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OPTION PRICING MODELS DRIVEN BY A FRACTIONAL LEVY PROCESS

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วิทยานิพนธ์ฉบับนี้พิจารณาการกระโดดของแบบจำลองแบลค-โชลเศษส่วน จากการที่ได้มีการศึกษาเชิงประสบการณ์หลายครั้งพบว่าราคาของสินทรัพย์และอัตราดอกเบี้ยมักจะแสดงพฤติกรรมแบบกระโดด การแพร่แบบกระโดดจัดได้ว่าเป็นกรณีพิเศษของกระบวนการเลวี ดังนั้นจึงมีการพัฒนาแบบจำลองแบลค-โชลแบบกระโดดขึ้น เพื่อให้แบบจำลองมีสมบัติหน่วยความจำระยะยาวจึงพิจารณาแบบจำลองตลาดที่ขับเคลื่อนแบบบราวเนียนเศษส่วน โดยใช้วิธีการประมาณ กล่าวคือการใช้การลู่เข้า L^2 ในปริภูมิของกึ่งมาร์ทิงเกลไปยังกระบวนการเศษส่วน

นอกจากนี้ได้มีการพิจารณาการกระโดดของแบบจำลองอัตราดอกเบี้ยเศษส่วนที่สำคัญคือแบบจำลองวาซิเชคเศษส่วน โดยได้หาผลเฉลยโดยประมาณของแบบจำลองดังกล่าวและพิสูจน์ว่าผลเฉลยโดยประมาณที่ได้ลู่เข้าสู่ผลเฉลยของแบบจำลองเดิม ผลเฉลยที่ได้สามารถประมาณให้ใกล้เคียงแค่ไหนก็ได้และง่ายต่อการคำนวณซึ่งแสดงถึงข้อได้เปรียบอย่างมากของวิธีการประมาณที่ใช้ในที่นี้

สุดท้ายได้แสดงวิถีตัวอย่างของแบบจำลองราคาแบลค-โชล แบบจำลองราคาแบลค-โชลแบบกระโดด และแบบจำลองราคาแบลค-โชลเศษส่วนแบบกระโดดเพื่อการประมาณค่าราคาหุ้นสามัญของบริษัทอุตสาหกรรมเคมีไทยเทียบกับข้อมูลเชิงประสบการณ์

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา__________________________

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FRACTIONAL BLACK-SCHOLES MODEL/ JUMPS DIFFUSION/ LEVY PROCESSES/ LONG MEMORY PROPERTY/ APPROXIMATE APPROACH/ SEMIMARTINGALE /FRACTIONAL VASICEK MODEL/ APPROXIMATE SOLUTION

In this thesis, the fractional Black-Scholes model is considered with jumps, since various empirical studies show that various asset prices and interest rates may exhibit a jumping behaviour. Jumps diffusion also form a subclass of Levy processes. Therefore, the Black-Scholes model with jumps was developed. For the model to have the long memory property, the market mathematical models driven by fractional Brownain motion are considered. The approximate approach is used: employing the L^2-convergence of semimartingales to fractional processes.

Moreover, an important jumps version of fractional interest rate is considered, namely, that of fractional Vasicek models. Their approximate solutions are derived and proved to converge its original solution. The solution of a fractional model with jumps can be approximated at any numerical exactitude and convention, a considerable advantage of the approximate approach.

Finally, sample paths of Black-Scholes pricing model, fractional Black-Scholes pricing model and fractional Black-Scholes pricing model with jumps for Thai Petrochemical Industry (TPI) are illustrated against the empirical data.

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Arthit Intarasit

Contents

Contents (Continued)

List of Figures

Figure **Page**

Chapter I

Overview

We introduce the classical Black & Scholes model, the need of studying fractional Brownian motion and the approximate approach. Secondly, we study the literature review of the Black & Sholes models with jumps. Finally we discuss the goals and objectives of this thesis.

1.1 Background

In their classic paper on the theory of option pricing, Black and Scholes (1973) present a mode of analysis that revolutionized the theory of corporate liability pricing. Their approach led to pricing formulas using, for the most part, only observable variables. In particular, their formulas do not require knowledge of either investors' tastes or their beliefs about expected returns on the underlying common stock. Moreover, under specific posed conditions, their formula was found to hold to avoid the creation of arbitrage possibilities.

To derive the option pricing formula, Black and Scholes assume "ideal conditions" in the market for stocks and options. These conditions are as follows.

1. *"Frictionless" markets: there are no transactions costs or differential taxes. Trading takes place continuously in time. Borrowing and short selling are allowed without restriction and with full proceeds available. The borrowing and lending rates are equal.*

2. *The short-term interest rate is known and constant through time.*

3. *The stock pays no dividends or other distributions during the life of the option.*

4. *The option is "European" in that it can only be exercised at the expiration date.*

5. *The stock price follows a "geometric" Brownian motion through time which produces a log-normal distribution for the stock price between any two points in time.(Merton, 1990)*

Following condition 5, we assume that the price $(S(t), t \geq 0)$ of a risky asset at time t is given by *geometric Brownian motion* of the form

$$S(t) \;=\; S(0)\; \exp\left(\left(\mu \;-\; \frac{1}{2}\sigma^2\right)t \;+\; \sigma Z(t)\right),\; \forall\, t \in [0,T], \tag{1.1}$$

where $(Z(t), t \geq 0)$ is the Brownian motion. $S(0)$ is a given random variable such that $ES^2(0) < \infty$, and μ and σ are constants. Equation (1.1) is called *the classical Black & Scholes pricing model.* The motivation for this assumption on $S(t)$ comes from the fact that $S(t)$ is the unique strong solution of the linear stochastic differential equation

$$S(t) \;=\; S(0) \;+\; \int_0^t \mu S(s)\,\mathrm{d}s \;+\; \int_0^t \sigma S(s)\,\mathrm{d}Z(s),\; \forall\, t \in [0,T],$$

which can be formally written as

$$\begin{aligned} \mathrm{d}S(t) \;&=\; \mu S(t)\,\mathrm{d}t \;+\; \sigma S(t)\,\mathrm{d}Z(t), \\ S(t)\big|_{t=0} \;&=\; S(0), \end{aligned} \tag{1.2}$$

where μ is also known as the *drift rate* or *rate of return* of the price $S(t)$ and σ as the *volatility* (which measures the standard deviation of the return $\mathrm{d}S(t)/S(t)$). Let us note that the Brownian motion $Z(t)$ is called the *driving process* of the stochastic differential equation (1.2) or, in other words, the stochastic differential equation (1.2) is *driven by* the Brownian motion $Z(t)$.

Recall that a Brownian motion $(B(t))$ is a stochastic process with the following properties:

1. Normal increments: $B(t) - B(s)$ *has a normal distribution with mean* 0 *and variance* $t - s$. *This with* $s = 0$ *implies that* $B(t) - B(0)$ *is a* $\mathcal{N}(0,t)$ *distribution.*

2. Independence of increments: $B(t) - B(s)$ *is independent of the past, that is, of* $B(u)$, $0 \leq u \leq s$.

3. Continuity of paths: $B(t)$, $t \geq 0$ *are continuous functions of* t.

If $\sigma = 0$, the equation (1.2) becomes an ordinary differential equation which describes an investment on a non-risky asset (e.g., a bank account). The initial capital $S(0)$ grows, from $t = 0$, continuously compounded with the interest rate μ as

$$S(t) \;=\; S(0)\; e^{\mu t}$$

at time t. If μ is a function of t, $S(0)$ grows from the initial time $t = 0$ according to:

$$S(t) \;=\; S(0)\; \exp\left(\int_0^t \mu(s)\; \mathrm{d}s\right)$$

at time t. On the other hand, if one knows the amount, say, $S(t)$ to be obtained at future time t, its present value can be also found by discounting it at the same rate of growth. That is, if it grows to become $S(t)$ and continuously compounded with the rate μ then its present value at $t = 0$ is

$$e^{-\mu t} S(t).$$

The value $e^{-\mu t} S(t)$ is called the *discounted value* or the *present value* of $S(t)$ at the rate μ.

Let us observe that the drift rate μ and the volatility σ could be some adapted stochastic processes so that the equation (1.2) could be in the form:

$$\begin{aligned} \mathrm{d}S(t) \;&=\; \mu(S(t), t)\; \mathrm{d}s \;+\; \sigma(S(t), t)\; \mathrm{d}Z(t), \\ S(t)\big|_{t=0} \;&=\; S(0). \end{aligned} \tag{1.3}$$

The strong solution (1.3) possesses the Markov property: its future behaviour depends only on its *immediate* previous values and not on values long time before. Thus, strong solutions of (1.3) have no memory. However in practice, the stock price $S(t)$ at t may have a long-range consequence: $S(t)$ may be a long memory process in the sense that, if $X = (X(t), t \geq 0)$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\rho(k) = E[X_1(X_{k+1} - X_k)]$ then

$$\sum_{k=0}^{\infty} \rho(k) \;=\; \infty.$$

Such a process X is said to have *long memory* or *long-range dependence* or *strong after effect*. This means that the process today may influence the process at some time in the future. In other words, the process long time ago may influence the process today.

We note that $S(t)$ is a *strong solution* to the Itô stochastic differential equation (1.3) such that, if for all $t > 0$, $S(t)$ is a function $F(t, (B(s), s \leq t))$ of the given Brownian motion $B(t)$, integrals $\int_0^t \mu(S(s), s)\mathrm{d}s$ and $\int_0^t \sigma(S(s), s)\mathrm{d}B(s)$ exist, and the following integral equation is satisfied:

$$\mathbb{P} \text{ a.s. } \forall\, t \geq 0, \quad S(t) \;=\; S(t) \;+\; \int_0^t \mu(S(s), s)\;\mathrm{d}s \;+\; \int_0^t \sigma(S(s), s)\;\mathrm{d}Z(s). \qquad (1.4)$$

1.2 The Need to Study Fractional Brownian Motion and the Approximate Approach

It is well-known that the Black & Scholes model is complete and free of arbitrage in the class of admissible strategies. This means that claims can be priced fairly and (in principle) one can even calculate the corresponding hedging portfolios. Hence, the Black & Scholes pricing model is very satisfactory from the theoretical point of view.

However, there is a problem with this model. It stipulates that the log-returns

$$\begin{aligned} R(t_k) \quad &:= \quad \log \frac{S(t_k)}{S(t_{k-1})} \\ &= \quad \left(\mu - \frac{\sigma^2}{2}\right)(t_k - t_{k-1}) + \sigma\big(Z(t_k) - Z(t_{k-1})\big) \end{aligned}$$

are normal and independent random variables.

The dependence structure of such log-returns have been studied using the so called *Hurst parameter* H. In the uncorrelated case one should have $H = \frac{1}{2}$. However, many studies have indicated that the Hurst indices were such that $H > \frac{1}{2}$. For example, Peter (1994) and Shiryaev (1998) found that the estimated Hurst index is $H = 0.642$ for the daily exchange rate between US Dollar and Japanese Yen between January 1972 and December 1990.

To overcome the independence assumption of the log-returns, it has been proposed to replace the Brownian motion by a fractional Brownian motion which captures the long-range dependency property measured by H. The first one to suggest this was Mandelbrot (1997).

In order to be able to apply fractional Brownian motion to study the market situations, we need a stochastic calculus for such a fractional Brownian motion (fBm), since for $H \neq \frac{1}{2}$, the fBm $Z_H(t)$ is neither a semimartingale nor a Markov process, meaning that the well developed stochastic calculus is not applicable. In particular, for $H > \frac{1}{2}$, it is a long memory process. In other words, the behavior of a real process after a given time t does not only depend on the situation at t but also of the whole history of the process up to time t.

Many authors have studied what a stochastic integral of a function with respect to fractional Brownian motion should mean. The most common constructions of such a stochastic integral are in two ways: the pathwise integral and the Skorohod (Wich-Ito) integral. Unfortunately, these two types of definitions do not allow economical interpretation and are difficult for numerics.

In 2002, Thao developed a theory for fractional Brownian motion which both allowed for economical interpretation and was convenient for numerics. He proposed another definition of *fractional stochastic integrals* motivated by the formula of integration by parts and an approximate approach to fractional Brownian motion. We will study this in more detail in chapter II.

1.3 Black & Scholes Models with Jumps

There are also empirical studies indicating that the *log-returns are not normal*. This is more evident, if the observation intervals $t_k - t_{k-1}$ are short.

The empirical literature has extensively reported on the non-normality of the log-returns especially on two features, which indeed are closely linked. First, it has been shown that the log-returns show excess kurtosis and skewness, inconsistent with the normality assumptions (see Mandelbrot (1963) and Fama (1965) for the early works. For more recent works, one can see Kon (1984), Jorion (1998) and Bate (1996)). Second, research has concentrated on the implied volatility smile or skewness (See Dumas et al. (1996) for a survey). Interestingly, this second fact is just another hint of the non-normality of the log-returns.

To overcome this critical point, namely the normality assumption of the log-returns, it has been proposed to replace Brownian motion by suitably chosen alternative Lévy processes, for instance, the hyperbolic Lévy motion, and more generally, one of the generalized hyperbolic Lévy motions (Prause (1999) and Eberlein (2001)).

Runggaldier (2002) has stated that most of the standard literature in Finance, in particular for pricing and hedging of contingent claims, is based on the assumption that the prices of the underlying assets follow a diffusion-type process, in particular a geometric Brownian motion (GBM) (see equation (1.1)). Documentation from various empirical studies shows that such models are inadequate, both in relation to their descriptive power, as well as for the mispricing that they might induce.

The contribution of this thesis concerns various generalizations of the basic GBM. We concentrate on the fact that returns of various asset prices and interest rates may exhibit jumping behavior and study possible superpositions of jump and diffusion processes, namely those called *jump-diffusion processes*. Jump-diffusion processes form a particular class of Levy processes. Our purpose here is not to study the general case of Levy driving processes, but rather to concentrate on the specific aspects of the subclass of jump-diffusions. Jump-diffusion models have also some intuitive appeal in that they let prices and interest rates change continuously most of the time, but take into account the fact that from time to time larger jumps may occur that cannot be adequately modeled by pure diffusion-type processes.

Among the earlier empirical studies documenting a jumping behavior in price and interest rates were Ball and Torous (1985) and Jorion (1988). There were also studies, such as Babbs and Webber (1997), putting forward specific sources of jumps in interest rates such as moves by central banks. On the other hand, a first approach developing further the basic Black and Scholes model with the inclusion of jumps appears to be that of Merton (1976).

1.4 Goals and Objectives of Thesis

The goal of this thesis to introduce a fractional Black-Scholes model with jumps using an approximate approach. We will compare the fractional Black-Scholes model with jumps with the classical Black-Scholes models and the Black-Scholes models with jumps, and conclude that in some cases, this model is superior in describing empirical observations and in its modeling flexibility. Moreover, we introduce the term structure, in this case, interest rates, for models with jumps as fractional Vasicek models with jumps. Finally, the sample paths of Thai Petrochemical (TPI) stock prices simulated by approcimate solution of classical Black-Scholes models, Black-Scholes models with jumps and fractional Black-Scholes models with jumps will be illustrated and compared to the empirical data.

The outline of this thesis is as follows. In Chapter II we introduce fractional stochastic calculus. Next, in Chapter III we recall some preliminary notions form stochastic analysis for jump-diffusion processes, such as a basic tools of the Poisson Process, counting processes, Poisson random measure, stock price models with jumps, and stochastic calculus for jump processes. We limit ourselves to those notions that be used in the sequel. In Chapter IV we then describe fractional Black-Scholes models with jumps using an approximate approach. Moreover, we describe fractional Vasicek models with jumps with using the same method. Finally, in Chapter V we describe simulation prices and results.

Chapter II

An Approach to Fractional Stochastic Calculus

In this chapter, we prepare mathematical tools for defining the stochastic integral with respect to fractional Brownian motion via integration by parts, since we will use the approximate approach: L^2-convergence of semimartingales to the fractional process.

We recall the definition of a martingale and semimartingale. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *filtration or information flow* on $(\Omega, \mathcal{F}, \mathbb{P})$ is a increasing family of σ-algebras $(\mathcal{F}_t, t \in [0, T])$, that is, $\forall t \geq s \geq 0,\ \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.

DEFINITION 2.1. Martingale *A stochastic process $(X(t), t \geq 0)$ is a* martingale *if it is integrable for any t, $E|X(t)| < \infty$, for any $s > 0$:*

$$E(X(t+s)|\mathcal{F}_t) \ = \ X(t), \quad a.s.,$$

where $\mathcal{F}_t$ is the information about the process up to time t, and the equality holds almost surely.

Note: if X and Y are two random variables, we write "$X = Y$ a.s." (almost surely) if $\mathbb{P}(\omega \in \Omega, X(\omega) = Y(\omega)) = 1$.

DEFINITION 2.2. Semimartingale *A stochastic process $(S(t), t \geq 0)$ is called a* semimartingale *if it can be represented as the following sum:*

$$S(t) \ = \ S(0) \ + \ M(t) \ + \ A(t),$$

where $(A(t), t \geq 0)$ is a process of bounded variation and $(M(t), t \geq 0)$ is a martingale, both defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$.

We consider the stochastic fractional integral

$$B(t) \;=\; \int_0^t (t-s)^{\alpha}\ \mathrm{d}Z(s),\ \ \alpha = H - \frac{1}{2},\ \ 0 < \alpha < \frac{1}{2}.$$

One can prove that $B(t)$ is not a semimartingale (see Shiryaev (1999)) but approximated by semimartingales as shown in Theorem 2.1.

