CONTINUOUS WAVELETS ASSOCIATED WITH MATRIX GROUPS

Wannapa Romero

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in Applied Mathematics
Suranaree University of Technology
Academic Year 2006
ISBN 974-533-587-8

เวฟเลตต่อเนื่องที่เกี่ยวเนื่องกับกลุ่มเมทริกซ์

นางวรรณนภา โรเมโร

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์
มหาวิทยาลัยเทคโนโลยีสุรนารี
ปีการศึกษา 2549

ISBN 974-533-587-8

CONTINUOUS WAVELETS ASSOCIATED WITH MATRIX GROUPS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

Thesis Examining Committee

(Assoc. Prof. Dr. Prapasri Asawakun)

Chairperson

(Asst. Prof. Dr. Eckart Schulz)

Member (Thesis Advisor)

(Asst. Prof. Dr. Arjuna Chaiyasena)

Member

(Assoc. Prof. Dr. Pairote Sattayatham)

Member

(Dr. Songkiat Sumetkijakan)

Member

(Assoc. Prof. Dr. Saowanee Rattanaphani)

Vice Rector for Academic Affairs

(Assoc. Prof. Dr. Sompong Thammathaworn)

Dean of Institute of Science

วรรณนภา โรเมโร : เวฟเลตต่อเนื่องที่เกี่ยวเนื่องกับกลุ่มเมทริกซ์ (CONTINUOUS WAVELET ASSOCIATED WITH MATRIX GROUPS) อาจารย์ที่ปรึกษา : Asst. Prof. Dr. Eckart Schulz, 104 หน้า. ISBN 974-533-587-8

งานวิจัยนี้ศึกษาการสร้างฟังก์ชันแอดมิซซิเบิลสำหรับการแปลงเวฟเลตต่อเนื่องที่เกี่ยวเนื่องกับกลุ่มย่อยแบบปิด H ของกลุ่มเชิงเส้นวางนัยทั่วไป โดยแนะนำภาคตัดขวางวางนัยทั่วไปสำหรับการกระทำของ H บนปริภูมิแบบยุคลิดและแสดงการหาฟังก์ชันแอดมิซซิเบิลจากภาคตัดขวางวางนัยทั่วไป ถ้ามีภาคตัดขวางวางนัยทั่วไปกระชับที่มีสมบัติว่าการส่งวงโคจรเป็นการส่งแบบเปิดและถ้าวงโคจรสอดคล้องกับเงื่อนไขบางประการแล้ว ฟังก์ชันแอดมิซซิเบิลที่เป็นแบนด์ลิมิตและปรับเรียบจะมีจริง การสร้างนี้ยังสามารถประยุกต์ไปยังกลุ่มของเมทริกซ์เฉียงซึ่งขึ้นอยู่กับพารามิเตอร์ p ตัว

ภาคตัดขวางวางนัยทั่วไปยังทำให้สามารถขยายการสร้างกรอบเวฟเลตวิยุตที่มีอยู่แล้วไปยังกลุ่ม H กับวงโคจรใดๆ ถ้ากลุ่ม H บรรจุกลุ่มย่อยกระชับร่วมและเป็นเซตวิยุตแล้ว กรอบกระชับจะมีจริง ได้มีการแสดงโดยยกตัวอย่างว่า การมีเมทริซ์แผ่ขยายไม่เป็นเงื่อนไขจำเป็นที่ทำให้กรอบกระชับมีจริง การสร้างเวฟเลตโดยการปริพันธ์จะขยายไปยังกลุ่ม H กับวงโคจรใดๆ

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา Wannapa Romero

ปีการศึกษา 2549 ลายมือชื่ออาจารย์ที่ปรึกษา

WANNAPA ROMERO : CONTINUOUS WAVELETS ASSOCIATED WITH MATRIX GROUPS. THESIS ADVISOR : ASST. PROF. ECKART SCHULZ, Ph.D. 104 PP. ISBN 974-533-587-8

CONTINUOUS WAVELET / WAVELET FRAME / INTEGRATED WAVELET / CROSS-SECTION

The construction of admissible functions for the continuous wavelet transform associated with a closed subgroup H of $GL_n(\mathbb{R})$ is discussed.

For this purpose, a generalized cross-section for the action of H on Euclidean space is introduced, and it is shown how to obtain admissible functions from generalized cross-sections. If there exists a compact generalized cross-section having the property that the orbit map is open, and if orbits satisfy some regularity condition, then smooth, bandlimited admissible functions exist. This construction is applied to p-parameter groups of diagonal matrices.

Generalized cross-sections also allow to extend the known construction of discrete wavelet frames to groups H with arbitrary orbit structure. In addition, if H contains a discrete, co-compact subgroup, then smooth, bandlimited tight frames exist. It is shown by example that the presence of an expanding matrix is not necessary for the existence of tight wavelet frames. Finally, the construction of integrated wavelets is extended to groups with arbitrary orbit structure.

School of Mathematics — Student's Signature Wannapa Romero

Academic Year 2006 — Advisor's Signature

ACKNOWLEDGEMENTS

My gratitude goes to Asst. Prof. Dr. Eckart Schulz for his outstanding guidance throughout the completion of this work and his commitment to excellent teaching at Suranaree University of Technology.

Heartfelt appreciation goes to Assoc. Prof. Dr. Prapasi Assawakun, Assoc. Prof. Dr. Pairote Sattayatham, Asst. Prof. Dr. Arjuna Chaiyasena, Prof. Dr. Sergey V. Meleshko, Assoc. Prof. Dr. Nikolay Moshkin, and Prof. Dr. Boris I. Kvasov for their quality lecturing and support during my stay at the School of Mathematics.

I would also like to thank Dr. Songkiat Sumetkijakan for his asserted comments which significantly improved the quality of this dissertation, and Prof. Keith F. Taylor for his valuable suggestions and the productive discussions we had.

Special mention deserve Dr. Jessada Tanthanuch for his assistance with the Latex software and Dr. Kanthima Thailert, Dr. Surattana Sungnul, and Dr. Boonlert Srihiran for their support and friendship. I would like to thank all friends at the school for their companionship.

I express my sincere appreciation to both the Ministry of University Affairs of Thailand (MUA) and Naresuan University for providing the necessary funds to support this study.

Finally, my deepest appreciation is to my parents, Mr. Iam and Mrs. Surin Ruangthanakorn, my aunt Miss Atchara Manakasikit, my brothers, and my husband David Romero for their support and encouragement during my studies.

Wannapa Romero

CONTENTS

	Page
ABSTRACT IN THAI	I
ABSTRACT IN ENGLISH	II
ACKNOWLEDGEMENTS	III
CONTENTS	IV
CHAPTER	
I INTRODUCTION	1
II BASIC BACKGROUND	7
2.1 Fourier Analysis	7
2.1.1 Preliminaries	7
2.1.2 The Fourier Transform	11
2.2 Topological Groups and Haar Measure	14
2.3 Group Representations	20
III CONTINUOUS WAVELETS	22
3.1 The Continuous Wavelet Transform	22
3.2 Frames	30
IV EXISTENCE OF SMOOTH ADMISSIBLE FUNCTIONS	32
4.1 Generalized Cross-Sections	32
4.2 Admissible Functions from Generalized Cross-Sections	62
V WAVELET FRAMES	75
5.1 Discretization	75
5.2 Integrated Wavelets	87

CONTENTS (Continued)

Page

5.2.1 Admissibility of Integrated Wavelets 89

5.2.2 Frames from Integrated Wavelets 90

VI CONCLUSION 98

REFERENCES 101

CURRICULUM VITAE 104

CHAPTER I

INTRODUCTION

In the classical continuous wavelet transform, as first introduced by Grossmann, Morlet, and Paul (1985, 1986), one begins with a square integrable function ψ defined on the real line, and considers the 2-parameter transforms of functions $f \in L^2(\mathbb{R})$ given by

$$(W_\psi f)(t,x) = 2^{-t/2} \int_{\mathbb{R}} f(y)\overline{\psi(2^{-t}y - x)}\, dy. \tag{1.1}$$

In the language of applications, if the function ψ is suitably well localized, then $(W_\psi f)(t,x)$ is understood to yield information of the *signal* f at location determined by x and at scale 2^t. Grossmann, Morlet, and Paul (1985, 1986) realized that this wavelet transform is connected to the term of group representations, and using the Duflo-Moore theorem for square integrable representations, classified those functions ψ which allow for the reconstruction of f,

$$f(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\psi f)(t,x) 2^{-t/2} \psi(2^{-t}y - x)\, dxdt$$

as a weak integral in $L^2(\mathbb{R})$. Mallat and Zhong (1992) considered such transforms with discrete dilation parameter k,

$$(W_\psi f)(k,x) = 2^{-k/2} \int_{\mathbb{R}} f(y)\overline{\psi(2^{-k}y - x)}\, dy.$$

The natural extension of this concept to functions f in $L^2(\mathbb{R}^n)$ is as follows. Given an invertible $n \times n$ matrix A and a fixed vector $\vec{x} \in \mathbb{R}^n$, consider the 2-parameter transform

$$(W_\psi f)(t,\vec{x}) = |\det(A)|^{-t/2} \int_{\mathbb{R}^n} f(\vec{y})\overline{\psi(A^{-t}\vec{y} - \vec{x})}\, d\vec{y} \tag{1.2}$$

$(t \in \mathbb{R}, \vec{x} \in \mathbb{R}^n)$, called the continuous wavelet transform. Similarly we may consider the 2-parameter transform

$$(W_\psi f)(k, \vec{x}) = |\det(A)|^{-k/2} \int_{\mathbb{R}^n} f(\vec{y})\overline{\psi(A^{-k}\vec{y} - \vec{x})}\, d\vec{y} \tag{1.3}$$

$(k \in \mathbb{Z}, \vec{x} \in \mathbb{R}^n)$, called the semi-discrete wavelet transform. In the continuous case, A^t must be defined for all real t, which requires that A is an exponential, that is, $A^t = e^{tB}$ for all t.

More generally, let H be a closed subgroup of $GL_n(\mathbb{R})$, called a dilation group. The *continuous wavelet transform* associated with H is defined by

$$(W_\psi f)(h, \vec{x}) = \langle f, \psi_{h,\vec{x}} \rangle = |\det(h)|^{-1/2} \int_{\mathbb{R}^n} f(\vec{y})\overline{\psi(h^{-1}\vec{y} - \vec{x})}\, d\vec{y}$$

for $h \in H, \vec{x} \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$, where $\psi_{h,\vec{x}}(\vec{y}) = |\det(h)|^{-1/2}\psi(h^{-1}\vec{y} - \vec{x})$.

In order to be able to reconstruct the function f from its wavelet transform, one wants the map W_ψ to be a multiple of a partial isometry of $L^2(\mathbb{R}^n)$ into $L^2(H \times \mathbb{R}^n)$, in which case one calls the vector ψ *admissible* or the group H *admissible*. In fact, if

$$\|W_\psi f\|^2_{L^2(H\times\mathbb{R}^n)} = c_\psi \|f\|^2_{L^2(\mathbb{R}^n)} \tag{1.4}$$

(c_ψ a positive constant) for all $f \in L^2(\mathbb{R}^n)$, then by the polarization identity,

$$\langle W_\psi f, W_\psi g \rangle_{L^2(H\times\mathbb{R}^n)} = c_\psi \langle f, g \rangle_{L^2(\mathbb{R}^n)}$$

for all $f, g \in L^2(\mathbb{R}^n)$, that is

$$\langle f, g \rangle = \frac{1}{c_\psi} \int_{\mathbb{R}^n} \int_H (W_\psi f)(h, \vec{x})\overline{\langle g, \psi_{h,\vec{x}} \rangle}\, d\mu(h) d\vec{x}$$

or

$$\langle f, g \rangle = \frac{1}{c_\psi} \int_{\mathbb{R}^n} \int_H \langle (W_\psi f)(h, \vec{x})\psi_{h,\vec{x}}, g \rangle\, d\mu(h) d\vec{x}$$

so that

$$f = \frac{1}{c_\psi} \int_{\mathbb{R}^n} \int_H (W_\psi f)(h, \vec{x})\psi_{h,\vec{x}}\, d\mu(h) d\vec{x} \tag{1.5}$$

as a weak integral in $L^2(\mathbb{R}^n)$. Here, μ denotes the left Haar measure on H and $d\vec{x}$ integration with respect to the Lebesgue measure λ on $\mathbb{R}^n$.

It is useful to formulate the wavelet transform in the language of group representations. Let $G = H \rtimes \mathbb{R}^n$ denote the semi direct product of H and $\mathbb{R}^n$ so that G is a closed subgroup of the affine group. It turns out that the Haar measure $d\nu$ on G is simply the product measure, that is $d\nu = d(\mu \times \lambda)$. Then $\pi : (h, \vec{x}) \mapsto \psi_{h,\vec{x}}$ constitutes a unitary representation of G on $L^2(\mathbb{R}^n)$. It follows from Duflo-Moore's theorem that if π is square integrable, then plenty of admissible functions exist. This idea was first used by Grossmann, Morlet, and Paul (1985, 1986) and Heil and Walnut (1989) for the one-dimensional wavelet transform (1.1).

To investigate properties of the wavelet transform, one usually works in Fourier space. The Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ is a Hilbert space isomorphism from $L^2(\mathbb{R}^n)$ onto $L^2(\widehat{\mathbb{R}^n})$ ($\widehat{\mathbb{R}^n}$ denoting Euclidean space with elements written as row vectors) taking the representation π to the representation $\rho = \mathcal{F} \circ \pi \circ \mathcal{F}^{-1}$ of G on $L^2(\widehat{\mathbb{R}^n})$, in fact

$$\rho_{(h,\vec{x})}\hat{\psi}(\vec{\gamma}) = |\det h|^{1/2} e^{-2i\pi\vec{\gamma}h\vec{x}}\hat{\psi}(\vec{\gamma}h).$$

The wavelet transform becomes thus

$$(W_\psi f)(h, \vec{x}) = |\det(h)|^{1/2} \int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma})\overline{\hat{\psi}(\vec{\gamma}h)} e^{2i\pi\vec{\gamma}h\vec{x}}\, d\vec{\gamma}.$$

Bernier and Taylor (1996) showed that if $\widehat{\mathbb{R}^n}$ decomposes into essentially a finite union of open, free H-orbits, then $L^2(\widehat{\mathbb{R}^n})$ decomposes into a corresponding direct sum of ρ-invariant subspaces on which ρ is square integrable, so that admissible vectors exist by Duflo-Moore's theorem. Führ (1996) and Fabec and Ólafasson (2003) generalized this observation to non-free open orbits, under the condition that the stabilizers of the orbit points are compact.

In many choices for H, the representation ρ is not square integrable, and one must use other means to find admissible functions, that is functions for which (1.4) holds. By Fourier transform arguments (Führ (1996), Weiss, and Wilson (2001)), one can show that ψ is admissible if and only if

$$\int_H |\hat{\psi}(\vec{\gamma}h)|^2 \, d\mu(h) = c_\psi = \text{constant} \tag{1.6}$$

for almost all $\vec{\gamma} \in \mathbb{R}^n$. Laugesen et al.(2002) have given a nearly complete characterization of groups H possessing admissible functions ψ. However, their construction is abstract, and does not yield an easy to understand function ψ. Larson et al.(2006) showed that in the particular cases (1.2) and (1.3), admissible functions ψ exist if and only if $|\det(A)| \neq 1$. They constructed $\hat{\psi}$ explicitly as the characteristic function of some measurable cross-section, so that ψ vanishes only very slowly at infinity.

For practical applications, however, one often requires admissible functions ψ with good localization properties. That is, one wishes ψ to be smooth and vanish rapidly at infinity. In the case of open orbits as discussed in Bernier and Taylor (1996) and Fabec and Ólafasson (2003), it is easy to construct such a function by choosing a $\hat{\psi}$ in class $C_c^\infty(\widehat{\mathbb{R}^n})$ whose support is contained in the union of all open orbits and then *averaging* the value of $\hat{\psi}$ over each orbit. For the cases (1.2) and (1.3), Schulz and Taylor have shown that there exists a ψ in the Schwartz space if and only if all eigenvalues of the matrix A lie either inside or outside of the unit circle, and they have presented a simple construction for $\hat{\psi}$. In the general case, however, this question is not solved yet.

An important problem is the discretization of the wavelet transform. Instead of reconstructing the function f from its wavelet transform by means of the weak integral (1.5) which is difficult to compute, one searches for discrete subsets

P of H and Γ of $\mathbb{R}^n$ so that

$$f = \sum_{k\in P}\sum_{\vec{x}\in\Gamma}(W_\psi f)(k,\vec{x})\psi_{k,\vec{x}} \tag{1.7}$$

with convergence in $L^2(\mathbb{R}^n)$. Such a collection of functions $\{\psi_{k,\vec{x}} : k \in P, \vec{x} \in \mathbb{R}^n\}$ is a particular case of a wavelet frame. The existence and construction of wavelet frames has attracted considerable attention, mainly, when H is a 1-parameter matrix group, or a group in low dimensions (Dai, Larson, and Speegle (1997), Wang (2002), Benedetto and Sumetkijakarn (2002), Laugensen (2002), Dai, Diao, Gu, and Han (2003), and Speegle (2003)). $\hat{\psi}$ is usually of the form χ_Ω for some measurable set Ω, so that ψ vanishes only slowly at infinity.

Bernier and Taylor (1996) and Fabec and Ólafasson (2003) have shown how to choose the sets P, Γ and the function ψ, in case of open orbits of H in $\widehat{\mathbb{R}^n}$. On the other hand, Heinlein (2003) has shown how to modify an admissible function ψ, given a partition $\{H_j\}$ of H and $\Gamma = \mathbb{Z}^n$, in order to obtain wavelet frames, using a construction called integrated wavelets.

In this thesis, we discuss the concrete construction of admissible functions, in particular of admissible functions with good smoothness and vanishing properties, for dilation groups with arbitrary orbit structure. The starting point is a generalized notion of cross-section, which we call an N-section. We show how to obtain admissible functions from an N-section S. If S is compact, the orbit map $(S,H) \mapsto S \cdot H$ is open, and orbits intersect S in some regular fashion, then smooth, bandlimited admissible functions exist. In order to apply this construction to p-parameter groups of diagonal matrices, we show that for these groups, there exist compact N-sections with open orbit map. We then present examples of smooth, bandlimited functions for some of these groups.

In the second part of the thesis, we show how the techniques of discretization discussed by Bernier and Taylor (1996), and also by Heinlein (2003) can also be adapted to the case of dilation groups with arbitrary orbit structure. In particular, if S is an N-section having the properties stated above, then the topological structure of H and that of the Euclidean space on which H acts are suitably compatible, and we can specify conditions on P, Γ and ψ so that $\{\psi_{k,\vec{x}} : k \in P, \vec{x} \in \Gamma\}$ is a wavelet frame. In the special case where P is a co-compact, discrete subgroup of H we obtain smooth and bandlimited Parseval frames, and reconstruction formula (1.7) holds directly. In general, one has the reconstruction

$$f = \sum_{k \in P} \sum_{\vec{x} \in \Gamma} \langle f, \psi_{k,\vec{x}} \rangle \widetilde{\psi}_{k,\vec{x}}$$

where $\{\widetilde{\psi}_{k,\vec{x}}\}$ is obtained from $\{\psi_{k,\vec{x}}\}$ by a positive operator, called the frame operator. We also show that the techniques of integrated wavelets can be applied to arbitrary dilation groups H; we even can allow that finitely many of the sets $\{H_j\}$ overlap.

This thesis is organized as follows. In chapter II the basic notation is introduced and the concepts from Fourier analysis, topological groups and their representations used in this thesis are reviewed. In chapter III, the continuous wavelet transform is reviewed from the group theoretic point of view, and those aspects of wavelet frames which are required for our work are discussed. Chapter IV is devoted to the discussion of N-sections, the construction of admissible functions from N-sections, and the construction of N-sections for p-parameter groups of diagonal matrices. In chapter V, the various methods for obtaining wavelet frames are discussed.

CHAPTER II

BASIC BACKGROUND

In this chapter, we review the mathematical concepts used in this thesis. We begin by discussing the Fourier transform in $L^2(\mathbb{R}^n)$ and its properties. We then review the basics of locally compact groups, and their applications. Throughout, it is assumed that the reader is familiar with the foundations of real analysis, such as measure theory and function spaces. Details can be found in Folland (1999).

2.1 Fourier Analysis

We begin by introducing the basic notations and reviewing the basic properties of the Fourier transform.

2.1.1 Preliminaries

Throughout this thesis, we shall be working on $\mathbb{R}^n$, and n will always refer to the dimension. If k is an $n \times n$ real matrix and $\vec{x} \in \mathbb{R}^n$, then $k\vec{x}$ is the column vector obtained by the usual matrix multiplication of k by $\vec{x}$ regarded as a column vector; while $\vec{\gamma}k$ denotes the product of a row vector $\vec{\gamma}$ with the $n \times n$ matrix k. We will consider the elements $\vec{x}$ in $\mathbb{R}^n$ as column vectors and we will use $\widehat{\mathbb{R}^n}$ for n-dimensional Euclidean space with the elements written as row vectors. If $\vec{x}, \vec{y} \in \mathbb{R}^n$, we set

$$\vec{x} \cdot \vec{y} = \sum_{j=1}^{n} x_j y_j, \qquad \|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}.$$

If $\vec{\gamma}$ is a row vector, then $\vec{\gamma} \cdot \vec{x}$ is just matrix multiplication, $\vec{\gamma} \cdot \vec{x} = \vec{\gamma}\vec{x}$.

It will be convenient to have a compact notation for partial derivatives. We shall write

$$\partial_j = \frac{\partial}{\partial x_j},$$

and for higher-order derivatives we use multi-index notation. A multi-index is an ordered n-tuple of nonnegative integers. If $\alpha = (\alpha_1, ..., \alpha_n)$ is multi-index, we set

$$|\alpha| = \sum_{j=1}^{n} \alpha_j, \quad \alpha! = \prod_{j=1}^{n} \alpha_j!, \quad \partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n},$$

and if $\vec{x} = (x_1, ..., x_n) \in \mathbb{R}^n$,

$$\vec{x}^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}.$$

(The notation $|\alpha| = \sum \alpha_j$ is inconsistent with the notation $\|\vec{x}\| = (\sum x_j^2)^{1/2}$, but the meaning will always be clear from the context.) Thus, for example, Taylor's formula for a function $f \in C^p(\mathbb{R}^n)$ reads

$$f(\vec{x}) = \sum_{|\alpha| \le p} (\partial^\alpha f)(\vec{x}_0) \frac{(\vec{x} - \vec{x}_0)^\alpha}{\alpha!} + R_p(\vec{x}), \qquad \lim_{x \to x_0} \frac{|R_p(\vec{x})|}{|\vec{x} - \vec{x}_0|^p} = 0,$$

and the product rule for derivatives becomes

$$\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} (\partial^\beta f)(\partial^\gamma g).$$

One subspace of $C^\infty(\mathbb{R}^n)$ will be of particular importance for us. That is the subspace $C_c^\infty(\mathbb{R}^n)$ of $C^\infty(\mathbb{R}^n)$ functions with compact support. The existence of nonzero functions in $C_c^\infty(\mathbb{R}^n)$ is not quite obvious; the standard construction is based on the fact that the function $f(t) = e^{-1/t} \chi_{(0,\infty)}(t)$ is $C^\infty(\mathbb{R})$ even at the origin. If we set

$$\psi(x) = f(1 - |x|^2) = \begin{cases} \exp[(|x|^2 - 1)^{-1}] & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1, \end{cases} \tag{2.1}$$

it follows that $\psi \in C^\infty(\mathbb{R})$, and $\text{supp}(\psi)$ is the closed unit ball.

We next investigate the continuity of translations on various function spaces. The following notation for translations will be used throughout this chapter and the next one. If f is a function on $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$, let

$$T_{\vec{y}}f(\vec{x}) = f(\vec{x} - \vec{y}).$$

We observe that $\|T_{\vec{y}}f\|_p = \|f\|_p$ for $1 \leq p \leq \infty$. A function f is called *uniformly continuous* if $\|T_{\vec{y}}f - f\|_\infty \to 0$ as $\|\vec{y}\| \to 0$.

Lemma 2.1. *If $f \in C_c(\mathbb{R}^n)$, then f is uniformly continuous.*

Proof. Given $\varepsilon > 0$, for each $\vec{x} \in \mathrm{supp}(f)$ there exists $\delta_{\vec{x}} > 0$ such that $|f(\vec{x} - \vec{y}) - f(\vec{x})| < \frac{1}{2}\varepsilon$ if $\|\vec{y}\| < \delta_{\vec{x}}$. Since $\mathrm{supp}(f)$ is compact, there exist $x_1, ..., x_N$ such that the balls of radius $\frac{1}{2}\delta_{x_j}$ about x_j cover $\mathrm{supp}(f)$. If $\delta = \frac{1}{2}\min\{\delta_{x_j}\}$, then one easily sees that $\|T_{\vec{y}}f - f\|_\infty < \varepsilon$ whenever $\|\vec{y}\| < \delta$. $\square$

Proposition 2.1. *If $1 \leq p < \infty$, then translation is continuous in the $L^p(\mathbb{R}^n)$ norm; that is, if $f \in L^p(\mathbb{R}^n)$ and $\vec{z} \in \mathbb{R}^n$, then $lim_{\vec{y}\to 0}\|T_{\vec{y}+\vec{z}}f - T_{\vec{z}}f\|_p = 0$.*

Proof. Since $T_{\vec{y}+\vec{z}} = T_{\vec{y}}T_{\vec{z}}$, by replacing f by $T_{\vec{z}}f$ it suffices to assume that $\vec{z} = 0$. First, if $g \in C_c(\mathbb{R}^n)$, for $\|\vec{y}\| \leq 1$ the functions $T_{\vec{y}}g$ are all supported in a common compact set K, so by Lemma 2.1,

$$\int_{\mathbb{R}^n} |T_{\vec{y}}g(\vec{x}) - g(\vec{x})|^p d\vec{x} \leq \|T_{\vec{y}}g - g\|_\infty^p \lambda(K) \to 0 \text{ as } \|\vec{y}\| \to 0.$$

Now suppose $f \in L^p(\mathbb{R}^n)$. Given $\varepsilon > 0$, there exists $g \in C_c(\mathbb{R}^n)$ with $\|g - f\|_p < \varepsilon/3$, so

$$\|T_{\vec{y}}f - f\|_p \leq \|T_{\vec{y}}(f - g)\|_p + \|T_{\vec{y}}g - g\|_p + \|g - f\|_p < \frac{2}{3}\varepsilon + \|T_{\vec{y}}g - g\|_p,$$

and $\|T_{\vec{y}}g - g\|_p < \varepsilon/3$ if $\|\vec{y}\|$ is sufficiently small. $\square$

Proposition 2.1 is false for $p = \infty$, as one should expect since the $L^\infty(\mathbb{R}^n)$ norm is closely related to the uniform norm.

In chapter V, we will deal with multiply periodic functions in $\mathbb{R}^n$, and for simplicity we shall take the fundamental period in each variable to be 1. That is, we define a function f on $\mathbb{R}^n$ to be periodic if $f(\vec{x}+\vec{m}) = f(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$ and $\vec{m} \in \mathbb{Z}^n$. Every periodic function is thus completely determined by its values on the unite cube

$$Q = [-\frac{1}{2}, \frac{1}{2})^n.$$

Periodic functions may be regarded as functions on the space $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}/\mathbb{Z})^n$ of cosets of $\mathbb{Z}^n$, which we call the n-dimensional torus and denote by $\mathbb{T}^n$. (When $n = 1$ we write $\mathbb{T}$ rather than $\mathbb{T}^1$.) $\mathbb{T}^n$ is a compact Hausdorff space; it may be identified with the set of all $\vec{z} = (z_1, ...z_n) \in \mathbb{C}^n$ such that $|z_j| = 1$ for all j, via the map

$$(x_1, x_2..., x_n) \mapsto (e^{2\pi i x_1}, e^{2\pi i x_2}, ..., e^{2\pi i x_n}).$$

On the other hand, for measure-theoretic purpose we identify $\mathbb{T}^n$ with the unit cube Q, and when we speak of Lebesgue measure on $\mathbb{T}^n$ we mean the measure induced on $\mathbb{T}^n$ by the Lebesgue measure on Q. In particular, $\lambda(\mathbb{T}^n) = 1$. Functions on $\mathbb{T}^n$ may be considered as periodic functions on $\mathbb{R}^n$ or as functions on Q; the point of view will be clear from the context when it matters.

Proposition 2.2. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ and in $C_0(\mathbb{R}^n)$.*

Proposition 2.3. (The $C^\infty(\mathbb{R}^n)$ Urysohn Lemma (Wade (1999)))
If $F \subset \mathbb{R}^n$ is compact and U is an open set containing F, then there exist $f \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le f \le 1$, $f = 1$ on F, and supp$(f) \subset U$.

2.1.2 The Fourier Transform

One of the fundamental principles of harmonic analysis is the exploitation of symmetry. To be more specific, if one is doing analysis on a space on which a group acts, it is a good idea to study functions (or other analytic objects) that transform in simple ways under the group action, and then try to decompose arbitrary functions as sums or integrals of these basic functions.

The spaces $\mathbb{R}^n$ and $\mathbb{T}^n$ are Abelian groups ($\mathbb{T}^n$ is a group under componentwise additive mod 1) under addition on $\mathbb{T}^n$ and act on themselves by translation. The building blocks of harmonic analysis on these spaces are the functions that transform under translation by multiplication by a factor of absolute value one, that is, functions f such that for each $\vec{x}$ there is a number $\phi(\vec{x})$ with $|\phi(\vec{x})| = 1$ such that $f(\vec{y}+\vec{x}) = \phi(\vec{x})f(\vec{y})$. If f and ϕ have this property, then $f(\vec{x}) = \phi(\vec{x})f(\vec{0})$, so f is completely determined by ϕ once $f(\vec{0})$ is given; moreover,

$$\phi(\vec{x})\phi(\vec{y})f(\vec{0}) = \phi(\vec{x})f(\vec{y}) = f(\vec{x}+\vec{y}) = \phi(\vec{x}+\vec{y})f(\vec{0}),$$

so that (unless $f = 0$) $\phi(\vec{x}+\vec{y}) = \phi(\vec{x})\phi(\vec{y})$. In short, to find all f's that transform as described above, it suffices to find all ϕ's of absolute value one that satisfy the functional equation $\phi(\vec{x}+\vec{y}) = \phi(\vec{x})\phi(\vec{y})$. Upon imposing the natural requirement that ϕ should be measurable, we have a complete solution to this problem.

Theorem 2.1. *If ϕ is a measurable function on $\mathbb{R}^n$ (resp. $\mathbb{T}^n$) such that $\phi(\vec{x}+\vec{y}) = \phi(\vec{x})\phi(\vec{y})$ and $|\phi| = 1$, there exists $\vec{\gamma} \in \mathbb{R}^n$ (resp. $\vec{\gamma} \in \mathbb{T}^n$) such that $\phi(\vec{x}) = e^{2\pi i\vec{\gamma}\cdot\vec{x}}$.*

The idea now is to decompose more or less arbitrary functions on $\mathbb{T}^n$ or $\mathbb{R}^n$ in terms of the exponentials $e^{2\pi i\vec{\gamma}\cdot\vec{x}}$. In the case of $\mathbb{T}^n$ this works out very simply for square integrable functions.

