การวิเคราะห์มัลติรีโซลูชันใน $L^2(\mathbb{R}^2)$

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MULTIRESOLUTION ANALYSIS IN $L^2(\mathbb{R}^2)$

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วิทยานิพนธ์นี้อภิปรายการขยายบทนิยามของการวิเคราะห์มัลติรีโซลูชันไปสู่ $L^2(\mathbb{R}^2)$ การเปลี่ยนขนาดถูกกำหนดโดยเมทริกซ์ทแยงมุมแบบเพิ่มขยาย A ที่มีส่วนประกอบเป็นจำนวนเต็ม เมื่อกำหนดเงื่อนไขเพิ่มให้สัมประสิทธิ์การปรับมาตราทำให้เราได้หมู่ของเวฟเลทแม่ r จำนวนโดยที่ $r = |\det A| - 1$ ดังนั้นหมู่ของฟังก์ชันที่คำนวณจากเวฟเลทแม่โดยการเลื่อนขนานด้วยสมาชิกของ $\mathbb{Z}^2$ และการเปลี่ยนขนาดด้วย A ยกกำลังจำนวนเต็ม ประกอบเป็นฐานหลักเชิงตั้งฉากปรกติของ $L^2(\mathbb{R}^2)$ จากนั้นเราขยายขั้นตอนวิธีเชิงรูปพีระมิดไปยัง $L^2(\mathbb{R}^2)$ ตามวิธีธรรมชาติ

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FOURIER SERIES / FOURIER TRANSFORM / WAVELET / MULTIRESOLUTION ANALYSIS / PYRAMIDAL ALGORITHM

This thesis discusses how to extend the definition of multiresolution analysis to $L^2(\mathbb{R}^2)$. Dilation is given by an expanding diagonal matrix A with integer entries. After imposing an additional condition on the scaling coefficients, we obtain a collection of r mother wavelets where $r = |\det A| - 1$, so that the family of functions obtained from the mother wavelets through translating by elements of $\mathbb{Z}^2$ and dilating by integer powers of A form an orthornormal basis of $L^2(\mathbb{R}^2)$. We then extend the pyramidal algorithm to $L^2(\mathbb{R}^2)$ in a natural way.

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Piyanut Puangjumpa

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CHAPTER I

INTRODUCTION

Since its beginning nearly two hundred years ago, Fourier analysis has become an important tool in various fields of science, engineering and mathematics, for example, in signal processing, image processing and solutions of differential equations. Its idea is to consider a given function as an infinite series or as on integral of basic, periodic functions.

For example, let $f(x)$ be a $2L$-periodic, locally integrable function. The Fourier coefficients of $f(x)$ are given by

$$\hat{f}(n) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-i\pi nx}{L}}\,dx.$$

If f is square integrable on $[-L, L]$, then as $\left\{\frac{1}{\sqrt{2L}}e^{\frac{i\pi nx}{L}}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2[-L, L]$, the Fourier coefficients $\hat{f}(n)$ are just constant multiples of the coefficients of f in this basis,

$$\hat{f}(n) = \frac{1}{\sqrt{2L}} < f, \frac{1}{\sqrt{2L}}e^{\frac{i\pi nx}{L}} >,$$

and we can reconstruct f from its Fourier coefficients,

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{\frac{i\pi nx}{L}}, \tag{1.1}$$

with convergence in $L^2[-L, L]$. In applications of signal processing for example, $\hat{f}(n)$ is interpreted as the contents of frequency $\frac{n}{2L}$ in the periodic signal $f(x)$.

There are some drawbacks with the Fourier series approach:

1. A function f, which is neither periodic nor compactly supported, cannot be expressed as an infinite series as in (1.1).

2. If we shift f by y and consider the function $g_y(x) = f(x-y)$, then the Fourier coefficients become $\hat{g}(n) = \hat{f}(n)e^{\frac{-i\pi ny}{L}}$. This means that the magnitude of the Fourier coefficients does not change, but only their phase does. From the magnitude of the Fourier coefficients we thus do not know where the support of f is located.

3. In particular, if $f(x)$ has steep gradients or is discontinuous at a point a, then we cannot deduce from the Fourier coefficients $\hat{f}(n)$ where this point is located.

These problems can be overcome by the wavelet series as introduced by Daubechies, Grossmann and Meyer (1986), and Daubechies (1988).

Fix $\psi \in L^2(\mathbb{R})$, and consider the family of functions $\{\psi_{(j,m)}\}_{j,m\in\mathbb{Z}}$, where

$$\psi_{(j,m)}(x) = \sqrt{2}\psi(2^j x - m)$$

is obtained from ψ by dilations and translations. The wavelet coefficients of $f \in L^2(\mathbb{R})$ are defined by the inner product

$$W_\psi f(j,m) = 2^{j/2} \int_{\mathbb{R}} f(x)\overline{\psi(2^j x - m)}\, dx.$$

If $\{\psi_{(j,m)}\}_{j,m\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$, then by Parseval's identity, f can be reconstructed from its wavelet coefficients by

$$f = \sum_{j,m\in\mathbb{Z}} W_\psi f(j,m)\psi_{(j,m)},$$

with convergence in the mean square norm. The problem now is to find a function ψ, called a mother wavelet, such that the family $\{\psi_{(j,m)}\}_{j,m\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. For this, one usually uses the method of multiresolution analysis, as first introduced by Mallat (1989).

A multiresolution of analysis on $L^2(\mathbb{R})$ is a sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying the following properties:

$(M1)\ :\ V_j \subseteq V_{j+1} \qquad \text{for all } j \in \mathbb{Z},$

$(M2)\ :\ \bigcup_{j\in\mathbb{Z}} V_j \quad \text{is dense in } L^2(\mathbb{R}),$

$(M3)\ :\ \bigcap_{j\in\mathbb{Z}} V_j = \{0\},$

$(M4)\ :\ f(x) \in V_0$ if and only if $f(2^j x) \in V_j$, for all j,

$(M5)$: There exists a function $\varphi(x) \in L^2(\mathbb{R})$, called the scaling function, such that the collection $\{\varphi(x-m)\}_{m\in\mathbb{Z}}$ of integer translates is an orthonormal basis of V_0.

It turns out that given a multiresolution analysis, the function ψ defined by

$$\psi = \sum_{m\in\mathbb{Z}} (-1)^m \overline{h_{1-m}} \varphi_{1,m},$$

where $h_m =< \varphi, \varphi_{1,m} >$, is a mother wavelet.

There is an efficient algorithm to compute the wavelet coefficients $W_\psi f(j,m)$, called the pyramidal algorithm. It makes use of the fact that, for each dilation parameter j, we have a decomposition

$$V_j = V_{j-1} \oplus W_{j-1},$$

where $W_{j-1} = \overline{\text{span}\{\psi_{(j-1,m)}\}}_{m\in\mathbb{Z}}$. Fix $f \in L^2(\mathbb{R})$ and let f_j denote the projection of f onto V_j. Since $\{\varphi_{(j,m)}\}_{m\in\mathbb{Z}}$ is an orthonormal basis of V_j, then

$$f_j = \sum_{m\in\mathbb{Z}} c_{j,m} \varphi_{(j,m)}$$

where $c_{j,m} =< f, \varphi_{(j,m)} >$ are called the scaling coefficients of f at scale j. On the other hand, since $\{\varphi_{(j-1,m)}\}_{m\in\mathbb{Z}}$ and $\{\psi_{(j-1,m)}\}_{m\in\mathbb{Z}}$ are orthonormal bases of V_{j-1} and W_{j-1}, respectively, then

$$f_j = \sum_{m\in\mathbb{Z}} c_{j-1,m} \varphi_{(j-1,m)} + \sum_{m\in\mathbb{Z}} d_{j-1,m} \psi_{(j-1,m)},$$

where $c_{j-1,m} =< f, \varphi_{(j-1,m)} >$ and $d_{j-1,m} = W_\psi f(j-1,m) =< f, \psi_{(j-1,m)} >$ are the scaling coefficients, respective wavelet coefficients, of f at scale $j-1$. The pyramidal algorithm allows their computations from the scaling coefficients at scale j by

$$\begin{aligned} c_{j-1,n} &= \sum_{m\in\mathbb{Z}} \overline{h_{m-2n}}\, c_{j,m}, \\ d_{j-1,n} &= \sum_{m\in\mathbb{Z}} (-1)^m \overline{h_{1-m+2n}}\, c_{j,m}. \end{aligned}$$

In reverse, the scaling coefficients at scale j can be obtained from the scaling and wavelet coefficients at scale $j-1$ by

$$c_{j,n} = \sum_{m\in\mathbb{Z}} \left(c_{j-1,m} h_{n-2m} + d_{j-1,m}(-1)^m \overline{h_{1-m+2n}} \right).$$

Note that if the scaling function φ has compact support, then only finitely many terms in these sums are nonzero.

A natural generalization of wavelets to $\mathbb{R}^n$ is as follows. Fix an expanding matrix $A \in GL_n(\mathbb{R})$ with integer entries, and find $\psi \in L^2(\mathbb{R}^n)$ such that $\{|\det A|^{j/2}$ $\psi(A^j x - m)\}_{j\in\mathbb{Z}, m\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. The wavelet coefficients of $f \in L^2(\mathbb{R}^n)$ are then defined by

$$W_\psi f(j,m) = |\det A|^{j/2} \int_{\mathbb{R}^n} f(x) \overline{\psi(A^j x - m)}\, dx,$$

for $x \in \mathbb{R}^n, j \in \mathbb{Z}, m \in \mathbb{Z}^n$. Then,

$$f(x) = \sum_{j\in\mathbb{Z}, m\in\mathbb{Z}^n} W_\psi f(j,m) |\det A|^{j/2} \psi(A^j x - m), \qquad (x \in \mathbb{R}^n)$$

with convergence in $L^2(\mathbb{R}^n)$. So the main problem is to find a function ψ whose dilates and translates form an orthonormal basis of $L^2(\mathbb{R}^n)$.

The questions which naturally arise are:

1. Can one generalize the concept of multiresolution analysis to this multi-dimensional setting, and obtain a function ψ so that $\{|\det A|^{j/2}\psi(A^j x - m)\}_{j\in\mathbb{Z}, m\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$?

2. If yes, is there still a simple algorithm to compute the wavelet coefficients generalizing the Pyramidal algorithm?

In this thesis, we extend the definition of multiresolution analysis to $L^2(\mathbb{R}^2)$. It turns out that we will need a family $\{\psi^r\}_{r\in\mathcal{R}}$ of mother wavelets, where the set $\mathcal{R}$ has cardinality $|\det A| - 1$. Furthermore, we need to impose one additional condition on the scaling coefficients $\{c_{1,m}\}_{m\in\mathbb{Z}^2}$ of the scaling function. We also show that the pyramidal algorithm extends to $L^2(\mathbb{R}^2)$ in a natural way. For ease of notation, we restrict our investigation to the space $L^2(\mathbb{R}^2)$, however, all our results should be easily extendable to $L^2(\mathbb{R}^n)$.

This thesis is organized as follows. In chapter II, we review the basic concepts and theorems from real analysis and Fourier analysis which are required in this thesis. In chapter III, we present the well-known construction of a wavelet basis from a multiresolution analysis. In chapter IV, we then extend the definition of multiresolution analysis to $L^2(\mathbb{R}^2)$, and prove that one can obtain a wavelet basis on $L^2(\mathbb{R}^2)$ from such a multiresolution analysis. We present as an example the construction of Haar wavelets on $L^2(\mathbb{R}^2)$.

CHAPTER II

BACKGROUND

In this chapter, we review the basic concepts from real analysis and Fourier Analysis used in this thesis. Detailed proofs can be found in most standard textbooks, such as Folland (1999) or Cohn (1980), for example.

2.1 The Lebesgue Integral

In this section, we review concepts from measure theory and the construction of the Lebesgue integral. We then introduce the function spaces which we will be working with.

Definition 2.1. *Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a σ-algebra if the following hold:*

1. $\emptyset \in \mathcal{M}, X \in \mathcal{M}$,
2. $\mathcal{S} \in \mathcal{M}$ *implies* $X \backslash S \in \mathcal{M}$,
3. $\mathcal{S}_1, \mathcal{S}_2, \ldots \in \mathcal{M}$ *implies* $\displaystyle\bigcup_{n=1}^{\infty} \mathcal{S}_n \in \mathcal{M}$.

The elements of $\mathcal{M}$ are called measurable sets and the pair $(X, \mathcal{M})$ is called a measurable space.

Definition 2.2. *Let $\mathcal{F}$ be a collection of subsets of X. There exists a smallest σ-algebra containing $\mathcal{F}$, called the σ-algebra generated by $\mathcal{F}$.*

Definition 2.3. *Let X be a topological space. The σ-*algebra *generated by the family of open sets is called the* Borel σ-algebra *on X, denoted $\mathcal{B}_X$. Its elements are called* $\mathcal{B}$-measurable sets *or* Borel sets.

Definition 2.4. *Let $\mathcal{M}$ be a σ-*algebra *of subsets of X. A measure on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ having the following properties:*

1. $\mu(\emptyset) = 0$,

2. *if $\{E_n\}_{n=1}^{\infty}$ is sequence of disjoint measurable set, then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

The triple $(X, \mathcal{M}, \mu)$ is called a measure space.

A measure space $(X, \mathcal{M}, \mu)$ is called complete *if whenever $E \subset A \in \mathcal{M}$ and $\mu(A) = 0$, then $E \in \mathcal{M}$ (and therefore $\mu(E) = 0$).*

*$(X, \mathcal{M}, \mu)$ is called a σ-*finite measure space *if there exists a countable collection $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ such that $\mu(E_n) < \infty \ \forall n$ and $X = \bigcup_{n=1}^{\infty} E_n$.*

The measure which we will be working with is the Lebesgue measure on $\mathbb{R}^n$.

Definition 2.5. *An n-*dimensional interval *in $\mathbb{R}^n$ is defined by*

$$I = I_1 \times I_2 \times \ldots \times I_n$$

*where $I_1, I_2, \ldots, I_n$ are intervals in $\mathbb{R}$. I is called open (or closed) if each I_i is open (or closed) in $\mathbb{R}$. If I is bounded, then its n-*dimensional volume *is defined by*

$$vol(I) = \prod_{i=1}^{n} (b_i - a_i)$$

where a_i, b_i are the left and right end points of I_i.

Definition 2.6. *Let $E \subset \mathbb{R}^n$ be an arbitrary set. Then*

$$\lambda^*(E) = \inf\{\sum_{i=1}^{\infty} vol(I_i) : I_i \text{ is an open } n\text{-interval}, E \subset \bigcup_{i=1}^{\infty} I_i\}$$

is called the Lebesgue outer measure *of E. A set $A \subset \mathbb{R}^n$ is called* Lebesgue-measurable *if for every $E \subset \mathbb{R}^n$,*

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^c).$$

Proposition 2.1. *Let $\mathcal{M}_\lambda = \{A \subset \mathbb{R}^n : A \text{ is Lebesgue-measurable}\}$ and set $\lambda(A) = \lambda^*(A)\ \forall A \in \mathcal{M}_\lambda$. Then*

1. *$\mathcal{M}_\lambda$ is a σ-*algebra*,*
2. *$(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$ is a complete σ-finite measure space,*
3. *$\mathcal{B}_{\mathbb{R}^n} \subset \mathcal{M}_\lambda$.*

We note that λ is called the *Lebesgue measure* and $\mathcal{M}_\lambda$ the *σ-algebra* of *Lebesgue measurable* subsets of $\mathbb{R}^n$. Next we define the notions of measurable function and Lebesgue integral. Throughout, $(X, \mathcal{M}, \mu)$ will denote a measure space.

Definition 2.7. Let $S \subset X$ with $S \in \mathcal{M}$. Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. We call a function $f : S \to \mathbb{R}^*$ extended real valued, and say that it is *$\mathcal{M}$-measurable* if for all $c \in \mathbb{R}$,

$$\{x \in S : f(x) \leq c\} \in \mathcal{M}.$$

A complex-valued function $f : S \to \mathbb{C}$ is called *$\mathcal{M}$-measurable* if $Re(f)$ and $Im(f)$ are *$\mathcal{M}$-measurable.*

Definition 2.8 (Lebesgue Integral)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. Let $\varphi_n = \sum_{k=1}^{n} \alpha_k \chi_k$ where $\chi_{A_k}(x) = \begin{cases} 1 & \text{if} \quad x \in A_k \\ 0 & \text{if} \quad x \notin A_k \end{cases}$, $A_k \in \mathcal{M}$ are disjoint, $\alpha_k \geq 0$. φ is called a *non-negative measurable simple function.* Its integral is defined to be
$$\int_X \varphi_n \, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k).$$
This integral is independent of the choice of the sets A_k.

2. Let $f : X \to [0, \infty]$ be $\mathcal{M}$-*measurable.* By the structure theorem for measurable function, there exists an increasing sequence $\{\varphi_n\}$ of non-negative, finite-valued measurable simple functions converging pointwise to f. We define the integral of f by
$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X \varphi_n \, d\mu.$$
This integral is independent of the choice of the functions φ_n, and is either a non-negative real number, or infinity.

3. Let $f : X \to \mathbb{R}^*$ be Lebesgue measurable and set $f^+ = \max\{0, f\}, f^- = -\min\{0, f\}$. Then f^+, f^- are measurable and non-negative. The *Lebesgue integral of* f is defined by
$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$
provided that $\int_X f^+ \, d\mu, \int_X f^- \, d\mu$ are not both ∞. f is called *integrable* if $\int_X f \, d\mu$ is defined and is finite.

4. A function $f : X \to \mathbb{C}$ is called *integrable* if $Re(f)$ and $Im(f)$ are integrable. In this case, the integral of f is defined by
$$\int_X f \, d\mu = \int_X Re(f) \, d\mu + i \int_X Im(f) \, d\mu.$$

5. If $E \subset X \in \mathcal{M}$, then we set

$$\int_E f\,d\mu = \int_X f\chi_E\,d\mu.$$

When we are working with the measure space $(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$, we often use the notation $\int_{\mathbb{R}^n} f(x)\,dx$ instead of $\int_{\mathbb{R}^n} f\,d\lambda$.

Definition 2.9. A set $E \in \mathcal{M}$ is called a *set of measure zero*, or *null set* if $\mu(E) = 0$.

Let P be a statement about the elements of X, and let $A \in \mathcal{M}$. We say that P holds μ-*almost everywhere* (a.e.) on A if there exists $E \in \mathcal{M}$ so that

1. $\{x \in A : P \textit{ does not hold}\} \subset E$.
2. $\mu(E) = 0$.

Note that if $(X, \mathcal{M}, \mu)$ is complete, then this is equivalent to

$$\mu\{x \in A : P \textit{ does not hold}\} = 0.$$

Next we define the spaces of functions used in this thesis. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f, g : X \to \mathbb{C}$, define $f \sim g$ if and only if $f(x) = g(x)$ a.e. on X. Then "$\sim$" defines an equivalence relation on the real, respectively complex vector space of measurable functions.

Definition 2.10. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. Then $L^p(X, \mathcal{M}, \mu)$ is the set of equivalence class of $\mathcal{M}$-*measurable* function $f : X \to \mathbb{C}$ such that $|f|^p$ is integrable. If $f \sim g$, then $\int_X |f|^p\,d\mu = \int_X |g|^p\,d\mu$. For ease of notation, we usually confuse a function f with its equivalence class in $L^p(X, \mathcal{M}, \mu)$, and simply write

$$L^p(X, \mathcal{M}, \mu) = \left\{ f : X \to \mathbb{C} : \int_X |f|^p\,d\mu < \infty \right\}.$$

Theorem 2.2. *For each p, $1 \leq p < \infty$, $L^p(X, \mathcal{M}, \mu)$ is a complex vector space, the number*

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

is a norm *on $L^p(X, \mathcal{M}, \mu)$, and $L^p(X, \mathcal{M}, \mu)$ is a Banach space in this norm.*

In this thesis, we will mainly deal with the following particular cases:

$$L^p[a,b] := L^p([a,b], \mathcal{M}_\lambda, \lambda) = \{f : [a,b] \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_{[a,b]} |f(x)|^p \, dx < \infty\},$$

$$L^p(\mathbb{R}) := L^p(X, \mathcal{M}, \mu) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_{\mathbb{R}} |f(x)|^p \, dx < \infty\}.$$

Similarly, we will consider

$$L^p[0,1]^n := \{f : [0,1]^n \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_{[0,1]^n} |f(x)|^p \, dx < \infty\},$$

$$L^p(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_{\mathbb{R}^n} |f(x)|^p \, dx < \infty\}.$$

Here, $[0,1]^n$ denotes the unit cube in $\mathbb{R}^n$, $[0,1]^n = \{x = (x_1, \ldots, x_n) : 0 \leq x_i \leq 1 \text{ for all } i\}$. A special case of the change of variables rule is:

Theorem 2.3. *Let A be an $n \times n$ invertible matrix. Then*

$$\int_{\mathbb{R}^n} f(x) \, dx = |\det A| \int_{\mathbb{R}^n} f(Ax) \, dx,$$

for all $f \in L^1(\mathbb{R}^n)$.

Theorem 2.4 (Monotone Convergence Theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\{f_n\}_{n=1}^\infty, f_n \geq 0$, a monotone nondecreasing sequence of measurable functions. Then $\{f_n\}_{n=1}^\infty$ converges pointwise to an extended real valued function. If $f(x) = \lim_{n\to\infty} f_n(x)$, then*

$$\int_X f(x) \, d\mu = \lim_{n\to\infty} \int_X f_n(x) \, d\mu.$$

Theorem 2.5 (Lebesgue Dominated Convergence Theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\{f_n\}_{n=1}^{\infty}$ a sequence of measurable functions. Suppose that*

1. *$\{f_n(x)\}_{n=1}^{\infty}$ converges a.e. on X.*
2. *There exists $g \in L^1(X, \mathcal{M}, \mu)$ such that*

$$|f_n(x)| \leq g(x) \text{ a.e on } X.$$

Set $f(x) = \lim_{n\to\infty} f_n(x)$. Then $f_n, f \in L^1(X, \mathcal{M}, \mu)$ and

$$\int_X f(x)\, d\mu = \lim_{n\to\infty} \int_X f_n(x)\, d\mu.$$

We will make use of the Lebesgue Dominated Convergence Theorem in the following way.

Corollary 2.6. *Let $f_n, f \in L^p(X, \mathcal{M}, \mu), 1 \leq p < \infty$, with $|f_n(x)| \leq |f(x)|\ \forall n$ a.e., and suppose $f_n(x) \to f(x)$ a.e. Then $f_n \to f$ in $\|\cdot\|_p$.*

Proof. By assumption, $|f_n(x) - f(x)|^p \to 0$ a.e. Furthermore,

$$|f_n(x) - f(x)|^p \leq \Big[|f_n(x)| + |f(x)|\Big]^p \leq 2^p |f(x)|^p \in L^1(X, \mathcal{M}, \mu).$$

Applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|f_n - f\|_p^p = \lim_{n\to\infty} \int |f_n(x) - f(x)|^p\, d\mu &= \int \lim_{n\to\infty} |f_n(x) - f(x)|^p\, d\mu \\ = \int 0\, d\mu = 0. \end{aligned}$$

□

Theorem 2.7. *Let $f_n, f \in L^p(X, \mathcal{M}, \mu), 1 \leq p \leq \infty$, such that $f_n \to f$ in the norm of $L^p(X)$. Then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that*

$$f_{n_k}(x) \to f(x)$$

a.e. on X.

Corollary 2.8 (Uniqueness of Limits)**.** *Let* $f_n, f \in L^p(X, \mathcal{M}, \mu), 1 \leq p \leq \infty$*, and* $g : X \to \mathbb{C}$ *be measurable. If* $f_n \to f$ *in the norm of* $L^p(X)$ *and* $f_n(x) \to g(x)$ *a.e., then* $f = g$ *a.e.*

Definition 2.11. Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous. The *support* of f is the set

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

We say that f has *compact support*, if $\operatorname{supp}(f)$ is a *compact set*. We say that f vanishes at infinity, if for each $\varepsilon > 0$ there exists a compact set K such that

$$|f(x)| < \varepsilon \quad \forall\, x \in \mathbb{R}^n \backslash K.$$

Definition 2.12. Let $p \in \{0, 1, 2, \ldots\}$. Set

1. $C^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is } p \text{ times continuously differentiable}\}$.

2. $C_c^p(\mathbb{R}^n) = \{f \in C^p(\mathbb{R}^n) : f \text{ has compact support}\}$.

3. $C^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is infinitely differentiable}\}$.

4. $C_c^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : f \text{ has compact support}\}$.

5. $C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : f \text{ is vanishes at infinity}\}$.

We obviously have that

$$C_c^\infty(\mathbb{R}^n) \subset C_c^p(\mathbb{R}^n) \subset C_0(\mathbb{R}^n).$$

These three spaces are normed linear spaces under the supremum norm,

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

Theorem 2.9. $(C_0(\mathbb{R}^n), \|\cdot\|_\infty)$ *is a Banach space, and* $C_c^\infty(\mathbb{R}^n)$ *is dense in* $C_0(\mathbb{R}^n)$*.*

Theorem 2.10. $C_c^\infty(\mathbb{R}^n)$ *is dense in* $L^p(\mathbb{R}^n)$*, for all* $1 \leq p < \infty$*, in the norm of* $L^p(\mathbb{R}^n)$*.*

Definition 2.13 (Step Functions)**.** Let $I \subset \mathbb{R}$ be an interval. A *step function* on I is a function of the form

$$\varphi = \sum_{k=1}^{n} c_k \chi_{I_k} \tag{2.1}$$

where $c_k \in \mathbb{C}$, and $\{I_k\}_{k=1}^{n}$ are pairwise disjoint bounded subintervals of I.