2.1 An Approximation of Fractional Brownian Motion

The following theorem was proved by Thao (2002). The result of this theorem will be frequently refered to throughout the text.

For every $\varepsilon > 0$ we define

$$B_{\varepsilon}(t) \;=\; \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}}\ \mathrm{d}Z(s),$$

$0 < H < 1\, H \neq \frac{1}{2}$.

***THEOREM* 2.1.** *The process* $(B_{\varepsilon}(t),\ t \geq 0)$ *is a semimartingale.*

PROOF: Consider the stochastic process $\varphi_{\varepsilon}(t)$ defined as

$$\varphi_{\varepsilon}(t) \;=\; \int_0^t (t-u+\varepsilon)^{\alpha-1}\ \mathrm{d}Z(u),$$

where $\alpha\ = H - \frac{1}{2}$ (then $-\frac{1}{2} < \alpha < \frac{1}{2}$, since $0 < H < 1$).

An application of the stochastic theorem of Fubini (Wade, 1999, p.395) give us:

$$\begin{aligned}
\int_0^t \varphi_{\varepsilon}(s)\ \mathrm{d}s \;&=\; \int_0^t \int_0^u (s-u+\varepsilon)^{\alpha-1}\ \mathrm{d}Z(u)\mathrm{d}s \\
&=\; \int_0^t \left(\int_u^t (s-u+\varepsilon)^{\alpha-1}\ \mathrm{d}s \right) \mathrm{d}Z(u) \\
&=\; \int_0^t \left(\frac{(t-u+\varepsilon)^{\alpha}}{\alpha} \;-\; \frac{\varepsilon^{\alpha}}{\alpha} \right) \mathrm{d}Z(u) \\
&=\; \frac{1}{\alpha}\left[\int_0^t (t-u+\varepsilon)^{\alpha}\ \mathrm{d}Z(u) - \int_0^t \varepsilon^{\alpha}\ \mathrm{d}Z(u) \right] \\
&=\; \frac{1}{\alpha}\left(B_{\varepsilon}(t) - \varepsilon^{\alpha} Z(t) \right).
\end{aligned}$$

Hence

$$B_{\varepsilon}(t) \;=\; \alpha \int_0^t \varphi_{\varepsilon}(s)\ \mathrm{d}s + \ \varepsilon^{\alpha} Z(t).$$

Since $\alpha \int_0^t \varphi_{\varepsilon}(s)\mathrm{d}s$ is bounded variation and $Z(t)$ is a martingale so $B_{\varepsilon}(t)$ is a semimartingale. □

THEOREM 2.2. *$B_\varepsilon(t)$ converges to $B(t)$ in $L^2(\Omega)$ when ε tends to 0. This convergence is uniform with respect to $t \in [0,T]$.*

PROOF: Consider the function $f(x) = (t-s+\varepsilon x)^\alpha$, direct calculation shows that,

$$f(1) = (t-s+\varepsilon)^\alpha,\; f(0) = (t-s)^\alpha, \text{ and } f'(x) = \alpha\varepsilon(t-s+\varepsilon x)^{\alpha-1}$$

By the Mean Value Theorem,

$$(t-s+\varepsilon)^\alpha - (t-s)^\alpha \;=\; \alpha\varepsilon(t-s+\varepsilon\theta)^{\alpha-1}$$

for some $0 \le \theta \le 1$. Thus

$$\begin{aligned} |(t-s+\varepsilon)^\alpha - (t-s)^\alpha| \;&\le\; |\alpha|\; \varepsilon \sup_{0\le\theta\le 1} |(t-s+\varepsilon\theta)^{\alpha-1}| \\ &=\; |\alpha|\; \varepsilon\; (t-s)^{\alpha-1}, \end{aligned} \tag{2.1}$$

where $\alpha = H - \frac{1}{2}$, $(0 < s < t)$. By virtue of Itô integration isometry we see that

$$\begin{aligned} E|B_\varepsilon(t) - B(t)|^2 \;&=\; E\left|\int_0^t \left[(t-s+\varepsilon)^\alpha - (t-s)^\alpha\right] \mathrm{d}Z(s)\right|^2 \\ &=\; \int_0^t \left|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\right|^2 \mathrm{d}s. \end{aligned} \tag{2.2}$$

(i) (Thao, 2003) If $\frac{1}{2} < H < 1$, that is, $0 < \alpha < \frac{1}{2}$ we have from (2.1)

$$\begin{aligned} \int_0^t |(t-s+\varepsilon)^\alpha - (t-s)^\alpha|^2 \mathrm{d}s \;&\le\; \alpha^2\varepsilon^2 \int_0^t |t-s|^{2\alpha-2}\,\mathrm{d}s \\ &=\; \alpha^2\varepsilon^2\left(\int_0^{t-\varepsilon} |t-s|^{2\alpha-2}\,\mathrm{d}s \;+\; \int_{t-\varepsilon}^t |t-s|^{2\alpha-2}\,\mathrm{d}s\right) \\ &\le\; \alpha^2\varepsilon^2\frac{\varepsilon^{2\alpha-1}}{1-2\alpha} + \alpha^2\varepsilon^2\frac{\varepsilon^{2\alpha-1}}{1-2\alpha} \\ &=\; C_1(\alpha)\;\varepsilon^{2\alpha+1} \to 0 \end{aligned} \tag{2.3}$$

as $\varepsilon \to 0$, where $C_1(\alpha) = \frac{2\alpha^2}{1-2\alpha} > 0$.

(ii) (Thao and Nquyen, 2002) If $0 < H < \frac{1}{2}$, that is, $-\frac{1}{2} < \alpha < 0$, we put $\alpha = -\beta$, so $0 < \beta < \frac{1}{2}$ and we have

$$\begin{aligned} |(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}| \;&\le\; \beta\varepsilon \sup_{0\le\theta\le 1} |(t-s+\theta\varepsilon)^{-\beta-1}| \\ &=\; \beta\varepsilon(t-s)^{-\beta-1}, \end{aligned} \tag{2.4}$$

From (2.2) we have

$$\begin{aligned} E|B_\varepsilon(t) - B(t)|^2 &= E\left| \int_0^t \left[(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right] \mathrm{d}Z(s) \right|^2 \\ &= \int_0^t \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 \mathrm{d}s \\ &= \int_0^{t-\varepsilon} |(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}|^2 \mathrm{d}s \\ &\quad + \int_{t-\varepsilon}^t |(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}|^2 \mathrm{d}s, \end{aligned} \tag{2.5}$$

The evaluation of (2.4) applied to the first term of (2.5) gives us

$$\int_0^{t-\varepsilon} \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 \mathrm{d}s^2 \;\leq\; \beta^2\varepsilon^2 \int_0^{t-\varepsilon} (t-s)^{-2\beta-2} \mathrm{d}s. \tag{2.6}$$

For the second term of the right hand side of (2.5) we have

$$\int_{t-\varepsilon}^t \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 \mathrm{d}s^2 \;\leq\; \int_{t-\varepsilon}^t (t-s)^{-2\beta} \mathrm{d}s. \tag{2.7}$$

It follows from (2.5), (2.6) and (2.7) that

$$E|B_\varepsilon(t) - B(t)|^2 \;\leq\; \beta^2\varepsilon^2 \int_0^{t-\varepsilon} (t-s)^{-2\beta-2} \mathrm{d}s + \int_{t-\varepsilon}^t (t-s)^{-2\beta} \mathrm{d}s. \tag{2.8}$$

After some calculations we get:

$$E|B_\varepsilon(t) - B(t)|^2 \;\leq\; C_2(\beta)\, \varepsilon^{1-2\beta} \to 0, \text{ as } \varepsilon \to 0, \tag{2.9}$$

where $C_2(\beta)$ is a positive constant depending only on β.

From (2.3) and (2.9) we see that in both cases ($H > \frac{1}{2}$ and $H < \frac{1}{2}$), there is an estimation for $||Z_\varepsilon(t) - Z(t)||^2 = E\big[|Z_\varepsilon(t) - Z(t)|^2\big]$ as follows:

$$||B_\varepsilon(t) - B(t)||^2 \;\leq C_3(\alpha)\, \varepsilon^{1+2\alpha}, \tag{2.10}$$

where $0 < \alpha < \frac{1}{2}$ for $\frac{1}{2} < H < 1$ and $-\frac{1}{2} < \alpha < 0$, for $0 < H < \frac{1}{2}$, and $C_3(\alpha) = \max\big\{C_1(\alpha), C_2(\beta)\big\}$ depending only on α $(= -\beta)$.

The relation (2.10) is valid for every $t \geq 0$, so

$$\sup_{0\leq t\leq T} ||B_\varepsilon(t) - B(t)|| \;\leq C(\alpha)\, \varepsilon^{\frac{1}{2}+\alpha} \to 0, \text{ as } \varepsilon \to 0,$$

where $C(\alpha) = \sqrt{C_3(\alpha)}$ which proves that $B_\varepsilon(t) \to B(t)$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$. □

2.2 An Approach to Fractional Stochastic Calculus

Let us consider a fractional stochastic dynamical system $(X(t),\ 0 \le t \le T)$ expressed by the following fractional stochastic differential equation:

$$\begin{aligned} \mathrm{d}X(t) &= b(X(t),t)\ \mathrm{d}t + \sigma(X(t),t)\ \mathrm{d}Z_H(t), \\ X(t)|_{t=0} &= X(0),\ t \in [0,T], \end{aligned} \tag{2.11}$$

where $X(0)$ is a given random variable, and

$$Z_H(t) = \frac{1}{\Gamma(1+\alpha)} \left[W(t) + \int_0^t (t-s)^{\alpha}\ \mathrm{d}Z(s) \right]. \tag{2.12}$$

In order to give (2.11) meaning, we have to define the fractional stochastic integral

$$\int_0^t f(s,\omega)\ \mathrm{d}Z_H(s).$$

However, in 2000 Alos et al have proposed to use

$$B(t) = \int_0^t (t-s)^{H-\frac{1}{2}}\ \mathrm{d}Z(s), \tag{2.13}$$

instead of $Z_H(t)$ in fractional stochastic calculus, since $W(t)$ (in equation(2.12)) has absolutely continuous trajectories and it is the term $B(t)$ that has long memory. Therefore, instead of (2.11), we consider the fractional stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) &= b(X(t),t)\ \mathrm{d}t + \sigma(X(t),t)\ \mathrm{d}B(t), \\ X(t)|_{t=0} &= X(0),\ t \in [0,T]. \end{aligned}$$

To define the fractional stochastic integral

$$\int_0^t f(s,\omega)\ \mathrm{d}B(s),$$

where $B(t)$ is given by (2.13) and $H \in (0,1)$, we follow the work by Thao (2002).

2.3 Fractional Stochastic Integration

Let a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^Z, t \ge 0), \mathbb{P})$ be given where $\mathcal{F}_t^Z$ is a σ-algebra generated by standard Brownian motion $(Z(t), t \ge 0)$. Suppose that $f(t)$ is a deterministic function of bounded variation on $[0,T]$ and the fractional process $B(t)$ is given as in (2.13):

$$B(t) \;=\; \int_0^t (t-s)^{\alpha}\, \mathrm{d}Z(s),\;\; \alpha \;=\; H - \frac{1}{2},\;\; 0 < H < 1.$$

Then the integral $\int_0^t B(s)\, \mathrm{d}f(s)$ is well defined in the sense of Riemann-Stieltjes for almost all ω.

***DEFINITION* 2.3.** *The fractional stochastic integral of $f(t)$ is a stochastic process $I(t)$ defined as*

$$I(t) \;:=\; \int_0^t f(s)\, \mathrm{d}B(s) \;=\; f(t)\; B(t) \;-\; \int_0^t B(s)\, \mathrm{d}f(s).$$

Now suppose $(f(t,\omega), t \geq 0)$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ whose sample paths are of bounded variation on $[0,T]$ for almost every $\omega \in \Omega$.

***DEFINITION* 2.4.** *The fractional stochastic integral of $f(t,\omega)$ is a stochastic process $I(t)$ defined as*

$$I(t) = \int_0^t f(s,\omega)\, \mathrm{d}B(s) = f(t,\omega)\; B(t) \;-\; \int_0^t B(s)\, \mathrm{d}f(s,\omega) \;-\; [f,B]_t, \tag{2.14}$$

where the notation $[.,.]$ stands for the quadratic variation of two processes given by a limit in probability:

$$[f,B]_t \;=\; \mathbb{P} \;-\; \lim_{max\; |t_{k+1}-t_k| \to 0} \sum_{k=0}^{n-1} \left[f(t_{k+1}) - f(t_k)\right]\left[B(t_{k+1}) - B(t_k)\right],$$

for all partitions $\{0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots < t_n = T\}$ of $[0,T]$.

***REMARK* 2.5.** (i) *The pathwise integral in the right hand side of (2.14) exists in the sense of Riemann-Stieltjes for almost all ω.*
(ii) *If the function $f(t,\omega)$ has absolutely continuous sample paths (for instance, if it is Lipschitzian with respect to t) then it is of bounded variation and so its integral $I(t) = \int_0^t f(s,\omega)\, \mathrm{d}B(s)$ exists.*

***THEOREM* 2.3.**

Suppose that the process $f(t, \omega)$ has continuous sample paths and of bounded variation on $[0, T]$ such that $E \int_0^t f^2(s, \omega)\ \mathrm{d}s\ <\ \infty$. Then the stochastic integral

$$I_\varepsilon(t)\ =\int_0^t f(s, \omega)\ \mathrm{d}B_\varepsilon(s),$$

where $B_\varepsilon(t)\ =\ \int_0^t (t - s + \varepsilon)^\alpha\ \mathrm{d}Z(s)$, $\alpha\ =\ H - \frac{1}{2}$, $0 < H < 1$, converges in $L^2(\Omega)$ as $\varepsilon \to 0$ to $I(t)\ =\ \int_0^t f(s, \omega)\ \mathrm{d}B(s)$ defined as in (2.14). This convergence is uniform with respect to $t\ \in\ [0, T]$.

PROOF: see Thao (2002). □

***REMARK* 2.6.** *Theorem 2.3 is proved for the L^2- convergence of $I(t) \to I_\varepsilon(t)$ in the case that f is of bounded variation. This motivates us to define the fractional stochastic integral for any stochastic process $f(t, \omega)$ as follows.*

***DEFINITION* 2.7.** *Let $f(t, \omega)$ be a stochastic process with continuous path. Then the fractional stochastic integral of $f(t, \omega)$ is defined by*

$$\int_0^t f(s, \omega)\ \mathrm{d}B(s)\ :=\ L^2 - \lim_{\varepsilon \to 0} \int_0^t f(s, \omega)\ \mathrm{d}B_\varepsilon(s),$$

whenever the limit exists in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, where $B(t) = \int_0^t (t - s)^{H - \frac{1}{2}}\ \mathrm{d}Z(s)$ and $B_\varepsilon(t) = \int_0^t (t - s + \varepsilon)^{H - \frac{1}{2}}\ \mathrm{d}Z(s)$ for $0 < H < 1$.

Chapter III

Diffusion Models with Jumps

In this chapter we begin by recalling basic definitions and results needed for the study of jump-diffusion models, limiting ourselves to the notions of the Poisson processes, counting processes, and Poisson random measures. Next, we study a stock-price model with jumps to understand the stochastic calculus for jump processes, assuming that we are familiar with the corresponding notions concerning diffusion processes. Moreover, we discuss the existence and uniqueness of a solution to the stochastic differential equation with jumps. Finally, in the last section we derive Itô's formula for jump-diffusion processes, one of the main tools in this thesis.

3.1 The Poisson Process: Definition and Properties

If the Brownian motion process is a basic model for cumulative small noise present continuously, the Poisson process is a basic model for the cumulative noise that occurs as a stock. It models phenomena where changes rarely occur, but when they do occur are large.

Recall that a random variable X has a Poisson distribution with parameter λ if it takes on nonnegative integer values $k \geq 0$ with probabilities

$$\mathbb{P}(X = k) \ = \ e^{-\lambda} \, \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

A Poisson process $N(t), t \geq 0$ is a stochastic process defined as follows:

DEFINITION 3.1. Poisson process *Let $\tau_1, \tau_2, \ldots$ be a sequence of independent, identically distributed exponentially random variables (defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$) with parameter λ, that is, $\mathbb{P}(\tau_1 > t) = e^{-\lambda t}$.*

Let

$$T_n = \sum_{i=1}^{n} \tau_i,$$

and

$$\begin{aligned} N(t) &= \sum_{n\geq 1} 1_{\{T_n \leq t\}} \\ &= \#\{n \geq 1,\ T_n \leq t\} \end{aligned}$$

Then the process $(N(t), t \geq 0)$ *is called a* Poisson process *with intensity* λ.

This process serves as a model for occurrence of independent rare events. The rate, the average number of events per unit of time, is denoted by λ. $N(t)$ counts the number of events that occurred up to time t, and $N(t) - N(s)$ gives the number of events that occurred in the time interval $(s, t]$.