Theorem 2.2. *Let $e_{\vec{m}}(\vec{x}) = e^{2\pi i\vec{m}\cdot\vec{x}}$. Then $\{e_{\vec{m}} : \vec{m} \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$.*

Proof. By Fubini's theorem it boils down to the fact that $\int_0^1 e^{2\pi imt}dt$ equals 1 if $m = 0$ and equals 0 otherwise. Next, since $e_{\vec{m}}e_{\vec{l}} = e_{\vec{m}+\vec{l}}$, the set of finite linear combinations of the $e_{\vec{m}}$'s is an algebra. It clearly separates points on $\mathbb{T}^n$; also, $e_0 = 1$ and $\overline{e}_{\vec{m}} = e_{-\vec{m}}$. Since $\mathbb{T}^n$ is compact, the Stone-Weierstrass theorem implies that this algebra is dense in $C(\mathbb{T}^n)$ in the uniform norm and hence in the $L^2(\mathbb{T}^n)$ norm, and $C(\mathbb{T}^n)$ is itself dense in $L^2(\mathbb{T}^n)$. It follows that $\{e_{\vec{m}}\}_{\vec{m}\in\mathbb{Z}^n}$ is a Hilbert space basis of $L^2(\mathbb{T}^n)$. $\square$

To restate this result: If $f \in L^2(\mathbb{T}^n)$, we define its *Fourier transform* $\hat{f}$, a function on $\mathbb{Z}^n$, by

$$\hat{f}(\vec{m}) = \langle f, e_{\vec{m}}\rangle = \int_{\mathbb{T}^n} f(\vec{x})e^{-2\pi i\vec{m}\cdot\vec{x}}d\vec{x},$$

and we call the series

$$\sum_{\vec{m}\in\mathbb{Z}^n} \hat{f}(\vec{m})e_{\vec{m}},$$

the Fourier series of f. The term *Fourier transform* is also used to denote the map $f \mapsto \hat{f}$. Theorem 2.2 then implies that the Fourier transform maps $L^2(\mathbb{T}^n)$ onto $l^2(\mathbb{Z}^n)$, that $\|\hat{f}\|_2 = \|f\|_2$ (Parseval's identity) and that the Fourier series of f converges to f in the $L^2(\mathbb{T}^n)$ norm.

The definition of $\hat{f}(\vec{m})$ makes sense if f is merely in $L^1(\mathbb{T}^n)$, and $|\hat{f}(\vec{m})| \leq \|f\|_1$, so the Fourier transform extends to a norm-decreasing map from $L^1(\mathbb{T}^n)$ to $l^\infty(\mathbb{Z}^n)$. (The Fourier series of an $L^1(\mathbb{R}^n)$ function may be quite badly behaved, but there are still methods for recovering f from $\hat{f}$ when $f \in L^1(\mathbb{R}^n)$). Interpolating between $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, one has the following result.

Theorem 2.3. (The Hausdorff-Young Inequality) *Suppose that* $1 \leq p \leq 2$ *and* q *is the conjugate exponent to* p*. If* $f \in L^p(\mathbb{T}^n)$*, then* $\hat{f} \in l^q(\mathbb{Z}^n)$ *and* $\|\hat{f}\|_q \leq \|f\|_p$*.*

The situation on $\mathbb{R}^n$ is more delicate. The formal analogue of Theorem 2.2

should be

$$f(\vec{x}) = \int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma}) e^{2\pi i \vec{\gamma}\vec{x}} d\vec{\gamma}, \quad \text{where } \hat{f}(\vec{\gamma}) = \int_{\mathbb{R}^n} f(\vec{x}) e^{-2\pi i \vec{\gamma}\vec{x}} d\vec{x}.$$

These relations turn out to be valid when suitably interpreted, but some care is needed. In the first place, the integral defining $\hat{f}(\vec{\gamma})$ is likely to diverge if $f \in L^2(\mathbb{R}^n)$. However, it certainly converges if $f \in L^1(\mathbb{R}^n)$. We therefore begin by defining the *Fourier transform* of $f \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}f(\vec{\gamma}) = \hat{f}(\vec{\gamma}) = \int_{\mathbb{R}^n} f(\vec{x}) e^{-2\pi i \vec{\gamma}\vec{x}} d\vec{x}.$$

(We use the notation $\mathcal{F}$ for the Fourier transform only where it is needed for clarity.) Clearly $\|\hat{f}\|_\infty \le \|f\|_1$, and $\hat{f}$ is continuous, thus, from the theorem below,

$$\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\widehat{\mathbb{R}^n}).$$

We summarize the elementary properties of $\mathcal{F}$ in a theorem.

Theorem 2.4. *Suppose $f, g \in L^1(\mathbb{R}^n)$.*

(a) If $\vec{x}^\alpha f \in L^1(\mathbb{R}^n)$ for $|\alpha| \le k$, then $\hat{f} \in C^k(\widehat{\mathbb{R}^n})$ and $\partial^\alpha \hat{f} = [\widehat{(-2\pi i \vec{x})^\alpha f}]$.

(b) If $f \in C^k(\mathbb{R}^n)$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for $|\alpha| \le k$, and $\partial^\alpha f \in C_0(\mathbb{R}^n)$ for $|\alpha| \le k-1$, then $(\widehat{\partial^\alpha f})(\vec{\gamma}) = (2\pi i \vec{\gamma})^\alpha \hat{f}(\vec{\gamma})$.

(c) The Riemann-Lebesgue Lemma: $\mathcal{F}(L^1(\mathbb{R}^n)) \subset C_0(\widehat{\mathbb{R}^n})$.

Parts (a) and (b) of Theorem 2.4 point to a fundamental property of the Fourier transform: Smoothness properties of f are reflected in the rate of decay of $\hat{f}$ at infinity, and vice versa. Parts (b) and (c) of this theorem are valid also on $\mathbb{T}^n$. We are now ready to invert the Fourier transform. If $f \in L^1(\mathbb{R}^n)$, we define

$$\check{f}(\vec{x}) = \hat{f}(-\vec{x}) = \int_{\widehat{\mathbb{R}^n}} f(\vec{\gamma}) e^{2\pi i \vec{\gamma}\vec{x}} d\vec{\gamma}.$$

Note that $\hat{f}$ need not be integrable. However, if $\hat{f} \in L^1(\widehat{\mathbb{R}^n})$ then we can reconstruct f from $\hat{f}$ as follows

$$(\hat{f})^{\vee}(\vec{x}) = \int_{\widehat{\mathbb{R}^n}} \int_{\mathbb{R}^n} f(\vec{y}) e^{-2\pi i \vec{\gamma}\vec{y}} e^{2\pi i \vec{\gamma}\vec{x}} d\vec{y} d\vec{\gamma}.$$

Theorem 2.5. (The Fourier Inversion Theorem) *If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\widehat{\mathbb{R}^n})$, then f agrees almost everywhere with a continuous function f_0, and $(\hat{f})^{\vee} = (\check{f})^{\wedge} = f_0$.*

Corollary 2.6. *If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} = 0$, then $f = 0$ a.e.. That is, the Fourier transform $\mathcal{F}$ is a one-to-one mapping.*

At last we are in a position to derive the analogue of theorem 2.2 on $\mathbb{R}^n$.

Theorem 2.7. (The Plancherel Theorem) *If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\hat{f} \in L^2(\widehat{\mathbb{R}^n})$; and $\mathcal{F}|_{(L^1(\mathbb{R}^n) \cup L^2(\mathbb{R}^n))}$ extends uniquely to a unitary isomorphism of $L^2(\mathbb{R}^n)$ onto $L^2(\widehat{\mathbb{R}^n})$.*

Theorem 2.8. (The Hausdorff-Young Inequality) *Suppose that $1 \leq p \leq 2$ and q is the conjugate exponent to p. If $f \in L^p(\mathbb{R}^n)$, then $\hat{f} \in L^q(\widehat{\mathbb{R}^n})$ and $\|\hat{f}\|_q \leq \|f\|_p$.*

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\widehat{\mathbb{R}^n})$, the inversion formula

$$f(\vec{x}) = \int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma}) e^{2\pi i \vec{\gamma}\vec{x}} d\vec{\gamma}$$

exhibits f as a superposition of the basic functions $e^{2\pi i \vec{\gamma}\vec{x}}$; it is often called the *Fourier integral* representation of f. This formula remains valid in spirit for all $f \in L^2(\mathbb{R}^n)$, although the integral (as well as the integral defining $\hat{f}$) may not converge pointwise.

2.2 Topological Groups and Haar Measure

The spaces $(\mathbb{R}^n, +)$ and $(\mathbb{T}^n, +)$ discussed above are typical representatives of locally compact groups which we will introduce now.

Definition 2.1. *A topological group is a group G endowed with a topology such that the group operations $(h,k) \mapsto hk$ and $h \mapsto h^{-1}$ are continuous from $G \times G$ and G to G. Examples include topological vector spaces (the group operation being addition), groups of invertible $n \times n$ real matrices (with the relative topology induced from $\mathbb{R}^{(n\times n)}$) and all groups equipped with the discrete topology. If G is a topological group, we denote the identity element of G by e, and for $A, B \subset G$ and $h \in G$ we define*

$$hA = \{hk : k \in A\}, \qquad Ah = \{kh : k \in A\},$$
$$A^{-1} = \{k^{-1} : k \in A\}, \qquad AB = \{kh : k \in A, h \in B\}.$$

We say that A is symmetric if $A = A^{-1}$.

Here are some of the basic properties of topological groups:

Proposition 2.4. *Let G be a topological group.*

(a) The topology of G is translation invariant: If U is open and $h \in G$, then Uh and hU are open.

(b) For every neighborhood U of e there exists a symmetric neighborhood V of e with $V \subset U$.

(c) For every neighborhood U of e there exists a neighborhood V of e with $VV \subset U$.

(d) If H is a subgroup of G then so is $\overline{H}$.

(e) Every open subgroup of G is also closed.

(f) If K_1, K_2 are compact subsets of G then so is K_1K_2.

Proof. (a) is equivalent to the continuity in each variable of the map $(h,k) \mapsto hk$, and (b) and (c) are equivalent to the continuity of $h \mapsto h^{-1}$ and $(h,k) \mapsto hk$ at the identity. For (d), if $h, k \in \overline{H}$, there exist nets $\{h_\alpha\}_{\alpha \in A}, \{k_\beta\}_{\beta \in B}$ in H that converge to h and k. Then $h_\alpha^{-1} \to h^{-1}$ and $h_\alpha k_\beta \to hk$, so h^{-1} and hk belong to $\overline{H}$. For (e), if H is an open subgroup, the cosets hH are open for all h, so that $G \setminus H = \cup_{h \notin H} hH$ is open and hence H is closed. Finally, (f) is true because K_1K_2 is the image of the compact set $K_1 \times K_2$ under the continuous map $(h,k) \mapsto hk$. $\square$

If f is a continuous function on the topological group G and $k \in G$, we define the left and the right translates of f through k by

$$L_k f(h) \;=\; f(k^{-1}h), \qquad R_k f(h) \;=\; f(hk).$$

(The point of using k^{-1} on the left and k on the right is to make $L_{kl} = L_k L_l$ and $R_{kl} = R_k R_l$.) f is called *left* (resp. *right*) *uniformly continuous* if for every $\varepsilon > 0$ there is a neighborhood V of e such that $\|L_k f - f\|_\infty < \varepsilon$ (resp. $\|R_k f - f\|_\infty < \varepsilon$) for $k \in V$. (Some authors reverse the roles of L_k and R_k in this definition.)

Proposition 2.5. *If $f \in C_c(G)$, then f is left and right uniformly continuous.*

Proof. We shall consider right uniform continuity; the proof on the left is the same. Let $K = \mathrm{supp}(f)$ and $\varepsilon > 0$. For each $h \in K$ there is a neighborhood U_h of e such that $|f(hk) - f(h)| < \frac{1}{2}\varepsilon$ for $k \in U_h$, and by Proposition 2.4(b,c) there is a symmetric neighborhood V_h of e with $V_h V_h \subset U_h$. Then $\{hV_h\}_{h \in K}$ covers K, so there exist $h_1, \ldots, h_n \in K$ such that $K \subset \cup_{j=1}^n h_j V_{h_j}$. Let $V = \cap_{j=1}^n V_{h_j}$; we claim that $|f(hk) - f(h)| < \varepsilon$ if $k \in V$. On the one hand, if $h \in K$, then for some j we have $h_j^{-1}h \in V_{h_j}$ and hence $hk = h_j(h_j^{-1}h)k \in h_j U_{h_j}$; therefore,

$$|f(hk) - f(h)| \leq |f(hk) - f(h_j)| + |f(h_j) - f(h)| < \varepsilon.$$

On the other hand, if $h \notin K$, then $f(h) = 0$, and either $f(hk) = 0$ (if $hk \notin K$) or $h_j^{-1}hk \in V_{h_j}$ for some j (if $hk \in K$); in the latter case $h_j^{-1}h = h_j^{-1}hkk^{-1} \in U_{h_j}$, so that $|f(h_j)| < \frac{1}{2}\varepsilon$ and hence $|f(hk)| < \varepsilon$. $\square$

A locally compact group is a topological group whose topology is locally compact and Hausdorff.

Suppose that G is a locally compact group. A Borel measure μ on G is called *left-invariant* (resp. *right-invariant*) if $\mu(hE) = \mu(E)$ (resp. $\mu(Eh) = \mu(E)$) for all $h \in G$ and E a Borel subset of G. A left (resp. right) Haar measure on G is a nonzero left-invariant (resp. right-invariant) Borel measure μ on G. For example, the Lebesgue measure is a (left and right) Haar measure on $\mathbb{R}^n$. The following proposition summarizes some elementary properties of Haar measures; in it, and in the sequel, we employ the notation

$$C_c^+ = \{\, f \in C_c(G) \ : \ f \geq 0 \ \text{ and } \ \|f\|_\infty > 0\}.$$

Proposition 2.6. *Let G be a locally compact group.*

(a) A Radon measure μ on G is a left Haar measure if and only if the measure $\tilde{\mu}$ defined by $\tilde{\mu}(E) = \mu(E^{-1})$ is a right Haar measure.

(b) A nonzero Radon measure μ on G is a left Haar measure if and only if $\int f(k^{-1}h)d\mu(h) = \int f(h)d\mu(h)$ for all $f \in C_c^+$, $k \in G$ if and only if $\int f(k^{-1}h)d\mu(h) = \int f(h)d\mu(h)$ for all $f \in L^1(G)$, $k \in G$.

(c) If μ is a left Haar measure on G, then $\mu(U) > 0$ for every nonempty open $U \subset G$ and $\int f(h)d\mu(h) > 0$ for all $f \in C_c^+$.

(d) If μ is a left Haar measure on G, then $\mu(K) < \infty$ for every $K \subset G$ compact.

Theorem 2.9. *Every locally compact group G possesses a left Haar measure. The left Haar measure is essentially unique, that is, if μ and ν are left Haar measures* *on G, there exists $c > 0$ such that $\mu = c\nu$. By symmetry, similar statements hold for a right Haar measure.*

If μ is a left Haar measure on G and $h \in G$, the measure $\mu_h(E) = \mu(Eh)$ is again a left Haar measure, because of the commutativity of left and right translations (i.e., the associative law). Hence, by theorem 2.9 there is a positive number $\Delta(h)$ such that $\mu_h = \Delta(h)\mu$. The function $\Delta : G \to (0, \infty)$ thus defined is independent of the choice of μ by theorem 2.9, it is called the *modular function* of G.

Proposition 2.7. *Δ is a continuous homomorphism from G into the multiplicative group of positive real numbers. Moreover, if μ is a left Haar measure on G, then for any $f \in L^1(G)$ and k in G we have*

$$\int f(hk)d\mu(h) = \Delta(k^{-1}) \int f(h)d\mu(h). \tag{2.2}$$

Proof. For any $h, k \in G$ and a Borel subset E of G of positive measure,

$$\Delta(hk)\mu(E) = \mu(Ehk) = \Delta(k)\mu(Eh) = \Delta(k)\Delta(h)\mu(E),$$

so Δ is a homomorphism from G to $(0, \infty)$. Also, since $\chi_E(hk) = \chi_{Ek^{-1}}(h)$,

$$\int \chi_E(hk)d\mu(h) = \mu(Ek^{-1}) = \Delta(k^{-1})\mu(E) = \Delta(k^{-1}) \int \chi_E(h)d\mu(h).$$

This proves (2.2) when $f = \chi_E$, and the general case follows by the definition of the integral. Finally, it is an easy consequence of proposition 2.5 that the map $k \mapsto \int f(hk)d\mu(h)$ is continuous for any $f \in C_c(G)$, so the continuity of Δ follows from (2.2). $\square$

Evidently, the left Haar measures on G are also right Haar measures precisely when Δ is identically 1, in which case G is called *unimodular*. Of course, every Abelian group is unimodular.

Proposition 2.8. *If G is compact, then G is unimodular.*

Proof. For any $h \in G$, obviously $G = Gh$. Hence if μ is a right Haar measure, we have $\mu(G) = \mu(Gh) = \Delta(h)\mu(G)$, and since $0 < \mu(G) < \infty$ we conclude that $\Delta(h) = 1$. □

We observed above that if μ is a left Haar measure, $\tilde{\mu}(E) = \mu(E^{-1})$ is a right Haar measure. We now show how to compute it in terms of μ and Δ.

Proposition 2.9. $d\tilde{\mu}(h) = \Delta(h)^{-1} d\mu(h)$.

Proof. By (2.2), if $f \in C_c(G)$,

$$\begin{aligned} \int f(h)\Delta(h)^{-1} d\mu(h) &= \Delta(k) \int f(hk)\Delta(hk)^{-1} d\mu(h) \\ &= \int f(hk)\Delta(h)^{-1} d\mu(h). \end{aligned}$$

Thus the Radon measure $\Delta^{-1} d\mu$ is right-invariant, so by theorem 2.9, $\Delta^{-1} d\mu = c d\tilde{\mu}$ for some $c > 0$. If $c \neq 1$, we can pick a symmetric neighborhood U of e in G such that $|\Delta(h)^{-1} - 1| < \frac{1}{2}|c-1|$ on U. But $\tilde{\mu}(U) = \mu(U)$, so

$$|c-1|\mu(U) = |c\tilde{\mu}(U) - \mu(U)| = \left| \int_U (\Delta(h)^{-1} - 1) d\mu(h) \right| < \frac{1}{2}|c-1|\mu(U),$$

a contradiction. Hence $c = 1$ and $d\tilde{\mu} = \Delta^{-1} d\mu$. □

Corollary 2.10. *Left and right Haar measures are mutually absolutely continuous.*

Definition 2.2. *Let G be a locally compact group, $\mathcal{H}$ a Hilbert space, and $F : G \to \mathcal{H}$ continuous. If there exists a vector $f \in \mathcal{H}$ such that*

$$\langle f, \psi \rangle = \int_G \langle F(g), \psi \rangle d\mu(g) \quad \forall \psi \in \mathcal{H} \tag{2.3}$$

then we say that $f = \int_G F(g) d\mu(g)$ as a weak integral in $\mathcal{H}$.

2.3 Group Representations

Definition 2.3. *Let G be a locally compact group and $\mathcal{H}$ be a Hilbert space. A representation π of G on $\mathcal{H}$ is a mapping satisfying:*

(a) $\pi : G \to \mathcal{U}(\mathcal{H})$. *($\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$.)*

(b) π *is a homomorphism:* $\pi_{hk} = \pi_h \pi_k$ *for all* $h, k \in G$.

(c) π *is continuous with respect to the strong operator topology of* $\mathcal{U}(\mathcal{H})$*, that is* $h \mapsto \pi_h \psi$ *is continuous for each* $\psi \in \mathcal{H}$.

Definition 2.4. *A representation π is called irreducible if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ which are invariant under π_h for each $h \in G$.*

Definition 2.5. *A representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ is called square integrable if*

(a) π *is irreducible.*

(b) There exists a vector $\psi \in \mathcal{H} \backslash \{0\}$ *such that* $\int_G |\langle \psi, \pi_h \psi \rangle|^2 d\mu(h) < \infty$ *where* μ *is the left Haar measure on G. That is, the function* $h \mapsto \langle \psi, \pi_h \psi \rangle$ *is square integrable. Such a vector ψ is called admissible.*

Theorem 2.11. *(Duflo-Moore Theorem) If π is a square-integrable representation of a locally compact group G on $\mathcal{H}$, then there exists a unique densely defined operator K on $\mathcal{H}$, self adjoint and positive which satisfies the following :*

(a) The set of admissible vectors in $\mathcal{H}$ coincides with the domain of K, that is dom $K = \{\psi \in \mathcal{H} \ : \ \psi$ *is admissible*$\}$.

(b) If ψ is an admissible vector and f is an arbitrary vector in $\mathcal{H}$, then

$$\|W_\psi f\|^2_{L^2(G)} = c_\psi \|f\|^2_{\mathcal{H}}$$

where $c_\psi = \|K\psi\|^2_{\mathcal{H}}$ *and* $W_\psi f(h) = \langle f, \pi_h \psi \rangle_{\mathcal{H}}$.

(c) If the group G is unimodular, then K is a multiple of the identity.

CHAPTER III

CONTINUOUS WAVELETS

In this chapter, we review the continuous wavelet transform from the group theoretic point of view. We also review the concept of frames which is essential for discretization of the continuous wavelet transform. Details can be found in Laugesen et al. (2002) and Hernandez and Weiss (1996).

3.1 The Continuous Wavelet Transform

In the most general sense, wavelets are defined by group representations as we explain now. Let $G^\sharp$ be the group which consists of all pairs $(h, \vec{x}) \in GL_n(\mathbb{R}) \times \mathbb{R}^n$ together with the group operation

$$(h, \vec{x}) \cdot (k, \vec{y}) = (hk, k^{-1}\vec{x} + \vec{y})$$

and the product topology. $G^\sharp$ is called the *affine group*. Then $\mathbb{R}^n$ is a closed normal subgroup of $G^\sharp$, and $G^\sharp/\mathbb{R}^n$ is isomorphic to $GL_n(\mathbb{R})$. This kind of group construction is called a semi-direct product, so $G^\sharp$ is called the *semi-direct product* of $GL_n(\mathbb{R})$ and $\mathbb{R}^n$, written $GL_n(\mathbb{R}) \rtimes \mathbb{R}^n$. Given a closed subgroup H of $GL_n(\mathbb{R})$, we consider the corresponding closed subgroup G of $G^\sharp$,

$$G = \{(h, \vec{x}) \in G^\sharp \ : \ h \in H, \vec{x} \in \mathbb{R}^n\}.$$

We identify H with the subgroup $\{(h, \vec{x}) \in G \ : \ h \in H, \vec{x} = 0\}$ and refer to it as the *dilation subgroup* of G, and $\mathbb{R}^n$ with the subgroup $\{(h, \vec{x}) \in G \ : \ h = e, \ \vec{x} \in \mathbb{R}^n\}$, and call it the *translation subgroup* of G. Thus G is the semi-direct product $H \rtimes \mathbb{R}^n$.

One easily checks that if μ is a left Haar measure for H and λ the Lebesgue measure on $\mathbb{R}^n$, then $\nu = \mu \times \lambda$ is a left Haar measure for G.

Next given $h \in H$ and a vector $\vec{x} \in \mathbb{R}^n$, define dilation, translation and modulation operators on $L^2(\mathbb{R}^n)$ by $(D_h f)(\vec{y}) = |\det h|^{-1/2} f(h^{-1}\vec{y})$, $(T_{\vec{x}} f)(\vec{y}) = f(\vec{y} - \vec{x})$ and $E_{\vec{x}} f(\vec{y}) = e^{2\pi i \vec{y}\cdot\vec{x}} f(\vec{y})$ for $f \in L^2(\mathbb{R}^n)$ and $\vec{y} \in \mathbb{R}^n$. The corresponding operators on $L^2(\widehat{\mathbb{R}^n})$ are defined similarly. For example, $(D_h f)(\vec{\gamma}) = |\det h|^{-1/2} f(\vec{\gamma} h^{-1})$ for all $f \in L^2(\widehat{\mathbb{R}^n})$. As for the modulation operator on $L^2(\widehat{\mathbb{R}^n})$, if $\vec{x} \in L^2(\mathbb{R}^n)$, we define $E_{\vec{x}} f(\vec{\gamma}) = e^{2\pi i \vec{\gamma}\vec{x}} f(\vec{\gamma})$. Using techniques from group representations (see Folland (1999), for example), it is easy to show that the mappings $h \mapsto D_h$, $\vec{x} \mapsto T_{\vec{x}}$ and $\vec{x} \mapsto E_{\vec{x}}$ are strongly continuous homomorphisms of the respective groups into the group of unitary operators on $L^2(\mathbb{R}^n)$ (respectively $L^2(\widehat{\mathbb{R}^n})$), that is, they are group representations (Continuity of $\vec{x} \mapsto T_{\vec{x}}$ was shown in proposition 2.1).

Proposition 3.1. *For $f \in L^2(\mathbb{R}^n), h \in H$ and $\vec{x} \in \mathbb{R}^n$*

(a) $\widehat{D_h f} = D_{h^{-1}} \hat{f}$

(b) $\widehat{T_{\vec{x}} f} = E_{-\vec{x}} \hat{f}$.

Proof. For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\widehat{D_h f}(\vec{\gamma}) &= \int_{\mathbb{R}^n} |\det h|^{-1/2} f(h^{-1}\vec{y}) e^{-2\pi i \vec{\gamma}\vec{y}} d\vec{y} \\
&= \int_{\mathbb{R}^n} |\det h|^{1/2} f(\vec{y}) e^{-2\pi i \vec{\gamma} h \vec{y}} d\vec{y} \\
&= |\det h|^{1/2} \hat{f}(\vec{\gamma} h) \\
&= D_{h^{-1}} \hat{f}(\vec{\gamma})
\end{aligned}$$

and also

$$\begin{aligned}
\widehat{T_{\vec{x}}f}(\vec{\gamma}) &= \int_{\mathbb{R}^n} f(\vec{y}-\vec{x})e^{-2\pi i\vec{\gamma}\vec{y}}d\vec{y} \\
&= \int_{\mathbb{R}^n} f(\vec{y})e^{-2\pi i\vec{\gamma}(\vec{y}+\vec{x})}d\vec{y} \\
&= e^{-2\pi i\vec{\gamma}\vec{x}}\int_{\mathbb{R}^n} f(\vec{y})e^{-2\pi i\vec{\gamma}\vec{y}}d\vec{y} \\
&= e^{-2\pi i\vec{\gamma}\vec{x}}\hat{f}(\vec{\gamma}) \\
&= E_{-\vec{x}}\hat{f}(\vec{\gamma}).
\end{aligned}$$

The assertion follows from density of $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ and continuity of all operators involved. $\square$

Since $T_{\vec{x}}D_h = D_h T_{h^{-1}\vec{x}}$, it follows that

$$\pi_{(h,\vec{x})} = D_h T_{\vec{x}}$$

defines a representation of G on $L^2(\mathbb{R}^n)$.

Given $\psi \in L^2(\mathbb{R}^n)$ and $(h,\vec{x}) \in G$, let us set

$$\psi_{h,\vec{x}}(\vec{y}) := (\pi_{(h,\vec{x})}\psi)(\vec{y}) = |\det h|^{-1/2}\psi(h^{-1}\vec{y}-\vec{x}).$$

Since the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\widehat{\mathbb{R}^n})$ is unitary, it induces a representation ρ of G on $L^2(\widehat{\mathbb{R}^n})$ by $\rho = \mathcal{F} \circ \pi \circ \mathcal{F}^{-1}$. Computing, we obtain

$$\begin{aligned}
\rho_{(h,\vec{x})}\hat{\psi} &= (\mathcal{F} \circ \pi_{(h,\vec{x})} \circ \mathcal{F}^{-1})(\hat{\psi}) \\
&= (\mathcal{F} \circ \pi_{(h,\vec{x})})(\psi) \\
&= \mathcal{F}(\pi_{(h,\vec{x})}(\psi)) \\
&= \mathcal{F}(D_h T_{\vec{x}}\psi) \\
&= D_{h^{-1}}E_{-\vec{x}}\mathcal{F}(\psi) \\
&= D_{h^{-1}}E_{-\vec{x}}\hat{\psi}
\end{aligned}$$

i.e. $\rho_{(h,\vec{x})}\hat{\psi}(\vec{\gamma}) = |\det h|^{1/2}\hat{\psi}(\vec{\gamma}h)e^{-2\pi i\vec{\gamma}h\vec{x}}$.

Definition 3.1. *Given $\psi \in L^2(\mathbb{R}^n)$, the (continuous) wavelet transform W_ψ induced by ψ and the group H is defined by*

$$W_\psi f(h, \vec{x}) = \langle f, \pi_{(h,\vec{x})}\psi\rangle = |\det h|^{-1/2} \int_{\mathbb{R}^n} f(\vec{y})\overline{\psi(h^{-1}\vec{y} - \vec{x})} d\vec{y}$$

for $f \in L^2(\mathbb{R}^n)$ and $(h, \vec{x}) \in G$.

The adjective *continuous* refers to the continuity of the translation group, consisting of all $\vec{x} \in \mathbb{R}^n$. The dilation group H, in contrast, is permitted to carry the discrete topology.

A goal in wavelet theory is to find a condition for ψ that guarantees that the mapping W_ψ is a multiple of a partial isometry,

$$\|W_\psi f\|^2_{L^2(G)} = c_\psi \|f\|^2_{L^2(\mathbb{R}^n)} \tag{3.1}$$

for all $f \in L^2(\mathbb{R}^n)$ and some constant $c_\psi > 0$. That is, one wants that

$$\int_G |\langle f, \psi_{h,\vec{x}}\rangle|^2 d\nu(h, \vec{x}) = \int_G |(W_\psi f)(h, \vec{x})|^2 d\nu(h, \vec{x}) = c_\psi \int_{\mathbb{R}^n} |f(\vec{y})|^2 d\vec{y} \tag{3.2}$$

for all $f \in L^2(\mathbb{R}^n)$. If this identity holds, then ψ is called *admissible*, and one has the Calderón reproducing formula

$$\begin{aligned} f &= \int_G (W_\psi f)(h, \vec{x})\psi_{h,\vec{x}} d\nu(h, \vec{x}) \\ &= \int_{\mathbb{R}^n} \int_H (W_\psi f)(h, \vec{x})\psi_{h,\vec{x}} d\mu(h) d\vec{x} \end{aligned} \tag{3.3}$$

as a weak integral in $L^2(\mathbb{R}^n)$, as shown in (1.5)

Theorem 3.1. (The admissibility condition, Laugesen et al. (2002)).
Equality (3.1) is valid for all $f \in L^2(\mathbb{R}^n)$ if and only if there exists $c_\psi > 0$ such that

$$\int_H |\hat{\psi}(\vec{\gamma} h)|^2 d\mu(h) = c_\psi \tag{3.4}$$

for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

Observe that by scaling ψ we may assume that $c_\psi = 1$. We want to identify all those groups H that possess a wavelet satisfying (3.4):

Definition 3.2. *We say that H is admissible if there exists a Borel measurable $g \in L^1(\widehat{\mathbb{R}^n})$ such that $g \geq 0$ and*

$$\int_H g(\vec{\gamma}h)d\mu(h) = 1 \tag{3.5}$$

for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

In terms of (3.4), if H is admissible, we pick ψ such that $|\hat{\psi}|^2 = g$. Then ψ is an admissible function. Conversely, given an admissible ψ, we set $g = \frac{1}{c_\psi}|\hat{\psi}|^2$ to see that H is admissible.

The fundamental result on admissibility given by Laugesen et al. (2002) involves the notation of the ε-*stabilizer*. Given $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ and $\varepsilon > 0$, the set

$$H^\varepsilon_{\vec{\gamma}} = \{h \in H : \|\vec{\gamma}h - \vec{\gamma}\| \leq \varepsilon\}$$

is called the ε-stabilizer of $\vec{\gamma}$. Similarly, the set $H_{\vec{\gamma}} \equiv H^0_{\vec{\gamma}} = \{h \in H : \vec{\gamma}h = \vec{\gamma}\}$ is referred to as the *stabilizer* of $\vec{\gamma}$. It is clear that $H_{\vec{\gamma}}$ and $H^\varepsilon_{\vec{\gamma}}$ are closed subsets of H, that $H_{\vec{\gamma}} = \bigcap_{\varepsilon>0} H^\varepsilon_{\vec{\gamma}}$, and that $H^{\varepsilon_1}_{\vec{\gamma}} \subset H^{\varepsilon_2}_{\vec{\gamma}}$ when $\varepsilon_1 \leq \varepsilon_2$.