Here we allow the sets I_k to be empty, or singletons. Furthermore, if $c_k = 0$ for some $k = k_0$ then we may drop the corresponding k_0-th term in (2.1). Let us set

$$St(I) = \{f : I \to \mathbb{C} \mid f \text{ is a step function}\}.$$

It is easy to see that $St(I)$ is a vector space.

Theorem 2.11. *Let $I \subset \mathbb{R}$ be an interval, $1 \leq p < \infty$. Then $St(I)$ is dense in $L^p(I)$. That is, each $f \in L^p(I)$ can be approximated arbitrarily by a step function in the norm of $L^p(I)$.*

The next theorem tells us when the order of integration in an iterated integral can be exchanged. It will be used throughout this thesis.

Theorem 2.12 (Fubini's Theorem)**.** *Let $f : \mathbb{R}^2 \to \mathbb{C}$ be measurable. Then*

1. *$f_y(x) = f(x, y)$ from $\mathbb{R} \to \mathbb{C}$ is measurable for each fixed $y \in \mathbb{R}$ and hence $x \longmapsto |f(x, y)|$ is measurable $\forall\, y \in \mathbb{R}$, $g_x(y) = f(x, y)$ from $\mathbb{R} \to \mathbb{C}$ is measurable for each fixed $x \in \mathbb{R}$ and hence $y \longmapsto |f(x, y)|$ is measurable $\forall\, x \in \mathbb{R}$,*

2. *The functions $h(y) = \int_{\mathbb{R}} |f_y(x)|\, dx = \int_{\mathbb{R}} |f(x, y)|\, dx$ and $k(x) = \int_{\mathbb{R}} |g_x(y)|\, dy = \int_{\mathbb{R}} |f(x, y)|\, dy$ are measurable from $\mathbb{R}$ to $\mathbb{C}$.*

If one of

i) $\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x, y)|\, dx\, dy$

ii) $\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x, y)| \, dy \, dx$

iii) $\int_{\mathbb{R}^2} |f(x, y)| \, d(x, y)$

is finite, then

a) $f_y(x) \in L^1(\mathbb{R})$ for almost all y

$g_x(y) \in L^1(\mathbb{R})$ for almost all x,

b) $h(y) := \int_{\mathbb{R}} f(x, y) \, dx \in L^1(\mathbb{R})$

$k(y) := \int_{\mathbb{R}} f(x, y) \, dy \in L^1(\mathbb{R})$

$f(x, y) \in L^1(\mathbb{R}^2)$

c) Double and iterated integrals are equal

$$\int_{\mathbb{R}^2} f(x, y) \, d(x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \, dx \, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \, dy \, dx.$$

2.2 Fourier Series in One Dimension

One recurrent theme in analysis is the decomposition of a given function into an infinite linear combination of basic functions. For example, the theory of analytic functions deals with functions which can be expressed by a power series, that is, as an infinite linear combination of monomials x^n.

In Fourier analysis, one decomposes functions into infinite linear combinations of trigonometric monomials $e^{2i\pi nx}$; the resulting series are called Fourier series. Since these monomials are periodic, the functions under discussion must also be periodic.

Definition 2.14. A function $f : \mathbb{R} \to \mathbb{C}$ is called *periodic*, if there exists $p \in \mathbb{R}$, $p \neq 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Any such p is called a *period* of f. The smallest such p, if it exists, is called the *basic period* of f.

While Fourier series can be defined for functions of any period, we will only deal with functions of period 1, and thus restrict the definition below to such functions. We set

$$L^p_1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \textit{ is Lebesgue measurable, } 1\textit{-periodic, } f|_{[0,1]} \in L^p[0,1]\}$$

with $\|f\|_p = (\int_0^1 |f(x)|^p\,dx)^{1/p}$. Obviously, $L^p_1(\mathbb{R})$ is isometrically isomorphic to $L^p[0,1]$.

Definition 2.15 (Fourier Series)**.** Let $f \in L^1[0,1]$ (or $f \in L^1_1(\mathbb{R})$). The function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ given by

$$\hat{f}(n) = \int_0^1 f(x)e^{-2i\pi nx}\,dx \tag{2.2}$$

is called the *Fourier transform* of f. The numbers $c_n = c_n(f) := \hat{f}(n)$ are called the Fourier coefficients of f and the formal series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2i\pi nx}$$

is called the Fourier series of f.

Let us set

$$c_0 = c_0(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} \mid \lim_{|n|\to\infty} f(n) = 0 \text{ with norm } \|f\|_\infty = \sup_{n\in\mathbb{Z}} |f(n)|\}$$
$$l^p = l^p(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} \mid \|f\|_p = \left[\sum_{n=-\infty}^{\infty} |f(n)|^p\right]^{1/p} < \infty\} \quad (1 \le p < \infty).$$

Note that $l^p = L^p(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu)$ where $\mathcal{P}(\mathbb{Z})$ is the power set of $\mathbb{Z}$, and μ is the counting measure. In particular, l^p is a Banach space. It is not difficult to prove that c_0 is a Banach space as well.

Theorem 2.13 (Properties of the Fourier Transform)**.** Let $f \in L^1[0,1]$ (or equivalently, $f \in L^1_1(\mathbb{R})$). Then

1. $\hat{f} \in c_0$,

2. $\|\hat{f}\|_\infty \leq \frac{1}{2\pi}\|f\|_1$,

3. The Fourier transform $\mathcal{F} : f \longmapsto \hat{f}$ from $L^1[0,1]$ to c_0 is linear and one-to-one,

4. $\widehat{T_y f}(n) = e^{-2i\pi ny}\hat{f}(n) \quad \forall\, y \in \mathbb{R}$. (Here, T_y denotes translation of an element f of $L_1^1(\mathbb{R})$ by y, $(T_y f)(x) = f(x-y)$.)

Ideally, we would like the Fourier series of f to converge to f in some way. The following theorems discuss pointwise convergence, uniform convergence, and convergence in the square mean.

Definition 2.16 (Piecewise Continuous Function)**.** Let $f : [a,b] \to \mathbb{C}$. We say that f is piecewise continuous on $[a,b]$ if there exist points $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ such that

1. f is continuous on $(x_{k-1}, x_k)\ \forall\, k = 1, \ldots, n$,

2. right-hand and left-hand limits of f at the partition points $x_0, x_1, \ldots, x_n$ exist, that is,
   $f(x_k^+) = \lim_{h\to 0^+} f(x_k + h)$ exists $\forall\, k = 0, \ldots, n-1$,
   $f(x_k^-) = \lim_{h\to 0^+} f(x_k - h)$ exists $\forall\, k = 1, \ldots, n$.

Definition 2.17 (One-Sided Derivatives)**.** Let $f : [a,b] \to \mathbb{C}$ and $x_0 \in (a,b)$. Assume that $f(x_0^+), f(x_0^-)$ exist. The limits

$$f'_R(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0^+)}{h} \qquad f'_L(x_0) = \lim_{h\to 0^+} \frac{f(x_0-h) - f(x_0^-)}{h}$$

if they exist, are called the *right-hand* and *left-hand* derivatives of f at x_0, respectively.

The next theorem shows that if f is differentiable on $[0,1]$, then its Fourier series will converge to f pointwise.

Theorem 2.14 (Pointwise Convergence of the Fourier Series). *Let $f \in L^1_1(\mathbb{R})$, $x_0 \in \mathbb{R}$ and suppose*

1. *$f(x_0^+), f(x_0^-)$ exist,*

2. *$f'_R(x_0), f'_L(x_0)$ exist.*

Then the Fourier series of f converges at x_0, and its limit is the average of the left-hand and right-hand limits of f at x_0,

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2i\pi n x_0} = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

In particular, if f is differentiable at x_0, then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2i\pi n x_0} = f(x_0).$$

Theorem 2.15 (Uniform Convergence of the Fourier Series). *Let $f \in L^1_1(\mathbb{R})$ (or $L^1[0,1]$) be such that $\hat{f} \in l^1$, i.e. $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Set $S_N(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{2i\pi n x}$. Then $\{S_N(x)\}_{N=0}^{\infty}$ converges uniformly on $\mathbb{R}$ to some (continuous) function g, and $f(x) = g(x)$ a.e.*

Theorem 2.16. *Let $\mathcal{H}$ be a Hilbert space, $\{e_n\}_{n=1}^{\infty}$ be an orthonormal subset of $\mathcal{H}$.*

1. *If $\{a_n\}_{n\in\mathbb{N}} \in l^2$, then $\sum_{n=1}^{\infty} a_n e_n$ converges in $\mathcal{H}$, and the limit is independent of the order of summation.*

2. *(Bessel's Inequality) Given $x \in \mathcal{H}$, set $c_n = < x, e_n >$. Then*

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \|x\|^2. \qquad (\textit{That is } \{c_n\}_{n\in\mathbb{N}} \in l^2.)$$

3. The following are equivalent

(a) $\overline{span\{e_n\}_{n=1}^{\infty}} = \mathcal{H}$*,*

(b) Every $x \in \mathcal{H}$ *is written uniquely as*

$$x = \sum_{n=1}^{\infty} c_n e_n = \sum_{n=1}^{\infty} < x, e_n > e_n$$

(i.e. $\{e_n\}_{n=1}^{\infty}$ *is an orthonormal basis or* $\mathcal{H}$*)*

(c) (Parseval's Identity)

$$\sum_{n=1}^{\infty} |c_n|^2 = \|x\|^2 \qquad \textit{for all } x \in \mathcal{H}.$$

Since $L^2[0,1] \subset L^1[0,1]$, the Fourier transform is also defined for $f \in L^2[0,1]$. It is well known that the collection $\{e_n\}_{n\in\mathbb{Z}}$ with $e_n(x) = e^{2i\pi nx}$ forms an orthonormal basis of the Hilbert space $L^2[0,1]$, and hence of $L^2_1(\mathbb{R})$.

Theorem 2.17. *Let* $f \in L^2[0,1]$ *(or* $f \in L^2_1(\mathbb{R})$*). Then*

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2i\pi nx}$$

with convergence in the mean square norm.

Proof. Observe that the Fourier coefficients of $f \in L^2[0,1]$ are of the form $\hat{f}(n) = \int_0^1 f(x) e^{-2i\pi nx}\,dx =< f, e_n >$. Since the collection $\{e_n\}_{n\in\mathbb{Z}}$ given by $e_n(x) = e^{2i\pi nx}$ form an orthonormal basis of $L^2[0,1]$, then by part 3. of theorem 2.16

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2i\pi nx},$$

with convergence in the norm $\|\cdot\|_2$. $\square$

Note that by theorem 2.16,

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \|f\|_2^2,$$

that is, the Fourier transform $f \to \hat{f}$ is an isometric isomorphism of $L^2[0,1]$ onto $l^2(\mathbb{Z})$.

2.3 Fourier Series in Several Dimensions

The concept of Fourier series has a natural extension to periodic functions on $\mathbb{R}^n$. Given vectors ξ, x in $\mathbb{R}^n$, $\xi \cdot x$ will denote the usual inner product,

$$\xi \cdot x =< \xi, x >= \sum_{i=1}^{n} \xi_i x_i$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x = (x_1, x_2, \ldots, x_n)$. If ξ is written as a row vector, and x as a column vector, then the inner product is simply multiplication of a $1 \times n$ matrix by a $n \times 1$ matrix, and we can write $\xi \cdot x = \xi x$. We also set $\underline{1} = \underbrace{(1, 1, \ldots, 1)}_{n-\text{ term}}$.

We call a function $f : \mathbb{R}^n \to \mathbb{C}$ *periodic*, if there exists $s \in \mathbb{R}^n$ such that $f(x + s) = f(x)$ for all $x \in \mathbb{R}^n$. The number s is called a period of f. We set

$$L^p_{\underline{1}}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \mid f \textit{ is Lebesgue measurable, } \underline{1}\textit{-periodic, } f|_{[0,1]^n} \in L^p[0,1]^n\},$$

with $\|f\|_p = (\int_{[0,1]^n} |f(x)|^p \, dx)^{1/p}$.

Definition 2.18. Given $f \in L^1[0,1]^n$ (or $L^1_{\underline{1}}(\mathbb{R}^n)$), the function $\hat{f} : \mathbb{Z}^n \to \mathbb{C}$ given by

$$\hat{f}(k) = \int_{[0,1]^n} f(x) e^{-2i\pi k \cdot x} \, dx,$$

where $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, is called the Fourier transform of f, and the formal series $\sum_{k \in \mathbb{Z}^n} \hat{f}(k) \, e^{2i\pi k \cdot x}$ is called the Fourier series of f.

Since $L^2[0,1]^n \subset L^1[0,1]^n$, the Fourier transform is defined for every $f \in L^2[0,1]^2$ (respectively $L^2_{\underline{1}}(\mathbb{R}^n)$). It is well known that the collection $\{e_k\}_{k \in \mathbb{Z}^n}$ with $e_k(x) = e^{2i\pi k \cdot x}$ forms an orthonormal basis of the Hilbert space $L^2_{\underline{1}}[0,1]^n$. Theorem 2.16 again shows that $f \to \hat{f}$ is an isometric isomorphism of $L^2[0,1]^n$ (respectively $L^2_{\underline{1}}(\mathbb{R}^n)$) onto $l^2(\mathbb{Z}^n)$, and

$$f(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) \, e^{2i\pi k \cdot x}, \qquad \forall \, f \in L^2[0,1]^n,$$

with convergence in the norm of $L^2[0,1]^n$ (respectively $L^2_{\underline{1}}(\mathbb{R}^n)$).

2.4 The Fourier Transform

For functions which are not periodic, one needs to replace the Fourier series by an integral.

Definition 2.19. Given $f \in L^1(\mathbb{R}^n)$, define a function $\hat{f}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2i\pi\xi\cdot x}\,dx \quad (\xi \in \mathbb{R}^n). \tag{2.3}$$

$\hat{f}$ is called the (continuous) *Fourier transform* of f. Similarly, the function

$$\check{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{2i\pi\xi\cdot x}\,dx \quad (\xi \in \mathbb{R}^n)$$

is called the *inverse* (continuous) *Fourier transform* of f.

Example 2.1 (The Fourier Transform of a Characteristic Function). Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a,b]$. If $\xi \neq 0$, then

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}} \chi_{[a,b]}(x)e^{-2i\pi\xi x}\,dx = \int_a^b e^{-2i\pi\xi x}\,dx = \left.\frac{e^{-2i\pi\xi x}}{-2i\pi\xi}\right|_a^b \\
&= -\frac{1}{2i\pi\xi}[e^{-2i\pi\xi b} - e^{-2i\pi\xi a}] \\
&= -\frac{1}{2i\pi\xi}e^{-i\pi\xi b}e^{-i\pi\xi a}[e^{-i\pi\xi b}e^{i\pi\xi a} - e^{-i\pi\xi a}e^{i\pi\xi b}] \\
&= \frac{e^{-i\pi\xi(b+a)}}{\pi\xi}\left[\frac{e^{i\pi\xi(b-a)} - e^{-i\pi\xi(b-a)}}{2i}\right] = \frac{1}{\pi\xi}e^{-i\pi\xi(b+a)}\sin\big(\pi\xi(b-a)\big),
\end{aligned}$$

while

$$\hat{f}(0) = \int_a^b 1\,dx = b - a.$$

Combining both cases, we obtain

$$\hat{f}(\xi) = (b-a)\mathrm{sinc}\big(\pi\xi(b-a)\big)e^{-i\pi\xi(b+a)}$$

where

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Note that $L^2(\mathbb{R}^n) \not\subset L^1(\mathbb{R}^n)$, hence the Fourier transform (2.3) need not be defined for a function $f \in L^2(\mathbb{R}^n)$. However, the next theorem says that the Fourier transform can be defined indirectly for such f. The inverse Fourier transform now takes the place of the Fourier series for reconstructing a function from its Fourier transform.

Theorem 2.18 (Plancherel's Theorem)**.** *The restriction of the Fourier transform $f \mapsto \hat{f}$ and of the inverse Fourier transform $f \mapsto \check{f}$ to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ are isometries*

$$\begin{aligned}
\mathcal{F} \;&:\; L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \\
\bar{\mathcal{F}} \;&:\; L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),
\end{aligned}$$

with respect to $\|\cdot\|_2$, and extend uniquely to surjective linear isometries

$$\begin{aligned}
\widetilde{\mathcal{F}} \;&:\; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \\
\widetilde{\bar{\mathcal{F}}} \;&:\; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),
\end{aligned}$$

satisfying $\widetilde{\bar{\mathcal{F}}}(\widetilde{\mathcal{F}}(f)) = \widetilde{\mathcal{F}}(\widetilde{\bar{\mathcal{F}}}(f)) = f, \quad \forall f \in L^2(\mathbb{R}^n)$.

We usually use the same symbols $\mathcal{F}$, respectively $\bar{\mathcal{F}}$ to denote these extensions, and call them the Fourier transform, respectively inverse Fourier transform, on $L^2(\mathbb{R}^n)$.

CHAPTER III

WAVELETS IN $L^2(\mathbb{R})$

In this chapter, we review the classic construction of wavelet bases in $L^2(\mathbb{R})$ by means of a multiresolution analysis. The reader may find further details and proofs in Mallat (1989), Daubechies (1992), Hernándes and Weiss (1996), and Walnut (2002). In chapter IV, these concepts will be extended to $L^2(\mathbb{R}^2)$. Thus some of the theorems in this section will be stated without proof, as their proofs will be given in the next chapter.

3.1 Multiresolution Analysis in $L^2(\mathbb{R})$

In wavelet analysis, one wants to find a function $\psi \in L^2(\mathbb{R})$ called a mother wavelet, such that the family of translates and dilates of ψ,

$$\{\psi_{(j,n)}\}_{j,n\in\mathbb{Z}} \text{ with } \psi_{(j,n)} = 2^{j/2}\psi(2^j x - n)$$

forms an orthonormal basis of $L^2(\mathbb{R})$. Such a basis is called a *wavelet* basis. This is usually achieved through a multiresolution analysis, as will be explained below. First we need to discuss some preliminary concepts.

The dilation, translation and modulation operators on $L^2(\mathbb{R})$ are defined by

$$(D_a f)(x) = \sqrt{a}f(ax) \qquad (a \in \mathbb{R}^+),$$

$$(T_b f)(x) = f(x-b) \qquad (b \in \mathbb{R}),$$

$$(E_c f)(x) = e^{2i\pi cx} f(x) \qquad (c \in \mathbb{R}).$$

One easily verifies that these are isometries on $L^2(\mathbb{R})$. Also, $D_{ab} = D_a D_b$ while $T_{c+d} = T_c T_d$ and $E_{c+d} = E_c E_d$, for $a, b \in \mathbb{R}^+,\ c, d \in \mathbb{R}$.

Definition 3.1. A *multiresolution analysis* (MRA) on $L^2(\mathbb{R})$ is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying the following properties:

$(M1)$: $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$,

$(M2)$: $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,

$(M3)$: $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

$(M4)$: the isometry D_{2^j} maps V_0 onto V_j, for each $j \in \mathbb{Z}$,

$(M5)$: there exists a function $\varphi(x) \in L^2(\mathbb{R})$, called the scaling function, such that the collection $\{T_m\varphi\}_{m\in\mathbb{Z}}$ of integer translates of φ is an orthonormal basis of V_0.

Observe that $(M4)$ is equivalent to $D_2 V_j = V_{j+1}$ for all j, this is easily proved by induction.

For each $m \in \mathbb{Z}$, set $\varphi_{(1,m)} = D_2 T_m \varphi$. Then by $(M4)$ an $(M5)$, the family $\{\varphi_{(1,m)}\}_{m\in\mathbb{Z}}$, is an orthonormal basis of V_1. Now, by $(M1)$, $\varphi \in V_1$ as well, so that we can express φ in term of this basis of V_1 by

$$\varphi = \sum_{m\in\mathbb{Z}} h_m \varphi_{(1,m)}, \qquad \text{where} \quad h_m =< \varphi, \varphi_{(1,m)} > . \tag{3.1}$$

The sequence $\{h_m\}_{m\in\mathbb{Z}} \in l^2(\mathbb{Z})$ is called the *scaling filter*. In fact by Parseval's identity,

$$\sum_{m\in\mathbb{Z}^2} |h_m|^2 = \|\varphi\|_2^2 = 1. \tag{3.2}$$

Many of the properties of functions discussed in wavelet analysis have elegant characterizations on the Fourier transform side. The next two theorems are examples of this correspondence. The first theorem characterizes those functions $g(x)$ for which the collection $\{g(x-m)\}_{m\in\mathbb{Z}}$ forms an orthonormal set. The second theorem characterizes those functions f which lie in the subspace of $L^2(\mathbb{R})$ spanned by such an orthonormal set.

Theorem 3.1. *Let $g \in L^2(\mathbb{R})$. Then the system of translates $\{T_m g\}_{m\in\mathbb{Z}}$ is an orthonormal set if and only if $\sum_{m\in\mathbb{Z}} |\hat{g}(\xi+m)|^2 = 1$ for almost all $\xi \in \mathbb{R}$.*

Theorem 3.2. *Let $g \in L^2(\mathbb{R})$ be such that the system of translates $\{T_m g\}_{m\in\mathbb{Z}}$ is an orthonormal set, and let $V = \overline{\operatorname{span}\{T_m g\}_{m\in\mathbb{Z}}}$. Let $f \in L^2(\mathbb{R})$ be arbitrary. Then $f \in V$ if and only if there exists $h \in L^2_1(\mathbb{R})$ such that $\hat{f}(\xi) = \hat{g}(\xi)h(\xi)$ a.e.*

Definition 3.2 (Mother Wavelet)**.** Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis for $L^2(\mathbb{R})$ with scaling function φ, and let $\{h_m\}_{m\in\mathbb{Z}}$ be the scaling filter. For each $m \in \mathbb{Z}$, set $g_m = (-1)^m \overline{h_{1-m}}\,(m \in \mathbb{Z})$. Then

$$\sum_{m\in\mathbb{Z}} |g_m|^2 = \sum_{m\in\mathbb{Z}} |h_{1-m}|^2 = \sum_{m\in\mathbb{Z}} |h_m|^2 < \infty,$$

so that $\{g_m\}_{m\in\mathbb{Z}} \in l^2(\mathbb{Z})$. In particular, $\{g_m\}$ defines an element ψ of V_1 by

$$\psi = \sum_{m\in\mathbb{Z}} g_m \varphi_{(1,m)}. \tag{3.3}$$

The sequence $\{g_m\}_{m\in\mathbb{Z}} \in l^2(\mathbb{Z})$ is called the *wavelet filter* and ψ is called the *mother wavelet.*

3.2 The Haar Wavelet

The simplest wavelet arising from a multiresolution analysis is the Haar wavelet. It was used by Haar in 1910 to obtain a basis for $L^2(\mathbb{R})$ directly, without the machinery of multiresolution analysis.

Example 3.1 (The Haar Wavelet). For each $n \in \mathbb{Z}$, let V_n denote the set of functions in $L^2(\mathbb{R})$ which are constant on intervals of length $\frac{1}{2^n}$ with dyadic end points, that is,

$$V_n = \left\{ f \in \mathrm{L}^2(\mathbb{R}) : f \text{ is constant a.e. on each interval } \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right] \; ; j \in \mathbb{Z} \right\}.$$

As scaling function φ we choose the characteristic function $\chi_{[0,1)}$ of the unit interval $[0, 1)$. Let us verify that the collection $\{V_n\}_{n\in\mathbb{Z}}$ is a multiresolution analysis.

First we verify that each V_n is a linear subspace of $L^2(\mathbb{R})$. Let $f, g \in V_n,\ \alpha, \beta$ be scalars. Consider an arbitrary interval $I_j := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]$, so that $f(x)$ and $g(x)$ are constant a.e. on I_j, say

$$f(x) = a_j \quad \text{a.e. on } I_j,$$

$$g(x) = b_j \quad \text{a.e. on } I_j.$$

Then, $(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha a_j + \beta b_j$ a.e. on I_j. This shows that $\alpha f + \beta g$ is constant a.e. on I_j, $\forall\, j$, that is, $\alpha f + \beta g \in V_n$. We have shown that V_n is a linear subspace of $L^2(\mathbb{R})$.

