13}$$

By Lemma A.5, system (4.13) has a unique mild solution $y(t)$.

Let $P : C([0, T_0], X) \to X$ be the Poincar mapping, defined by

$$Px = y(T_0) \tag{4.14}$$

Finally, we consider the following system,

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad t \geq 0 \\ x(0) = Px, \end{cases} \tag{4.15}$$

which by Lemma A.5 also has a unique mild solution $x(t)$.

Let $Q : C([0, T_0], X) \to C([0, T_0], X)$ be a mapping defined by

$$Qx = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds. \tag{4.16}$$

We are now in a position to state and prove the basic tool for the proof existence of periodic mild solution.

Theorem 4.1.4. *System (4.12) has a T_0-periodic mild solution if and only if the mapping Q has a fixed point.*

Proof. Let x be a T_0-periodic mild solution of system (4.12). Then x is clearly a T_0-periodic mild solution of system (4.13). Since x is T_0-periodic mild solution, $x(0) = x(T_0)$. Therefore $x(0) = x(T_0) = Px$, where x satisfy (4.15) and so $Qx = x$. Conversely, let x be a fixed point of Q. By definition, x satisfies (4.13) and since $x(0) = y(0)$. By Lemma A.5 , show that $x(t) \equiv y(t)$ and hence $x(T_0) = y(T_0)$. Since $Qx = x$, it follows from (4.15) that $x(0) = Px = y(T_0) = x(T_0)$. That is, $x(0) = x(T_0)$. The function $\psi(t) := x(t + T_0)$ is also a mild solution of (4.12).

Since f is T_0-periodic, $\dot{\psi}(t) = \dot{x}(t+T_0) = Ax(t+T_0) + f(t+T_0, x(t+T_0)) = A\psi(t) + f(t, \psi(t))$. Therefore $x(t) = x(t+T_0)$ for all $t \geq 0$. i.e., system (4.12) has a T_0-periodic mild solution. This completes the proof. □

Theorem 4.1.5. *If assumptions (A1) and (A5) hold, then system (4.1) has a unique T_0-periodic mild solution $x \in PC([0,T_0], X)$ and there exists a constant $\beta > 0$ such that*

$$\|x\|_{PC} \leq \beta.$$

Proof. Consider the following semilinear system without impulses,

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad t \in \gamma, \\ x(\tau) = \varphi, \quad \varphi \in X, \end{cases} \tag{4.17}$$

where γ is any subinterval of $[0, T_0]$ and τ is the left end point of γ.
Define $B = \{y \mid y \in C(\gamma, X),\ y(\tau) = \varphi\}$ and a mapping $F : B \to B$ by $y = Fx$, where y is a mild solution of the following system,

$$\begin{cases} \dot{y}(t) = Ay(t) + f(t, x(t)), \quad t \in \gamma, \\ y(\tau) = \varphi, \end{cases} \tag{4.18}$$

Similar to the proof Theorem 4.1.1, we can show that F is continuous and compact on B. Next, we want to show that there is a constant $\beta_1 > 0$ such that

$$\|x\|_{C(\gamma,X)} \leq \beta_1$$

for all $x \in B$ and $\lambda \in [0,1]$ satisfying $x = \lambda F x$.
Let $x \in C(\gamma, X)$. We consider the operator equation

$$x = \lambda F x, \quad \lambda \in [0,1]. \tag{4.19}$$

If x is a mild solution of equation (4.19), then we have

$$\begin{aligned}\|x(t)\|_X &\le \lambda\|T(t)\|_{\mathcal{L}(X)}\|x_0\|_X + \lambda\int_0^t \|T(t-s)\|_{\mathcal{L}(X)}\|f(s,x(s))\|_X ds \\ &\le \lambda M_1(\|x_0\|_X + K_1T_0) \\ &\le M_1(\|x_0\|_X + K_1T_0) := \beta_1, \quad \lambda\in[0,1],\end{aligned}$$

where $M_1 = \sup_{0\le s\le t\le T_0}\|T(t-s)\|_{\mathcal{L}(X)}$ and $\|f(t,x)\|_X \le K_1$.

That is, there exists a constant $\beta_1 > 0$ such that $\|x\|_{C(\gamma,X)} \le \beta_1$ for all $x\in B$ and $x = \lambda Fx$ where $\lambda\in[0,1]$. This shows that all mild solutions of (4.19) are bounded independently of $\lambda\in[0,1]$. By Leray-Schauder's fixed point theorem, F has a fixed point $x\in C(\gamma,X)$ which is a mild solution of system (4.17).

Next, we must show that a mild solution is unique on $C(\gamma,X)$. Suppose that x, y are mild solutions of system (4.17) on $C(\gamma,X)$. By Theorem 2.3.2(1), there exist constants $K\ge 1$ and $\omega\ge 0$ such that $\|T(t)\|_{\mathcal{L}(X)} \le Ke^{\omega t}$. Then we have

$$\begin{aligned}\|x(t)-y(t)\|_X &\le \int_0^t \|T(t-s)\|_{\mathcal{L}(X)}\|f(s,x(s))-f(s,y(s))\|_X ds \\ &\le Ke^{\omega t}\int_0^t e^{-\omega s}\|x(s)-y(s)\|_X ds.\end{aligned}$$

It follows from Gronwall Lemma, we obtain $\|x(t)-y(t)\| = 0$ for all $t\in\gamma$. That is, $x = y$. Therefore, system (4.17) has a unique mild solution. Further by Theorem 4.1.4 which implies that a mild solution is T_0-periodic.

Now we consider the partition of interval $I = [0,T_0]$. We define $\gamma_k = (\tau_{k-1},\tau_k]$, $k = 1,2,\ldots,\sigma$, where $\tau_0 = 0$ and $\tau_\sigma = T_0$. Consider any arbitrary interval, say, γ_k and $x(\tau_k^+) = (I+B_k)x(\tau_k)\in X$. By Theorem 4.1.1 and similar procedure of Theorem A.6, we obtain $x\in PC([0,T_0],X)$ is a unique mild solution of system (4.1) and given by

$$x(t) = \begin{cases} x_1(t), & 0 \le t \le \tau_1, \\ x_k(t), & \tau_{k-1} < t \le \tau_k, \ k = 2, 3, \dots, \sigma. \end{cases}$$

x is just T_0-periodic mild solution of system (4.1). By assumption (A5), one can verify the priori estimate of solution of system (4.1) that

$$\begin{aligned} \|x(t)\|_X \le & \ \|U(t,0)\|_{\mathcal{L}(X)} \|x_0\|_X + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|f(s,x(s))\|_X ds \\ \le & \ M\|x_0\|_X + MK_1T_0 := \beta, \end{aligned}$$

where $M = \sup\limits_{0 \le s \le t \le T_0} \|U(t,s)\|_{\mathcal{L}(X)}$ and $\|f(t,x)\|_X \le K_1$.

That is, there exists a constant $\beta > 0$ such that $\|x\|_{PC} \le \beta$. □

4.2 Semilinear Impulsive Periodic Control Systems

We consider the following semilinear impulsive periodic control systems

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) + u(t), & t \ne \tau_k, \\ \Delta x(t) = B_k x(t) + c_k, & t = \tau_k, \end{cases} \tag{4.20}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Suppose that system (4.20) satisfy the assumptions (A1), (A2) and (A5).

4.2.1 Definitions of Solutions

Definition 4.2.1. A function $x \in PC([0,\infty), X)$ is said to be a *mild solution* of impulsive system (4.20) with initial condition $x(0) = x_0 \in X$ if x is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)[f(s,x(s)) + u(s)]ds + \sum_{0<\tau_k<t} U(t,\tau_k)c_k. \tag{4.21}$$

Definition 4.2.2. A function $x \in PC([0,\infty); X)$ is said to be a *periodic mild solution* of system (4.20) if it is a mild solution and there exists $T_0 > 0$ such that $x(t + T_0) = x(t)$ for all $t \ge 0$.

Definition 4.2.3. A function $x \in PC([0,\infty);X)$ is said to be a T_0*-periodic mild solution* of system (4.20) if it is a mild solution and $x(t+T_0)=x(t)$ for all $t \geq 0$.

4.2.2 Existence and Uniqueness of Periodic Mild Solutions

At first, we consider the following semilinear systems without impulses

$$\begin{cases} \dot{x}(t) &= Ax(t)+f(t,x(t))+u(t), \\ x(0) = x_0, \end{cases} \tag{4.22}$$

where A is the infinitesimal generator a compact semigroup $\{T(t), t>0\}$ in X, $f:[0,\infty)\times X \to X$ and $u:[0,\infty)\to X$.

Definition 4.2.4. A function $x \in C([0,T_0],X)$ is said to be a *mild solution* of system (4.22) if x is given by

$$x(t) = \ T(t)x_0 + \int_0^t T(t-s)[f(s,x(s))+u(s)]ds.$$

Theorem 4.2.1. *Suppose (a1)* A *be the infinitesimal generator of a compact semigroup* $\{T(t), t>0\}$, *(a2)* $f:[0,\infty)\times X \to X$ *is continuous and map a bounded set of* $[0,\infty)\times X$ *into a bounded set of* X *and (a3)* $u:[0,\infty)\to X$ *is continuous and map a bounded set of* $[0,\infty)$ *into a bounded set of* X*. Then for every* $x_0 \in X$ *system (4.22) has a mild solution* $x \in C([0,t_{\max}),X)$, *where* $[0,t_{\max})$ *is a maximal interval solution of existence. Further, if* $t_{\max}<\infty$, *then* $\lim\limits_{t\to t_{\max}} \|x(t)\|_X = \infty$.

Proof. Similar to the proof of Theorem 4.1.1. □

Theorem 4.2.2. *Suppose* A *be the infinitesimal generator of a compact semigroup* $\{T(t), t>0\}$*. If assumptions (A1), (A2) and (A5) hold, then system (4.20) has a unique mild solution on* $x \in PC([0,T_0],X)$.

Proof. For $t \in [0, \tau_1]$, Theorem 4.2.1 implies that system

$$\begin{cases} \dot{x}(t) &= Ax(t) + f(t, x(t)) + u(t), \quad 0 < t \le \tau_1, \\ x(0) = x_0, \end{cases} \tag{4.23}$$

has a mild solution on $I_1 = [0, \tau_1]$ which satisfies

$$x_1(t) = \ T(t)x_0 + \int_0^t T(t-s)[f(s, x_1(s)) + u(s)]ds, \quad t \in [0, \tau_1]. \tag{4.24}$$

Now, define

$$x_1(\tau_1) = T(\tau_1)x_0 + \int_0^{\tau_1} T(\tau_1 - s)[f(s, x_1(s)) + u(s)]ds, \tag{4.25}$$

so that $x_1(\cdot)$ is left continuous at τ_1.

Next, on $I_2 = (\tau_1, \tau_2]$, consider system

$$\begin{cases} \dot{x}(t) &= Ax(t) + f(t, x(t)) + u(t), \quad \tau_1 < t < \tau_2, \\ x_1(\tau_1^+) &= (I + B_1)x_1(\tau_1) + c_1. \end{cases} \tag{4.26}$$

Since $x_1 \in X$, we can use Theorem 4.2.1 again to get a mild solution on $(\tau_1, \tau_2]$ which satisfying

$$x_2(t) = T(t - \tau_1)x_1(\tau_1^+) + \int_{\tau_1}^t [T(t-s)f(s, x_2(s)) + u(s)]ds. \tag{4.27}$$

Now, define $x_2(\tau_2)$ accordingly so that $x_2(\cdot)$ is left continuous at τ_2. It is easy to see that Theorem 4.2.1 can be applied to interval $(\tau_1, \tau_2]$ to verify that $x_2(\tau_2) \in X$. Repeat the procedure above, use step-by-step approach on intervals $I_k = (\tau_{k-1}, \tau_k]$, $k = 3, 4, \ldots, \sigma$ $(\tau_\sigma = T_0)$ to get a mild solutions

$$x_k(t) = \ T(t - \tau_{k-1})x_{k-1}(\tau_{k-1}^+) + \int_{\tau_{k-1}}^t [T(t-s)f(s, x_k(s)) + u(s)]ds.$$

for $t \in (\tau_{k-}, \tau_k]$ and define $x_k(\tau_k)$ accordingly with $x_k(\cdot)$ left continuous at τ_k and $x_k(\tau_k) \in X$, $k = 1, 2, \ldots, \sigma$.

Thus we obtain $x \in PC([0,T_0],X)$ is a mild solution of system (4.20) and given by.

$$x(t) = \begin{cases} x_1(t), & 0 \le t \le \tau_1, \\ x_k(t), & \tau_{k-1} < t \le \tau_k,\ k = 2,3,\ldots,\sigma. \end{cases}$$

Next, by mathematical induction to show that (4.33) is satisfied on $[0,T_0]$. First, (4.33) is satisfied on $[0,\tau_1]$. If (4.33) is satisfied on $(\tau_{k-1},\tau_k]$, then for $t \in (\tau_k,\tau_{k+1}]$,

$$\begin{aligned} x(t) = \;& x_{k+1}(t) = T(t-\tau_k)x_k(\tau_k^+) + \int_{\tau_k}^{t} T(t-s)[f(s,x_{k+1}(s)) + u(s)]ds \\ = \;& T(t-\tau_k)[(I+B_k)x(\tau_k) + c_k] + \int_{\tau_k}^{t} T(t-s)[f(s,x_{k+1}(s)) + u(s)]ds \\ = \;& T(t-\tau_k)(I+B_k)\Big[U(\tau_k,0)x_0 + \int_0^{\tau_k} U(\tau_k,s)[f(s,x(s)) + u(s)]ds \\ & + \sum_{0<\tau_i<\tau_k} T(\tau_k-\tau_i)c_i\Big] + T(t-\tau_k)c_k + \int_{\tau_k}^{t} T(t-s)[f(s,x_{k+1}(s)) + u(s)]ds \\ = \;& U(t,0)x_0 + \int_0^{\tau_k} U(t,s)[f(s,x(s)) + u(s)]ds + \sum_{0<\tau_i<\tau_k} U(t,\tau_i)c_i \\ & U(t,\tau_k)c_k + \int_{\tau_k}^{t} U(t,s)[f(s,x(s)) + u(s)]ds \\ = \;& U(t,0)x_0 + \int_0^{t} U(t,s)f(s,x(s))ds + \sum_{0<\tau_i<t} U(t,\tau_i)c_i. \end{aligned}$$

Thus (4.33) is also true on $(\tau_k,\tau_{k+1}]$. Therefore (4.33) is true on $[0,T_0]$.