***DEFINITION* 3.2.** *(i)* $X(t-,\omega) = \lim_{s\to t} X(s,\omega),\ s < t,$ *for each* $\omega \in \Omega$.
(ii) $X(t-) = \lim_{s\to t} X(s)$ $\mathbb{P}$ *almost surely.*
(iii) A stochastic process $X(t)$ *is cadlag if it* $\mathbb{P}$ *almost surely has sample paths which are right continuous, with left limits.*

We list the following properties of Poisson process:

PROPOSITION* 3.1. *Properties of Poisson processes

Let $(N(t), t \geq 0)$ *be a Poisson process.*

1. *For any* $t > 0$, $N(t)$ *is almost surely finite.*

2. *For any* ω, *the sample path* $t \to N(t,\omega)$ *is piecewise constant and increases by jumps of size* 1.

3. *The sample paths* $t \mapsto N(t)$ *are right continuous with left limit (cadlag).*

4. *For any* $t > 0$, $N(t-) = N(t)$ *with probability* 1.

5. $(N(t), t \geq 0)$ *is continuous in probability:*

$$N(s) \xrightarrow[s\to t]{\mathbb{P}} N(t) \quad \forall t > 0.$$

6. *For any $t > 0$, N(t) follows a Poisson distribution with parameter λt :*

$$\mathbb{P}\big(N(t) = n\big) \;=\; e^{-\lambda t}\,\frac{(\lambda t)^n}{n!} \quad \forall n \in \mathbb{N}.$$

7. *The characteristic function of $N(t)$ is given by*

$$E\Big[e^{iuN(t)}\Big] \;=\; \exp\big\{\lambda t(e^{iu} - 1)\big\},\ \forall u\ \in \mathbb{R}.$$

8. *$(N(t), t \geq 0)$ has independent increments: for any $t_1 < t_2 < \ldots < t_n, N(t_n) - N(t_{n-1}), \ldots, N(t_2) - N(t_1), N(t_1)$ are independent random variables.*

9. *The increments of $(N(t), t \geq 0)$ are homogeneous: for any $t > s$, $N(t) - N(s)$ has the same distribution as $N(t - s)$.*

10. *$(N(t), t \geq 0)$ has the Markov property:*

$$\forall t > s,\ E\big[f(N(t))|N(u),\ u \leq s\big] \;=\; E\big[f(N(t))|N(s)\big].$$

11. *$N(t)$ is a semimartingale.*

PROOF: The proof of *1-10* can be found in (Cont and Tankov, 2004, p.48). The following is a proof of *11.*

Let us consider the process $M(t) = N(t) - \lambda t$, Since $N(t+s) - N(t)$ is independent of $\mathcal{F}_t$, then

$$E\Big[N(t+s) - N(t)\Big|\mathcal{F}_t\Big] \;=\; E\Big[N(t+s) - N(t)\Big]. \tag{3.1}$$

Moreover, since $\sigma(N(t)) \subset \mathcal{F}_t$ then

$$E\Big[N(t)\Big|\mathcal{F}_t\Big] \;=\; N(t). \tag{3.2}$$

Therefore,

$$\begin{aligned} E\Big[N(t+s)-\lambda(t+s)\Big|\mathcal{F}_t\Big] & \\ &= E\Big[N(t+s) - N(t) + N(t) - \lambda(t+s)\Big|\mathcal{F}_t\Big] \\ &= E\Big[N(t+s) - N(t)\Big|\mathcal{F}_t\Big] \;+\; E\Big[N(t)\Big|\mathcal{F}_t\Big] \;-\; \lambda(t+s) \end{aligned}$$

(since conditional expectation is linear).

From (3.1) and (3.2),

$$\begin{aligned} E\Big[N(t+s)-N(t)\Big|\mathcal{F}_t\Big] + E\Big[N(t)\Big|\mathcal{F}_t\Big] \;-\; \lambda(t+s) \\ &= E\Big[N(t+s)-N(t)\Big] \;+\; N(t) \;-\; \lambda(t+s) \\ &= \lambda(t+s) \;-\; \lambda t \;+\; N(t) \;-\; \lambda(t+s) \\ &= N(t) \;-\; \lambda t. \end{aligned}$$

Thus $M(t)$ is a martinagale and since λt is of bounded variation so $N(t) = M(t)+\lambda t$ is a semimartingale by difinition. □

3.2 Counting Processes

The counting process is a generalization of the Poisson process. We first introduce the idea of the point process.

DEFINITION 3.3. Point processes A point process *is a sequence of random variables* $(T_n, n \geq 1)$ *over time* $[0, t]$ *such that*

$$0 = T_0 < T_1 < T_2 < \dots$$

with $\mathbb{P}(T_n \to \infty) = 1$.

DEFINITION 3.4. Counting process *Let* $(T_n, n \geq 0)$ *be a point process. The associated* counting process $(X(t), t \geq 0)$ *is given by*

$$X(t) \;=\; \sum_{n\geq 1} 1_{\{T_n \leq t\}} \;=\; \#\{n \geq 1, T_n \leq t\}.$$

$X(t)$ *is simply the number of random times (or jumps)* $(T_n, n \geq 1)$ *occurring in* $[0, t]$.

The condition $\mathbb{P}(T_n \to \infty) = 1$ guarantees that, with probability 1, $X(t)$ is finite for any $t \geq 0$. Like the Poisson process, $(X(t), t \geq 0)$ is a cadlag process with piecewise constant trajectories: its sample paths move by jumps of size $+1$.

If the random times (T_n) are constructed as partial sums of a sequence of independent identically distributed exponential random variables, then $X(t)$ is a Poisson process. For a general counting process, the sequence of random times (T_n) can have any distribution and dependence structure. The following proposition shows that the only counting processes with independent stationary increments are Poisson process:

***PROPOSITION* 3.2.**

Let $(X(t), t \geq 0)$ *be a counting process with stationary independent increments. Then* $(X(t), t \geq 0)$ *is a poisson process.*

PROOF: see Cont and Tankov (2004). □

3.3 Definition of the Poisson Random Measure

The poisson process $(N(t), t \geq 0)$ was defined in section (3.1) as a counting process: if $T_1, T_2, \ldots$ is the sequence of jump times of N, then $N(t)$ is simply the number of jumps between 0 and t:

$$N(t) \; = \; \#\{n \geq 1, \; T_n \in [0, T]\}.$$

Similarly, if $t > s$ then

$$N(t) - N(s) \; = \; \#\{n \geq 1, \; T_n \in [s, t]\}.$$

The jump times $T_1, T_2, \ldots$ form a random configuration of points on $[0, \infty)$ and the Poisson process $N(t)$ counts the number of such points in the interval $[0, t]$. This counting procedure defines a *measure* N on $[0, \infty)$: for any Borel measurable set $A \subset \mathbb{R}^+$,

$$N(\omega, A) \; = \; \#\{n \geq 1, T_n(\omega) \in A\}.$$

$N(\omega, \cdot)$ is a positive, integer valued measure on the Borel subsets of $\mathbb{R}^+$. We note that $N(\cdot, A)$ is finite with probability 1 for any bounded set $A \subset \mathbb{R}^+$. Note that the measure $N(\omega, \cdot)$ depends on ω: it is thus a *random measure*. The intensity λ of the Poisson process determines the *average* value of the random measure $N(\cdot, A)$, that is,

$$E[N(\cdot, A)] \; = \; \lambda |A|$$

where $|A|$ is the Lebesgue measure of A.

$N(\omega, \cdot)$ is called *Poisson random measure* associated to the Poisson process $N(t)$. The Poisson process $N(t)$ may be expressed in terms of the random measure N in the following way:

$$N(\omega, t) \; = \; N(\omega, [0, t]) \; = \; \int_{[0,t]} N(\omega, \mathrm{d}s).$$

Conversely, the Poisson random measure N can also be viewed as the "derivative" of the Poisson process. Recall that each trajectory $t \mapsto N(\omega, t)$ of a Poisson process is an increasing step function. Hence its derivative (in the sense of distributions) is a positive measure on the Borel set of $\mathbb{R}^+$. In fact, it is simply the superposition of Dirac masses located at the jump times:

$$\frac{d}{\mathrm{d}t} N(\omega, t) \ = \ \sum_{n \geq 1} \delta_{T_n(\omega)}(\cdot) \ =: \ N(\omega, \cdot).$$

Hence for any predictable process $f(\omega, s)$, the stochastic integral with respect to the Poisson random measure N admits, for any $t \in \mathbb{R}^+$, the form

$$\int_0^t f(\cdot, s) \ N(\cdot, \mathrm{d}s) \ = \ \sum_{n \geq 1} f(T_n) 1_{\{T_n \leq t\}}(\cdot) \ = \ \sum_{n=1}^{N(\cdot, t)} f(T_n),$$

or in compact form as follows:

$$\int_0^t f(s) \ \mathrm{d}N(s) \ = \ \sum_{n=1}^{N(t)} f(T_n). \tag{3.3}$$

Note that if a filtration $(\mathcal{F}_t, t \geq 0)$ is given, a process f_t is called *predictable* (with respect to this filtration) if for each t, f_t is $\mathcal{F}_{t-1}$-measurable, that is, the value of the process f at time t is determined by the information up to and including time $t - 1$.

We will assume that the T_n's correspond to the jump times of a Poisson process $N(t)$ and that Y_n is a sequence of indentically distributed random variables with values in $(-1, \infty)$. Let a process $S(t)$ be a predictable. At time T_n, the jump of the dynamics of $S(t)$ is given by

$$S(T_n) - S(T_n-) = S(T_n-) Y_n, \tag{3.4}$$

which, by the assumption $Y_n > -1$, leads always to positive values of the prices.

If $F(S, t)$ is a $C^{2,1}$-function [1] , then it follows from (3.3) that

$$\int_0^t [F(S(s-)(1 + Y_s), s) - F(S(s-), s)] \ \mathrm{d}N(s) \ = \ \sum_{n=1}^{N(t)} [F(S(T_n), T_n) - F(S(T_n-), T_n)]. \tag{3.5}$$

where Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation. An application of equation (3.5) to the function $F(S, t) = S$ for $S \geq 0$ yields

[1]This means that F is C^2 in the variable s, and C^1 in the variable t.

$$\int_0^t [S(s-)(1+Y_s) - S(s-)]\ \mathrm{d}N(s) \ = \ \sum_{n=1}^{N(t)} [S(T_n) - S(T_n-)],$$

or

$$\int_0^t S(s-)Y_s\ \mathrm{d}N(s) \ = \ \sum_{n=1}^{N(t)} [S(T_n) - S(T_n-)]. \tag{3.6}$$

Applying (3.4) to the right side of equation (3.6), we have

$$\int_0^t S(s-)Y_s\ \mathrm{d}N(s) \ = \ \sum_{n=1}^{N(t)} S(T_n-)Y_n. \tag{3.7}$$

3.4 The Stock-Price Model with Jumps

The objective of this section is to model a financial market in which there is one riskless asset (with price $S(t) = e^{rt}$, at time t) and one risky asset whose price jumps at the proportions $Y_1, \ldots, Y_n, \ldots$, at some times $T_1, \ldots, T_n, \ldots$ and which, between any two jumps, follows the Black-Scholes model. Moreover, we will assume that the T_n's correspond to the jump times of a Poisson process. To be more rigorous, let us consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which we define a standard Brownian motion $(Z(t), t \geq 0)$, a Poisson process $(N(t), t \geq 0)$ with intensity λ and a sequence $(Y_n, n \geq 1)$ of independent, identically distributed random variables taking values in $(-1, +\infty)$. We will assume that the σ-algebras generated respectively by $(Z(t), t \geq 0)$, $(N(t), t \geq 0)$ and $(Y_n, n \geq 1)$ are independent.

For all $t \geq 0$, let us denote by $\mathcal{F}_t$ the σ-algebra generated by the random variables $Z(s)$, $N(s)$ for $s \leq t$ and $Y_n 1_{\{n \leq N(t)\}}$ for $n \geq 1$ where $1_{\{n \leq N(t)\}}$ is the indicator function defined as: if $N(t) \geq n$, then $1_{\{n \leq N(t)\}} = 1$ and if $N(t) > n$, then $1_{\{n > N(t)\}} = 0$. It can be shown that $(Z(t), t \geq 0)$ is a standard Brownian motion with respect to the filtration $(\mathcal{F}_t, t \geq 0)$, that $(N(t), t \geq 0)$ is a process adapted to this filtration and that, for all $t > s$, $N(t) - N(s)$ is independent of the σ-algebra $\mathcal{F}_s$. Because the random variables $Y_n 1_{\{n \leq N(t)\}}$ are $\mathcal{F}_t$-measurable, we deduce that, at time t, the relative amplitudes of the jumps taking place before t are known. Note as well that the T_n's are stopping times of $(\mathcal{F}_t, t \geq 0)$, since $(T_n \leq t) = (N(t) \geq n) \in \mathcal{F}_t$.

The dynamics of $S(t)$, the price of the risky asset at time t, can now be described. The process $(S(t), t \geq 0)$ is an adapted, right-continuous process such that on the time intervals $[T_n, T_{n+1})$,

$$\mathrm{d}S(t) \;=\; S(t)\Big(\mu \;\mathrm{d}t \;+\; \sigma \;\mathrm{d}Z(t)\Big), \tag{3.8}$$

while at $t = T_n$ the jump of $S(t)$ is given by

$$\Delta S_n \;=\; S(T_n) - S(T_n-) = S(T_n-)Y_n.$$

Thus

$$S(T_n) \;=\; S(T_n-)(1 + Y_n),$$

which, by the assumption of $Y_n > -1$, leads always to positive values of the prices. By using the standard Itô formula, the solution of (3.8) on the interval $[0, T_1)$ is

$$S(t) \;=\; S(0)\;\exp\left(\Big(\mu - \frac{\sigma^2}{2}\Big)t \;+\; \sigma Z(t)\right).$$

Consequently, the left-hand limit at T_1 is given by

$$S(T_1-) \;:=\; \lim_{u\to T_1} S(u) \;=\; S(0)\exp\left(\Big(\mu - \frac{\sigma^2}{2}\Big)T_1 \;+\; \sigma Z(T_1)\right)$$

and

$$S(T_1) \;=\; S(0)(1 + Y_1)\exp\left(\Big(\mu - \frac{\sigma^2}{2}\Big)T_1 \;+\; \sigma Z(T_1)\right).$$

Then, for $t \;\in\; [T_1, T_2)$,

$$\begin{aligned} S(t) \;&=\; S(T_1)\exp\left(\Big(\mu - \tfrac{\sigma^2}{2}\Big)(t - T_1) \;+\; \sigma(Z(t) - Z(T_1))\right) \\ &=\; S(0)(1 + Y_1)\exp\left(\Big(\mu - \tfrac{\sigma^2}{2}\Big)t \;+\; \sigma Z(t)\right). \end{aligned}$$

Repeating this scheme, we obtain

$$S(t) \;=\; S(0)\left[\prod_{n=1}^{N(t)} (1 + Y_n)\right]\;\exp\left(\Big(\mu - \frac{\sigma^2}{2}\Big)t \;+\; \sigma Z(t)\right), \tag{3.9}$$

with the convention $\displaystyle\prod_{n=1}^{0} = 1$.

Using the formula (3.7), $S(t)$ can be given in the following equivalent representations

$$\begin{aligned} S(t) = & \; S(0)\exp\left[\Big(\mu - \frac{\sigma^2}{2}\Big)t \;+\; \sigma Z(t)\right]\left[\prod_{n=1}^{N(t)}(1+Y_n)\right] \\ = & \; S(0)\exp\left[\Big(\mu - \frac{1}{2}\sigma^2\Big)t \;+\; \sigma Z(t) \;+\; \log\left(\prod_{n=1}^{N(t)}(1+Y_n)\right)\right] \\ = & \; S(0)\exp\left[\Big(\mu - \frac{1}{2}\sigma^2\Big)t \;+\; \sigma Z(t) + \sum_{n=1}^{N(t)}\log(1+Y_n)\right] \\ = & \; S(0)\exp\left[\Big(\mu - \frac{1}{2}\sigma^2\Big)t \;+\; \sigma Z(t) + \int_0^t \log(1+Y_s)\;\mathrm{d}N(s)\right], \end{aligned}$$

where Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation.

The process $(S(t), t \geq 0)$ in equation (3.9) is right-continuous, adapted and has only finitely many discontinuities on each interval $[0, t]$. We also prove the following.