Next we show that V_n is closed in $L^2(\mathbb{R})$. Let $\{f_k\}$ be a sequence in V_n converging to some $f \in L^2(\mathbb{R})$. We need to show that $f \in V_n$, that is, f is constant a.e. an each interval I_j. Fix such on interval. Now each f_k is constant a.e. on I_j, say $f_k(x) = c_k$ a.e. on I_j. We claim that $\{c_k\}_{k=1}^{\infty}$ converges in $\mathbb{C}$. Note that $\int_{I_j} |f_k - f|^2 \le \int_{\mathbb{R}} |f_k - f|^2 \to 0$ by assumption, that is $f_k \to f$ in $L^2(I_j)$. In particular, $\{f_k\}$ is Cauchy. That is, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|f_k - f_l\|_{L^2(I_j)} < \varepsilon \cdot \lambda(I_j)^{1/2} \qquad \forall\, k, l \ge N.$$

But

$$\|f_k - f_l\|_{L^2(I_j)} = \left[\int_{I_j} |f_k - f_l|^2\right]^{1/2} = \left[\int_{I_j} |c_k - c_l|^2\right]^{1/2} = |c_k - c_l|\lambda(I_j)^{1/2}$$

which shows that

$$|c_k - c_l| < \varepsilon \qquad \forall\, k, l \ge N$$

and proves the claim. As the sequence $\{c_k\}$ is Cauchy, it converges to some $c \in \mathbb{C}$. That is,

$$f_k(x) = c_k \to c \ \text{ a.e. on } I_j \text{ as } k \to \infty.$$

On the other hand,

$$f_k(x) \to f(x) \text{ in } L^2(I_j) \text{ as } k \to \infty.$$

By uniqueness of limits, $f(x) = c =$ constant a.e. on I_j. We have shown that f is constant on each interval I_j, that is, $f \in V_n$ as well, and hence V_n is closed.

Next we verify that the requirements of a multiresolution analysis are satisfied.

$(M1)$. Let $f \in V_n$ for some n. Then $f(x)$ is constant a.e. on each interval $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]$. We split this interval into two equal halves, namely

1. $\left[\dfrac{j}{2^n}, \dfrac{j}{2^n} + \dfrac{1}{2^{n+1}}\right] = \left[\dfrac{2j}{2^{n+1}}, \dfrac{2j+1}{2^{n+1}}\right]$
   and
2. $\left[\dfrac{j}{2^n} + \dfrac{1}{2^{n+1}}, \dfrac{j+1}{2^n}\right] = \left[\dfrac{2j+1}{2^{n+1}}, \dfrac{2j+2}{2^{n+1}}\right]$.

Obviously, $f(x)$ is constant a.e. on both halves. Since every interval $\left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right]$ is of either form 1. or 2. for some j, depending on whether k is even or odd, it follows that $f(x)$ is constant a.e. on every interval $\left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right]$. That is, $f \in V_{n+1}$, so that $(M1)$ holds.

$(M2)$. Let $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$ be given. We must find $n \in \mathbb{Z}$ and $g \in V_n$ such that $\|f - g\|_2 < \varepsilon$. By theorem 2.11, there exists a step function

$$h = \sum_{k=1}^{N} c_k \chi_{I_k} \qquad (I_k \text{ disjoint intervals})$$

such that $\|f - h\|_2 < \frac{\varepsilon}{2}$. Without loss of generality, we may assume that the intervals I_k are half-open, i.e.

$$h = \sum_{k=1}^{N} c_k \chi_{[a_k, b_k)} \qquad \text{where} \quad I_k = [a_k, b_k).$$

We now modify the endpoints a_k, b_k to suitable dyadic endpoints. Set $M = \max\limits_{k=1\ldots N} |c_k|$, and pick n sufficiently large such that

1. $\dfrac{M\sqrt{2N}}{2^{n/2}} < \dfrac{\varepsilon}{2}$

2. $b_k > a_k + \dfrac{2}{2^n}$ for all $k = 1 \ldots N$.

Next pick $i_k, j_k \in \mathbb{Z}$ such that $\frac{i_k - 1}{2^n} \le a_k < \frac{i_k}{2^n}$, $\frac{j_k}{2^n} \le b_k < \frac{j_k+1}{2^n}$. Note that by 2., $i_k < j_k$. Set $g = \sum_{k=1}^{N} c_k \chi_{\left[\frac{i_k}{2^n}, \frac{j_k}{2^n}\right)} \in V_n$. Then $g(x)$ and $h(x)$ only differ on the intervals $\left[a_k, \frac{i_k}{2^n}\right)$ and $\left[\frac{j_k}{2^n}, b_k\right)$ of length less than $\frac{1}{2^n}$, and on these intervals, $h(x) = c_k$ while $g(x) = 0$. Thus,

$$\begin{aligned}
\|h - g\|_2 &= \left[\int_{\mathbb{R}} |h(x) - g(x)|^2\, dx\right]^{\frac{1}{2}} \\
&= \left[\sum_{k=1}^{N} \left(\int_{a_k}^{\frac{i_k}{2^n}} |h(x) - g(x)|^2\, dx + \int_{\frac{j_k}{2^n}}^{b_k} |h(x) - g(x)|^2\, dx\right)\right]^{\frac{1}{2}} \\
&= \left[\sum_{k=1}^{N} \left(\int_{a_k}^{\frac{i_k}{2^n}} |c_k - 0|^2\, dx + \int_{\frac{j_k}{2^n}}^{b_k} |c_k - 0|^2\, dx\right)\right]^{\frac{1}{2}} \\
&\le \left[\sum_{k=1}^{N} \left(\int_{a_k}^{\frac{i_k}{2^n}} M^2\, dx + \int_{\frac{j_k}{2^n}}^{b_k} M^2\, dx\right)\right]^{\frac{1}{2}} \\
&\le \left[\sum_{k=1}^{N} \left(M^2 \frac{1}{2^n} + M^2 \frac{1}{2^n}\right)\right]^{\frac{1}{2}} \\
&= \left[M^2 N \frac{2}{2^n}\right]^{\frac{1}{2}} = \frac{M\sqrt{2N}}{2^{n/2}} < \frac{\varepsilon}{2}.
\end{aligned}$$

By the triangle inequality,

$$\|f - g\|_2 \le \|f - h\|_2 + \|h - g\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(M2)$ holds.

$(M3)$. Suppose to the contrary that there exists $f \in \bigcap_{n\in\mathbb{Z}} V_n$, $f \neq 0$. Then in particular $f \in V_n$ for all n. Since $f \neq 0$, we can pick a measurable subset E of $\mathbb{R}$, such that $\lambda(E) > 0$ and $f(x) \neq 0$, $\forall\, x \in E$. For each $i \in \mathbb{Z}$, set $E_i = [i, i+1) \cap E$. Since $\bigcup_{i\in\mathbb{Z}} E_i = E$, and the collection $\{E_i\}$ is disjoint, then $0 < \lambda(E) = \sum_{i\in\mathbb{Z}} \lambda(E_i)$. That is, there exists $i_0 \in \mathbb{Z}$ such that $\lambda(E_{i_0}) > 0$. Since $f \in V_0$ then $f(x)$ is constant a.e. on $[i_0, i_0+1]$, say $f(x) = a$ for almost all $x \in [i_0, i_0+1]$. Now E_{i_0} is a subset of nonzero measure of $[i_0, i_0+1]$, and $f(x) \neq 0$, $\forall\, x \in E_{i_0}$, hence $a \neq 0$. So we have shown that $f(x) = a \neq 0$ a.e. on $[i_0, i_0+1]$. In the following we assume that $i_0 \geq 0$, if $i_0 < 0$ we proceed similarly. Now as $f \in V_n \; \forall\, n < 0$, it follows that $f(x)$ is constant a.e. on each interval $\left[0, \frac{1}{2^n}\right) = [0, 2^{-n})$ $(n < 0)$. For sufficiently large negative n, $[i_0, i_0+1) \subset [0, 2^{-n})$. Since $f \in V_n$, $f(x)$ is constant a.e. on $[0, 2^{-n})$. But $f(x) = a$ a.e. on $[i_0, i_0+1)$, hence $f(x) = a$ a.e. on $[0, 2^{-n})$. Replacing n by $-n$, we have shown $f(x) = a \neq 0$ on $[0, 2^n)$ for sufficiently large $n > 0$. Then

$$\begin{aligned} \|f\|_2^2 &= \int_{\mathbb{R}} |f(x)|^2 \, dx \geq \int_0^{2^n} |f(x)|^2 \, dx = \int_0^{2^n} |a|^2 \, dx \\ &= |a|^2 2^n \to \infty \text{ as } n \to \infty \quad (a \neq 0), \end{aligned}$$

contradicting the fact that $f \in L^2(\mathbb{R})$. Thus, $(M3)$ holds.

$(M4)$. Note that for each n

$$\begin{aligned} f(x) \in V_n \iff & f(x) = \text{ constant } = a_j \quad \text{for almost all } x,\ \frac{j}{2^n} \leq x \leq \frac{j+1}{2^n} \quad \forall\, j, \\ \iff & f(2x) = a_j \quad \text{for almost all } 2x,\ \frac{j}{2^n} \leq 2x \leq \frac{j+1}{2^n} \quad \forall\, j, \\ \iff & f(2x) = a_j \quad \text{for almost all } x,\ \frac{j}{2^{n+1}} \leq x \leq \frac{j+1}{2^{n+1}} \quad \forall\, j, \\ \iff & f(2x) \in V_{n+1}, \\ \iff & \sqrt{2} f(2x) = (D_2 f)(x) \in V_{n+1}. \end{aligned}$$

This shows that $D_2 V_n = V_{n+1} \; \forall\, n$, hence $(M4)$ holds.

$(M5)$. First we show that $V_0 = \overline{\text{span}\{T_m \varphi : m \in \mathbb{Z}\}}$. To see this, let $f \in V_0$. Obviously,

$$f(x) = \lim_{N \to \infty} \chi_{[-N,N)}(x) f(x) \quad \text{pointwise a.e.}$$

By the corollary to the Lebesgue Dominated Convergence Theorem,

$$f = \lim_{N \to \infty} \chi_{[-N,N)} f \quad \text{in } \| \cdot \|_2. \tag{3.4}$$

Since $f \in V_0$, $\chi_{[-N,N)}f$ is constant a.e. on all intervals $[m, m+1)$, hence $\chi_{[-N,N)}f$ is a step function, $\chi_{[-N,N)}(x)f(x) = \sum_{m=-N}^{N-1} c_m\chi_{[m,m+1)}(x)$ where c_m is the value a.e. of f on $[m, m+1)$. Thus, (3.4) becomes

$$f(x) = \lim_{N\to\infty} \sum_{m=-N}^{N-1} c_m\chi_{[m,m+1)}(x) = \sum_{m=-\infty}^{\infty} c_m\chi_{[m,m+1)}(x) \tag{3.5}$$

with convergence in $L^2(\mathbb{R})$. On the other hand,

$$T_m\varphi(x) = \chi_{[0,1)}(x-m) = \chi_{[m,m+1)}(x). \tag{3.6}$$

Thus (3.5) becomes.

$$f(x) = \sum_{m=-\infty}^{\infty} c_m T_m\varphi(x).$$

This shows that $f \in \overline{\text{span}\{T_m\varphi\}_{m\in\mathbb{Z}}}$, that is, $V_0 \subset \overline{\text{span}\{T_m\varphi\}_{m\in\mathbb{Z}}}$. To show the reverse inclusion note that by (3.6), $T_m\varphi \in V_0\ \forall\, m$. But V_0 is a closed subspace of $L^2(\mathbb{R})$, hence $\overline{\text{span}\{T_m\varphi\}_{m\in\mathbb{Z}}} \subset V_0$.

It is left to show that $\{T_m\varphi\}_{m\in\mathbb{Z}}$ is an orthonormal set. By (3.6) we have $< T_m\varphi, T_k\varphi >=< \chi_{[m,m+1)}(x), \chi_{[k,k+1)}(x) >= \int_{\mathbb{R}} \chi_{[m,m+1)}(x)\cdot\chi_{[k,k+1)}(x)\,dx = \delta_{m,k}$. Thus $(M5)$ holds.

This shows that the collection $\{V_n\}_{n=-\infty}^{\infty}$ with scaling function $\varphi = \chi_{[0,1)}$ is indeed a multiresolution analysis.

Next we need to compute the scaling filter. Since

$$\varphi_{(1,m)}(x) = D_2T_m\varphi(x) = \sqrt{2}\varphi(2x-m)$$

then

$$\begin{aligned} h_m =< \varphi, \varphi_{(1,m)} > &= \int_{\mathbb{R}} \chi_{[0,1)}(x)\cdot\overline{\sqrt{2}\chi_{[0,1)}(2x-m)}\,dx \\ &= \int_0^1 \sqrt{2}\chi_{[0,1)}(2x-m)\,dx \\ &= \int_0^1 \sqrt{2}\chi_{[\frac{m}{2},\frac{m+1}{2})}(x)\,dx = \begin{cases} \frac{1}{\sqrt{2}} & \text{if} \quad m = 0,1 \\ 0 & \quad\text{else,} \end{cases} \end{aligned}$$

so that the scaling filter $\{h_m\}_{m\in\mathbb{Z}}$ has exactly two nonzero terns. The wavelet filter is then given by

$$g_m = (-1)^m \overline{h_{1-m}} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if} \quad m = 0, \\ -\frac{1}{\sqrt{2}} & \text{if} \quad m = 1, \\ 0 & \quad\text{else,} \end{cases}$$

so that $\{g_m\}_{m\in\mathbb{Z}}$ also has only two nonzero terms. The Haar wavelet is thus defined by

$$\psi = \sum_{m\in\mathbb{Z}} g_m \varphi_{(1,m)} = \frac{1}{\sqrt{2}}\varphi_{(1,0)} - \frac{1}{\sqrt{2}}\varphi_{(1,1)} = \frac{1}{\sqrt{2}}(\varphi_{(1,0)} - \varphi_{(1,1)}).$$

That is,

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2}}\left(\sqrt{2}\varphi(2x) - \sqrt{2}\varphi(2x-1)\right) \\ &= \chi_{\left[0,\frac{1}{2}\right)}(x) - \chi_{\left[\frac{1}{2},1\right)}(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < \frac{1}{2}, \\ -1 & \text{if} \quad \frac{1}{2} \le x < 1, \\ 0 & \quad\text{else.} \end{cases} \end{aligned}$$

The Haar scaling functions and wavelet are shown in figure 3.1.

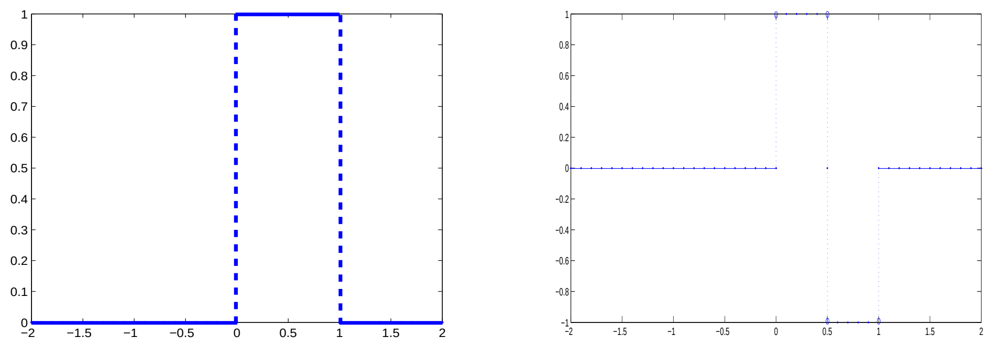

Figure 3.1 The Haar scaling function and the Haar wavelet

3.3 Wavelet Decomposition of $L^2(\mathbb{R})$

We now show how to construct an orthonormal wavelet basis of $L^2(\mathbb{R})$ from a multiresolution analysis.

First we show that any sequence $\{V_j\}_{j\in\mathbb{Z}}$ of subspaces of $L^2(\mathbb{R})$ satisfying $(M1)$-$(M3)$ gives rise to a decomposition

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$$

of $L^2(\mathbb{R})$ into orthogonal subspaces. Then we show that for each j, the family

$$\{\psi_{(j,n)} : j, n \in \mathbb{Z}\}$$

where $\psi_{(j,n)} = 2^{j/2}\psi(2^j x - n)$ forms an orthonormal basis of W_j.

While we are only interested in the space $L^2(\mathbb{R})$, we will separate out those parts of the construction which apply to a general Hilbert space.

So let $\mathcal{H}$ be a Hilbert space, and $\{V_j\}_{j\in\mathbb{Z}}$ be a collection of closed subspaces satisfying $(M1)$-$(M3)$. For each n, let W_n denote the orthogonal complement of V_n in V_{n+1}, so that

$$V_{n+1} = V_n \oplus W_n. \tag{3.7}$$

Start with some space V_k. Then

$$L^2(\mathbb{R}) = V_k \oplus V_k^{\perp}.$$

Applying (3.7) to $n = k-1$, we obtain $V_k = V_{k-1} \oplus W_{k-1}$, and hence

$$L^2(\mathbb{R}) = V_{k-1} \oplus W_{k-1} \oplus V_k^{\perp}.$$

Applying (3.7) again to $n = k-2$, we obtain

$$L^2(\mathbb{R}) = V_{k-2} \oplus W_{k-2} \oplus W_{k-1} \oplus V_k^{\perp}.$$

Continuing inductively, we obtain

$$L^2(\mathbb{R}) = V_{k-i} \oplus W_{k-i} \oplus W_{k-i+1} \oplus \ldots \oplus W_{k-1} \oplus V_k^{\perp},$$

for each $i \geq 1$. Replace $k-i$ by m and $k-1$ by n,

$$L^2(\mathbb{R}) = V_m \oplus W_m \oplus W_{m+1} \oplus \ldots \oplus W_{n-1} \oplus W_n \oplus V_{n+1}^{\perp},$$

for each $m \leq n,\ m, n \in \mathbb{Z}$. That is,

$$L^2(\mathbb{R}) = V_m \oplus \bigoplus_{j=m}^{n} W_j \oplus V_{n+1}^{\perp}. \tag{3.8}$$

As $m \to -\infty$ and $n \to \infty$, we expect that the spaces V_m and $V_{n+1}^{\perp}$ shrink to zero. To prove this fact, we introduce the following operators. For each $n \in \mathbb{Z}$, let P_n denote the orthogonal projection of $L^2(\mathbb{R})$ onto V_n. That is,

$$\begin{aligned} P_n f &= f \quad \forall\, f \in V_n, \\ P_n g &= 0 \quad \forall\, g \in V_n^{\perp}. \end{aligned}$$

The projections P_n are called *approximation operators.* Also, let Q_n denote the orthogonal projection of $L^2(\mathbb{R})$ onto W_n. That is,

$$\begin{aligned} Q_n f &= f \quad \forall\, f \in W_n, \\ Q_n g &= 0 \quad \forall\, g \in W_n^{\perp}. \end{aligned}$$

The projections Q_n are called the *detail operators.* Observe that by (3.7)

$$P_{n+1} = P_n + Q_n \qquad \text{or} \qquad Q_n = P_{n+1} - P_n. \tag{3.9}$$

The following lemma says that every $f \in \mathcal{H}$ can be arbitrarily approximated by an element of V_n, by choosing n sufficiently large. Also, when n is large negative, then the part of f "living" in V_n is small.

Lemma 3.3. *For all $f \in \mathcal{H}$,* *1.* $\lim_{n\to\infty} P_n f = f$,

2. $\lim_{n\to-\infty} P_n f = 0$.

Proof. 1. Let $\varepsilon > 0$ be given. By $(M2)$, $\bigcup_{n=-\infty}^{\infty} V_n$ is dense in $\mathcal{H}$. So there exists $g \in \bigcup_{n=-\infty}^{\infty} V_n$ such that $\|f-g\| < \frac{\varepsilon}{2}$. Pick N such that $g \in V_N$. By $(M1)$ then $g \in V_n \, \forall\, n \geq N$, so that $P_n g = P_N g = g$.

$$\begin{aligned}
\|f - P_n f\|_2 &\leq \|f-g\|_2 + \|g - P_n f\|_2 = \|f-g\|_2 + \|P_n g - P_n f\|_2 \\
&= \|f-g\|_2 + \|P_n(g-f)\|_2 \leq \|f-g\|_2 + \|g-f\|_2 \quad (\|P\| = 1) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for all $n \geq N$. Since ε was arbitrary, it follows that

$$\lim_{n\to\infty} \|f - P_n f\|_2 = 0.$$

2. We claim that $\{\|P_n f\|\}_{n=-1}^{-\infty}$ is monotone decreasing. Recall that $P_{n+1} = P_n + Q_n$, so that $P_{n+1}f = \underbrace{P_n f}_{\in V_n} + \underbrace{Q_n f}_{\in W_n}$. Since $V_n \perp W_n$, we have by the Pythagorean equality,

$$\|P_{n+1}f\|^2 = \|P_n f\|^2 + \|Q_n f\|^2 \geq \|P_n f\|^2$$

which proves the claim.

By the claim, and since $\|P_n(f)\|_2 \geq 0 \; \forall\, n$, it follows that $\{\|P_n f\|\}_{n=-1}^{-\infty}$ converges. That is, there exists $L \geq 0$ such that

$$L = \lim_{n\to-\infty} \|P_n f\|.$$

Next we claim that $\{P_n f\}_{n=-1}^{-\infty}$ is Cauchy. Let $\varepsilon > 0$ be given. Pick $N < 0$ such that

$$L^2 \leq \|P_n f\|_2^2 < L^2 + \varepsilon^2 \tag{3.10}$$

for all $n \leq N$. Now let $m < n \leq N$. Since $V_m \subset V_n,\ P_m f \in V_m,\ P_n f \in V_n$, there exists $g_{nm} \in V_n,\ g_{nm} \perp P_m f$ such that

$$P_n f = P_m f + g_{nm}.$$

By (3.10) $L^2 \leq \|P_m f + g_{nm}\|^2 < L^2 + \varepsilon^2$, or by the Pythagorean identity, $L^2 \leq \|P_m f\|^2 + \|g_{nm}\|^2 < L^2 + \varepsilon^2$. Since $\|P_m f\|^2 \geq L^2$ then $\|g_{nm}\|^2 < \varepsilon^2$ That is, $\|P_n f - P_m f\|_2 = \|g_{nm}\|_2 < \varepsilon \quad \forall\, m < n \leq N$. This proves the claim. Thus $f_0 := \lim_{n \to -\infty} P_n f$ exists in $\mathcal{H}$. It is left to show that $f_0 = 0$.

Let m be arbitrary. If $n < m$, then $V_n \subset V_m$ so that $P_n f \in V_n \subset V_m$, i.e. $P_n f \in V_m\ \forall n \leq m$. Since V_m is a closed subspace, then $f_0 = \lim_{n \to -\infty} P_n f \in V_m$. As m was arbitrary, thus, $f_0 \in \displaystyle\bigcap_{m=-\infty}^{\infty} V_m = \{0\}$. Hence, $f_0 = 0$.

$\square$

Theorem 3.4. *Let $\{V_j\}_{j \in \mathbb{Z}}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$ satisfying $(M1) - (M3)$. Then $\mathcal{H} = \displaystyle\bigoplus_{j=-\infty}^{\infty} W_j$ where W_j is defined by (3.7).*

Proof. Let $f \in \mathcal{H}$ be given. Given $\varepsilon > 0$, by Lemma 3.3 we can pick $N \in \mathbb{N}$ such that

1. $\|f - P_n f\|_2 < \frac{\varepsilon}{2}, \quad \forall\, n \geq N,$

2. $\|P_m f\|_2 < \frac{\varepsilon}{2}. \quad \forall\, m \leq -N.$

Note that $\forall\, m, n \in \mathbb{N},\ m < n$, we have by (3.9),

$$\sum_{j=m}^{n} Q_j f = \sum_{j=m}^{n} [P_{j+1} f - P_j f] = P_{n+1} f - P_m f$$

and hence $\forall\, n \geq N$, and $m \leq -N$,

$$\begin{aligned} \|f - \sum_{j=m}^{n} Q_j f\|_2 &= \|f - [P_{n+1} f - P_m f]\|_2 \\ &\leq \|f - P_{n+1} f\|_2 + \|P_m f\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As ε is arbitrary, it follows that

$$f = \sum_{j=-\infty}^{\infty} Q_j f.$$

Since $Q_j f \in W_j$ and the spaces $\{W_j\}_{j\in\mathbb{Z}}$ are mutually orthogonal, it follows that $f \in \bigoplus_{j=-\infty}^{\infty} W_j$. But $f \in \mathcal{H}$ was arbitrary, hence

$$\mathcal{H} = \bigoplus_{j=-\infty}^{\infty} W_j.$$

□

Theorem 3.5. *Let $\mathcal{H}$ be a Hilbert space, and $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces satisfying $(M1)$-$(M3)$. Let W_j be defined as in (3.7). Suppose that*

1. *there exists a linear isometry D on $\mathcal{H}$ mapping V_j to V_{j+1} for all j,*
2. *$\{\psi_m\}_{m=1}^{\infty}$ is an orthonormal basis of W_0.*

Then

$$\{D^j\psi_m : j \in \mathbb{Z},\, n \in \mathbb{N}\}$$

is an orthonormal basis of $\mathcal{H}$.