Theorem 3.2. (Laugesen et al. (2002))

(a) If H is admissible, then $\Delta \not\equiv |\det|$ and the stabilizer of $\vec{\gamma}$ is compact for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

(b) If $\Delta \not\equiv |\det|$ and for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ there exists an $\varepsilon > 0$ such that the ε-stabilizer of $\vec{\gamma}$ is compact, then H is admissible.

This theorem is quite useful for determining the admissibility of particular groups H. For example it is clear that no compact group H can be admissible since in this case $\Delta \equiv |\det| \equiv 1$.

The above discussion applies more generally to σ-compact, locally compact groups H possessing a representation $\varphi : H \to GL_n(\mathbb{R})$. The wavelet transform of $f \in L^2(\mathbb{R}^n)$ is then defined by

$$(W_\psi f)(h, \vec{x}) = \langle f, \pi_{(\varphi(h),\vec{x})}\psi \rangle$$

and the Calderón reproducing formula becomes

$$f = \int_{\mathbb{R}^n} \int_H \langle f, \psi_{h,\vec{x}} \rangle \psi_{h,\vec{x}} d\mu(h) d\lambda(\vec{x})$$

where $\psi_{h,\vec{x}} = \pi_{(\varphi(h),\vec{x})}\psi$.

We will apply this generalization to the case where $H = \mathbb{R}^p$. Recall that given an exponential matrix $A = e^M$, one defines $A^s = e^{sM}$ for each real number s. Then that the mapping $s \mapsto A^s$ is a continuous homomorphism of $\mathbb{R}$ into $GL_n(\mathbb{R})$. Thus, if we fix commuting $n \times n$ matrices $A_1 = e^{M_1}, A_2 = e^{M_2}, ..., A_p = e^{M_p}$ and set $\varphi(s_1, s_2, ..., s_p) = A_1^{s_1} A_2^{s_2} \cdots A_p^{s_p}$, then φ is a continuous homomorphism of $\mathbb{R}^p$ into $GL_n(\mathbb{R})$, called a p-parameter group of matrices or a p-parameter subgroup of $GL_n(\mathbb{R})$.

We finish this section by reviewing some fundamental concepts of transformation groups. (For further details, see for example Kawakubo (1991)).

Definition 3.3. *Let X be a set, H a group. By a (right) H-action we mean a map*

$$\varphi : X \times H \to X$$

satisfying

(a) $\varphi(x, e) = x \quad \forall x \in X$ *where e denotes the identity of H*

(b) $\varphi((x, h_1), h_2) = \varphi(x, h_1 h_2) \quad \forall x \in X,\ h_1, h_2 \in H.$

The triple (X, H, φ) is also called a transformation group, and X is called a H-set.

It is often convenient to denote $\varphi(x,h)$ by $x\cdot h$. Then (a) and (b) become

(a′) $x\cdot e = x \quad \forall x\in H$

(b′) $(x\cdot h_1)\cdot h_2 = x\cdot(h_1h_2) \quad \forall x\in X,\, h_1,h_2\in H$.

If X is a topological space and H a topological group, then one also requires that the map φ be continuous, and calls X a H-space. In this case,

(a) φ is an open map

(b) for fixed $h\in H$, the map $x\mapsto x\cdot h$ is a homeomorphism of X onto X.

Given $x\in X$, the set $\mathcal{O}(x) = x\cdot H = \{x\cdot h \,:\, h\in H\}$ is called the *orbit* of x. The *stabilizer* of $x\in X$ is the set $H_x = \{h\in H \,:\, x\cdot h = x\}$. It is a closed subgroup of H provided that X is a T_1-space. The orbit $\mathcal{O}(x)$ is called *free* if $H_x = \{e\}$.

Example 3.1. Let $X = \widehat{\mathbb{R}^n}$, and H a closed subgroup of $GL_n(\mathbb{R})$. Then the map

$$\varphi : \widehat{\mathbb{R}^n}\times H \to \widehat{\mathbb{R}^n}$$

given by matrix multiplication,

$$\varphi(\vec{\gamma},h) = \vec{\gamma}\cdot h := \vec{\gamma}h$$

turns $\widehat{\mathbb{R}^n}$ into an H-space. We have $\mathcal{O}(\vec{\gamma}) = \{\vec{\gamma}h \,:\, h\in H\}$.

Example 3.2. Let $X = \widehat{\mathbb{R}^n}$, and $H = \mathbb{R}^p$ and M a fixed $n\times p$ matrix. Then the map

$$\varphi : \widehat{\mathbb{R}^n}\times \mathbb{R}^p \to \widehat{\mathbb{R}^n}$$

given by

$$\varphi(\vec{\gamma},s) = \vec{\gamma}\cdot s = \vec{\gamma} + Ms$$

turns $\widehat{\mathbb{R}^n}$ into an $\mathbb{R}^p$-space. The action is free if and only if rank $M = p$

Example 3.3. Every topological group H is itself an H-space through the action

$$\varphi(h_1, h_2) = h_1 \cdot h_2 := h_1 h_2 \qquad (h_1,\, h_2 \in H)$$

determined by group operation. Note that there exists only one orbit which is free.

We will make use of the following observation :

Lemma 3.1. *Let X be an H-space, and $S \subset X$ be given.*

(1) Let $U \subset H$ be open, and $h_0 \in H$. Then $S \cdot U$ is open if and only if $S \cdot Uh_0$ is open.

(2) Let $\{B_\alpha\}_{\alpha \in \Lambda}$ be a neighborhood base of e. Then $S \cdot U$ is open for all open subsets U of H if and only if $S \cdot B_\alpha$ is open for all $\alpha \in \Lambda$.

Proof. (1) Note that

$$\begin{aligned} S \cdot (Uh_0) &= \{x \cdot hh_0 \,:\, x \in S,\, h \in U\} \\ &= \{(x \cdot h) \cdot h_0 \,:\, x \in S,\, h \in U\} \\ &= (S \cdot U) \cdot h_0. \end{aligned}$$

Since the map $x \mapsto x \cdot h_0$ is a homeomorphism, the assertion follows.

(2) Suppose, $S \cdot B_\alpha$ is open for all $\alpha \in \Lambda$. Let $U \subset H$ be open. Since for each $h_0 \in H$, the collection $\{B_\alpha h_0\}_{\alpha \in \Lambda}$ is a neighborhood base of h_0, we can write

$$U = \underset{h \in U}{\cup} B_{\alpha_h} h$$

where $\alpha_h \in \Lambda$. Then

$$S \cdot U = \underset{h \in U}{\cup} S \cdot (B_{\alpha_h} h) = \underset{h \in U}{\cup} (S \cdot B_{\alpha_h}) \cdot h.$$

Since each $S \cdot B_{\alpha_h}$ is open, it follows that $S \cdot U$ is open. The reverse implication is obvious. $\square$

3.2 Frames

In practical computations, it is much easier to work with series than with integrals. Thus we would like to replace the integral (3.3) by

$$f = \sum_{k \in P} \sum_{\vec{x} \in \Gamma} \langle f, \psi_{k,\vec{x}} \rangle \psi_{k,\vec{x}} \tag{3.6}$$

for some discrete subsets P and Γ of H and $\mathbb{R}^n$ respectively, with convergence in $L^2(\mathbb{R}^n)$. As the functions $\{\psi_{k,\vec{x}}\}_{k \in P, \vec{x} \in \Gamma}$ need not be orthogonal, one needs to use the concept of frames, which is a generalization of Hilbert space bases.

Definition 3.4. *Let H be a Hilbert space. A collection of elements $\{\psi_j : j \in J\}$ in H is called a frame if there exist constants a and b, $0 < a \leq b < \infty$, such that*

$$a\|f\|^2 \leq \sum_{j \in J} |\langle f, \psi_j \rangle|^2 \leq b\|f\|^2 \qquad \text{for all } f \in H.$$

The constants a and b are called frame bounds. If $a = b$ then we say that the frame is tight. If $a = b = 1$, then it is called a Parseval frame.

Note : Any orthonormal basis in a Hilbert space is a Parseval frame. On the other hand, even a Parseval frame need not be a basis:

Example 3.4. Let $H = \mathbb{C}^2$ and take

$$\psi_1 = (0, \sqrt{\frac{2}{3}}), \quad \psi_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}), \quad \psi_3 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}).$$

Then, for $f = (f_1, f_2) \in \mathbb{C}^2$ we have

$$\begin{aligned} \sum_{j=1}^{3} |\langle f, \psi_j \rangle|^2 &= \frac{2}{3}|f_2|^2 + |\frac{1}{\sqrt{2}} f_1 + \frac{1}{\sqrt{6}} f_2|^2 + |\frac{1}{\sqrt{2}} f_1 - \frac{1}{\sqrt{6}} f_2|^2 \\ &= \|f\|^2. \end{aligned}$$

Therefore, $\{\psi_j\}_{j=1}^3$ is a Parseval frame, but obviously is not a basis for $\mathbb{C}^2$.

Theorem 3.3. *Let* $\{\psi_j : j \in J\}$ *be a frame for a Hilbert space* H *with frame bounds* a *and* b*. Then there exists a frame* $\{\widetilde{\psi}_j : j \in J\}$*, called the dual frame, which allows reconstruction of* $f \in H$ *by*

$$f = \sum_{j \in J} \langle f, \psi_j \rangle \widetilde{\psi}_j.$$

In fact, $\widetilde{\psi}_j = S^{-1}\psi_j$ for all j, where S is some positive bounded linear operator on H, called the frame operator. In case where $\{\psi_j : j \in J\}$ is a tight frame, S is a multiple of the identity, $S = aI$.

CHAPTER IV

EXISTENCE OF SMOOTH ADMISSIBLE FUNCTIONS

Given an admissible group H, the problem of finding admissible wavelets ψ with desired properties, such as smoothness for example, remains. One way of obtaining an admissible function is through the use of a cross-section for the action of H on $\widehat{\mathbb{R}^n}$. In this chapter, we introduce a generalized concept of cross-section, which we call *an almost cross-section* or *an N-section*. We then discuss the existence of N-sections for p-parameter groups of diagonal matrices with various properties, such as boundedness or compactness, for example. We show how to obtain smooth, bandlimited admissible functions from N-sections which are sufficiently well behaved with respect to the topology of $\widehat{\mathbb{R}^n}$. Throughout this chapter, we will work in $\widehat{\mathbb{R}^n}$, that is, vectors will be written as row vectors. For any set J, $\sharp J$ will denote the cardinality of J.

4.1 Generalized Cross-Sections

Definition 4.1. *A Borel set $S \subset \widehat{\mathbb{R}^n}$ is called an almost cross-section or an N-section for the action of H on $\widehat{\mathbb{R}^n}$ if*

(a) $\bigcup_{h\in H} Sh = \widehat{\mathbb{R}^n} \setminus E$ *where E is a set of measure zero.*

(b) $N := \sup_{\vec{\gamma}\in S}(\sharp\{k \in H \,:\, \vec{\gamma}k \in S\}) < \infty$.

If $N = 1$ then S is called a cross-section.

The first property says that almost every orbit intersects the set S, while the second property states that an orbit intersects the set S at most N times.

Notation : Let S be a subset of $\widehat{\mathbb{R}^n}$. For $\vec{\gamma}_0 \in \widehat{\mathbb{R}^n}$, we set

$$S_\varepsilon(\vec{\gamma}_0) = \{\vec{\gamma} \in S \, : \, \|\vec{\gamma} - \vec{\gamma}_0\| < \varepsilon\} = B_\varepsilon(\vec{\gamma}_0) \cap S.$$

Furthermore, $\overline{S}_\varepsilon(\vec{\gamma}_0)$ will denote the closure of $S_\varepsilon(\vec{\gamma}_0)$ in S. Recall that by theorem 3.2, a sufficient, although not necessary condition for admissibility of a group H is that ε-stabilizers be compact, for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$. The groups discussed in this chapter have this property :

Proposition 4.1. *Let H be a closed subgroup of $GL_n(\mathbb{R})$. Suppose there exists an N-section S such that the map $\Theta : S \times H \to \widehat{\mathbb{R}^n}$ defined by $\Theta : (\vec{\gamma}, h) \mapsto \vec{\gamma}h$ is open. Then for every $\vec{\gamma} \in SH$, there exists $\varepsilon > 0$ such that the ε-stabilizer $H^\varepsilon_{\vec{\gamma}}$ is compact.*

Proof. First let $\vec{\gamma}_0 \in S$. By assumption, $S \cap \mathcal{O}(\vec{\gamma}_0)$ is a finite set, say

$$S \cap \mathcal{O}(\vec{\gamma}_0) = \{\vec{\gamma}_0, \vec{\gamma}_1, ..., \vec{\gamma}_m\}. \tag{4.1}$$

Choose $\varepsilon' > 0$ such that

$$\vec{\gamma}_i \notin B_{\varepsilon'}(\vec{\gamma}_0) \tag{4.2}$$

for all $i = 1, 2, ..., m$. Now as the map Θ is continuous, there exist $\delta > 0$ and a compact neighborhood $\overline{U}$ of e in H, such that

$$S_\delta(\vec{\gamma}_0)\overline{U} \subset B_{\varepsilon'}(\vec{\gamma}_0).$$

Note that $\delta < \varepsilon'$, and by choice of ε',

$$S_\delta(\vec{\gamma}_0)\overline{U} \cap \mathcal{O}(\vec{\gamma}_0) = \vec{\gamma}_0\overline{U}, \tag{4.3}$$

for if $\vec{\eta}_0 \in S_\delta(\vec{\gamma}_0), u_0 \in \overline{U}$ are such that $\vec{\eta}_0 u_0 \in \mathcal{O}(\vec{\gamma}_0)$, then by (4.1) $\vec{\eta}_0 = \vec{\gamma}_i$ for some i, and hence by (4.2), $i = 0$. Now as Θ is an open map, we can pick $\varepsilon > 0$

such that

$$\overline{B}_\varepsilon(\vec{\gamma}_0) \subset S_\delta(\vec{\gamma}_0)\overline{U}.$$

Observe that by (4.3), if $\vec{\gamma} \in \overline{B}_\varepsilon(\vec{\gamma}_0) \cap \mathcal{O}(\vec{\gamma}_0)$, then $\vec{\gamma}$ is of the form $\vec{\gamma}_0 u_0$, for some $u_0 \in \overline{U}$. So if $h \in H^\varepsilon_{\vec{\gamma}_0}$, that is, $\vec{\gamma}_0 h \in B_\varepsilon(\vec{\gamma}_0)$, then

$$\vec{\gamma}_0 h = \vec{\gamma}_0 u$$

for some $u \in \overline{U}$ or

$$\vec{\gamma}_0 h u^{-1} = \vec{\gamma}_0.$$

That is, $hu^{-1} \in H_{\vec{\gamma}_0}$, or $h \in H_{\vec{\gamma}_0} u$. It follows that

$$H^\varepsilon_{\vec{\gamma}_0} = \{h \in H \,:\, \|\vec{\gamma}_0 h - \vec{\gamma}_0\| \leq \varepsilon\} \subset H_{\vec{\gamma}_0}\overline{U}.$$

Now as $H^\varepsilon_{\vec{\gamma}_0}$ is closed, and $H_{\vec{\gamma}_0}\overline{U}$ is compact, it follows that $H^\varepsilon_{\vec{\gamma}_0}$ is compact in H. Next let $\vec{\gamma}_0 \in SH$ be arbitrary. Pick $\vec{\eta}_0 \in S$, $h_0 \in H$ such that

$$\vec{\eta}_0 = \vec{\gamma}_0 h_0.$$

By the above, there exists $\tilde{\varepsilon} > 0$ such that $H^{\tilde{\varepsilon}}_{\vec{\eta}_0} = \{h \in H \,:\, \|\vec{\eta}_0 h - \vec{\eta}_0\| \leq \tilde{\varepsilon}\}$ is compact. Set $\varepsilon = \frac{\tilde{\varepsilon}}{\|h_0\|}$. Now if $\|\vec{\gamma}_0 h - \vec{\gamma}_0\| \leq \varepsilon$ then

$$\|\vec{\gamma}_0 h_0 h_0^{-1} h h_0 - \vec{\gamma}_0 h_0\| \leq \|\vec{\gamma}_0 h_0 h_0^{-1} h - \vec{\gamma}_0\| \, \|h_0\| \leq \tilde{\varepsilon}.$$

That is

$$\|\vec{\eta}_0 h_0^{-1} h h_0 - \vec{\eta}_0\| \leq \tilde{\varepsilon}$$

so that $h_0^{-1} h h_0 \in H^{\tilde{\varepsilon}}_{\vec{\eta}_0}$, or equivalently, $h \in h_0 H^{\tilde{\varepsilon}}_{\vec{\eta}_0} h_0^{-1}$. We have shown that $H^\varepsilon_{\vec{\gamma}_0} \subset h_0 H^{\tilde{\varepsilon}}_{\vec{\eta}_0} h_0^{-1}$, which is a compact set. Thus, $H^\varepsilon_{\vec{\gamma}_0}$ is itself compact. □

In particular case, if S is a cross-section, the next proposition says ε-stabilizers of close points lie in a common compact set, for ε sufficiently small.

Proposition 4.2. *Let H be a closed subgroup of $GL_n(\mathbb{R})$. Suppose there exists an N-section S such that*

(a) the map $\Theta : S \times H \to \widehat{\mathbb{R}^n}$ given by $(\vec{\gamma}, h) \mapsto \vec{\gamma}h$ is open

(b) for each $\vec{\gamma} \in S$, there exist $\delta > 0$ and a compact neighborhood V of the identity e in H such that if $\vec{\eta}_0, \vec{\eta}_1 \in S_\delta(\vec{\gamma})$ and $\vec{\eta}_1 = \vec{\eta}_0 v$ for some $v \in H$, then $v \in V$.

Then for each $\vec{\gamma} \in SH$ there exist an open neighborhood $B_\varepsilon(\vec{\gamma}), \tilde{\varepsilon} > 0$ and $D \subset H$ compact such that

$$H_{\vec{\eta}}^{\tilde{\varepsilon}} \subset D$$

for all $\vec{\eta} \in B_\varepsilon(\vec{\gamma})$.

Proof. Suppose first that $\vec{\gamma} \in S$, and let δ and V be as in the assumption. Since Θ is continuous and open, there exist $\delta_1, \delta_2 > 0$ and compact neighborhoods U and W of e in H, and $\varepsilon, \varepsilon_2 > 0$ such that

$$S_{\delta_1}(\vec{\gamma}) \subset S_{\delta_2}(\vec{\gamma})$$

$$W \subset U$$

$$B_\varepsilon(\vec{\gamma}) \subset S_{\delta_1}(\vec{\gamma})W \subset B_{\frac{\varepsilon_2}{2}}(\vec{\gamma}) \tag{4.4}$$

and

$$B_{\varepsilon_2}(\vec{\gamma}) \subset S_{\delta_2}(\vec{\gamma})U \subset B_\delta(\vec{\gamma}). \tag{4.5}$$

Note that $\delta_1 < \frac{\varepsilon_2}{2}$ and $\delta_2 < \delta$.

Set $\tilde{\varepsilon} = \frac{\varepsilon}{2M}$ where $M = \sup_{w \in W} \|w^{-1}\| < \infty$. Now let $\vec{\eta} \in B_\varepsilon(\vec{\gamma})$ be fixed but arbitrary, and $h \in H_{\vec{\eta}}^{\tilde{\varepsilon}}$. Then

$$\|\vec{\eta}h - \vec{\eta}\| < \tilde{\varepsilon}.$$

Now by (4.4), $\vec{\eta} = \vec{\eta}_0 w$ for some $\vec{\eta}_0 \in S_{\delta_1}(\vec{\gamma})$, $w \in W$. Then

$$\begin{aligned}\|\vec{\eta}_0 w h w^{-1} - \vec{\eta}_0\| &= \|\vec{\eta}_0 w h w^{-1} - \vec{\eta}_0 w w^{-1}\| \\ &\leq \|\vec{\eta}_0 w h - \vec{\eta}_0 w\| \, \|w^{-1}\| \\ &\leq \|\vec{\eta} h - \vec{\eta}\| \, M \\ &< \tilde{\varepsilon} M = \frac{\varepsilon}{2}\end{aligned}$$

so that

$$\|\vec{\eta}_0 w h w^{-1} - \vec{\gamma}\| \leq \frac{\varepsilon}{2} + \|\vec{\eta}_0 - \vec{\gamma}\| < \frac{\varepsilon}{2} + \delta_1 < \frac{\varepsilon}{2} + \frac{\varepsilon_2}{2} \leq \varepsilon_2.$$

It follows from (4.5) that

$$\vec{\eta}_0 w h w^{-1} = \vec{\eta}_1 u$$

where $\vec{\eta}_1 \in S_{\delta_2}(\vec{\gamma}) \cap \mathcal{O}(\vec{\eta}_0)$, $u \in U$. By assumption (b), $whw^{-1}u^{-1} \in V$ or

$$h \in W^{-1} V U W =: D.$$

We have shown that

$$H_{\vec{\eta}}^{\tilde{\varepsilon}} \subset D \qquad \forall \, \vec{\eta} \in B_{\varepsilon}(\vec{\gamma})$$

where D is compact in H.

Next let $\vec{\gamma} \in SH$ be arbitrary, say $\vec{\gamma} = \vec{\gamma}_0 h_0$ for some $\vec{\gamma}_0 \in S,\ h_0 \in H$. Let ε, $\tilde{\varepsilon}$, D be as above, for $\vec{\gamma}_0$. Pick $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(\vec{\gamma}) h_0^{-1} \subset B_{\varepsilon}(\vec{\gamma}_0)$ and set $\tilde{\varepsilon}_1 = \frac{\tilde{\varepsilon}}{\|h_0^{-1}\|}$. Now let $\vec{\eta} \in B_{\varepsilon_1}(\vec{\gamma})$ be arbitrary. If $h \in H$ is such that

$$\|\vec{\eta} h - \vec{\eta}\| < \tilde{\varepsilon}_1,$$

then

$$\|\vec{\eta} h_0^{-1} h_0 h h_0^{-1} - \vec{\eta} h_0^{-1}\| \leq \|\vec{\eta} h - \vec{\eta}\| \, \|h_0^{-1}\| < \tilde{\varepsilon}.$$

Now since $\vec{\eta} h_0^{-1} \in B_{\varepsilon}(\vec{\gamma}_0)$ it follows from the first part that $h_0 h h_0^{-1} \in D$, or equivalently, $h \in h_0^{-1} D h_0 =: D_1$. We have shown that

$$H_{\vec{\eta}}^{\tilde{\varepsilon}_1} \subset D_1 \qquad \forall \vec{\eta} \in B_{\varepsilon_1}(\vec{\gamma})$$

hence the proposition follows. $\square$

If H contains an expanding matrix, then every N-section may be modified to be bounded, and bounded away from zero:

Proposition 4.3. *Let H be a closed subgroup of $GL_n(\mathbb{R})$. Suppose*

(a) there exists an N-section S for the action of H on $\widehat{\mathbb{R}^n}$,

(b) H contains an expanding matrix.

Then there exist an N-section S' and $M > 1$ such that

$$1 < \|\vec{\gamma}\| < M \tag{4.6}$$

for all $\vec{\gamma} \in S'$.

Proof. Let $A \in H$ be expanding. We partition $\widehat{\mathbb{R}^n}\backslash\{0\}$ into annuli $B_m = \{\vec{\gamma} \in \widehat{\mathbb{R}^n} : 2^m < \|\vec{\gamma}\| \leq 2^{m+1}\,,\, m \in \mathbb{Z}\}$. Then $\overline{B}_m = \{\vec{\gamma} \in \widehat{\mathbb{R}^n} : 2^m \leq \|\vec{\gamma}\| \leq 2^{m+1}\,,\, m \in \mathbb{Z}\}$ is compact. Next we split each B_m into small pieces, and translate each piece into the annulus $\{\vec{\xi} \in \widehat{\mathbb{R}^n} : 1 < \|\vec{\xi}\| \leq \|A\|\}$. In fact, for $\vec{\gamma} \in \overline{B}_m$, as A is expanding,

$$\lim_{k\to\infty} \|\vec{\gamma}A^k\| = \infty \qquad (k \in \mathbb{Z}).$$

Thus, there exists a smallest $k = k^m_{\vec{\gamma}}$ such that $\|\vec{\gamma}A^k\| > 1$ for all $k \geq k^m_{\vec{\gamma}}$. Then $\|\vec{\gamma}A^{k^m_{\vec{\gamma}}-1}\| \leq 1$, and hence

$$1 < \|\vec{\gamma}A^{k^m_{\vec{\gamma}}}\| \leq \|\vec{\gamma}A^{k^m_{\vec{\gamma}}-1}\| \cdot \|A\| \leq \|A\|,$$

i.e.

$$1 < \|\vec{\gamma}A^{k^m_{\vec{\gamma}}}\| \leq \|A\|.$$

Pick any $M > \|A\|$. Since the annulus $\{\vec{\xi} \in \widehat{\mathbb{R}^n} : 1 < \|\vec{\xi}\| < M\}$ is open, there exists a neighborhood $V_{\vec{\gamma}}$ of $\vec{\gamma}A^{k^m_{\vec{\gamma}}}$ such that $V_{\vec{\gamma}} \subset \{\vec{\xi} \in \widehat{\mathbb{R}^n} : 1 < \|\vec{\xi}\| < M\}$. Let

$U_{\vec{\gamma}} = V_{\vec{\gamma}} A^{-k^m_{\vec{\gamma}}}$. Then $U_{\vec{\gamma}}$ is an open neighborhood of $\vec{\gamma}$. Now $\{U_{\vec{\gamma}}\}_{\vec{\gamma}\in\overline{B}_m}$ is an open cover of $\overline{B}_m$, hence there exists a finite subcover $\{U_{\vec{\gamma}_i(m)}\}_{i=1}^{l_m}$ of $\overline{B}_m$. Set

$$\begin{aligned}
U_1^m &= B_m \cap U_{\vec{\gamma}_1(m)} \\
U_2^m &= [B_m \cap U_{\vec{\gamma}_2(m)}] \setminus U_{\vec{\gamma}_1(m)} \\
&\vdots \\
U_{l_m}^m &= [B_m \cap U_{\vec{\gamma}_{l_m}(m)}] \setminus \cup_{i=1}^{l_m-1} U_{\vec{\gamma}_i(m)}.
\end{aligned}$$

Then $\{U_i^m\}_{i=1}^{l_m}$ is a partition of B_m into disjoint Borel sets, and $U_i^m A^{k^m_{\vec{\gamma}_i(m)}} \subset V_{\vec{\gamma}_i(m)} \subset \{\vec{\xi} \in \widehat{\mathbb{R}^n} \,:\, 1 < \|\vec{\xi}\| < M\}$. By this process, we obtain a countable collection of disjoint Borel sets $\{U_i^m \,:\, m \in \mathbb{Z},\, i = 1, 2, ..., l_m\}$, and numbers $k^m_{\vec{\gamma}_i(m)}$ such that

1. $\underset{m\in\mathbb{Z}}{\cup} \overset{l_m}{\underset{i=1}{\cup}} U_i^m = \widehat{\mathbb{R}^n} \setminus \{0\}$
2. $U_i^m A^{k^m_{\vec{\gamma}_i(m)}} \subset \{\vec{\xi} \in \widehat{\mathbb{R}^n} \,:\, 1 < \|\vec{\xi}\| < M\}$ for all $i = 1, 2, ..., l_m$ and $m \in \mathbb{Z}$.

Finally, we set

$$S' = \underset{m\in\mathbb{Z}}{\cup} \overset{l_m}{\underset{i=1}{\cup}} (S \cap U_i^m) A^{k^m_{\vec{\gamma}_i(m)}}.$$

It is easy to check that S' is an N-section, and by construction, $S' \subset \{\vec{\xi} \in \widehat{\mathbb{R}^n} \,:\, 1 < \|\vec{\xi}\| < M\}$. $\square$

Remark: It is natural to think that in order for bounded N-sections to exist, the group H must contain an expanding matrix. Example 5.2 shows that this is not the case.

We want a compact N-section S which has the property that the orbit map $\Theta : S \times H \to \widehat{\mathbb{R}^n}$ is open. In the remainder of this section, we will construct such a section for p-parameter groups of diagonal matrices.

Let H_p be a p-parameter group of diagonal $n \times n$ matrices, i.e. $H_p = \{A_1^{s_1} A_2^{s_2} \cdots A_p^{s_p} \,:\, s_i \in \mathbb{R}\ \forall i\}$ satisfying

1. $p \leq n$

2. A_j is diagonal, say $A_j = \text{diag}[a_{1j}, a_{2j}, ..., a_{nj}]$ where $a_{ij} > 0$.

The condition $a_{ij} > 0$ is equivalent to every matrix A_j being exponential, while the first condition is necessary for admissibility of H_p, as will be made clear below. Observe that since $\mathbb{R}^p$ has no non-trivial compact subgroups, H_p admissible will imply that stabilizers are trivial for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$, in particular, the map $s \in \mathbb{R}^p \mapsto A^s$ is one-to-one. We thus may identify H_p with $\mathbb{R}^p$. Furthermore, $\det A > 0$ for all $A \in H_p$.

Notation : For $s = (s_1, s_2, ..., s_p) \in \mathbb{R}^p$, A^s will mean $A_1^{s_1} A_2^{s_2} \cdots A_p^{s_p}$. (Below, s_i may denote an element in $\mathbb{R}^p$, or a scalar in $\mathbb{R}$. The correct meaning will be clear from the context).

Let $\vec{\gamma} \in \widehat{\mathbb{R}^n}$. For $\vec{\eta} \in \mathcal{O}(\vec{\gamma})$ there exists $s = (s_1, s_2, ..., s_p) \in \mathbb{R}^p$ such that

$$\vec{\eta} = \vec{\gamma} A_1^{s_1} A_2^{s_2} \cdots A_p^{s_p}.$$

Preferring to write vectors in column form,

$$\vec{\eta}^{\mathsf{T}} = A_1^{s_1} A_2^{s_2} \cdots A_p^{s_p} \vec{\gamma}^{\mathsf{T}}$$

i.e.

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} = \begin{bmatrix} a_{11}^{s_1} a_{12}^{s_2} \cdots a_{1p}^{s_p} & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & a_{n1}^{s_1} a_{n2}^{s_2} \cdots a_{np}^{s_p} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}.$$

That is, for each $i = 1, 2, ..., n$,

$$\eta_i = a_{i1}^{s_1} a_{i2}^{s_2} \cdots a_{ip}^{s_p} \gamma_i. \tag{4.7}$$

Thus, corresponding components of $\vec{\gamma}$ and $\vec{\eta}$ have the same sign. Assume first that $\vec{\gamma}$ lies in $(\widehat{\mathbb{R}^+})^n$. Then $\vec{\eta} \in (\widehat{\mathbb{R}^+})^n$ and we can linearize equation (4.7) by taking the natural logarithm,

$$\ln \eta_i = \sum_{j=1}^{p} b_{ij} s_j + \ln \gamma_i \quad \text{where } b_{ij} = \ln a_{ij}.$$

In matrix notation,

$$\begin{bmatrix} \ln \eta_1 \\ \ln \eta_2 \\ \vdots \\ \ln \eta_n \end{bmatrix} = \underbrace{\begin{bmatrix} b_{11} & \dots & b_{1p} \\ b_{21} & \dots & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{np} \end{bmatrix}}_{M} \begin{bmatrix} s_1 \\ \vdots \\ s_p \end{bmatrix} + \begin{bmatrix} \ln \gamma_1 \\ \ln \gamma_2 \\ \vdots \\ \ln \gamma_n \end{bmatrix}.$$

Or in short,

$$\ln \vec{\eta} = Ms + \ln \vec{\gamma}.$$

(So by convention, $\ln \vec{\gamma}$ will always be a column vector). It is now clear that stabilizers are trivial a.e. if and only if $\ker M = \{0\}$, hence necessarily $p \leq n$. Observe that the mapping $\Psi : (\widehat{\mathbb{R}^+})^n \to \mathbb{R}^n$ given by $\vec{\gamma} \mapsto \ln \vec{\gamma}$ is a homeomorphism. If $\vec{\gamma}$ lies in any other octant, then we can linearize in a similar way. However, by symmetry we may always assume in what follows that $\vec{\gamma}$ lies in the first octant $(\widehat{\mathbb{R}^+})^n$. (Strictly speaking, the word octant is only correct if $n = 3$, we will however use it for general n).

