RADIATIVE SYMMETRY BREAKING IN SCALE INVARIANT SINGLET EXTENSION OF TYPE II SEESAW MODEL

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การเสียสมมาตรแบบแผ่รังสีในส่วนขยายด้วยซิงเล็ตแบบที่ไม่แปรเปลี่ยนตามสเกลของแบบจำลอง SEESAW แบบที่สอง

นายบายู ดีกันตารา

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy in Physics.

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คำถามเกี่ยวกับมวลนิวทริโนและจุดกำเนิดของการเสียสมมาตรอิเล็กโทรวีคเป็นปริศนาสำคัญสองประการในฟิสิกส์อนุภาค การสร้างมวลนิวทริโนต้องการฟิสิกส์ใหม่ที่นอกเหนือจากแบบจำลองมาตรฐานและยังเสนอแนะให้มีการทบทวนฟิสิกส์ของการเสียสมมาตรอีกครั้ง จุดมุ่งหมายของวิทยานิพนธ์นี้คือเพื่อศึกษาการเสียสมมาตรแบบแผ่รังสีในส่วนขยายที่ไม่ขึ้นกับสเกลของแบบจำลองมวลนิวทริโน SEESAW แบบที่สอง ในงานวิจัยนี้เราได้หาเงื่อนไขขอบเขตจากด้านล่างสำหรับศักย์ของสเกลาร์ของแบบจำลองในลักษณะทั่วไปเป็นครั้งแรก พร้อมทั้งหาจุดต่ำสุดของศักย์ภายใต้กรอบแนวคิดของ Gildener และ Weinberg เมื่อกำหนดเงื่อนไขขอบเขตจากด้านล่างแล้วเราสามารถหาชุดของค่าคู่ควบแบบควอร์ติกของสเกลาร์ ที่ทำให้เกิดการเสียสมมาตรอิเล็กโทรวีคแบบแผ่รังสีที่ระดับหนึ่งลูปได้และมวลของฮิกส์ที่คล้ายกับแบบจำลองมาตรฐานมีค่าสอดคล้องกับค่าจากการทดลอง การคำนวณชุดสมการรีนอมัลไลเซชันหนึ่งลูปสำหรับค่าคู่ควบแบบควอร์ติกของสเกลาร์ได้ถูกดำเนินการเพื่อแสดงให้เห็นว่าค่าคู่ควบยังคงเป็นค่ารบกวนไปจนถึงสเกลแพลงค์ด้วยเงื่อนไข $|\lambda_i| < 4\pi$ ซึ่งพบว่าการวิวัฒน์ของค่าคู่ควบตัวเองของฮิกส์ที่คล้ายกับแบบจำลองมาตรฐาน λ_H สามารถถูกป้องกันไม่ได้มีค่าเป็นลบที่พลังงานสูงได้ตราบที่มวลของสเกลอนมีค่าไม่กี่กิกะอิเล็กตรอนโวลต์

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา ____________

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BAYU DIRGANTARA : RADIATIVE SYMMETRY BREAKING IN SCALE INVARIANT SINGLET EXTENSION OF TYPE II SEESAW MODEL. THESIS ADVISOR : WARINTORN SREETHAWONG, Ph.D. 102 PP.

SCALE INVARIANT/RADIATIVE SYMMETRY BREAKING/SINGLET EXTENSION OF TYPE II SEESAW MODEL

The questions of neutrino mass and the origin of electroweak symmetry breaking are two major puzzles in particle physics. The neutrino mass generation requires new physics beyond the Standard Model (SM) and also suggests the reconsideration of physics of symmetry breaking. The aim of this thesis is to study the radiative symmetry breaking of singlet scalar extension of type II seesaw neutrino mass model. In this work, we derive bounded from below conditions for the scalar potential of the model in full generality for the first time. A novel framework of Gildener-Weinberg is utilized in minimizing the multiscalar potential. Upon imposing the bounded from below conditions, we find sets of scalar quartic couplings that can radiatively realize electroweak symmetry breaking at one-loop level. The SM-like Higgs mass is required to be consistent with the experimental value. The calculation of one-loop renormalization group equations of scalar quartic couplings are also performed in order to show that they remain perturbative all the way to the Planck scale with the condition $|\lambda_i| < 4\pi$. It is found that the evolution of the SM-Higgs doublet self coupling λ_H can be prevented from being negative at high energy as long as the mass of the scalon is a few GeV.

School of Physics — Student's Signature ______

Academic Year 2020 — Advisor's Signature W. Sreethawong

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CHAPTER I

INTRODUCTION

The most outstanding achievement of a theoretical framework based on quantum field theory to date is the Standard Model (SM) of particle physics. It almost becomes unquestionably right in describing the behavior of fundamental constituents of nature. The often famous quoted prediction of the Standard Model is the electron magnetic moment which agrees with experiments to unprecedented accuracy. Since its construction around the 70's, two well-known predictions of the SM have been verified experimentally. The first one is the existence of neutral current interaction (Hasert et al., 1973b; Hasert et al., 1973a) which implies the presence of a new neutral gauge boson (Arnison et al., 1983). The second one is the long-sought scalar Higgs boson for which its existence is extremely crucial for the SM to be a consistent theory (Aad et al., 2012a; Chatrchyan et al., 2012a). Despite these successful predictions, it is undeniable that this theory is also suffering from frustrating problems that physicists have been struggling to settle. The non-zero neutrino masses, the presence of dark matter and dark energy, the baryon-antibaryon asymmetry, the problem of the SM Higgs mass, the origin of the electroweak scalar potential, the stability of electroweak potential, etc., are several drawbacks that the SM cannot address. These pressing issues impel us to consider the SM as an effective theory which is a low energy manifestation of some UV-complete theory. Motivated by this fact, there has been a huge number of proposals for physics beyond the SM. New types of particles are usually added either to the fermion sector or to the scalar sector or even to both sectors. Even more so, a new type of interaction is commonly introduced by enlarging the gauge symmetry of the SM.