Proof. By the previous theorem, $\mathcal{H} = \bigoplus_{j=-\infty}^{\infty} W_j$. We thus must show that $\{D^j\psi_m : m \in \mathbb{N}\}$ is an orthonormal basis of W_j, for all j. By induction, one easily verifies that D^j maps V_0 isometrically onto V_j, for all $j \in \mathbb{Z}$. Then

$$\begin{aligned} V_{j+1} = D^{j+1}(V_0) = D^j(V_1) &= D^j(V_0 \oplus W_0) = D^j(V_0) \oplus D^j(W_0) \\ &= V_j \oplus D^j(W_0) \qquad \text{for all } j, \end{aligned}$$

which shows that $D^j(W_0)$ is the orthogonal complement of V_j in V_{j+1}, that is,

$$D^j(W_0) = W_j.$$

Since $\{\psi_m\}_{m=1}^{\infty}$ is an orthonormal basis of W_0, it follows that $\{D^j\psi_m\}_{m=1}^{\infty}$ is an orthonormal basis of W_j. This proves the theorem. □

Let us return to the multiresolution analysis of $L^2(\mathbb{R})$, with scaling function $\varphi \in V_0$ and wavelet $\psi \in W_0$. In order to show that the collection $\{\psi_{(j,n)} : j, n \in \mathbb{Z}\}$ is a basis of $L^2(\mathbb{R})$, by theorem 3.5, we only need to show that the collection $\{\psi_n\}_{n\in\mathbb{Z}}$, where $\psi_n(x) = T_n\psi(x) = \psi(x-n)$ is an orthonormal basis of W_0. This follows from the particular interplay between dilation and translation. For the proof, we need to work with Fourier transforms.

Let us first describe the Fourier transform of the functions φ and ψ. Suppose, $f \in V_1$. Since $\{D_2T_m\varphi\}_{m\in\mathbb{Z}}$ is an orthonormal basis of V_1, then by theorem 2.16,

$$f = \sum_{m\in\mathbb{Z}} c_m D_2 T_m \varphi$$

where

$$c_m =< f, D_2T_m\varphi >,$$

and $\{c_m\}_{m\in\mathbb{Z}}$ is a square integrable sequence. Since the Fourier transform $\mathcal{F}$ is a linear isometry on $L^2(\mathbb{R})$, and $\widehat{D_a g} = D_{a^{-1}}\hat{g}$, $\widehat{T_m g} = E_{-m}\hat{g}$ for all $g \in L^2(\mathbb{R})$, then

$$\hat{f} = \mathcal{F}(\sum_{m\in\mathbb{Z}} c_m D_2T_m\varphi) = \sum_{m\in\mathbb{Z}} c_m \mathcal{F}(D_2T_m\varphi) = \sum_{m\in\mathbb{Z}} c_m D_{2^{-1}}E_{-m}\hat{\varphi}.$$

That is,

$$\hat{f}(\xi) = \sum_{m\in\mathbb{Z}} c_m \frac{1}{\sqrt{2}}\hat{\varphi}\left(\frac{\xi}{2}\right) e^{-i\pi\xi m}$$

with convergence in $L^2(\mathbb{R})$. So if we set

$$m_f(\xi) = \frac{1}{\sqrt{2}} \sum_{m\in\mathbb{Z}} c_m e^{-2i\pi\xi m} \tag{3.11}$$

then

$$\hat{f}(\xi) = m_f\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right). \tag{3.12}$$

Note that since $\{e^{-2i\pi\xi m}\}_{m\in\mathbb{Z}}$ is an orthonormal basis of $L^2_1(\mathbb{R})$, the series (3.11) defining $m_f(\xi)$ converges in $L^2_1(\mathbb{R})$. If $f = \varphi$, respectively $f = \psi$, then we denote

m_f by m_0, repectively, m_1. That is, by (3.1) and (3.3),

$$\hat{\varphi}(\xi) = m_0\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right) \tag{3.13}$$

where

$$m_0(\xi) = \frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}} h_m e^{-2i\pi\xi m},$$

and

$$\hat{\psi}(\xi) = m_1\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right) \tag{3.14}$$

where

$$m_1(\xi) = \frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}} g_m e^{-2i\pi\xi m}.$$

There is a relation between m_1 and m_0 as follows. By definition of the wavelet filter,

$$m_1(\xi) = \frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}}(-1)^m\overline{h_{1-m}}e^{-2i\pi\xi m}.$$

Replacing m by $1-m$,

$$\begin{aligned}
m_1(\xi) &= \frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}}(-1)^{1-m}\overline{h_m}e^{-2i\pi\xi(1-m)} \\
&= \frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}} -(-1)^m\overline{h_m}\overline{e^{-2i\pi\xi m}}e^{-2i\pi\xi} \\
&= e^{-2i\pi\left(\xi+\frac{1}{2}\right)}\overline{\left[\frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}} h_m e^{-2i\pi\left(\xi+\frac{1}{2}\right)m}\right]} \\
&= e^{-2i\pi(\xi+\frac{1}{2})}\,\overline{m_0\left(\xi+\frac{1}{2}\right)}.
\end{aligned} \tag{3.15}$$

Theorem 3.6. *Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis with scaling function φ and mother wavelet ψ defined by (3.3). Then $\{T_m\psi\}_{m\in\mathbb{Z}}$ is an orthonormal basis W_0.*

Proof. We need to show that

1. $\{T_m\psi\}_{m\in\mathbb{Z}}$ is orthonormal,

2. $T_m\psi \in W_0$ for all $m \in \mathbb{Z}$,

3. $W_0 = \overline{\text{span}\,\{T_m\psi\}_{m\in\mathbb{Z}}}$.

Since $\{T_m\varphi\}_{m\in\mathbb{Z}}$ is an orthonormal basis of V_0, we have by theorem 3.1 that

$$\sum_{m\in\mathbb{Z}} |\hat{\varphi}(\xi+m)|^2 = 1 \quad \text{a.e.} \tag{3.16}$$

Note that, by (3.13),

$$\sum_{m\in\mathbb{Z}} |\hat{\varphi}(\xi+m)|^2 = \sum_{m\in\mathbb{Z}} \left| m_0\left(\frac{\xi+m}{2}\right)\hat{\varphi}\left(\frac{\xi+m}{2}\right)\right|^2.$$

Now each $m \in \mathbb{Z}$ can be written uniquely as $m = 2n+s \quad (n \in \mathbb{Z}, s \in \{0,1\})$. So we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |\hat{\varphi}(\xi+m)|^2 &= \sum_{s=0}^{1}\sum_{n\in\mathbb{Z}} \left| m_0\left(\frac{\xi+2n+s}{2}\right)\hat{\varphi}\left(\frac{\xi+2n+s}{2}\right)\right|^2 \\ &= \sum_{s=0}^{1}\sum_{n\in\mathbb{Z}} \left| m_0\left(\frac{\xi+s}{2}+n\right)\hat{\varphi}\left(\frac{\xi+s}{2}+n\right)\right|^2.\end{aligned}$$

Since m_0 is 1-periodic, we have $m_0(\xi+n) = m_0(\xi)$ for all $n \in \mathbb{Z}$ and almost all ξ, and hence

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |\hat{\varphi}(\xi+m)|^2 &= \sum_{s=0}^{1} \left| m_0\left(\frac{\xi+s}{2}\right)\right|^2 \underbrace{\sum_{n\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{\xi+s}{2}+n\right)\right|^2}_{=1 \text{ a.e. by theorem 3.1}} \\ &= \sum_{s=0}^{1} \left| m_0\left(\frac{\xi+s}{2}\right)\right|^2 \\ &= \left| m_0\left(\frac{\xi}{2}\right)\right|^2 + \left| m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)\right|^2 \quad \text{a.e.}\end{aligned} \tag{3.17}$$

Using the same computations as above, but, with $\hat{\psi}$ instead of $\hat{\varphi}$, we obtain applying (3.14) and (3.15)

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} |\hat{\psi}(\xi+m)|^2 &= \sum_{s=0}^{1} \left| m_1\left(\frac{\xi+s}{2}\right)\right|^2 \\
&= \sum_{s=0}^{1} \left| e^{-2i\pi\left(\frac{\xi+s}{2}+\frac{1}{2}\right)} m_0\left(\frac{\xi+s}{2}+\frac{1}{2}\right)\right|^2 \\
&= \sum_{s=0}^{1} \left| m_0\left(\frac{\xi+s}{2}+\frac{1}{2}\right)\right|^2 \\
&= \sum_{s=0}^{1} \left| m_0\left(\frac{(\xi+1)+s}{2}\right)\right|^2 . \qquad (3.18)
\end{aligned}$$

Combining (3.16) , (3.17) and (3.18), we have

$$\sum_{m\in\mathbb{Z}} |\hat{\psi}(\xi+m)|^2 = \sum_{m\in\mathbb{Z}} |\hat{\varphi}((\xi+1)+m)|^2 = 1 \quad \text{a.e.} \qquad (3.19)$$

We remark that (3.18) and (3.19) imply that $|m_0(\xi)| \leq 1$ and $|m_1(\xi)| \leq 1$ a.e., and hence $m_0, m_1 \in L^\infty(\mathbb{R})$ as well.

It follows from theorem 3.1 that the collection $\{T_m\psi\}_{m\in\mathbb{Z}}$ is an orthonormal set. Next we need to show that $T_m\psi \in W_0$ for all m. Since $V_1 = V_0 \oplus W_0$, and $\{T_m\varphi\}_{m\in\mathbb{Z}}$ is an orthonormal basis of V_0, it is enough to show that $T_k\varphi \perp T_m\psi$ for all $m, k \in \mathbb{Z}$. Now observe that

$$< T_k\varphi, T_m\psi > = < \varphi, T_{m-k}\psi > \qquad (3.20)$$

for all $k, m \in \mathbb{Z}$. So it is enough to show that

$$< \varphi, T_k\psi > = 0$$

for all $k \in \mathbb{Z}$. Now by Plancherel's Theorem and (3.15),

$$\begin{aligned}
< \varphi, T_k\psi > &= < \hat{\varphi}, E_{-k}\hat{\psi} > \\
&= \int_{\mathbb{R}} \hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)e^{-2i\pi\xi k}}\, d\xi \\
&= \int_{\mathbb{R}} m_0\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right)\overline{m_1\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right)} e^{2i\pi\xi k}\, d\xi \\
&= \int_{\mathbb{R}} m_0\left(\frac{\xi}{2}\right) e^{2i\pi\left(\frac{\xi}{2}+\frac{1}{2}\right)} m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)\left|\hat{\varphi}\left(\frac{\xi}{2}\right)\right|^2 e^{2i\pi\xi k}\, d\xi.
\end{aligned}$$

Next replace ξ by 2ξ,

$$\begin{aligned}< \varphi, T_k\psi > &= 2\int_{\mathbb{R}} e^{2i\pi\left[2\xi k+\xi+\frac{1}{2}\right]} m_0\left(\xi\right) m_0\left(\xi+\frac{1}{2}\right) \left|\hat{\varphi}\left(\xi\right)\right|^2 d\xi \\ &= 2\sum_{l\in\mathbb{Z}}\int_l^{l+1} e^{2i\pi\left[2\xi k+\xi+\frac{1}{2}\right]} m_0\left(\xi\right) m_0\left(\xi+\frac{1}{2}\right) \left|\hat{\varphi}\left(\xi\right)\right|^2 d\xi.\end{aligned}$$

Replace ξ by $\xi + l$ inside the integral, and obtain

$$< \varphi, T_k\psi >= 2\sum_{l\in\mathbb{Z}}\int_0^1 e^{2i\pi\left[2\xi k+2lk+\xi+l+\frac{1}{2}\right]} m_0\left(\xi+l\right) m_0\left(\xi+l+\frac{1}{2}\right) \left|\hat{\varphi}\left(\xi+l\right)\right|^2 d\xi.$$

Now as m_0 is 1-periodic, and since $e^{2i\pi m} = 1$ for all $m \in \mathbb{Z}$, then

$$\begin{aligned}< \varphi, T_k\psi > &= 2\int_0^1 e^{2i\pi\left[2\xi k+\xi+\frac{1}{2}\right]} m_0\left(\xi\right) m_0\left(\xi+\frac{1}{2}\right) \underbrace{\sum_{l\in\mathbb{Z}} \left|\hat{\varphi}\left(\xi+l\right)\right|^2}_{=1 \text{ a.e. by theorem 3.1}} \ d\xi \\ &= 2\left[\int_0^{\frac{1}{2}} e^{2i\pi\left[2\xi k+\xi+\frac{1}{2}\right]} m_0\left(\xi\right) m_0\left(\xi+\frac{1}{2}\right) d\xi \right. \\ &\qquad \left. + \int_{\frac{1}{2}}^1 e^{2i\pi\left[2\xi k+\xi+\frac{1}{2}\right]} m_0\left(\xi\right) m_0\left(\xi+\frac{1}{2}\right) d\xi\right]\end{aligned}$$

Replace ξ by $\xi + \frac{1}{2}$ in the 2nd integral, and obtain

$$\begin{aligned}< \varphi, T_k\psi > &= 2\left[\int_0^{\frac{1}{2}} e^{2i\pi\left[2\xi k+\xi+\frac{1}{2}\right]} m_0\left(\xi\right) m_0\left(\xi+\frac{1}{2}\right) d\xi \right. \\ &\qquad \left. + \int_0^{\frac{1}{2}} e^{2i\pi\left[2\xi k+\xi+\frac{1}{2}\right]} (-1) m_0\left(\xi\right) m_0\left(\xi+\frac{1}{2}\right) d\xi\right] \\ &= 0 \qquad (3.21)\end{aligned}$$

This shows that $T_k\varphi \perp T_m\psi$ for all $m, k \in \mathbb{Z}$. That is, $\overline{\text{span}\{T_m\psi\}_{m\in\mathbb{Z}}} \subset W_0$.

It is left to show that $W_0 \subset \overline{\text{span}\{T_m\psi\}_{m\in\mathbb{Z}}}$. Since $V_1 = V_0 \oplus W_0$, and $\{T_m\varphi\}_{m\in\mathbb{Z}}$ is an orthonormal basis of V_0, it is enough to show that

$$V_1 \subset \overline{\text{span}\{T_m\varphi, T_m\psi\}_{m\in\mathbb{Z}}}.$$

For this, we will use theorem 3.2. Let $f \in V_1$ be arbitrary. Recall from (3.12) that $\hat{f}(\xi) = m_f\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right)$, where $m_f\left(\xi\right) = \frac{1}{\sqrt{2}}\sum_{m\in\mathbb{Z}} c_m e^{-2i\pi\xi m}$.

We claim that there exist functions $h_0, h_1 \in L_1^2(\mathbb{R}^2)$ such that

$$m_f\left(\frac{\xi}{2}\right) = h_0(\xi) m_0\left(\frac{\xi}{2}\right) + h_1(\xi) m_1\left(\frac{\xi}{2}\right) \tag{3.22}$$

with convergence in $L_1^2(\mathbb{R})$. If (3.22) holds, then

$$\begin{aligned} \hat{f}(\xi) &= m_f\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right) = h_0(\xi)m_0\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right) + h_1(\xi)m_1\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right) \\ &= h_0(\xi)\hat{\varphi}(\xi) + h_1(\xi)\hat{\psi}(\xi) \quad \text{a.e.} \end{aligned}$$

so that $f = f_0 + f_1$, with $\widehat{f_0} = h_0(\xi)\hat{\varphi}(\xi)$ and $\widehat{f_1} = h_1(\xi)\hat{\psi}(\xi)$. By theorem 3.2, $f_0 \in \overline{\text{span}\{T_m\varphi\}}$ and $f_1 \in \overline{\text{span}\{T_m\psi\}}$, so we will have shown that

$$V_1 \subset \overline{\text{span}\{T_m\varphi, T_m\psi\}_{m\in\mathbb{Z}}},$$

and the proof will be complete. To prove the claim, change ξ to $\xi + 1$ in (3.22),

$$\begin{aligned} m_f\left(\frac{\xi}{2}+\frac{1}{2}\right) &= h_0(\xi+1)\, m_0\left(\frac{\xi}{2}+\frac{1}{2}\right) + h_1(\xi+1)\, m_1\left(\frac{\xi}{2}+\frac{1}{2}\right) \\ &= h_0(\xi)\, m_0\left(\frac{\xi}{2}+\frac{1}{2}\right) + h_1(\xi)\, m_1\left(\frac{\xi}{2}+\frac{1}{2}\right) \qquad \text{a.e.} \end{aligned} \tag{3.23}$$

by periodicity of the required h_0 and h_1. Equations (3.22) and (3.25) can be written as a matrix equation,

$$\begin{bmatrix} m_f\left(\frac{\xi}{2}\right) \\ m_f\left(\frac{\xi}{2}+\frac{1}{2}\right) \end{bmatrix} = \underbrace{\begin{bmatrix} m_0\left(\frac{\xi}{2}\right) & m_1\left(\frac{\xi}{2}\right) \\ m_0\left(\frac{\xi}{2}+\frac{1}{2}\right) & m_1\left(\frac{\xi}{2}+\frac{1}{2}\right) \end{bmatrix}}_{T} \begin{bmatrix} h_0(\xi) \\ h_1(\xi) \end{bmatrix} \tag{3.24}$$

We now show that T is unitary. Note that,

$$TT^* = \begin{bmatrix} m_0\left(\frac{\xi}{2}\right) & m_1\left(\frac{\xi}{2}\right) \\ m_0\left(\frac{\xi}{2}+\frac{1}{2}\right) & m_1\left(\frac{\xi}{2}+\frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} \overline{m_0\left(\frac{\xi}{2}\right)} & \overline{m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)} \\ \overline{m_1\left(\frac{\xi}{2}\right)} & \overline{m_1\left(\frac{\xi}{2}+\frac{1}{2}\right)} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$$

By (3.15) and (3.17),

$$\alpha_1 = \left|m_0\left(\frac{\xi}{2}\right)\right|^2 + \left|m_1\left(\frac{\xi}{2}\right)\right|^2 = \left|m_0\left(\frac{\xi}{2}\right)\right|^2 + \left|m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)\right|^2 = 1 \quad \text{a.e.}$$

and similarly,

$$\alpha_4 = \left|m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)\right|^2 + \left|m_1\left(\frac{\xi}{2}+\frac{1}{2}\right)\right|^2 = \left|m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)\right|^2 + \left|m_0\left(\frac{\xi}{2}\right)\right|^2 = 1 \quad \text{a.e.}$$

Again, by (3.15), and periodicity of m_0,

$$\begin{aligned}
\alpha_2 &= m_0\left(\frac{\xi}{2}\right)\overline{m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)} + m_1\left(\frac{\xi}{2}\right)\overline{m_1\left(\frac{\xi}{2}+\frac{1}{2}\right)} \\
&= m_0\left(\frac{\xi}{2}\right)\overline{m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)} + e^{-2i\pi\left(\frac{\xi}{2}+\frac{1}{2}\right)}\overline{m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)}\,\overline{e^{-2i\pi\left(\frac{\xi}{2}+1\right)}m_0\left(\frac{\xi}{2}+1\right)} \\
&= m_0\left(\frac{\xi}{2}\right)\overline{m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)} + e^{i\pi}m_0\left(\frac{\xi}{2}\right)\overline{m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)} = 0 \text{ a.e.}
\end{aligned}$$

Finally, $\alpha_3 = \overline{\alpha_2} = 0$ as well. This shows that $TT^* = I$ a.e., that is, T is unitary a.e., so that T is invertible and $T^{-1}(\xi) = T^*(\xi)$ a.e.. Hence, system (3.24) has a solution, namely

$$\begin{bmatrix} h_0(\xi) \\ h_1(\xi) \end{bmatrix} = T^* \begin{bmatrix} m_f\left(\frac{\xi}{2}\right) \\ m_f\left(\frac{\xi}{2}+\frac{1}{2}\right) \end{bmatrix} \quad \text{a.e. } \xi.$$

To see that $h_0, h_1 \in L^2_1(\mathbb{R})$, note that

$$\begin{aligned}
h_0(\xi) &= m_f\left(\frac{\xi}{2}\right)\overline{m_0\left(\frac{\xi}{2}\right)} + m_f\left(\frac{\xi}{2}+\frac{1}{2}\right)\overline{m_0\left(\frac{\xi}{2}+\frac{1}{2}\right)} \\
h_1(\xi) &= m_f\left(\frac{\xi}{2}\right)\overline{m_1\left(\frac{\xi}{2}\right)} + m_f\left(\frac{\xi}{2}+\frac{1}{2}\right)\overline{m_1\left(\frac{\xi}{2}+\frac{1}{2}\right)}
\end{aligned} \tag{3.25}$$

Now be periodicity of m_f and m_1,

$$\begin{aligned}
h_1(\xi+1) &= m_f\left(\frac{\xi+1}{2}\right)\overline{m_1\left(\frac{\xi+1}{2}\right)} + m_f\left(\frac{\xi+1}{2}+\frac{1}{2}\right)\overline{m_1\left(\frac{\xi+1}{2}+\frac{1}{2}\right)} \\
&= m_f\left(\frac{\xi}{2}\right)\overline{m_1\left(\frac{\xi}{2}\right)} + m_f\left(\frac{\xi}{2}+\frac{1}{2}\right)\overline{m_1\left(\frac{\xi}{2}+\frac{1}{2}\right)} = h_1(\xi)
\end{aligned}$$

and similarly,

$$h_0(\xi+1) = h_0(\xi).$$

As remarked earlier, $|m_1(\xi)| \le 1$ a.e. Since $m_f \in L^2_1(\mathbb{R})$, it follows from (3.25) that $h_1(\xi) \in L^2_1(\mathbb{R})$. By a similar argument, $h_0(\xi) \in L^2_1(\mathbb{R})$. This proves the claim and the theorem. $\square$

Combining theorems 3.3, 3.4 and 3.5 we have shown:

Corollary 3.7. *Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis with scaling function φ, and let ψ be the mother wavelet defined by (3.3). Then $\{\psi_{(j,m)}\}_{j,m\in\mathbb{Z}}$, where $\psi_{(j,m)} = D_{2^j}T_m\psi$, is an orthonormal basis of $L^2(\mathbb{R})$.*

Example 3.2 (The Haar Wavelet, Continued). Let us compute the wavelet basis in case of the Haar wavelet

$$\psi(x) = \chi_{\left[0,\frac{1}{2}\right)}(x) - \chi_{\left[\frac{1}{2},1\right)}(x)$$

of example 3.1. By theorem 3.6, the functions $\{\psi_n\}_{n\in\mathbb{Z}}$, with

$$\begin{aligned}\psi_n(x) = (T_n\psi)(x) = \psi(x-n) &= \chi_{\left[0,\frac{1}{2}\right)}(x-n) - \chi_{\left[\frac{1}{2},1\right)}(x-n)\\ &= \chi_{\left[n,n+\frac{1}{2}\right)}(x) - \chi_{\left[n+\frac{1}{2},n+1\right)}(x)\end{aligned}$$

form an orthonormal basis of W_0. Applying the map $(D_2)^j = D_{2^j}$, we see that the functions $\{\psi_{(j,n)}\}_{n\in\mathbb{Z}}$ given by

$$\psi_{(j,n)} = (D_{2^j}\psi_n)(x) = 2^{j/2}\psi_n(2^jx) = 2^{j/2}\psi(2^jx-n)$$

form an orthonormal basis of W_j, and in fact

$$\begin{aligned}\psi_{(j,n)}(x) &= 2^{j/2}\chi_{\left[n,n+\frac{1}{2}\right)}(2^jx) - 2^{j/2}\chi_{\left[n+\frac{1}{2},n+1\right)}(2^jx)\\ &= 2^{j/2}\chi_{\left[\frac{2n}{2^{j+1}},\frac{2n+1}{2^{j+1}}\right)}(x) - 2^{j/2}\chi_{\left[\frac{2n+1}{2^{j+1}},\frac{2n+2}{2^{j+1}}\right)}(x).\end{aligned} \tag{3.26}$$

Thus, the collection $\{\psi_{(j,n)} : j,n\in\mathbb{Z}\}$ with $\psi_{j,n}$ given as in (3.26) is an orthonormal basis of $L^2(\mathbb{R})$, called the *Haar basis* or the *Haar system.*

Let us study the spaces W_j in detail. Since $\{\psi_{(0,k)}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_0, every $f \in W_0$ is the limit in the norm of $L^2(\mathbb{R})$ of a sequence f_n of finite linear combinations of the form

$$\sum_{k=m}^{l} c_k\left(\chi_{\left[k,k+\frac{1}{2}\right)} - \chi_{\left[k+\frac{1}{2},k+1\right)}\right) \tag{3.27}$$

and arguing as example 3.1, a pointwise limit as well. Now by (3.27), each f_n is constant on intervals $\left[k, k+\frac{1}{2}\right)$ and $\left[k+\frac{1}{2}, k+1\right)$ and on the second interval, takes on the negative of its value on the first interval. This property carries over to the pointwise limit. Hence

$$\begin{aligned} W_0 \;=\; & \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is constant on all intervals } \left[\frac{i}{2}, \frac{i+1}{2}\right) \; (i \in \mathbb{Z}), \\ & \text{and the values of } f \text{ on } \left[\frac{2k}{2}, \frac{2k+1}{2}\right) \text{ and } \left[\frac{2k+1}{2}, \frac{2k+2}{2}\right) \\ & \text{have same absolute values, but opposite signs}\} \end{aligned}$$

Applying the map D_{2^j} as in (3.26), then

$$\begin{aligned} W_j \;=\; & \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is constant on all intervals } \left[\frac{i}{2^{j+1}}, \frac{i+1}{2^{j+1}}\right) \; (i \in \mathbb{Z}) \\ & \text{and the values of } f \text{ on } \left[\frac{2k}{2^{j+1}}, \frac{2k+1}{2^{j+1}}\right) \text{ and } \left[\frac{2k+1}{2^{j+1}}, \frac{2k+2}{2^{j+1}}\right) \\ & \text{have equal absolute values, but opposite signs}\}. \end{aligned}$$

CHAPTER IV

MULTIRESOLUTION ANALYSIS IN $L^2(\mathbb{R}^2)$

In this chapter, we extend the definition of multiresolution analysis to $L^2(\mathbb{R}^2)$. From such a multiresolution analysis, a wavelet decomposition of $L^2(\mathbb{R}^2)$ can be constructed as outlined in the previous chapter. We then give an example which extends the Haar wavelets of example 3.1 to $L^2(\mathbb{R}^2)$.