Notation : For $1 \leq i_1 < i_2 < \cdots < i_p \leq n$, let $\vec{i} = (i_1, i_2, ..., i_p)$. $M_{\vec{i}}$ will denote the $p \times p$ matrix containing rows $i_1, i_2, ..., i_p$ of M, $\vec{\gamma}_{\vec{i}}$ the p-vector containing the entries $i_1, i_2, ..., i_p$ of an n-vector $\vec{\gamma}$ and $\Delta_{\vec{i}} = \det M_{\vec{i}}$. Set

$$J = \{\vec{i} = (i_1, i_2, ..., i_p) \,:\, 1 \leq i_1 < i_2 < ... < i_p \leq n \text{ and } \Delta_{\vec{i}} \neq 0\}.$$

Also, given $\vec{\gamma} \in \mathbb{R}^n, \vec{\gamma}_{[s,t]}$ will denote the vector consisting of the s-th to t-th components of $\vec{\gamma}$, i.e. if $\vec{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_n)$, then $\vec{\gamma}_{[s,t]} = (\gamma_s, \gamma_{s+1}, ..., \gamma_t)$ for $1 \leq s \leq t \leq n$. Similarly, $M_{[s,t]}$ will denote the $[(t-s)+1] \times p$ matrix containing rows s to t of M.

After linearization, we obtain the following proposition.

Proposition 4.4. *The following are equivalent:*

(a) H_p *is admissible.*

(b) There exists j *such that* $\det A_j \neq 1$, *and* $rank(M) = p$.

(c) There exists j *such that* $\det A_j \neq 1$, *and* $J \neq \emptyset$.

Proof. $(b) \Leftrightarrow (c)$

$$\begin{aligned} rank(M) = p &\Leftrightarrow M \text{ has } p \text{ linearly independent rows, say rows } i_1, i_2, ..., i_p \\ &\Leftrightarrow \Delta_{(i_1, i_2, ..., i_p)} \neq 0. \end{aligned}$$

$(a) \Rightarrow (b)$ Suppose H_p is admissible. Then by theorem 3.2,

1. there exists $\widetilde{A} \in H_p$ such that $\det \widetilde{A} \neq 1$
2. the stabilizer of $\vec{\gamma}$, $H_{\vec{\gamma}}$, is trivial a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

Let $s = (s_1, s_2, ..., s_p) \in \mathbb{R}^p$ be such that $\widetilde{A} = A^s = A_1^{s_1} A_2^{s_2} \cdots A_p^{s_p}$. Since $\det \widetilde{A} \neq 1$ then $\det A_j \neq 1$ for some j. Now choose a point $\vec{\gamma} \in (\widehat{\mathbb{R}^+})^n$ with trivial stabilizer. Then

$$\begin{array}{lrll} & \vec{\gamma} A^s = \vec{\gamma} & \text{implies} & s = 0 \\ \text{equivalently,} & Ms + \ln \vec{\gamma} = \ln \vec{\gamma} & \text{implies} & s = 0 \\ \text{hence} & Ms = 0 & \text{implies} & s = 0 \end{array}$$

which is equivalent to $rank(M) = p$.

$(b) \Rightarrow (a)$ By theorem 3.2 it is enough to show that for every $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ whose entries are nonzero, there exists $\varepsilon > 0$ such that the ε-stabilizer $H^{\varepsilon}_{\vec{\gamma}}$ is compact. Since Ψ

is a homeomorphism, this is equivalent to compactness of

$$\begin{aligned}\widetilde{H}^{\varepsilon}_{\vec{\gamma}} &:= \{s \in \mathbb{R}^p \,:\, \|\Psi(\vec{\gamma}A^s) - \Psi(\vec{\gamma})\| \leq \varepsilon\} \\ &= \{s \in \mathbb{R}^p \,:\, \|(Ms + \ln\vec{\gamma}) - \ln\vec{\gamma}\| \leq \varepsilon\} \\ &= \{s \in \mathbb{R}^p \,:\, \|Ms\| \leq \varepsilon\}\end{aligned}$$

for some $\varepsilon > 0$ and every $\vec{\gamma} \in \mathbb{R}^n$.

Let $\varepsilon > 0$ be arbitrary. Because M has rank p, M defines an invertible linear transformation of $\mathbb{R}^p$ onto a closed subspace V of $\mathbb{R}^n$, hence M defines a homeomorphism of $\mathbb{R}^p$ onto V. Since $\{y : \|y\| \leq \varepsilon\}$ is compact in V, so is its pre-image $\{s : \|Ms\| \leq \varepsilon\}$. Hence H_p is admissible. $\square$

The next theorem shows the existence of unbounded cross-sections.

Theorem 4.1. *Let H_p be as above. Given $\vec{i} \in J$, set*

$$T_{\vec{i}} := \{\vec{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_n) \in \widehat{\mathbb{R}^n} \,:\, \gamma_{i_j} = \pm 1 \textit{ for } j = 1, 2, ..., p\}.$$

Then

(a) $T_{\vec{i}}$ is a cross-section for the action of H_p (or equivalently, of $\mathbb{R}^p$) on $\widehat{\mathbb{R}^n}$

(b) the map $\Theta : (\vec{\gamma}, s) \mapsto \vec{\gamma}A^s$ is a homeomorphism of $T_{\vec{i}} \times \mathbb{R}^p$ onto $\{\vec{\xi} \in \widehat{\mathbb{R}^n} :$ $\xi_{i_j} \neq 0,\ j = 1, 2, ..., p\}$.

Proof. After suitably exchanging basis vectors in $\widehat{\mathbb{R}^n}$, we may assume that $i_1 = 1$, $i_2 = 2$, ..., $i_p = p$. Since $(\widehat{\mathbb{R} \setminus \{0\}})^p \times \widehat{\mathbb{R}^{n-p}}$ is the disjoint union of open sets of the form $S_1 \times S_2 \times ... \times S_p \times \widehat{\mathbb{R}^{n-p}}$, where $S_k = \widehat{\mathbb{R}^+}$ or $S_k = \widehat{\mathbb{R}^-}$, $k = 1, 2, ..., p$, and orbits stay in each of these sets, by symmetry we only need consider the set

$$T^{\circ}_{\vec{i}} := \{(1, 1, ..., 1, \gamma_{p+1}, ..., \gamma_n) \,:\, \gamma_i \in \mathbb{R}\}. \tag{4.8}$$

We need to show that

1. $T^{\circ}_{\vec{i}}$ is a cross-section for the action of H_p on $W_0 := (\widehat{\mathbb{R}^+})^p \times \widehat{\mathbb{R}^{n-p}}$.

2. Θ is a homeomorphism of $T^{\circ}_{\vec{i}} \times \mathbb{R}^p$ onto W_0.

Define a mapping $\Phi : W_0 \to \mathbb{R}^n$ by

$$\Phi(\vec{\gamma}) = (\ln \gamma_1, \ln \gamma_2, ..., \ln \gamma_p, \gamma_{p+1}, ..., \gamma_n)^{\mathsf{T}}.$$

Then

$$\Phi(\vec{\gamma}) = \ln \vec{\gamma}_{[1,p]} \oplus \vec{\gamma}_{[p+1,n]}$$

where $\vec{\gamma}_{[p+1,n]}$ consists of the last $n-p$ entries of $\vec{\gamma}$, now written as a column vector. Note that Φ is a homeomorphism. Thus the action of H_p on W_0 induces an action $\vec{\xi} \mapsto \vec{\xi} \cdot s$ of $\mathbb{R}^p$ on $\mathbb{R}^n$ given by

$$\vec{\xi} \cdot s = \Phi(\vec{\gamma}) \cdot s := \Phi(\vec{\gamma} A^s)$$

where $\vec{\xi} = \Phi(\vec{\gamma})$. Then

$$\begin{aligned} \vec{\xi} \cdot s &= \Phi(\vec{\gamma} A^s)_{[1,p]} \oplus \Phi(\vec{\gamma} A^s)_{[p+1,n]} \\ &= \left(M_{[1,p]} s + \begin{bmatrix} \ln \gamma_1 \\ \ln \gamma_2 \\ \vdots \\ \ln \gamma_p \end{bmatrix} \right) \oplus A_0^s \begin{bmatrix} \gamma_{p+1} \\ \gamma_{p+2} \\ \vdots \\ \gamma_n \end{bmatrix} \end{aligned}$$

that is,

$$\vec{\xi} \cdot s = (M_{[1,p]} s + \vec{\xi}_{[1,p]}) \oplus A_0^s \vec{\xi}_{[p+1,n]} \tag{4.9}$$

where A_0^s is the $(n-p) \times (n-p)$ matrix obtained from A^s by cutting the first p rows and columns. By assumption, $M_{[1,p]}$ is invertible. Since Φ is a homeomorphism, it is enough to show :

1. $\Phi(T^{\circ}_{\vec{i}}) = \{(0, 0, ..., 0, \gamma_{p+1}, ..., \gamma_n)^{\mathsf{T}} : \gamma_i \in \mathbb{R}\}$ is a cross-section for the action $\theta : (\vec{\gamma}, s) \mapsto \vec{\gamma} \cdot s$ of $\mathbb{R}^p$ on $\mathbb{R}^n$.

2. θ is a homeomorphism of $\Phi(T^\circ_{\vec{i}}) \times \mathbb{R}^p$ onto $\mathbb{R}^n$.

First we show that $\Phi(T^\circ_{\vec{i}})$ is a cross-section. Let $\vec{\eta} \in \mathbb{R}^n$ be given. Set $s = M^{-1}_{[1,p]}\vec{\eta}_{[1,p]}$, and let $\vec{\xi}$ be the vector given by $\vec{\xi}_{[1,p]} = [0, 0, ..., 0]$, and $\vec{\xi}_{[p+1,n]} = A_0^{-s}\vec{\eta}_{[p+1,n]}$. Then by (4.9)

$$\begin{aligned}\vec{\xi} \cdot s &= (M_{[1,p]}s + \vec{\xi}_{[1,p]}) \oplus A_0^s\vec{\xi}_{[p+1,n]} \\ &= (M_{[1,p]}(M^{-1}_{[1,p]}\vec{\eta}_{[1,p]})) \oplus A_0^s(A_0^{-s}\vec{\eta}_{[p+1,n]}) \\ &= \vec{\eta}_{[1,p]} \oplus \vec{\eta}_{[p+1,n]} = \vec{\eta}.\end{aligned}$$

In particular, θ is surjective.

On the other hand, suppose there exist $\vec{\xi}, \vec{\gamma} \in \Phi(T^\circ_{\vec{i}})$ and $s \in \mathbb{R}^p$ such that $\vec{\xi} \cdot s = \vec{\gamma}$. Then

$$\begin{aligned}M_{[1,p]}s + \vec{\xi}_{[1,p]} &= \vec{\gamma}_{[1,p]}, \\ A_0^s\vec{\xi}_{[p+1,n]} &= \vec{\gamma}_{[p+1,n]}.\end{aligned}$$

Since $\vec{\xi}_{[1,p]} = \vec{\gamma}_{[1,p]} = \vec{0}$, and $M_{[1,p]}$ is invertible, the first identity gives $s = 0$. Then the second identity gives $\vec{\xi}_{[p+1,n]} = \vec{\gamma}_{[p+1,n]}$, hence $\vec{\xi} = \vec{\gamma}$, that is θ is one-to-one. It follows that $\Phi(T^\circ_{\vec{i}})$ is a cross-section for the action of $\mathbb{R}^p$ on $\mathbb{R}^n$.

Since θ is continuous, one-to-one and surjective, we are left to show that θ is an open map. Since basic neighborhoods in $\Phi(T^\circ_{\vec{i}}) \times \mathbb{R}^p$ are of the form $S_\delta(\vec{\gamma}_0) \times B_\varepsilon(s_0)$, it suffices to show that $S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(s_0)$ is open in $\mathbb{R}^n$, for each basic open neighborhood $S_\delta(\vec{\gamma}_0)$ of $\vec{\gamma}_0 \in \Phi(T^\circ_{\vec{i}})$ and open ball $B_\varepsilon(s_0)$ in $\mathbb{R}^p$. Since $B_\varepsilon(s_0) = B_\varepsilon(0) + s_0$, we may assume by lemma 3.1 that $s_0 = 0$. So let $\vec{\gamma}_0 \in \Phi(T^\circ_{\vec{i}})$, $\delta > 0$ and $\varepsilon > 0$ be given. Then

$$(\vec{\gamma}_0)_{[1,p]} = \vec{0}$$

and

$$S_\delta(\vec{\gamma}_0) = \{\vec{\gamma} \in \mathbb{R}^n \,:\, \vec{\gamma}_{[1,p]} = \vec{0},\ \|\vec{\gamma}_{[p+1,n]} - (\vec{\gamma}_0)_{[p+1,n]}\| < \delta\}.$$

Next let $\vec{\xi} \in S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0)$ be arbitrary. We need to find an open neighborhood $B_{\tilde{\delta}}(\vec{\xi})$ in $\mathbb{R}^n$ which is contained in $S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0)$.

Now

$$\vec{\xi} = \vec{\gamma}_1 \cdot s_1$$

for some $\vec{\gamma}_1 \in S_\delta(\vec{\gamma}_0),\ s_1 \in B_\varepsilon(0)$. That is,

$$\vec{\xi}_{[1,p]} = M_{[1,p]} s_1 \quad \text{and} \quad \vec{\xi}_{[p+1,n]} = A_0^{s_1}(\vec{\gamma}_1)_{[p+1,n]}.$$

Now let $\delta_1 = \delta - \|(\vec{\gamma}_1)_{[p+1,n]} - (\vec{\gamma}_0)_{[p+1,n]}\| > 0$. It follows that if $\vec{\eta} \in \mathbb{R}^{n-p}$ and $\|\vec{\eta} - (\vec{\gamma}_1)_{[p+1,n]}\| < \delta_1$, then

$$\|\vec{\eta} - (\vec{\gamma}_0)_{[p+1,n]}\| \leq \|\vec{\eta} - (\vec{\gamma}_1)_{[p+1,n]}\| + \|(\vec{\gamma}_1)_{[p+1,n]} - (\vec{\gamma}_0)_{[p+1,n]}\| < \delta. \tag{4.10}$$

Similarly, we let $\varepsilon_1 = \varepsilon - \|s_1\|$. It follows that if $s \in \mathbb{R}^p$ and $\|s - s_1\| < \varepsilon_1$, then

$$\|s\| \leq \|s - s_1\| + \|s_1\| < \varepsilon. \tag{4.11}$$

Now as the map $(\vec{\eta}, s) \mapsto A_0^{-s}\vec{\eta}$ from $\mathbb{R}^{n-p} \times \mathbb{R}^p$ into $\mathbb{R}^{n-p}$ is continuous, there exist $\tilde{\delta} > 0$ and $\tilde{\varepsilon} > 0$ such that

$$\|\vec{\eta} - \vec{\xi}_{[p+1,n]}\| < \tilde{\delta} \quad \text{and} \quad \|s - s_1\| < \tilde{\varepsilon}$$

imply

$$\|A_0^{-s}\vec{\eta} - A_0^{-s_1}\vec{\xi}_{[p+1,n]}\| < \delta_1$$

that is

$$\|A_0^{-s}\vec{\eta} - (\vec{\gamma}_1)_{[p+1,n]}\| < \delta_1$$

so that by (4.10) with $\vec{\eta}$ replaced by $A_0^{-s}\vec{\eta}$,

$$\|A_0^{-s}\vec{\eta} - (\vec{\gamma}_0)_{[p+1,n]}\| < \delta. \tag{4.12}$$

Reducing $\tilde{\varepsilon}$ if necessary, we may assume that $\tilde{\varepsilon} < \varepsilon_1$. Now as $M_{[1,p]}$ is an invertible matrix, it defines a homeomorphism of $\mathbb{R}^p$ onto $\mathbb{R}^p$, so reducing $\tilde{\delta}$ if necessary,

$$\|M(s - s_1)\| < \tilde{\delta} \quad \text{implies} \quad \|s - s_1\| < \tilde{\varepsilon}. \tag{4.13}$$

Now let $\vec{\eta} \in B_{\tilde{\delta}}(\vec{\xi})$ be arbitrary. Then

$$\|\vec{\eta}_{[1,p]} - \vec{\xi}_{[1,p]}\| < \tilde{\delta} \quad \text{and} \quad \|\vec{\eta}_{[p+1,n]} - \vec{\xi}_{[p+1,n]}\| < \tilde{\delta}. \tag{4.14}$$

As $M_{[1,p]}$ is invertible, $\vec{\eta}_{[1,p]} = M_{[1,p]}s$ for some $s \in \mathbb{R}^p$. Then

$$\|M_{[1,p]}(s - s_1)\| = \|\vec{\eta}_{[1,p]} - \vec{\xi}_{[1,p]}\| < \tilde{\delta},$$

so that by (4.13), $\|s - s_1\| < \tilde{\varepsilon} < \varepsilon_1$, hence by (4.11), $\|s\| < \varepsilon$. Then by (4.12),

$$\|A_0^{-s}\vec{\eta}_{[p+1,n]} - (\vec{\gamma}_0)_{[p+1,n]}\| < \delta.$$

So if we set $\vec{\eta}_0 := \vec{0}_{[1,p]} \oplus A_0^{-s}\vec{\eta}_{[p+1,n]}$, then $\vec{\eta}_0 \in S_\delta(\vec{\gamma}_0)$. It now follows that

$$\vec{\eta}_0 \cdot s = M_{[1,p]}s \oplus \vec{\eta}_{[p+1,n]} = \vec{\eta},$$

and hence $\vec{\eta} \in S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0)$. We have shown that

$$B_{\tilde{\delta}}(\vec{\xi}) \subset S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0).$$

This shows that θ is an open mapping, and proves the proposition. $\square$

The cross-section $T_{\vec{i}}$ in proposition 4.1 is unbounded. We would like to obtain a bounded cross-section having the property (b). In what follows, we can nearly achieve this, in fact we obtain a set S which is an almost cross-section provided that H_p contains an expanding matrix. The idea is to show that each orbit intersects at least one of the cross-sections $T_{\vec{i}}$, within some bounded set.

For each $\vec{i} \in J$, set

$$S_{\vec{i}} = \{\vec{\gamma} \in \widehat{\mathbb{R}^n} \,:\, 0 < |\gamma_i| \le 1 \;\forall\; i \text{ and } |\gamma_{i_j}| = 1 \;\forall\; j = 1, 2, ..., p\}$$

$$S^+_{\vec{i}} = \{\vec{\gamma} \in \widehat{\mathbb{R}^n} \,:\, 0 < \gamma_i \le 1 \;\forall\; i \text{ and } \gamma_{i_j} = 1 \;\forall\; j = 1, 2, ..., p\}$$

$$S'_{\vec{i}} = \{\vec{\gamma} \in \widehat{\mathbb{R}^n} \,:\, 0 \le |\gamma_i| \le 1 \,\forall\, i,\; \gamma_i = 0 \;\exists\; i,\; \text{ and } |\gamma_{i_j}| = 1 \;\forall\; j = 1, 2, ..., p\}$$

$$S := \bigcup_{\vec{i} \in J} S_{\vec{i}} \cup S'_{\vec{i}}.$$

Note that S is compact, while $\cup_{\vec{i}\in J} S_{\vec{i}}$ is not. Thus, while the sets $S'_{\vec{i}}$ have measure zero, we need to include them in order to obtain a compact N-section.

We let $\Theta : S \times \mathbb{R}^p \to SH_p$ denote the continuous map given by

$$\Theta(\vec{\gamma}, s) = \vec{\gamma} A^s.$$

Theorem 4.2. *Let H_p be as in theorem 4.1. If in addition, H_p contains an expanding matrix, then S is a compact almost cross-section, and Θ is an open mapping.*

We split the proof into 2 parts. In the first part we show that S is an almost cross-section. In the second part we show that Θ is an open mapping.

Proposition 4.5. *S is an almost cross-section for the action of H_p on $\widehat{\mathbb{R}^n}$.*

Proof. Observe that $S \subset \cup_{\vec{i}\in J} T_{\vec{i}}$, hence each orbit intersects S at most $\sharp J$ times. We thus must show that the orbit of almost every $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ intersects S. In fact, we will show that the orbit of almost every $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ intersects $\cup_{\vec{i}\in J} S_{\vec{i}}$.

The major parts of the proof are lemmas 4.1 and 4.2 below. Let us first linearize and introduce some new notations. Without loss of generality, that is after change of basis in $\mathbb{R}^p$ we may assume that A_1 is expanding. As noted earlier, orbits remain within octants, so by symmetry, we only need to show that $\cup_{\vec{i}\in J} S^+_{\vec{i}}$ is an almost cross-section for the action of H_p on the subset $(\widehat{\mathbb{R}^+})^n$ of $\widehat{\mathbb{R}^n}$. Let $\vec{\gamma} \in (\widehat{\mathbb{R}^+})^n$. Then for each $\vec{\eta} \in \mathcal{O}(\vec{\gamma})$, there exists $s = (s_1, s_2, ..., s_p) \in \mathbb{R}^p$ such that

$$\vec{\eta} = \vec{\gamma} A_1^{s_1} A_2^{s_2} \cdots A_p^{s_p}.$$

Applying the map Ψ defined earlier,

$$\ln \vec{\eta} = Ms + \ln \vec{\gamma}.$$

In what follows, we will drop the logarithm, that is replace a vector $\Psi(\vec{\eta}) = \ln \vec{\eta}$ by $\vec{\eta}$, and simply write

$$\vec{\eta} = Ms + \vec{\gamma}.$$

Thus, the action $\vec{\gamma} \mapsto \vec{\gamma} \cdot s$ of $\mathbb{R}^p$ on $\mathbb{R}^n$ is given by the family of transformations

$$T_{\vec{\gamma}} : \mathbb{R}^p \longrightarrow \mathbb{R}^n$$

defined by

$$\vec{\eta} = T_{\vec{\gamma}}(s) = Ms + \vec{\gamma}$$

where $s \in \mathbb{R}^p$. Thus, $\mathcal{O}(\vec{\gamma}) = \{T_{\vec{\gamma}}(s) \,:\, s \in \mathbb{R}^p\} = \text{Range}(T_{\vec{\gamma}})$.

Since A_1 is expanding, the entries b_{i1} in the first column of M are all positive. Hence, if s_j, $j = 2, 3, ..., p$ remain fixed, then for each i, $1 \leq i \leq n$,

$$\begin{aligned} \lim_{s_1 \to -\infty} \eta_i &= \lim_{s_1 \to -\infty} (Ms + \vec{\gamma})_i \\ &= \lim_{s_1 \to -\infty} b_{i1}s_1 + \sum_{j=2}^{p} b_{ij}s_j + \gamma_i \\ &= -\infty, \end{aligned}$$

that is each component of $\vec{\eta}$ tends to $-\infty$ as s_1 goes to $-\infty$. In particular, there exists $s \in \mathbb{R}^p$ such that

$$\eta_i = (T_{\vec{\gamma}}s)_i < 0 \tag{4.15}$$

for all $i = 1, 2, ..., n$.

By an $n - r$ coordinate plane, we will mean the set

$$P_{(i_1,\ldots,i_r)} = \{\vec{\gamma} \in \mathbb{R}^n \,:\, \gamma_i = 0 \;\forall\; i \in \{i_1, i_2, ..., i_r\}\,\},$$

for a fixed set of indices $\{i_1, ..., i_r\}$. By a non-positive $n - r$ coordinate plane, we will mean the set

$$P^{\circ}_{(i_1,\ldots,i_r)} = \{\vec{\gamma} \in P_{(i_1,\ldots,i_r)} \,:\, \gamma_i \leq 0 \;\forall\; i \notin \{i_1, i_2, ..., i_r\}\,\}.$$

By a negative $n-r$ coordinate plane, we will mean the set

$$P^{-}_{(i_1,\ldots,i_r)} = \{\vec{\gamma} \in P_{(i_1,\ldots,i_r)} \,:\, \gamma_i < 0 \;\forall\; i \notin \{i_1, i_2, ..., i_r\}\,\}.$$

Observe that for $\vec{i} \in J$, $\Psi(S^{+}_{\vec{i}})$ is the non-positive $n-p$ plane

$$P^{\circ}_{\vec{i}} = \{\vec{\gamma} \in \mathbb{R}^n \,:\, \gamma_i \leq 0 \;\forall\; i,\; \gamma_i = 0 \text{ for } i = i_j,\; j = 1, 2, ..., p\}.$$

Recall that by an affine map T we mean a map of some finite dimensional vector space V into $\mathbb{R}^n$ of the form

$$T(s) = M(s) + \vec{\gamma}$$

for some fixed vector $\vec{\gamma} \in \mathbb{R}^n$ and linear mapping $M : V \to \mathbb{R}^n$. It is convenient to identify M with a matrix, then

$$T(s) = Ms + \vec{\gamma}.$$

We will write Ts instead of $T(s)$. Note that Range(T) is a connected set. Also, T is trivial if and only if $\ker(M) = V$.

Lemma 4.1. *Let $T : V \to \mathbb{R}^n$ be a non-trivial affine map. Suppose there exists $s \in \mathbb{R}^p$ such that*

$$(Ts)_i < 0 \tag{4.16}$$

for all $i = 1, 2, ..., n$. Then there exists $\tilde{s} \in V$ such that

(a) $(T\tilde{s})_{i_0} = 0$ for some $i_0 \in \{1, 2, ..., n\}$

(b) $(T\tilde{s})_i \leq 0$ for all $i = 1, 2, ..., n$.

Proof. We first show that there exists $s_0 \in V$ so that $(Ts_0)_i > 0$ for at least one i. In fact, as $\ker(M) \neq V$, there exists $s_1 \in V$ such that $Ms_1 \neq 0$. In particular,

$(Ms_1)_i \neq 0$ for some i. Pick $m \in \mathbb{Z}$ so that $m(Ms_1)_i > -\gamma_i$ and set $s_0 = ms_1$. Then

$$(Ts_0)_i = (M(ms_1))_i + \gamma_i = m(Ms_1)_i + \gamma_i > 0. \tag{4.17}$$

Next set

$$\mathcal{U} = \{\vec{\xi} \in \mathbb{R}^n \, : \, \xi_i < 0 \; \forall \; i\}.$$

Then

$$\text{bdry}(\mathcal{U}) = \{\vec{\xi} \in \mathbb{R}^n \, : \, \xi_i \leq 0 \; \forall \; i \text{ and } \xi_i = 0 \; \exists \; i\}$$

and

$$\overline{\mathcal{U}}^c = \{\vec{\xi} \in \mathbb{R}^n \, : \, \xi_i > 0 \; \exists \; i\}.$$

Observe that $\mathbb{R}^n$ is the disjoint union of these three sets. Set

$$O_1 = \mathcal{U} \cap \text{Range}(T)$$

$$O_2 = \text{bdry}(\mathcal{U}) \cap \text{Range}(T)$$

$$O_3 = \overline{\mathcal{U}}^c \cap \text{Range}(T),$$

so Range(T) is the disjoint union of O_1, O_2, and O_3. As Range(T) carries the subspace topology of $\mathbb{R}^n$, O_1 and O_3 are open sets in Range(T). Now suppose to the contrary, that there exists no $\tilde{s} \in V$ satisfying (a) and (b). This is equivalent to $O_2 = \emptyset$. Then by (4.16) and (4.17), Range(T) is the disjoint union of two nonempty open sets, contradicting connectedness of Range(T). This prove the lemma. $\square$

Lemma 4.2. *Let H_p be as above, $\vec{\gamma} \in \mathbb{R}^n$ and $T = T_{\vec{\gamma}}$ the corresponding affine map,*

$$Ts = Ms + \vec{\gamma}.$$

Then there exist $k \geq p$, a collection of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $s \in \mathbb{R}^p$ such that *(a)* $(Ts)_i = 0$ *for all* $i \in \{i_1, i_2, ..., i_k\}$

(b) $(Ts)_i < 0$ *for all* $i \notin \{i_1, i_2, ..., i_k\}$.

That is, $\vec{\eta} = Ts$ *lies in some negative* $n - k$ *coordinate plane. Furthermore, there exists a subset* $\{i_{j_1}, i_{j_2}, ..., i_{j_p}\}$ *of* $\{i_1, i_2, ..., i_k\}$ *such that* $\Delta_{(i_{j_1}, i_{j_2}, ..., i_{j_p})} \neq 0$.

Proof. We proceed by induction on the co-dimension of the coordinate plane. Throughout this proof, we will switch the standard basis vectors in $\mathbb{R}^n$ suitably, so that (a) and (b) become

(a′) $(Ts)_i = 0$ for $i = 1, 2, ..., k$

(b′) $(Ts)_i < 0$ for $i = k + 1, k + 2, ..., n$.

Initial step : As shown in (4.15), there exists $s \in \mathbb{R}^p$ such that $(Ts)_i < 0$ for all i. Applying lemma 4.1, followed by a suitable switch of basis vectors in $\mathbb{R}^n$, it follows that there exists $s_0 \in \mathbb{R}^p$ such that

(a) $(Ts_0)_1 = 0$

(b) $(Ts_0)_i \leq 0$ for all i.

Induction step : Assume we have found $s_0 \in \mathbb{R}^p$ and $k \geq 1$ so that after switching basis vectors,

(c) $(Ts_0)_i = 0$ for $i = 1, 2, ..., k$

(d) $(Ts_0)_i \leq 0$ for $i = k + 1, k + 2, ..., n$.

If $(Ts_0)_i = 0$ for some $i \in \{k+1, k+2, ..., n\}$, then after exchanging the i-th basis vector in $\mathbb{R}^n$ with the $(k + 1)$-st vector, it follows that

(c′) $(Ts_0)_i = 0$ for $i = 1, 2, ..., k + 1$

(d′) $(Ts_0)_i \leq 0$ for $i = k+2, k+3, ..., n$.

Repeating this process, we may assume that (d) above is replaced by

(d″) $(Ts_0)_i < 0$ for $i = k+1, k+2, ..., n$.

Now set

$$V_k = \{s \in \mathbb{R}^p \, : \, (Ms)_i = 0 \text{ for } i = 1, 2, ..., k\}.$$

Then V_k is a linear subspace of $\mathbb{R}^p$. If $V_k = \{0\}$ we stop; this can only happen if $k \geq p$, for if $k < p$, then the kernel V_k of the linear transformation defined by the $k \times p$ matrix $M_{[1,k]}$ is always nontrivial.

If $V_k \neq \{0\}$, we consider the affine mapping

$$T' : V_k \to \mathbb{R}^{n-k}$$

given by

$$T's = M_{[k+1,n]}s + \vec{\gamma}'$$

where $\vec{\gamma}' = (Ts_0)_{[k+1,n]}$.

For convenience, we will consider $\mathbb{R}^{n-k}$ as a subspace of $\mathbb{R}^n$, so $\vec{\gamma}'$ is a vector in $\mathbb{R}^n$ whose first k entries are zero; $\gamma'_1 = \gamma'_2 = \cdots = \gamma'_k = 0$. Observe that by assumption (d″), $\gamma'_i < 0$ for $i = k+1, k+2, ..., n$.