Next, we want to show that a mild solution is unique on $PC([0,T_0],X)$. Suppose that x, y are mild solutions of system (4.20) on $PC([0,T_0],X)$. Then by Corollary 3.2.4, we have

$$\begin{aligned} \|x(t)-y(t)\|_X \le \;& \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \|f(s,x(s)) - f(s,y(s))\|_X ds \\ \le \;& Ke^{\nu t}\int_0^t e^{-\nu s}\|x(s)-y(s)\|_X ds. \end{aligned}$$

It follows from Gronwall Lemma, we obtain $\|x(t)-y(t)\| = 0$ for all $t \in [0,T_0]$. That is, $x = y$. Therefore, system (4.20) has a unique mild solution. This completes the proof. □

To study the semilinear impulsive periodic control systems (4.20), define the operator $\Omega : C([0,T_0],X) \to C([0,T_0],X)$ by

$$\Omega x = T(t)x_0 + \int_0^t T(t-s)[f(s,x(s)) + u(s)]ds. \tag{4.28}$$

Analogous to Theorem 4.1.4 , system (4.22) has a T_0-periodic mild solution if and only if the following operator equation has a fixed points

$$x = \Omega x.$$

Theorem 4.2.3. *If assumptions (A1), (A2) and (A5) hold, then system (4.20) has a unique T_0-periodic mild solution $x \in PC([0,T_0],X)$ and there exists a constant $\beta > 0$ such that*

$$\|x\|_{PC} \leq \beta.$$

Proof. Consider the following semilinear control system without impulses,

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t,x(t)) + u(t), \quad t \in \gamma, \\ x(\tau) = \varphi, \quad \varphi \in X, \end{cases} \tag{4.29}$$

where γ is any subinterval of $[0,T_0]$ and τ is the left end point of γ. Define $B = \{y \mid y \in C(\gamma,X),\ y(\tau) = \varphi\}$ and a mapping $F : B \to B$ by $y = Fx$, where y is a solution of the following system,

$$\begin{cases} \dot{y}(t) = Ay(t) + f(t,x(t)) + u(t), \quad t \in \gamma, \\ y(\tau) = \varphi, \end{cases} \tag{4.30}$$

Similar to the proof of Theorem 4.1.1, we can show that F is continuous and compact on B. Next, we want to show that there is a constant $\beta_1 > 0$ such that

$$\|x\|_{C(\gamma,X)} \leq \beta_1$$

for all $x \in B$ and $\lambda \in [0,1]$ satisfying $x = \lambda Fx$.

Let $x \in C(\gamma, X)$. We consider the operator equation

$$x = \lambda F x, \quad \lambda \in [0,1]. \tag{4.31}$$

If x is a mild solution of equation (4.31), then we have

$$\begin{aligned}
\|x(t)\|_X \leq\ & \lambda \|T(t)\|_{\mathcal{L}(X)} \|x_0\|_X + \lambda \int_0^t \|T(t-s)\|_{\mathcal{L}(X)} \Big[\|f(s,x(s))\|_X + \|u(s)\|_X \Big] ds \\
\leq\ & \lambda M_1 \Big(\|x_0\|_X + (K_1 + K_3) T_0 \Big) \\
\leq\ & M_1 \Big(\|x_0\|_X + (K_1 + K_3) T_0 \Big) := \beta_1, \quad \lambda \in [0,1],
\end{aligned}$$

where $M_1 = \sup\limits_{0 \leq s \leq t \leq T_0} \|T(t-s)\|_{\mathcal{L}(X)}$ and $K_3 = \sup\limits_{s \in [0,T_0]} \|u(s)\|_X$.

That is, there exists a constant $\beta_1 > 0$ such that $\|x\|_{C(\gamma,X)} \leq \beta_1$ for all $x \in B$ and $x = \lambda F x$ where $\lambda \in [0,1]$. This shows that all mild solution of (4.31) are bounded independently of $\lambda \in [0,1]$. By Leray-Schauder's fixed point theorem, F has a fixed point $x \in C(\gamma, X)$ which is a mild solution of system (4.29). Next, we want to show that a mild solution is unique on $C(\gamma, X)$. Suppose that x, y are mild solutions of system (4.29) on $C(\gamma, X)$. By Theorem 2.3.2(1), there exist constants $K \geq 1$ and $\omega \geq 0$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq K e^{\omega t}$. Then we have

$$\begin{aligned}
\|x(t) - y(t)\|_X \leq\ & \int_0^t \|T(t-s)\|_{\mathcal{L}(X)} \|f(s,x(s)) - f(s,y(s))\|_X ds \\
\leq\ & K e^{\omega t} \int_0^t e^{-\omega s} \|x(s) - y(s)\|_X ds.
\end{aligned}$$

It follows from Gronwall Lemma, we obtain $\|x(t) - y(t)\| = 0$ for all $t \in \gamma$. That is, $x = y$. Therefore, system (4.29) has a unique mild solution. Furthermore, a mild solution is a T_0-periodic mild solution.

Now we consider the partition of interval $I = [0, T_0]$. We define $\gamma_k = (\tau_{k-1}, \tau_k]$, $k = 1, 2, \ldots, \sigma$, where $\tau_0 = 0$ and $\tau_\sigma = T_0$. Consider any arbitrary interval, say, γ_k and $x(\tau_k^+) = (I + B_k) x(\tau_k) + c_k \in X$. By Theorem 4.2.1 and similar procedure of Theorem 4.2.2, we obtain $x \in PC([0, T_0], X)$ is a unique mild solution of system (4.20) and given by

$$x(t) = \begin{cases} x_1(t), & 0 \leq t \leq \tau_1, \\ x_k(t), & \tau_{k-1} < t \leq \tau_k, \; k = 2, 3, \ldots, \sigma. \end{cases}$$

x is just T_0-periodic mild solution of system (4.20). By assumption (A2) and (A5), one can verify the priori estimate of solution of system (4.20) that

$$\begin{aligned} \|x(t)\|_X \leq & \; \|U(t,0)\|_{\mathcal{L}(X)} \|x_0\|_X + \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \Big[\|f(s,x(s))\|_X + \|u(s)\|_X \Big] ds \\ \leq & \; M\Big(\|x_0\|_X + (K_1 + K_3)T_0 \Big) := \beta, \end{aligned}$$

where $M = \sup_{0 \leq s \leq t \leq T_0} \|U(t,s)\|_{\mathcal{L}(X)}$ and $K_3 = \sup_{s \in [0,T_0]} \|u(s)\|_X$.

That is, there exists a constant $\beta > 0$ such that $\|x\|_{PC} \leq \beta$. □

4.3 Semilinear Impulsive Periodic Control Systems With Parameter Perturbations

We consider the semilinear impulsive periodic control system with parameter perturbations as the following

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t,x(t)) + u(t) + p(t,x(t),\xi), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k + q_k(x(t),\xi), & t = \tau_k, \end{cases} \tag{4.32}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. In addition to assumptions (A1), (A2), (A4) and (A5), we introduce the following assumption

Assumption (A6) ;

(A6.1) The Fréchet derivative $\frac{\partial}{\partial x} f(t,x)$ exists in $[0,\infty) \times X$. For each $y \in X$, $t \mapsto \frac{\partial}{\partial x} f(t,x)y$ is strongly measurable, $x \mapsto \frac{\partial}{\partial x} f(t,x)y$ is continuous. For every $\rho > 0$, there exists a constant $K_3(\rho) > 0$ such that

$$\left\| \frac{\partial}{\partial x} f(t,x) \right\|_{\mathcal{L}(X)} \leq K_3(\rho)$$

for all $t \geq 0$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

(A6.2) $p(t,x,\xi)$ and $q_k(x,\xi)$ satisfy Lipschitz conditions, i.e. for any (t,x,ξ), $(t,y,\xi) \in [0,\infty) \times \mathcal{B}_\rho \times [0,\xi_0]$, there exists a constant $N(\xi) > 0$ such that

$$\|p(t,x,\xi) - p(t,y,\xi)\|_X \leq N(\xi)\|x-y\|_X$$

and
$$\|q_k(x,\xi) - q_k(y,\xi)\|_X \leq N(\xi)\|x-y\|_X.$$

(A6.3) $B_k \in \mathcal{L}(X)$ and there exists constant $h_k(\rho) > 0$ such that

$$\|B_k(x) - B_k(y)\|_X \leq h_k(\rho)\|x-y\|_X,$$

for all $k \in \mathbb{N}$ and all $x, y \in X$ such that $\|x\|_X, \|y\|_X \leq \rho$.

(A6.4) The Fréchet derivative $\dfrac{\partial}{\partial x}B_k(x)$ exists in X. For every $\rho > 0$, there exists a constant $\bar{h}_k(\rho) > 0$ such that

$$\left\|\frac{\partial}{\partial x}B_k(x)\right\|_{\mathcal{L}(X)} \leq \bar{h}_k(\rho)$$

for all $t \geq 0$, $k \in \mathbb{N}$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

4.3.1 Definitions of Solutions

Definition 4.3.1. A function $x \in PC([0,\infty), X)$ is said to be a *mild solution* of impulsive system (4.32) with initial condition $x(0) = x_0 \in X$ if x is given by

$$\begin{aligned} x(t) = \;& U(t,0)x_0 + \int_0^t U(t,s)[f(s,x(s)) + u(s) + p(s,x(s),\xi)]ds \\ & + \sum_{0<\tau_k<t} U(t,\tau_k)[c_k + q_k(x(\tau_k),\xi)]. \end{aligned} \tag{4.33}$$

Definition 4.3.2. A function $x \in PC([0,\infty), X)$ is said to be a *periodic mild solution* of system (4.32) if it is a mild solution and there exists $T_0 > 0$ such that $x(t+T_0) = x(t)$ for all $t \geq 0$.

Definition 4.3.3. A function $x \in PC([0,\infty), X)$ is said to be a T_0*-periodic mild solution* of system (4.32) if it is a mild solution and $x(t+T_0) = x(t)$ for all $t \geq 0$.

4.3.2 Existence and Uniqueness of Periodic Mild Solutions

First, we consider the following reference system

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{4.34}$$

and assume that $x_{T_0}(t)$ is a T_0- periodic mild solution of the reference system (4.34) which satisfies

$$x_{T_0}(t) = U(t, 0)x_0 + \int_0^t U(t, s)f(s, x(s))ds. \tag{4.35}$$

Next, we consider the following variation system

$$\begin{cases} \dot{x}(t) = Ax(t) + \frac{\partial}{\partial x} f(t, x_{T_0}(t))x(t), & t \neq \tau_k, \\ \Delta x(t) = \frac{\partial}{\partial x} B_k(x_{T_0}(t))x(t), & t = \tau_k, \end{cases} \tag{4.36}$$

and assume that the variation system (4.36) has only trivial solution.