4.1 Definition of Multiresolution Analysis in $L^2(\mathbb{R}^2)$

The dilation, translation and modulation operators on $L^2(\mathbb{R})$ introduced in section 3.2 can be generalized to $L^p(\mathbb{R}^n)$ in a natural way. In fact, let A be an $n \times n$ invertible matrix. We define operators on $L^p(\mathbb{R}^n)$ by

$$(D_A f)(x) = |\det A|^{1/p} f(Ax), \qquad \textit{(dilation)}$$

$$(T_y f)(x) = f(x - y), \qquad \textit{(translation)}$$

$$(E_\gamma f)(x) = e^{2i\pi\gamma\cdot x} f(x), \qquad \textit{(modulation)}$$

where $x, y, \gamma \in \mathbb{R}^n, f \in L^p(\mathbb{R}^n)$. One easily checks that these operators are all isometries on $L^p(\mathbb{R}^n)$. For example, for all $f, g \in L^p(\mathbb{R}^n)$, scalars α, β, and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} D_A(\alpha f + \beta g)(x) &= |\det A|^{1/p}(\alpha f + \beta g)(Ax) = |\det A|^{1/p}(\alpha f(Ax) + \beta g(Ax)) \\ &= \alpha|\det A|^{1/p} f(Ax) + \beta|\det A|^{1/p} g(Ax) \\ &= \alpha D_A f(x) + \beta D_A g(x). \end{aligned}$$

That is, $D_A(\alpha f + \beta g) = \alpha D_A f + \beta D_A g$, so that D_A is linear. Next notice that

$$\|D_A f\|_p^p = \int_{\mathbb{R}^n} \left[|\det A|^{1/p} |f(Ax)| \right]^p dx = |\det A| \int_{\mathbb{R}^n} |f(Ax)|^p \, dx.$$

Next replace x by $A^{-1}x$,

$$\|D_A f\|_p^p = \int_{\mathbb{R}^n} |f(x)|^p \, dx = \|f\|_p^p,$$

which shows that D_A is an isometry. Observe that

$$\begin{aligned} (D_A D_B f)(x) &= |\det A|^{1/p} (D_B f)(Ax) = |\det B|^{1/p} |\det A|^{1/p} f(ABx) \\ &= |\det BA|^{1/p} f(BAx) = (D_{BA} f)(x) \end{aligned}$$

that is, $D_A D_B = D_{BA}$. In particular, for each $f \in L^p(\mathbb{R}^n)$,

$$D_A(D_{A^{-1}} f) = D_{A^{-1}A} f = D_I f = f,$$

which shows that D_A is surjective.

It is easy to see that $T_x T_y = T_{x+y}$ and $E_\gamma E_\xi = E_{\gamma+\xi}$ for all $x, y, \gamma, \xi \in \mathbb{R}^n$. Furthermore,

$$< T_x f, g > = < f, T_{-x} g >$$

$$< E_\gamma f, g > = < f, E_{-\gamma} g >$$

for all $f, g \in L^2(\mathbb{R}^n)$.

Observe that $\widehat{D_A f} = D_{A^{-1}} \hat{f}$, $\widehat{T_x f} = E_{-x} \hat{f}$ and $\widehat{E_\gamma f} = T_\gamma \hat{f}$, for all $f \in L^2(\mathbb{R}^n)$. For example, consider the dilation D_A. If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then

$$\begin{aligned} \widehat{D_A f}(\xi) &= \int_{\mathbb{R}} (D_A f)(x) e^{-2i\pi\xi x} \, dx \\ &= \int_{\mathbb{R}} |\det A|^{1/2} f(Ax) e^{-2i\pi\xi x} \, dx, \end{aligned}$$

since the scalar product $\xi \cdot x$ is multiplication of the row vector ξ by the column vector x. Next replace x by $A^{-1}x$,

$$\begin{aligned} \widehat{D_A f}(\xi) &= \int_{\mathbb{R}^n} |\det A|^{1/2} |\det A|^{-1} f(x) e^{-2i\pi\xi A^{-1}x}\, dx \\ &= |\det A|^{-1/2} \hat{f}(\xi A^{-1}) = D_{A^{-1}} \hat{f}(\xi), \end{aligned}$$

where dilation is now defined by $(D_A\hat{f})(\xi) = |\det A|^{1/2}\hat{f}(\xi A)$. Since the Fourier transform is continuous, and $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, it follows that $\widehat{D_A f} = D_{A^{-1}}\hat{f}$ for all $f \in L^2(\mathbb{R}^n)$.

In order to clearly distinguish between vectors and scalars, in the following we will denote elements of $\mathbb{R}^2$ written as column vectors by $\underline{x}$ or $\underline{y}$. Elements of $\mathbb{R}^2$ written as row vectors will be denoted by $\underline{\xi}$. Similarly, elements of $\mathbb{Z}^2$ written as column vectors will be denoted by $\underline{a}, \underline{k}, \underline{m}$ or $\underline{n}$, while elements of $\mathbb{Z}^2$ written as row vectors will be denoted by $\underline{i}, \underline{j}, \underline{r}$ or $\underline{s}$. For example, $\underline{x} = (x_1, x_2)^T$, $\underline{\xi} = (\xi_1, \xi_2)$, $\underline{m} = (m_1, m_2)^T$, $\underline{r} = (r_1, r_2)$. Furthermore, $\underline{0}$ will denote the zero vector, and $\underline{1}$ the vector $(1, 1)$.

Definition 4.1. Fix a matrix

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

where $a_1, a_2 \in \{2, 3, 4, \ldots\}$. A *multiresolution analysis* on $L^2(\mathbb{R}^2)$ is a sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^2)$ satisfying the following properties:

$(M1)$: $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$,

$(M2)$: $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^2)$,

$(M3)$: $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

$(M4)$: D_A is an isometry of V_j onto V_{j+1}, for all j,

$(M5)$: There exists $\varphi \in V_0$ such that the family of translates of φ, $\{T_{\underline{m}}\varphi\}_{\underline{m}\in\mathbb{Z}^2}$, is an orthonormal basis of V_0. Such a φ is called a *scaling function*.

Note that by $(M4)$ and $(M5)$, the collection $\{D_A T_{\underline{m}}\varphi\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal basis of V_1. By $(M1)$, $\varphi \in V_1$ as well. Let $h_{\underline{m}}$ denote the coefficients of φ in this basis of V_1, that is,

$$h_{\underline{m}} =< \varphi, D_A T_{\underline{m}}\varphi > .$$

Then $\{h_{\underline{m}}\}_{\underline{m}\in\mathbb{Z}^2} \in l^2(\mathbb{Z}^2)$, and in fact by Parseval's identity,

$$\sum_{\underline{m}\in\mathbb{Z}^2} |h_{\underline{m}}|^2 = \|\varphi\|_2^2 = 1. \tag{4.1}$$

This sequence is called the *scaling filter*.

In case of higher dimensional wavelets, we need to impose an additional condition on the scaling function.

$(M6) : \sum_{\underline{m}\in\mathbb{Z}^2} \overline{h_{\underline{m}}} h_{\underline{m}+A\underline{k}} e^{2i\pi\underline{r}A^{-1}\cdot\underline{m}} = 0$

for all $\underline{k}, \underline{r}$ unless $\underline{k} = \underline{r} = 0$, and

$$h_{\underline{m}} = \overline{h_{\underline{a}-\underline{m}}}$$

for all $\underline{m}$, where $\underline{k}, \underline{m} \in \mathbb{Z}^2$, $\underline{r} = (r_1, r_2)$, $r_1 \in \{0, \ldots, a_1 - 1\}$, $r_2 \in \{0, \ldots, a_2 - 1\}$.

4.2 Wavelets from a Multiresolution Analysis

Given a multiresolution analysis, we now construct a family of wavelet functions as follows. Set $\mathcal{R} = \{\underline{r} = (r_1, r_2) \in \mathbb{Z}^2 : \ 0 \leq r_1 < a_1, \ 0 \leq r_2 < a_2, \ \underline{r} \neq \underline{0}\}$. Then $\mathcal{R}$ has cardinality $a_1 a_2 - 1$. Furthermore, set $\mathcal{R}_0 = \mathcal{R} \cup \{\underline{0}\}$ and $\underline{a} = (a_1 - 1, a_2 - 1)$. For each $\underline{r} \in \mathcal{R}$, set

$$g^{\underline{r}}_{\underline{m}} = e^{2i\pi\underline{r}A^{-1}\cdot\underline{m}} \overline{h_{\underline{a}-\underline{m}}} \qquad (\underline{m} \in \mathbb{Z}^2)$$

and

$$\psi^{\underline{r}} = \sum_{\underline{m}\in\mathbb{Z}^2} g^{\underline{r}}_{\underline{m}} D_A T_{\underline{m}}\varphi. \tag{4.2}$$

For each $\underline{r} \in \mathcal{R}$, the sequence $\{g^{\underline{r}}_{\underline{m}}\}_{m\in\mathbb{Z}^2}$ is called the *wavelet filter* corresponding to $\psi^{\underline{r}}$. Observe that since $\{h_{\underline{m}}\}_{\underline{m}\in\mathbb{Z}^2}$ is square integrable, the sequences $\{g^{\underline{r}}_{\underline{m}}\}_{m\in\mathbb{Z}^2}$ are square integrable as well. It follows that the sequence (4.2) defining $\psi^{\underline{r}}$ converges in $L^2(\mathbb{R}^2)$, and in fact, that $\psi^{\underline{r}} \in V_1$, $\forall\, \underline{r} \in \mathcal{R}$. For ease of notation, set $\{g^{\underline{0}}_{\underline{m}}\} = \{h_{\underline{m}}\}$ and $\psi^{\underline{0}} = \varphi$.

Let us find the Fourier transform of the function $\psi^{\underline{r}}$. Suppose, $f \in V_1$. Since $\{D_A T_{\underline{m}}\varphi\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal basis of V_1, then by Parseval's theorem,

$$f = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} D_A T_{\underline{m}}\varphi$$

where

$$c_{\underline{m}} =< f, D_A T_{\underline{m}}\varphi >,$$

and $\{c_{\underline{m}}\}_{\underline{m}\in\mathbb{Z}^2}$ is a square integrable sequence. Since the Fourier transform $\mathcal{F}$ is a linear isometry on $L^2(\mathbb{R}^2)$, then

$$\hat{f} = \mathcal{F}(\sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} D_A T_{\underline{m}}\varphi) = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} \mathcal{F}(D_A T_{\underline{m}}\varphi) = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} D_{A^{-1}} E_{-\underline{m}}\hat{\varphi}.$$

That is,

$$\hat{f}(\underline{\xi}) = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} \frac{1}{\sqrt{|\det A|}} \hat{\varphi}(\underline{\xi}A^{-1}) e^{-2i\pi\underline{\xi}A^{-1}\cdot\underline{m}}$$

with convergence in $L^2(\mathbb{R}^2)$. So if we set

$$m_f(\underline{\xi}) = \frac{1}{\sqrt{|\det A|}} \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}} \tag{4.3}$$

then

$$\hat{f}(\underline{\xi}) = m_f(\underline{\xi}A^{-1})\hat{\varphi}(\underline{\xi}A^{-1}). \tag{4.4}$$

Note that since $\{e^{-2i\pi\underline{\xi}\cdot\underline{m}}\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal basis of $L^2_1(\mathbb{R}^2)$, the series (4.3) defining $m_f(\underline{\xi})$ converges in $L^2_1(\mathbb{R}^2)$. If $f = \psi^{\underline{r}}$, we denote $m_{\psi^{\underline{r}}}$ simply by $m_{\underline{r}}$, that is,

$$\widehat{\psi^{\underline{r}}}(\underline{\xi}) = m_{\underline{r}}(\underline{\xi}A^{-1})\hat{\varphi}(\underline{\xi}A^{-1})$$

where

$$m_{\underline{r}}(\underline{\xi}) = \frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} g^{\underline{r}}_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}. \qquad (\underline{r} \in \mathcal{R})$$

In case $\underline{r} = \underline{0}$ we have $\psi^{\underline{0}} = \varphi$, so

$$\hat{\varphi}(\underline{\xi}) = m_{\underline{0}}(\underline{\xi}A^{-1})\hat{\varphi}(\underline{\xi}A^{-1}) \tag{4.5}$$

where

$$m_{\underline{0}}(\underline{\xi}) = \frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}.$$

There is a relation between $m_{\underline{r}}$ and $m_{\underline{0}}$ as follows. By definition of the wavelet filter,

$$m_{\underline{r}}(\underline{\xi}) = \frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} e^{2i\pi\underline{r}A^{-1}\cdot\underline{m}} h_{\underline{a}-\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}.$$

Replacing $\underline{m}$ by $\underline{a} - \underline{m}$

$$\begin{aligned}
m_{\underline{r}}(\underline{\xi}) &= \frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} e^{2i\pi\underline{r}A^{-1}\cdot(\underline{a}-\underline{m})} \overline{h_{\underline{m}}} e^{-2i\pi\underline{\xi}\cdot(\underline{a}-\underline{m})} \\
&= \frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} e^{2i\pi\underline{r}A^{-1}\cdot\underline{a}} e^{-2i\pi\underline{\xi}\cdot\underline{a}} \overline{h_{\underline{m}}} e^{-2i\pi\underline{r}A^{-1}\cdot\underline{m}} e^{2i\pi\underline{\xi}\cdot\underline{m}} \\
&= e^{-2i\pi(\underline{\xi}-\underline{r}A^{-1})\cdot\underline{a}} \overline{\left[\frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}} e^{-2i\pi(\underline{\xi}-\underline{r}A^{-1})\cdot\underline{m}} \right]} \\
&= e^{-2i\pi(\underline{\xi}-\underline{r}A^{-1})\cdot\underline{a}}\, \overline{m_{\underline{0}}(\underline{\xi} - \underline{r}A^{-1})}. \qquad (\underline{r} \in \mathcal{R})
\end{aligned} \tag{4.6}$$

Similarly, by $(M6)$,

$$\begin{aligned}
m_{\underline{0}}(\underline{\xi}) &= \frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} \overline{h_{\underline{m}}} e^{-2i\pi\underline{\xi}\cdot\underline{m}} \\
&= \frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} \overline{h_{\underline{a}-\underline{m}}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}.
\end{aligned}$$

Replacing $\underline{m}$ by $\underline{a} - \underline{m}$

$$\begin{aligned}
m_{\underline{0}}(\underline{\xi}) &= \frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot(\underline{a}-\underline{m})} \\
&= e^{-2i\pi\underline{\xi}\cdot\underline{a}} \left[\overline{\frac{1}{\sqrt{a_1 a_2}} \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}} \right] \\
&= e^{-2i\pi\underline{\xi}\cdot\underline{a}}\, \overline{m_{\underline{0}}(\underline{\xi})}.
\end{aligned}$$

Next let W_n denote the orthogonal complement of V_n in V_{n+1}, that is,

$$V_{n+1} = V_n \oplus W_n \qquad (n \in \mathbb{Z}). \tag{4.7}$$

If $n = 0$, this becomes $V_1 = V_0 \oplus W_0$. Recall that $\{T_{\underline{m}}\varphi\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal basis of V_0 by $(M5)$. To generalize the usual wavelet construction, we need to show that $\{T_{\underline{m}}\psi^{\underline{r}}\}_{m\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}}$ is an orthonormal basis of W_0.

For this, we will need the following two theorems, generalizing theorems 3.1 and 3.2. In the following, given $N \in \mathbb{N}$ we set $I_N = \{\underline{m} = (m_1, m_2) \in \mathbb{Z}^2 : -N \leq m_1, m_2 \leq N\}$.

Theorem 4.1. *Let $g \in L^2(\mathbb{R}^2)$. Then the system of translates $\{T_{\underline{m}}g\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal set if and only if $\sum_{\underline{m}\in\mathbb{Z}^2} |\hat{g}(\underline{\xi} + \underline{m})|^2 = 1$ for almost all $\underline{\xi} \in \mathbb{R}^2$.*

Proof. Set $h(\underline{\xi}) := \sum_{\underline{m}\in\mathbb{Z}^2} |\hat{g}(\underline{\xi} + \underline{m})|^2$. We claim that $g(\underline{\xi}) \in L^1[0,1]^2$. In fact, for all $\underline{\xi} \in \mathbb{R}^2$,

$$h(\underline{\xi}) = \lim_{N\to\infty} \sum_{\underline{m}\in I_N} |\hat{g}(\underline{\xi} + \underline{m})|^2.$$

By the Monotone Convergence Theorem,

$$\begin{aligned}
\int_{[0,1]^2} h(\underline{\xi})\, d\underline{\xi} &= \int_{[0,1]^2} \lim_{N\to\infty} \sum_{\underline{m}\in I_N} |\hat{g}(\underline{\xi} + \underline{m})|^2\, d\underline{\xi} \\
&= \lim_{N\to\infty} \sum_{\underline{m}\in I_N} \int_{[0,1]^2} |\hat{g}(\underline{\xi} + \underline{m})|^2\, d\underline{\xi}.
\end{aligned}$$

Replace $\underline{\xi}$ by $\underline{\xi} - \underline{m}$ inside the integral, and obtain

$$\begin{aligned}
\int_{[0,1]^2} h(\underline{\xi})\, d\underline{\xi} &= \lim_{N\to\infty} \sum_{\underline{m}\in I_N} \int_{\underline{m}+[0,1]^2} |\hat{g}(\underline{\xi})|^2\, d\underline{\xi} \\
&= \sum_{\underline{m}\in\mathbb{Z}^2} \int_{\underline{m}+[0,1]^2} |\hat{g}(\underline{\xi})|^2\, d\underline{\xi} \\
&= \int_{\mathbb{R}^2} |\hat{g}(\underline{\xi})|^2\, d\underline{\xi} < \infty \text{ as } g \in L^2(\mathbb{R}^2).
\end{aligned}$$

This proves the claim. Observe that if $f_1, f_2 \in L^2(\mathbb{R}^2)$, then

$$< T_{\underline{k}} f_1, T_{\underline{l}} f_2 >=< f_1, T_{-\underline{k}} T_{\underline{l}} f_2 >=< f_1, T_{\underline{l}-\underline{k}} f_2 > . \tag{4.8}$$

Hence $\{T_{\underline{k}} g\}_{k\in\mathbb{Z}^2}$ is orthonormal, if and only if $< g, T_{\underline{k}} g >= \delta_{\underline{k},\underline{0}}$. Now

$$\begin{aligned}
< g, T_{\underline{k}} g > &=< \hat{g}, \widehat{T_{\underline{k}} g} >=< \hat{g}, E_{-\underline{k}} \hat{g} > \\
&= \int_{\mathbb{R}^2} \hat{g}(\underline{\xi}) e^{2i\pi \underline{\xi}\cdot\underline{k}} \overline{\hat{g}(\underline{\xi})}\, d\underline{\xi} = \int_{\mathbb{R}^2} |\hat{g}(\underline{\xi})|^2 e^{2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi} \\
&= \sum_{\underline{m}\in\mathbb{Z}^2} \int_{\underline{m}+[0,1]^2} |\hat{g}(\underline{\xi})|^2 e^{-2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi} \\
&= \lim_{N\to\infty} \sum_{\underline{m}\in I_N} \int_{\underline{m}+[0,1]^2} |\hat{g}(\underline{\xi})|^2 e^{-2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi}.
\end{aligned}$$

Replace $\underline{\xi}$ by $\underline{\xi} + \underline{m}$ inside the integral, and obtain

$$\begin{aligned}
< g, T_{\underline{k}} g > &= \lim_{N\to\infty} \sum_{\underline{m}\in I_N} \int_{[0,1]^2} |\hat{g}(\underline{\xi}+\underline{m})|^2 e^{-2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi} \\
&= \lim_{N\to\infty} \int_{[0,1]^2} \sum_{\underline{m}\in I_N} |\hat{g}(\underline{\xi}+\underline{m})|^2 e^{-2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi}
\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}
< g, T_{\underline{k}} g > &= \int_{[0,1]^2} \lim_{N\to\infty} \sum_{\underline{m}\in I_N} |\hat{g}(\underline{\xi}+\underline{m})|^2 e^{-2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi} \\
&= \int_{[0,1]^2} \sum_{\underline{m}\in\mathbb{Z}^2} |\hat{g}(\underline{\xi}+\underline{m})|^2 e^{-2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi} \\
&= \int_{[0,1]^2} h(\underline{\xi}) e^{-2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi} = \hat{h}(\underline{k}).
\end{aligned}$$

Recall that the Fourier transform is 1-1, and it is easy to check that if $f = 1$, then $\hat{f}(\underline{k}) = \delta_{\underline{k},\underline{0}}$. Hence $\{T_{\underline{k}} g\}_{\underline{k}\in\mathbb{Z}^2}$ is orthonormal if and only if $\hat{h}(\underline{k}) = \delta_{\underline{k},\underline{0}}$ if and only if $h(\underline{\xi}) = 1$ a.e. We thus have shown that $\{T_{\underline{k}} g\}_{\underline{k}\in\mathbb{Z}^2}$ is an orthonormal system if and only if $\sum_{\underline{m}\in\mathbb{Z}^2} |\hat{g}(\underline{\xi}+\underline{m})|^2 = 1$ a.e. □

Theorem 4.2. *Let $g \in L^2(\mathbb{R}^2)$ be such that the system of translates $\{T_{\underline{m}} g\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal set, and let $V = \overline{\mathrm{span}\{T_{\underline{m}} g\}_{\underline{m}\in\mathbb{Z}^2}}$. Let $f \in L^2(\mathbb{R}^2)$ be arbitrary. Then $f \in V$ if and only if there exists $h \in L^2_1(\mathbb{R}^2)$ such that $\hat{f}(\underline{\xi}) = \hat{g}(\underline{\xi})h(\underline{\xi})$ a.e.*

Proof. ($\Longrightarrow$) Suppose, $f \in V$. Since $\{T_{\underline{m}}g\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal basis of V,

$$f = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} T_{\underline{m}} g,$$

where $c_{\underline{m}} =< f, T_{\underline{m}} g >$ and by theorem 2.16 $\sum_{\underline{m}\in\mathbb{Z}^2} |c_{\underline{m}}|^2 = \|f\|_2^2 < \infty$. In particular, $\{c_{\underline{m}}\}_{\underline{m}\in\mathbb{Z}^2} \in l^2(\mathbb{Z}^2)$. Apply the Fourier transform,

$$\begin{aligned}\hat{f} &= \mathcal{F}\left(\sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} T_{\underline{m}} g\right) = \mathcal{F}\left(\lim_{N\to\infty} \sum_{\underline{m}\in I_N} c_{\underline{m}} T_{\underline{m}} g\right) \\ &= \lim_{N\to\infty} \mathcal{F}\left(\sum_{\underline{m}\in I_N} c_{\underline{m}} T_{\underline{m}} g\right) = \lim_{N\to\infty} \sum_{\underline{m}\in I_N} c_{\underline{m}} \widehat{T_{\underline{m}} g} \\ &= \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} E_{-\underline{m}} \hat{g}.\end{aligned}$$

Thus,

$$\hat{f}(\underline{\xi}) = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} \left(E_{-\underline{m}}\hat{g}\right)(\underline{\xi}) = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}} \hat{g}(\underline{\xi}). \tag{4.9}$$

Let $h(\underline{\xi}) = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}} \in L_1^2(\mathbb{R}^2)$. Then

$$h_N(\underline{\xi}) := \sum_{\underline{m}\in I_N} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}} \to h(\underline{\xi}) \quad \text{in} \quad \|\cdot\|_{L^2[0,1]^2} \text{ as } N \to \infty.$$

By theorem 2.7, there exists a subsequence $\{h_{N_k}\}_{k\in\mathbb{N}}$ such that

$$h_{N_k}(\underline{\xi}) \to h(\underline{\xi}) \quad \text{a.e. on } \mathbb{R}^2.$$

Then

$$\hat{g}(\underline{\xi}) h_{N_k}(\underline{\xi}) \to \hat{g}(\underline{\xi}) h(\underline{\xi}) \quad \text{a.e. on } \mathbb{R}^2$$

while by (4.9)

$$\begin{aligned}\hat{g}(\underline{\xi}) h_{N_k}(\underline{\xi}) &= \hat{g}(\underline{\xi}) \sum_{\underline{m}\in I_{N_k}} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}} \\ &= \sum_{\underline{m}\in I_{N_k}} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}} \hat{g}(\underline{\xi}) \to \hat{f}(\underline{\xi}) \quad \text{in } \|\cdot\|_2 \quad \text{as } k \to \infty.\end{aligned}$$

By uniqueness of limits,

$$\hat{f}(\underline{\xi}) = \hat{g}(\underline{\xi})h(\underline{\xi}) \quad \text{a.e.}$$

($\Longleftarrow$) Suppose, there exists $h \in L^2_1(\mathbb{R}^2)$ such that $\hat{f}(\underline{\xi}) = \hat{g}(\underline{\xi})h(\underline{\xi})$ a.e. Since $h \in L^2[0,1]^2$, and $\{e^{-2i\pi\underline{\xi}\cdot\underline{m}}\}_{\underline{m}\in\mathbb{Z}^2}$ is a basis for $L^2[0,1]^2$, there exists a unique sequence $\{c_{\underline{m}}\}_{\underline{m}\in\mathbb{Z}^2} \in l^2(\mathbb{Z}^2)$ such that

$$h(\underline{\xi}) = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}$$

where $c_{\underline{m}} =< h, e^{-2i\pi\underline{\xi}\cdot\underline{m}} >_{L^2[0,1]^2}$. Set

$$f_N = \sum_{\underline{m}\in I_N} c_{\underline{m}} T_{\underline{m}} g \in V$$

and

$$f_0 = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} T_{\underline{m}} g \in V. \tag{4.10}$$

Claim $f = f_0$.