Since $T'(0) = \vec{\gamma}'$, we can apply lemma 4.1 to obtain $s_1 \in V_k$ satisfying, after a switch of basis vectors,

$$(T's_1)_{k+1} = 0 \quad \text{and} \quad (T's_1)_i \leq 0$$

for all $i \in \{k+1, k+2, ..., n\}$. Now set $\tilde{s}_0 = s_0 + s_1$. Then

$$T\tilde{s}_0 = T(s_0 + s_1) = M(s_0 + s_1) + \vec{\gamma} = Ts_0 + Ms_1.$$

Consider the various components of $T\tilde{s}_0$:

If $1 \leq i \leq k$, then by assumption (c), $(Ts_0)_i = 0$, while as $s_1 \in V_k$, $(Ms_1)_i = 0$. Thus, $(T\tilde{s}_0)_i = 0$.

If $i = k+1$, then

$$(Ms_1)_{k+1} = (T's_1)_{k+1} - \gamma'_{k+1} = -\gamma'_{k+1} = -(Ts_0)_{k+1}$$

so that

$$(T\tilde{s}_0)_{k+1} = (Ts_0)_{k+1} - (Ts_0)_{k+1} = 0.$$

If $k+2 \leq i \leq n$, then

$$(T\tilde{s}_0)_i = (Ts_0)_i + (Ms_1)_i = \gamma'_i + [(T's_1)_i - \gamma'_i] = (T's_1)_i \leq 0.$$

Thus, we have shown that

$$\begin{array}{lcl} (T\tilde{s}_0)_i = 0 & \text{for} & 1 \leq i \leq k+1 \\ (T\tilde{s}_0)_i \leq 0 & \text{for} & k+2 \leq i \leq n, \end{array}$$

that is, (c) and (d), and hence (c$'$) and (d$'$) hold for a large value of k.

By induction, the first assertion follows. Note that we stop when $V_k = \{0\}$, hence $M_{[1,k]}$ is a rank p matrix. Thus, there exist p rows $i_{j_1}, i_{j_2}, ..., i_{j_p}$ among the rows of $M_{[1,k]}$ such that $\Delta_{(i_{j_1}, i_{j_2}, ..., i_{j_p})} \neq 0$. This proves the lemma. $\square$

It follows from lemmas 4.1 and 4.2 that for every $\vec{\gamma} \in \mathbb{R}^n$ there exist $\vec{i} \in J$ and $s \in \mathbb{R}^p$ such that $\eta_i = 0$ for $i \in \{i_1, i_2, ..., i_p\}$, and $\eta_i \leq 0 \ \forall i$, where

$$\vec{\eta} = Ms + \vec{\gamma}. \tag{4.18}$$

That is, $\vec{\eta} \in P^{\circ}_{\vec{i}}$.

Exponentiating (4.18), that is applying the map Ψ^{-1}, we obtain that for every $\vec{\gamma} \in (\widehat{\mathbb{R}^+})^n$, there exist $\vec{i} \in J$ and $s \in \mathbb{R}^p$ such that

$$\vec{\eta} = \vec{\gamma} A^s \in S^+_{\vec{i}}.$$

Thus, $\mathcal{O}(\vec{\gamma})$ intersects $\cup_{\vec{i} \in J} S^+_{\vec{i}}$ at least once, for all $\vec{\gamma} \in (\widehat{\mathbb{R}^+})^n$. By symmetry, it follows that $\mathcal{O}(\vec{\gamma})$ intersects $\cup_{\vec{i} \in J} S_{\vec{i}}$ at least one, for each $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ whose components are nonzero. This proves proposition 4.5. $\square$

Proposition 4.6. *The mapping* $\Theta : S \times \mathbb{R}^p \to \widehat{\mathbb{R}^n}$ *is open.*

Proof. Again, we need to show that given basic open neighborhoods $S_\delta(\vec{\gamma}_0)$ and $B_\varepsilon(0)$ in S and $\mathbb{R}^p$, respectively, the set $S_\delta(\vec{\gamma}_0)B_\varepsilon(0)$ is open in $\widehat{\mathbb{R}^n}$, provided that δ and ε are sufficiently small. For convenience, we will use the maximum norms in $\mathbb{R}^n$, respectively $\mathbb{R}^p$, and choose matrix norms as corresponding operator norms.

Now for each $\vec{i} \in J$, set

$$S^\circ_{\vec{i}} = \{\vec{\gamma} \in S_{\vec{i}} \cup S'_{\vec{i}} : |\gamma_i| < 1 \ \forall\, i \notin \{i_1, i_2, ..., i_p\}\}$$

and

$$S^\circ = \cup_{\vec{i} \in J} S^\circ_{\vec{i}}.$$

Thus, S° contains those elements of S which have exactly p coordinates of absolute value one. Each set $S^\circ_{\vec{i}}$ is open in S, and the sets $S^\circ_{\vec{i}}$ are mutually disjoint. To see this, let $\vec{\gamma} \in S^\circ_{\vec{i}}$ be given, for some $\vec{i} \in J$. Choose $\delta > 0$ so that

$$\max_{i \notin \{i_1, i_2, ..., i_p\}} |\gamma_i| + \delta < 1.$$

Then if $\vec{\eta} \in S_\delta(\vec{\gamma})$, we have

$$|\eta_i| \le |\eta_i - \gamma_i| + |\gamma_i| < \delta + (1 - \delta) = 1$$

for all $i \notin \{i_1, i_2, ..., i_p\}$. Since at least p coordinates of $\vec{\eta} \in S$ must have absolute value one, it follows that

$$|\eta_i| = 1$$

for $i \in \{i_1, i_2, ..., i_p\}$ and since $|\eta_i - \gamma_i| < \delta < 1$, then

$$\eta_i = \gamma_i$$

for $i \in \{i_1, i_2, ..., i_p\}$. This shows that $\vec{\eta} \in S^\circ_{\vec{i}}$ as well, hence $S^\circ_{\vec{i}}$ is open in S. Disjointness of the sets $\{S^\circ_{\vec{i}}\}_{\vec{i} \in J}$ follows again from the fact that if $\vec{\gamma} \in S^\circ_{\vec{i}}$, where $\vec{i} = (i_1, i_2, ..., i_p)$ then

$$|\gamma_i| = 1$$

for $i \in \{i_1, i_2, ..., i_p\}$ while

$$|\gamma_i| < 1$$

for $i \notin \{i_1, i_2, ..., i_p\}$.

Now let $\vec{\gamma}_0 \in S$ be given. Suppose first that $\vec{\gamma}_0 \in S^\circ$. By the above discussion, there exist $\delta > 0$ and $\vec{i} \in J$ such that

$$S_\delta(\vec{\gamma}_0) \subset S^\circ_{\vec{i}}.$$

Since $S^\circ_{\vec{i}}$ is an open subset of the cross-section $T_{\vec{i}}$ of proposition 4.1, then so is $S_\delta(\vec{\gamma}_0)$. It then follows from proposition 4.1 that $S_\delta(\vec{\gamma}_0)B_\varepsilon(0)$ is open in $\widehat{\mathbb{R}^n}$, for every $\varepsilon > 0$.

Now suppose, $\vec{\gamma}_0 \notin S^\circ$. Then more than p of the components of $\vec{\gamma}_0$ have absolute value one, say after suitably exchanging the standard basis vectors in $\widehat{\mathbb{R}^n}$,

$$|(\vec{\gamma}_0)_i| = 1 \qquad \text{for } i = 1, 2, ..., k$$

$$|(\vec{\gamma}_0)_i| < 1 \qquad \text{for } i = k+1, k+2, ..., n$$

for some $k > p$. Let us set

$$J_0 = \{\vec{i} = (i_1, i_2, ..., i_p) \in J \, : \, i_p \leq k\}.$$

Then $\vec{\gamma}_0 \in S_{\vec{i}} \cup S'_{\vec{i}}$ if and only if $\vec{i} \in J_0$. Observe that if $\vec{\gamma} \in S_\delta(\vec{\gamma}_0)$ and $\delta < 1$, then

$$|\gamma_i - (\vec{\gamma}_0)_i| < 1$$

for $i = 1, 2, ..., k$ and hence

$$\operatorname{sgn} \gamma_i = \operatorname{sgn} (\vec{\gamma}_0)_i$$

for $i = 1, 2, ..., k$. Choosing $\delta < 1$, we can thus linearize the first k components. As always, by symmetry and since $(\widehat{\mathbb{R}^+})^k \times \widehat{\mathbb{R}^{n-k}}$ is open in $\widehat{\mathbb{R}^n}$ we may assume that $(\vec{\gamma})_i > 0$ for $i = 1, 2, ..., k$, and define a homeomorphism

$$\Phi : (\widehat{\mathbb{R}^+})^k \times \widehat{\mathbb{R}^{n-k}} \to \mathbb{R}^n$$

by $\Phi(\vec{\gamma}) = (\ln\gamma_1, \ln\gamma_2, ..., \ln\gamma_k, \gamma_{k+1}, ..., \gamma_n)^\mathsf{T}$, similar to the proof of theorem 4.1. Correspondingly, the action of $\mathbb{R}^p$ on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ becomes

$$\vec{\xi} \cdot s = (M_{[1,k]}s + \vec{\xi}_{[1,k]}) \oplus A_1^s \vec{\xi}_{[k+1,n]}$$

where A_1^s is the matrix obtained from A^s by cutting the first k rows and columns. Now for each $\vec{i} \in J_0$, set

$$S_{\vec{i}}^k = \{\vec{\gamma} \in S_{\vec{i}} \cup S'_{\vec{i}} : 0 < \gamma_i \leq 1 \text{ for } 1 \leq i \leq k, \ |\gamma_i| < 1 \text{ for } k+1 \leq i \leq n\}$$
$$S_k = \{\vec{\gamma} \in \widehat{\mathbb{R}^n} : \gamma_i = 1 \text{ for } 1 \leq i \leq k, |\gamma_i| < 1 \text{ for } k+1 \leq i \leq n\}.$$

Then $\vec{\gamma}_0 \in S_k \subset S_{\vec{i}}^k \subset S$ for all $\vec{i} \in J_0$. Furthermore,

$$\begin{aligned}\Phi(S_{\vec{i}}^k) = \big\{\vec{\xi} \in \mathbb{R}^n : \ & \xi_i \leq 0 \text{ for } 1 \leq i \leq k, \\ & \xi_i = 0 \text{ for } i \in \{i_1, i_2, ..., i_p\}, \\ & |\xi_i| < 1 \text{ for } k+1 \leq i \leq n\big\}\end{aligned}$$

and

$$\Phi(S_k) = \{\vec{\xi} \in \mathbb{R}^n : \vec{\xi}_{[1,k]} = \vec{0}, |\xi_i| < 1 \text{ for } k+1 \leq i \leq n\}.$$

In the following, we will identify vectors $\vec{\eta} \in (\widehat{\mathbb{R}^+})^k \times \widehat{\mathbb{R}^{n-k}}$ with their images $\Phi(\vec{\eta})$, and in particular, $\vec{\gamma}_0$ with $\Phi(\vec{\gamma}_0)$.

First we determine how small δ and ε need to be. Set

$$d = \|(\vec{\gamma}_0)_{[k+1,n]}\| = \max_{k+1 \leq i \leq n} |(\vec{\gamma}_0)_i|.$$

Then $d < 1$. Since the map $(\vec{\alpha}, s) \mapsto A_1^s \vec{\alpha}$ from $(\mathbb{R}^{n-k} \times \mathbb{R}^p) \to \mathbb{R}^{n-k}$ is continuous, there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that

$$B_{\delta_0}((\vec{\gamma}_0)_{[k+1,n]}) \cdot B_{\varepsilon_0}(0) \subset B_r((\vec{\gamma}_0)_{[k+1,n]}) \tag{4.19}$$

where $r = 1 - d > 0$, $B_{\delta_0}((\vec{\gamma}_0)_{[k+1,n]})$ denoting an open ball in $\mathbb{R}^{n-k}$. In particular, $\delta_0 < r$. In what follows, we choose $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$ arbitrarily. Let

$\vec{\eta} \in S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0)$ be arbitrary. We need to find an open neighborhood V of $\vec{\eta}$ in $\mathbb{R}^n$, with $V \subset S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0)$. Now

$$\vec{\eta} = \vec{\gamma} \cdot s_0$$

for some $\vec{\gamma} \in S_\delta(\vec{\gamma}_0),\ \|s_0\| < \varepsilon$. Since

$$\|\vec{\gamma}_{[k+1,n]} - (\vec{\gamma}_0)_{[k+1,n]}\| \leq \|\vec{\gamma} - \vec{\gamma}_0\| < \delta < \delta_0,$$

it follows from (4.19) that

$$\|\vec{\eta}_{[k+1,n]} - (\vec{\gamma}_0)_{[k+1,n]}\| < r \tag{4.20}$$

and also,

$$\begin{aligned} \|\vec{\gamma}_{[k+1,n]}\| &\leq \|\vec{\gamma}_{[k+1,n]} - (\vec{\gamma}_0)_{[k+1,n]}\| + \|(\vec{\gamma}_0)_{[k+1,n]}\| \\ &< \delta + d \\ &< r + (1-r) \\ &= 1. \end{aligned} \tag{4.21}$$

Thus, the p components of $\vec{\gamma}$ (to be precise, $\Phi^{-1}(\vec{\gamma})$) which have absolute value one must be among the first k components. That is, $\Phi^{-1}(\vec{\gamma}) \in S^k_{\vec{i}}$ for some $\vec{i} \in J_0$. After exchanging some of the first k basis vectors in $\mathbb{R}^n$, we may assume that

1. $\vec{i} = (1, 2, ..., p)$
2. $\gamma_i = 0$ for $1 \leq i \leq q$ where $p \leq q \leq k$
3. $\gamma_i < 0$ for $q + 1 \leq i \leq k$.

   Recall also that by (4.21),

4. $|\gamma_i| < 1$ for $k + 1 \leq i \leq n$.

Note that by 1., $M_{[1,p]}$ is invertible.

As before, for each $\vec{i} \in J_0$, let $M_{\vec{i}}$ denote the invertible $p \times p$ matrix obtained by selecting rows $i_1, i_2, ..., i_p$ of M. Similarly, given a vector $\vec{\zeta}$, let $\vec{\zeta}_{\vec{i}}$ denote the vector obtained by selecting the entries $i_1, i_2, ..., i_p$ of $\vec{\zeta}$. We make the following two observations : Suppose, $\vec{\rho} = M_{[1,k]}s + \vec{\zeta}$ for some $\vec{\rho}$, $\vec{\zeta} \in \mathbb{R}^k$, $s \in \mathbb{R}^p$.

1. If $\vec{\zeta}_{\vec{i}} = 0$ for some $\vec{i} \in J_0$, then $\vec{\rho}_{\vec{i}} = M_{\vec{i}}s$, and hence

$$\|s\| = \|(M_{\vec{i}})^{-1}\vec{\rho}_{\vec{i}}\| \leq \|(M_{\vec{i}})^{-1}\| \, \|\vec{\rho}_{\vec{i}}\| \leq N\|\vec{\rho}\| \tag{4.22}$$

   where $N = \max\limits_{\vec{i} \in J_0}(\|(M_{\vec{i}})^{-1}\|, 1)$

2.

$$\|\vec{\rho} - \vec{\zeta}\| \leq \|M\| \, \|s\| \leq K\|s\| \tag{4.23}$$

   where $K = \max(\|M\|, 1)$.

Now using again continuity of the map $(\vec{\alpha}, s) \mapsto A_1^s\vec{\alpha}$, there exist $r_1 > 0$ and $\varepsilon_1 > 0$ such that if $\|\vec{\alpha} - \vec{\gamma}_{[k+1,n]}\| < r_1$ and $\|s\| < \varepsilon_1$, then

$$\|A_1^s\vec{\alpha} - \vec{\gamma}_{[k+1,n]}\| < \delta - \|\vec{\gamma} - \vec{\gamma}_0\|. \tag{4.24}$$

Let us set $\tilde{\varepsilon} = \min\{\frac{1}{2}(\varepsilon - \|s_0\|), \frac{1}{2}(\delta - \|\vec{\gamma} - \vec{\gamma}_0\|), \varepsilon_1\} > 0$.

We are now ready to specify the required neighborhood V by setting

$$\begin{aligned} V = \{\vec{\beta} \in \mathbb{R}^n \ : \vec{\beta} = \vec{\xi} \cdot s = (M_{[1,k]}s + \vec{\xi}_{[1,k]}) \oplus A_1^s\vec{\xi}_{[k+1,n]}, \\ \|s - s_0\| < \tilde{\varepsilon}, \ \ \vec{\xi}_{[1,p]} = 0, \\ \|\vec{\xi}_{[p+1,k]} - \vec{\gamma}_{[p+1,k]}\| < \frac{\tilde{\varepsilon}}{KN}, \\ \|\vec{\xi}_{[k+1,n]} - \vec{\gamma}_{[k+1,n]}\| < r_1\}. \end{aligned} \tag{4.25}$$

Then $\vec{\eta} \in V$ (simply choose $s = s_0$, $\vec{\xi} = \vec{\gamma}$). Furthermore, V is an open neighborhood of $\vec{\eta}$ in $\mathbb{R}^n$. In fact, applying Φ^{-1} we have

$$\Phi^{-1}(V) = \Gamma(\vec{\gamma})B_{\tilde{\varepsilon}}(s_0)$$

where $\Gamma(\vec{\gamma})$ is an open neighborhood around $\Phi^{-1}(\vec{\gamma})$ in the cross-section $T_{\vec{i}}$ of proposition 4.1. (To be precise, $\Gamma(\vec{\gamma}) = \{\vec{\varrho} : \vec{\varrho}_{[1,p]} = \vec{\gamma}_{[1,p]} = \vec{1}_{[1,p]},\ \vec{\varrho}_{[p+1,k]} \in \exp(B_{\frac{\tilde{\varepsilon}}{KN}}(\vec{\gamma}_{[p+1,k]})),\ \vec{\varrho}_{[k+1,n]} \in B_{r_1}(\vec{\gamma}_{[k+1,n]})\}$).

Now let $\vec{\beta} \in V$ be arbitrary, say

$$\vec{\beta} = \vec{\xi} \cdot s_1 = (M_{[1,k]}s_1 + \vec{\xi}_{[1,k]}) \oplus A_1^{s_1}\vec{\xi}_{[k+1,n]}$$

for some $\vec{\xi}$ and s_1 as in (4.25). First consider the part of $\vec{\beta}$ living in $\mathbb{R}^q$,

$$\vec{\beta}_{[1,q]} = M_{[1,q]}s_1 + \vec{\xi}_{[1,q]}.$$

By assumption on M, and lemma 4.2, the orbit of $\vec{\xi}_{[1,q]}$ intersects some non-positive $q-p$ plane in $\mathbb{R}^q$. That is, there exist $\hat{s} \in \mathbb{R}^p$ and $\vec{\zeta}_{[1,q]} \in \mathbb{R}^q$, with $\zeta_i \le 0$ and $\zeta_i = 0$ for $1 \le i_1 \le i_2 \le ... \le i_p \le q$, such that

$$\vec{\xi}_{[1,q]} = M_{[1,q]}\hat{s} + \vec{\zeta}_{[1,q]} \tag{4.26}$$

and $\Delta_{\vec{j}} \ne 0$, where $\vec{j} = (i_1, i_2, ..., i_p)$. Thus, $\vec{j} \in J_0$. Now as $\vec{\zeta}_{\vec{j}} = 0$, then by (4.22) and (4.25),

$$\|\hat{s}\| \le N\|\vec{\xi}_{[1,q]}\| < N\frac{\tilde{\varepsilon}}{KN} = \frac{\tilde{\varepsilon}}{K} \le \tilde{\varepsilon} \le \varepsilon_1 \tag{4.27}$$

where we have used the fact that $\vec{\gamma}_{[1,q]} = 0$. We define the remaining components of $\vec{\zeta}$ by

$$\vec{\zeta}_{[q+1,k]} = \vec{\xi}_{[q+1,k]} - M_{[q+1,k]}\hat{s}$$
$$\vec{\zeta}_{[k+1,n]} = A_1^{-\hat{s}}\vec{\xi}_{[k+1,n]}$$

so that $\vec{\xi} = \vec{\zeta} \cdot \hat{s}$. Then by (4.23) and (4.27),

$$\|\vec{\xi}_{[1,k]} - \vec{\zeta}_{[1,k]}\| \le \|M\| \cdot \|\hat{s}\| < K\frac{\tilde{\varepsilon}}{K} = \tilde{\varepsilon}$$

so that by (4.25) and choice of $\tilde{\varepsilon}$,

$$\begin{aligned}\|\vec{\gamma}_{[1,k]} - \vec{\zeta}_{[1,k]}\| &\leq \|\vec{\gamma}_{[1,k]} - \vec{\xi}_{[1,k]}\| + \|\vec{\xi}_{[1,k]} - \vec{\zeta}_{[1,k]}\| \\ &< \frac{\tilde{\varepsilon}}{KN} + \tilde{\varepsilon} \\ &\leq 2\tilde{\varepsilon} \leq \delta - \|\vec{\gamma} - \vec{\gamma}_0\|\end{aligned}$$

while also, by choice of $\tilde{\varepsilon}$ and (4.24)

$$\|\vec{\gamma}_{[k+1,n]} - \vec{\zeta}_{[k+1,n]}\| = \|\vec{\gamma}_{[k+1,n]} - A_1^{\hat{s}}\vec{\xi}_{[k+1,n]}\| < \delta - \|\vec{\gamma} - \vec{\gamma}_0\|$$

so that

$$\|\vec{\gamma} - \vec{\zeta}\| < \delta - \|\vec{\gamma} - \vec{\gamma}_0\|. \tag{4.28}$$

Thus,

$$\begin{aligned}\|\vec{\gamma}_0 - \vec{\zeta}\| &\leq \|\vec{\gamma}_0 - \vec{\gamma}\| + \|\vec{\gamma} - \vec{\zeta}\| \\ &< \|\vec{\gamma}_0 - \vec{\gamma}\| + (\delta - \|\vec{\gamma}_0 - \vec{\gamma}\|) \\ &= \delta.\end{aligned}$$

Finally, set $s = s_1 + \hat{s}$. Then by (4.27) and choice of $\tilde{\varepsilon}$,

$$\begin{aligned}\|s\| &\leq \|s_1 - s_0\| + \|s_0\| + \|\hat{s}\| \\ &< \tilde{\varepsilon} + \|s_0\| + \tilde{\varepsilon} \\ &= 2\tilde{\varepsilon} + \|s_0\| \\ &\leq (\varepsilon - \|s_0\|) + \|s_0\| = \varepsilon.\end{aligned}$$

Then

$$\vec{\beta} = \vec{\xi} \cdot s_1 = (\vec{\zeta} \cdot \hat{s}) \cdot s_1 = \vec{\zeta} \cdot (\hat{s} + s_1) = \vec{\zeta} \cdot s \in S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0).$$

As $\vec{\beta} \in V$ was arbitrary, it follows that $V \subset S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0)$. This shows that $S_\delta(\vec{\gamma}_0) \cdot B_\varepsilon(0)$ is open, and hence Θ is an open map. $\square$

Example 4.1. Let $H_2 = \{A^s = A_1^{s_1} A_2^{s_2} = \begin{bmatrix} 2^{s_1}3^{s_2} & 0 & 0 \\ 0 & 2^{s_1}3^{s_2} & 0 \\ 0 & 0 & 4^{s_2} \end{bmatrix} : s = (s_1, s_2) \in \mathbb{R}^2\}$.

Let $\vec{\gamma} \in \widehat{\mathbb{R}^3}$. Then $\vec{\eta} \in \mathcal{O}(\vec{\gamma})$ if and only if $\vec{\eta} = \vec{\gamma} A^s$ for some $s \in \mathbb{R}^2$.

By symmetry we may assume that $\vec{\gamma} \in (\widehat{\mathbb{R}^+})^3$, and we linearize the above equation so

$$\vec{\eta} \in \mathcal{O}(\vec{\gamma}) \Leftrightarrow \ln \vec{\eta} = Ms + \ln \vec{\gamma}$$

for some $s \in \mathbb{R}^2$ where $M = \begin{bmatrix} \ln 2 & \ln 3 \\ \ln 2 & \ln 3 \\ 0 & \ln 4 \end{bmatrix}$.

We see $\Delta_{(1,2)} = 0$ and $\Delta_{(1,3)} = \Delta_{(2,3)} \neq 0$. Using the notation of the theorem, we have

$$\begin{aligned} S_{(1,3)} &= \{(\pm 1, \gamma_2, \pm 1) \in \widehat{\mathbb{R}^3} : 0 < |\gamma_2| \leq 1\} \\ S'_{(1,3)} &= \{(\pm 1, 0, \pm 1)\} \\ S_{(2,3)} &= \{(\gamma_1, \pm 1, \pm 1) \in \widehat{\mathbb{R}^3} : 0 < |\gamma_1| \leq 1\} \\ S'_{(2,3)} &= \{(0, \pm 1, \pm 1)\}. \end{aligned}$$

Thus

$$\begin{aligned} S &= S_{(1,3)} \cup S'_{(1,3)} \cup S_{(2,3)} \cup S'_{(2,3)} \\ &= \{(\pm 1, \gamma_2, \pm 1) \in \widehat{\mathbb{R}^3} : 0 \leq |\gamma_2| \leq 1\} \cup \{(\gamma_1, \pm 1, \pm 1) \in \widehat{\mathbb{R}^3} : 0 \leq |\gamma_1| \leq 1\} \end{aligned}$$

which is the union of two squares, the square with vertices $(1, -1, 1), (1, 1, 1), (-1, 1, 1), (-1, -1, 1)$ in the horizontal plane $\gamma_3 = 1$ and the square with vertices $(1, -1, -1), (1, 1, -1), (-1, 1, -1), (-1, -1, -1)$ in the plane $\gamma_3 = -1$. By proposition 4.6, $S_\delta U$ is open in $\widehat{\mathbb{R}^3}$ for U open in $\mathbb{R}^2$ and S_δ open in S.

4.2 Admissible Functions from Generalized Cross-Sections

Let H be a closed subgroup of $GL_n(\mathbb{R})$. If H is admissible, then by theorem 3.2, the function $\frac{|\det|}{\Delta}$ is not constant. Thus, in what follows we make the standing assumption that there exists $h_0 \in H$ with $\frac{|\det h_0|}{\Delta(h_0)} < 1$.

Now suppose, there exists an N-section S for the action of H on $\widehat{\mathbb{R}^n}$. Let us show how to obtain admissible functions from S. Pick a set V of finite, positive measure in H and set $\Omega = SV \subset \widehat{\mathbb{R}^n}$. We consider the function $g = \chi_\Omega$ and compute

$$\sigma(\vec{\gamma}) := \int_H |g(\vec{\gamma}h)|^2 d\mu(h).$$

Let $\vec{\gamma} \in SH$ be given. Then $S \cap \mathcal{O}(\vec{\gamma})$ is a finite set, say $\{\vec{\gamma}_1, \vec{\gamma}_2, ..., \vec{\gamma}_m\}$. Pick elements $l_1, l_2, ..., l_m \in H$ such that $\vec{\gamma} = \vec{\gamma}_i l_i$ $(i = 1, 2, ..., m)$.

Since S is an N-section, the stabilizer $H_{\vec{\gamma}_i}$ of each $\vec{\gamma}_i$ is finite, say

$$\begin{aligned} H_{\vec{\gamma}_i} &= \{h \in H \,:\, \vec{\gamma}_i h = \vec{\gamma}_i\} \\ &= \{h_1^{(i)}, h_2^{(i)}, ..., h_{n_i}^{(i)}\}. \end{aligned}$$

Then

$$\sum_{i=1}^{m} n_i \leq N.$$

Observe that for $h \in H$,

$$\begin{aligned} \vec{\gamma}_i h \in \vec{\gamma}_i V &\Leftrightarrow \vec{\gamma}_i h = \vec{\gamma}_i v \quad \text{for some } v \in V \\ &\Leftrightarrow \vec{\gamma}_i = \vec{\gamma}_i v h^{-1} \quad \text{for some } v \in V \\ &\Leftrightarrow v h^{-1} = h_j^{(i)} \quad \text{for some } v \in V,\ j \in \{1, 2, ..., n_i\} \\ &\Leftrightarrow h \in \bigcup_{j=1}^{n_i} [h_j^{(i)}]^{-1} V \\ &\Leftrightarrow h \in \bigcup_{j=1}^{n_i} h_j^{(i)} V. \end{aligned}$$

Since the Haar measure μ is left-invariant we have for each $i = 1, 2, ..., m$,

$$\begin{aligned}
\int_H \chi_{\vec{\gamma}_i V}(\vec{\gamma}h)d\mu(h) &= \int_H \chi_{\vec{\gamma}_i V}(\vec{\gamma}_i l_i h)d\mu(h) \\
&= \int_H \chi_{\vec{\gamma}_i V}(\vec{\gamma}_i h)d\mu(h) \\
&= \int_{\overset{n_i}{\underset{j=1}{\cup}} h_j^{(i)} V} 1 d\mu(h) \\
&\leq \sum_{j=1}^{n_i} \mu(h_j^{(i)} V) \\
&= \sum_{j=1}^{n_i} \mu(V) \\
&= n_i \mu(V)
\end{aligned}$$

so that for each i,

$$\begin{aligned}
0 < \mu(V) = \int_V 1 d\mu(h) &\leq \int_{\overset{n_i}{\underset{j=1}{\cup}} h_j^{(i)} V} 1 d\mu(h) \\
&= \int_H \chi_{\vec{\gamma}_i V}(\vec{\gamma}h)d\mu(h) \\
&\leq \int_H \chi_{\overset{m}{\underset{i=1}{\cup}} \vec{\gamma}_i V}(\vec{\gamma}h)d\mu(h) \\
&\leq \sum_{i=1}^{m} \int_H \chi_{\vec{\gamma}_i V}(\vec{\gamma}h)d\mu(h) \\
&= \sum_{i=1}^{m} n_i \mu(V) \leq N\mu(V).
\end{aligned}$$

Now as $\Omega \cap \mathcal{O}(\vec{\gamma}) = \overset{m}{\underset{i=1}{\cup}} \vec{\gamma}_i V$, it follows that

$$\chi_\Omega(\vec{\gamma}h) = \chi_{\overset{m}{\underset{i=1}{\cup}} \vec{\gamma}_i V}(\vec{\gamma}h)$$

for all $h \in H$, and hence

$$0 < \mu(V) \leq \int_H \chi_\Omega(\vec{\gamma}h)d\mu(h) \leq N\mu(V) \tag{4.29}$$

so that

$$0 < \mu(V) \leq \sigma(\vec{\gamma}) \leq N\mu(V)$$

for all $\vec{\gamma} \in SH$. Observe that if S is a cross-section, then $\sigma(\vec{\gamma}) = \mu(V)$ a.e. $\vec{\gamma}$. Now, if we set

$$\varphi(\vec{\gamma}) = \frac{g(\vec{\gamma})}{\sqrt{\sigma(\vec{\gamma})}}$$

for all $\vec{\gamma} \in SH$ then φ satisfies the admissibility condition,

$$\int_H |\varphi(\vec{\gamma}h)|^2 d\mu(h) = \frac{1}{\sigma(\vec{\gamma})} \int_H |g(\vec{\gamma}h)|^2 \mu(h) = 1$$

for all $\vec{\gamma} \in SH$. Thus, if Ω has finite Lebesgue measure, as happens when S is bounded and V precompact for example, then g and hence φ is square integrable so that $\check{\varphi}$ is an admissible function for H.