Theorem 4.3.1. *Let assumption (A1), (A2), and (A4)-(A6) holds. Suppose $x_{T_0}(t)$ be a T_0-periodic mild solution of the reference system (4.34) satisfies*

$$\rho_0 = \sup_{t \in [0, T_0]} \|x_{T_0}(t)\|_X.$$

Assume that

1. *system (4.36) has only trivial solution,*

2. *let $\xi_0 > 0$ and $\varepsilon_o \in (0, \rho - \rho_0)$ such that $\eta < 1$ with*

$$\eta := M\Big([K_2(\varepsilon_0) + K_3(\varepsilon_0)]T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0)]\sigma + [T_0 + \sigma] \sup_{\xi \in [0, \xi_0]} N(\xi)\Big)$$

where

$$\begin{aligned} M &= \sup_{0 \le s \le t \le T_0} \|U(t, s)\|_{\mathcal{L}(X)}, \\ \bar{h}_k(\varepsilon_0) &= \sup_{k \in \mathbb{N},\, \|y\| \le \varepsilon_0} \left\| \frac{\partial}{\partial x} B_k(x_{T_0}(\tau_k) + y(\tau_k)) \right\|_X, \end{aligned}$$

3. the following inequality is valid

$$\sup_{t\in[0,T_0],\,|\xi|\le\xi_0}\Bigg\| U(t,0)x_0+\int_0^t U(t,s)[u(s)+p(s,x_{T_0}(s),\xi)]ds$$
$$+\sum_{0\le\tau_k<T_0} U(t,\tau_k)[c_k+q_k(x_{T_0}(\tau_k),\xi)]\Bigg\|_X \le \varepsilon_0(1-\eta).$$

Then for any constant $\rho>\rho_0>0$, there exists a sufficiently small $\xi_0>0$ such that for every fixed $\xi\in[0,\xi_0]$ system (4.32) has a unique T_0-periodic mild solution $x^{\xi}_{T_0}(t)$ satisfying

$$\|x^{\xi}_{T_0}(t)-x_{T_0}(t)\|<\varepsilon_0 \quad \textit{for all} \quad t\ge 0 \tag{4.37}$$

and $$\lim_{\xi\to 0} x^{\xi}_{T_0}(t)=x_{T_0}(t) \qquad \textit{uniformly on } t.$$

Proof. Let $x(t)=x_{T_0}(t)+y(t)$, then we can change system (4.32) into

$$\dot y(t)=Ay(t)+\frac{\partial}{\partial x}f(t,x_{T_0}(t))y(t)+o(t,y(t))+u(t)+p(s,x_{T_0}(t)+y(t),\xi),\ t\ne\tau_k, \tag{4.38}$$
$$\Delta y(t)=\frac{\partial}{\partial x}B_k(x_{T_0}(t))y(t)+o_k(y(t))+c_k+q_k(x_{T_0}(t)+y(t),\xi), \qquad t=\tau_k,$$

where

$$\begin{aligned} o(t,y(t)) &= f(t,x_{T_0}(t)+y(t))-f(t,x_{T_0}(t))-\frac{\partial}{\partial x}f(t,x_{T_0}(t))y(t)\\ o_k(y(t)) &= B_k(x_{T_0}(t)+y(t))-B_k(x_{T_0}(t))-\frac{\partial}{\partial x}B_k(x_{T_0}(t))y(t)\end{aligned} \tag{4.39}$$

Let $PC_{T_0}([0,T_0];X):=\{x\in PC([0,T_0];X)\ \mid\ x(0)=x(T_0)\ \}$
with norm

$$\|x\|_{PC_{T_0}}=\sup_{t\in[0,T_0]}\|x(t)\|_X.$$

Let us define

$$\mathcal{B}:=\mathcal{B}(\varepsilon_0)=\{y\in PC_{T_0}([0,T_0];X)\ \mid\ \|y\|_{PC_{T_0}}\le\varepsilon_0\ \} \tag{4.40}$$

and an operator $\Omega : \mathcal{B} \to PC_{T_0}([0,T_0]\,;\,X)$ such that

$$\begin{aligned}\Omega(x)(t) := U(t,0)x_0 + \int_0^t U(t,s)\Big[o(s,y(s)) + u(s) + p(s,x_{T_0}(s)+y(s),\xi)\Big]ds \\ + \sum_{0\le\tau_k<t} U(t,\tau_k)[o_k(y(\tau_k)) + c_k + q_k(x_{T_0}(\tau_k)+y(\tau_k),\xi)].\end{aligned} \tag{4.41}$$

If $y \in \mathcal{B}$, then

$$\begin{aligned}\|x\|_{PC_{T_0}} &= \|x_{T_0}+y\|_{PC_{T_0}} \\ &\le \|x_{T_0}\|_{PC_{T_0}} + \|y\|_{PC_{T_0}} \\ &\le \rho_0 + \varepsilon_0 \\ &\le \rho_0 + (\rho-\rho_0) = \rho.\end{aligned}$$

From equation (4.41), we have

$$\begin{aligned}\Omega(x_{T_0})(t) = {}& U(t,0)x_0 + \int_0^t U(t,s)\Big[u(s) + p(s,x_{T_0}(s),\xi)\Big]ds \\ &+ \sum_{0\le\tau_k<t} U(t,\tau_k)[c_k + q_k(x_{T_0}(\tau_k),\xi)].\end{aligned} \tag{4.42}$$

For any $x, x_{T_0} \in \mathcal{B}$, then we have

$$\begin{aligned}&\|\Omega(x)-\Omega(x_{T_0})\|_{PC_{T_0}} \\ &\le \int_0^t \|U(t,s)\|_{\mathcal{L}(X)}\|o(s,y(s)) + p(s,x_{T_0}(s)+y(s),\xi) - p(s,x_{T_0}(s),\xi)\|_X\,ds \\ &+ \sum_{0\le\tau_k<t}\|U(t,\tau_k)\|_{\mathcal{L}(X)}\|o_k(y(\tau_k)) + q_k(x_{T_0}(\tau_k)+y(\tau_k),\xi) - q_k(x_{T_0}(\tau_k),\xi)\|_X \\ &\le \int_0^t \|U(t,s)\|_{\mathcal{L}(X)}\left(\left\|f(t,x_{T_0}(s)+y) - f(t,x_{T_0}(s)) - \frac{\partial}{\partial x}f(t,x_{T_0}(s))y\right\|_X\right. \\ &\left. + \|p(s,x_{T_0}(s)+y(s),\xi) - p(s,x_{T_0}(s),\xi)\|_X\right)ds \\ &+ \sum_{0\le\tau_k<t}\|U(t,\tau_k)\|_{\mathcal{L}(X)}\left(\left\|B_k(x_{T_0}(\tau_k)+y) - B_k(x_{T_0}(\tau_k)) - \frac{\partial}{\partial x}B_k(x_{T_0}(\tau_k))y\right\|_X\right. \\ &\left. + \|q_k(x_{T_0}(\tau_k)+y(\tau_k),\xi) - q_k(x_{T_0}(\tau_k),\xi)\|_X\right)\end{aligned}$$

$$\leq \int_0^t \|U(t,s)\|_{\mathcal{L}(X)} \Bigg(\|f(t, x_{T_0}(s)+y) - f(t, x_{T_0}(s))\| + \left\| \frac{\partial}{\partial x} f(t, x_{T_0}(s))y \right\|_X$$

$$+\|p(s, x_{T_0}(s)+y(s), \xi) - p(s, x_{T_0}(s), \xi)\|_X \Big) ds$$

$$+ \sum_{0 \leq \tau_k < t} \|U(t,\tau_k)\|_{\mathcal{L}(X)} \big(\|B_k(x_{T_0}(\tau_k)+y) - B_k(x_{T_0}(\tau_k))\|$$

$$+ \left\| \frac{\partial}{\partial x} B_k(x_{T_0}(\tau_k))y \right\|_X + \|q_k(x_{T_0}(\tau_k)+y(\tau_k), \xi) - q_k(x_{T_0}(\tau_k), \xi)\|_X \Big)$$

$$\leq M\Big([K_2(\varepsilon_0) + K_3(\varepsilon_0) + N(\xi)]T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0) + N(\xi)]\sigma \Big) \|x - x_{T_0}\|_{PC_{T_0}}.$$

Let us choose $\xi_0 > 0$ and $\varepsilon_0 \in (0, \rho - \rho_0)$ such that $\eta < 1$ with

$$\eta := M\Big([K_2(\varepsilon_0) + K_3(\varepsilon_0)]T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0)]\sigma + [T_0 + \sigma] \sup_{\xi \in [0,\xi_0]} N(\xi) \Big). \quad (4.43)$$

So
$$\|\Omega(x) - \Omega(x_{T_0})\|_{PC_{T_0}} \leq \eta \|x - x_{T_0}\|_{PC_{T_0}} \quad (4.44)$$

It follows from (4.42), (4.44,) and assumption (3) that

$$\begin{aligned} \|\Omega(x)\|_{PC_{T_0}} &\leq \|\Omega(x) - \Omega(x_{T_0})\|_{PC_{T_0}} + \|\Omega(x_{T_0})\|_{PC_{T_0}} \\ &\leq \eta \|x - x_{T_0}\|_{PC_{T_0}} + \varepsilon_0(1-\eta) \\ &\leq \eta\varepsilon_0 + \varepsilon_0(1-\eta) = \varepsilon_0 \end{aligned}$$

from which we know that $\Omega(x) \in \mathcal{B}$, then $\Omega : \mathcal{B} \to \mathcal{B}$ is a contraction mapping. Therefore, there exists a unique fixed point $y_1(t) \in \mathcal{B}$. From the fact that $y_1(t)$ is a solution of system (4.38), we know $x^{\xi}_{T_0}(t) = x_{T_0}(t) + y_1(t)$ is a T_0- periodic mild solution of (4.32) and satisfies

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\| = \|y_1(t)\| < \varepsilon_0.$$

So we have $\lim_{\xi \to 0} x^{\xi}_{T_0} = x_{T_0}(t)$ uniformly on t.

This completes the proof. □

CHAPTER V

APPLICATIONS

In this chapter, to illustrate the application of our work, we apply Theorem 4.1.5 to prove the existence of periodic mild solution of systems governed by semilinear partial differential equations of parabolic types with impulses.

The first part of this chapter is about basic concepts of Sobolev spaces and related results. The second part consists of our example that we introduce constructively to show how our abstract results can be applied.

5.1 Terminology

In the following we use $y = (y_1, y_2, \ldots, y_n)$ to be a variable point in the n-dimensional Euclidean space $\mathbb{R}^n$. For any two such points $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ we set $y \cdot z = \sum_{i=1}^{n} y_i z_i$ and $|y|^2 = y \cdot y$.

An n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called a multi-index and we define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \quad \text{for} \quad y = (y_1, y_2, \ldots, y_n).$$

Denoting $D_k = \dfrac{\partial}{\partial y_k}$ and $D = (D_1, D_2, \ldots, D_n)$ we have

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial y_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}}.$$

Let Ω be a fixed domain in $\mathbb{R}^n$ with boundary and closure $\bar{\Omega}$. Assume that $\partial\Omega$ is sufficiently smooth, e.g., $\partial\Omega$ is of the class C^k for some suitable $k \geq 0$, this means that for each point $y \in \partial\Omega$ there is a ball $\mathcal{B}$ with center at y such that $\partial\Omega \cap \mathcal{B}$ can be represent in the form $y_i = \varphi(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ for some i, and φ is a $k-$times continuously differentiable function.

For a nonnegative integer m, we denote by $C^m(\Omega)(\text{resp.}\, C^m(\bar{\Omega}))$ the set of all $m-$times continuously differentiable real-valued or complex-valued functions in $\Omega(\text{resp.}\, \bar{\Omega},$ by $C_0^m(\Omega)$ the subspace of $C^m(\Omega)$ consisting of those functions which have compact support in Ω.

For $x \in C^m(\Omega)$ and $1 \leq p < \infty$, we define

$$\|x\|_{m,p} = \left(\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha x|^p dy \right)^{\frac{1}{p}}. \tag{5.1}$$

Also for $p = 2$ and $u, v \in C^m(\Omega)$, we define

$$\langle u, v \rangle_m = \int_\Omega \sum_{|\alpha| \leq m} D^\alpha u D^{\bar{\alpha}} u dy. \tag{5.2}$$

Let $\hat{C}_p^m(\Omega)$ be the subset of $C^m(\Omega)$ consisting of those function x for which $\|x\|_{m,p} < \infty$. We define $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ to be the completions in the norm $\|\cdot\|_{m,p}$ of $\hat{C}_p^m(\Omega)$ and $C_0^m(\Omega)$, respectively. The space $W^{m,p}(\Omega)$ consists of function $x \in L^p(\Omega)$ whose derivatives $D^\alpha x$ in the sense of distribution, of order $|\alpha| \leq m$ are in $L^p(\Omega)$ and $W_0^{m,p}(\Omega)$ is the closure of $C_0^m(\Omega)$, in $W^{m,p}(\Omega)$.

It is well known that $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are Banach spaces with the usual norm $\|\cdot\|_{m,p}$. Then $W^{m,p}(\Omega)$ is separable, uniformly convex and hence reflexive. Let

$$H^m(\Omega) = W^{m,2}(\Omega) \quad \text{and} \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

The spaces $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces with the scalar product $\langle \cdot, \cdot \rangle$ is given by (5.2). The following embedding theorem describes various relations among the above spaces.

Theorem 5.1.1. *(Sobolev) The following relations among* $W^{m,p}(\Omega)$, $C^m(\Omega)$ *and* $L^p(\Omega)$ *hold ;*

1. $W^{m,p}(\Omega) \subset W^{m,r}(\Omega)$ *if* $1 \le r \le p$ *and the embedding is continuous.*

2. $W^{m,r}(\Omega) \subset W^{j,p}(\Omega)$ *if* $1 \le r,\ p < \infty$, j *and* m *are integers such that* $0 \le j \le m$ *and* $\dfrac{1}{p} > \dfrac{1}{r} + \dfrac{j}{n} - \dfrac{m}{n}$ *and the embedding is compact.*

3. $W^{m,p}(\Omega) \subset L^{\frac{np}{n-mp}}(\Omega)$ *if* $mp \le n$ *and there exists a constant* c_1 *such that*
$$\|x\|_{0,\frac{np}{n-mp}} \le c_1 \|x\|_{m,p} \quad \textit{for} \quad x \in W^{m,p}(\Omega).$$

4. $W^{m,p}(\Omega) \subset C^k(\bar{\Omega})$ *if* $0 \le k \le m - \dfrac{n}{p}$ *and there exists a constant* c_2 *such that*
$$\sup\{|D^\alpha x(y)|\,;\, |\alpha| \le k,\ y \in \bar{\Omega}\} \le c_2 \|x\|_{m,p} \quad \textit{for} \quad x \in W^{m,p}(\Omega).$$

5. *(Poincaré Inequality) There exists a constant* $c = c(\Omega)$ *such that*
$$\inf_{k \in \mathbb{R}} \|x + k\|_{0,2} \le c(\Omega) \|\nabla x\|_{0,2} \quad \textit{for} \quad x \in H_0^1(\Omega).$$

Since $\partial\Omega$ *is smooth,* $C^\infty(\Omega)$ *is dense in* $W_0^{m,p}(\Omega)$ *and* $L_2(\Omega)$, $W_0^{m,p}(\Omega)$ *is dense in* $L_2(\Omega)$. *From Sobolev's embedding theorem, we have that the embeddings*
$$C^\infty(\Omega) \hookrightarrow W_0^{m,p}(\Omega) \hookrightarrow L_2(\Omega).$$

For any $\sigma = k + \eta > 0$, where k is a nonnegative integer and $\eta \in (0,1)$, $C^\sigma(\bar{\Omega})$ denotes the Banach space consisting of those functions belonging to $C^k(\bar{\Omega})$ whose derivatives $D^\alpha x$ of order $|\alpha| = k$ satisfy a uniform Hölder condition with exponent η. The norm in this space is defined as
$$\|x\|_{C^\sigma(\bar{\Omega})} = \|x\|_{C^k(\bar{\Omega})} + \sum_{|\alpha|=k} [D^\alpha x]_\eta\,,$$
with
$$[v]_\eta = \sup_{y,\,z \in \Omega,\, y \ne z} \frac{|v(y) - v(z)|}{|y - z|^\eta}.$$

5.2 Example

In the following, we give some examples of the existence of periodic mild solution of semilinear impulsive periodic systems.