In fact, $\widehat{f_N}(\underline{\xi}) = \sum_{\underline{m}\in I_N} c_{\underline{m}} \left(E_{-\underline{m}}\hat{g}\right)(\underline{\xi}) = \sum_{\underline{m}\in I_N} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}\hat{g}(\underline{\xi})$. Note that by theorem 4.1, $|\hat{g}(\xi)| \leq 1$ a.e. Let h_N, h be as in the first part of the proof. Then

$$\begin{aligned}
\|f - f_N\|_2^2 &= \|\hat{f} - \widehat{f_N}\|_2^2 = \int_{\mathbb{R}^2} |\hat{f}(\underline{\xi}) - \widehat{f_N}(\underline{\xi})|^2 \, d\,\underline{\xi} \\
&= \int_{\mathbb{R}^2} |\hat{g}(\underline{\xi})h(\underline{\xi}) - \sum_{\underline{m}\in I_N} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}\hat{g}(\underline{\xi})|^2 \, d\,\underline{\xi} \\
&= \int_{\mathbb{R}^2} |\hat{g}(\underline{\xi})|^2 |h(\underline{\xi}) - h_N(\underline{\xi})|^2 d\,\underline{\xi} \\
&\leq \int_{\mathbb{R}^2} |h(\underline{\xi}) - h_N(\underline{\xi})|^2 d\,\underline{\xi} \\
&= \|h - h_N\|_2^2 \to 0 \text{ as } N \to \infty.
\end{aligned}$$

This proves the claim. and that by (4.10), $f = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} T_{\underline{m}} g \in V$. $\square$

Theorem 4.3. *Let $\{\psi^{\underline{r}}\}_{\underline{r}\in\mathcal{R}}$ be defined as in (4.2), and W_0 as in (4.7). Then $\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}}$ is an orthonormal basis of W_0.*

Proof. We need to show that

1. $\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}}$ is an orthonormal collection,

2. $T_{\underline{m}}\psi^{\underline{r}} \in W_0 \qquad$ for all $\underline{m} \in \mathbb{Z}^2, \underline{r} \in \mathcal{R}$,

3. $W_0 = \overline{\operatorname{span}\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}}}$.

For 1. and 2., we will in fact show that $\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}_0}$ is orthonormal by using theorem 4.1. Since $\{T_{\underline{m}}\varphi\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal basis of V_0, we have, by theorem 4.1, that

$$\sum_{\underline{m}\in\mathbb{Z}^2} |\hat{\varphi}(\underline{\xi}+\underline{m})|^2 = 1 \qquad \text{a.e.} \tag{4.11}$$

However note that by (4.5),

$$\sum_{\underline{m}\in\mathbb{Z}^2} |\hat{\varphi}(\underline{\xi}+\underline{m})|^2 = \sum_{\underline{m}\in\mathbb{Z}^2} |m_{\underline{0}}\left((\underline{\xi}+\underline{m})A^{-1}\right)\hat{\varphi}\left((\underline{\xi}+\underline{m})A^{-1}\right)|^2.$$

Now each $\underline{m} \in \mathbb{Z}^2$ can be written uniquely as $\underline{m} = \underline{n}A + \underline{s} \quad (\underline{n} \in \mathbb{Z}^2, \underline{s} \in \mathcal{R}_0)$. So we have

$$\begin{aligned}\sum_{\underline{m}\in\mathbb{Z}^2} |\hat{\varphi}(\underline{\xi}+\underline{m})|^2 &= \sum_{\underline{s}\in\mathcal{R}_0}\sum_{\underline{n}\in\mathbb{Z}^2} |m_{\underline{0}}((\underline{\xi}+\underline{n}A+\underline{s})A^{-1})\hat{\varphi}((\underline{\xi}+\underline{n}A+\underline{s})A^{-1})|^2 \\ &= \sum_{\underline{s}\in\mathcal{R}_0}\sum_{\underline{n}\in\mathbb{Z}^2} |m_{\underline{0}}((\underline{\xi}+\underline{s})A^{-1}+\underline{n})\hat{\varphi}((\underline{\xi}+\underline{s})A^{-1}+\underline{n})|^2.\end{aligned}$$

Since $m_{\underline{0}}$ is $\underline{1}$-periodic, we have $m_{\underline{0}}(\underline{\xi}+\underline{n}) = m_{\underline{0}}(\underline{\xi})$ for all $\underline{n} \in \mathbb{Z}^2$, and hence

$$\begin{aligned}\sum_{\underline{m}\in\mathbb{Z}^2} |\hat{\varphi}(\underline{\xi}+\underline{m})|^2 &= \sum_{\underline{s}\in\mathcal{R}_0} |m_{\underline{0}}((\underline{\xi}+\underline{s})A^{-1})|^2 \underbrace{\sum_{\underline{n}\in\mathbb{Z}^2} |\hat{\varphi}((\underline{\xi}+\underline{s})A^{-1}+\underline{n})|^2}_{=1 \text{ a.e. by theorem 4.1}} \\ &= \sum_{\underline{s}\in\mathcal{R}_0} |m_{\underline{0}}((\underline{\xi}+\underline{s})A^{-1})|^2 \quad \text{a.e.}\end{aligned} \tag{4.12}$$

Using the same computations as above, but with $\widehat{\psi^{\underline{r}}}$ instead of $\hat{\varphi}$ we obtain applying

(4.6),

$$\begin{aligned}
\sum_{\underline{m}\in\mathbb{Z}^2} |\widehat{\psi^{\underline{r}}}(\underline{\xi}+\underline{m})|^2 &= \sum_{\underline{s}\in\mathcal{R}_0} |m_{\underline{r}}((\underline{\xi}+\underline{s})A^{-1})|^2 \\
&= \sum_{\underline{s}\in\mathcal{R}_0} |e^{-2i\pi((\underline{\xi}+\underline{s})A^{-1}-\underline{r}A^{-1})} \cdot \overline{m_{\underline{0}}((\underline{\xi}+\underline{s})A^{-1}-\underline{r}A^{-1})}|^2 \\
&= \sum_{\underline{s}\in\mathcal{R}_0} |m_{\underline{0}}((\underline{\xi}-\underline{r}+\underline{s})A^{-1})|^2. \qquad (4.13)
\end{aligned}$$

Hence by (4.11) and (4.12), we have

$$\sum_{\underline{m}\in\mathbb{Z}^2} |\widehat{\psi^{\underline{r}}}(\underline{\xi}+\underline{m})|^2 = \sum_{\underline{m}\in\mathbb{Z}^2} |\hat{\varphi}(\underline{\xi}-\underline{r}+\underline{m})|^2 = 1 \text{ a.e.} \qquad (4.14)$$

We remark that (4.13) and (4.14) imply that $|m_{\underline{r}}(\underline{\xi})| \leq 1$ a.e. for all $\underline{r} \in \mathcal{R}_0$, and hence $m_{\underline{r}} \in L^\infty(\mathbb{R}^2)$ as well. Now observe that

$$< T_{\underline{k}}\psi^{\underline{r}}, T_{\underline{l}}\psi^{\underline{s}} > = < \psi^{\underline{r}}, T_{\underline{l}-\underline{k}}\psi^{\underline{s}} > \qquad (4.15)$$

for all $\underline{k}, \underline{l} \in \mathbb{Z}^2$, $\underline{r}, \underline{s} \in \mathcal{R}_0$. So in order to show that $\{T_{\underline{k}}\psi^{\underline{r}}\}_{\underline{k}\in\mathbb{Z}^2, \underline{r}\in\mathcal{R}_0}$ is orthonormal, it is enough to show that

$$< \psi^{\underline{r}}, T_{\underline{k}}\psi^{\underline{s}} > = \delta_{\underline{k},\underline{0}}\delta_{\underline{r},\underline{s}}.$$

Now by (4.6),

$< \psi^{\underline{r}}, T_{\underline{k}}\psi^{\underline{s}} > = < \widehat{\psi^{\underline{r}}}, E_{-\underline{k}}\widehat{\psi^{\underline{s}}} >$

$$\begin{aligned}
&= \int_{\mathbb{R}^2} \widehat{\psi^{\underline{r}}}(\underline{\xi})\overline{\widehat{\psi^{\underline{s}}}(\underline{\xi})}e^{-2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi} \\
&= \int_{\mathbb{R}^2} m_{\underline{r}}\left(\underline{\xi}A^{-1}\right)\hat{\varphi}\left(\underline{\xi}A^{-1}\right)\overline{m_{\underline{s}}\left(\underline{\xi}A^{-1}\right)\hat{\varphi}\left(\underline{\xi}A^{-1}\right)}e^{2i\pi\underline{\xi}\cdot\underline{k}}\, d\underline{\xi} \\
&= \int_{\mathbb{R}^2} e^{-2i\pi(\underline{\xi}A^{-1}-\underline{r}A^{-1})\cdot\underline{a}}e^{2i\pi(\underline{\xi}A^{-1}-\underline{s}A^{-1})\cdot\underline{a}}e^{2i\pi\underline{\xi}\cdot\underline{k}}\overline{m_{\underline{0}}\left(\underline{\xi}A^{-1}-\underline{r}A^{-1}\right)} \\
&\qquad\qquad m_{\underline{0}}\left(\underline{\xi}A^{-1}-\underline{s}A^{-1}\right)|\hat{\varphi}\left(\underline{\xi}A^{-1}\right)|^2\, d\underline{\xi} \\
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}\int_{\mathbb{R}^2} e^{2i\pi\underline{\xi}\cdot\underline{k}}\,\overline{m_{\underline{0}}\left(\underline{\xi}A^{-1}-\underline{r}A^{-1}\right)}\, m_{\underline{0}}\left(\underline{\xi}A^{-1}-\underline{s}A^{-1}\right)|\hat{\varphi}\left(\underline{\xi}A^{-1}\right)|^2\, d\underline{\xi}.
\end{aligned}$$

Next replace $\underline{\xi}$ by $\underline{\xi}A$,

$< \psi^{\underline{r}}, T_{\underline{k}}\psi^{\underline{s}} >$

$$
\begin{aligned}
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}|\det A|\int_{\mathbb{R}^2} e^{2i\pi\underline{\xi}A\cdot\underline{k}}\overline{m_{\underline{0}}\left(\underline{\xi}-\underline{r}A^{-1}\right)}m_{\underline{0}}\left(\underline{\xi}-\underline{s}A^{-1}\right)|\hat{\varphi}\left(\underline{\xi}\right)|^2\,d\underline{\xi} \\
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}a_1a_2\sum_{\underline{l}\in\mathbb{Z}^2}\int_{\underline{l}+[0,1]^2} e^{2i\pi\underline{\xi}A\cdot\underline{k}}\overline{m_{\underline{0}}\left(\underline{\xi}-\underline{r}A^{-1}\right)}m_{\underline{0}}\left(\underline{\xi}-\underline{s}A^{-1}\right)|\hat{\varphi}\left(\underline{\xi}\right)|^2\,d\underline{\xi}.
\end{aligned}
$$

Replace $\underline{\xi}$ by $\underline{\xi}+\underline{l}$ inside the integral, and obtain

$$
\begin{aligned}
< \psi^{\underline{r}}, T_{\underline{k}}\psi^{\underline{s}} > &= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}a_1a_2\sum_{\underline{l}\in\mathbb{Z}^2}\int_{[0,1]^2} e^{2i\pi(\underline{\xi}+\underline{l})A\cdot\underline{k}}\overline{m_{\underline{0}}\left((\underline{\xi}+\underline{l})-\underline{r}A^{-1}\right)} \\
&\qquad m_{\underline{0}}\left((\underline{\xi}+\underline{l})-\underline{s}A^{-1}\right)|\hat{\varphi}\left(\underline{\xi}+\underline{l}\right)|^2\,d\underline{\xi}.
\end{aligned}
$$

Now as $m_{\underline{0}}$ is $\underline{1}$-periodic, since $e^{2i\pi\underline{l}A\cdot\underline{k}}=1$ for all $\underline{k},\underline{l}\in\mathbb{Z}^2$, and since A has integer entries, then

$< \psi^{\underline{r}}, T_{\underline{k}}\psi^{\underline{s}} >$

$$
\begin{aligned}
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}a_1a_2\int_{[0,1]^2} e^{2i\pi\underline{\xi}A\cdot\underline{k}}\overline{m_{\underline{0}}\left(\underline{\xi}-\underline{r}A^{-1}\right)}m_{\underline{0}}\left(\underline{\xi}-\underline{s}A^{-1}\right) \\
&\qquad \underbrace{\sum_{\underline{l}\in\mathbb{Z}^2}|\hat{\varphi}\left(\underline{\xi}+\underline{l}\right)|^2}_{=1 \text{ a.e. by theorem 4.1}}\;d\underline{\xi} \\
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}a_1a_2\int_{[0,1]^2} e^{2i\pi\underline{\xi}A\cdot\underline{k}}\overline{m_{\underline{0}}\left(\underline{\xi}-\underline{r}A^{-1}\right)}m_{\underline{0}}\left(\underline{\xi}-\underline{s}A^{-1}\right)\,d\underline{\xi} \\
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}a_1a_2\int_{[0,1]^2} e^{2i\pi\underline{\xi}A\cdot\underline{k}}\frac{1}{\sqrt{a_1a_2}}\left(\sum_{\underline{m}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}e^{2i\pi(\underline{\xi}-\underline{r}A^{-1})\cdot\underline{m}}\right) \\
&\qquad \frac{1}{\sqrt{a_1a_2}}\left(\sum_{\underline{n}\in\mathbb{Z}^2}h_{\underline{n}}e^{-2i\pi(\underline{\xi}-\underline{s}A^{-1})\cdot\underline{n}}\right)d\underline{\xi} \\
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}\int_{[0,1]^2} e^{2i\pi\underline{\xi}A\cdot\underline{k}}\sum_{\underline{m}\in\mathbb{Z}^2}\sum_{\underline{n}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}h_{\underline{n}}e^{2i\pi\underline{\xi}\cdot(\underline{m}-\underline{n})}e^{2i\pi(\underline{s}A^{-1}\cdot\underline{n}-\underline{r}A^{-1}\cdot\underline{m})}\,d\underline{\xi} \\
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}\sum_{\underline{m}\in\mathbb{Z}^2}\sum_{\underline{n}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}h_{\underline{n}}\underbrace{\int_{[0,1]^2} e^{2i\pi\underline{\xi}\cdot(\underline{m}-\underline{n}+A\underline{k})}\,d\underline{\xi}}_{\delta_{\underline{n},\underline{m}+A\underline{k}}}e^{2i\pi(\underline{s}A^{-1}\cdot\underline{n}-\underline{r}A^{-1}\cdot\underline{m})} \\
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}}\sum_{\underline{m}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}h_{\underline{m}+A\underline{k}}e^{2i\pi[\underline{s}A^{-1}\cdot(\underline{m}+A\underline{k})-\underline{r}A^{-1}\cdot\underline{m}]}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}} \sum_{\underline{m}\in\mathbb{Z}^2} \overline{h_{\underline{m}}} h_{\underline{m}+A\underline{k}} e^{2i\pi[(\underline{s}-\underline{r})A^{-1}\cdot\underline{m}+\underline{s}\cdot\underline{k}]} \qquad\qquad (e^{2i\pi\underline{s}\cdot\underline{k}}=1)\\
&= e^{2i\pi(\underline{r}-\underline{s})A^{-1}\cdot\underline{a}} \underbrace{\sum_{\underline{m}\in\mathbb{Z}^2} \overline{h_{\underline{m}}} h_{\underline{m}+A\underline{k}} e^{2i\pi(\underline{s}-\underline{r})A^{-1}\cdot\underline{m}}}_{=0 \quad \text{unless } \underline{k}=\underline{0},\ \underline{s}=\underline{r} \text{ by } (M6)}\\
&= \delta_{\underline{k},\underline{0}}\cdot\delta_{\underline{r},\underline{s}} \underbrace{\sum_{\underline{m}\in\mathbb{Z}^2} |h_{\underline{m}}|^2}_{=\|\varphi\|_2=1} = \delta_{\underline{k},\underline{0}}\cdot\delta_{\underline{r},\underline{s}},
\end{aligned}
$$

where we have used $(M6)$ and (4.1).

This shows that $\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}_0}$ is an orthogonal family. In particular, $T_{\underline{m}}\psi^{\underline{r}} \perp T_{\underline{k}}\psi^{\underline{0}} = T_{\underline{k}}\varphi$ for all $\underline{m},\underline{k}\in\mathbb{Z}^2, \underline{r}\in\mathcal{R}$. Since $V_1 = V_0\oplus W_0$ and $\{T_{\underline{k}}\varphi\}$ is an orthonormal basis of V_0, it follows that $T_{\underline{m}}\psi^{\underline{r}}\in W_0$ for all $\underline{r}\in\mathcal{R}$. That is, $\overline{\operatorname{span}\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}}} \subset W_0$.

If is left to show that $W_0\subset \overline{\operatorname{span}\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}}}$. Since $V_1 = V_0\oplus W_0$, and $\{T_{\underline{m}}\psi^{\underline{0}}\}_{\underline{m}\in\mathbb{Z}^2}$ is an orthonormal basis of V_0, it is enough to show that

$$V_1 \subset \overline{\operatorname{span}\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}_0}}.$$

Let $f\in V_1$ be arbitrary. Recall from (4.4) that $\hat{f}(\underline{\xi}) = m_f\left(\underline{\xi}A^{-1}\right)\hat{\varphi}\left(\underline{\xi}A^{-1}\right)$, where $m_f\left(\underline{\xi}\right) = \frac{1}{\sqrt{a_1a_2}}\sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} e^{-2i\pi\underline{\xi}\cdot\underline{m}}$.

We claim that there exist functions $h_{\underline{r}}\in L^2_1(\mathbb{R}^2)$ such that

$$m_f\left(\underline{\xi}A^{-1}\right) = \sum_{\underline{r}\in\mathcal{R}_0} h_{\underline{r}}(\underline{\xi}) m_{\underline{r}}\left(\underline{\xi}A^{-1}\right), \tag{4.16}$$

with convergence in $L^2_1(\mathbb{R}^2)$. If (4.16) holds, then

$$\hat{f}(\underline{\xi}) = m_f\left(\underline{\xi}A^{-1}\right)\hat{\varphi}\left(\underline{\xi}A^{-1}\right) = \sum_{\underline{r}\in\mathcal{R}_0} h_{\underline{r}}(\underline{\xi}) m_{\underline{r}}\left(\underline{\xi}A^{-1}\right)\hat{\varphi}\left(\underline{\xi}A^{-1}\right) = \sum_{\underline{r}\in\mathcal{R}_0} h_{\underline{r}}(\underline{\xi})\widehat{\psi^{\underline{r}}}(\underline{\xi}),$$

so that by theorem 4.2, $f = \sum_{\underline{r}\in\mathcal{R}_0} f^{\underline{r}}$, with $f^{\underline{r}}\in\overline{\operatorname{span}\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2}}$ for each $\underline{r}$, that is, $f\in\overline{\operatorname{span}\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}_0}}$. We thus will have shown that

$$V_1 \subset \overline{\operatorname{span}\{T_{\underline{m}}\psi^{\underline{r}}\}_{\underline{m}\in\mathbb{Z}^2,\underline{r}\in\mathcal{R}_0}},$$

and the proof will be complete. To prove the claim, change $\underline{\xi}$ to $\underline{\xi}+\underline{s}$ in (4.16),

$$\begin{aligned} m_f\left(\underline{\xi}A^{-1}+\underline{s}A^{-1}\right) &= \sum_{\underline{r}\in\mathcal{R}_0} h_{\underline{r}}(\underline{\xi}+\underline{s})\, m_{\underline{r}}\left((\underline{\xi}+\underline{s})A^{-1}\right) \qquad (\underline{s}\in\mathcal{R}_0) \\ &= \sum_{\underline{r}\in\mathcal{R}_0} h_{\underline{r}}(\underline{\xi})\, m_{\underline{r}}\left((\underline{\xi}+\underline{s})A^{-1}\right) \qquad \text{a.e.} \end{aligned} \tag{4.17}$$

by expected periodicity of $h_{\underline{r}}$. Equations (4.17) can be written as a matrix equation,

$$\begin{bmatrix} m_f\left(\underline{\xi}A^{-1}\right) \\ \vdots \\ m_f\left((\underline{\xi}+\underline{a})A^{-1}\right) \end{bmatrix} = T \begin{bmatrix} h_{\underline{0}}(\underline{\xi}) \\ \vdots \\ h_{\underline{a}}(\underline{\xi}) \end{bmatrix} \tag{4.18}$$

where we have fixed an order for the elements of $\mathcal{R}_0$ so that $\underline{0}\le\underline{i}\le\underline{a}$ for all $\underline{i}\in\mathcal{R}_0$, and T is the $a_1a_2\times a_1a_2$ matrix,

$$T=(t_{\underline{i},\underline{j}})=\begin{bmatrix} m_{\underline{0}}\left(\underline{\xi}A^{-1}\right) & \cdots & m_{\underline{a}}\left(\underline{\xi}A^{-1}\right) \\ \vdots & \ddots & \vdots \\ m_{\underline{0}}\left((\underline{\xi}+\underline{a})A^{-1}\right) & \cdots & m_{\underline{a}}\left((\underline{\xi}+\underline{a})A^{-1}\right) \end{bmatrix}.$$

Then

$$T^*=(\bar{t}_{\underline{j},\underline{i}})=\begin{bmatrix} \overline{m_{\underline{0}}\left(\underline{\xi}A^{-1}\right)} & \cdots & \overline{m_{\underline{0}}\left((\underline{\xi}+\underline{a})A^{-1}\right)} \\ \vdots & \ddots & \vdots \\ \overline{m_{\underline{a}}\left(\underline{\xi}A^{-1}\right)} & \cdots & \overline{m_{\underline{a}}\left((\underline{\xi}+\underline{a})A^{-1}\right)} \end{bmatrix}.$$

We now show that T is unitary. Let $TT^*=\begin{bmatrix} \alpha_{(\underline{0},\underline{0})} & \cdots & \alpha_{(\underline{0},\underline{a})} \\ \vdots & \ddots & \vdots \\ \alpha_{(\underline{a},\underline{0})} & \cdots & \alpha_{(\underline{a},\underline{a})} \end{bmatrix}$. Then by (4.6),

$$\alpha_{(\underline{i},\underline{j})} = \sum_{\underline{r}\in\mathcal{R}_0} m_{\underline{r}}\left((\underline{\xi}+\underline{i})A^{-1}\right)\overline{m_{\underline{r}}\left((\underline{\xi}+\underline{j})A^{-1}\right)}$$

$$\begin{aligned}
&= \sum_{\underline{r}\in\mathcal{R}_0} e^{-2i\pi\left((\underline{\xi}+\underline{i})A^{-1}-\underline{r}A^{-1}\right)\cdot\underline{a}}\overline{e^{-2i\pi\left((\underline{\xi}+\underline{j})A^{-1}-\underline{r}A^{-1}\right)\cdot\underline{a}}m_{\underline{0}}\left((\underline{\xi}+\underline{i})A^{-1}-\underline{r}A^{-1}\right)} \\
&\qquad m_{\underline{0}}\left((\underline{\xi}+\underline{j})A^{-1}-\underline{r}A^{-1}\right) \\
&= \sum_{\underline{r}\in\mathcal{R}_0} e^{-2i\pi\left((\underline{i}-\underline{j})A^{-1}\right)\cdot\underline{a}}\overline{m_{\underline{0}}\left((\underline{\xi}-\underline{r}+\underline{i})A^{-1}\right)}m_{\underline{0}}\left((\underline{\xi}-\underline{r}+\underline{j})A^{-1}\right) \\
&= \sum_{\underline{r}\in\mathcal{R}_0} e^{-2i\pi\left((\underline{i}-\underline{j})A^{-1}\right)\cdot\underline{a}}\left(\frac{1}{\sqrt{a_1a_2}}\sum_{\underline{m}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}e^{2i\pi(\underline{\xi}-\underline{r}+\underline{i})A^{-1}\cdot\underline{m}}\right) \\
&\qquad \left(\frac{1}{\sqrt{a_1a_2}}\sum_{\underline{n}\in\mathbb{Z}^2}h_{\underline{n}}e^{-2i\pi(\underline{\xi}-\underline{r}+\underline{j})A^{-1}\cdot\underline{n}}\right) \text{ a.e} \\
&= \sum_{\underline{r}\in\mathcal{R}_0}\frac{1}{a_1a_2}e^{-2i\pi\left((\underline{i}-\underline{j})A^{-1}\right)\cdot\underline{a}}\sum_{\underline{m}\in\mathbb{Z}^2}\sum_{\underline{n}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}h_{\underline{n}}e^{2i\pi(\underline{\xi}-\underline{r})A^{-1}\cdot(\underline{m}-\underline{n})}e^{2i\pi\underline{i}A^{-1}\cdot\underline{m}}e^{-2i\pi\underline{j}A^{-1}\cdot\underline{n}} \\
&= \frac{1}{a_1a_2}e^{-2i\pi\left((\underline{i}-\underline{j})A^{-1}\right)\cdot\underline{a}}\sum_{\underline{m}\in\mathbb{Z}^2}\sum_{\underline{n}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}h_{\underline{n}}e^{2i\pi\underline{\xi}A^{-1}\cdot(\underline{m}-\underline{n})}e^{2i\pi\underline{i}A^{-1}\cdot\underline{m}}e^{-2i\pi\underline{j}A^{-1}\cdot\underline{n}} \\
&\qquad \sum_{\underline{r}\in\mathcal{R}_0} e^{2i\pi\underline{r}A^{-1}\cdot(\underline{n}-\underline{m})}.
\end{aligned}$$