If S is unbounded and H contains an expanding matrix, one can always modify S to a bounded N-section by proposition 4.3. On the other hand, if H does not contain an expanding matrix, then this may not be possible, and one needs to modify the above construction. Partition $\widehat{\mathbb{R}^n}$ into a collection $\{T_i\}_{i=1}^{\infty}$ of bounded, measurable sets. Set $\Omega_i = \Omega \cap T_i$ and pick a sequence $\{r_i\}_{i=1}^{\infty}$ of integers so that

$$M := \sum_{i=1}^{\infty} \lambda(\Omega_i) \left[\frac{|\det h_0|}{\Delta(h_0)} \right]^{r_i} < \infty.$$

Consider the function

$$g(\vec{\gamma}) = \left[\sum_{i=1}^{\infty} \Delta(h_0)^{-r_i} \chi_{\tilde{\Omega}_i}(\vec{\gamma}) \right]^{1/2}$$

where $\widetilde{\Omega}_i = \Omega_i h_0^{r_i}$. Then

$$\begin{aligned}
\sigma(\vec{\gamma}) :&= \int_H |g(\vec{\gamma}h)|^2 d\mu(h) \\
&= \int_H \sum_{i=1}^{\infty} \Delta(h_0)^{-r_i} \chi_{\tilde{\Omega}_i}(\vec{\gamma}h) d\mu(h) \\
&= \sum_{i=1}^{\infty} \int_H \Delta(h_0)^{-r_i} \chi_{\tilde{\Omega}_i}(\vec{\gamma}h) d\mu(h).
\end{aligned}$$

Now

$$\begin{aligned}\chi_{\widetilde{\Omega}_i}(\vec{\gamma}h) = 1 \quad &\Leftrightarrow \quad \vec{\gamma}h \in \widetilde{\Omega}_i = \Omega_i h_0^{r_i} \\ &\Leftrightarrow \quad \vec{\gamma}hh_0^{-r_i} \in \Omega_i \\ &\Leftrightarrow \quad \chi_{\Omega_i}(\vec{\gamma}hh_0^{-r_i}) = 1\end{aligned}$$

so that

$$\begin{aligned}\sigma(\vec{\gamma}) &= \sum_{i=1}^{\infty} \int_H \Delta(h_0)^{-r_i} \chi_{\Omega_i}(\vec{\gamma}hh_0^{-r_i}) d\mu(h) \\ &= \sum_{i=1}^{\infty} \int_H \chi_{\Omega_i}(\vec{\gamma}h) d\mu(h) \\ &= \int_H \chi_{\Omega}(\vec{\gamma}h) d\mu(h)\end{aligned}$$

by disjointness of the collection $\{\Omega_i\}$, while by (4.29)

$$0 < \mu(V) \leq \sigma(\vec{\gamma}) \leq N\mu(V)$$

for all $\vec{\gamma} \in SH$ and hence the function $\varphi(\vec{\gamma}) = \frac{g(\vec{\gamma})}{\sqrt{\sigma(\vec{\gamma})}}$ again satisfies the admissibility condition. Furthermore, since

$$\begin{aligned}\int_{\widehat{\mathbb{R}^n}} |g(\vec{\gamma})|^2 d\vec{\gamma} = \int_{\widehat{\mathbb{R}^n}} \sum_{i=1}^{\infty} \Delta(h_0)^{-r_i} \chi_{\widetilde{\Omega}_i}(\vec{\gamma}) d\vec{\gamma} \\ = \sum_{i=1}^{\infty} \Delta(h_0)^{-r_i} \lambda(\widetilde{\Omega}_i) \\ = \sum_{i=1}^{\infty} \frac{|\det(h_0)|^{r_i}}{\Delta(h_0)^{r_i}} \lambda(\Omega_i) = M,\end{aligned}$$

it follows that $\varphi \in L^2(\widehat{\mathbb{R}^n})$, and hence $\check{\varphi}$ is an admissible function for H. Note that φ may be unbounded, and may have unbounded support. We thus have shown :

Proposition 4.7. *Let H be a closed subgroup of $GL_n(\mathbb{R})$, and suppose*

(a) there exists an N-section S for the action of H on $\widehat{\mathbb{R}^n}$

(b) there exists $h_0 \in H$ such that $|\det h_0| \neq \Delta(h_0)$.

Then H is admissible.

The function φ above is obtained from the characteristic function of the set Ω, and thus does not have good smoothness properties. However, under some additional assumptions on S, we can obtain $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Theorem 4.3. *Let H be a closed subgroup of $GL_n(\mathbb{R})$. Suppose there exist N-sections S_0 and S satisfying*

(a) $S_0 \subset S$

(b) S_0 is compact

(c) the map $\Theta : S \times H \to \widehat{\mathbb{R}^n}$ given by $(\vec{\gamma}, h) \mapsto \vec{\gamma}h$ is open

(d) there exists a compact neighborhood K of e in H such that

$$\{h \in H \,:\, Sh \cap S \neq \emptyset\} \subset K.$$

Then there exists an admissible function ψ with $\hat{\psi} \in C_c^\infty(\widehat{\mathbb{R}^n})$.

Proof. Let us first construct ψ. Pick open neighborhoods U and V of e in H such that

$$e \in V \subset \overline{V} \subset U,$$

and $\overline{U}$ is compact. Since S_0 and $\overline{V}$ are compact, then so is $S_0\overline{V}$. Also, by assumption (c), SU is open in $\widehat{\mathbb{R}^n}$. Thus, by theorem 2.3 there exists $g \in C_c^\infty(\widehat{\mathbb{R}^n})$ such that

1. $0 \leq g \leq 1$

2. $g(\vec{\gamma}) = 1$ for all $\vec{\gamma} \in S_0\overline{V}$

3. $g(\vec{\gamma}) = 0$ for all $\vec{\gamma} \in \widehat{\mathbb{R}^n} \setminus SU$.

As before, we set

$$\sigma(\vec{\gamma}) = \int_H |g(\vec{\gamma}h)|^2 d\mu(h).$$

Now as $\chi_{_{S_0\overline{V}}} \leq g^2 \leq \chi_{_{SU}}$, we obtain from (4.29) that

$$\begin{aligned} 0 < \mu(\overline{V}) &\leq \int_H \chi_{_{S_0\overline{V}}}(\vec{\gamma}h)d\mu(h) \\ &\leq \int_H |g(\vec{\gamma}h)|^2 d\mu(h) \\ &\leq \int_H \chi_{_{SU}}(\vec{\gamma}h)d\mu(h) \\ &\leq N\,\mu(U) < \infty \end{aligned}$$

for all $\vec{\gamma} \in SH$. That is, there exist $m > 0$, $M > 0$ such that

$$m \leq \sigma(\vec{\gamma}) \leq M$$

for all $\vec{\gamma} \in SH$. Next set

$$\varphi(\vec{\gamma}) = \begin{cases} \frac{g(\vec{\gamma})}{\sqrt{\sigma(\vec{\gamma})}} & \text{if } \vec{\gamma} \in SH, \\ 0 & \text{else.} \end{cases} \tag{4.30}$$

Obviously, $\mathrm{supp}(\varphi) \subset SU$, φ is bounded hence square integrable, and

$$\int_H |\varphi(\vec{\gamma}h)|^2 d\mu(h) = 1$$

for all $\vec{\gamma} \in SH$, that is, $\psi = \check{\varphi}$ is an admissible function for H. It is thus left to show that φ is infinitely differentiable.

Lemma 4.3. *For each $\vec{\gamma} \in SH$, there exist an open neighborhood W of $\vec{\gamma}$ in SH, and a compact subset F of H such that for all $\vec{\eta} \in W$, $\{h \in H : \vec{\eta}h \in SU\} \subset F$.*

Proof. Let $\vec{\gamma} \in SH$ be given, say $\vec{\gamma} = \vec{\gamma}_0 h_0$ with $\vec{\gamma}_0 \in S$, $h_0 \in H$. Pick an open neighborhood Z of e containing K, with $\overline{Z}$ compact. Then by assumption (d),

$$\{h \in H : \vec{\eta}_0 h \in S\} \subset Z \tag{4.31}$$

for all $\vec{\eta}_0 \in S$. Set $W = SZh_0$. Then W is an open neighborhood of $\vec{\gamma}$ in SH. Now let $\vec{\eta} \in W$ be arbitrary, say $\vec{\eta} = \vec{\eta}_0 z h_0$ for some $\vec{\eta}_0 \in S$ and $z \in Z$. Suppose, $h \in H$ is such that $\vec{\eta}h \in SU$. Then

$$\vec{\eta}_0 z h_0 h = \vec{\eta}_j u \tag{4.32}$$

for some $\vec{\eta}_j \in S \cap \mathcal{O}(\vec{\eta}_0)$, and $u \in U$. Equivalently, $\vec{\eta}_j = \vec{\eta}_0 z h_0 h u^{-1}$. Then by (4.31),

$$z h_0 h u^{-1} \in Z. \tag{4.33}$$

so that

$$h \in h_0^{-1} Z^{-1} Z U \subset h_0^{-1} \overline{Z}^{-1} \overline{Z}\,\overline{U}.$$

Setting $F = h_0^{-1} \overline{Z}^{-1} \overline{Z}\,\overline{U}$, the lemma follows. ☐

Lemma 4.4. *Let $f \in C_c^1(\widehat{\mathbb{R}^n})$ be such that supp$(f) \subset SU$. Set*

$$\sigma(\vec{\gamma}) = \int_H f(\vec{\gamma}h) d\mu(h). \tag{4.34}$$

Then each partial derivative $\frac{\partial \sigma}{\partial \gamma_i}$ exists on SH, is continuous and

$$\frac{\partial \sigma(\vec{\gamma})}{\partial \gamma_i} = \int_H \sum_{j=1}^{n} f_{,j}(\vec{\gamma}h) h_{ij} d\mu(h)$$

where $h = (h_{ij})$.

Proof. Recall that for each $h \in GL_n(\mathbb{R})$, the Jacobian matrix of the map $\vec{\gamma} \mapsto \vec{\gamma}h$ is the matrix h itself. Thus by the chain value,

$$\begin{aligned} \frac{\partial f(\vec{\gamma}h)}{\partial \gamma_i} &= \sum_{j=1}^{n} \frac{\partial f(\vec{\xi})}{\partial \xi_j}\Big|_{\vec{\xi}=\vec{\gamma}h} \cdot \frac{\partial (\vec{\gamma}h)_j}{\partial \gamma_i} \\ &= \sum_{j=1}^{n} f_{,j}(\vec{\gamma}h) h_{ij}. \end{aligned}$$

Now each partial derivative $f_{,j}$ has the same property as f, namely $f_{,j} \in C_c^\infty(\widehat{\mathbb{R}^n})$ and supp$(f_{,j}) \subset SU$. In particular, there exists $M > 0$ such that

$$|f_{,j}(\vec{\gamma})| \leq M$$

for all $\vec{\gamma} \in \widehat{\mathbb{R}^n}, j = 1, 2, ..., n$. Given $\vec{\gamma}_0 \in SH$, let W be an open neighborhood of $\vec{\gamma}_0$ and $F \subset H$ be as in lemma 4.3. Then $f_{,j}(\vec{\gamma}h) = 0$ for all $\vec{\gamma} \in W$, $h \notin F$, $j = 1, 2, ..., n$ and hence

$$\frac{\partial f(\vec{\gamma}h)}{\partial \gamma_i} = 0$$

for all $\vec{\gamma} \in W,\ h \notin F$. Hence, $\left|\frac{\partial f(\vec{\gamma}h)}{\partial \gamma_i}\right| \leq M(\sup_{h\in F}\|h\|)\chi_F(h)$ for all $\vec{\gamma} \in W$.

It now follows from Leibnitz's theorem that $\sigma(\vec{\gamma})$ is differentiable at $\vec{\gamma} = \vec{\gamma}_0$, the partial derivatives are continuous at $\vec{\gamma}_0$, and

$$\begin{aligned}
\frac{\partial \sigma(\vec{\gamma})}{\partial \gamma_i}\Big|_{\vec{\gamma}=\vec{\gamma}_0} &= \int_H \frac{\partial f(\vec{\gamma}h)}{\partial \gamma_i}\Big|_{\vec{\gamma}=\vec{\gamma}_0} d\mu(h) \\
&= \int_H \sum_{j=1}^{n} f_{,j}(\vec{\gamma}_0 h) h_{ij} d\mu(h) \\
&= \sum_{j=1}^{n} h_{ij} \int_H f_{,j}(\vec{\gamma}_0 h) d\mu(h).
\end{aligned}$$

As $\vec{\gamma}_0 \in SH$ was arbitrary the lemma follows. Observe that $f_{,j}$ is supported on SU. □

Return to the proof of the theorem. Applying lemma 4.4 to $f = g^2$, it follows that $\sigma(\vec{\gamma})$ and all its first partial order derivatives exist on SH, are continuous, and

$$\frac{\partial \sigma(\vec{\gamma})}{\partial \gamma_i}\Big|_{\vec{\gamma}=\vec{\gamma}_0} = \sum_{j=1}^{n} h_{ij} \int_H f_{,j}(\vec{\gamma}_0 h) d\mu(h).$$

Since the integrand satisfies the same assumptions as f, we can apply the lemma again and obtain that all second order partial derivatives of σ exist on SH, in fact

$$\begin{aligned}
\frac{\partial^2 \sigma(\vec{\gamma})}{\partial \gamma_k \partial \gamma_i}\Big|_{\vec{\gamma}=\vec{\gamma}_0} &= \int_H \frac{\partial^2 f(\vec{\gamma}h)}{\partial \gamma_k \partial \gamma_i}\Big|_{\vec{\gamma}=\vec{\gamma}_0} d\mu(h) \\
&= \sum_{l=1}^{n}\sum_{j=1}^{n} h_{ij} h_{kl} \int_H f_{,jl}(\vec{\gamma}_0 h) d\mu(h).
\end{aligned}$$

Again each integrand satisfies the same assumptions as f. Continuing inductively, it follows that partial derivatives of all orders exist on SH, and for any multi-index

α,

$$D^\alpha \sigma(\vec{\gamma})_{|\vec{\gamma}=\vec{\gamma}_0} = \int_H D^\alpha f(\vec{\gamma}h)_{|\vec{\gamma}=\vec{\gamma}_0} d\mu(h).$$

Hence $\sigma \in C^\infty(SH)$. Since $m \leq \sigma(\vec{\gamma}) \leq M$ on SH, then $\varphi \in C_c^\infty(SH)$, in fact, $\text{supp}(\varphi) \subset SU$. Now if $\vec{\gamma}_0 \notin SH$, and since $S\overline{U}$ is compact, we can pick an open neighborhood W of $\vec{\gamma}_0$ such that $W \cap SU = \emptyset$. Thus, $g(\vec{\gamma}) = 0$ on W, and hence $\varphi(\vec{\gamma}) = 0$ for all $\vec{\gamma} \in W$. In particular, all partial derivatives $D^\alpha \varphi(\vec{\gamma}_0)$ exist, and are zero. It follows that $\varphi \in C_c^\infty(\widehat{\mathbb{R}^n})$, and the theorem is proved. □

Example 4.2. Let H_2 be a 2-parameter group of expanding diagonal matrices in $\mathbb{R}^3$, say

$$H_2 = \{\text{diag}[\alpha_1^s\beta_1^t, \alpha_2^s\beta_2^t, \alpha_3^s\beta_3^t] \,:\, \alpha_i, \beta_i > 1,\, (s,t) \in \mathbb{R}^2\}.$$

Setting $a_i = \ln \alpha_i$, $b_i = \ln \beta_i$, then

$$M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

with a_i, $b_i > 0$.

Let us suppose that all 2×2 subdeterminants are nonzero, that is $\Delta_{(1,2)} \neq 0$, $\Delta_{(1,3)} \neq 0$, $\Delta_{(2,3)} \neq 0$. Since all entries of M are positive, one readily checks that after suitably exchanging basis vectors in $\mathbb{R}^3$, these three subdeterminants all become positive. Thus by theorem 4.1, the set

$$\widetilde{S} = S_{(1,2)} \cup S_{(1,3)} \cup S_{(2,3)}$$

with $S_{(1,2)} = \{(\pm 1, \pm 1, \gamma_3) \,:\, |\gamma_3| \leq 1\}$, $S_{(1,3)} = \{(\pm 1, \gamma_2, \pm 1) \,:\, |\gamma_2| \leq 1\}$, $S_{(2,3)} = \{(\gamma_1, \pm 1, \pm 1) \,:\, |\gamma_1| \leq 1\}$ is a compact N-section for H_2 with open orbit map $\Theta : \widetilde{S} \times \mathbb{R}^2 \to \mathbb{R}^3$.

However, $\widetilde{S}$ is not a cross-section. To see this, as usual we consider only points $\vec{\gamma}$ in the first *octant*, and linearize by applying the map Ψ as in the discussion following (4.7). Note that $\widetilde{S}$ is a cross-section if and only if none of the following equations has a solution,

$$M\vec{s}+\begin{bmatrix}\gamma_1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\\gamma_2\\0\end{bmatrix},M\vec{s}+\begin{bmatrix}\gamma_1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\\gamma_3\end{bmatrix},M\vec{s}+\begin{bmatrix}0\\\gamma_2\\0\end{bmatrix}=\begin{bmatrix}0\\0\\\gamma_3\end{bmatrix} \tag{4.35}$$

with $\gamma_1, \gamma_2, \gamma_3 < 0$, $\vec{s} = (s, t) \in \mathbb{R}^2$. The three equations have solutions

$$\gamma_2 = -\frac{\Delta_{(2,3)}}{\Delta_{(1,3)}}\gamma_1, \quad \gamma_3 = \frac{\Delta_{(2,3)}}{\Delta_{(1,2)}}\gamma_1, \quad \gamma_3 = -\frac{\Delta_{(1,3)}}{\Delta_{(1,2)}}\gamma_2 \tag{4.36}$$

respectively. Since all subdeterminants are positive the first and third equations have no solutions satisfying $\gamma_i < 0$ for all i. However, the second equation

$$\gamma_3 = \frac{\Delta_{(2,3)}}{\Delta_{(1,2)}}\gamma_1 \qquad (\gamma_1,\, \gamma_3 < 0) \tag{4.37}$$

has a solution for any choice of $\gamma_1 < 0$. Thus, $\widetilde{S}$ is a 2-section, but not a cross-section.

Note that $\widetilde{S}$ does not satisfy the regularity condition (d) of theorem 4.3. To see this, we solve the second equations in (4.35) and (4.36) for $\vec{s} = (s, t)$.

$$\begin{aligned} a_1 s + b_1 t + \gamma_1 &= 0 \\ a_2 s + b_2 t &= 0 \\ a_3 s + b_3 t &= \frac{\Delta_{(2,3)}}{\Delta_{(1,2)}}\gamma_1 \end{aligned}$$

give us that

$$\begin{aligned} s &= -\frac{\gamma_1}{\Delta_{(1,3)}}\left[b_3 + b_1\frac{\Delta_{(2,3)}}{\Delta_{(1,2)}}\right] \\ t &= -\frac{a_2}{b_2}s \end{aligned} \tag{4.38}$$

Since $b_1, b_3 > 0$, then $s \to \infty$ as $\gamma_1 \to -\infty$. Hence, $\{\vec{s} = (s, t) \in \mathbb{R}^2 \,:\, \widetilde{S} \cap \widetilde{S} \cdot \vec{s} \neq \emptyset\}$ is unbounded.

To overcome this problem, observe that by (4.37), every orbit which intersects $S_{(1,2)}$ also intersects $S_{(2,3)}$. Thus we may remove parts of $S_{(2,3)}$ (or equivalently, $S_{(1,2)}$) and still have an N-section. In fact, if we set

$$S_0 = S_{(1,2)} \cup S_{(1,3)}$$

then S_0 is a compact cross-section. Also, set

$$S = S_{(1,2)} \cup S_{(1,3)} \cup \{(\gamma_1, \pm 1, \pm 1) \,:\, 1 - \varepsilon < |\gamma_1| \leq 1\}$$

for some $0 < \varepsilon < 1$. It follows from the computations in the proof of proposition 4.6 that the 2-section S has the property that the orbit map $\Theta : S \times \mathbb{R}^2 \to \widehat{\mathbb{R}^3}$ is open. Furthermore, (4.38) shows that $\vec{s}$ remains in some bounded set as long as γ_1 remains bounded; it follows that $\{\vec{s} \in \mathbb{R}^2 \,:\, S \cap S \cdot \vec{s} \neq \emptyset\}$ is bounded.

Hence by theorem 4.3, one can construct admissible functions ψ with $\hat{\psi} \in C_c^\infty(\mathbb{R}^n)$ from the pair of 2-sections S_0 and S.

We conclude this chapter by presenting the classification of abelian 2-parameter subgroups of $GL_3(\mathbb{R})$ and describe the existence of cross-sections. Note that the matrices involved are not necessarily diagonal. The interested reader may easily verify the details of the proof which we omit for brevity.

Example 4.3. For fixed commuting exponential matrices $A, B \in GL_3(\mathbb{R})$, $A = e^M$, $B = e^N$, we define a 2-parameter group

$$H_\varphi : \mathbb{R}^2 \to GL_3(\mathbb{R})$$

by $H_\varphi(t, s) = A^t B^s$. Up to a change of basis in $\mathbb{R}^3$, that is up to conjugation by an invertible matrix, there are 4 distinct possibilities.

Case 1 : $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ and $B = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ $\lambda_i > 0$, $a_i > 0$.

This is the situation discussed in theorem 4.1. There exists a cross-section

T if and only if $\ln \lambda_i \ln a_j - \ln \lambda_j \ln a_i \neq 0$ for some $i, j \in \{1,2,3\}$ with $i \neq j$.

A possible choice of cross-section is $T = \{\vec{\gamma} \in \widehat{\mathbb{R}^3} \,:\, \gamma_1, \gamma_2 \in \{-1,1\}\}$

Case 2 : $A = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ and $B = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ $\lambda_i > 0,\ a_i > 0.$

There exists a cross-section T if and only if $a_2\lambda_1 \mathrm{ln}\lambda_1 - a_1 \mathrm{ln} a_1 \neq 0$.

A possible choice of cross-section is $T = \{\vec{\gamma} \in \widehat{\mathbb{R}^3} \,:\, \gamma_1 \in \{-1,1\},\ \gamma_2 = 0\}$

Case 3 : $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ and $B = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix}$ $\lambda_1 > 0,\ a_1 > 0.$

There exists a cross-section T if and only if $a_2\lambda \mathrm{ln}\lambda - a_1 \mathrm{ln} a_1 \neq 0$.

A possible choice of cross-section is $T = \{\vec{\gamma} \in \widehat{\mathbb{R}^3} \,:\, \gamma_1 \in \{-1,1\},\ \gamma_2 = 0\}$

Case 4 : $A = \begin{pmatrix} \lambda_1 \cos\theta & \lambda_1 \sin\theta & 0 \\ -\lambda_1 \sin\theta & \lambda_1 \cos\theta & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ and

$B = \begin{pmatrix} a_1 \cos\beta & a_1 \sin\beta & 0 \\ -a_1 \sin\beta & a_1 \cos\beta & 0 \\ 0 & 0 & a_2 \end{pmatrix}$ $\lambda_i > 0,\ a_i > 0,\ 0 \leq \theta,\ \beta < 2\pi$ with either $\theta \neq 0$ or $\beta \neq 0$. Set $\Delta_1 = \theta \ln a_1 - \beta \ln \lambda_1$, $\Delta_2 = \ln \lambda_2 \ln a_1 - \ln a_2 \ln \lambda_1$ and $u = \lambda_2^{\frac{\ln a_1}{\Delta_1}} a_2^{-\frac{\ln \lambda_1}{\Delta_1}}$. There exists a cross-section T if and only if $\Delta_i \neq 0$ for all $i = 1,2$. A possible choice of cross-section is $T = \{\vec{\gamma} \in \widehat{\mathbb{R}^3} \,:\, \gamma_1 = 1,\ \gamma_2 = 0,\ 1 \leq |\gamma_3| < u^{2\pi}\}$ provided that $u > 1$ (if $u < 1$ we replace u by u^{-1}).

In all 4 cases, existence of a cross-section is equivalent to stabilizers being compact a.e. which is equivalent to stabilizers being trivial a.e. which in turn is equivalent to existence of compact ε-stabilizer a.e. Thus applying theorem 3.2, the following are equivalent:

1. H_p is admissible

2. there exists a cross-section T, and at least one of A and B has determinant different from 1.

CHAPTER V

WAVELET FRAMES

In this chapter, we discuss how to reconstruct a function from its wavelet transform by a series instead of the weak integral (3.3). That is, we want to find countable subsets P of H, and Γ of $\mathbb{R}^n$ such that $\{\psi_{k,\vec{x}}\}_{k\in P,\vec{x}\in\Gamma}$ is a frame for $L^2(\mathbb{R}^n)$, where $\psi_{k,\vec{x}} = D_k T_{\vec{x}}\psi$. Such frames are called *wavelet frames.*

Discretization of the translation parameter is usually achieved by requiring the wavelet ψ to be bandlimited. Discretization of the dilation parameter can be achieved by either choosing P to be a *separated* subset of H, (Bernier and Taylor (1996)) and specifying conditions for the support of the Fourier transform of an admissible ψ, or by using a subset F of H and a discrete subset P so that the collection $\{kF\}_{k\in P}$ partitions H, and modifying an admissible ψ; these are the integrated wavelets in Heinlein (2003). Both of the above methods were presented in the literature for the case that the orbits are open. We show how these techniques can be applied to groups H with arbitrary orbit structure, provided that there exist bounded almost cross-sections.

5.1 Discretization

In Bernier and Taylor (1996), discretization was achieved in the case of free open orbits, using the fact that the Lebesgue measure on each orbit is equivalent to the measure transferred onto it from the group. We now show that their approach can also be used in the case of general orbits, provided that there exists a bounded almost cross-section. The important ingredient is to find a subset F_1 of $\widehat{\mathbb{R}^n}$ having the property that $a < \mu(F_1 \cap \mathcal{O}(\vec{\gamma})) \leq b$ for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$, where μ denotes the Haar measure of H transferred onto the orbit $\mathcal{O}(\vec{\gamma})$.

Definition 5.1. *A subset P of H is called separated if there exists a neighborhood V of the identity e in H such that $Vk \cap Vl = \emptyset$ for $l \neq k$ and $l, k \in P$. We say that P is separated by V.*

The following lemma was used in Bernier and Taylor (1996) without proof; we present its proof for completeness.

Lemma 5.1. *Let P be a separated subset of H and D a compact subset of H. Then there exists $M_0 \in \mathbb{N}$ such that*

$$\sharp\{p \in P : Dp \cap Dk \neq \emptyset\} \leq M_0 \tag{5.1}$$

for all $k \in P$.

Proof. We first prove the following claim : Let K be a fixed compact subset of H. For each $k \in P$, define $P_k = \{p \in P : p \in Kk\}$. Then there exists $M_K \in \mathbb{N}$ such that

$$\sharp P_k \leq M_K$$

for all $k \in H$.

Proof of the claim : Let V be an open, relatively compact neighborhood of e separating P. Then for each $k \in P$,

$$P_k \subset \{p \in P : Vp \subset VKk\}.$$

Let μ denote the right Haar measure on H. As $\overline{V}$ and K are compact, then $\overline{V}K$ is compact, and hence $\mu(VK) < \infty$.

Now we have $\cup_{p\in P_k} Vp \subset VKk$. Since this is a disjoint union, then

$$\sum_{p\in P_k} \mu(Vp) \leq \mu(VKk)$$

$$\text{or} \qquad \sum_{p\in P_k} \mu(V) \leq \mu(VK)$$

$$\text{or} \qquad \mu(V)\sharp(P_k) \leq \mu(VK).$$

Thus

$$\sharp P_k \leq \frac{\mu(VK)}{\mu(V)} =: M_K.$$

This proves the claim.

Next let $k \in P$ be arbitrary. Then for each $p \in P$,

$$\begin{aligned} Dp \cap Dk \neq \emptyset &\Leftrightarrow \exists\, q, l \in D \quad \text{such that} \quad qp = lk \\ &\Leftrightarrow \exists\, q, l \in D \quad \text{such that} \quad p = q^{-1}lk \\ &\Leftrightarrow p \in D^{-1}Dk \end{aligned}$$

so

$$\{p \in P : Dp \cap Dk \neq \emptyset\} = \{p \in P : p \in D^{-1}Dk\}$$

and thus by the claim,

$$\sharp\{p \in P : Dp \cap Dk \neq \emptyset\} = \sharp\{p \in P : p \in D^{-1}Dk\} \leq M_{D^{-1}D} < \infty.$$

This proves the lemma. □

Definition 5.2. *A frame generator is a pair (P, F) where P is a separated subset of H and F is a pre-compact subset of H such that $\cup_{k\in P} Fk = H$.*

We now describe how to obtain a frame $\{\psi_{k,\vec{m}}\}_{k\in P,\vec{m}\in\Gamma}$ in $L^2(\mathbb{R}^n)$ from a frame generator (P, F) under the presence of a bounded N-section S which satisfies property (d) of theorem 4.3.

Let D be a pre-compact subset of H such that $F \subset D$. Set $F_1 = SF$ and $D_1 = SD$. Since S is bounded, then so is SD, hence we can pick an n-dimensional parallelepiped R with $D_1 \subseteq R$. By an n-dimensional parallelepiped we mean an affine image of the unit cube,

$$R = \vec{\gamma}_0 + [-\frac{1}{2}, \frac{1}{2}]^n B$$

for some $\vec{\gamma}_0 \in \widehat{\mathbb{R}^n}$ and $B \in GL_n(\mathbb{R})$. Set $\Gamma = B^{-1}\mathbb{Z}^n$ and $\delta(B) = |\det B|$. Since $\{e^{2i\pi\vec{\gamma}\vec{m}}\}_{\vec{m}\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2[-\frac{1}{2}, \frac{1}{2}]^n$, it follows that $\{e_{\vec{m}}(\vec{\gamma})\}_{\vec{m}\in\Gamma}$ where $e_{\vec{m}}(\vec{\gamma}) = \frac{1}{\sqrt{\delta(B)}} e^{2i\pi\vec{\gamma}\vec{m}}$ is an orthonormal basis for $L^2(R)$.

Lemma 5.2. There exists $M \in \mathbb{N}$ such that

$$\sharp\{p \in P : D_1 p \cap D_1 k \neq \emptyset\} \leq M$$

for all $k \in P$.

Proof. Suppose $\vec{\gamma} \in D_1 p \cap D_1 k$ for some $p, k \in P$. Then $\vec{\gamma} = \vec{\gamma}_0 d_0 p = \vec{\gamma}_1 d_1 k$ for some $d_0, d_1 \in D$, $\vec{\gamma}_0, \vec{\gamma}_1 \in S$, or equivalently, $\vec{\gamma}_1 = \vec{\gamma}_0 d_0 p k^{-1} d_1^{-1}$. By assumption on S, there exists a compact neighborhood K of e in H, independent of $\vec{\gamma}_0, \vec{\gamma}_1$, such that

$$d_0 p k^{-1} d_1^{-1} \in K$$

so that $d_0 p \in K d_1 k$ that is,

$$Dp \cap KDk \neq \emptyset.$$

Applying lemma 5.1 to the set $\widetilde{D} = KD$, the assertion follows. □

In the following, let $\psi \in L^2(\mathbb{R}^n)$ satisfy the following conditions:

1. support of $\hat{\psi} \subset D_1$.
2. $a = \inf\{|\hat{\psi}(\vec{\gamma})| : \vec{\gamma} \in F_1\} > 0$.
3. $b = \sup\{|\hat{\psi}(\vec{\gamma})| : \vec{\gamma} \in D_1\} < \infty$.

Such a ψ certainly exists, for example, let $\hat{\psi}$ be the characteristic function of F_1.

Theorem 5.1. *With the notation which has been established above,* $\{D_{k^{-1}}T_{\vec{m}}\psi : k \in P, \vec{m} \in \Gamma\}$ *is a frame for* $L^2(\mathbb{R}^n)$ *with frame bounds* $\delta(B)a^2$ *and* $\delta(B)Mb^2$.