***THEOREM* 3.3.** *For all $t \geq 0$, $(S(t), t \geq 0)$ in equation (3.9) satisfies:*

$$\mathbb{P}\ a.s. \quad S(t) \;=\; S(0) \;+\; \int_0^t S(s)\Big(\mu \mathrm{d}s \;+\; \sigma \mathrm{d}Z(s)\Big) \;+\; \sum_{n=1}^{N(t)} S(T_n-)Y_n, \tag{3.10}$$

or, in differential from

$$\mathbb{P}\ a.s. \quad \mathrm{d}S(t) \;=\; S(t)(\mu \mathrm{d}t \;+\; \sigma \mathrm{d}Z(t)) \;+\; S(t-)Y_t\;\mathrm{d}N(t). \tag{3.11}$$

PROOF: Let $\Delta S_n = S(T_n) - S(T_n-) = S(T_n-)Y_n$. Then (3.10) can be written in the following form:

$$\mathbb{P}\text{ a.s.} \quad S(t) \;=\; S(0) \;+\; \int_0^t S(s)\Big(\mu \mathrm{d}s \;+\; \sigma \mathrm{d}Z(s)\Big) \;+\; \sum_{n=1}^{N(t)} \Delta S_n, \tag{3.12}$$

We choose the function $f(x, s) = \log x$. Direct calculation shows that

$$f_x = \frac{1}{x}, \qquad f_{xx} = -\frac{1}{x^2}, \quad \text{and} \quad f_s = 0.$$

We note that $f(x, t)$ is a $C^{2,1}$ function if $x > 0$. Assume that $S(t)$ in (3.10) is nonnegative. Applying the Itô formula for jump-diffusion processes (see Theorem 3.5 in section 3.6) to $f(x, t) = \log x$, we obtain

$$\log S(t) \;=\; \log S(0)\left(\mu - \frac{1}{2}\sigma^2\right)t \;+\; \sigma Z(t) \;+\; \sum_{n=1}^{N(t)}\log(1+Y_n).$$

Thus,

$$S(t) \;=\; S(0))\left[\prod_{n=1}^{N(t)}(1+Y_n)\right]\exp\left(\left(\mu-\frac{1}{2}\sigma^2\right) \;+\; \sigma Z(t)\right).$$

Hence, (3.9) holds as asserted. □

3.5 Jump-diffusion Stochastic Differential Equations

The model (3.12) discussed in the last section contains the basic features of financial models including jump effects that have been investigated since the pioneering work of Merton (1976). It is often more convenient, theoretically at least, to include the jump mechanism in the differential equation itself. For models like those in the last section, this idea gives rise to a jump-diffusion stochastic differential equation. In the scalar case, the general form of a jump-diffusion stochastic differential equation reads

$$\mathrm{d}X(t) \;=\; a(X(t),t)\;\mathrm{d}t \;+\; b(X(t),t)\;\mathrm{d}Z(t) \;+\; c(X(t-),t)\;\mathrm{d}N(t), \tag{3.13}$$

where $a(x,t)$ is the drift coefficient, $b(x,t)$ the diffusion coefficient, and $c(x,t)$ the jump magnitude coefficient. As before $Z(t)$ is a Brownian motion and $N(t)$ is a Poisson process.

The jump-diffusion stochastic differential equation (3.13) is interpreted as a stochastic integral equation

$$\begin{aligned} X(t) \;=\; X(t_0) \;&+\; \int_{t_0}^{t} a(X(s),s)\;\mathrm{d}s \;+\; \int_{t_0}^{t} b(X(s),s)\;\mathrm{d}Z(s) \\ &+\; \int_{t_0}^{t} c(X(s-),s)\;\mathrm{d}N(s), \end{aligned}$$

where the first integral is a deterministic Riemann integral, the second is a stochastic Ito integral and the third is a stochastic integral with respect to a Poisson process or, more generally, Poisson random measure. The existence and uniqueness of a solution process $X(t)$ of equation (3.13) follows under the usual growth restriction, uniform Lipschitz, and smoothness conditions on the coefficient functions a, b and c, leading to the next theorem.

Let the filtered probability space $(\omega, \mathcal{F}_t, (\mathcal{F}_t^Z, t \geq 0), \mathbb{P})$ satisfy the usual conditions. As usual, $(Z(t), t \geq 0)$ is a standard Brownian motion. To ensure the existence of the stochastic integrals and the existence and uniqueness of a solution of (3.13), the following conditions are required refer need to as the *standard assumptions* in the sequel. We assume the **standard assumptions** as follows:

$$
\begin{aligned}
&(i) \quad \int_0^t \Big| a(X(s,\omega), s) \Big|^2 \mathrm{d}s \ < \ \infty, \quad \omega \in \Omega \\
&(ii) \quad \int_0^t \Big| c(X(s,\omega), s) \Big|^2 \mathrm{d}s \ < \ \infty, \quad \omega \in \Omega \\
&(iii) \quad |a(x,s)|^2 + |b(x,s)|^2 \ \leq \ C\Big(1 + |x|^2\Big) \\
&(iv) \quad |a(x_1,s) - a(x_2,s)| \ \leq L|x_1 - x_2| \\
&(v) \quad |b(x_1,s) - b(x_2,s)|^2 \ \leq L^2 |x_1 - x_2|^2
\end{aligned}
$$

where C and L are some constant.

THEOREM 3.4. Existence and Uniqueness of a solution to the stochastic differential equation *If the functions $a(x,s)$, $b(x,s)$ and $c(x,s)$ satisfy the standard assumptions, then the stochastic differential eqatuion (3.13) has a unique solution $X(t)$ which is a cadlag process and adapted to the filtration $(\mathcal{F}_t, t \geq 0)$.*

PROOF: See Gihman and Skorohod (1979), Theorem 3.4 p.138 and p.156. □

We recall the definition of compound Poisson processes to use in next section.

DEFINITION 3.5. Compound Poisson process *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process $X(t)$ defined as*

$$X(t) \ = \ \sum_{n=1}^{N(t)} Y_n$$

where jumps sizes Y_n are independent, identically distributed with distribution f and $(N(t))$ is a Poisson process with intensity λ, independent from $(Y_n, n \geq 1)$.

3.6 Itô's Formula for Diffusions with Jumps

THEOREM 3.5. Itô's formula for jump-diffusion processes

Let X be a diffusion process with jumps, defined as the sum of drift term, a Brownian stochastic integral and a compound Poisson process:

$$X(t) \;=\; X(0) + \int_0^t b(s)\;\mathrm{d}s \;+\; \int_0^t \sigma(s)\;\mathrm{d}Z(s) \;+\; \sum_{n=1}^{N(t)} \;\Delta X_n.$$

Here $b(t)$ and $\sigma(t)$ are continuous nonanticipating processes with

$$E\left[\int_0^\tau \sigma^2(t)\;\mathrm{d}t\right] \;<\; \infty.$$

$\Delta X_n = X(T_n) - X(T_n-)$ are the jump sizes and $N(t)$ is the number of jumps that can be represented as the value at t of a counting process.

Then, for any $C^{2,1}$ function, $f : \mathbb{R} \times [0,T] \to \mathbb{R}$, the process $Y(t) = f(X(t),t)$ can be represented as:

$$\begin{aligned} f(X(t),t) \;-\; & f(X(0),0) \\ &= \int_0^t \left[\frac{\partial f}{\partial s}(X(s),s) \;+\; \frac{\partial f}{\partial x}(X(s),s)\; b(s)\right] \mathrm{d}s \\ &\quad + \;\frac{1}{2}\int_0^t \sigma^2(s)\;\frac{\partial^2 f}{\partial x^2}(X(s),s)\;\mathrm{d}s \;+\; \int_0^t \frac{\partial f}{\partial x}(X(s),s)\;\sigma(s)\;\mathrm{d}Z(s) \\ &\quad + \sum_{n=1}^{N(t)} \left[f\big(X(T_n),T_n\big) \;+\; f(X(T_n-),T_n)\right]. \end{aligned}$$

In differential notation:

$$\begin{aligned} dY(t) = \;& \frac{\partial f}{\partial t}(X(t),t)\;\mathrm{d}t \;+\; b(t)\;\frac{\partial f}{\partial x}(X(t),t)\;\mathrm{d}t \\ & + \;\frac{\sigma^2(t)}{2}\;\frac{\partial^2 f}{\partial x^2}(X(t),t)\;\mathrm{d}t \;+\; \frac{\partial f}{\partial x}(X(t),t)\;\sigma(t)\;\mathrm{d}Z(t) \\ & + \;\left[f\big(X(T_n),T_n)\big) \;+\; f(X(T_n-),T_n)\right]\;\mathrm{d}N(t). \end{aligned}$$

PROOF:

Define $Y(t) = f(X(t), t)$ where $f \in C^{2,1}(\mathbb{R})$ and denote by $T_n,\ n = 1, 2, \ldots, N(T)$ the jump times of X. On (T_n, T_{n+1}), X evolves according to

$$\mathrm{d}X(t) \ = \ b(t)\ \mathrm{d}t \ + \ \sigma(t)\ \mathrm{d}Z(t).$$

Applying the Itô formula in the Brownian case, we obtain:

$$\begin{aligned}
Y(T_{n+1}-) - Y(T_n) & \\
&= \int_{T_n}^{T_{n+1}-} \frac{\partial f}{\partial s}(X(s), s)\ \mathrm{d}s \ + \ \int_{T_n}^{T_{n+1}-} \frac{\partial f}{\partial x}(X(s), s)\ \mathrm{d}X(s) \\
&\qquad + \ \frac{1}{2}\int_{T_n}^{T_{n+1}-} \sigma^2(s)\ \frac{\partial^2 f}{\partial x^2}(X(s), s)\ \mathrm{d}s \\
&= \int_{T_n}^{T_{n+1}-} \frac{\partial f}{\partial s}(X(s), s)\ \mathrm{d}s \\
&\qquad + \ \int_{T_n}^{T_{n+1}-} \frac{\partial f}{\partial x}(X(s), s)\ \Big(b(s)\ \mathrm{d}s \ + \ \sigma(s)\ \mathrm{d}Z(s)\Big) \\
&\qquad\quad + \ \frac{1}{2}\int_{T_n}^{T_{n+1}-} \sigma^2(s)\ \frac{\partial^2 f}{\partial x^2}(X(s), s)\ \mathrm{d}s \\
&= \int_{T_n}^{T_{n+1}-} \left[\frac{\partial f}{\partial s}(X(s), s) \ + \ \frac{\partial f}{\partial x}(X(s), s)\ b(s)\ \right] \mathrm{d}s \\
&\qquad + \ \int_{T_n}^{T_{n+1}-} \frac{\partial f}{\partial x}(X(s), s)\ \sigma(s)\ \mathrm{d}Z(s) \\
&\qquad\quad + \ \frac{1}{2}\int_{T_n}^{T_{n+1}-} \sigma^2(s)\ \frac{\partial^2 f}{\partial x^2}(X(s), s)\ \mathrm{d}s.
\end{aligned}$$

If a jump of size $\Delta X(t)$ occurs, the resulting change in $Y(t)$ is given by $f(X(t-) + \Delta X(t), t) - f(X(t-), t)$. The total change in $Y(t)$ can therefore be written as the sum of these two contributions:

$$\begin{aligned}
f(X(t), t) \ - \ f(X(0), 0) & \\
&= \int_0^t \Big[\frac{\partial f}{\partial s}(X(s), s) \ + \ \frac{\partial f}{\partial x}(X(s), s)\ b(s)\Big]\ \mathrm{d}s \\
&\quad + \ \frac{1}{2}\int_0^t \sigma^2(s)\ \frac{\partial^2 f}{\partial x^2}(X(s), s)\ \mathrm{d}s \ + \ \int_0^t \frac{\partial f}{\partial x}(X(s), s)\ \sigma(s)\ \mathrm{d}Z(s) \\
&\qquad + \ \sum_{0 \le s \le t, \Delta X(s) \neq 0} \Big[f(X(s-) + \Delta X(s), s) - f(X(s-), s)\Big].
\end{aligned}$$

Since $f(X(t-)+\Delta X(t)) - f(X(t-),t) = f(X(t),t) - f(X(t-),t)$ and the number of jumps is finite in a finite time interval $[0,t]$, one obtain an equivalent expression:

$$\begin{aligned} f(X(t),t) \;-\; & f(X(0),0) \\ &= \int_0^t \Big[\frac{\partial f}{\partial s}(X(s),s) \;+\; \frac{\partial f}{\partial x}(X(s),s)\; b(s)\Big]\, \mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \sigma^2(s)\, \frac{\partial^2 f}{\partial x^2}(X(s),s)\, \mathrm{d}s \;+\; \int_0^t \frac{\partial f}{\partial x}(X(s),s)\; \sigma(s)\, \mathrm{d}Z(s) \\ &\qquad + \sum_{\{n\geq 1,\ T_n\leq t\}} \Big[f(X(T_n),T_n) - f(X(T_n-),T_n)\Big]. \end{aligned}$$

For the last term on the right we have equivalent representations

$$\begin{aligned} f(X(t),t) \;-\; & f(X(0),0) \\ &= \int_0^t \Big[\frac{\partial f}{\partial s}(X(s),s) \;+\; \frac{\partial f}{\partial x}(X(s),s)\; b(s)\Big]\, \mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \sigma^2(s)\, \frac{\partial^2 f}{\partial x^2}(X(s),s)\, \mathrm{d}s \;+\; \int_0^t \frac{\partial f}{\partial x}(X(s),s)\; \sigma(s)\, \mathrm{d}Z(s) \\ &\qquad + \sum_{n=1}^{N(t)} \Big[f(X(T_n),T_n) - f(X(T_n-),T_n)\Big]. \end{aligned}$$

This completes the proof. □

Chapter IV

A Fractional Model with Jumps

In this chapter, a fractional Black-Scholes model with jumps and term structure models with jumps will be studied. First an approximate approach to a fractional Black-Scholes model with jumps is introduced. Then we will consider the fractional Vasicek model with jumps and its corresponding approximate model. Finally, the solution to the approximate model will be found and proved that it converges in $L^2(\Omega)$ to the solution of the original model.

4.1 A Fractional Stock-Price Model with Jumps

Let us consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which we define a standard Brownian motion $(Z(t), t \geq 0)$, a Poisson process $(N(t), t \geq 0)$ with intensity λ and a sequence $(Y_n, n \geq 1)$ of independent, identically distributed random variables taking values in $(-1, +\infty)$. We will assume that the σ-algebras generated respectively by $(Z(t), t \geq 0)$, $(N(t), t \geq 0)$ and $(Y_n, n \geq 1)$ are independent.

The objective of this section is to model a financial market in which there is one riskless asset (with price $S(t) = e^{rt}$, at time t) and one risky asset whose price jumps at random time $T_1, \ldots T_n, \ldots$ with the relative/proportional change in its value at a jump time is given by $Y_1, \ldots, Y_n, \ldots$ respectively. We may then assume that, between two jump times, the price $S(t)$ follows a *fractional Black-Scholes model* for a fractional process $B(t)$, that is, T_n are the jump times of a Poisson process $N(t)$ with intensity λ and that Y_n is a sequence of identically distributed random variables with values in $(-1, \infty)$. This description can be formalized on the intervals $[T_n, T_{n+1})$ by letting:

$$\mathrm{d}S(t) \;=\; S(t)\big(\mu \;\mathrm{d}t \;+\; \sigma \;\mathrm{d}B(t)\big). \tag{4.1}$$

At $t = T_n$ the jump of $S(t)$ is given by

$$\Delta S_n \;=\; S(T_n) - S(T_n-) \;=\; S(T_n-)Y_n$$

so that

$$S(T_n) \;=\; S(T_n-)(1+Y_n)$$

which, by the assumption $Y_n > -1$, leads always to positive values of the prices.

Now, we consider the *fractional Black-Scholes model with jumps* defined similarly to (3.11) by the following stochastic differential equation

$$\begin{aligned} \mathrm{d}S(t) \;&=\; S(t)\,\big(\mu\,\mathrm{d}t + \;\sigma\,\mathrm{d}B(t)\big) \;+\; S(t-)Y_t\,\mathrm{d}N(t), \\ S(t)|_{t=0} \;&=\; S(0). \end{aligned} \tag{4.2}$$

Here $B(t) = \int_0^t (t-s)^{H-\frac{1}{2}}\mathrm{d}Z(s)$ and H is the Hurst index, $0 < H < 1$.

The corresponding approximate model of (4.2) is defined for each $\varepsilon > 0$ by

$$\begin{aligned} \mathrm{d}S_\varepsilon(t) \;&=\; S_\varepsilon(t)\,\big(\mu\,\mathrm{d}t \;+\; \sigma\,\mathrm{d}B_\varepsilon(t)\big) \;+\; S_\varepsilon(t-)Y_t\,\mathrm{d}N(t), \\ S_\varepsilon(t)|_{t=0} \;&=\; S(0)\ \text{(same initial condition as in (4.2))}, \end{aligned} \tag{4.3}$$

where $B_\varepsilon(t) = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}}\mathrm{d}Z(s)$. We can prove that:

(i) $B_\varepsilon(t)$ is a semimartingale and $B_\varepsilon(t) \to B(t)$, in $L^2(\Omega)$, $t \in [0,T]$, as $\varepsilon \to 0$ (this assertion is mentioned already in Chapter I)

(ii) The solution $S_\varepsilon(t)$ of (4.3) converges in $L^2(\Omega)$ to the exact solution $S(t)$ of (4.2) as $\varepsilon \to 0$.

Furthermore, the convergence mentioned in (i) and (ii) are uniform with respect to t. That is we have the following theorem.

***THEOREM* 4.1.** *The solution of (4.3), for $||S(0)||^2 = E|S(0)|^2 < \infty$, is given by*

$$S_\varepsilon(t) \;=\; S(0)\exp\left(-\frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma\varepsilon^\alpha Z(t) + \int_0^t H_\varepsilon(s)\,\mathrm{d}s + \int_0^t \log(1+Y_s)\,\mathrm{d}N(s)\right),$$

where $\sigma = H - \frac{1}{2}$,

$$H_\varepsilon(t) = \mu \;+\; \alpha\sigma\int_0^t (t-s+\varepsilon)^\alpha\,\mathrm{d}Z(s).$$

Furthermore, for $H > 1/2$ the stochastic process $S_*(t)$ defined by

$$S_*(t) \;=\; S(0)\exp\left(\mu t \;+\; \sigma B(t) \;+\; \int_0^t \log(1+Y_s)\;\mathrm{d}N(s)\right)$$

is the limit in $L^2(\Omega)$ of $S_\varepsilon(t)$ as $\varepsilon \to 0$. This limit is uniform with respect to $t \in [0,T]$.