Example 5.2.1. Consider

$$\begin{aligned} \frac{\partial x(t,y)}{\partial t} &= \frac{\partial^2 x(t,y)}{\partial y^2} + f_1(t, x(t,y)), \quad y \in \Omega,\ t \neq \tau_k,\ t \in [0, T_0], \\ \Delta x(t_k, y) &= B_k x(t_k, y), \quad t = \tau_k,\ k = 0, 1, 2, \ldots, \sigma, \\ x(0, y) &= x(T_0, y), \quad \text{on } \Omega, \\ x(t,y)|_{\partial\Omega} &= 0, \quad t \in [0, T_0], \end{aligned} \tag{5.3}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ and $\Omega \subset \mathbb{R}^n$ is a bounded open domain with C^2-boundary. Take $X = L^2(\Omega)$, $A := \dfrac{\partial^2}{\partial y^2}$ with domain $D(A) = \{H^2(\Omega) \cap H_0^1(\Omega)\}$. We suppose that f_1 satisfies the following assumption :

(F) $f_1 : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is an operator such that $f_1(t + T_0, x) = f_1(t, x)$ and $t \mapsto f_1(t, x)$ is strongly measurable. For every $\rho > 0$ there exist constants $M_1,\ M_2 > 0$ such that

$$|f_1(t, x)| \leq M_1$$

and

$$|f_1(t, x) - f_1(t, y)| \leq M_2 |x - y|,$$

for all $t \geq 0$ and all $x, y \in \mathbb{R}$.

Define $x(t)(y) = x(t, y)$ and $B_k x(t_k) = f_k(t_k) x(t_k)$ where $f_k \in L^\infty(\Omega)$. Then it is clearly that B_k satisfies assumption (A1.3). Given $z \in X$, define mappings $f : [0, T_0] \times X \to X$ by

$$f(t, z)(y) = f_1(t, z(y)).$$

In order to apply Theorem 4.1.5 we have to show that f take value in X and satisfies assumption (A5).

Lemma 5.2.2. $f(t, z)$ satisfies assumption (A5) in Section 4.1.

Proof. Since $f(t + T_0, z)(y) = f_1(t + T_0, z(y)) = f_1(t, z(y)) = f(t, z)(y)$, so that $f(t + T_0, z) = f(t, z)$. Next, we show that f takes value in X. Since $f(t, z)(y) = f_1(t, z(y))$, we obtain for almost all t that

$$\begin{aligned}\int_\Omega |f(t,z)(y)|^2 dy = &\int_\Omega |f_1(t, z(y))|^2 dy \\ \leq &\int_\Omega M_1^2 dy < \infty,\end{aligned}$$

since Ω is bounded. This show that $f(t, z) \in L^2(\Omega)$ for almost all t and all $z \in X$, so that f is well defined on $[0, T_0] \times X$. Next, we show that $f(t, \cdot)$ satisfies a Lipschitz condition. For any z_1, $z_2 \in L^2(\Omega)$, we have

$$\begin{aligned}\|f(t,z_1) - f(t,z_2)\|^2_{L^2(\Omega)} = &\int_\Omega |f(t,z_1)(y) - f(t,z_2)(y)|^2 dy \\ = &\int_\Omega |f_1(t, z_1(y)) - f_1(t, z_2(y))|^2 dx \\ \leq &M_2^2 \int_\Omega |z_1(y) - z_2(y)|^2 dy = M_2^2 \|z_1 - z_2\|^2_{L^2(\Omega)}\end{aligned}$$

for almost all t. Hence $f(t, \cdot)$ satisfies a Lipschitz condition for almost all t. This shows that f satisfies assumption (A5). □

Thus system (5.3) can be written as

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), & t \neq \tau_k, \\ \Delta x(t_k) = B_k x(t_k), & t = \tau_k, \ \ k = 0, 1, 2, \ldots, \sigma, \\ x(0) = x(T_0). \end{cases} \tag{5.4}$$

Obviously, it satisfies all the assumptions given in our former Theorem 4.1.5, our result can be applied to system (5.3).

Example 5.2.3. Consider

$$
\begin{aligned}
\frac{\partial x(t,y)}{\partial t} &= \frac{\partial^2 x(t,y)}{\partial y^2} + \sin t, \quad y \in \Omega,\ t \neq \tau_k,\ t \in [0, T_0],\\
\Delta x(t_k, y) &= -x(t_k, y), \quad t = \tau_k,\ k = 0, 1, 2, \ldots, \sigma,\\
x(0, y) &= x(T_0, y), \quad \text{on } \Omega,\\
x(t, y)|_{\partial\Omega} &= 0, \quad t \in [0, T_0],
\end{aligned}
\tag{5.5}
$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ and $\Omega \subset \mathbb{R}^n$ is a bounded open domain with C^2-boundary. Take $X = L^2(\Omega)$, $A := \dfrac{\partial^2}{\partial y^2}$ with domain $D(A) = \{H^2(\Omega) \cap H_0^1(\Omega)\}$. Define $x(t)(y) = x(t, y)$, $f(t, x) = \sin t$ and $B_k x(t_k) = -x(t_k)$. Then it is clearly that B_k and f satisfy assumption (A1.3) and (A5), respectively. Thus system (5.5) can be written as (5.4). Since it satisfies all the assumptions given in our former Theorem 4.1.5, our result can be applied to system (5.5).

CHAPTER VI

CONCLUSIONS

6.1 Thesis Summary

In this thesis, we have studied the existence of periodic mild solutions for linear and semilinear impulsive periodic systems with impulses, in these case where the operator involved is the infinitesimal generator of C_0-semigroup.

6.1.1 Problems

This thesis has considered the following problems :

1. Linear periodic systems with impulses :

1.1 Existence of periodic mild solutions for the homogenous linear impulsive periodic systems.

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{6.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$.

1.2 Existence of periodic mild solutions for the nonhomogenous linear impulsive periodic control systems.

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k, & t = \tau_k, \end{cases} \tag{6.2}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$.

1.3 Existence of periodic mild solutions for the linear impulsive control systems with parameter perturbations.

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t) + p(t, x(t), \xi), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k + q_k(x(t), \xi), & t = \tau_k, \end{cases} \tag{6.3}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$.

2. Semilinear periodic systems with impulses :

2.1 Existence of periodic mild solutions for the semilinear impulsive systems

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{6.4}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$.

2.2 Existence of periodic mild solutions for the semilinear impulsive control systems.

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x) + u(t), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k, & t = \tau_k, \end{cases} \tag{6.5}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$.

2.3 Existence of periodic mild solutions for the semilinear impulsive control systems with parameter perturbations.

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) + u(t) + p(t, x(t), \xi), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k + q_k(x(t), \xi), & t = \tau_k, \end{cases} \tag{6.6}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$.

6.1.2 Assumptions

Assumption (A1) ;

(A1.1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots,\quad \tau_k \to \infty$ as $k \to \infty$ and there exists a positive integer σ such that $\tau_{k+\sigma} = \tau_k + T_0$ for all $k \in \mathbb{N}$.

(A1.2) A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X.

(A1.3) $B_k \in \mathcal{L}(X)$ such that $B_{k+\sigma} = B_k$.

Assumption (A2) ;

(A2.1) A is the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$ in X.

(A2.2) $u \in PC([0,\infty), X)$ such that $u(t + T_0) = u(t)$.

(A2.3) $c_k \in X$ and $c_{k+\sigma} = c_k$ for all $k \in \mathbb{N}$.

Assumption (A3) ;

(A3.1) A^* is the infinitesimal generator of the adjoint semigroup $\{T^*(t), t \geq 0\}$ in X^*.

(A3.2) $B_k^* \in \mathcal{L}(X^*)$ such that $B_{k+\sigma}^* = B_k^*$ for all $k \in \mathbb{N}$.

Assumption (A4) ;

(A4.1) $p(\cdot, x, \xi) \in PC([0,\infty), X)$ such that $p(t + T_0, x, \xi) = p(t, x, \xi)$ for all $(t, x, \xi) \in [0, \infty) \times \mathcal{B}_\rho \times [0, \xi_0]$.

(A4.2) $q_k \in C(\mathcal{B}_\rho \times [0, \xi_0], X)$ such that $q_{k+\sigma}(x, \xi) = q_k(x, \xi)$ for all $k \in \mathbb{N}$

(A4.3) For each $(t, x, \xi) \in [0, \infty) \times \mathcal{B}_\rho \times [0, \xi_0]$, there exists a nonnegative function $\chi(\xi)$ such that

$$\lim_{\xi \to 0} \chi(\xi) = \chi(0) = 0$$

and
$$\|p(t, x, \xi)\|_X \leq \chi(\xi), \quad \|q_k(x, \xi)\|_X \leq \chi(\xi) \tag{6.7}$$

for all $k \in \mathbb{N}$.

Assumption (A5) ;

(A5) $f : [0,\infty) \times X \to X$ is an operator such that $f(t+T_0, x) = f(t,x)$ and $t \mapsto f(t,x)$ is strongly measurable. For every $\rho > 0$, there exist constants $K_1(\rho)$, $K_2(\rho) > 0$ such that

$$\|f(t,x)\|_X \leq K_1(\rho)$$

and
$$\|f(t,x) - f(t,y)\|_X \leq K_2(\rho)\|x-y\|_X,$$

for all $t \geq 0$ and all $x, y \in X$ such that $\|x\|_X$, $\|y\|_X \leq \rho$.

Assumption (A6) ;

(A6.1) The Fréchet derivative $\frac{\partial}{\partial x} f(t,x)$ exists in $[0,\infty) \times X$. For each $y \in X$, $t \mapsto \frac{\partial}{\partial x} f(t,x)y$ is strongly measurable, $x \mapsto \frac{\partial}{\partial x} f(t,x)y$ is continuous. For every $\rho > 0$, there exists a constant $K_3(\rho) > 0$ such that

$$\left\| \frac{\partial}{\partial x} f(t,x) \right\|_{\mathcal{L}(X)} \leq K_3(\rho)$$

for all $t \geq 0$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

(A6.2) $p(t,x,\xi)$ and $q_k(x,\xi)$ satisfy Lipschitz conditions,i.e. for any (t,x,ξ), $(t,y,\xi) \in [0,\infty) \times \mathcal{B}_\rho \times [0,\xi_0]$, there exists a constant $N(\xi) > 0$ such that

$$\|p(t,x,\xi) - p(t,y,\xi)\|_X \leq N(\xi)\|x-y\|_X$$

and
$$\|q_k(x,\xi) - q_k(y,\xi)\|_X \leq N(\xi)\|x-y\|_X.$$

(A6.3) $B_k \in \mathcal{L}(X)$ and there exists constant $h_k(\rho) > 0$ such that

$$\|B_k(x) - B_k(y)\|_X \leq h_k(\rho)\|x-y\|_X,$$

for all $k \in \mathbb{N}$ and all $x, y \in X$ such that $\|x\|_X$, $\|y\|_X \leq \rho$.

(A6.4) The Fréchet derivative $\frac{\partial}{\partial x} B_k(x)$ exists in X. For every $\rho > 0$, there exists a constant $\bar{h}_k(\rho) > 0$ such that

$$\left\| \frac{\partial}{\partial x} B_k(x) \right\|_{\mathcal{L}(X)} \leq \bar{h}_k(\rho)$$

for all $t \geq 0$, $k \in \mathbb{N}$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

6.1.3 Results

The main results of this thesis are summarized as follows :

Theorem 6.1.1. *Let assumption (A1) hold. The system (6.1) has a periodic mild solution if and only if the operator* $U(T_0, 0)$ *has a fixed point* $x_0 \in X$.

Theorem 6.1.2. *Let assumption (A1) hold. Furthermore, assume that A is the infinitesimal generator of a compact semigroup* $\{T(t), t > 0\}$ *in* X. *Then system (6.1) either has a unique trivial solution or have finitely many linearly independent nontrivial periodic mild solutions in* $PC([0, \infty), X)$.