Proof. For each $f \in L^2(\mathbb{R}^n), k \in P$ and $\vec{m} \in \Gamma$,

$$\begin{aligned}\langle f, D_{k^{-1}}T_{\vec{m}}\psi\rangle_{L^2(\mathbb{R}^n)} &= \langle \hat{f}, D_k E_{-\vec{m}}\hat{\psi}\rangle_{L^2(\widehat{\mathbb{R}^n})} \\ &= \int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma})\delta^{-1/2}(k)\bar{\hat{\psi}}(\vec{\gamma}k^{-1})e^{2i\pi\vec{\gamma}k^{-1}\vec{m}}d\vec{\gamma} \\ &= \int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma}k)\delta^{1/2}(k)\bar{\hat{\psi}}(\vec{\gamma})e^{2i\pi\vec{\gamma}\vec{m}}d\vec{\gamma}.\end{aligned}$$

As $\operatorname{supp}(\hat{\psi}) \subset D_1 \subset R$, we have by Parseval's identity

$$\begin{aligned}&\sum_{k\in P}\sum_{\vec{m}\in\Gamma}\left|\langle f, D_{k^{-1}}T_{\vec{m}}\psi\rangle_{L^2(\mathbb{R}^n)}\right|^2 \\ &\qquad= \sum_{k\in P}\sum_{\vec{m}\in\Gamma}\left|\int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma}k)\delta^{1/2}(k)\bar{\hat{\psi}}(\vec{\gamma})e^{2i\pi\vec{\gamma}\vec{m}}d\vec{\gamma}\right|^2 \\ &\qquad= \delta(B)\sum_{k\in P}\delta(k)\sum_{\vec{m}\in\Gamma}\left|\int_R \hat{f}(\vec{\gamma}k)\bar{\hat{\psi}}(\vec{\gamma})e_{\vec{m}}(\vec{\gamma})d\vec{\gamma}\right|^2 \\ &\qquad= \delta(B)\sum_{k\in P}\delta(k)\sum_{\vec{m}\in\Gamma}\left|\langle e_{\vec{m}}(\vec{\gamma}), \bar{\hat{f}}(\vec{\gamma}k)\hat{\psi}(\vec{\gamma})\rangle_{L^2(R)}\right|^2 \\ &\qquad= \delta(B)\sum_{k\in P}\delta(k)\,\|\bar{\hat{f}}(\cdot k)\hat{\psi}(\cdot)\|^2_{L^2(R)} \\ &\qquad= \delta(B)\sum_{k\in P}\delta(k)\int_R |\bar{\hat{f}}(\vec{\gamma}k)|^2|\hat{\psi}(\vec{\gamma})|^2 d\vec{\gamma} \\ &\qquad= \delta(B)\sum_{k\in P}\delta(k)\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma}k)|^2|\hat{\psi}(\vec{\gamma})|^2 d\vec{\gamma} \\ &\qquad= \delta(B)\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2\sum_{k\in P}|\hat{\psi}(\vec{\gamma}k^{-1})|^2 d\vec{\gamma}. \qquad (5.2)\end{aligned}$$

Since S is an N-section, $\widehat{\mathbb{R}^n} \setminus SH$ is a set of measure zero. Now if $\vec{\gamma} \in SH = \cup_{k\in P}SFk$, then $\vec{\gamma} \in SFk_0$ for some $k_0 \in P$, so $\vec{\gamma}k_0^{-1} \in SF = F_1$ and hence $\sum_{k\in P}|\hat{\psi}(\vec{\gamma}k^{-1})|^2 \geq a^2$. On the other hand, if for some $k \in P$, $\vec{\gamma}k^{-1} \in D_1$, then $\vec{\gamma} \in D_1k$. By lemma 5.2 this is only possible for at most M values of $k \in P$. Since $|\hat{\psi}(\vec{\gamma}k^{-1})| \leq b$ for any of those values, then $\sum_{k\in P}|\hat{\psi}(\vec{\gamma}k^{-1})|^2 \leq Mb^2$. It follows that

$$\delta(B)a^2\|f\|^2_{L^2(\mathbb{R}^n)} \leq \sum_{k\in P}\sum_{\vec{m}\in\Gamma} |\langle f, D_{k^{-1}}T_{\vec{m}}\psi\rangle|^2 \leq \delta(B)Mb^2\|f\|^2_{L^2(\mathbb{R}^n)}$$

which means that $\{D_{k^{-1}}T_{\vec{m}}\psi : k \in P, \vec{m} \in \Gamma\}$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds $\delta(B)a^2$ and $\delta(B)Mb^2$. □

Remark:

1. In general, the frames obtained this way are not tight. However, if S is a cross-section and the collection $\{Fk\}_{k\in P}$ is disjoint, we can choose $D = F$ and $\hat{\psi} = \frac{1}{\sqrt{\delta(B)}}\chi_{D_1}$. Then $M = 1$ and $a = b = \frac{1}{\sqrt{\delta(B)}}$, so that $\{D_{k^{-1}}T_{\vec{m}}\psi : k \in P, \vec{m} \in \Gamma\}$ is a Parseval frame.

2. If S is a cross-section and H contains an expanding matrix, then by the proof of proposition 4.3, given $B \in GL_n(\mathbb{R})$, we can modify S so that $SD \subset [-\frac{1}{2}, \frac{1}{2}]^n B$. Thus, for each frame generator (P, F) and each lattice Γ in $\mathbb{R}^n$, there exists $\psi \in L^2(\mathbb{R}^n)$ such that $\{D_{k^{-1}}T_{\vec{m}}\psi : k \in P, \vec{m} \in \Gamma\}$ is a frame for $L^2(\mathbb{R}^n)$.

Corollary 5.2. *Let H be a closed subgroup of $GL_n(\mathbb{R})$, and suppose there exists a pair of N-sections S_0 and S satisfying the assumptions of theorem 4.3. Let P be a separated subset of H. Then there exist $\psi \in L^2(\mathbb{R}^n)$ with $\hat{\psi} \in C_c^\infty(\widehat{\mathbb{R}^n})$, and a lattice Γ in $\mathbb{R}^n$, such that $\{D_{k^{-1}}T_{\vec{m}}\psi : k \in P, \vec{m} \in \Gamma\}$ is a frame for $L^2(\mathbb{R}^n)$.*

Proof. Pick F and D with $\overline{F}$ compact, $\overline{F} \subset D$ and D open. Then by assumption, $F_1 = S_0\overline{F}$ is compact and $D_1 = SD$ is open in $\widehat{\mathbb{R}^n}$, so applying Urysohn's lemma (theorem 2.3), there exists $\psi \in L^2(\mathbb{R}^n)$ with $\hat{\psi} \in C_c^\infty(\widehat{\mathbb{R}^n})$, and

(a) $0 \leq \hat{\psi} \leq 1$

(b) $\hat{\psi}(\vec{\gamma}) = 1$ for all $\vec{\gamma} \in F_1$

(c) $\hat{\psi}(\vec{\gamma}) = 0$ for all $\vec{\gamma} \notin D_1$.

Choosing the matrix B so that $\mathrm{supp}(\hat{\psi}) \subset [-\frac{1}{2}, \frac{1}{2}]^n B$ and Γ as in the proof of the theorem, the assertion follows. $\square$

Let H be a separable locally compact group and P a discrete subgroup such that H/P is compact. Using basic topological arguments, one easily shows that there exists a Borel set $F \subset H$ with the following properties:

1. $F \cap Fk = \emptyset$ for all $k \in P$, $k \neq e$
2. $\overline{F}$ is compact
3. $\bigcup\limits_{k \in P} Fk = H$.

We call F a fundamental domain for the set P.

In this particular case we can obtain Parseval frames :

Corollary 5.3. *Let H be a closed subgroup of $GL_n(\mathbb{R})$. Suppose there exist*

(a) a co-compact discrete subgroup P of H

(b) a bounded N-section S for the action of H on $\widehat{\mathbb{R}^n}$ satisfying condition (d) of theorem 4.3.

Then there exists a Parseval frame $\{D_k T_{\vec{m}} \varphi \,:\, k \in P,\, \vec{m} \in \Gamma\}$.

Proof. Let F be a fundamental domain for P. Then (P, F) is a frame generator. Choose D, $\hat{\psi}$, B, and Γ as in the theorem. Now as P is a group, we can average the values of $\hat{\psi}$ over P-orbits, similar to section 4.2. Set

$$\sigma(\vec{\gamma}) = \sum_{k \in P} |\hat{\psi}(\vec{\gamma} k^{-1})|^2$$

for all $\vec{\gamma} \in SH$. Then

$$a^2 \leq \sigma(\vec{\gamma}) \leq b^2 M$$

for all $\vec{\gamma} \in SH$. Now let

$$\hat{\varphi}(\vec{\gamma}) = \begin{cases} \frac{1}{\sqrt{|\delta(B)|}}\left(\frac{\hat{\psi}(\vec{\gamma})}{\sqrt{\sigma(\vec{\gamma})}}\right) & \text{if } \vec{\gamma} \in SH, \\ 0 & \text{if } \vec{\gamma} \notin SH. \end{cases}$$

Since $\sigma(\vec{\gamma}k^{-1}) = \sigma(\vec{\gamma})$ for all $k \in P$, we have for all $\vec{\gamma} \in SH$,

$$\begin{aligned} \sum_{k \in P} |\hat{\varphi}(\vec{\gamma}k^{-1})|^2 &= \frac{1}{\delta(B)} \sum_{k \in P} \frac{|\hat{\psi}(\vec{\gamma}k^{-1})|^2}{\sigma(\vec{\gamma}k^{-1})} \\ &= \frac{1}{\delta(B)\sigma(\vec{\gamma})} \sum_{k \in P} |\hat{\psi}(\vec{\gamma}k^{-1})|^2 \\ &= \frac{1}{\delta(B)}. \end{aligned}$$

Since $\operatorname{supp}(\hat{\varphi}) = \operatorname{supp}(\hat{\psi})$, it follows from (5.2) with ψ replaced by φ that

$$\begin{aligned} \sum_{k \in P} \sum_{\vec{m} \in \Gamma} \left| \langle f, D_{k^{-1}} T_{\vec{m}} \varphi \rangle_{L^2(\mathbb{R}^n)} \right|^2 &= \delta(B) \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 \sum_{k \in P} |\hat{\varphi}(\vec{\gamma}k^{-1})|^2 d\vec{\gamma} \\ &= \|\hat{f}\|_2^2. \end{aligned}$$

Replacing k^{-1} by k, it follows that $\{D_k T_{\vec{m}} \varphi \,:\, k \in P, \vec{m} \in \Gamma\}$ is a Parseval frame. $\square$

If in addition, all assumptions of theorem 4.3 are satisfied, then we obtain smooth, bandlimited Parseval frames.

Corollary 5.4. *Let H be a closed subgroup of $GL_n(\mathbb{R})$. Suppose there exist*

(a) a co-compact discrete subgroup P of H,

(b) N-sections S_0 and S satisfying the assumptions of theorem 4.3.

Then there exist $\psi \in L^2(\mathbb{R}^n)$ with $\hat{\psi} \in C_c^\infty(\widehat{\mathbb{R}^n})$, and a lattice Γ in $\mathbb{R}^n$ so that $\{D_k T_{\vec{m}} \psi \,:\, k \in P, \vec{m} \in \Gamma\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$.

Proof. Let F be a fundamental domain for P. Pick $D \subset H$ open with $\overline{F} \subset D$ and $\overline{D}$ compact. Then $F_1 := S_0\overline{F}$ is compact in $\widehat{\mathbb{R}^n}$, and $D_1 := SD$ is open.

Now by using Urysohn's lemma (theorem 2.3), there exists $g \in C_c^\infty(\widehat{\mathbb{R}^n})$ such that

1. $0 \leq g \leq 1$
2. $g(\vec{\gamma}) = 1$ for all $\vec{\gamma} \in F_1$
3. $g(\vec{\gamma}) = 0$ for all $\vec{\gamma} \notin D_1$.

Set as before

$$\sigma(\vec{\gamma}) = \sum_{k \in P} |g(\vec{\gamma}k)|^2 \tag{5.3}$$

for all $\vec{\gamma} \in SH$. We now claim that locally, this is a finite sum.

For first let $\vec{\gamma} \in D_1$ be arbitrary. Let $k \in P$ be such that $g(\vec{\gamma}k) \neq 0$. Then $\vec{\gamma}k \in D_1$. Since $\vec{\gamma} \in D_1$ and $\vec{\gamma}k \in D_1$ and both lie in the same H-orbit, then

$$\vec{\gamma} = \vec{\gamma}_0 d_0 \quad \text{and} \quad \vec{\gamma}k = \vec{\gamma}_1 d_1$$

for some $\vec{\gamma}_0$, $\vec{\gamma}_1 \in S$, d_0, $d_1 \in D$. It follows that $\vec{\gamma}_1 = \vec{\gamma}k d_1^{-1} = \vec{\gamma}_0 d_0 k d_1^{-1}$. Then by assumption (d) of theorem 4.3,

$$d_0 k d_1^{-1} = h$$

for some $h \in K$ so that $k = d_0^{-1} h d_1 \in \overline{D}^{-1} K \overline{D}$. Now as $K_1 := P \cap \overline{D}^{-1} K \overline{D}$ is a compact subset of discrete group, it is finite, say $\sharp K_1$=M. Thus for all $\vec{\gamma} \in D_1$,

$$\sum_{k \in P} |g(\vec{\gamma}k)|^2 = \sum_{k \in K_1} |g(\vec{\gamma}k)|^2, \tag{5.4}$$

a finite sum.

Next let $\vec{\gamma} \in SH$ be arbitrary. Since

$$SH = S(\underset{k \in P}{\cup} Fk) = \underset{k \in P}{\cup} (SF)k = \underset{k \in P}{\cup} F_1 k,$$

there exists $k_0 \in P$ such that $\vec{\gamma} \in F_1 k_0$. Then $D_1 k_0$ is an open neighborhood of $\vec{\gamma}$ in SH. So if $\vec{\eta} \in D_1 k_0$, then $\vec{\eta} k_0^{-1} \in D_1$ and hence by (5.4),

$$\begin{aligned}
\sum_{k\in P} |g(\vec{\eta}k)|^2 &= \sum_{k\in P} |g(\vec{\eta}k_0^{-1}k)|^2 \\
&= \sum_{k\in K_1} |g(\vec{\eta}k_0^{-1}k)|^2 \\
&= \sum_{k\in k_0^{-1}K_1} |g(\vec{\eta}k)|^2,
\end{aligned}$$

a sum of at most M terms. This proves the claim.

Now pick $B \in GL_n(\mathbb{R})$ such that $\operatorname{supp}(g) \subset [-\frac{1}{2}, \frac{1}{2}]^n B$, and set $\Gamma = B^{-1}\mathbb{Z}^n$. Since the sum (5.3) is locally finite, it follows that $\sigma(\vec{\gamma}) \in C^\infty(\widehat{\mathbb{R}^n})$. If we thus set

$$\hat{\psi}(\vec{\gamma}) = \begin{cases} \frac{1}{\sqrt{\delta(B)}}\left(\frac{g(\vec{\gamma})}{\sqrt{\sigma(\vec{\gamma})}}\right) & \text{if } \vec{\gamma} \in SH, \\ 0 & \text{if } \vec{\gamma} \notin SH \end{cases}$$

then since $\sigma(\vec{\gamma}) \geq 1$ for all $\vec{\gamma} \in SH$, it follows that $\hat{\psi} \in C_c^\infty(\widehat{\mathbb{R}^n})$, and also for $\vec{\gamma} \in SH$,

$$\begin{aligned}
\sum_{k\in P} |\hat{\psi}(\vec{\gamma}k)|^2 &= \frac{1}{\delta(B)} \sum_{k\in P} \frac{|g(\vec{\gamma}k)|^2}{\sigma(\vec{\gamma}k)} \\
&= \frac{1}{\delta(B)} \sum_{k\in P} \frac{|g(\vec{\gamma}k)|^2}{\sigma(\vec{\gamma})} \\
&= \frac{1}{\delta(B)} \cdot \frac{1}{\sigma(\vec{\gamma})} \cdot \sigma(\vec{\gamma}) \\
&= \frac{1}{\delta(B)}.
\end{aligned}$$

By (5.2) in the proof of the theorem, we conclude that $\{D_k T_{\vec{m}}\psi : k \in P, \vec{m} \in \Gamma\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$. $\square$

Example 5.1. Consider the group $H_2 = \{A^{\vec{s}} = \begin{bmatrix} \alpha_1^s\beta_1^t & 0 & 0 \\ 0 & \alpha_2^s\beta_2^t & 0 \\ 0 & 0 & \alpha_3^s\beta_3^t \end{bmatrix} : \alpha_i, \beta_i > 1, \vec{s} = (s,t) \in \mathbb{R}^2\}$ of example 4.2, with 2-sections S_0 and S contained in $\{\vec{\gamma} \in$

$\mathbb{R}^3 \, : \, |\gamma_i| \leq 1 \, \forall \, i\}$. Let $P = \mathbb{Z}^2$, then P is a co-compact discrete subgroup of $\mathbb{R}^2$, with fundamental domain $[k-1, k) \times [k-1, k)$ for any real number k.

Pick k such that $\alpha_i^{k+1} \beta_i^{k+1} < \frac{1}{2}$ for $i = 1, 2$. It follows that if we set

$$F = [k-1, k) \times [k-1, k) \quad \text{and}$$

$$D = (k-2, k+1) \times (k-2, k+1),$$

then $D_1 = SD \subset R = [-\frac{1}{2}, \frac{1}{2}]^3$. We can thus apply corollary 5.4 to obtain ψ with $\hat{\psi} \in C_c^\infty(\mathbb{R}^3)$, such that $\{D_{A^{\vec{k}}} T_{\vec{m}} \, : \, \vec{k} \in \mathbb{Z}^2, \, \vec{m} \in \mathbb{Z}^3\}$ is a Parseval frame for $L^2(\mathbb{R}^3)$.

The next example shows that without an expanding matrix, we also obtain a bounded cross-section and a Parseval frame.

Example 5.2. Fix $\beta > 1$. Let

$$H_\beta = \{ \begin{pmatrix} \alpha^\beta & 0 & b \\ 0 & \alpha & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \, : \, \alpha > 0, b \in \mathbb{R}\}.$$

One easily checks that H_β is a closed subgroup of $GL_3(\mathbb{R})$. In fact

$$H_\beta \sim \{(\alpha, b) \, : \, \alpha > 0, b \in \mathbb{R}\}$$

with group operation $(\alpha, b) \cdot (\alpha', b') = (\alpha\alpha', \alpha^\beta b' + \frac{b}{\alpha'})$. A straightforward computation shows that $S = \{(\pm x, \pm(1-x), 0) \in \widehat{\mathbb{R}^3} \, : \, 0 < x \leq 1\}$ is a cross section for the continuous action of H_β on $\mathbb{R}^3$. Next set

$$D = \{ \begin{pmatrix} 2^{t\beta} & 0 & 2^{t\beta} b \\ 0 & 2^t & 0 \\ 0 & 0 & \frac{1}{2^t} \end{pmatrix} \, : \, -\frac{1}{2} \leq b < \frac{1}{2}, -\frac{1}{2} \leq t < \frac{1}{2}\}$$

$$\sim \{(2^t, 2^{t\beta} b) \, : \, -\frac{1}{2} \leq b < \frac{1}{2}, -\frac{1}{2} \leq t < \frac{1}{2}\}$$

$$P = \{ \begin{pmatrix} 2^{n\beta} & 0 & 2^{-n}m \\ 0 & 2^n & 0 \\ 0 & 0 & \frac{1}{2^n} \end{pmatrix} : m, n \in \mathbb{Z}\}$$

$$\sim \{(2^n, 2^{-n}m) : m, n \in \mathbb{Z}\}.$$

Since D contains an open neighborhood of e, the next claim shows that P is a separated subset of H.

Claim : $Dk \cap Dl = \emptyset$ whenever $k \neq l$ and $k, l \in P$.

Proof of Claim : Let $k, l \in P$ and $k \neq l$ where $k = (2^n, 2^{-n}m)$ and $l = (2^{\tilde{n}}, 2^{-\tilde{n}}\tilde{m})$ and suppose there exist $d = (2^t, 2^{\beta t}b)$ and $\tilde{d} = (2^{\tilde{t}}, 2^{\beta\tilde{t}}\tilde{b})$ in D such that

$$dk = \tilde{d}l. \tag{5.5}$$

Then (5.5) gives

$$(2^{t+n}, \frac{2^{\beta t}m}{2^n} + \frac{2^{\beta t}b}{2^n}) = (2^{\tilde{t}+\tilde{n}}, \frac{2^{\beta\tilde{t}}\tilde{m}}{2^{\tilde{n}}} + \frac{2^{\beta\tilde{t}}\tilde{b}}{2^{\tilde{n}}}).$$

Comparing the first components, $2^{t+n} = 2^{\tilde{t}+\tilde{n}}$, that is $t - \tilde{t} = \tilde{n} - n \in \mathbb{Z}$. Since $-1 < t - \tilde{t} < 1$, then $t - \tilde{t}$ must be zero, this implies $n = \tilde{n}$ and $t = \tilde{t}$.

Thus, comparing the second components, we obtain $\frac{2^{\beta t}m}{2^n} + \frac{2^{\beta t}b}{2^n} = \frac{2^{\beta\tilde{t}}\tilde{m}}{2^{\tilde{n}}} + \frac{2^{\beta\tilde{t}}\tilde{b}}{2^{\tilde{n}}}$. Multiplying the equation by $\frac{2^n}{2^{\beta t}}$, we get $m + b = \tilde{m} + \tilde{b}$, that is $b - \tilde{b} = \tilde{m} - m \in \mathbb{Z}$. Since $-1 < b - \tilde{b} < 1$ so $b - \tilde{b}$ must be zero. This implies $\tilde{m} = m$ and $b = \tilde{b}$ which is a contradiction to $k \neq l$.

Hence the claim is proved.

Set

$$D_1 = SD = \{(\pm 2^{\beta t}x, \pm 2^t(1-x), \pm 2^{\beta t}bx) : 0 < x \leq 1, -\frac{1}{2} \leq b < \frac{1}{2}, -\frac{1}{2} \leq t < \frac{1}{2}\}.$$

Since

$$\begin{aligned}
DP &= \{(2^t, 2^{t\beta}b)(2^n, 2^{-n}m) \,:\, -\frac{1}{2} \le b < \frac{1}{2}, -\frac{1}{2} \le t < \frac{1}{2}, \text{ and } m, n \in \mathbb{Z}\} \\
&= \{(2^{t+n}, 2^{t\beta}2^{-n}m + 2^{t\beta}b2^{-n}) \,:\, -\frac{1}{2} \le b < \frac{1}{2}, -\frac{1}{2} \le t < \frac{1}{2}, \text{ and } m, n \in \mathbb{Z}\} \\
&= \{(2^{t+n}, 2^{t\beta-n}(m+b) \,:\, -\frac{1}{2} \le b < \frac{1}{2}, -\frac{1}{2} \le t < \frac{1}{2}, \text{ and } m, n \in \mathbb{Z}\} \\
&= \{(2^{t+n}, 2^{t\beta-n}\tilde{b} \,:\, -\frac{1}{2} \le t < \frac{1}{2}, \ \tilde{b} \in \mathbb{R}, \text{ and } n \in \mathbb{Z}\} \\
&= \{(2^s, b) \,:\, s, b \in \mathbb{R}\} \\
&= \{(\alpha, b) \,:\, \alpha > 0, \ b \in \mathbb{R}\} \\
&= H,
\end{aligned}$$

it follows from the fact that S is a cross section that

1. $D_1 P = SDP = SH = \widehat{\mathbb{R}^3} \setminus E$ for some set E of measure zero.

2. $D_1 k \cap D_1 l \neq \emptyset$ implies $Dk \cap Dl \neq \emptyset$ which by the claim implies $k = l$ for $k, l \in P$.

Thus D_1 is a bounded cross-section for the action of P on $\widehat{\mathbb{R}^3}$.

Set $B = 2^{\beta+1} I_3$, $R = [-\frac{1}{2}, \frac{1}{2}]^3 B = [-2^\beta, 2^\beta]^3$ and $\Gamma = B^{-1}\mathbb{Z}^3$. Then $\{e_{\vec{m}}\}_{\vec{m}\in\Gamma}$ is an orthonormal basis for $L^2(R)$, where $e_{\vec{m}}(\vec{\gamma}) = \frac{1}{\sqrt{2^{3(\beta+1)}}} e^{2i\pi\vec{\gamma}\vec{m}}$.

Let $\psi \in L^2(\mathbb{R}^3)$ be such that $\hat{\psi} = \sqrt{2^{3(\beta+1)}}\chi_{D_1}$. By the remark following theorem 5.1, $\{D_{k^{-1}} T_{\vec{m}} \psi \,:\, k \in P, \vec{m} \in \Gamma\}$ is a Parseval frame for $L^2(\mathbb{R}^3)$.

5.2 Integrated Wavelets

A different approach to discretizing the dilation group H by local averaging of wavelet coefficients yielding so called *integrated wavelets* was introduced by Heinlein (2003). For this one looks at the continuous wavelet transform as a continuous partition of unity in Fourier space generated by the wavelet. Loosely speaking, by the admissibility condition,

$$\hat{f}(\vec{\gamma}) = \frac{1}{c_\psi}\left(\int_H |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h)\right) \cdot \hat{f}(\vec{\gamma}) \tag{5.6}$$

for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$. The idea of integrated wavelets is to partition H into a family $\{H_j\}$ of mutually disjoint sets. Setting

$$\widehat{\Psi^j}(\vec{\gamma}) = \left[\frac{1}{c_\psi}\int_{H_j} |\hat{\psi}(\vec{\gamma}h)|^2 dh\right]^{1/2},$$

then (5.6) becomes

$$\hat{f}(\vec{\gamma}) = \sum_j |\widehat{\Psi^j}(\vec{\gamma})|^2 \hat{f}(\vec{\gamma}).$$

In Heinlein (2003), it was assumed that orbits are free and open. We show that this idea may be applied to arbitrary orbit structure, and that finitely many of the sets $\{H_j\}$ may overlap.

Definition 5.3. *Let J be a discrete countable index set. We call a family $\{H_j\}_{j\in J}$ of subsets of H a detail decomposition, if it is a partition with respect to measurability, i.e. $\mu(H \setminus \bigcup_{j\in J} H_j) = 0$ and $\mu(H_i \cap H_j) = 0$ for all $i \neq j$ in J.*

The integrated wavelet with respect to an admissible wavelet ψ and a detail decomposition $\{H_j\}_{j\in J}$ is the family $\{\Psi^j\}_{j\in J}$ defined in Fourier space by

$$|\widehat{\Psi^j}(\vec{\gamma})|^2 := \frac{1}{c_\psi}\int_{H_j} |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h), \tag{5.7}$$

for $j \in J$ and a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

Observe that $\sum_{j\in J} |\widehat{\Psi^j}(\vec{\gamma})|^2 = 1$ a.e.$\vec{\gamma} \in \widehat{\mathbb{R}^n}$. In particular, $\widehat{\Psi^j} \in L^\infty(\widehat{\mathbb{R}^n})$. Furthermore, the integrated wavelet Ψ^j is not yet completely defined because no phase is given.

If each H_j is compact and ψ is band-limited, its integrated wavelet is also band-limited.

In the following we call both, the function Ψ^j and the family $\{\Psi^j\}_{j\in J}$, integrated wavelet. We call the mapping $W^I_\psi : f \mapsto W^I_\psi f$ given by

$$W^I_\psi f(j,\vec{x}) = \langle f, T_{\vec{x}}\Psi^j\rangle \qquad (j \in J\ ,\ \vec{x} \in \mathbb{R}^n)$$

the integrated wavelet transform.

5.2.1 Admissibility of Integrated Wavelets

Integrated wavelets generate a partition of unity in Fourier space,

$$\sum_{j\in J} |\widehat{\Psi^j}(\vec{\gamma})|^2 = 1 \quad \text{a.e. } \vec{\gamma} \in \widehat{\mathbb{R}^n}, \tag{5.8}$$

which can be seen from the admissibility condition (5.7). This is the key to reconstruction.

Theorem 5.5. *Let $\{H_j\}_{j\in J}$ be a detail decomposition of H, $\psi \in L^2(\mathbb{R}^n)$ admissible, and $\{\Psi^j\}_{j\in J}$ the corresponding integrated wavelet. Then the integrated wavelet transform W^I_ψ is an isometry of $L^2(\mathbb{R}^n)$ into $L^2(J\times\mathbb{R}^n)$. In fact,*

$$\sum_{j\in J}\int_{\mathbb{R}^n} |\langle f, T_{\vec{x}}\Psi^j\rangle|^2 d\vec{x} = \|f\|^2_{L^2(\mathbb{R}^n)}$$

for all $f \in L^2(\mathbb{R}^n)$. Reconstruction in $L^2(\mathbb{R}^n)$ is given by

$$f = \sum_{j\in J}\int_{\mathbb{R}^n} W^I_\psi(j,\vec{x}) T_{\vec{x}}\Psi^j d\vec{x}$$

as a week integral.

Proof. For $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\langle f, T_{\vec{x}}\Psi^j\rangle_{L^2(\mathbb{R}^n)} &= \langle \hat{f}, E_{-\vec{x}}\widehat{\Psi^j}\rangle_{L^2(\widehat{\mathbb{R}^n})} \\
&= \int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma})\overline{\widehat{\Psi^j}}(\vec{\gamma})e^{2i\pi\vec{\gamma}\vec{x}}d\vec{\gamma} \\
&= \int_{\widehat{\mathbb{R}^n}} \Phi_j(\vec{\gamma})e^{2i\pi\vec{\gamma}\vec{x}}d\vec{\gamma} \\
&= \check{\Phi}_j(\vec{x})
\end{aligned}$$

where $\Phi_j = \hat{f}\widehat{\widehat{\Psi^j}} \in L^1(\widehat{\mathbb{R}^n}) \cap L^2(\widehat{\mathbb{R}^n})$. Thus, by Plancherel's theorem and (5.8),

$$\begin{aligned}
\sum_{j\in J}\int_{\mathbb{R}^n} \left|\langle f, T_{\vec{x}}\Psi^j\rangle\right|^2 d\vec{x} &= \sum_{j\in J}\int_{\mathbb{R}^n} |\check{\Phi}_j(\vec{x})|^2 d\vec{x} \\
&= \sum_{j\in J} \|\check{\Phi}_j\|^2_{L^2(\mathbb{R}^n)} \\
&= \sum_{j\in J} \|\Phi_j\|^2_{L^2(\widehat{\mathbb{R}^n})} \\
&= \sum_{j\in J}\int_{\widehat{\mathbb{R}^n}} |\Phi_j(\vec{\gamma})|^2 d\vec{\gamma} \\
&= \sum_{j\in J}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 |\widehat{\Psi^j}(\vec{\gamma})|^2 d\vec{\gamma} \\
&= \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 \sum_{j\in J} |\widehat{\Psi^j}(\vec{\gamma})|^2 d\vec{\gamma} \\
&= \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

That is, the integrated wavelet transform is an isometry of $L^2(\mathbb{R}^n)$ into $L^2(J\times\mathbb{R}^n)$. Now the usual weak reconstruction function gives us

$$\langle f, g\rangle = \sum_{j\in J}\int_{\mathbb{R}^n} \langle W^I_\psi(j,\vec{x}) T_{\vec{x}}\Psi^j, g\rangle d\vec{x}$$

for all $g \in L^2(\mathbb{R}^n)$. That is,

$$f = \sum_{j\in J}\int_{\mathbb{R}^n} W^I_\psi(j,\vec{x}) T_{\vec{x}}\Psi^j d\vec{x}.$$

weakly in $L^2(\mathbb{R}^n)$. □

5.2.2 Frames from Integrated Wavelets

In the previous section we have only discretized the dilation parameter given by the group H. In this section we discretize the translation group $\mathbb{R}^n$. This is done using the standard techniques discussed in Heil and Walnut (1989) and Bernier and Taylor (1996). In generality, it can only be done for bandlimited functions.