PROOF: Replacing $\mathrm{d}B_\varepsilon(t) = \alpha\varphi_\varepsilon(t)\mathrm{d}t + \varepsilon^\alpha \mathrm{d}Z(t)$, where $\varphi_\varepsilon(t) = \int_0^t (t-s+\varepsilon)^{\alpha-1}\mathrm{d}Z(s)$, in equation (4.3), we obtain

$$\mathrm{d}S_\varepsilon(t) \;=\; [\mu + \alpha\sigma\varphi_\varepsilon(t)]S_\varepsilon(t)\;\mathrm{d}t \;+\; \sigma\varepsilon^\alpha S_\varepsilon(t)\;\mathrm{d}Z(t) \;+\; S_\varepsilon(t-)Y_t\;\mathrm{d}N(t), \tag{4.4}$$

or,

$$\begin{aligned}\frac{\mathrm{d}S_\varepsilon(t)}{S_\varepsilon(t)} \;&=\; [\mu + \alpha\sigma\varphi_\varepsilon(t)]\;\mathrm{d}t \;+\; \sigma\varepsilon^\alpha\;\mathrm{d}Z(t) \;+\; \left(\frac{S_\varepsilon(t-)}{S_\varepsilon(t)}\right)Y_t\;\mathrm{d}N(t)\\ &=\; H_\varepsilon(t)\;\mathrm{d}t \;+\; \sigma\varepsilon^\alpha\;\mathrm{d}Z(t) \;+\; \left(\frac{S_\varepsilon(t-)}{S_\varepsilon(t)}\right)Y_t\;\mathrm{d}N(t) \qquad (4.5)\end{aligned}$$

where $H_\varepsilon(t) = \mu t + \alpha\sigma\varphi_\varepsilon(t)$. Moreover, we write equation (4.4) in integral from as

$$\int_0^t \mathrm{d}S_\varepsilon(t) \;=\; \int_0^t H_\varepsilon(s)S_\varepsilon(s)\;\mathrm{d}s \;+\; \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)\;\mathrm{d}Z(s) \;+\; \int_0^t S_\varepsilon(s-)Y_s\;\mathrm{d}N(s)$$

Thus,

$$S_\varepsilon(t) \;=\; S(0) + \int_0^t H_\varepsilon(s)S_\varepsilon(s)\;\mathrm{d}s + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)\;\mathrm{d}Z(s) + \int_0^t S_\varepsilon(s-)Y_s\;\mathrm{d}N(s)$$

Using the formula (3.7), $S_\varepsilon(t)$ can be given in the following equivalent representations

$$S_\varepsilon(t) \;=\; S(0) + \int_0^t H_\varepsilon(s)S_\varepsilon(s)\;\mathrm{d}s + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)\;\mathrm{d}Z(s) + \sum_{n=1}^{N(t)} S_\varepsilon(T_n-)Y_n. \tag{4.6}$$

Let $\Delta S_\varepsilon(T_n) = S_\varepsilon(T_n) - S_\varepsilon(T_n-) = S_\varepsilon(T_n-)Y_n$. Equation (4.6) can then be written:

$$S_\varepsilon(t) \;=\; S(0) \;+\; \int_0^t H_\varepsilon(s)S_\varepsilon(s)\;\mathrm{d}s \;+\; \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)\;\mathrm{d}Z(s) \;+\; \sum_{n=1}^{N(t)} \Delta S_\varepsilon(T_n).$$

Choosing the function $f(x,s) = \log x$ for $x = S_\varepsilon(t) > 0$, direct calculation shows that

$$f_x = \frac{1}{x}, \qquad f_{xx} = -\frac{1}{x^2}, \quad \text{and} \quad f_s = 0.$$

An application of the Itô formula for jump-diffusion processes (see Theorem 3.5 in Chapter II) gives:

$$
\begin{aligned}
\log S_\varepsilon(t) = \; & \log S(0) \; + \; \int_0^t \left(0 + \Big(\frac{1}{S_\varepsilon(s)}\Big) \cdot \Big(H_\varepsilon(s) S_\varepsilon(s)\Big)\right) \,\mathrm{d}s \\
& + \; \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 S_\varepsilon^2(s) \left(- \frac{1}{S_\varepsilon(s)}\right)^2 \,\mathrm{d}s \\
& + \; \int_0^t \left(\frac{1}{S_\varepsilon(s)}\right) (\sigma\varepsilon^\alpha) S_\varepsilon(s) \;\mathrm{d}Z(s) \\
& + \; \sum_{n=1}^{N(t)} \Big[\log\big(S_\varepsilon(T_n-) + \Delta S_\varepsilon(T_n)\big) - \log\big(S_\varepsilon(T_n-)\big)\Big] \\
= \; & \log S(0) \; + \; \int_0^t H_\varepsilon(s) \,\mathrm{d}s \; - \; \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 \,\mathrm{d}s \; + \; \int_0^t \sigma\varepsilon^\alpha \,\mathrm{d}Z(s) \\
& + \; \sum_{n=1}^{N(t)} \left[\log \left(\frac{S_\varepsilon(T_n-)(1+Y_n)}{S_\varepsilon(T_n-)} \right) \right] \\
= \; & \log S(0) \; + \; \int_0^t \Big(H_\varepsilon(s) \,\mathrm{d}s \; + \; \sigma\varepsilon^\alpha \,\mathrm{d}Z(s) \Big) - \; \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 \,\mathrm{d}s \\
& + \; \sum_{n=1}^{N(t)} \log(1+Y_n)
\end{aligned}
\tag{4.7}
$$

Using formula (3.7), equation (4.7) can be given in the following equivalent representations

$$
\begin{aligned}
\log S_\varepsilon(t) = \; & \log S(0) \; + \; \int_0^t \Big(H_\varepsilon(s) \,\mathrm{d}s \; + \; \sigma\varepsilon^\alpha \,\mathrm{d}Z(s) \Big) - \; \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 \,\mathrm{d}s \\
& + \; \int_0^t \log(1+Y_s) \,\mathrm{d}N(s) \\
\overset{by (4.5)}{=} \; & \log S(0) \; + \; \left(\int_0^t \frac{\mathrm{d}S_\varepsilon(s)}{S_\varepsilon(s)} \; - \; \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s \,\mathrm{d}N(s) \right) \; - \; \frac{1}{2}\sigma^2\varepsilon^{2\alpha} t \\
& + \; \int_0^t \log(1+Y_s) \,\mathrm{d}N(s) \\
= \; & \log S(0) \; + \; \int_0^t \frac{\mathrm{d}S_\varepsilon(s)}{S_\varepsilon(s)} \; - \; \frac{1}{2}\sigma^2\varepsilon^{2\alpha} t \; + \; \int_0^t \log(1+Y_s) \,\mathrm{d}N(s) \\
& - \; \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s \,\mathrm{d}N(s).
\end{aligned}
$$

Here Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation. Thus

$$
\int_0^t \frac{\mathrm{d}S_\varepsilon(s)}{S_\varepsilon(s)} \; = \; \log \frac{S_\varepsilon(t)}{S(0)} \; + \; \frac{1}{2}\sigma^2\varepsilon^{2\alpha} t \; - \; \int_0^t \log(1+Y_s) \,\mathrm{d}N(s) \; + \; \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s \,\mathrm{d}N(s).
\tag{4.8}
$$

Combining (4.8) and (4.5) we get

$$\log\frac{S_\varepsilon(t)}{S(0)} + \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t - \int_0^t \log(1+Y_s)\,\mathrm{d}N(s) + \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right)Y_s\,\mathrm{d}N(s)$$
$$= \int_0^t H_\varepsilon(s)\,\mathrm{d}s + \sigma\varepsilon^\alpha Z(t) + \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right)Y_s\,\mathrm{d}N(s).$$

Hence, the solution of (4.3) is

$$S_\varepsilon(t) = S(0)\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma\varepsilon^\alpha Z(t) + \int_0^t H_\varepsilon(s)\,\mathrm{d}s + \int_0^t \log(1+Y_s)\,\mathrm{d}N(s)\right). \tag{4.9}$$

On the other hand,

$$\int_0^t H_\varepsilon(s)\,\mathrm{d}s = \mu t + \alpha\sigma\int_0^t \varphi_\varepsilon(s)\,\mathrm{d}s,$$

and it follows from the semimartigale expression of $B_\varepsilon(t)$ (see the proof of Theorem 2.1) that

$$\int_0^t \varphi_\varepsilon(s)\,\mathrm{d}s = \frac{1}{\alpha}\Big(B_\varepsilon(t) - \varepsilon^\alpha Z(t)\Big).$$

Therefore

$$\int_0^t H_\varepsilon(s)\,\mathrm{d}s = \mu t + \sigma B_\varepsilon(t) - \sigma\varepsilon^\alpha Z(t).$$

Hence, from equation (4.9), we obtain

$$S_\varepsilon(t) = S(0)\exp\left(\mu t - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma B_\varepsilon(t) + \int_0^t \log(1+Y_s)\,\mathrm{d}N(s)\right).$$

One can see that if $\varepsilon \to 0$ and $\alpha = H - \frac{1}{2} > 0$ then $\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t \to 0$, so we have shown (Theorem 2.2) that $B_\varepsilon(t) \to B(t)$ in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$. In fact, let us consider the process $S_*(t)$ defined as

$$S_*(t) = S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_s)\,\mathrm{d}N(s)\right).$$

We are now ready to show that $S_*(t)$ is the limit of $S_\varepsilon(t)$ in $L^2(\Omega)$ as $\varepsilon \to 0$. We have

$$S_\varepsilon(t) - S_*(t)$$
$$= S(0)\exp\left(\mu t - \frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma B_\varepsilon(t) + \int_0^t \log(1+Y_s)\,\mathrm{d}N(s)\right)$$
$$- S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_s)\,\mathrm{d}N(s)\right)$$

Thus

$$
\begin{aligned}
S_\varepsilon(t) \;-\; S_*(t) & \\
&= S(0)\exp\left(\mu t \;+\; \sigma B(t) \;+\; \int_0^t \log(1+Y_s)\ \mathrm{d}N(s)\right)\cdot \\
&\qquad \left[\exp\left(\;-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t \;+\; \sigma(B_\varepsilon(t)-B(t))\right)-1\right] \\
&= S(0)\underbrace{\exp\left(\mu t \;+\; \sigma B(t)\right)}_{(4.10.1)}\cdot\underbrace{\exp\left(\int_0^t \log(1+Y_s)\ \mathrm{d}N(s)\right)}_{(4.10.2)}\cdot \\
&\qquad \underbrace{\left[\exp\left(\;-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t \;+\; \sigma(B_\varepsilon(t)-B(t))\right)-1\right]}_{(4.10.3)}
\end{aligned}
\tag{4.10}
$$

Denoting the norm in $L^2(\Omega)$ by $||\cdot||$, we see that by a hypothesis of the theorem,

$$
||S(0)||^2 \;=\; ES(0)^2 \;<\; \infty.
$$

Moreover, from (4.10.1) we note that

$$
\left|\left|\exp\left(\mu t \;+\; \sigma B(t)\right)\right|\right| \le \exp(\mu t)\cdot\exp\Big(\sigma||B(t)||\Big) \le \exp(\mu T)\cdot\exp\left(\sigma\frac{T^{\alpha+\frac{1}{2}}}{\sqrt{2\alpha+1}}\right)
\tag{4.11}
$$

since, by virtue of the Itô integration isometry,

$$
||B(t)||^2 \;=\; E\left[\int_0^t (t-s)^\alpha\ \mathrm{d}Z(s)\right]^2 \;=\; E\int_0^t (t-s)^{2\alpha}\ \mathrm{d}s \;=\; \frac{t^{2\alpha+1}}{2\alpha+1}.
$$

Secondly, from (4.10.2) we compute

$$
\begin{aligned}
\left|\left|\exp\left(\int_0^t \log(1+Y_s)\ \mathrm{d}N(s)\right)\right|\right| &\;=\; \left|\left|\exp\left(\sum_{n=1}^{N(t)}\log(1+Y_n)\right)\right|\right| \\
&\;=\; \left|\left|\sum_{n=1}^{N(t)}(1+Y_n)\right|\right| \;\le\; K,
\end{aligned}
\tag{4.12}
$$

K a constant. This is due to the finite number of jumps in the finite interval $[0,t]$. Finally, we compute term (4.10.3) on the right hand side of (4.10). It follows from the relation $e^A-1 \;=\; A+\mathrm{o}(A)$ that we have

$$\Bigg|\Bigg|\underbrace{\exp\Bigg(-\frac{1}{2}(\sigma\varepsilon^{\alpha})^2t+\sigma(B_\varepsilon(t)-B(t))\Bigg)-1}_{(4.10.3)}\Bigg|\Bigg|$$

$$\leq \ \Bigg|\Bigg|-\frac{1}{2}(\sigma\varepsilon^{\alpha})^2t+\sigma(B_\varepsilon(t)-B(t))\Bigg|\Bigg|+\mathrm{o}\Bigg(\Bigg|\Bigg|-\frac{1}{2}(\sigma\varepsilon^{\alpha})^2t+\sigma(B_\varepsilon(t)-B(t))\Bigg|\Bigg|\Bigg)$$

$$\leq \ \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t+\sigma||B_\varepsilon(t)-B(t)||+\mathrm{o}\Bigg(\frac{1}{2}(\sigma\varepsilon^{\alpha})^2t+||\sigma(B_\varepsilon(t)-B(t))||\Bigg)$$

It follows from Theorem 2.2 that $||B_\varepsilon(t)-B(t)||\leq C(\alpha)\varepsilon^{\frac{1}{2}+\alpha}$ and $\alpha = H-\frac{1}{2}>0$ (since $H\geq\frac{1}{2}$). Hence

$$\Bigg|\Bigg|\exp\Bigg(-\frac{1}{2}(\sigma\varepsilon^{2\alpha})t+\sigma(B_\varepsilon(t)-B(t))\Bigg)-1\Bigg|\Bigg| \leq \ \frac{1}{2}\sigma^2\varepsilon^{2\alpha}T+\sigma C(\alpha)\varepsilon^{\frac{1}{2}+\alpha} \ + \ \mathrm{o}\Bigg(\frac{1}{2}\sigma^2\varepsilon^{2\alpha}T+\sigma C(\alpha)\varepsilon^{\frac{1}{2}+\alpha}\Bigg). \tag{4.13}$$

The right hand side of (4.13) does not depend on t and approaches zero when $\varepsilon\to 0$. Therefore, one can see from (4.11), (4.12), and, (4.13), that $\big|\big|S_\varepsilon(t)\to S_*(t)\big|\big|$ in $L^2(\Omega)$ as $\varepsilon\to 0$ and the convergence is uniform with respect to t. □

***COLLORARY* 4.2.** *The process $S_*(t)$ is the unique solution of the fractional stock pricing model with jump (4.2).*

PROOF: The uniqueness of this solution follows from that of the L^2-limit. If $S_*(t)$ and $S_*^{'}(t)$ are limits of $S_\varepsilon(t)$ in the $L^2(\Omega)$,

$$||S_*(t)-S_*^{'}(t)|| \ \leq ||S_*(t)-S_\varepsilon(t)|| \ + \ ||S_\varepsilon(t)-S_*^{'}(t)||\to 0, \quad \text{as} \quad \varepsilon\to 0.$$

□

4.2 Introduction to a Fractional Vasicek Model with Jumps

In continuous time $t \geq 0$, the standard definition of the bank interest rate $(r(t), t \geq 0)$ is based on the relation

$$\mathrm{d}B(t) \;=\; r(t)B(t)\;\mathrm{d}t$$

where $(B(t), t \geq 0)$ is a bank account. Clearly,

$$r(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\big(\ln B(t)\big)$$

and

$$B(t) \;=\; B(0)\exp\left(\int_0^t r(s)\;\mathrm{d}s\right).$$

The *interest rate* (also called *short rate* of interest, *spot rate* or *instantaneous rate* of interest), in fact, reflects the price of *borrowing/investing* money from/in the bank. The concept of interest rate plays an even more important role in the "indirect data" of the evaluation of share prices. This explains why there are a variety of models with interest rate $(r(t), t \geq 0)$ described by diffusion equations

$$\mathrm{d}r(t) \;=\; \mu(r(t),t)\;\mathrm{d}t \;+\; \sigma(r(t),t)\;\mathrm{d}Z(t), \tag{4.14}$$

where $(\mu(x,t), t \geq 0)$, $(\sigma(x,t), t \geq 0)$ are given stochastic processes and $(Z(t), t \geq 0)$ is a standard Brownian motion. It is known that the solution of (4.14) is always a Markov process that has no memory. So the model (4.14) is not appropriate since, in the financial markets, each value of $r(t)$ can influence upon its behavior in some time range. Moreover, we know from above that bond prices may indeed jump. This reason disqualifies (4.14) as a valid model.

Recall the classical Vasicek model:

$$\mathrm{d}r(t) \;=\; (b - ar(t))\;\mathrm{d}t \;+\; \sigma\;\mathrm{d}Z(t),$$

a, σ positive constants and b any real number.