Theorem 6.1.3. *If system (6.1) has only trivial solution , then system (6.2) has a unique* T_0*-periodic mild solution*

$$\begin{aligned} x_{T_0}(t) = \; & U(t,0)[I - U(T_0,0)]^{-1} \left(\int_0^{T_0} U(T_0,s)u(s)\, ds \right. \\ & \left. + \sum_{0 \le \tau_k < T_0} U(T_0,\tau_k)c_k \right) + \int_0^t U(t,s)u(s)ds \\ & + \sum_{0 \le \tau_k < t} U(t,\tau_k)c_k. \end{aligned} \tag{6.8}$$

Theorem 6.1.4. *Assume that (A1) and (A2) hold. Furthermore, assume that* X *is a Hilbert space and* $u \in L^1_{loc}([0, \infty), X)$. *If the system (6.1) have* m *linearly independent periodic mild solutions* $x^1, x^2, \ldots, x^m$ *with* $1 \le m \le n$ *where* x^i *are periodic mild solutions of the system (6.1) corresponding to initial conditions* $x^i(0) = x^i_0,\ i = 1, 2, \ldots, m$, *then*

1. *the adjoint system (A.19) also have* m *linearly independent periodic mild solutions* $y^1, y^2, \ldots, y^m$.

2. *system (6.2) has a* T_0*-periodic mild solution if and only if*

$$\langle\, y\, ,\, z\, \rangle = 0, \tag{6.9}$$

where $y \in X^*$ *satisfying*

$$[I - U^*(T_0, 0)]y = 0 \tag{6.10}$$

and $\quad z := \int_0^{T_0} U(T_0, s)u(s)ds + \sum_{0 \le \tau_k < T_0} U(T_0, \tau_k)c_k,$

or if and only if

$$\int_0^{T_0} \langle\, y(s),\, u(s)\,\rangle ds + \sum_{0 \le \tau_k < T_0} \langle\, y(\tau_k),\, c_k\,\rangle = 0. \tag{6.11}$$

Furthermore, let $x_a(t)$ be a particular T_0-periodic mild solution of system (6.2), each T_0-periodic mild solution of system (6.2) has the the form

$$x(t) = x_a(t) + \sum_{i=1}^{m} \alpha_i x^i(t),$$

where $\alpha_i,\ i = 1, 2, \ldots, m,$ are constants.

Theorem 6.1.5. *If system (6.2) has a bounded mild solution, then it has at least one* T_0*-periodic mild solution.*

Corollary 6.1.6.

1. *Assume that system (6.2) has no* T_0*-periodic mild solution, then all of its solutions are unbounded for* $t \ge 0$.

2. *Assume that system (6.2) has a unique bounded mild solution for* $t \ge 0$, *then this solution is* T_0*-periodic.*

Theorem 6.1.7. *Let assumptions (A1), (A2) and (A4) hold. Assume that*

1. *system (6.1) has only trivial solution,*

2. *the following inequality is valid*

$$\rho_0 = \sup_{t \in [0, T_0]} \|x_{T_0}(t)\|_X < \rho \tag{6.12}$$

where ρ *be any positive real number,*

3. *$p(t,x,\xi)$ and $q_k(x,\xi)$ satisfy Lipschitz conditions, i.e. for any (t,x,ξ), $(t,y,\xi)\in[0,\infty)\times\mathcal{B}_\rho\times[0,\xi_0]$, there exists a constant $N(\xi)>0$ such that*

$$\|p(t,x,\xi)-p(t,y,\xi)\|_X \le N(\xi)\|x-y\|_X$$

and

$$\|q_k(x,\xi)-q_k(y,\xi)\|_X \le N(\xi)\|x-y\|_X.$$

Then for any constant $\rho>\rho_0>0$, there exists a sufficiently small $\xi_0>0$ such that for every fixed $\xi\in[0,\xi_0]$ system (6.3) has a unique T_0-periodic mild solution $x^{\xi}_{T_0}(t)$ satisfying

$$\|x^{\xi}_{T_0}(t)-x_{T_0}(t)\|_X < \rho-\rho_0 \tag{6.13}$$

and

$$\lim_{\xi\to 0} x^{\xi}_{T_0}(t) = x_{T_0}(t) \tag{6.14}$$

uniformly on t.

Theorem 6.1.8. *Let assumptions (A1), (A2) and (A4) hold. Assume that*

1. *system (6.1) has only trivial solution,*

2. *the following inequality is valid*

$$\rho_0 = \sup_{t\in[0,\infty]} \|x_{T_0}(t)\|_X < \rho \tag{6.15}$$

Then for any constant $\rho>\rho_0>0$, there exists a sufficiently small $\xi_0>0$ such that for every fixed $\xi\in[0,\xi_0]$ system (6.3) has a unique T_0-periodic mild solution $x^{\xi}_{T_0}(t)$ satisfying

$$\|x^{\xi}_{T_0}(t)-x_{T_0}(t)\|_X \le \rho-\rho_0. \tag{6.16}$$

Theorem 6.1.9. *If assumptions (A1) and (A5) hold, then system (6.4) has a unique T_0-periodic mild solution $x \in PC([0,T_0],X)$ and there exists a constant $\beta > 0$ such that*

$$\|x\|_{PC} \leq \beta.$$

Theorem 6.1.10. *If assumptions (A1), (A2) and (A5) hold, then system (6.5) has a unique T_0-periodic mild solution $x \in PC([0,T_0],X)$ and there exists a constant $\beta > 0$ such that*

$$\|x\|_{PC} \leq \beta.$$

Theorem 6.1.11. *Let assumption (A1), (A2) and (A4)-(A6) holds. Suppose $x_{T_0}(t)$ be a T_0-periodic mild solution of the reference system (4.34) satisfies*

$$\rho_0 = \sup_{t\in[0,T_0]} \|x_{T_0}(t)\|_X.$$

Assume that

1. *system (4.36) has only trivial solution,*

2. *let $\xi_0 > 0$ and $\varepsilon_o \in (0, \rho - \rho_0)$ such that $\eta < 1$ with*

$$\eta := M\Big([K_2(\varepsilon_0) + K_3(\varepsilon_0)]T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0)]\sigma + [T_0 + \sigma] \sup_{\xi\in[0,\xi_0]} N(\xi)\Big)$$

   where

$$\begin{aligned} M = & \sup_{0\leq s\leq t\leq T_0} \|U(t,s)\|_{\mathcal{L}(X)}, \\ \bar{h}_k(\varepsilon_0) = & \sup_{k\in\mathbb{N},\, \|y\|\leq\varepsilon_0} \left\| \frac{\partial}{\partial x} B_k(x_{T_0}(\tau_k) + y(\tau_k)) \right\|_X, \end{aligned}$$

3. *the following inequality is valid*

$$\begin{aligned} \sup_{t\in[0,T_0],\, |\xi|\leq\xi_0} \Bigg\| & U(t,0)x_0 + \int_0^t U(t,s)[u(s) + p(s, x_{T_0}(s), \xi)]ds \\ & + \sum_{0\leq\tau_k<T_0} U(t,\tau_k)[c_k + q_k(x_{T_0}(\tau_k),\xi)] \Bigg\|_X \leq \varepsilon_0(1-\eta). \end{aligned}$$

Then for any constant $\rho > \rho_0 > 0$, *there exists a sufficiently small* $\xi_0 > 0$ *such that for every fixed* $\xi \in [0, \xi_0]$ *system (6.6) has a unique* T_0*-periodic mild solution* $x^{\xi}_{T_0}(t)$ *satisfying*

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\| < \varepsilon_0 \quad \textit{for all} \quad t \geq 0 \tag{6.17}$$

and

$$\lim_{\xi \to 0} x^{\xi}_{T_0}(t) = x_{T_0}(t) \qquad \textit{uniformly on } t.$$

6.1.4 Applications

All results of the abstract framework in this thesis can be applied to semilinear partial differential equations of parabolic types with impulses. The example concerning second order semilinear parabolic impulsive differential equation was given. We prove the existence of periodic mild solutions.

6.1.5 Suggestion for Further Work

We should observe that further problems can be considered. For instance, how to deal with the relaxation and optimal control for impulsive periodic control problem. Discuss existence of almost periodic solution for linear and semilinear impulsive control of almost periodic systems. Furthermore, we can consider other application problems and computation algorithm. We will continue to study in this field.

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APPENDICES

APPENDIX A

PUBLICATION

IMPULSIVE PERIODIC CONTROL SYSTEM WITH PARAMETER PERTURBATIONS*

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Abstract : In this paper, we study the existence of periodic solution for impulsive periodic control system with parameter perturbations on infinite dimensional space, in these cases where the differential operator involved is the infinitesimal generator of C_0-semigroup.

Keywords : Impulsive differential equation, semigroup, periodic solution, Banach space.

2000 Mathematics Subject Classification : 34A37, 34G10, 34K13.

A.1 Introduction

The impulsive differential equations appear to a natural framework for mathematical modelings of several real world phenomena. For instance, systems with impulse effects have applications in physics, in biotechnology, in population dynamics, in optimal control and so on. For an introduction to the theory of impulsive systems, we refer the reader to see in [4]. In the framework of impulsive differential equations, some exis-

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tence result of periodic solutions for impulsive periodic control systems with parameter perturbations on finite dimensional space has been studied by many authors in [2] and [6].

However, the investigation of the existence of periodic solutions for impulsive periodic control systems with parameter perturbations on infinite dimensional space have not been study. We apply the semigroup theory (see [1] and [5]) and fixed point theorems (see [3] and [7]) for impulsive systems, we establish conditions for ensuring that the system has a unique periodic solution.

The organization of this paper is as follows. Firstly, in Section 2, we introduce some definition of impulsive evolution operator and prove the existence of periodic solution for homogeneous linear impulsive periodic system by using fixed point theorem and Fredholm alternative theorem. In Section 3, nonhomogeneous linear impulsive periodic control system is investigated, we prove the existence of periodic solution by using properties of compact operators and boundedness of solution. Finally, in Section 4, we prove the existence of periodic solution for impulsive periodic control system with parameter perturbations by using fixed point theorems.

A.2 Impulsive Evolution Operator and Homogeneous Linear Impulsive Periodic System

Throughout this paper X will denote a Banach space with norm $\|\cdot\|_X$ and $\mathcal{L}(X)$ denote the space of all bounded linear operators on X. Let $PC([0,T_0]\,;\,X)$ be the space of all functions $x:[0,T_0]\to X$, $x(t)$ is continuous at $t\neq\tau_k$, left continuous at $t=\tau_k$ and the right limit $x(\tau_k^+)$ exists for $k=1,2,\ldots,\sigma$, where $0=\tau_0<\tau_1<\tau_2<\ldots<\tau_{\sigma-1}<\tau_\sigma=T_0<\infty$, which is a Banach space with the norm

$$\|x\|_{PC}=\sup_{t\in[0,T_0]}\|x(t)\|_X.$$

In this paper, we study the existence of periodic solutions for impulsive periodic control systems with parameter perturbations on infinite dimensional space,

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t) + p(t,x,\xi), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k + q_k(x,\xi), & t = \tau_k,\ \ k \in \mathbb{N} \end{cases} \tag{A.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$. Suppose that the system (A.1) satisfy the following assumptions (A1), (A2), and (A3).

(A1.1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots,\ \tau_k \to \infty$ as $k \to \infty$ and there exists a positive integer σ such that $\tau_{k+\sigma} = \tau_k + T_0$ for all $k \in \mathbb{N}$.

(A1.2) A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X.

(A1.3) $B_k \in \mathcal{L}(X)$ such that $B_{k+\sigma} = B_k$ for all $k \in \mathbb{N}$.

(A2.1) $u \in PC([0,\infty), X)$ such that $u(t+T_0) = u(t)$ for all $t \geq 0$.

(A2.2) $c_k \in X$ such that $c_{k+\sigma} = c_k$ for all $k \in \mathbb{N}$.

(A3.1) For each $\rho > 0$ and $x \in \mathcal{B}_\rho := \{x \in X \mid \|x\|_X \leq \rho\}$. $p(\cdot, x, \xi) \in PC([0,\infty), X)$ such that $p(t+T_0, x, \xi) = p(t,x,\xi)$ for all $(t,x,\xi) \in [0,\infty) \times \mathcal{B}_\rho \times [0,\xi_0]$.

(A3.2) $q_k \in C(\mathcal{B}_\rho \times [0,\xi_0], X)$ such that $q_{k+\sigma}(x,\xi) = q_k(x,\xi)$ for all $k \in \mathbb{N}$ and $(x,\xi) \in \mathcal{B}_\rho \times [0,\xi_0]$.

(A3.3) there exists a nonnegative function $\chi(\xi)$ such that

$$\|p(t,x,\xi)\|_X \leq \chi(\xi),\ \|q_k(x,\xi)\|_X \leq \chi(\xi) \quad \text{and} \quad \lim_{\xi \to 0} \chi(\xi) = \chi(0) = 0 \tag{A.2}$$

for all $k \in \mathbb{N}$ and $(t,x,\xi) \in [0,\infty) \times \mathcal{B}_\rho \times [0,\xi_0]$.

For the system (A.1), we give the following definition.

Definition A.1. Let Assumption (A1) hold. An operator value function $U(t,s)$ with values in $\mathcal{L}(X)$, defined on the triangle $\Delta \equiv \{\, 0 \leq s \leq t \leq a \,\}$ with $t, s \in (\tau_{k-1}, \tau_k]$

for all $k \in \mathbb{N}$, given by

$$U(t,s) = \begin{cases} T(t-s), & \tau_{k-1} \le s \le t \le \tau_k, \\ T(t-\tau_k)(I+B_k)T(\tau_k - s), & \tau_{k-1} < s \le \tau_k < t \le \tau_{k+1}, \\ T(t-\tau_k)\left[\prod_{j=i+1}^{k}(I+B_j)T(\tau_j-\tau_{j-1})\right](I+B_i)T(\tau_i - s), & \\ \qquad \text{for } i<k,\ \tau_{i-1} < s \le \tau_i < \ldots < \tau_k < t \le \tau_{k+1}. & \end{cases} \tag{A.3}$$

is called an *impulsive evolution operator.*

Proposition A.1. *Let assumption (A1) hold and* $\{U(t,s), 0 \le s \le t \le a\}$ *be a family of impulsive evolution operators. For each fixed* $T_0 = \tau_\sigma > 0$, *then the following are satisfied :*

(i) $U(t,t) = I$, *the identity operator on* X *;*

(ii) $U(t,s) = U(t,r)U(r,s)$ *for all* $0 \le s \le r \le t \le a$ *;*

(iii) $U(t+KT_0, s+KT_0) = U(t,s)$ *for all* $K \in \mathbb{N}$ *and* $0 \le s \le t \le T_0$ *with* $T_0 \le a$.