Theorem 5.6. *Let $\{H_j\}_{j\in J}$ be a detail decomposition of H and $\psi \in L^2(\mathbb{R}^n)$ admissible. Let $R = \vec{\gamma}_0 + [-\frac{1}{2}, \frac{1}{2}]^n B$ be a parallelepiped, where $B \in GL_n(\mathbb{R})$. Set $L^2(R)^\vee = \{f \in L^2(\mathbb{R}^n) : supp(\hat{f}) \subset R\}$ and $\Gamma = B^{-1}\mathbb{Z}^n$. Then*

$$\sum_{j\in J}\sum_{\vec{m}\in\Gamma} |\langle f, T_{\vec{m}}\Psi^j\rangle|^2 = \delta(B)\,\|f\|^2. \tag{5.9}$$

Proof. For $f \in L^2(R)^\vee$,

$$\begin{aligned}
\langle f, \frac{1}{\sqrt{\delta(B)}} T_{\vec{m}}\Psi^j\rangle_{L^2(\mathbb{R}^n)} &= \langle \hat{f}, \frac{1}{\sqrt{\delta(B)}} E_{-\vec{m}}\widehat{\Psi^j}\rangle_{L^2(\widehat{\mathbb{R}^n})} \\
&= \int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma})\overline{\widehat{\Psi^j}}(\vec{\gamma})\frac{1}{\sqrt{\delta(B)}} e^{2i\pi\vec{\gamma}\vec{m}} d\vec{\gamma} \\
&= \int_R \Phi_j(\vec{\gamma})\frac{1}{\sqrt{\delta(B)}} e^{2i\pi\vec{\gamma}\vec{m}} d\vec{\gamma} \\
&= \langle \Phi_j, e_{\vec{m}}\rangle_{L^2(R)}
\end{aligned}$$

where $\Phi_j = \hat{f}\overline{\widehat{\Psi^j}} \in L^1(R) \cap L^2(R)$. Thus by Parseval's identity

$$\begin{aligned}
\sum_{j\in J}\sum_{\vec{m}\in\Gamma} \left|\langle f, \frac{1}{\sqrt{\delta(B)}} T_{\vec{m}}\Psi^j\rangle\right|^2 &= \sum_{j\in J}\sum_{\vec{m}\in\Gamma} \left|\langle \Phi_j, e_{\vec{m}}\rangle\right|^2 \\
&= \sum_{j\in J} \|\Phi_j\|^2_{L^2(R)} \\
&= \sum_{j\in J} \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 |\widehat{\Psi^j}(\vec{\gamma})|^2 d\vec{\gamma} \\
&= \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 \sum_{j\in J} |\widehat{\Psi^j}(\vec{\gamma})|^2 d\vec{\gamma} \\
&= \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

□

Remark: Since $\psi \notin L^2(R)^\vee$ we have not yet obtained frames for $L^2(R)^\vee$. However, if we set $\widehat{\widetilde{\Psi}^j} = P(\widehat{\Psi^j})$ where P is the projection of $L^2(\widehat{\mathbb{R}^n})$ onto $L^2(R)$ given by $P(\hat{f}) = \hat{f}\chi_R$, then (5.9) holds for $\{\widetilde{\Psi}^j\}$, so that $\{\widetilde{\Psi}^j\}$ generates a tight frame for the shift-invariant subspace $L^2(R)^\vee$ of $L^2(\mathbb{R}^n)$.

To obtain frames for non-bandlimited functions, the detail decomposition must be compatible with the group structure, and ψ must be bandlimited. One can allow some of the sets of the detail decomposition to overlap, however, in this case one does not obtain a Parseval frame. Again, we may start with a frame generator (P, F). To keep with the notation used by Heinlein (2003), we consider sets of the form $\{kF\}_{k\in P}$ instead of $\{Fk\}_{k\in P}$, that is, we replace P used in the previous section by P^{-1}, and F by F^{-1}.

Theorem 5.7. *Let $P \subset H$ be countable and $F \subset H$ be pre-compact such that*

(a) $\bigcup_{k\in P} kF = H \setminus E$ *where E is a set of measure zero,*

(b) $M := \sup_{k\in P} \sharp\{l \in P \,:\, kF \cap lF \neq \emptyset\} < \infty$.

Let $\psi \in L^2(\mathbb{R}^n)$ be admissible and bandlimited, and let Ψ^0 be the integrated wavelet defined by

$$|\widehat{\Psi^0}(\vec{\gamma})|^2 = \frac{1}{c_\psi}\int_F |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h).$$

Then there exists a lattice $\Gamma = B^{-1}\mathbb{Z}^n$ $(B \in GL_n(\mathbb{R}))$ so that $\{\frac{1}{\sqrt{\delta(B)}} D_k T_{\vec{m}} \Psi^0 \,:\, k \in P, \vec{m} \in \Gamma\}$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds 1 and M.

Proof. By assumption, $(\mathrm{supp}(\hat{\psi}))F^{-1}$ is bounded. Let $R = \vec{\gamma}_0 + [-\frac{1}{2}, \frac{1}{2}]^n B$ where $\vec{\gamma}_0 \in \widehat{\mathbb{R}^n}$ and $B \in GL_n(\mathbb{R})$, be any parallelepiped containing $(\mathrm{supp}(\hat{\psi}))F^{-1}$, and set $\Gamma = B^{-1}\mathbb{Z}^n$. Then for $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
&\sum_{k\in P}\sum_{\vec{m}\in\Gamma} \big|\langle f, \frac{1}{\sqrt{\delta(B)}} D_k T_{\vec{m}} \Psi^0\rangle\big|^2 \\
&\qquad = \sum_{k\in P}\sum_{\vec{m}\in\Gamma} \big|\langle \hat{f}, \frac{1}{\sqrt{\delta(B)}} D_{k^{-1}} E_{-\vec{m}} \widehat{\Psi^0}\rangle\big|^2 \\
&\qquad = \sum_{k\in P}\sum_{\vec{m}\in\Gamma} \big|\langle D_k\hat{f}, \frac{1}{\sqrt{\delta(B)}} E_{-\vec{m}} \widehat{\Psi^0}\rangle\big|^2 \\
&\qquad = \sum_{k\in P}\sum_{\vec{m}\in\Gamma} \big|\int_{\widehat{\mathbb{R}^n}} D_k\hat{f}(\vec{\gamma})\overline{\widehat{\Psi^0}}(\vec{\gamma})\frac{1}{\sqrt{\delta(B)}} e^{2i\pi\vec{\gamma}\vec{m}} d\vec{\gamma}\big|^2
\end{aligned}$$

$$= \sum_{k\in P} \delta(k)^{-1} \sum_{\vec{m}\in\Gamma} \left| \int_R \hat{f}(\vec{\gamma}k^{-1})\overline{\widehat{\Psi^0}}(\vec{\gamma})\frac{1}{\sqrt{\delta(B)}}e^{2i\pi\vec{\gamma}\vec{m}}d\vec{\gamma}\right|^2$$
$$= \sum_{k\in P} \delta(k)^{-1} \sum_{\vec{m}\in\Gamma} \left|\langle e_{\vec{m}}(\vec{\gamma}), \bar{\hat{f}}(\vec{\gamma}k^{-1})\widehat{\Psi^0}(\vec{\gamma})\rangle_{L^2(R)}\right|^2$$
$$= \sum_{k\in P} \delta(k)^{-1} \|\bar{\hat{f}}(\vec{\gamma}k^{-1})\widehat{\Psi^0}(\vec{\gamma})\|^2_{L^2(R)}$$
$$= \sum_{k\in P} \delta(k)^{-1} \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma}k^{-1})|^2|\widehat{\Psi^0}(\vec{\gamma})|^2 d\vec{\gamma}$$
$$= \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 \sum_{k\in P} |\widehat{\Psi^0}(\vec{\gamma}k)|^2 d\vec{\gamma}. \tag{5.10}$$

We now show that

$$1 \leq \sum_{k\in P} |\widehat{\Psi^0}(\vec{\gamma}k)|^2 \leq M \tag{5.11}$$

for almost all $\vec{\gamma}$.

To show the left inequality, observe that

$$\begin{aligned}\sum_{k\in P} |\widehat{\Psi^0}(\vec{\gamma}k)|^2 &= \sum_{k\in P} \frac{1}{c_\psi} \int_{kF} |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h)\\ &\geq \frac{1}{c_\psi}\int_H |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h)\\ &= 1.\end{aligned}$$

To show the right inequality, we index the elements of P, say $P = \{k_i\}_{i=1}^{\infty}$. Then

$$\begin{aligned}&\sum_{k\in P}|\widehat{\Psi^0}(\vec{\gamma}k)|^2\\ &\quad= \frac{1}{c_\psi}\sum_{k\in P}\int_{kF} |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h)\\ &\quad= \frac{1}{c_\psi}\sum_{i=1}^{\infty}\int_{k_iF} |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h)\\ &\quad= \frac{1}{c_\psi}\left[\int_{k_1F} + \left[\int_{k_2F\setminus k_1F} + \int_{k_2F\cap k_1F}\right] + \left[\int_{k_3F\setminus(k_1F\cup k_2F)} + \int_{k_3F\cap(k_1F\cup k_2F)}\right] + \cdots\right.\\ &\qquad \left. + \left[\int_{k_iF\setminus\cup_{j=1}^{i-1}k_jF} + \int_{k_iF\cap(\cup_{j=1}^{i-1}k_jF)}\right] + \cdots + \cdots\right]|\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{c_\psi}\Bigg[\int_{k_1F} + \Bigg[\int_{k_2F\setminus k_1F} + \int_{k_1F\cap k_2F}\Bigg] \\
&\quad + \Bigg[\int_{k_3F\setminus(k_1F\cup k_2F)} + \int_{k_1F\cap k_3F} + \int_{[k_2F\setminus k_1F]\cap k_3F}\Bigg] \\
&\quad + \Bigg[\int_{k_4F\setminus(k_1F\cup k_2F\cup k_3F)} + \int_{k_1F\cap k_4F} + \int_{(k_2F\setminus k_1F)\cap k_4F} + \int_{[k_3F\setminus(k_1F\cup k_2F)]\cap k_4F}\Bigg] + \cdots \\
&\quad + \Bigg[\int_{k_iF\setminus\cup_{j=1}^{i-1}k_jF} + \int_{k_1F\cap k_iF} + \int_{(k_2F\setminus k_1F)\cap k_iF} + \int_{[k_3F\setminus(k_1F\cup k_2F)]\cap k_iF} + \cdots \\
&\quad + \int_{[k_mF\setminus\cup_{j=1}^{m-1}k_jF]\cap k_iF} + \cdots + \int_{[k_{i-1}F\setminus\cup_{j=1}^{i-2}k_jF]\cap k_iF}\Bigg] + \cdots\Bigg]|\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h) \\
&= \quad \frac{1}{c_\psi}\Bigg[\int_{k_1F} + \int_{k_2F\setminus k_1F} + \int_{k_3F\setminus(k_1F\cup k_2F)} + \int_{k_4F\setminus\cup_{j=1}^{3}k_jF} + \cdots + \int_{k_iF\setminus\cup_{j=1}^{i-1}k_jF} + \cdots \\
&\quad + \int_{k_1F\cap k_2F} + \int_{k_1F\cap k_3F} + \int_{k_1F\cap k_4F} + \cdots + \int_{k_1F\cap k_iF} + \cdots \\
&\quad + \int_{(k_2F\setminus k_1F)\cap k_3F} + \int_{(k_2F\setminus k_1F)\cap k_4F} + \cdots + \int_{(k_2F\setminus k_1F)\cap k_iF} + \cdots \\
&\quad + \int_{[k_3F\setminus(k_1F\cup k_2F)]\cap k_4F} + \int_{[k_3F\setminus(k_1F\cup k_2F)]\cap k_5F} + \cdots + \int_{[k_3F\setminus(k_1F\cup k_2F)]\cap k_iF} + \cdots \\
&\quad + \cdots + \cdots \\
&\quad + \int_{[k_mF\setminus\cup_{j=1}^{m-1}k_jF]\cap k_{m+1}F} + \cdots + \int_{[k_mF\setminus\cup_{j=1}^{m-1}k_jF]\cap k_iF} + \cdots\Bigg]|\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h) \\
&\le \frac{1}{c_\psi}\Bigg[\int_H + (M-1)\int_{k_1F} + (M-1)\int_{k_2F\setminus k_1F} + (M-1)\int_{k_3F\setminus(k_1F\cup k_2F)} \\
&\quad + \cdots + (M-1)\int_{k_m\setminus\cup_{j=1}^{m-1}k_jF} + \cdots\Bigg]|\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h) \\
&= \frac{1}{c_\psi}\Bigg[\int_H + (M-1)\Bigg[\int_{k_1F} + \int_{k_2F\setminus k_1F} + \int_{k_3F\setminus(k_1F\cup k_2F)} \\
&\quad + \cdots + \int_{k_m\setminus\cup_{j=1}^{m-1}k_jF} + \cdots\Bigg]\Bigg]|\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h) \\
&= \frac{1}{c_\psi}\Bigg[\int_H + (M-1)\int_H\Bigg]|\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h) \\
&= \frac{1}{c_\psi} M \int_H |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h) \\
&= M \qquad \left(\text{since} \int_H |\hat{\psi}(\vec{\gamma}h)|^2 d\mu(h) = c_\psi\right).
\end{aligned}
$$

This proves (5.11). It follows from (5.10) that

$$\|f\|_2^2 \leq \sum_{k\in P}\sum_{\vec{m}\in\mathbb{Z}^n} \left|\langle f, \frac{1}{\sqrt{\delta(B)}} D_k T_{\vec{m}} \Psi^0\rangle\right|^2 \leq M\|f\|_2^2$$

for all $f \in L^2(\mathbb{R}^n)$. $\square$

Remark : Observe that if P is a co-compact discrete subgroup of H, and F a fundamental domain, then this theorem applies with $M = 1$. One thus has a method for obtaining Parseval frames different from corollary 5.4.

If ψ is not bandlimited, we have the following generalization of theorem 5.1.6 from Heil and Walnut (1989).

Theorem 5.8. *Let $P \subset H$ be countable and $F \subset H$ be such that*

(a) $\bigcup_{k\in P} kF = H \setminus E$ where E is a set of measure zero,

(b) $M := \sup_{k\in P} \sharp\{l \in P \,:\, kF \cap lF \neq \emptyset\} < \infty$.

Let $|\Psi^0(\vec{\gamma})|^2 = \frac{1}{c_\psi}\int_F |\psi(\vec{\gamma}h)|^2 d\mu(h)$ be the integrated wavelet with respect to an admissible $\psi \in L^2(\mathbb{R}^n)$. Set

$$\beta(\vec{\xi}) := \sup_{\vec{\gamma}\in\widehat{\mathbb{R}^n}} \sum_{k\in P} |\widehat{\Psi^0}(\vec{\gamma}k)||\widehat{\Psi^0}(\vec{\gamma}k+\vec{\xi})|$$

for $\vec{\xi} \in \widehat{\mathbb{Z}^n}$ and suppose that

$$K := \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}} [\beta(\vec{\xi})\beta(-\vec{\xi})]^{1/2} < 1.$$

Then $\{D_k T_{\vec{m}} \Psi^0 \,:\, \vec{m} \in \mathbb{Z}^n \,,\, k \in P\}$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds $1-K$ and $M+K$.

Proof. For $f \in L^2(\mathbb{R}^n)$

$$\begin{aligned}\langle f, D_k T_{\vec{m}}\Psi^0\rangle &= \langle \hat{f}, D_{k^{-1}} E_{-\vec{m}} \widehat{\Psi^0}\rangle \\ &= \langle D_k \hat{f}, E_{-\vec{m}}\widehat{\Psi^0}\rangle\end{aligned}$$

$$= \int_{\widehat{\mathbb{R}^n}} (D_k\hat{f})(\vec{\gamma})\overline{\widehat{\Psi^0}}(\vec{\gamma})e^{2i\pi\vec{\gamma}\vec{m}}d\vec{\gamma}$$
$$= \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}} \int_{[0,1]^n} (D_k\hat{f})(\vec{\gamma}+\vec{\xi})\overline{\widehat{\Psi^0}}(\vec{\gamma}+\vec{\xi})e^{2i\pi\vec{\gamma}\vec{m}}d\vec{\gamma}$$
$$= \Big\langle \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}} (D_k\hat{f})(\cdot+\vec{\xi})\overline{\widehat{\Psi^0}}(\cdot+\vec{\xi}), e^{-2i\pi\cdot\vec{m}}\Big\rangle_{L^2([0,1]^n)}.$$

Thus, by Parseval's identity

$$\begin{aligned}
&\sum_{k\in P}\sum_{\vec{m}\in\mathbb{Z}^n} \big|\langle f, D_kT_{\vec{m}}\Psi^0\rangle\big|^2 \\
&= \sum_{k\in P} \Big\| \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}} (D_k\hat{f})(\cdot+\vec{\xi})\overline{\widehat{\Psi^0}}(\cdot+\vec{\xi})\Big\|^2_{L^2([0,1]^n)} \\
&= \sum_{k\in P}\int_{[0,1]^n} \Big|\sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}} (D_k\hat{f})(\vec{\gamma}+\vec{\xi})\overline{\widehat{\Psi^0}}(\vec{\gamma}+\vec{\xi})\Big|^2 d\vec{\gamma} \\
&= \sum_{k\in P}\Big\langle \sum_{\vec{\eta}\in\widehat{\mathbb{Z}^n}} (D_k\hat{f})(\vec{\gamma}+\vec{\eta})\overline{\widehat{\Psi^0}}(\vec{\gamma}+\vec{\eta}), \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}} (D_k\hat{f})(\vec{\gamma}+\vec{\xi})\overline{\widehat{\Psi^0}}(\vec{\gamma}+\vec{\xi})\Big\rangle \\
&= \sum_{k\in P}\sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}}\int_{\widehat{\mathbb{R}^n}} (D_k\hat{f})(\vec{\gamma})\overline{\widehat{\Psi^0}}(\vec{\gamma})\overline{(D_k\hat{f})(\vec{\gamma}+\vec{\xi})}\widehat{\Psi^0}(\vec{\gamma}+\vec{\xi})d\vec{\gamma} \\
&= \sum_{k\in P}\Big[\int_{\widehat{\mathbb{R}^n}} \big|(D_k\hat{f})(\vec{\gamma})\widehat{\Psi^0}(\vec{\gamma})\big|^2 d\vec{\gamma} \\
&\quad + \underbrace{\sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}}\int_{\widehat{\mathbb{R}^n}} (D_k\hat{f})(\vec{\gamma})\overline{\widehat{\Psi^0}}(\vec{\gamma})\overline{(D_k\hat{f})(\vec{\gamma}+\vec{\xi})}\widehat{\Psi^0}(\vec{\gamma}+\vec{\xi})d\vec{\gamma}}_{\text{Rest}}\Big] \qquad (5.12)
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k\in P}\int_{\widehat{\mathbb{R}^n}} \big|(D_k\hat{f})(\vec{\gamma})\widehat{\Psi^0}(\vec{\gamma})\big|^2 d\vec{\gamma} &= \sum_{k\in P}|\det k|^{-1}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma}k^{-1})\widehat{\Psi^0}(\vec{\gamma})|^2 d\vec{\gamma} \\
&= \sum_{k\in P}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2|\widehat{\Psi^0}(\vec{\gamma}k)|^2 d\vec{\gamma} \\
&= \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2\sum_{k\in P}|\widehat{\Psi^0}(\vec{\gamma}k)|^2 d\vec{\gamma}
\end{aligned}$$

we have, proceeding as in the proof of theorem 5.6 that

$$\|f\|_2^2 \le \sum_{k\in P}\int_{\widehat{\mathbb{R}^n}} \big|(D_k\hat{f})(\vec{\gamma})\widehat{\Psi^0}(\vec{\gamma})\big|^2 d\vec{\gamma} \le M\|f\|_2^2. \qquad (5.13)$$

Now we estimate the remainder in (5.12)

$$
\begin{aligned}
|\mathrm{Rest}| &\le \sum_{k\in P}\sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} \int_{\widehat{\mathbb{R}^n}} \left|(D_k\hat{f})(\vec{\gamma})\overline{\widehat{\Psi^0}}(\vec{\gamma})\overline{(D_k\hat{f})(\vec{\gamma}+\vec{\xi})}\widehat{\Psi^0}(\vec{\gamma}+\vec{\xi})\right| d\vec{\gamma} \\
&\le \sum_{k\in P}\sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} \left\| |(D_k\hat{f})(\vec{\gamma})|\ |\widehat{\Psi^0}(\vec{\gamma})|^{1/2}|\widehat{\Psi^0}(\vec{\gamma}+\vec{\xi})|^{1/2}\right\|_{L^2(\widehat{\mathbb{R}^n})} \\
&\qquad \times \left\| |(D_k\hat{f})(\vec{\gamma}+\vec{\xi})||\widehat{\Psi^0}(\vec{\gamma})|^{1/2}|\widehat{\Psi^0}(\vec{\gamma}+\vec{\xi})|^{1/2}\right\|_{L^2(\widehat{\mathbb{R}^n})} \\
&= \sum_{k\in P}\sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} \left(\int_{\widehat{\mathbb{R}^n}} |\det k|^{-1}|\hat{f}(\vec{\gamma}k^{-1})|^2|\widehat{\Psi^0}(\vec{\gamma})|\ |\widehat{\Psi^0}(\vec{\gamma}+\vec{\xi})| d\vec{\gamma}\right)^{1/2} \\
&\qquad \left(\int_{\widehat{\mathbb{R}^n}} |\det k|^{-1}|\hat{f}((\vec{\gamma}+\vec{\xi})k^{-1})|^2|\widehat{\Psi^0}(\vec{\gamma})|\ |\widehat{\Psi^0}(\vec{\gamma}+\vec{\xi})| d\vec{\gamma}\right)^{1/2} \\
&= \sum_{k\in P}\sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} \left(\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2|\widehat{\Psi^0}(\vec{\gamma}k)|\ |\widehat{\Psi^0}(\vec{\gamma}k+\vec{\xi})| d\vec{\gamma}\right)^{1/2} \\
&\qquad \left(\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2|\widehat{\Psi^0}(\vec{\gamma}a-\vec{\xi})|\ |\widehat{\Psi^0}(\vec{\gamma}k)| d\vec{\gamma}\right)^{1/2} \\
&\le \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} \left(\sum_{k\in P}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2|\widehat{\Psi^0}(\vec{\gamma}k)|\ |\widehat{\Psi^0}(\vec{\gamma}k+\vec{\xi})| d\vec{\gamma}\right)^{1/2} \\
&\qquad \left(\sum_{k\in P}\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2|\widehat{\Psi^0}(\vec{\gamma}k-\vec{\xi})||\widehat{\Psi^0}(\vec{\gamma}k)| d\vec{\gamma}\right)^{1/2} \\
&= \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} \left(\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2\sum_{k\in P}|\widehat{\Psi^0}(\vec{\gamma}k)||\widehat{\Psi^0}(\vec{\gamma}k+\vec{\xi})| d\vec{\gamma}\right)^{1/2} \\
&\qquad \left(\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2\sum_{k\in P}|\widehat{\Psi^0}(\vec{\gamma}k)||\widehat{\Psi^0}(\vec{\gamma}k-\vec{\xi})| d\vec{\gamma}\right)^{1/2} \\
&\le \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} \left(\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2\beta(\vec{\xi}) d\vec{\gamma}\right)^{1/2}\left(\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2\beta(-\vec{\xi}) d\vec{\gamma}\right)^{1/2} \\
&= \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} (\beta(\vec{\xi}))^{1/2}\Big(\int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 d\vec{\gamma}\Big)(\beta(-\vec{\xi}))^{1/2} \\
&= \|f\|_2^2 \sum_{\vec{\xi}\in\widehat{\mathbb{Z}^n}\setminus\{0\}} (\beta(\vec{\xi})\beta(-\vec{\xi}))^{1/2} \\
&< K\|f\|_2^2.
\end{aligned}
$$

From here and (5.13), the assertion follows. $\square$

CHAPTER VI

CONCLUSION

In this thesis, we have discussed two main topics : how to construct admissible functions from almost cross-sections, and how to obtain wavelet frames, both in the case of an arbitrary dilation subgroup H of $GL_n(\mathbb{R})$.

In chapter 4, we introduced the notion of an N-section for the action of H on $\widehat{\mathbb{R}^n}$, generalizing the concept of cross-section. Starting from an N-section S, we showed how to construct an admissible function ψ from S, provided that H satisfies the condition that $|\det| \neq \Delta$. If H contains an expanding matrix, then one can modify S to be a bounded set, and the construction yields a bandlimited admissible ψ. In theorem 4.3 we showed how to obtain smooth, bandlimited admissible functions, provided that there exists a compact N-section having the property that the orbit map $(S, H) \to S \cdot H$ is open, and that orbits intersect S in some regular fashion, as expressed by property (d) of theorem 4.3.

We then showed that if H is a p-parameter group of diagonal matrices, then there exists a compact N-section with open orbit map, and we presented an example where theorem 4.3 applies.

Wavelet frames are of interest as they allow reconstruction of a function f in $L^2(\mathbb{R}^n)$ from its wavelet transform by a series. Two approaches have been described in the literature, identifying the support of the Fourier transform of a frame generator (Bernier and Taylor, 1996) or integrating an admissible ψ over tiles in the orbits (Heinlein, 2003) leading to integrated wavelets $\{\Psi^j\}$. Both approaches are valid in the case of free, open H-orbits in $\widehat{\mathbb{R}^n}$. We have generalized both approaches to groups with arbitrary orbit structure. The starting point here is a separated subset P of H. Given a bounded N-section S whose orbits satisfy the regularity condition (d) of theorem 4.3, we can find a lattice Γ in $\mathbb{R}^n$, and specify conditions on a function $\psi \in L^2(\mathbb{R}^n)$ so that the collection $\{\psi_{k,\vec{x}}\}_{k\in P,\vec{x}\in\Gamma}$ is a frame for $L^2(\mathbb{R}^n)$. If S is a cross-section, and there exists $F \subset H$ pre-compact, such that the collection $\{Fk\}_{k\in P}$ tiles H, then Parseval frames exist. If P is a discrete co-compact subgroup of H, then smooth, bandlimited Parseval frames exist. We showed by example that H containing an expanding matrix is not necessary for the existence of bounded N-sections and Parseval frames. Similarly, we showed that given a bandlimited admissible function ψ, the integrated wavelet Ψ^0 has the property that $\{\Psi^0_{k,\vec{x}}\}_{k\in P,\vec{x}\in\Gamma}$ forms a wavelet frame. Again, if the collection $\{kF\}_{k\in P}$ tiles H, we can obtain a Parseval frame. For ψ not bandlimited, we have specified a condition on $\hat{\psi}$ which guarantees that $\{\Psi^0_{k,\vec{x}}\}$ is a wavelet frame.

REFERENCES

REFERENCES

Benedetto, J.J. and Sumetkijakan, S. (2002), **A fractal set constructed from a class of wavelet sets**, Contemporary Mathematics, Vol. 313, pp. 19-35.

Bernier, D. and Taylor, K.F. (1996), **Wavelets from square-integrable representations**, SIAM J. Math. Anal., Vol. 27, pp. 594-608.

Boggess, A. and Narcowich F.J. (2001), **A First Course in Wavelets with Fourier Analysis**, Prentice-Hall.

Curtis, M.L. (1984), **Matrix Groups**, Springer-Verlag.

Dai, X., Diao, Y., Gu, Q., and Han, D. (2003), **The existence of subspace wavelet sets**, J. Comp. Appl. Math., Vol. 155, pp. 83-90.

Dai, X., Larson, D., and Speegle, D. (1997), **Wavelet sets in** $\mathbb{R}^n$, J. Fourier Anal. Appl., Vol. 3, pp. 451-456.

Daubechies, I. (1992), **Ten Lectures on Wavelets**, Society for Industrial and Applied Mathematics, Philadelphia.

Donald, C.L. (1997), **Measure Theory**, Birkhäuser.

Duflo, M. and Moore, C.C. (1976), **On the regular representations of a nonunimodular locally compact group**, J. Funct. Anal., Vol. 27, pp. 209-243.

Fabec, R. and Ólafasson, G.(2003), **The Continuous wavelet transform and symmetric spaces**, Acta Applicanda Math., Vol. 87, pp. 261-277.

Folland, G.B.(1999), **Real Analysis**, Canada, John Wiley&Sons.

Führ, H. (1996), **Wavelet frames and admissibility in higher dimensions**, J. of Math. Phys., Vol. 37, pp. 6353-6366.

Grossmann, A., Morlet, J., and Paul, T. (1985), **Transforms associated to square integrable group representations I**, General results. J. Math. Phys., Vol. 26, pp. 2473-2479.

Grossmann, A., Morlet, J., and Paul, T. (1986), **Transforms associated to square integrable group representations II**, Examples. Ann. Inst. Henri Poincare: Phys. Theor., Vol. 45, pp. 293-309.

Heil, C.E. and Walnut, D.F. (1989), **Continuous and discrete wavelet transforms**, SIAM Review, Vol. 31, pp. 628-666.

Heinlein, P. (2003), **Discretizing continuous wavelet transforms using integrated wavelets**, Appl. Comput. Harmon. Anal., Vol. 14, pp. 238-256.

Hernandez E. and Weiss G. (1996), **A First Course on Wavelets**, CRC Press.

Kawakubo, K. (1991), **The Theory of Transformaion Groups**, Oxford University Press.

Larson, D., Schulz, E., Speegle, D., and Taylor, K.F. (2006), **Explicit cross sections of singly generated group actions**, Harmonic Analysis and Applications: In Honor of John J. Benedetto, Christopher Heil (ed), Springer, pp. 209-230.

Laugensen, R.S. (2002), **Translational averaging for completeness, characterization, and oversampling of wavelets**, Collect. Math., Vol. 53 , pp. 211-249.

Laugesen, R.S., Weaver, N., Weiss, G., and Wilson, E.N. (2002), **A characterization of the higher dimensional groups associated with continuous wavelets**, J. Geom. Anal., Vol. 12, pp. 89-102.

Mallat, S. and Zhong, S. (1992), **Characterization of signals from multiscale edges**, IEEE Trans. PAMI, Vol. 14, pp. 710-732.

Schulz, E. and Taylor, K.F., **Smooth waveletes generated by dilations**, preprint.

Speegle, D. (2003), **On the existence of wavelets for non-expansive dilation matrices**, Collect. Math., Vol 54, pp. 163-179.

Weiss, G. and Wilson, E.N.(2001), **The Mathematical Theory of Wavelets**, Proceedings of the NATO-ASI meeting *Harmonic Analysis 2000-a celebration*, Kluwer, pp. 329-366.

Wade, W.R. (1999), **An Introduction to Analysis**, Prentice-Hall.

Wang, Y. (2002), **Wavelets, tilings and spectral sets**, Duke Math. J., Vol. 114, pp. 43-51.

CURRICULUM VITAE

NAME: Wannapa Romero. **GENDER:** Female. **NATIONALITY:** Thai.

EDUCATION BACKGROUND:

- Bachelor of Science in Mathematics, Naresuan University, Thailand, 1994 - 1997.

- Master of Science in Mathematics, Chiang Mai University, Thailand, 1998 - 2000.

WORK EXPERIENCES:

- Lecturer, Department of Mathematics, Naresuan University, Phitsanulok, Thailand, June 2000- September 2001.

- Long Term Visitor, Department of Mathematics and Statistics, Dalhousie University, Nova Scotia, Canada, January 2004 - September 2004.

PRESENTATIONS:

- *Continuous Wavelets Associated with Matrix Groups*, The First Mathematics and Physical Science Graduate Congress, December 6-8, 2005, Faculty of Science, Chulalongkorn University, Bangkok, Thailand.

- *Continuous Wavelets Associated with Matrix Groups*, International Workshop: Modeling and Simulation in Applied Mathematics, January 20-21, 2006, School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima, Thailand.

- *Cross-sections and Continuous Wavelets Associated with Matrix Groups*, Current Trends in Harmonic Analysis and Its Applications: Wavelets and Frames, May 18-20, 2006, Department of Mathematics, University of Colorado at Boulder, Colorado, United States of America.

SCHOLARSHIP:

- The Ministry of University Affairs of Thailand (MUA), 2002-2004.