Note that for $\underline{l}\in\mathbb{Z}^2$,

$$\begin{aligned}
\sum_{\underline{r}\in\mathcal{R}_0} e^{2i\pi\underline{r}A^{-1}\cdot\underline{l}} &= \sum_{r_1=0}^{a_1-1}\sum_{r_2=0}^{a_2-1} e^{2i\pi r_1l_1/a_1}e^{2i\pi r_2l_2/a_2} = \left(\sum_{r_1=0}^{a_1-1}e^{2i\pi r_1l_1/a_1}\right)\left(\sum_{r_2=0}^{a_2-1}e^{2i\pi r_2l_2/a_2}\right) \\
&= a_1\delta_{l_1,k_1a_1}\cdot a_2\delta_{l_2,k_2a_2} = a_1a_2\delta_{\underline{l},A\underline{k}}, \quad (\underline{k}=(k_1,k_2)\in\mathbb{Z}^2) \qquad (4.19)
\end{aligned}$$

so that with $\underline{l}=\underline{n}-\underline{m}$,

$$\begin{aligned}
\alpha_{(\underline{i},\underline{j})} &= e^{-2i\pi\left((\underline{i}-\underline{j})A^{-1}\right)\cdot\underline{a}}\sum_{\underline{k}\in\mathbb{Z}^2}\sum_{\underline{m}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}h_{\underline{m}+A\underline{k}}e^{2i\pi\underline{\xi}A^{-1}\cdot(\underline{m}-\underline{m}-A\underline{k})} \\
&\qquad e^{2i\pi\underline{i}A^{-1}\cdot\underline{m}}e^{-2i\pi\underline{j}A^{-1}\cdot(\underline{m}+A\underline{k})} \\
&= e^{-2i\pi\left((\underline{i}-\underline{j})A^{-1}\right)\cdot\underline{a}}\sum_{\underline{k}\in\mathbb{Z}^2}e^{-2i\pi\underline{\xi}\cdot\underline{k}}\underbrace{\sum_{\underline{m}\in\mathbb{Z}^2}\overline{h_{\underline{m}}}h_{\underline{m}+A\underline{k}}e^{2i\pi(\underline{i}-\underline{j})A^{-1}\cdot\underline{m}}}_{=0\text{ unless }\underline{k}=\underline{0},\underline{i}=\underline{j}\quad\text{by }(M6)} \quad (e^{-2i\pi(\underline{j}\cdot\underline{k})}=1) \\
&= \delta_{\underline{i},\underline{j}}\underbrace{\sum_{\underline{m}\in\mathbb{Z}^2}|h_{\underline{m}}|^2}_{=\|\varphi\|_2=1} = \delta_{\underline{i},\underline{j}}.
\end{aligned}$$

This shows that $TT^* = I$ a.e., that is, T is unitary a.e., so that T is invertible and $T^{-1}(\underline{\xi}) = T^*(\underline{\xi})$ a.e.. Hence, system (4.18) has a solution, namely

$$\begin{bmatrix} h_{\underline{0}}(\underline{\xi}) \\ \vdots \\ h_{\underline{a}}(\underline{\xi}) \end{bmatrix} = T^* \begin{bmatrix} m_f\left(\underline{\xi}A^{-1}\right) \\ \vdots \\ m_f\left((\underline{\xi}+\underline{a})A^{-1}\right) \end{bmatrix} \text{ a.e.}$$

Computing this product we obtain

$$h_{\underline{s}}(\underline{\xi}) = \sum_{\underline{r}\in\mathcal{R}_0} \overline{m_{\underline{s}}\left((\underline{\xi}+\underline{r})A^{-1}\right)} m_f\left((\underline{\xi}+\underline{r})A^{-1}\right) \text{ a.e,} \tag{4.20}$$

for all $\underline{s} \in \mathcal{R}_0$. Next we must show that the functions $h_{\underline{s}}$ are periodic.

$$\begin{aligned} h_{\underline{s}}(\underline{\xi}+\underline{1}) &= \sum_{\underline{r}\in\mathcal{R}_0} \overline{m_{\underline{s}}\left((\underline{\xi}+\underline{1}+\underline{r})A^{-1}\right)} m_f\left((\underline{\xi}+\underline{1}+\underline{r})A^{-1}\right) \\ &= \sum_{r_1=0}^{a_1-1}\sum_{r_2=0}^{a_2-1} \overline{m_{\underline{s}}\left(\frac{\xi_1+1+r_1}{a_1}, \frac{\xi_2+1+r_2}{a_2}\right)} \\ &\qquad\qquad m_f\left(\frac{\xi_1+1+r_1}{a_1}, \frac{\xi_2+1+r_2}{a_2}\right). \end{aligned}$$

Set $\widetilde{\underline{r}} = \underline{r} + \underline{1}$, so that $\widetilde{\underline{r}} = (\widetilde{r_1}, \widetilde{r_2})$ with $\widetilde{r_1} = r_1 + 1$, $\widetilde{r_2} = r_2 + 1$. Then

$$h_{\underline{s}}(\underline{\xi}+\underline{1}) = \sum_{\widetilde{r_1}=1}^{a_1}\sum_{\widetilde{r_2}=1}^{a_2} \overline{m_{\underline{s}}\left(\frac{\xi_1+\widetilde{r_1}}{a_1}, \frac{\xi_2+\widetilde{r_2}}{a_2}\right)} m_f\left(\frac{\xi_1+\widetilde{r_1}}{a_1}, \frac{\xi_2+\widetilde{r_2}}{a_2}\right).$$

Observe that if $\widetilde{r_1} = a_1$, we obtain

$$\begin{aligned} \overline{m_{\underline{s}}\left(\frac{\xi_1+a_1}{a_1}, \cdot\right)} m_f\left(\frac{\xi_1+a_1}{a_1}, \cdot\right) &= \overline{m_{\underline{s}}\left(\frac{\xi_1}{a_1}+1, \cdot\right)} m_f\left(\frac{\xi_1}{a_1}+1, \cdot\right) \\ &= \overline{m_{\underline{s}}\left(\frac{\xi_1}{a_1}, \cdot\right)} m_f\left(\frac{\xi_1}{a_1}, \cdot\right) \end{aligned}$$

as $m_{\underline{s}}$ and m_f are 1-periodic. Similarly,

$$\overline{m_{\underline{s}}\left(\cdot, \frac{\xi_2+a_2}{a_2}\right)} m_f\left(\cdot, \frac{\xi_2+a_2}{a_2}\right) = \overline{m_{\underline{s}}\left(\cdot, \frac{\xi_2}{a_2}\right)} m_f\left(\cdot, \frac{\xi_2}{a_2}\right).$$

Thus

$$h_{\underline{s}}(\underline{\xi}+\underline{1}) = \sum_{\widetilde{r_1}=0}^{a_1-1}\sum_{\widetilde{r_2}=0}^{a_2-1} \overline{m_{\underline{s}}\left(\frac{\xi_1+\widetilde{r_1}}{a_1}, \frac{\xi_2+\widetilde{r_2}}{a_2}\right)} m_f\left(\frac{\xi_1+\widetilde{r_1}}{a_1}, \frac{\xi_2+\widetilde{r_2}}{a_2}\right) = h_{\underline{s}}(\underline{\xi}) \text{ a.e.}$$

As remarked earlier, $m_{\underline{s}} \in L^\infty(\mathbb{R}^2)$ for all $\underline{s} \in \mathcal{R}_0$, and in fact $|m_{\underline{s}}(\xi)| \leq 1$ a.e. Since $m_f \in L_1^2(\mathbb{R}^2)$ it follows from (4.20) that $h_{\underline{s}}(\underline{\xi}) \in L_1^2(\mathbb{R}^2)$ for all $\underline{s} \in \mathcal{R}_0$. This proves the claim and the theorem. $\square$

4.3 The Haar Wavelet in $L^2(\mathbb{R}^2)$

In this section, we explain how Haar wavelets can be defined on $L^2(\mathbb{R}^2)$ by using theorem 4.3. As usual when constructing a multiresolution analysis, we will introduce the scaling function first.

Fix a matrix $A = \left[\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right]$ with $a_1, a_2 \in \{2, 3, \ldots\}$ and let φ be the characteristic function of the unit square, $\varphi = \chi_{[0,1)^2}$. Since the space V_0 should be spanned by the translates of φ, we set

$$V_0 = \{f \in L^2(\mathbb{R}^2) : f \text{ is constant on each square } \underline{m} + [0,1)^2,\ \underline{m} \in \mathbb{Z}^2\}.$$

Let us first check that the translates $\{T_{\underline{m}}\varphi\}_{\underline{m}\in\mathbb{Z}^2}$ form an orthonormal basis of V_0. Arguing as in (4.15), we first need to show that

$$< \varphi, T_{\underline{m}}\varphi >= \delta_{\underline{m},\underline{0}}.$$

Note that

$$(T_{\underline{m}}\varphi)(x) = \varphi(\underline{x} - \underline{m}) = \chi_{[0,1)^2}(\underline{x} - \underline{m}) = \chi_{\underline{m}+[0,1)^2}(x), \tag{4.21}$$

so that

$$(T_{\underline{m}}\varphi)(x) = \begin{cases} 1 & \text{if } \underline{x} \in \underline{m} + [0,1)^2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} < \varphi, T_{\underline{m}}\varphi > &= \int_{\mathbb{R}^2} \varphi(\underline{x})\overline{(T_{\underline{m}}\varphi)(\underline{x})}\, d\underline{x} = \int_{\mathbb{R}^2} \chi_{[0,1)^2}(\underline{x})\chi_{\underline{m}+[0,1)^2}(\underline{x})\, d\underline{x} \\ &= \int_{[0,1)^2} \chi_{\underline{m}+[0,1)^2}(\underline{x})\, d\underline{x} = \begin{cases} \int_{[0,1)^2} 1\, d\underline{x} & \text{if } \underline{m} = \underline{0} \\ \int_{[0,1)^2} 0\, d\underline{x} & \text{if } \underline{m} \neq \underline{0} \end{cases} = \delta_{\underline{m},\underline{0}}. \end{aligned}$$

Next we need to show that $V_0 = \overline{\operatorname{span}\{T_{\underline{m}}\varphi\}_{\underline{m}\in\mathbb{Z}^2}}$. Arguing exactly as in example 3.1, part $(M1)$, one shows that V_0 is a closed linear subspace of $L^2(\mathbb{R}^2)$, and obviously, $T_{\underline{m}}\varphi = \chi_{\underline{m}+[0,1)^2} \in V_0$ for all $\underline{m}$. Now let $f \in V_0$. As

$$f\chi_{[-N,N)\times[-N,N)}(x) \to f(x) \quad \text{pointwise a.e.,}$$

as $N \to \infty$, then by the corollary to the Lebesgue Dominated Convergence theorem,

$$f = \lim_{N\to\infty} f\chi_{[-N,N)\times[-N,N)} = \lim_{N\to\infty} \sum_{j=-N}^{N-1} \sum_{k=-N}^{N-1} f\chi_{[j,j+1)\times[k,k+1)}$$

in the norm of $L^2(\mathbb{R}^2)$. As f is constant on each interval $[j, j+1) \times [k, k+1)$, say f has value $c_{j,k}$ on this interval, then

$$\begin{aligned} f &= \lim_{N\to\infty} \sum_{j=-N}^{N-1} \sum_{k=-N}^{N-1} c_{j,k}\chi_{[j,j+1)\times[k,k+1)} \\ &= \lim_{N\to\infty} \sum_{j=-N}^{N-1} \sum_{k=-N}^{N-1} c_{j,k}\chi_{(j,k)+[0,1)^2} \\ &= \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}}\chi_{\underline{m}+[0,1)^2} = \sum_{\underline{m}\in\mathbb{Z}^2} c_{\underline{m}} T_{\underline{m}}\varphi \end{aligned}$$

where we have set $\underline{m} = (j,k)^T$. This shows that $\{T_{\underline{m}}\varphi\}_{\underline{m}\in\mathbb{Z}^2}$ spans V_0. Thus, $(M5)$ holds.

It is now natural to define the subspaces V_n of $L^2(\mathbb{R}^n)$ by

$$V_n = D_{A^n} V_0. \tag{4.22}$$

Then obviously $V_n = D_A(D_{A^{n-1}}V_0) = D_A V_{n-1}$, so that, $(M4)$ holds.

Let us now show that $(M1)$ holds. Note that

$$V_n = \{f \in L^2(\mathbb{R}^2) : f \text{ is constant on each rectangle } A^{-n}\underline{m} + A^{-n}[0,1)^2,\ \underline{m} \in \mathbb{Z}^2\}.$$

To see this, note that

$$\begin{aligned} f \in V_n &\iff D_{A^{-n}} f \in V_0, \\ &\iff f(A^{-n}\cdot) \in V_0, \\ &\iff f(A^{-n}x) = f(A^{-n}y) \quad \text{whenever}\, x, y \in \underline{m} + [0,1)^2 \,\text{for some}\, \underline{m}, \\ &\iff f(\tilde{x}) = f(\tilde{y}) \quad \text{whenever}\, \tilde{x}, \tilde{y} \in A^{-n}(\underline{m} + [0,1)^2) \,\text{for some}\, \underline{m}, \end{aligned}$$

where we have set $\tilde{x} = A^{-n}x, \tilde{y} = A^{-n}y$. If $n \geq 0$, then each of these rectangles $A^{-n}\underline{m} + A^{-n}[0,1)^2$ is of the form

$$R_1 = \{\underline{x} = (x_1, x_2)^T : \frac{m_1}{a_1{}^n} \leq x_1 \leq \frac{m_1 + 1}{a_1{}^n}, \frac{m_2}{a_2{}^n} \leq x_2 \leq \frac{m_2 + 1}{a_2{}^n}\}$$

and is thus contained in a rectangle whose vertices have integer coordinates,

$$R_2 = \{\underline{x} = (x_1, x_2)^T : \widetilde{m_1} \leq x_1 \leq \widetilde{m_1} + 1, \widetilde{m_2} \leq x_2 \leq \widetilde{m_2} + 1\}$$

for some $\widetilde{m_1}, \widetilde{m_2} \in \mathbb{Z}$. Thus, if f is constant on each rectangle of form R_2, then it will be constant on each rectangle of form R_1. Choosing $n = 1$, it follows that

$$V_0 \subset V_1.$$

Using (4.22) and induction, one now obtains that $V_n \subset V_{n+1}$ for all n. Thus, $(M1)$ holds.

Next we show that $\bigcup_{n=-\infty}^{\infty} V_n$ is dense in $L^2(\mathbb{R}^2)$. Let $f \in L^2(\mathbb{R}^2)$, and $\varepsilon > 0$ be given. Since $C_c(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$, we can pick $g \in C_c(\mathbb{R}^2)$ such that

$$\|f - g\|_2 < \frac{\varepsilon}{2}. \tag{4.23}$$

Since $\mathrm{supp}(g)$ is compact, there exists a finite subset M of $\mathbb{Z}^2$ such that $\mathrm{supp}(g) \subset K := \bigcup_{\underline{m} \in M} \underline{m} + [0,1)^2$, and g is uniformly continuous. Thus, there exists $\delta > 0$ such that

$$\|\underline{x} - \underline{y}\|_\infty < \delta \Longrightarrow |g(\underline{x}) - g(\underline{y})| < \frac{\varepsilon}{2\sqrt{\lambda(K)}}.$$

Let n be sufficiently large, $n \geq 0$, such that

$$\|\underline{x} - \underline{y}\|_\infty < \delta \quad \forall\, \underline{x}, \underline{y} \in A^{-n}[0,1)^2 = [0, {a_1}^{-n}] \times [0, {a_2}^{-n}].$$

Then also

$$\|\underline{x} - \underline{y}\|_\infty < \delta \quad \forall\, \underline{x}, \underline{y} \in A^{-n}\underline{m} + A^{-n}[0,1)^2,\ \forall\, \underline{m} \in \mathbb{Z}^2.$$

Let $M_1 = \{\underline{m} \in \mathbb{Z}^2 : (A^{-n}\underline{m} + A^{-n}[0,1)^2) \subset K\}$. Then $K = \bigcup_{\underline{m} \in M_1} A^{-n}\underline{m} + A^{-n}[0,1)^2$, so if $\underline{x} \in \operatorname{supp}(g)$, then $\underline{x} \in A^{-n}\underline{m} + A^{-n}[0,1)^2$ for some $\underline{m} \in M_1$. For each $\underline{m} \in M_1$, pick a point $x_{\underline{m}} \in A^{-n}\underline{m} + A^{-n}[0,1)^2$, and set

$$h = \sum_{\underline{m} \in M_1} g(x_{\underline{m}}) \chi_{A^{-n}(\underline{m} + [0,1)^2)} \in V_n$$

so that

$$h(\underline{x}) = g(x_{\underline{m}}) \quad \forall\, \underline{x} \in A^{-n}(\underline{m} + [0,1)^2).$$

If $\underline{x} \in A^{-n}(\underline{m} + [0,1)^2)$, for some $\underline{m} \in M_1$, then by choice of n, $\|\underline{x} - x_{\underline{m}}\|_\infty < \delta$, and hence

$$|h(\underline{x}) - g(\underline{x})| = |g(x_{\underline{m}}) - g(\underline{x})| < \frac{\varepsilon}{2\sqrt{\lambda(K)}},$$

while if $x \notin K$ then $h(x) = g(x) = 0$. Thus,

$$\|h - g\|_2^2 = \int_K |h(\underline{x}) - g(\underline{x})|^2\, d\underline{x} \leq \int_K \frac{\varepsilon^2}{4\lambda(K)}\, d\underline{x} = \frac{\varepsilon^2}{4} \cdot \frac{\lambda(K)}{\lambda(K)} = \frac{\varepsilon^2}{4}$$

so that

$$\|h - g\|_2 \leq \frac{\varepsilon}{2}. \tag{4.24}$$

By (4.23) and (4.24),

$$\|f - h\|_2 = \|f - g\|_2 + \|g - h\|_2 < \varepsilon,$$

where $h \in V_n$ for some $n \geq 0$. As ε was arbitrary, $(M2)$ follows.

Now we prove that $(M3)$ holds. Suppose to the contrary, that there exists $f \in \bigcap_{n\in\mathbb{Z}} V_n,\ f \neq 0$. Then $\int_{\mathbb{R}^2} |f|^2\, d\underline{x} > 0$, so there exist a set E of nonzero measure, and a constant $c > 0$ such that $f(x) \geq c$ for all $x \in E$. Now for each $\underline{m} \in \mathbb{Z}^2, n \in \mathbb{Z}$, set

$$E_{\underline{m}}^{(n)} = E \cap A^{-n}(\underline{m} + [0,1)^2).$$

As $\mathbb{R}^2$ is the disjoint union of $\{A^{-n}(\underline{m} + [0,1)^2)\}_{\underline{m}\in\mathbb{Z}^2}$ we have

$$E = \bigcup_{\underline{m}\in\mathbb{Z}^2} E_{\underline{m}}^{(n)},$$

for each n, and

$$\sum_{\underline{m}\in\mathbb{Z}^2} \lambda(E_{\underline{m}}^{(n)}) = \lambda(E) > 0.$$

Hence, there must exist $\underline{m}_n$ such that

$$\lambda(E_{\underline{m}_n}^{(n)}) > 0.$$

Since $f(x) \geq c$ on $E_{\underline{m}_n}^{(n)}$, and f is constant a.e. on $A^{-n}(\underline{m}_n + [0,1)^2)$, then

$$f(x) \geq c \quad \text{a.e. on } A^{-n}(\underline{m}_n + [0,1)^2).$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^2} |f(x)|^2\, d\underline{x} &\geq \int_{A^{-n}(\underline{m}_n+[0,1)^2)} |f(x)|^2\, d\underline{x} \geq \int_{A^{-n}(\underline{m}_n+[0,1)^2)} c^2\, d\underline{x} \\
&= c^2 \cdot \lambda(A^{-n}(\underline{m}_n + [0,1)^2)) = c^2 \cdot |\det A|^{-n} \to \infty \text{ as } n \to -\infty,
\end{aligned}$$

contradicting the assumption that $f \in L^2(\mathbb{R}^2)$. Hence, $(M3)$ holds.

Finally, we verify that $(M6)$ holds. Note that

$$\begin{aligned}
h_{\underline{m}} &=< \varphi, D_A T_{\underline{m}} \varphi >= \int_{\mathbb{R}^2} \varphi(\underline{x}) \overline{|\det A|^{1/2} \varphi(A\underline{x} - \underline{m})}\, d\underline{x} \\
&= \int_{\mathbb{R}^2} \chi_{[0,1)^2}(\underline{x}) \overline{\sqrt{a_1 a_2} \chi_{[0,1)^2}(A\underline{x} - \underline{m})}\, d\underline{x} \\
&= \sqrt{a_1 a_2} \int_{[0,1)^2} \chi_{\underline{m}+[0,1)^2}(A\underline{x})\, d\underline{x}.
\end{aligned} \tag{4.25}$$

$$
\begin{array}{cc|c|c|c|}
\cline{3-5}
 & a_1-1 & \frac{1}{\sqrt{a_1a_2}} & \cdots & \frac{1}{\sqrt{a_1a_2}} \\
\cline{3-5}
m_1 & \vdots & \vdots & \ddots & \vdots \\
\cline{3-5}
 & 0 & \frac{1}{\sqrt{a_1a_2}} & \cdots & \frac{1}{\sqrt{a_1a_2}} \\
\cline{3-5}
\multicolumn{2}{c}{} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{a_2-1} \\
\multicolumn{2}{c}{} & \multicolumn{3}{c}{m_2}
\end{array}
$$

Table 4.1 The scaling filter $\{h_{\underline{m}}\}$.