(iv) $U(t,0) = U(\bar{t},0)[U(T_0,0)]^M$ *where* $t = \bar{t} + MT_0$ *for all* $\bar{t} \in [0,T_0]$ *and* $M \in \mathbb{N} \cup \{0\}$.

Corollary A.2. *Let assumption (A1) hold and* $\{U(t,s) : 0 \le s \le t \le a\}$ *be a family of impulsive evolution operators, then*

$$\sup_{0 \le s \le t \le a} \|U(t,s)\|_{\mathcal{L}(X)} < \infty \quad \textit{for all} \quad a > 0.$$

Definition A.2. A function $x \in PC([0,\infty);X)$ is said to be *a mild solution* of the system (A.1) with initial condition $x(0) = x_0$ if x is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)[u(s) + p(s,x,\xi)]ds + \sum_{0 \le \tau_k < t} U(t,\tau_k)[c_k + q_k(x,\xi)]. \tag{A.4}$$

Definition A.3. A function $x \in PC([0,\infty);X)$ is said to be *a periodic solution* of the system (A.1) if there exists $T_0 > 0$ such that $x(t+T_0) = x(t)$ for all $t \ge 0$.

Definition A.4. Function $x \in PC([0,\infty);X)$ is said to be *a T_0-periodic solution* of the system (A.1) if $x(t+T_0) = x(t)$ for all $t \geq 0$.

First, we consider the homogeneous linear impulsive periodic system,

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k,\ k \in \mathbb{N}. \end{cases} \tag{A.5}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ and satisfies the assumption (A1).

For the system (A.5), we give the following definition.

Definition A.5. A function $x \in PC([0,\infty);X)$ is said to be *a mild solution* of the system (A.5) with initial condition $x(0) = x_0$ if x is given by

$$x(t) = U(t,0)x_0$$

where

$$U(t,0) = \begin{cases} T(t), & 0 \leq t \leq \tau_1, \\ T(t-\tau_k)\left[\prod_{j=1}^{k}(I+B_j)T(\tau_j - \tau_{j-1})\right], & \tau_k < t \leq \tau_{k+1}, \end{cases} \tag{A.6}$$

for all $k \in \mathbb{N}$.

Remark A.1. If $\{T(t), t > 0\}$ is a compact semigroup in X, then $U(t,0)$ is a compact operator. Particularly, $U(T_0,0)$ is also a compact operator.

Theorem A.3. *Let assumption (A1) hold. The system (A.5) has a periodic solution if and only if the operator $U(T_0,0)$ has a fixed point $x_0 \in X$.*

Proof. Let $x(t)$ be a periodic solution of system (A.5). Suppose $x(0) = x_0$ be the initial condition of system (A.5), then $x(T_0) = x(0) = x_0$. Since $x(T_0) = U(T_0,0)x_0$, then $x_0 = U(T_0,0)x_0$. That is, the operator $U(T_0,0)$ has a fixed point $x_0 \in X$. Conversely, assume that x_0 be a fixed point of $U(T_0,0)$. Use x_0 as the initial condition of system (A.5), then the solution is $x(t) = U(t,0)x_0$ where $t = \bar{t} + MT_0$ for all $\bar{t} \in [0,T_0]$ and

$M \in \mathbb{N} \cup \{0\}$. By assumption and Proposition A.1 (4), we have $x(t) = x(\bar{t} + MT_0) = U(\bar{t}, 0)[U(T_0, 0)]^M x_0 = U(\bar{t}, 0)x_0 = x(\bar{t})$. Hence x is a periodic solution of system (A.5). □

Theorem A.4. *Let assumption (A1) hold. Furthermore, assume that A is the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$ in X. Then system (A.5) either has a unique trivial solution or have finitely many linearly independent nontrivial periodic solutions in $PC([0, \infty), X)$.*

Proof. Since $U(T_0, 0) : X \to X$ is a compact linear operator, then by applying Fredholm alternative theorem (see[3]), we obtain $U(T_0, 0)$ satisfy Fredholm alternative that either (a) or (b) holds: (a)The homogenous equations $[I - U(T_0, 0)]x = 0$ have only the trivial solution $x = 0$. That is, $U(T_0, 0)$ has only a unique fixed point $x = 0$ (i.e., by theorem A.3, this means that system (A.5) has a unique trivial solution). (b) The homogenous equations $[I - U(T_0, 0)]x = 0$ have nontrivial solutions, then all of linearly independent nontrivial solutions are finite. Suppose all of nontrivial solutions $x_0^1, x_0^2, \ldots, x_0^m$ be such that $[I - U(T_0, 0)]x_0^i = 0$, $i = 1, 2, \ldots, m$. So $x_0^1, x_0^2, \ldots, x_0^m$ are fixed points of $U(T_0, 0)$. Again by Theorem A.3, this means that system (A.5) have periodic solutions, say $x^1, x^2, \ldots, x^m$ where x^i are the solutions of system (A.5) corresponding to initial conditions $x^i(0) = x_0^i$, $i = 1, 2, \ldots, m$. Hence the number of linearly independent nontrivial periodic solutions of system (A.5) are finite. □

A.3 Nonhomogeneous Linear Impulsive Periodic Control System

We consider the following nonhomogeneous linear impulsive periodic control system,

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t) + c_k, & t = \tau_k, \ k \in \mathbb{N} \end{cases} \tag{A.7}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ and A is the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$ in X. Suppose that system (A.7) satisfy the assumptions (A1) and (A2).

For system (A.7), we give the following definition.

Definition A.6. A function $x \in PC([0,\infty), X)$ is said to be a *mild solution* of system (A.7) with initial condition $x(0) = x_0$ and the input $u \in L^1_{loc}([0,\infty), X)$ if x is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)u(s)ds + \sum_{0 \le \tau_k < t} U(t,\tau_k)c_k, \tag{A.8}$$

for all $k \in \mathbb{N}$.

To be able to apply the method in Pazy [5], we also need the following lemma.

Lemma A.5. ([5]) Consider the nonhomogeneous initial value problem

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t), \quad t > 0; \\ x(0) = x_0. \end{cases} \tag{A.9}$$

If $u \in L^1_{loc}([0,\infty), X)$, then for every $x_0 \in X$ the initial value problem (A.9) has a unique solution which satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-s)u(s)ds, \quad 0 \le t \le T_0. \tag{A.10}$$

Theorem A.6. *If assumptions (A1) and (A2) hold, then system (A.7) has a unique mild solution* $x \in PC([0, T_0], X)$.

Proof. For $t \in [0, \tau_1]$, Lemma A.5 implies that system

$$\dot{x}(t) = Ax(t) + u(t), \quad 0 \le t \le \tau_1, \quad x(0) = x_0, \tag{A.11}$$

has a unique mild solution on $I_1 = [0, \tau_1]$ which satisfies

$$x_1(t) = T(t)x_0 + \int_0^t T(t-s)u(s)ds, \quad t \in [0, \tau_1]. \tag{A.12}$$

Now, define

$$x_1(\tau_1) = T(\tau_1)x_0 + \int_0^{\tau_1} T(\tau_1 - s)u(s)ds, \tag{A.13}$$

so that $x_1(\cdot)$ is left continuous at τ_1.

Next, on $I_2 = (\tau_1, \tau_2]$, consider system

$$\dot{x}(t) = Ax(t) + u(t), \quad \tau_1 < t \leq \tau_2, \quad x_1(\tau_1^+) = (I + B_1)x_1(\tau_1) + c_1, \tag{A.14}$$

Since $x_1 \in X$, we can use Lemma A.5 again to get a unique mild solution on $(\tau_1, \tau_2]$ which satisfying

$$x_2(t) = T(t - \tau_1)\left[(I + B_1)x_1(\tau_1) + c_1\right] + \int_{\tau_1}^{t} T(t - s)u(s)ds. \tag{A.15}$$

Now, define $x_2(\tau_2)$ accordingly so that $x_2(\cdot)$ is left continuous at τ_2.

It is easily seen that Lemma A.5 can be applied to interval $(\tau_1, \tau_2]$ to verify that $x_2(\tau_2) \in X$. It is also easily seen that this procedure can be repeated on $I_k = (\tau_{k-1}, \tau_k],\ k = 3, 4, \ldots, \sigma\ (\tau_\sigma = T_0)$ to get a mild solutions

$$x_k(t) = \quad T(t - \tau_{k-1})\left[(I + B_{k-1})x_{k-1}(\tau_{k-1}) + c_{k-1}\right] + \int_{\tau_{k-1}}^{t} T(t - s)u(s)ds.$$

for $t \in (\tau_{k-}, \tau_k]$ and define $x_k(\tau_k)$ accordingly with $x_k(\cdot)$ left continuous at τ_k and $x_k(\tau_k) \in X,\ k = 1, 2, \ldots, \sigma$.

Thus we obtain $x \in PC([0, T_0], X)$ is a unique mild solution of system (A.7) and given by.

$$x(t) = \begin{cases} x_1(t), & 0 \leq t \leq \tau_1, \\ x_k(t), & \tau_{k-1} < t \leq \tau_k,\ k = 2, 3, \ldots, \sigma. \end{cases}$$

□

Next, by mathematical induction to show that (A.8) is satisfied on $[0, T_0]$. First, (A.8) is satisfied on $[0, \tau_1]$. If (A.8) is satisfied on $(\tau_{k-1}, \tau_k]$, then for $t \in (\tau_k, \tau_{k+1}]$,

$$\begin{aligned} x(t) &= x_{k+1}(t) = T(t - \tau_k)\left[(I + B_k)x_k(\tau_k) + c_k\right] + \int_{\tau_k}^{t} T(t - s)u(s)ds \\ &= T(t - \tau_k)(I + B_k)x(\tau_k) + T(t - \tau_k)c_k + \int_{\tau_k}^{t} T(t - s)u(s)ds \\ &= T(t - \tau_k)(I + B_k)\Big[U(\tau_k, 0)x_0 + \int_0^{\tau_k} U(\tau_k, s)u(s)ds + \sum_{0 \leq \tau_i < \tau_k} U(\tau_k, \tau_i)c_i\Big] \\ &\quad + T(t - \tau_k)c_k + \int_{\tau_k}^{t} T(t - s)u(s)ds \end{aligned}$$

$$= U(t,0)x_0 + \int_0^{\tau_k} U(t,s)u(s)ds + \int_{\tau_k}^{t} U(t,s)u(s)ds$$

$$+ \textstyle\sum_{0\le\tau_i<\tau_k} U(t,\tau_i)c_i + U(t,\tau_k)c_k$$

$$= U(t,0)x_0 + \int_0^{t} U(t,s)u(s)ds + \sum_{0\le\tau_i<t} U(t,\tau_i)c_i.$$

Thus (A.8) is also true on $(\tau_k, \tau_{k+1}]$. Therefore (A.8) is true on $[0, T_0]$.

If $x(t)$ is T_0-periodic solution of system (A.7), then we have $x(T_0) = x(0)$; namely,

$$[I - U(T_0,0)]x(0) = \int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k. \tag{A.16}$$

We consider into 2 cases.

Case 1 : $[I - U(T_0,0)]^{-1}$exists

Theorem A.7. *Let assumptions (A1) and (A2) hold. Assume that $[I - U(T_0,0)]^{-1}$ exists and system (A.5) has no nontrivial periodic solution , then system (A.7) has a unique T_0-periodic solution*

$$x_{T_0}(t) = U(t,0)[I - U(T_0,0)]^{-1}\Bigg(\int_0^{T_0} U(T_0,s)u(s)\,ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\Bigg) + \int_0^{t} U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k. \tag{A.17}$$

Proof. Suppose that $[I - U(T_0,0)]^{-1}$ exists and system (A.5) has only trivial solution. Then (A.16) gives

$$x(0) = [I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\right) := x_0.$$

Substitute $x(0) = x_0$ into equation (A.8), we get

$$x(t) = U(t,0)[I - U(T_0,0)]^{-1}\Bigg(\int_0^{T_0} U(T_0,s)u(s)\,ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\Bigg) + \int_0^{t} U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k. \tag{A.18}$$

which is a mild solution of system (A.7).

Next, we want to show that a mild solution is unique and is T_0-periodic. Suppose that $y(t) = x(t+T_0)$ is a mild solution of system (A.7).