Let us consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which we define a standard Brownian motion $(Z(t), t \geq 0)$, a Poisson process $(N(t), t \geq 0)$ with intensity λ and a sequence $(Y_n, n \geq 1)$ of independent, identically distributed random variables taking values in

$(-1, +\infty)$. We will assume that the σ-algebras generated respectively by $(Z(t), t \geq 0)$, $(N(t), t \geq 0)$ and $(Y_n, n \geq 1)$ are independent.

The objective of this section is to model an interest rate with jumps at random times $T_1, \ldots T_n, \ldots$ and suppose that the relative/proportional change in its value at a jump time is given by $Y_1, \ldots, Y_n, \ldots$ respectively. We may then assume that, between two jump times, the bond price $r(t)$ follows a *fractional Vasicek model* for a fractional process $B(t)$: T_n are the jump times of a Poisson process $N(t)$ with intensity λ and that Y_n is a sequence of identically distributed random variables with value in $(-1, \infty)$. This description can be formalized by letting, on the intervals $[T_n, T_{n+1})$,

$$\mathrm{d}r(t) \;=\; (b - ar(t))\; \mathrm{d}t \;+\; \sigma \mathrm{d}B(t),$$

where a, σ are positive constants and b is any real number, while at $t = T_n$ the jump of $S(t)$ is given by

$$\Delta r_n \;=\; r(T_n) - r(T_n-) \;=\; r(T_n-)Y_n,$$

so that

$$r(T_n) \;=\; r(T_n-)(1 + Y_n),$$

which, by the assumption $Y_n > -1$, leads always to positive values of the prices.

Now, we consider the *fractional Vasicek model with jumps* which is defined by the following stochastic differential equation

$$\begin{aligned} \mathrm{d}r(t) \;&=\; (b - ar(t))\; \mathrm{d}t \;+\; \sigma\; \mathrm{d}B(t) \;+\; Y_t\; \mathrm{d}N(t), \\ r(t)|_{t=0} \;&=\; r(0), \end{aligned} \tag{4.15}$$

where Y_t denotes the piecewise constant, left-continuous time interpolation of the sequce of value at a jump time Y_n and $B(t) = \int_0^t (t-s)^\alpha\; \mathrm{d}Z(s)$, $\alpha = H - \frac{1}{2}$, $H \in (0,1)$ and $r(0)$ is a given square integrable random variable.

The stochastic differential equation (4.15) is interpreted as a stochastic integral equation:

$$r(t) \;=\; \int_0^t (b - ar(s))\; \mathrm{d}s \;+\; \sigma B(t) \;+\; \int_0^t Y_s\; \mathrm{d}N(s).$$

4.3 Approximate Fractional Vasicek Model with Jumps

We have seen that the driving process $B(t)$ of (4.15) is not a semimartingale unless $H = \frac{1}{2}$ (Theorem (2.1)). Hence in order to solve (4.15) we consider the approximate equation

$$\mathrm{d}r_\varepsilon(t) \;=\; (b - ar_\varepsilon(t))\;\mathrm{d}s \;+\; \sigma\;\mathrm{d}B_\varepsilon(t) \;+\; Y_t\;\mathrm{d}N(t), \tag{4.16}$$

where $B_\varepsilon(t) = \int_0^t (t-s-\varepsilon)^\alpha \mathrm{d}Z(s)$ is a semimartingale (Theorem 2.1). In fact $B_\varepsilon(t)$ can be expressed (see the proof of Theorem 2.1) as

$$B_\varepsilon(t) \;=\; \alpha \int_0^t \varphi_\varepsilon(s)\;\mathrm{d}s \;+\; \varepsilon^\alpha Z(t), \tag{4.17}$$

where $\varphi_\varepsilon(t) = \int_0^t (t-s-\varepsilon)^{\alpha-1}\mathrm{d}Z(s)$. Writing (4.17) in differential form:

$$\mathrm{d}B_\varepsilon(t) \;=\; \alpha\varphi_\varepsilon(t) \;+\; \varepsilon^\alpha\;\mathrm{d}Z(t)$$

and substituting it into (4.16), we obtain

$$\begin{aligned} \mathrm{d}r_\varepsilon(t) \;&=\; (b - ar_\varepsilon(t))\;\mathrm{d}t \;+\; \sigma\big(\alpha\varphi_\varepsilon(t)\;\mathrm{d}t \;+\; \varepsilon^\alpha\;\mathrm{d}Z(t)\big) \;+\; Y_t\;\mathrm{d}N(t), \\ r_\varepsilon(t)|_{t=0} \;&=\; r(0), \end{aligned} \tag{4.18}$$

where $r(0)$ is given at time $t = 0$. We rewrite (4.18) as:

$$\begin{aligned} \mathrm{d}r_\varepsilon(t) \;&=\; [(b - ar_\varepsilon(t)) \;+\; \sigma\varphi(t)]\mathrm{d}t \;+\; \sigma\varepsilon^\alpha\;\mathrm{d}Z(t) \;+\; Y_t\;\mathrm{d}N(t) \\ r_\varepsilon(t)|_{t=0} \;&=\; r(0). \end{aligned} \tag{4.19}$$

where $\varphi(t) = \alpha\varphi_\varepsilon(t)$. We will solve the approximate model (4.19) (by Theorem 4.5) and then we prove that its solution converges in L^2 to the solution of the original model (by Theorem 4.7).

The following two lemmas resulted from the Theorem (3.4) in Chapter II, which show the existence and uniqueness of the solution of stochastic differential equations (4.15) and (4.16) (Indeed (4.19)).

***LEMMA* 4.3.** *All coefficients of the stochastic differential equation (4.15) satisfies the standard assumptions of the theorem (3.4). Thus there exists a unique solution of the stochastic differential equation (4.15).*

PROOF: Rewrite equation (4.15) in integral form:

$$r(t) \;=\; r(0) \;+\; \int_0^t (b - ar(s))\;\mathrm{d}s \;+\; \sigma B(t) \;+\; \int_0^t Y_s\;\mathrm{d}N(s).$$

Since $B(t) = \int_0^t (t-s)^\alpha \mathrm{d}Z(s)$, where $\alpha = H - \frac{1}{2}$ and $\frac{1}{2} \le H < 1$ (that is $0 \le \alpha < \frac{1}{2}$), we obtain

$$r(t) \;=\; r(0) \;+\; \int_0^t (b - ar(s))\;\mathrm{d}s \;+\; \int_0^t \sigma(t-s)^\alpha\;\mathrm{d}Z(s) \;+\; \int_0^t Y_s\;\mathrm{d}N(s) \qquad (4.20)$$

According to Theorem (3.4) in Chapter II, comparing the coefficients of (4.20) with (3.5), we have $a(r,s) = b - ar$, $b(r,s) = \sigma(t-s)^\alpha$ and $c(r,s) = Y_s$, where a, σ are positive constants, b is any real number and $r = r(s)$.

We will check standard assumptions one at a time: Firstly, since b and a are constant, with r integrable on each finite time interval, assumption (i) holds.

Secondly, we know that the process Y_s can only jump finitely many times in each finite time interval. By hypothesis in section 4.2, when Y_n is the relative change in its value at a jump time T_n and Y_s is obtained from Y_n by a piecewise constant and left continuous time interpolation, then the trajectory of Y_s is piecewise constant. Thus Y_s is finitely integrable on each finite time interval.

Thirdly, we will check that the function $a(r,s)$ and $b(r,s)$ are bounded by the some constant C. Let us consider, for $0 < s < t \le T$. Since $\alpha = H - \frac{1}{2}$ and $\frac{1}{2} \le H < 1$, then $0 \le \alpha < \frac{1}{2}$:

$$\begin{aligned}
|a(r,s)|^2 + |b(r,s)|^2 &= |b - ar|^2 + |\sigma(t-s)^\alpha|^2 \\
&\le (|b| + |ar|)^2 + |\sigma|^2 |(t-s)^\alpha|^2 \\
&\le b^2 + 2a|b||r| + a^2 r^2 + \sigma^2 (t-s)^{2\alpha}, \qquad (4.21) \\
&\quad \Big(\text{since } a, r \ge 0, b \in \mathbb{R}, (t-s) > 0 \text{ and } \alpha \in [0, \frac{1}{2})\Big) \\
&= b^2 + 2a|b|r + \sigma^2 (t-s)^{2\alpha} + a^2 r^2.
\end{aligned}$$

In the case $\alpha = 0$, it easy to see that $C := (2b^2 + \sigma^2)a^2$ and in the case $0 < \alpha < \frac{1}{2}$, since

$$(t-s)^{2\alpha} \;=\; e^{2\alpha \log(t-s)} \;\le\; e^{2\alpha \log(t)} \le e^{2\alpha \log(T)} \;=:\; C_1.$$

Thus (4.21) become

$$\begin{aligned} b^2 + 2a|b|r + \sigma^2(t-s)^{2\alpha} + a^2r^2 &\le b^2 + 2a^2b^2r^2 + C_1\sigma^2 + a^2r^2 \\ &\le (2b^2 + C_1\sigma^2)a^2 + (2b^2 + C_1\sigma^2)a^2r^2 \\ &=: C[1+r^2], \end{aligned}$$

where $C := (2b^2 + C_1\sigma^2)a^2$. Fourthly, to check that fuction $a(r,s)$ satisfy a uniform Lipschitz-condition with some constant L, direct calculation show us,

$$|a(r_1(s),s) - a(r_2(s),s)| = |b - ar_1 - b + ar_2| = |-ar_1 + ar_2| = a|r_1 - r_2|,$$

that is $L = a$. Finally, obviously

$$|b(r_1(s),s) - b(r_2(s),s)| = |\sigma(t-s)^\alpha - \sigma(t-s)^\alpha| = 0 \le L^2|r_1 - r_2|^2.$$

Thus all standard assumptions hold.

Existence and uniqueness of solution of stochastic differential equation (4.15) follows from above that all coefficients of the stochastic differential equation (4.15) satisfies all standard assumptions of Theorem 3.4, hence there exists a unique solution $r(t)$ of the stochastic differential equation (4.15). □

Now, consider the stochastic differential equation (4.16). As in the proof of Lemma 4.3, if we replace $r(t)$ by $r_\varepsilon(t)$ and $\sigma(t-s)^\alpha$ by $\sigma(t-s-\varepsilon)^\alpha$, then the following lemma holds:

***LEMMA* 4.4.** *All coefficients of the stochastic differential equation (4.16) satisfies standard assumptions of the theorem (3.4). Thus there exists a unique solution of the stochastic differential equation (4.16).*

***REMARK* 4.1.** *As above we can represent the stochastic differential equation (4.16) by (4.19). Hence, it follows from lemma 4.4 that there exists a unique solution of the stochastic differential equation (4.19).*

The following theorem yields the solution $r_\varepsilon(t)$ to the problem (4.19). In fact:

***THEOREM* 4.5.** *The solution of the approximate model (4.19) is given by:*

$$\begin{aligned} r_\varepsilon(t) &= \frac{b}{a} + \left(r(0) - \frac{b}{a}\right)e^{-at} + \sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)}\, \mathrm{d}Z(s) \\ &\qquad + \sigma \int_0^t \varphi(s) e^{-a(t-s)}\, \mathrm{d}s + \int_0^t Y_s e^{-a(t-s)}\, \mathrm{d}N(s), \end{aligned}$$

where $0 \le \alpha < \frac{1}{2}$.

PROOF: Let us consider first the stochastic differential equation

$$\begin{aligned} \mathrm{d}x(t) &= (b - ax(t))\, \mathrm{d}t + \sigma\varepsilon^\alpha\, \mathrm{d}Z(t) + Y_t\, \mathrm{d}N(t), \\ x(t)|_{t=0} &= x(0). \end{aligned} \tag{4.22}$$

Set $u(t) = b - ax(t)$. Hence $u(0) = b - a(0)$, $\mathrm{d}x(t) = -\frac{\mathrm{d}u(t)}{a}$ and (4.22) becomes

$$-\frac{\mathrm{d}u(t)}{a} = u(t)\, \mathrm{d}t + \sigma\varepsilon^\alpha\, \mathrm{d}Z(t) + Y_t\, \mathrm{d}N(t),$$

or

$$\mathrm{d}u(t) = -au(t)\, \mathrm{d}t - a\sigma\varepsilon^\alpha\, \mathrm{d}Z(t) - a\, Y_t\, \mathrm{d}N(t). \tag{4.23}$$

The equation (4.23) is in fact the classical stochastic Langevin equation with jumps. To solve this equation, consider the process $v(t) = u(t)e^{at}$. Using the differential of the product rule and the stochastic differential equation (4.23), we have

$$\begin{aligned} \mathrm{d}v(t) &= e^{at}\, \mathrm{d}u(t) + ae^{at}u(t)\, \mathrm{d}t \\ &= e^{at}\big(\mathrm{d}u(t) + au(t)\, \mathrm{d}t\big) \\ &= e^{at}\big(-a\sigma\varepsilon^\alpha\, \mathrm{d}Z(t) - aY_t\, \mathrm{d}N(t)\big). \end{aligned}$$

This gives

$$v(t) - v(0) = -a\sigma\varepsilon^\alpha \int_0^t e^{as}\, \mathrm{d}Z(s) - a\int_0^t Y_s e^{as}\, \mathrm{d}N(s).$$

Now the solution for x(t) (i.e. the solution of the classical stochastic Langevin equation with jumps) is:

$$u(t) = u(0)e^{-at} - a\sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)}\, \mathrm{d}Z(s) - a\int_0^t Y_s e^{-a(t-s)}\, \mathrm{d}N(s). \tag{4.24}$$

***REMARK* 4.2.** *This solution is a strong solution, and the process u(t) in (4.23) is called* Ornstein Uhlenbeck process with jumps.

Substituting u(t) as a function of x(t), (4.24) becomes

$$b - ax(t) \;=\; \big(b - ax(0)\big)e^{-at} \;-\; a\sigma\varepsilon^{\alpha}\int_0^t e^{-a(t-s)}\,\mathrm{d}Z(s) \;-\; a\int_0^t Y_s e^{-a(t-s)}\,\mathrm{d}N(s),$$

or

$$x(t) \;=\; \frac{b}{a} - \left(\frac{b}{a} - x(0)\right)e^{-at} + \sigma\varepsilon^{\alpha}\int_0^t e^{-a(t-s)}\,\mathrm{d}Z(s) + \int_0^t Y_s e^{-a(t-s)}\,\mathrm{d}N(s). \quad (4.25)$$

Let us consider further an ordinary differential equation:

$$\begin{aligned} \mathrm{d}y(t) &= -ay(t)\,\mathrm{d}t + \sigma\varphi(t)\,\mathrm{d}t, \\ y(t)|_{t=0} &= y(0). \end{aligned} \quad (4.26)$$

Solving (4.26), we get

$$y(t) \;=\; y(0)e^{-at} \;+\; \sigma\int_0^t \varphi(s)e^{-a(t-s)}\,\mathrm{d}s.$$

Now, let $z(t) := r_\varepsilon(t)$ where $z(t) = x(t) + y(t)$ and $x(0) = y(0) = \frac{r(0)}{2}$. Then

$$\begin{aligned} z(t) &= x(t) + y(t) \\ &= \frac{b}{a} - \left(\frac{b}{a} - x(0)\right)e^{-at} + \sigma\varepsilon^{\alpha}\int_0^t e^{-a(t-s)}\,\mathrm{d}Z(s) + \int_0^t Y_s e^{-a(t-s)}\,\mathrm{d}N(s) \\ &\qquad + y(0)e^{-at} + \sigma\int_0^t \varphi(s)e^{-a(t-s)}\,\mathrm{d}s \\ &= \frac{b}{a} - \frac{b}{a}e^{-at} + (x(0)+y(0))e^{-at} + \sigma\varepsilon^{\alpha}\int_0^t e^{-a(t-s)}\,\mathrm{d}Z(s) \\ &\qquad + \int_0^t Y_s e^{-a(t-s)}\,\mathrm{d}N(s) + \sigma\int_0^t \varphi(s)e^{-a(t-s)}\,\mathrm{d}s \\ &= \frac{b}{a} + \left(r(0) - \frac{b}{a}\right)e^{-at} + \sigma\varepsilon^{\alpha}\int_0^t e^{-a(t-s)}\,\mathrm{d}Z(s) \\ &\qquad + \sigma\int_0^t \varphi(s)e^{-a(t-s)}\,\mathrm{d}s + \int_0^t Y_s e^{-a(t-s)}\,\mathrm{d}N(s), \end{aligned} \quad (4.27)$$

the solution of (4.19). To check this, we note that

$$\begin{aligned} \mathrm{d}z(t) &= \mathrm{d}x(t) + \mathrm{d}y(t) \\ &= (b - ax(t))\,\mathrm{d}t + \sigma\varepsilon^{\alpha}\,\mathrm{d}Z(t) + Y_t\,\mathrm{d}N(t) \\ &\qquad - ay(t)\,\mathrm{d}t + \sigma\varphi(t)\,\mathrm{d}t \\ &= (b - az(t) + \sigma\varphi(t))\,\mathrm{d}t + \sigma\varepsilon^{\alpha}\,\mathrm{d}Z(t) + Y_t\,\mathrm{d}N(t), \end{aligned}$$

with $z(0) = x(0) + y(0) = r(0)$, which is indeed the solution of problem (4.19). Therefore, by uniqueness of the solution of (4.19) (Remark 4.1) we get (4.27) is the solution to (4.19). □

A natural question arises as to whether the solution $r_\varepsilon(t)$ of (4.16) would converge to the solution $r(t)$ of (4.15). The following theorem will be used in the proof of Theorem 4.7.