Now if $\underline{m} = (m_1, m_2)^T$ then

$$
\begin{aligned}
A\underline{x} \in \underline{m} + [0,1)^2 \iff & (a_1x_1, a_2x_2) \in [m_1, m_1+1) \times [m_2, m_2+1) \\
\iff & \begin{cases} m_1 \le a_1x_1 < m_1+1 \\ m_2 \le a_2x_2 < m_2+1 \end{cases} \iff \begin{cases} \frac{m_1}{a_1} \le x_1 < \frac{m_1+1}{a_1} \\ \frac{m_2}{a_2} \le x_2 < \frac{m_2+1}{a_2} \end{cases}.
\end{aligned}
$$

Since $0 \le x_1, x_2 < 1$ on the set of integration, the integral will be nonzero only if $0 \le m_1 < a_1$, $0 \le m_2 < a_2$, that is, for $\underline{m} \in \mathcal{R}_0$. For $\underline{m} \in \mathcal{R}_0$ we have

$$
h_{\underline{m}} = \int_{\frac{m_1}{a_1}}^{\frac{m_1+1}{a_1}} \int_{\frac{m_2}{a_2}}^{\frac{m_2+1}{a_2}} \sqrt{a_1a_2}\, dx_2 dx_1 = \sqrt{a_1a_2}\left(\frac{1}{a_1}\right)\left(\frac{1}{a_2}\right) = \frac{1}{\sqrt{a_1a_2}}.
$$

Hence,

$$
h_{\underline{m}} = \begin{cases} \frac{1}{\sqrt{a_1a_2}} & m \in \mathcal{R}_0 \\ 0 & m \notin \mathcal{R}_0 \end{cases}.
$$

Now for each $\underline{m} \in \mathcal{R}_0$ and $\underline{k} \in \mathbb{Z}^2$, $\underline{m} + A\underline{k} \in \mathcal{R}_0$ if and only if $\underline{k} = \underline{0}$, and hence

$$
\begin{aligned}
\sum_{\underline{m} \in \mathbb{Z}^2} \overline{h_{\underline{m}}} h_{\underline{m}+A\underline{k}} e^{2i\pi \underline{j} A^{-1} \cdot \underline{m}} &= \sum_{\underline{m} \in \mathcal{R}_0} \overline{h_{\underline{m}}} h_{\underline{m}+A\underline{k}} e^{2i\pi \underline{j} A^{-1} \cdot \underline{m}} = \sum_{\underline{m} \in \mathcal{R}_0} \frac{1}{a_1a_2} \delta_{\underline{k},\underline{0}} e^{2i\pi \underline{j} A^{-1} \cdot \underline{m}} \\
&= \frac{1}{a_1a_2} \delta_{\underline{k},\underline{0}} \sum_{\underline{m} \in \mathcal{R}_0} e^{2i\pi \underline{j} A^{-1} \cdot \underline{m}} = \frac{1}{a_1a_2} \delta_{\underline{k},\underline{0}} \cdot \delta_{\underline{j},\underline{0}} a_1a_2 = \delta_{\underline{k},\underline{0}} \cdot \delta_{\underline{j},\underline{0}}
\end{aligned}
$$

for $\underline{j} \in \mathcal{R}_0$ arguing as in (4.19). Observe that $\underline{a} - \underline{m} \in \mathcal{R}_0$ if and only if $\underline{m} \in \mathcal{R}_0$, hence

$$
h_{\underline{a}-\underline{m}} = \begin{cases} \frac{1}{\sqrt{a_1a_2}} & \underline{a} - \underline{m} \in \mathcal{R}_0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{\sqrt{a_1a_2}} & \underline{m} \in \mathcal{R}_0 \\ 0 & \text{otherwise} \end{cases} = h_{\underline{m}} = \overline{h_{\underline{m}}}.
$$

	$a_1 - 1$	$\frac{e^{2i\pi r_1(a_1-1)/a_1}}{\sqrt{a_1 a_2}}$	$\frac{e^{2i\pi r_1(a_1-1)/a_1} e^{2i\pi r_2/a_2}}{\sqrt{a_1 a_2}}$	$\cdots$	$\frac{e^{2i\pi r_1(a_1-1)/a_1} e^{2i\pi r_2(a_2-1)/a_2}}{\sqrt{a_1 a_2}}$
m_1	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	1	$\frac{e^{2i\pi r_1/a_1}}{\sqrt{a_1 a_2}}$	$\frac{e^{2i\pi r_1/a_1} e^{2i\pi r_2/a_2}}{\sqrt{a_1 a_2}}$	$\vdots$	$\frac{e^{2i\pi r_1/a_1} e^{2i\pi r_2(a_2-1)/a_2}}{\sqrt{a_1 a_2}}$
	0	$\frac{1}{\sqrt{a_1 a_2}}$	$\frac{e^{2i\pi r_2/a_2}}{\sqrt{a_1 a_2}}$	$\cdots$	$\frac{e^{2i\pi r_2(a_2-1)/a_2}}{\sqrt{a_1 a_2}}$
		0	1	$\cdots$	$a_2 - 1$
			m_2		

Table 4.2 The wavelet filter $\{g_{\underline{m}}^{\underline{r}}\}$.

Thus, $(M6)$ holds.

The wavelet filter is now given by

$$g_{\underline{m}}^{\underline{r}} = e^{2i\pi \underline{r} A^{-1} \cdot \underline{m}} \overline{h_{\underline{a}-\underline{m}}} = \begin{cases} \frac{1}{\sqrt{a_1 a_2}} e^{2i\pi \underline{r} A^{-1} \cdot \underline{m}} & \underline{r} \in \mathcal{R}, \underline{m} \in \mathcal{R}_0 \\ 0 & \underline{r} \in \mathcal{R}, \underline{m} \notin \mathcal{R}_0 \end{cases}.$$

Note that $h_{\underline{m}} = 0$, $g_{\underline{m}}^{\underline{r}} = 0$ for all $\underline{m} \notin \mathcal{R}_0$, and thus the wavelets (as well as the scaling function) are finite sums,

$$\psi^{\underline{r}} = \sum_{\underline{m} \in \mathbb{Z}^2} g_{\underline{m}}^{\underline{r}} D_A T_{\underline{m}} \varphi = \sum_{\underline{m} \in \mathcal{R}_0} \frac{1}{\sqrt{a_1 a_2}} e^{2i\pi \underline{r} A^{-1} \cdot \underline{m}} \chi_{A^{-1}(\underline{m} + [0,1)^2)}.$$

4.4 The Pyramidal Algorithm

Let $A = \left[\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right]$ with $a_1, a_2 \in \{2, 3, \ldots\}$ and let φ be a scaling function for a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^2)$. Let W_j denote the orthogonal complement of V_j in V_{j+1}, so that $V_{j+1} = V_j \oplus W_j$ for all j.

In section 4.2, we have shown that the functions $\{T_{\underline{k}} \psi^{\underline{r}}\}_{\underline{k} \in \mathbb{Z}^2, \underline{r} \in \mathcal{R}}$ where $\psi^{\underline{r}}$ are defined by (4.2) form an orthonormal basis of W_0. It now follows from theorems 3.4 and 3.5 that we have a decomposition

$$L^2(\mathbb{R}^2) = \bigoplus_{j \in \mathbb{Z}} W_j$$

where $W_{j+1} = D_A W_j$ for all j, and

$$\{\psi^{\underline{r}}_{(j,\underline{k})}\}_{\underline{k}\in\mathbb{Z}^n,\underline{r}\in\mathcal{R}} \quad \text{with} \quad \psi^{\underline{r}}_{(j,\underline{k})} = D_{A^j} T_{\underline{k}} \psi^{\underline{r}}$$

form an orthonormal basis of W_j, so that

$$\{\psi^{\underline{r}}_{(j,\underline{k})} : \underline{r} \in \mathcal{R}, j \in \mathbb{Z}, \underline{k} \in \mathbb{Z}^n\} \tag{4.26}$$

is a wavelet basis of $L^2(\mathbb{R}^2)$. By theorem 2.16, each $f \in L^2(\mathbb{R}^2)$ can be written uniquely as

$$f = \sum_{\underline{r}\in\mathcal{R}} \sum_{j\in\mathbb{Z}} \sum_{\underline{k}\in\mathbb{Z}^2} d^{\underline{r}}_{j,\underline{k}} \psi^{\underline{r}}_{(j,\underline{k})}$$

where the wavelet coefficients $d^{\underline{r}}_{j,\underline{k}}$ are given by

$$d^{\underline{r}}_{j,\underline{k}} =< f, \psi^{\underline{r}}_{(j,\underline{k})} > .$$

In practical applications, one can compute only finitely many wavelet coefficients. Recall by (3.8) that for any $m < n$,

$$L^2(\mathbb{R}^2) = V_n \oplus V_n^{\perp} = V_m \oplus \bigoplus_{j=m}^{n} W_j \oplus V_n^{\perp}.$$

One usually fixes some choice of m and n, and computes the wavelet coefficients for $m \leq j \leq n$ only. Let $f \in L^2(\mathbb{R}^2)$ be given. Fix $\varepsilon > 0$. By lemma 3.3, we can choose n and m such that

1. $\|f - P_n f\|_2 < \dfrac{\varepsilon}{2}$
2. $\|P_m f\|_2 < \dfrac{\varepsilon}{2}$

where P_n denotes the orthogonal projection of $L^2(\mathbb{R}^2)$ onto V_n. Set $f_n = P_n f \in V_n$. Since $W_j \subset V_n$ for all $j < n$ then $\psi^{\underline{r}}_{(j,\underline{k})} \in V_n$ for all $j < n$, and the wavelet coefficients of f are given by

$$d^{\underline{r}}_{j,\underline{k}} =< f, \psi^{\underline{r}}_{(j,\underline{k})} >=< f, P_n \psi^{\underline{r}}_{(j,\underline{k})} >=< P_n f, \psi^{\underline{r}}_{(j,\underline{k})} >=< f_n, \psi^{\underline{r}}_{(j,\underline{k})} > .$$

Similarly, for each $j \leq n$, let $c_{j,\underline{k}}$ denote the scaling coefficients of f_n in the basis $\{\varphi_{(j,\underline{k})}\}_{\underline{k}\in\mathbb{Z}^2}$ of V_j, where $\varphi_{(j,\underline{k})} = D_{A^j}T_{\underline{k}}\varphi$, so that

$$c_{j,\underline{k}} =< f_n, \varphi_{(j,\underline{k})} >=< f, P_n\varphi_{(j,\underline{k})} >=< P_n f, \varphi_{(j,\underline{k})} >=< f_n, \varphi_{(j,\underline{k})} >$$

since $V_j \subset V_n$ for $j \leq n$. That is, the scaling coefficients and wavelet coefficients of f and f_n are identical for $j \leq n$.

The pyramidal algorithm now allows to compute the scaling and wavelet coefficients of f at level $j-1$ from the scaling coefficients at level j as follows. The important ingredient is the observation that for all $g \in L^2(\mathbb{R}^2)$,

$$\begin{aligned}(T_{\underline{k}}D_A g)(\underline{x}) &= (D_A g)(\underline{x}-\underline{k}) = |\det A|^{1/2} g(A(\underline{x}-\underline{k})) = |\det A|^{1/2} g(A\underline{x} - A\underline{k}) \\ &= |\det A|^{1/2}(T_{A\underline{k}}g)(A\underline{x}) = (D_A T_{A\underline{k}} g)(x),\end{aligned}$$

that is, $T_{\underline{k}}D_A = D_A T_{A\underline{k}}$. Hence,

$$\begin{aligned}\varphi_{(j-1,\underline{k})} &= D_{A^{j-1}}T_{\underline{k}}\varphi = D_{A^{j-1}}T_{\underline{k}}\left(\sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}}\varphi_{(1,\underline{m})}\right) = D_{A^{j-1}}T_{\underline{k}}\left(\sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}} D_A T_{\underline{m}}\varphi\right) \\ &= \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}} D_{A^{j-1}}T_{\underline{k}}D_A T_{\underline{m}}\varphi = \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}} D_{A^j}T_{\underline{m}+A\underline{k}}\varphi.\end{aligned}$$

Next replace $\underline{m}$ by $\underline{m} - A\underline{k}$,

$$\varphi_{(j-1,\underline{k})} = \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}-A\underline{k}} D_{A^j}T_{\underline{m}}\varphi = \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}-A\underline{k}}\varphi_{(j,\underline{m})}. \tag{4.27}$$

Next, $\psi^{\underline{r}} = \sum_{\underline{m}\in\mathbb{Z}^2} g^{\underline{r}}_{\underline{m}}\varphi_{(1,\underline{m})}$ gives in the same way

$$\begin{aligned}\psi^{\underline{r}}_{(j-1,\underline{k})} &= D_{A^{j-1}}T_{\underline{k}}\psi^{\underline{r}} = D_{A^{j-1}}T_{\underline{k}}\left(\sum_{\underline{m}\in\mathbb{Z}^2} g^{\underline{r}}_{\underline{m}}\varphi_{(1,\underline{m})}\right) = \sum_{\underline{m}\in\mathbb{Z}^2} g^{\underline{r}}_{\underline{m}-A\underline{k}}\varphi_{(j,\underline{m})} \\ &= \sum_{\underline{m}\in\mathbb{Z}^2} e^{2i\pi\underline{r}A^{-1}\cdot(\underline{m}-A\underline{k})}\overline{h_{\underline{a}-(\underline{m}-A\underline{k})}}\varphi_{(j,\underline{m})} = \sum_{\underline{m}\in\mathbb{Z}^2} e^{2i\pi\underline{r}A^{-1}\cdot\underline{m}}\overline{h_{\underline{a}-(\underline{m}-A\underline{k})}}\varphi_{(j,\underline{m})}.\end{aligned} \tag{4.28}$$

Suppose, we have computed the scaling coefficients of f, $\{c_{j,\underline{k}}\}_{\underline{k}\in\mathbb{Z}^2}$ for some j. Then

$$c_{j-1,\underline{k}} =< f, \varphi_{(j-1,\underline{k})} >=< f, \sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}-A\underline{k}}\varphi_{(j,\underline{m})} >= \sum_{\underline{m}\in\mathbb{Z}^2} \overline{h_{\underline{m}-A\underline{k}}} < f, \varphi_{(j,\underline{m})} >$$

$$= \sum_{\underline{m}\in\mathbb{Z}^2} \overline{h_{\underline{m}-A\underline{k}}} c_{j,\underline{m}}.$$

Similarly, (4.28) gives

$$d^{r}_{j-1,\underline{k}} =< f, \psi^{r}_{(j-1,\underline{k})} >=< f, \sum_{\underline{m}\in\mathbb{Z}^2} e^{2i\pi\underline{r}A^{-1}\cdot\underline{m}}\overline{h_{\underline{a}-(\underline{m}-A\underline{k})}}\varphi_{(j,\underline{m})} >$$

$$= \sum_{\underline{m}\in\mathbb{Z}^2} e^{-2i\pi\underline{r}A^{-1}\cdot\underline{m}} h_{\underline{a}-(\underline{m}-A\underline{k})} < f, \varphi_{j,\underline{m}} >= \sum_{\underline{m}\in\mathbb{Z}^2} e^{-2i\pi\underline{r}A^{-1}\cdot\underline{m}} h_{\underline{a}-(\underline{m}-A\underline{k})} c_{\underline{j},\underline{m}}$$

$$= \sum_{\underline{m}\in\mathbb{Z}^2} \overline{g^{r}_{\underline{m}-A\underline{k}}} c_{\underline{j},\underline{m}}.$$

Conversely, we can reconstruct $c_{\underline{j},\underline{m}}$ from $c_{j-1,\underline{k}}$ and $\{d^{r}_{j-1,\underline{k}}\}_{\underline{r}\in\mathcal{R}}$ as follows.

Since $V_j = V_{j-1} \oplus W_{j-1}$ then $P_j = P_{j-1} + Q_{j-1}$ so that $P_j f = P_{j-1} f + Q_{j-1} f$. Thus, by (4.27) and (4.28),

$$\sum_{\underline{m}\in\mathbb{Z}^2} < f, \varphi_{(j,\underline{m})} > \varphi_{(j,\underline{m})}$$

$$= \sum_{\underline{k}\in\mathbb{Z}^2} < f, \varphi_{(j-1,\underline{k})} > \varphi_{(j-1,\underline{k})} + \sum_{\underline{r}\in\mathcal{R}} \sum_{\underline{k}\in\mathbb{Z}^2} < f, \psi^{r}_{(j-1,\underline{k})} > \psi^{r}_{(j-1,\underline{k})}$$

$$= \sum_{\underline{k}\in\mathbb{Z}^2} c_{j-1,\underline{k}} \left(\sum_{\underline{m}\in\mathbb{Z}^2} h_{\underline{m}-A\underline{k}}\varphi_{(j,\underline{m})} \right) + \sum_{\underline{r}\in\mathcal{R}} \sum_{\underline{k}\in\mathbb{Z}^2} d^{r}_{j-1,\underline{k}} \left(\sum_{\underline{m}\in\mathbb{Z}^2} g^{r}_{\underline{m}-A\underline{k}}\varphi_{(j,\underline{m})} \right)$$

$$= \sum_{\underline{m}\in\mathbb{Z}^2} \sum_{\underline{k}\in\mathbb{Z}^2} \left(c_{j-1,\underline{k}} h_{\underline{m}-A\underline{k}} + \sum_{\underline{r}\in\mathcal{R}} d^{r}_{j-1,\underline{k}} g^{r}_{\underline{m}-A\underline{k}} \right) \varphi_{(j,\underline{m})}.$$

Since $\{\varphi_{(j,\underline{m})}\}_{\underline{m}\in Z^2}$ is an orthonormal set, the coefficients on both sides must be identical

$$< f, \varphi_{(j,\underline{m})} >= \sum_{\underline{k}\in\mathbb{Z}^2} \left(c_{j-1,\underline{k}} h_{\underline{m}-A\underline{k}} + \sum_{\underline{r}\in\mathcal{R}} d^{r}_{j-1,\underline{k}} g^{r}_{\underline{m}-A\underline{k}} \right)$$

that is,

$$c_{j,\underline{m}} = \sum_{\underline{k}\in\mathbb{Z}^2} \left(c_{j-1,\underline{k}} h_{\underline{m}-A\underline{k}} + \sum_{\underline{r}\in\mathcal{R}} d^{r}_{j-1,\underline{k}} g^{r}_{\underline{m}-A\underline{k}} \right).$$

4.5 Practical Computations

We now present two examples illustrating the technique of wavelet analysis by means of the Haar wavelet. The first example shows the wavelet coefficients of a function f at various scales, while the second example illustrates how wavelets can be used in data compression.

For the first example, let f be a function whose support is contained in the unit square $[0,1]\times[0,1]$. We choose the dilation matrix $A = \left[\begin{smallmatrix}3 & 0\\ 0 & 3\end{smallmatrix}\right]$. Because of space limitations, we only compute and display the wavelet coefficients at scales $j = 0$ and $j = 1$. As outlined in the previous section, we must begin by computing the scaling coefficients $c_{2,\underline{m}}$ at scale $j = 2$. Since the functions $\varphi_{(2,\underline{m})}$ are constant on the squares $I_m = [\frac{m_1}{9}, \frac{m_1+1}{9}] \times [\frac{m_2}{9}, \frac{m_2+1}{9}]$, it is convenient to sample the values of f on these squares, and store the sampled values in a 9×9 matrix. The integral can then be approximated as sums over each of the squares I_m. In this example, we use a function f whose sampled matrix is tridiagonal,

$$f_{\text{samp}} = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}$$

The scaling coefficients $c_{2,\underline{m}}$ are listed in table 4.3. Since f is supported on the unit square, the scaling coefficients for the non-listed values of $\underline{m} = (m_1, m_2)$ are zero. Some of the scaling and wavelet coefficients at levels $j = 1$ and $j = 0$ are listed in tables 4.4 to 4.13, those which are not listed are zero.

Next we show how to apply wavelet analysis to data compression. It is well known that the Haar wavelet is not particularly suitable to this task as it is has

$m_1 \backslash m_2$	0	1	2	3	...	7	8
0	0.1111	0.1111	0	0	...	0	0
1	0.1111	0.1111	0.1111	0	...	0	0
2	0	0.1111	0.1111	0.1111	...	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$
7	0	0	0	0	...	0.1111	0.1111
8	0	0	0	0	...	0.1111	0.1111

Table 4.3 The scaling coefficients $c_{2,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	0.2593	0.0370	0
1	0.0370	0.2593	0.0370
2	0	0.0370	0.2593

Table 4.4 The scaling coefficients $c_{1,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	$-0.0185 - 0.0321i$	0.0370	0
1	$-0.0185 + 0.0321i$	$-0.0185 - 0.0321i$	0.0370
2	0	$-0.0185 + 0.0321i$	$-0.0185 - 0.0321i$

Table 4.5 The wavelet coefficients $d^{(0,1)}_{1,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	$-0.0185 + 0.0321i$	0.0370	0
1	$-0.0185 - 0.0321i$	$-0.0185 + 0.0321i$	0.0370
2	0	$-0.0185 - 0.0321i$	$-0.0185 + 0.0321i$

Table 4.6 The wavelet coefficients $d^{(0,2)}_{1,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	$-0.0185 - 0.0321i$	$-0.0185 + 0.0321i$	0
1	0.0370	$-0.0185 - 0.0321i$	$-0.0185 + 0.0321i$
2	0	0.0370	$-0.0185 - 0.0321i$

Table 4.7 The wavelet coefficients $d^{(1,0)}_{1,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	$0.0370 - 0.0642i$	$-0.0185 + 0.0321i$	0
1	$-0.0185 + 0.0321i$	$0.0370 - 0.0642i$	$-0.0185 + 0.0321i$
2	0	$-0.0185 + 0.0321i$	$0.0370 - 0.0642i$

Table 4.8 The wavelet coefficients $d^{(1,1)}_{1,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	0.0370	$-0.0185 + 0.0321i$	0
1	$-0.0185 - 0.0321i$	0.0370	$-0.0185 + 0.0321i$
2	0	$-0.0185 - 0.0321i$	0.0370

Table 4.9 The wavelet coefficients $d^{(1,2)}_{1,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	$-0.0185 + 0.0321i$	$-0.0185 - 0.0321i$	0
1	0.0370	$-0.0185 + 0.0321i$	$-0.0185 - 0.0321i$
2	0	0.0370	$-0.0185 + 0.0321i$

Table 4.10 The wavelet coefficients $d^{(2,0)}_{1,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	0.0370	$-0.0185 - 0.0321i$	0
1	$-0.0185 + 0.0321i$	0.0370	$-0.0185 - 0.0321i$
2	0	$-0.0185 + 0.0321i$	0.0370

Table 4.11 The wavelet coefficients $d^{(2,1)}_{1,\underline{m}}$.

$m_1 \backslash m_2$	0	1	2
0	$0.0370 + 0.0642i$	$-0.0185 - 0.0321i$	0
1	$-0.0185 - 0.0321i$	$0.0370 + 0.0642i$	$-0.0185 - 0.0321i$
2	0	$-0.0185 - 0.0321i$	$0.0370 + 0.0642i$

Table 4.12 The wavelet coefficients $d^{(2,2)}_{1,\underline{m}}$.

$c_{0,\underline{m}}$	0.3086

$d^{(0,1)}_{0,\underline{m}}$	$-0.0062 - 0.0107i$
$d^{(0,2)}_{0,\underline{m}}$	$-0.0062 + 0.0107i$
$d^{(1,0)}_{0,\underline{m}}$	$-0.0062 - 0.0107i$
$d^{(1,1)}_{0,\underline{m}}$	$0.0123 - 0.0214i$

$d^{(1,2)}_{0,\underline{m}}$	0.2346
$d^{(2,0)}_{0,\underline{m}}$	$-0.0062 + 0.0107i$
$d^{(2,1)}_{0,\underline{m}}$	0.2346
$d^{(2,2)}_{0,\underline{m}}$	$0.0123 + 0.0214i$

Table 4.13 The scaling coefficients and wavelet coefficients $c_{0,\underline{0}}$ and $d^{(r)}_{0,\underline{0}}$.

jumps, so that we cannot achieve outstanding compression ratios. Nevertheless, it is an easy example which can illustrate this technique.

We consider the black-and-white picture of figure 4.1, downloaded from `http://horum.homeunix.net/~carl/wavelet`. This picture is commonly used in papers on image processing. The picture used is in bitmap format, as a 512×512 array, each of whose entries represents a pixel whose gray-level is given by a 8-bit unsigned integer. We consider this picture as a function supported on the unit square, and choose the dilation matrix $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. We begin with computing the scaling coefficients at scale $j = 9$, and then compute the wavelet coefficients at scales $j = 8, \ldots, 0$ using the pyramidal algorithm. All wavelets coefficients whose absolute values are below a threshold ε are set to zero, and then the image is reconstructed from the modified wavelet coefficients.

The resulting bitmap file is then compressed using the 'bzip2' utility. Table 4.14 shows the compressed file size at various threshold levels, and figures 4.1 show the reconstructed images. Note that $\varepsilon = 0$ corresponds to the original image.

threshold ε	file size	compressed file size	compression ratio
0	263,222	203,299	22.8%
0.02	263,222	135.220	48.6%
0.05	263,222	87,458	66.8%
0.1	263,222	43,279	83.6%
0.2	263,222	18,471	93.0%
0.5	263,222	14,931	94.3%

Table 4.14 Compressed file sizes.

Figure 4.1 Reconstructed images at thresholds $\varepsilon = 0, \varepsilon = 0.02, \varepsilon = 0.05, \varepsilon = 0.1, \varepsilon = 0.2$ and $\varepsilon = 0.5$.

CHAPTER V

CONCLUSION

The objective of this thesis was to investigate whether the concept of multiresolution analysis can be extended from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$, to construct wavelets from the scaling function, and to find an efficient algorithm for computing the wavelet coefficients. We have obtained the following results :

1. We have given a definition of multiresolution analysis for $L^2(\mathbb{R}^2)$. For this, we needed to impose the additional condition $(M6)$ on the scaling filter.

2. Starting from a multiresolution analysis, we have defined a family of mother wavelets $\{\psi^{\underline{r}}\}_{\underline{r}\in\mathcal{R}}$ in (4.2).

3. By proving and using theorem 4.3, we have shown that this family of mother wavelets gives rise to an orthonormal wavelet basis of $L^2(\mathbb{R}^2)$, of the form $\{\psi^{\underline{r}}_{(j,\underline{m})} : \underline{r} \in \mathcal{R},\ j \in \mathbb{Z},\ \underline{m} \in \mathbb{Z}^2\}$, where $\psi^{\underline{r}}_{(j,\underline{m})}$ is as defined in (4.26) .

4. In section 4.3 we have constructed Haar wavelets on $L^2(\mathbb{R}^2)$ as an application of theorem 4.3.

5. In section 4.4 we have extended the pyramidal algorithm to the wavelet basis of $L^2(\mathbb{R}^2)$ discussed in this thesis. This algorithm helps to quickly compute the wavelet and scaling coefficients of a function f.

For simplicity, in chapter IV we have only worked in $L^2(\mathbb{R}^2)$. However, all definitions, constructions and proofs carry over to $L^2(\mathbb{R}^n)$ in an obvious way. The notation used in this thesis should allow the reader to easily do so.

The condition $(M6)$ imposed on the scaling function of the multiresolution analysis appears restrictive. Thus, further study to find additional concrete examples of scaling functions satisfying this condition is warranted.

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