By Proposition A.1(3), we obtain

$$\begin{aligned}
y(t) = x(t+T_0) &= U(t+T_0,0)x_0 + \int_0^{t+T_0} U(t+T_0,s)u(s)ds \\
&\quad + \sum_{0\le\tau_k<t+T_0} U(t+T_0,\tau_k)c_k \\
&= U(t+T_0,T_0)U(T_0,0)x_0 + \int_0^{T_0} U(t+T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(t+T_0,\tau_k)c_k \\
&\quad + \int_{T_0}^{t+T_0} U(t+T_0,s)u(s)ds + \sum_{T_0\le\tau_k<t+T_0} U(t+T_0,\tau_k)c_k \\
&= U(t,0)U(T_0,0)x_0 + \int_0^{T_0} U(t+T_0,T_0)U(T_0,s)u(s)ds \\
&\quad + \sum_{0\le\tau_k<T_0} U(t+T_0,T_0)U(T_0,\tau_k)c_k + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k \\
&= U(t,0)U(T_0,0)x_0 + U(t,0)\int_0^{T_0} U(T_0,s)u(s)ds + U(t,0)\sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k \\
&\quad + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k \\
&= U(t,0)\left[U(T_0,0)x_0 + \int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\right] \\
&\quad + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k \\
&= U(t,0)x(T_0) + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k \\
&= U(t,0)y(0) + \int_0^t U(t,s)u(s)ds + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k.
\end{aligned}$$

This implies that $y(t)$ is also a solution. By Corollary A.5 implies that $y(t) = x(t+T_0) = x(t)$ for all $t \ge 0$. So $x(t)$ is a T_0-periodic solution of system (A.7), which is exactly (A.17). This completes the proof. □

Case 2 : $[I - U(T_0,0)]^{-1}$ does not exists

In this case, system (A.5) has nontrivial T_0-periodic solutions. Let us construct the following adjoint equation of system (A.5),

$$\begin{cases} \dot{y}(t) &= -A^* y, \quad t \neq \tau_k, \\ -\Delta y(t) &= B_k^* y(t), \quad t = \tau_k,\ k = 1, 2, \ldots, \sigma \end{cases} \tag{A.19}$$

where A^* is the adjoint operator of A, $0 < \tau_1 < \tau_2 < \ldots < \tau_{\sigma-1} < \tau_\sigma = T_0$ and $\Delta y(\tau_k) = y(\tau_k^+) - y(\tau_k^-)$. Suppose that system (A.19) satisfies the following assumption (A4).

(A4.1) A^* is the infinitesimal generator of the adjoint semigroup $\{T^*(t), t \geq 0\}$ in X^*;

(A4.2) $B_k^* \in \mathcal{L}(X^*)$ such that $B_{k+\sigma}^* = B_k^*$ for all $k \in \mathbb{N}$.

Definition A.7. A function $y \in PC([0, T_0], X)$ is said to be a *periodic solution* of system (A.19) with initial condition $y(T_0) = y(0)$ if y is given by

$$y(t) = U^*(T_0, t) y(0), \quad 0 \leq t \leq T_0, \tag{A.20}$$

where

$$U^*(T_0, t) = \begin{cases} T^*(T_0 - t), & \tau_{\sigma-1} < t \leq \tau_\sigma = T_0, \\ T^*(\tau_i - t)(I + B_i^*) \left[\prod_{j=i+1}^{k} (I + B_j) T(\tau_j - \tau_{j-1}) \right]^* T^*(T_0 - \tau_k) & \\ & 0 \leq \tau_{i-1} < t \leq \tau_i \leq \tau_\sigma = T_0, \end{cases} \tag{A.21}$$

for all $i = 1, 2, \ldots, \sigma - 1$.

Theorem A.8. *Let assumptions (A1) and (A2) hold. Furthermore, assume that X is a Hilbert space and $u \in L^1_{loc}([0, \infty), X)$. If system (A.5) have m linearly independent periodic solutions $x^1, x^2, \ldots, x^m$ with $1 \leq m \leq n$ where x^i are periodic solutions of system (A.5) corresponding to initial conditions $x^i(0) = x_0^i$ for all $i = 1, 2, \ldots, m$, then*

(i) *the adjoint system (A.19) also have m linearly independent periodic solutions $y^1, y^2, \ldots, y^m$;*

(ii) *system (A.7) has a T_0-periodic solution if and only if*

$$\langle\, y\,,\, z\,\rangle = 0, \tag{A.22}$$

where $\langle\, y\,,\, z\,\rangle$ the pairing of an element $y \in X^*$ with an element $z \in X$ such that

$$[I - U^*(T_0, 0)]y = 0 \tag{A.23}$$

and $z := \int_0^{T_0} U(T_0, s)u(s)ds + \sum_{0 \le \tau_k < T_0} U(T_0, \tau_k)c_k$, *or if and only if*

$$\int_0^{T_0} \langle\, y(s),\, u(s)\,\rangle ds + \sum_{0 \le \tau_k < T_0} \langle\, y(\tau_k),\, c_k\,\rangle = 0. \tag{A.24}$$

Furthermore, let $x_a(t)$ be a particular T_0-periodic solution of system (A.7), then each T_0-periodic solution of system (A.7) has the form

$$x(t) = x_a(t) + \sum_{i=1}^{m} \alpha_i x^i(t),$$

where $\alpha_i,\ i = 1, 2, \ldots, m$ are constants.

Proof. (i) Suppose system (A.5) have m linearly independent periodic solutions $x^1, x^2, \ldots, x^m$ with $1 \le m \le n$ where x^i are periodic solutions of system (A.5) corresponding to initial conditions $x^i(0) = x^i_0$, for all $i = 1, 2, \ldots, m$. By Theorem A.3, this means that the equations

$$[I - U(T_0, 0)]x^i_0 = 0 \tag{A.25}$$

have fixed points $x^1_0, x^2_0, \ldots, x^m_0$. Then from Theorem 8.6-3 [3], we know that the following adjoint equations of (A.25)

$$[I - U^*(T_0, 0)]y^i_0 = 0, \quad \text{where} \quad y^i_0 = y^i(0) \tag{A.26}$$

also have m linearly independent solutions $y^1_0, y^2_0, \ldots, y^m_0$. So $y^1_0, y^2_0, \ldots, y^m_0$ are fixed points of $U^*(T_0, 0)$. Again by Theorem A.3, this means that system (A.19) have periodic solutions, say $y^1,\ y^2, \dots,\ y^m$ where y^i are periodic solutions of system (A.19) corresponding to initial conditions $y^i(0) = y^i_0$, for all $i = 1, 2, \dots, m$.

(ii) System (A.7) has a T_0-periodic solution $x(t)$ if and only if the equation

$$[I - U(T_0, 0)]x(0) = \int_0^{T_0} U(T_0, s)u(s)ds + \sum_{0 \le \tau_k < T_0} U(T_0, \tau_k)c_k := z \tag{A.27}$$

has a solution $x(0)$. It follows from Theorem 8.5-1 [3], that the above condition is equivalent to

$$\langle\, y\,,\, z\,\rangle = 0, \tag{A.28}$$

for all $y \in X^*$ satisfying

$$[I - U^*(T_0, 0)]y = 0 \tag{A.29}$$

From equation (A.28), we obtain

$$\begin{aligned}
\langle\, y\,,\, z\,\rangle = 0 \quad &\Leftrightarrow \quad \langle\, y, \int_0^{T_0} U(T_0, s)u(s)ds + \sum_{0 \le \tau_k < T_0} U(T_0, \tau_k)c_k\,\rangle = 0 \\
&\Leftrightarrow \quad \int_0^{T_0} \langle\, y,\, U(T_0, s)u(s)\,\rangle ds + \sum_{0 \le \tau_k < T_0} \langle\, y,\, U(T_0, \tau_k)c_k\,\rangle = 0 \\
&\Leftrightarrow \quad \int_0^{T_0} \langle\, U^*(T_0, s)y,\, u(s)\,\rangle ds + \sum_{0 \le \tau_k < T_0} \langle\, U^*(T_0, \tau_k)y,\, c_k\,\rangle = 0 \\
&\Leftrightarrow \quad \int_0^{T_0} \langle\, y(s),\, u(s)\,\rangle ds + \sum_{0 \le \tau_k < T_0} \langle\, y(\tau_k),\, c_k\,\rangle = 0,
\end{aligned}$$

from which we immediately have (A.24). This completes the proof. □

The following theorem guarantee the existence of periodic solution. The proof is based on boundedness property.

Theorem A.9. *If system (A.7) has a bounded solution, then it has at least one T_0-periodic solution.*

Proof. Assume that $x(t)$ is a bounded solution of system (A.7). Then for any $t \geq 0$, we have

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)u(s)ds + \sum_{0 \leq \tau_k < t} U(t,\tau_k)c_k,$$

where $x(0) = x_0$ and

$$x(T_0) = U(T_0,0)x_0 + \int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0 \leq \tau_k < T_0} U(T_0,\tau_k)c_k.$$

Define $z := \int_0^{T_0} U(T_0,s)u(s)ds + \sum_{0 \leq \tau_k < T_0} U(T_0,\tau_k)c_k$, then

$$x(T_0) = U(T_0,0)x_0 + z.$$

We know that the function $x(t+T_0)$ is also a solution of system (A.7) for $t \in [0,T_0]$ and its value at $t=0$ is $x(T_0)$. So

$$x(t+T_0) = U(t,0)x(T_0) + \int_0^t U(t,s)u(s)ds + \sum_{0 \leq \tau_k < t} U(t,\tau_k)c_k$$

and

$$x(2T_0) = U(T_0,0)x(T_0) + z = U^2(T_0,0)x_0 + [U(T_0,0)+I]z.$$

Proceeding by this way, we get

$$x(mT_0) = U^m(T_0,0)x_0 + \sum_{i=0}^{m-1} U^i(T_0,0)z \quad \text{for all } m \in \mathbb{N}. \tag{A.30}$$

By contradiction, we assume that (A.7) has no T_0-periodic solution. This means that the periodicity condition

$$x(T_0) = U(T_0,0)x_0 + z = x_0 \tag{A.31}$$

has no solution, i.e., the equation

$$[I - U(T_0,0)]x_0 = z \tag{A.32}$$

has no solution. Then from Theorem 8.5-1 [3], we know that there is $y \in X^*$ such that

$$[I - U^*(T_0, 0)]y = 0 \quad \text{and} \quad \langle\, y\,,\, z\rangle \neq 0. \tag{A.33}$$

The first condition means that $U^*(T_0, 0)y = y$, hence

$$U^{*^m}(T_0, 0)y = y, \quad \text{for all } \ m \in \mathbb{N}. \tag{A.34}$$

Assume that $\langle\, y\,,\, z\rangle = \gamma \neq 0$. Then from equation (A.30), we have

$$\begin{aligned}
\langle\, y\,,\, x(mT_0)\,\rangle &= \langle\, y\,,\, U^m(T_0, 0)x_0\,\rangle + \sum_{i=0}^{m-1}\langle\, y\,,\, U^i(T_0, 0)z\rangle \\
&= \langle\, U^{*^m}(T_0, 0)y\,,\, x_0\,\rangle + \sum_{i=0}^{m-1}\langle\, U^{*^i}(T_0, 0)y\,,\, z\rangle \\
&= \langle\, y\,,\, x_0\,\rangle + \sum_{i=0}^{m-1}\langle\, y\,,\, z\rangle \\
&= \langle\, y\,,\, x_0\,\rangle + m\gamma.
\end{aligned}$$

Letting $m \to \infty$, then

$$\lim_{m\to\infty}\langle\, y\,,\, x(mT_0)\,\rangle = \infty. \tag{A.35}$$

Since $x(t)$ is bounded solution and $y \in X^*$, then

$$|\langle\, y\,,\, x(mT_0)\,\rangle| \;\leq\; \|y\|_{X^*}\|x(mT_0)\|_X \;\leq\; M\|y\|_{X^*} \;<\; \infty.$$

It's contradiction to (A.35). Consequently, the assumption is not true and system (A.7) has at least one T_0-periodic solution. □

Corollary A.10.

(i) *Assume that system (A.7) has no T_0-periodic solution, then all of its solutions are unbounded for $t \geq 0$.*

(ii) *Assume that system (A.7) has a unique bounded solution for $t \geq 0$, then this solution is T_0-periodic.*

A.4 Impulsive Periodic Control System with Parameter Perturbations

In this section, we will find sufficient conditions for the existence of T_0-periodic solutions of system (A.1), by using the fixed point theorems of an operator acting in a Banach space (see [7]). We assume that system (A.5) has only trivial solution. Let $\xi = 0$, then system (A.1) has the same form as system (A.7) because it follows from (A.2) that $p(t, x, 0) = 0$ and $q_k(x, 0) = 0$. It follows from Theorem A.7, that system (A.1) has a T_0-periodic solution ;

$$\begin{aligned} x_{T_0}(t) &= U(t,0)[I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)u(s)\,ds \right. \\ &\quad \left. + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)c_k\right) + \int_0^t U(t,s)u(s)ds \\ &\quad + \sum_{0\le\tau_k<t} U(t,\tau_k)c_k, \end{aligned} \tag{A.36}$$

where $U(t, s)$ is defined in (A.3). Then we have the following theorem to show that for small ξ system (A.1) has a T_0-periodic solution which is closed to $x_{T_0}(t)$.

Theorem A.11. *Under assumption (A1)-(A3). Let A be the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$ in X. Assume that*

(i) system (A.5) has only trivial solution ;

(ii) the following inequality is valid

$$\rho_0 = \sup_{t\in[0,T_0]} \|x_{T_0}(t)\|_X < \rho \tag{A.37}$$

where ρ be any positive real number ;

(iii) $p(t, x, \xi)$ and $q_k(x, \xi)$ satisfy Lipschitz conditions, i.e. for any (t, x, ξ), $(t, y, \xi) \in [0,\infty) \times \mathcal{B}_\rho \times [0, \xi_0]$, there exists a constant $N(\xi) > 0$ such that

$$\|p(t, x, \xi) - p(t, y, \xi)\|_X \le N(\xi)\|x - y\|_X$$

and $$\|q_k(x,\xi) - q_k(y,\xi)\|_X \le N(\xi)\|x-y\|_X.$$

Then for any constant $\rho > \rho_0 > 0$, *there exists a sufficiently small* $\xi_0 > 0$ *such that for every fixed* $\xi \in [0,\xi_0]$ *system (A.1) has a unique* T_0*-periodic mild solution* $x^{\xi}_{T_0}(t)$ *satisfying*

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\|_X < \rho - \rho_0 \tag{A.38}$$

and

$$\lim_{\xi\to 0} x^{\xi}_{T_0}(t) = x_{T_0}(t) \tag{A.39}$$

uniformly on t.