THEOREM 4.6. Gronwall's lemma *If, for* $0 \le t \le T$, $f(t) \ge 0$ *and* $g(t) \ge 0$ *are continuous functions such that the inequality*

$$f(t) \;\le\; K + L\int_0^t g(s)f(s)\ \mathrm{d}s$$

holds on $0 \le t \le T$, *with* K *and* L *positive constants, then*

$$f(t) \;\le\; K\exp\left(L\int_0^t g(s)\ \mathrm{d}s\right)$$

on $0 \le t \le T$.

PROOF: see Klebaner (1998) □

4.4 Convergence

Suppose that $r(t)$ and $r_\varepsilon(t)$ are solutions of (4.15) and (4.16), respectively:

$$\mathrm{d}r(t) \;=\; (b - ar(t))\ \mathrm{d}t \;+\; \sigma\ \mathrm{d}B(t) \;+\; Y_t\ \mathrm{d}N(t), \quad 0 \le t \le T,$$

and

$$\mathrm{d}r_\varepsilon(t) \;=\; (b - ar_\varepsilon(t))\ \mathrm{d}t \;+\; \sigma\ \mathrm{d}B_\varepsilon(t) \;+\; Y_t\ \mathrm{d}N(t), \quad 0 \le t \le T.$$

THEOREM 4.7. $r_\varepsilon(t)$ *converges to* $r(t)$ *uniformly in* $L^2(\Omega)$ *as* $\varepsilon \to 0$.

PROOF: Since

$$r(t) - r_\varepsilon(t) \;=\; -a\int_0^t (r(s) - r_\varepsilon(s))\ \mathrm{d}s \;+\; \sigma(B(t) - B_\varepsilon(t)),$$

then

$$||r(t) - r_\varepsilon(t)|| \;\le\; \left|\left|a\int_0^t (r(s) - r_\varepsilon(s))\ \mathrm{d}s\right|\right| \;+\; \sigma||B(t) - B_\varepsilon(t)||, \tag{4.28}$$

where $||\cdot||$ denotes the norm in $L^2(\Omega)$. Since $B_\varepsilon(t)$ converges to $B(t)$ in $L^2(\Omega)$ where ε tends to zero and this convergence is uniform with respect to $t \in [0,T]$ (Theorem 2.2).

We have

$$\sup_{0\le t\le T} ||B(t) - B_\varepsilon(t)|| \;\le\; C(\alpha)\varepsilon^{\alpha+\frac{1}{2}},$$

where $0 < \alpha < \frac{1}{2}$ and $C(\alpha)$ depends on α (see the proof of Theorem 2.2). Therefore (4.28) becomes

$$||r(t) - r_\varepsilon(t)|| \;\le\; \sigma C(\alpha)\varepsilon^{\alpha+\frac{1}{2}} \;+\; a\int_0^t ||r(s) - r_\varepsilon(s)||\, \mathrm{d}s. \tag{4.29}$$

A standard application of Gronwall's lemma (Theorem 4.6) to equation (4.29) with

$$f(t) = ||r(t) - r_\varepsilon(t)||,\; g(t) = 1,\; K = (\alpha)\varepsilon^{\alpha+\frac{1}{2}},\; \text{and } L = a$$

gives

$$||r(t) - r_\varepsilon(t)|| \;\le\; e^{at}\sigma C(\alpha)\varepsilon^{\alpha+\frac{1}{2}}.$$

It follows that

$$\sup_{0\le t\le T} ||r(t) - r_\varepsilon(t)|| \;\le\; e^{aT}\sigma C(\alpha)\varepsilon^{\alpha+\frac{1}{2}} \;\to\; 0,$$

as $\varepsilon \;\to\; 0$. The proof is completed. □

It is known that fractional Brownian motion differs from standard Brownian motion. One of the differences is that increments of fractional Brownian motion are dependent and exhibit some long memory and may indeed cause jumps. Thus the process $r_\varepsilon(t)$ long time ago may have influence upon its behavior today and may indeed cause jumps. Hence, the fractional Vasicek model with jumps (4.15) reflects the situation in real market more precisely than the classical one.

4.5 Conclusion

We began with a fractional Black-Scholes model with jumps, and have completely solved this model with our approximation approach by Itô formula for jump-diffusion processes by showing that its solution converges to the exact solution.

Finally, we have seen that after introducing a fractional Vasicek model with jumps, we have completely solved this model with our approximate approach by establishing and solving the approximately fractional model with jumps and by showing that its solution converges to the exact solution. These results respond to increasing demands in practice from considering the long range consequences of interest values with jumps.

Chapter V

Applications and Conclustions

In this chapter, the empirical historical data of the Thai Petrochemical Industry (TPI) are simulated by classical Black-Scholes model, Black-Scholes model with jumps and by the approximate fractional Balck-Scholes model with jumps. These paths are illustrated against the empirical data. As expected, the result of simulation indicats that the our pricing model give a better fit with the empirical data.

5.1 Introduction

This chapter will show an application of the approximate fractional Black-Scholes model with jumps. We note that TPI stock prices are a suitable data since the TPI open-prices (Figure 5.1) exhibit many instances of jump-up and jump-down. We estimate parameters from experiment of experience with many samples, so we do not show numerical schemes. We wish not to compare our model to the classical Black-Scholes model or Black-Scholes model with jumps, but rather to illustrate the power of our method. Although in our case of TPI, the results seem to show that the approximate fractional Black-Scholes model with jumps display a better fit than the classical model, nevertheless, it is too premature to declare the results true for other cases.

5.2 Stock-Price Simulation

In this section we give the formula simulation of stock prices. Firstly, we show an approximate solution of the fractional Black-Scholes pricing model:

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z(t)\right), \qquad \forall t \in [0.T]. \tag{5.1}$$

where the random source is an approximate fractional Brownian process $B_\varepsilon(t)$. Secondly, the prices simulated by the classical Black-Scholes pricing model with jumps:

$$S(t) \;=\; S(0)\;\exp\left(\left(\mu \;-\; \frac{\sigma^2}{2}\right)t \;+\; \sigma Z(t) \;+\; \sum_{n=1}^{N(t)}(1+Y_n)\right), \tag{5.2}$$

Finally, the prices simulated by our approximate solution to the fractional Black-Scholes pricing model with jumps:

$$S_\varepsilon(t) \;=\; S(0)\;\exp\left(\left(\mu \;-\; \frac{1}{2}(\sigma\varepsilon^\alpha)^2\right)t \;+\; \sigma B_\varepsilon(t) \;+\; \sum_{n=1}^{N(t)}(1+Y_n)\right), \tag{5.3}$$

For comparative purposed, we compute the average relative percentage error (ARPE):

$$ARPE \;=\; \frac{1}{N}\sum_{k=1}^{N}\frac{|X_k - Y_k|}{X_k}\cdot 100$$

where N is the number of prices, $X = (X_k, k \geq 1)$ is the market prices and $Y = (Y_k, k \geq 1)$ is the model prices.

***REMARK* 5.1.**

(i) For a sample path of fractional process, let us recall that an approximate fractional Brownian motion used in this thesis is

$$B_\varepsilon(t) \;=\; \int_0^t (t - s + \varepsilon)^\alpha \;\mathrm{d}Z(s), \tag{5.4}$$

where $\alpha = H - \frac{1}{2}$, and the Hurst parameter $H \in (0,1)$. Using the same idea of simulation of standard Brownian motion, a sample path of the fractional process (5.4) can be simulated, for fixed $t \geq 0$, as

$$\begin{aligned} B_\varepsilon(t) \;&\simeq\; \sum_{k=1}^{N}(t - \frac{k}{N}t + \varepsilon)^\alpha\left[Z((k+1)\frac{t}{N}) - Z(k\frac{t}{N})\right] \\ &=\; \sum_{k=1}^{N}(t - \frac{k}{N}t + \varepsilon)^\alpha\sqrt{\frac{t}{N}}\left[Z((k+1)) - Z(k)\right] \\ &=\; \sqrt{\frac{t}{N}}\sum_{k=1}^{N}(t - \frac{k}{N}t + \varepsilon)^\alpha g_k \end{aligned}$$

where $g_k \sim \mathcal{N}(0,1)$.

(ii) The explicit solution (5.1) - (5.3) contains three stochastic processes as inputs.

i) Brownian motion process $Z(t), \geq 0$.

ii) The Poisson process $N(t)$ and random sequence $(T_n, Y_n)_{n\geq 1}$, of points in $[0,T] \times [-1,\infty)$.

5.3 Thai Petrochemical Industry (TPI)-Price Simulation

The paper of Cyganowski, Grünce and Kloeden (2002) describes the use of MAPLE for jump-diffusion stochastic differential equations, in particular for the derivation of numerical schemes and contains an implementation of schemes in MAPLE. We used and developed a scheme that was appropriate for our research.

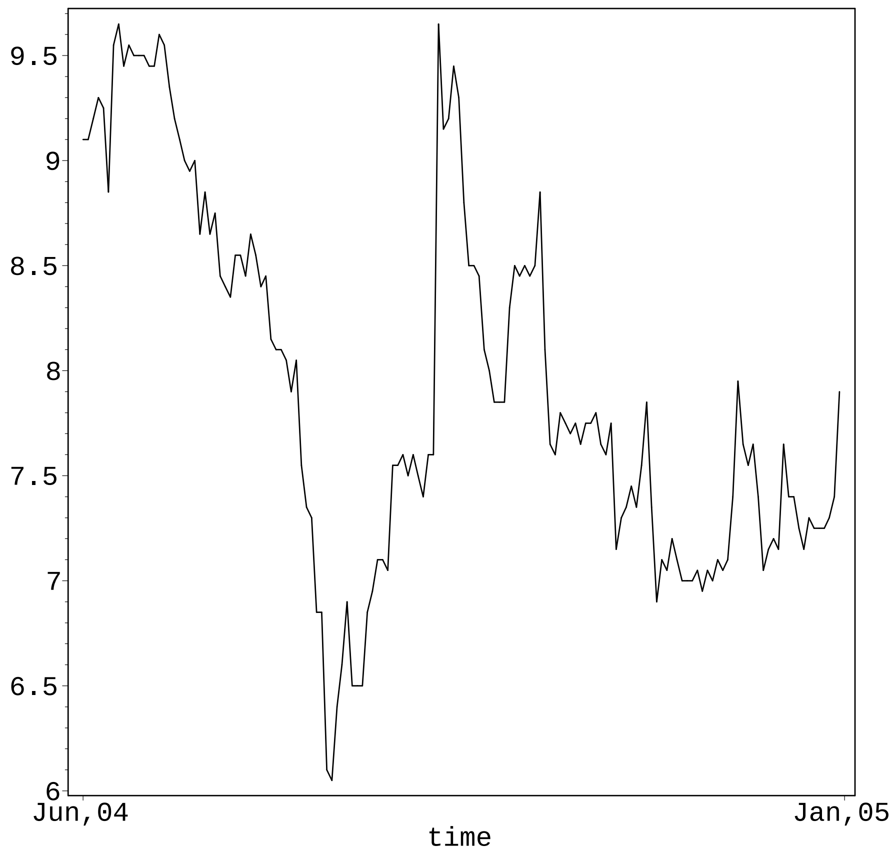

Figure 5.1: TPI open-prices.

An Figure 5.1 shows the daily prices of data set consists of 150 open-prices of TPI starting from June 9, 2004 to January 7, 2005. The historical stock prices was obtained from `http://finance.yahoo.com`.

Let the ARPE by the model (5.1), (5.2) and (5.3) be denoted by ARPE(B), ARPE(BJ) and ARPE(FBJ), respectively. With $\mu = -0.0000725$, $\sigma = 0.3025$, $H = 0.50001$ $\varepsilon = 0.000001$ and parameter for jumps as $\mu_j = 0.00007624$, $\sigma_j = 0.0003679$, $\lambda = 55.46$ and $\gamma = 1$ are fixed. We worked out 500 trails and stored the ARPE(B), ARPE(BJ) and ARPE(FBJ), for each sampling. The results show the following:

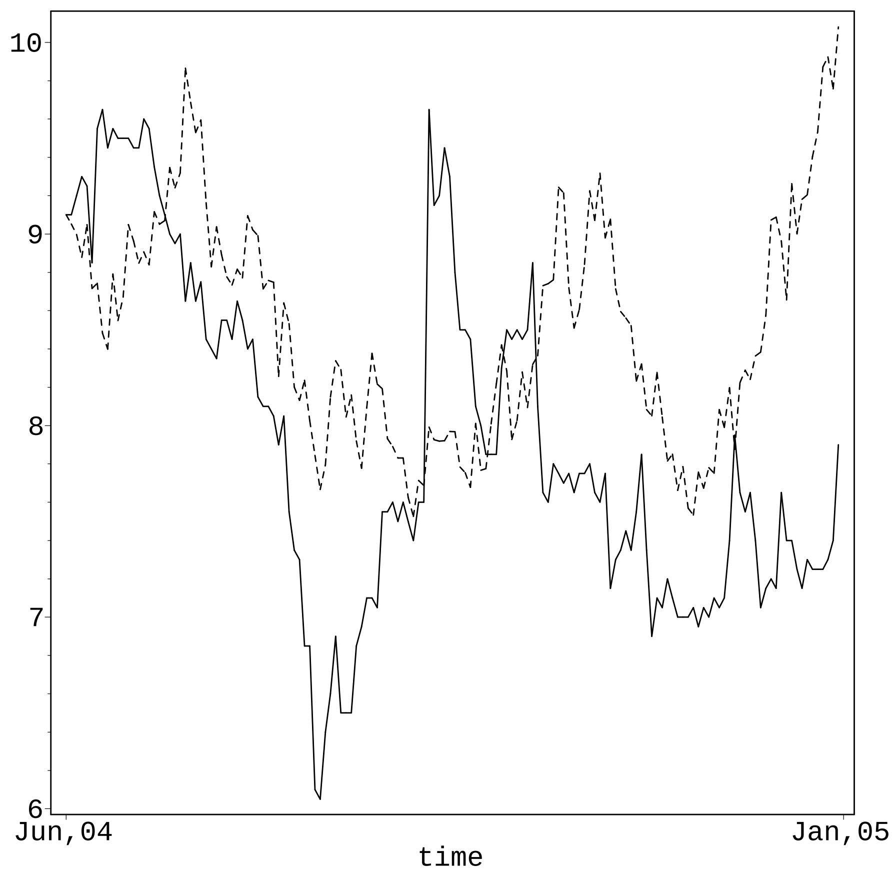

Figure 5.2: TPI open-prices as simulated by classical Black-Scholes pricing model (solid = Empirical data, gray dashed = Simulated by $S(t) = S(0)\exp((\mu - \frac{\sigma^2}{2})t + \sigma Z(t))$, ARPE(B) = 23.69%, and variance = 0.02656)

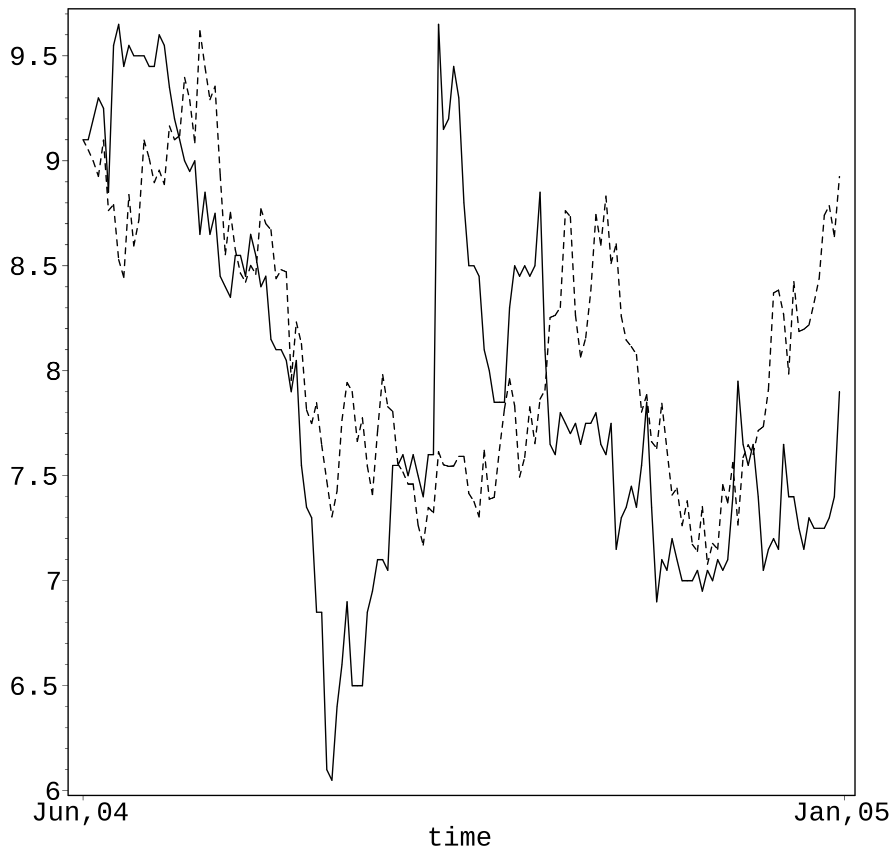

Figure 5.3: TPI open-prices as simulated by Black-Scholes pricing model with jumps (solid = Empirical data, gray dashed = Simulated by $S(t) = S(0)\exp((\mu - \frac{\sigma^2}{2})t + \sigma Z(t) + \sum_{n=1}^{N(t)}(1 + Y_n))$, ARPE(BJ) = 19.64%, and variance = 0.01546)

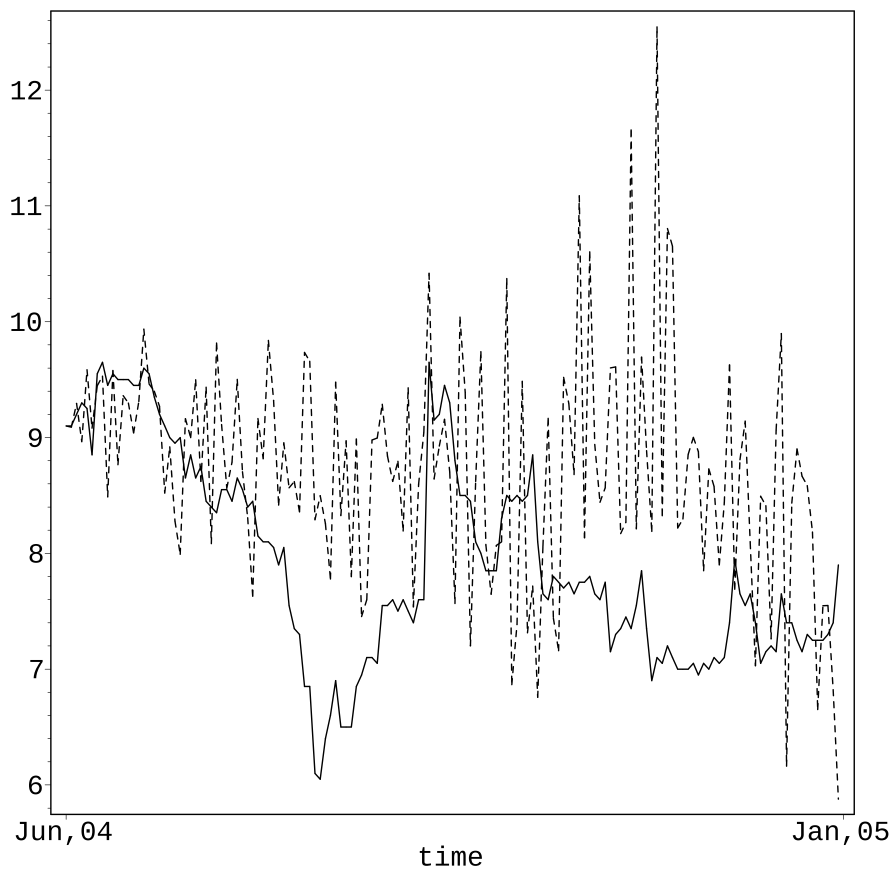

Figure 5.4: TPI open-prices as simulated by fractional Black-Scholes pricing model with jumps (solid = Empirical data, gray dashed = Simulated by $S_\varepsilon(t) = S(0)\exp((\mu - \frac{1}{2}(\sigma\varepsilon^\alpha)^2)t + \sigma B_\varepsilon(t) + \sum_{n=1}^{N(t)}(1+Y_n))$, ARPE(BJ) = 13.54%, and variance = 0.00033)

One can see in case of TPI open-prices (Figures 5.2, 5.3, and 5.4) that the fractional Black-Scholes pricing model with jumps gives better fit with data than classical Black-Scholes pricing model and Black-Scholes pricing model with jumps.

5.4 Suggestions for Further Research

We should observe that further problems can be considered. For instance, one can discuss the arbitrage opportunities in fractional Black-Scholes model with jumps, formulate European or American options pricing formulae, study portfolio optimization for the fractional Black-Scholes model with jumps, and model calibration. More details follow.

An arbitrage opportunity exists if there is a feasible transaction requiring no investment that produces a positive payoff with certainty. To rule out the possibility of arbitrage, zero wealth must be an absorbing state. That is, if an investor's total wealth (including capitalized future wage income, gifts, bequests, and welfare payments) reaches zero, then it remains there. Rogers (1997) show that there exist arbitrage opportunities with fractional Brownian motion although he did not consider the geometric fractional Brownian motion. So, if we add a jump term in fractional Black-Sholes model, it is not enough to exclude arbitrage opportunities in the fractional pricing model with jumps.

Options are generally defined as a "contract between two parties in which one party has the right but not the obligation to do someting, usually to buy or sell some underlying asset". So options are a type of derivative assets since an asset derives its value from some other asset. *Call options* are contracts giving the option holder the right to buy something, while *put options*, conversely, entitle the holder to sell something. The *European option* is such that the option can only be exercised on the expiration date while The *American option* allows the option to be exercised at any time during the life of the option.

Several articles have already focused on the valuation of European options when the underlying value follows a jump diffusion process. Merton (1976) was the first obtain a closed form solution. However work on the analytical valuation formulas for European or American options when the price of the underlying asset evolves as fractional Black-Scholes model with jumps is conspicuous by its absence. In fact the problem of the American option valuation, especially useful for real options, is more complex.

Finally, indeed, the option pricing formula comes from the absence of arbitrage, together with the fact that the option payoff $f(S(T))$ can be replicated by a *hedge portfolio* consisting of stocks and bonds in suitable weights. The option value is the value of the hedge portfolio at time zero, that is, the initial capital needed to establish this replicating portfolio. What does this impact *portfolio optimization* if the asset price is evaluated by the fractional Black-Scholes model with jumps ? Much more work in this direction needs to be done.

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Appendix

Option Pricing Model Driven by a fractional Levy Process

Simulation Part

A MAPLE worksheet by Arthit Intarasit, October 2004

This program is run on Maple V Release 5.1 Version 5.1. We used and developed a schemes of Cyganoski, Grune and Kloedn (2002).

```
> restart:
  with(stats): with(plots): with(plottools):
  ### plotsetup(inline,plotoptions="width=200in,height=300in");
  ### WARNING: persistent store makes one-argument readlib
  obsolete
  readlib(randomize)():
  # ---- Loading data
         # ---- Read data for X-Sever
  emp_data  :=
  readdata("/home/arthit/simulation/tpi01.txt",[float]):
         # ---- Read data for work station
  buffer    := emp_data:

  # ----- Setting amount of data and value of first data
  emp_amount := nops(emp_data):
  emp_data   := array(1..emp_amount):
  for k from 1 to emp_amount do
      emp_data[emp_amount - k + 1] := buffer[k]:
  od:

  # ----- Setting about structure of approximation
  T       := floor(emp_amount/100):
  steps   := emp_amount:
```

```
# ----- Plotting data of TPI
X0  := emp_data[1]:
X   := [0,X0]:

for n from 1 to (steps-1) do
    t := n*T/steps:
    X := X, [t,emp_data[n]]:
od:

[X]:
a := plot([X],color=black,linestyle=1):

# ----- Evaluates jump times
jumps := proc(lambda::algebraic, T::algebraic, mu::algebraic,
sigma::algebraic)
  local i, j, tau, t, again, U, Ulist;
  again := true;
  t[0]  := 0;
  for i from 0 while again=true do
    tau := stats[random, exponential[lambda]](1):
      if (t[i]+tau <= T) then
      t[i+1] := t[i]+tau:
    else
      again := false:
    fi:
  od:

  for j from 1 to i-1 do
    if sigma = 0 then U:=1:
    else U:=exp(stats[random, normald[mu,sigma]](1))-1: fi:
    if (j = 1) then Ulist := [t[j],U]:
    else Ulist := Ulist, [t[j],U]:   fi:
  od;

  if (i = 1) then []:
  else [Ulist]: fi:
end:

# ----- Evaluates jump sum
jumpsum := proc(t::algebraic, U::list, gamm::algebraic)
    local i, j, nj, sum, again;
    sum :=0: j := 0:
    nj  :=nops(U):
```

```
  if (nj>0) then
    again := true:
    for i from 1 while (again) do
      if (t>=U[i,1]) then j := i:
      else again := false: fi:
      if (i=nj) then again := false: fi:
    od:
  fi:

  for i from 1 to j do
    sum := sum + ln(1 + gamm * U[i,2]):
  od:
  sum:
end:

# ----- Evaluates a path Brownian motion
W_path := proc(T,n)
  local w, h, t, alist:
  w     := 0:
  t     := 0:
  h     := T/n:
  alist := [0,w]:
  from 1 to n do
    t := t + h:
    w := w + random[normald[0,sqrt(h)]](1):
    alist := alist,[t,w]:
  od:
  [alist]:
end:

# ----- Evaluates a path Brownian motion
Wt := proc(W,t,T,n)
  local i, dt:
  i  := floor(n*t/T):
  if (i=n) then W[n+1,2]
  else
    dt := t*n/T -i:
    dt*W[i+2,2] + (1-dt)*W[i+1,2]:
  fi:
end:
```

```
# ----- Evaluates a path fractional Brownian motion
fWt := proc(W,t,T,n,steps,e)
  local h, k, fWt, wiener:
  k   := 0:
  fWt := 0:
  h   := T:
  wiener  :=  Wt(W,t,T,W_steps)-Wt(W,t-h,T,W_steps):
  from 1 to steps do
    k   :=  k + 1:
    fWt :=  fWt +
(((t-t/steps+e)^alpha)*sqrt(t/steps*1.0)*wiener):
  od:
fWt:
end:

# ------ Gernerate a discrete path of a Wiener process on [0,T]
with n = 1000 steps
randomize():

W_steps := 1000 :
W       := W_path(T,W_steps):
mu      := -0.0000725  : sigma  := 0.3075:

gamm := 1000           : lambda := 15.25:
muG  := -0.00001       : sigmaG := 0.00001:

randomize():
U     := jumps(lambda,T,muG,sigmaG):
lines := [seq(line([U[i,1],0], [U[i,1],gamm*U[i,2]],
color=black, linestyle=1), i=1..nops(U))]:

St  :=  [0,X0]:
Se  :=  [0,X0]:
Xj  :=  [0,X0]:

e                :=  0.000001:
Hurst_parameter  :=  0.50001:
alpha            := Hurst_parameter-(1/2):

for n from 1 to (steps-1) do
    t  :=  n*T/steps;

    St :=  St,[t,
    X0*exp((mu-sigma^2/2)*t+sigma*Wt(W,t,T,W_steps))]:
```

```
     Xj :=  Xj,[t,

  X0*exp((mu-sigma^2/2)*t+sigma*Wt(W,t,T,W_steps)+jumpsum(t,U,gamm
  ))]:

     Se :=  Se,[t,

  X0*exp((mu-((sigma*(e^Hurst_parameter))^2)*(1/2))*t+sigma*fWt(W,
  t,T,W_steps,steps,e)
     +jumpsum(t,U,gamm))]:
  od:

  [Xj]:
  [St]:
  [Se]:

  b := plot([St],color=black,linestyle=2):
  c := plot([Xj],color=black,linestyle=2):
  d := plot([Se],color=black,linestyle=2):

  plotsetup(ps,plotoutput=‘/home/arthit/simulation/tpi1801_0.ps‘,
  plotoptions="height=380,width=380,noborder,portrait");
  plots[display]({a},titlefont=[HELVETICA,14],numpoints=1000,
  labels=["time",""],font=[COURIER,10],axes=boxed,xtickmarks=[0="J
  un,04",T="Jan,05"]);

  plotsetup(ps,plotoutput=‘/home/arthit/simulation/tpi1801_1.ps‘,
  plotoptions="height=380,width=380,noborder,portrait");
  plots[display]({a,b},titlefont=[HELVETICA,14],numpoints=1000,
  labels=["time",""],font=[COURIER,10],axes=boxed,xtickmarks=[0="J
  un,04",T="Jan,05"]);

  plotsetup(ps,plotoutput=‘/home/arthit/simulation/tpi1801_2.ps‘,
  plotoptions="height=380,width=380,noborder,portrait");
  plots[display]({a,c},titlefont=[HELVETICA,14],numpoints=1000,
  labels=["time",""],font=[COURIER,10],axes=boxed,xtickmarks=[0="J
  un,04",T="Jan,05"]);

  plotsetup(ps,plotoutput=‘/home/arthit/simulation/tpi1801_3.ps‘,
  plotoptions="height=380,width=380,noborder,portrait");
  plots[display]({a,d},titlefont=[HELVETICA,14],numpoints=1000,
  labels=["time",""],font=[COURIER,10],axes=boxed,xtickmarks=[0="J
  un,04",T="Jan,05"]);

Warning, new definition for transform
Warning, computation interrupted
```

```
>


  mu   := -0.0000725   : sigma  := 0.3075:
  e    :=  0.000001:

  Hurst_parameter  :=  0.50001:
  alpha            := Hurst_parameter-(1/2):

> loop_num := 500:
  for k from 1 to loop_num do

      randomize():
      W_steps := 1000 :
      W       := W_path(T,W_steps):

      randomize():
      U      :=  jumps(lambda,T,muG,sigmaG):
      lines  :=  [seq(line([U[i,1],0], [U[i,1],gamm*U[i,2]],
  color=black, linestyle=1), i=1..nops(U))]:


  ################### Black & Scholes pricing models
      St  :=  [0,X0]:
      for n from 1 to (steps-1) do
          t  :=  n*T/steps;
          St :=  St,[t,
          X0*exp((mu-sigma^2/2)*t+sigma*Wt(W,t,T,W_steps))]:
      od:
      [St]:

      if k = 1 then
          Y  :=  [St]:
       else
          Y  :=  Y,[St]:
      fi:

  #################### Black & Scholes pricing models with jumps
      Xj  :=  [0,X0]:
      for n from 1 to (steps-1) do
          t  :=  n*T/steps;
          Xj :=  Xj,[t,

  X0*exp((mu-sigma^2/2)*t+sigma*Wt(W,t,T,W_steps)+jumpsum(t,U,gamm
  ))]:
      od:
      [Xj]:
```

```
    if k = 1 then
        Yj  :=  [Xj]:
     else
        Yj  :=  Yj,[Xj]:
    fi:

#################### fractional Black & Scholes pricing models
with jumps

    Se  :=  [0,X0]:
    for n from 1 to (steps-1) do
        t  :=  n*T/steps;
        Se :=  Se,[t,

X0*exp((mu-((sigma*(e^Hurst_parameter))^2)*(1/2))*t+(sigma*fWt(W
,t,T,W_steps,steps,e))
       +jumpsum(t,U,gamm))]:
    od:
    [Se]:

    if k = 1 then
        Ye  :=  [Se]:
     else
        Ye  :=  Ye,[Se]:
    fi:

od:

# ------ Compute the root mean-square error (RMSE):
RMSE         := array(1..loop_num):
RMSE_final   := 0.0:

RMSE_j       := array(1..loop_num):
RMSE_final_j := 0.0:

RMSE_e       := array(1..loop_num):
RMSE_final_e := 0.0:

for k from 1 to loop_num do
    num   := 0.0:
    num_j := 0.0:
    num_e := 0.0:

    byff  :=  0.0:
```

```
    RMSE_buff   := 0.0:
    RMSE_buff_j := 0.0:
    RMSE_buff_e := 0.0:

    for l from 2 to steps do
        Z     :=  Y[k,l]:
        buff  :=  abs(X[l,2]-Z[2])/X[l,2]:
        num   :=  num + buff:
    od:

    for l from 2 to steps do
        Zj      :=  Yj[k,l]:
        buff    :=  abs(X[l,2]-Zj[2])/X[l,2]:
        num_j   :=  num_j + buff:
    od:

    for l from 2 to steps do
        Ze      :=  Ye[k,l]:
        buff    :=  abs(X[l,2]-Ze[2])/X[l,2]:
        num_e   :=  num_e + buff:
    od:

    RMSE_buff   :=  num/(steps-1):
    RMSE_buff_j :=  num_j/(steps-1):
    RMSE_buff_e :=  num_e/(steps-1):
    RMSE[k]    := RMSE_buff:
    RMSE_final := RMSE_final + RMSE_buff:

    RMSE_j[k]    := RMSE_buff_j:
    RMSE_final_j := RMSE_final_j + RMSE_buff_j:

    RMSE_e[k]    := RMSE_buff_e:
    RMSE_final_e := RMSE_final_e + RMSE_buff_e:
od:

RMSE_final    :=  RMSE_final/loop_num;
RMSE_final_j  :=  RMSE_final_j/loop_num;
RMSE_final_e  :=  RMSE_final_e/loop_num;

for i from 1 to loop_num do
    if i = 1 then
        RMSE_List :=  RMSE[i]:
    else
        RMSE_List := RMSE_List, RMSE[i]:
    fi:
od:
```

```
for i from 1 to loop_num do
    if i = 1 then
        RMSE_List_j :=  RMSE_j[i]:
    else
        RMSE_List_j := RMSE_List_j, RMSE_j[i]:
    fi:
od:

for i from 1 to loop_num do
    if i = 1 then
        RMSE_List_e :=  RMSE_e[i]:
    else
        RMSE_List_e := RMSE_List_e, RMSE_e[i]:
    fi:
od:

[RMSE_List];
[RMSE_List_j];
[RMSE_List_e];
```

$$RMSE\_final := .2367878178$$

$$RMSE\_final\_j := .1964470666$$

$$RMSE\_final\_e := .1354346378$$

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>
```

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