Proof. Let $PC_{T_0}([0,\infty),X) := \big\{x \in PC([0,\infty),X) \mid x(t+T_0) = x(t), \forall t \ge 0\big\}$. Moreover, $PC_{T_0}([0,T_0],X)$ is a Banach space with the norm

$$\|x\|_{PC_{T_0}} = \sup_{t\in[0,T_0]} \|x(t)\|_X.$$

Let us define

$$\begin{aligned} \mathcal{B} &:= \mathcal{B}(x_{T_0},\rho_1) = \{x \in PC_{T_0}([0,T_0],X) \mid \|x - x_{T_0}\|_{PC_{T_0}} \le \rho_1 := \rho - \rho_0\} \\ L_1 &= \sup_{0\le s\le t\le T_0} \|U(t,s)\|_{\mathcal{L}(X)} \\ L_2 &= \|[I - U(T_0,0)]^{-1}\|_{\mathcal{L}(X)} \end{aligned} \tag{A.40}$$

and an operator $\Omega : \mathcal{B} \to PC_{T_0}([0,T_0]\,,\,X)$ such that

$$\begin{aligned} \Omega(x)(t) := \quad & U(t,0)[I - U(T_0,0)]^{-1}\left(\int_0^{T_0} U(T_0,s)[u(s) + p(s,x(s),\xi)]\,ds\right. \\ & \left. + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[c_k + q_k(x(\tau_k),\xi)]\right) + \int_0^t U(t,s)[u(s) \\ & + p(s,x(s),\xi)]ds + \sum_{0\le\tau_k<t} U(t,\tau_k)[c_k + q_k(x(\tau_k),\xi)]. \end{aligned} \tag{A.41}$$

From (A.37) and (A.40), we know that if $x \in \mathcal{B}$, then

$$\|x\|_{PC_{T_0}} \le \|x - x_{T_0}\|_{PC_{T_0}} + \|x_{T_0}\|_{PC_{T_0}} \le \rho_1 + \rho_0 = \rho. \tag{A.42}$$

For any $x, y \in \mathcal{B}$, we have

$$
\begin{aligned}
\|\Omega(x)-\Omega(y)\|_{PC_{T_0}} &= \sup_{t\in[0,T_0]} \|U(t,0)[I-U(T_0,0)]^{-1} \\
&\quad \Big(\int_0^{T_0} U(T_0,s)[p(s,x(s),\xi)-p(s,y(s),\xi)]\,ds \\
&\quad + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)[q_k(x(\tau_k),\xi)-q_k(y(\tau_k),\xi)]\Big) \\
&\quad + \int_0^t U(t,s)[p(s,x(s),\xi)-p(s,y(s),\xi)]ds \\
&\quad + \sum_{0\le\tau_k<t} U(t,\tau_k)[q_k(x(\tau_k),\xi)-q_k(y(\tau_k),\xi)]\|_X \\
&\le LN(\xi)\|x-y\|_{PC_{T_0}},
\end{aligned}
\tag{A.43}
$$

where $\qquad L = L_1^2L_2T_0 + L_1^2L_2\sigma + L_1T_0 + L_1\sigma \qquad$ and

$$
\begin{aligned}
\|\Omega(x_{T_0})-x_{T_0}\|_{PC_{T_0}} &= \sup_{t\in[0,T_0]} \|U(t,0)[I-U(T_0,0)]^{-1} \\
&\quad \Big(\int_0^{T_0} U(T_0,s)p(s,x_{T_0}(s),\xi)\,ds \\
&\quad + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)q_k(x_{T_0}(\tau_k),\xi)\Big) \\
&\quad + \int_0^t U(t,s)p(s,x_{T_0}(s),\xi)ds \\
&\quad + \sum_{0\le\tau_k<t} U(t,\tau_k)q_k(x_{T_0}(\tau_k),\xi)\|_X \\
&\le L\chi(\xi).
\end{aligned}
\tag{A.44}
$$

Let us choose $\xi_0 > 0$ such that

$$
\eta = L\sup_{|\xi|\le\xi_0} N(\xi) < 1, \qquad L\sup_{|\xi|\le\xi_0}\chi(\xi) \le \rho_1(1-\eta). \tag{A.45}
$$

Assume that $\xi \in [0,\xi_0]$, then it follows from (A.43), (A.44) and (A.45) that

$$
\begin{aligned}
\|\Omega(x)-\Omega(y)\|_{PC_{T_0}} &\le \eta\|x-y\|_{PC_{T_0}}, \\
\|\Omega(x_{T_0})-x_{T_0}\|_{PC_{T_0}} &\le \rho_1(1-\eta).
\end{aligned}
\tag{A.46}
$$

This means that $\Omega : \mathcal{B} \to \mathcal{B}$ is a contraction mapping, so Ω has a unique fixed point $x^{\xi}_{T_0} \in \mathcal{B}$ satisfy

$$\begin{aligned} x^{\xi}_{T_0}(t) = \ & U(t,0)[I - U(T_0,0)]^{-1} \left(\int_0^{T_0} U(T_0,s)[u(s) + p(s, x^{\xi}_{T_0}(s), \xi)]\, ds \right. \\ & \left. + \sum_{0 \le \tau_k < T_0} U(T_0, \tau_k)[c_k + q_k(x^{\xi}_{T_0}(\tau_k), \xi)] \right) \\ & + \int_0^t U(t,s)[u(s), p(s, x^{\xi}_{T_0}(s), \xi)] ds \\ & + \sum_{0 \le \tau_k < t} U(t, \tau_k)[c_k + q_k(x^{\xi}_{T_0}(\tau_k), \xi)]. \end{aligned} \tag{A.47}$$

It is clear that $x^{\xi}_{T_0}(t)$ is a T_0-periodic solution of system (A.1) and satisfies estimate (A.38). Since we know that $\Omega(x^{\xi}_{T_0})(t) = x^{\xi}_{T_0}(t)$ for all $t \in [0, T_0]$.

Then $\quad \|x^{\xi}_{T_0}(t) - x_{T_0}(t)\|_X = \|\Omega(x^{\xi}_{T_0})(t) - x_{T_0}(t)\|_X \le L\chi(\xi).$

Letting $\xi \to 0$, we obtain (A.39). This completes the proof. □

The following definition and lemma will be used in the proof of Theorem A.13.

Definition A.8. A set $\mathcal{S} \subset PC([0,T_0],X)$ is *quasiequicontinuous* in $[0,T_0]$ if for any $\delta > 0$ there exists $\varepsilon > 0$ such that if $x \in \mathcal{S}$, $t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap [0, T_0]$, $k \in \mathbb{N}$ and $|t_1 - t_2| < \varepsilon$, then $\|x(t_1) - x(t_2)\|_X < \delta$.

Lemma A.12. A set $\mathcal{S} \subset PC([0,T_0],X)$ is relatively compact if and only if

(i) $\mathcal{S}$ is bounded for each $x \in \mathcal{S}$,

(ii) $\mathcal{S}$ is quasiequicontinuous in $[0, T_0]$.

Theorem A.13. *Under assumption (A1)-(A3). Let A be the infinitesimal generator of a compact semigroup $\{T(t), t > 0\}$ in X. Assume that*

(ii) system (A.5) has only trivial solution;

(i) the following inequality is valid

$$\rho_0 = \sup_{t\in[0,\infty]} \|x_{T_0}(t)\|_X < \rho. \tag{A.48}$$

Then for any constant $\rho > \rho_0 > 0$, *there exists a sufficiently small* $\xi_0 > 0$ *such that for every fixed* $\xi \in [0, \xi_0]$ *system (A.1) has a unique* T_0*-periodic mild solution* $x^{\xi}_{T_0}(t)$ *satisfying*

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\|_X \le \rho - \rho_0. \tag{A.49}$$

Proof. As in the proof of Theorem A.11, we determine successively the number $\rho_1 = \rho - \rho_0$, the Banach space $PC_{T_0}([0, T_0], X)$, the set $\mathcal{B} := \mathcal{B}(x_{T_0}; \rho_1)$ and the operator $\Omega : \mathcal{B} \to PC_{T_0}([0, T_0], X)$ is defined in (A.41). Obviously, $\mathcal{B}$ is a non-empty bounded closed and convex set. It follows from equation (A.42) that if $x \in \mathcal{B}$, then $\|x\|_{PC_{T_0}} \le \rho$. For any $x \in \mathcal{B}$, we have

$$\begin{aligned}
\|\Omega(x_{T_0}) - x_{T_0}\|_{PC_{T_0}} = & \sup_{t\in[0,T_0]} \|U(t,0)[I - U(T_0,0)]^{-1} \\
& \Big(\int_0^{T_0} U(T_0,s)p(s, x_{T_0}(s), \xi)\, ds \\
& + \sum_{0\le\tau_k<T_0} U(T_0,\tau_k)q_k(x_{T_0}(\tau_k), \xi)\Big) \\
& + \int_0^t U(t,s)p(s, x_{T_0}(s), \xi)ds \\
& + \sum_{0\le\tau_k<T_0} U(t,\tau_k)q_k(x_{T_0}(\tau_k), \xi)\|_X \\
\le & \Big(L_1^2 L_2 T_0 + L_1^2 L_2 \sigma + L_1 T_0 + L_1\sigma\Big)\chi(\xi).
\end{aligned}$$

So

$$\|\Omega(x_{T_0}) - x_{T_0}\|_{PC_{T_0}} \le L\chi(\xi). \tag{A.50}$$

where $L = L_1^2 L_2 T_0 + L_1^2 L_2\sigma + L_1 T_0 + L_1\sigma$.

Let us choose $\xi \in [0, \xi_0]$ such that

$$L \sup_{\xi\in[0,\xi_0]} \chi(\xi) \le \rho_1. \tag{A.51}$$

Then for $\xi \in [0, \xi_0]$, we have

$$\|\Omega(x_{T_0}) - x_{T_0}\|_{PC_{T_0}} \le L\chi(\xi) \le \rho_1, \tag{A.52}$$

From which we know that $\Omega(x) \in \mathcal{B}$ and therefore $\Omega : \mathcal{B} \to \mathcal{B}$.

It follows from (A.36), (A.48) and (A.52) that

$$\|\Omega(x_{T_0})\|_{PC_{T_0}} \le \|\Omega(x_{T_0}) - x_{T_0}\|_{PC_{T_0}} + \|x_{T_0}\|_{PC_{T_0}} \le \rho_1 + \rho_0 = \rho. \tag{A.53}$$

That is, the set $\mathcal{B}$ is uniformly bounded.

Let $x \in \mathcal{B}_\rho$ and $t_1, t_2 \in (\tau_{i-1}, \tau_i] \cap [0, T_0]$, $i = 1, 2, \ldots, \sigma$, where $\tau_0 = 0$ and $\tau_\sigma = T_0$.

For $0 < \varepsilon < t_1 < t_2 \le T_0$, then we have

$$\begin{aligned}
\|(\Omega x)(t_1) - (\Omega x)(t_2)\|_X \le \; & \|U(t_1, 0) - U(t_2, 0)\|_{\mathcal{L}(X)} \|[I - U(T_0, 0)]^{-1}\|_{\mathcal{L}(X)} \\
& \Big(\int_0^{T_0} \|U(T_0, s)\|_{\mathcal{L}(X)} \|u(s) + p(s, x(s), \xi)\|_X \, ds \\
& + \sum_{0 \le \tau_k < T_0} \|U(T_0, \tau_k)\|_{\mathcal{L}(X)} \|c_k + q_k(x(\tau_k), \xi)\|_X \Big) \\
& + \int_0^{t_1 - \epsilon} \|U(t_1, s) - U(t_2, s)\|_{\mathcal{L}(X)} \|u(s) + p(s, x(s), \xi)\|_X ds \\
& + \int_{t_1 - \epsilon}^{t_1} \|U(t_1, s) - U(t_2, s)\|_{\mathcal{L}(X)} \|u(s) + p(s, x(s), \xi)\|_X ds \\
& + \int_{t_1}^{t_2} \|U(t_2, s)\|_{\mathcal{L}(X)} \|u(s) + p(s, x(s), \xi)\|_X ds \\
& + \sum_{0 \le \tau_k < t} \|U(t_1, \tau_k) - U(t_2, \tau_k)\|_{\mathcal{L}(X)} \|c_k + q_k(x(\tau_k), \xi)\|_X .
\end{aligned}$$

from which we know that for any $\delta > 0$, there exists $\varepsilon > 0$ such that if $t_1 - t_2 < \varepsilon$, then $\|\Omega(x)(t_1) - \Omega(x)(t_2)\|_X < \delta$. Thus $\mathcal{B}$ is quasiequicontinuous and by Lemma A.12, we know that the following set is relatively compact in $\mathcal{B}$;

$$\mathcal{S} = \{y \in \mathcal{B} \mid y = \Omega(x), \; x \in \mathcal{B}\}. \tag{A.54}$$

Applying Schuader's fixed point theorem, it follows that the operator Ω has a fixed point $x_{T_0}^{\xi} \in \mathcal{B}$ and satisfies equation (A.47). It is clear that $x_{T_0}^{\xi}(t)$ is a T_0-periodic solution of system (A.1). $\square$

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