With the discovery of the scalar Higgs, the next question that we have to answer is the origin of electroweak symmetry breaking. It is well known that the successful implementation of spontaneous symmetry breaking in the SM electroweak sector is due to the presence of the dimensionful mass parameter μ^2. The right choice of a sign of this parameter determines a non-trivial structure of the vacuum. Remarkably, the absence of this parameter renders the SM Lagrangian scale-invariant. However, this feature holds only at the classical level. When higher-order corrections are included, the scale dependence of the running couplings will destroy this property. Nevertheless, by embracing the argument by Bardeen (Bardeen, 1995), imposing classical scale invariance on the theory is still of great benefit.

In this thesis, we focus on the neutrino mass and the origin of electroweak symmetry breaking issues. Particularly, we study a scale invariant version of an appealing type II seesaw neutrino model with an additional singlet scalar. We study the possibility of successful radiative electroweak symmetry breaking in this model.

This thesis is organized as follows. In Chapter II, we present a short introduction to the Standard Model as well as its important features, such as spontaneous symmetry breaking by the Higgs mechanism and accidental lepton and baryon symmetries. In Chapter III, we firstly give a brief review on three canonical seesaw models, and elaborate on the type II seesaw model. Next, effective action and effective potential which are basic tools for the study of radiative symmetry breaking are presented, and calculations for toy models are demonstrated and discussed. Lastly, the method of Gildener and Weinberg for finding the broken symmetry solution of multiscalar potential is described. In Chapter IV, firstly we describe the scalar potential of the scale invariant singlet extension of type II seesaw model. We derive the bounded from below conditions of the potential in full generality and minimize the potential. Subsequently, the spectra of scalar bosons are given. Next, we calculate the one-loop renormalization group equations (RGEs) for all quartic couplings. Finally, we numerically solve the stationary equations of the potential, and find sets of quartic couplings that realize radiative electroweak symmetry breaking at one-loop level while also satisfying theoretical constrains. In the last Chapter, conclusions and discussions are given. In Appendix A, we provide detailed derivation of BFB conditions. The Feynman rules for scalar-scalar interactions are given in Appendix B.

CHAPTER II

STANDARD MODEL OF PARTICLE PHYSICS

In this chapter, we give a brief overview of some important features of the standard model (SM) of particle physics. Long before the construction of the SM, the Fermi theory (Fermi, 1934) was a successful effective field theory that can explain the neutron beta decay at low energy regime. The interaction between quark and lepton is described by current-current contact interaction Hamiltonian

$$\mathcal{H}_{\text{int}} \;=\; -\frac{G_F}{\sqrt{2}} J_\mu(x) J(x)^{\mu\dagger} \tag{2.1}$$

where the current is given by

$$J_\mu \;=\; \bar{u}\gamma_\mu(1-\gamma_5)d + \bar{\nu}_e\gamma_\mu(1-\gamma_5)e\,, \tag{2.2}$$

and G_F is the Fermi coupling constant.

The $V-A$ structure of this weak current (Sudarshan and Marshak, 1958; Feynman and Gell-Mann, 1958) comes from the experimental fact that weak interaction violates parity (maximally violated), i.e. it treats a left-handed particle differently from a right-handed one. The four-fermion interaction can also explain muon decay with the same strength G_F. Indeed, its most precise experimental determination is obtained from measurements of the muon lifetime,

$$G_F \;=\; 1.16638\times10^{-5}\,\text{GeV}^{-2}. \tag{2.3}$$

Note that this coupling has negative mass dimension. This poses a problem for Fermi theory as a valid description of high energy interaction, or in the modern language we say that Fermi theory is non-renormalizable.

The early development of the SM was coming from an effort to build a gauge theory of weak interaction for which the $V - A$ Fermi theory is the low energy realization. The feeble interaction of weak interaction can be well attributed to the massiveness of force carrier being exchanged. However, massive gauge mediators will ruin the renormalizability of the theory. In order to build a gauge theory of weak interaction, Glashow, Weinberg, and Salam proposed the unification of weak and electromagnetic interactions (Glashow, 1961; Weinberg, 1967; Salam, 1968). Later, it was proved by t'Hooft and Veltman ('t Hooft and Veltman, 1972) that the theory which is built based on gauge principle is renormalizable. The electroweak gauge boson masses are generated via the so called Higgs mechanism (Higgs, 1964; Englert and Brout, 1964). The gauge symmetry, which is respected at the Lagrangian level, is spontaneously broken by the ground state of theory. This occurs when a scalar Higgs field develops a non-zero vacuum expectation value (VEV). Being interacted with the Higgs field, the gauge bosons acquire mass proportional to the Higgs VEV.

In the following sections, two main ingredients of the SM: a local gauge invariance and the Higgs mechanism are described. Next, gauge interactions after electroweak symmetry breaking is presented. Lastly, the lepton and baryon number conservations are discussed in the last section.

2.1 The SM Lagrangian

In this section, the field content of the SM together with their interactions are presented.

2.1.1 The SM Particle Content

The interactions of fundamental particles are dictated by the gauge symmetry of SM. The SM is built based on $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group. In quantum chromodynamics (QCD), $SU(3)_C$ symmetry governs the strong interaction between colored particles. Matter particles that carry $SU(3)_C$ color charge are known as quarks. Each quark-type is labeled by three color quantum numbers: R, G and B. Strong interactions among quarks proceed by exchanging force carriers called gluons. They are also colored particles with eight color states. Unlike photon, gluon can interact with eath other. This self-interaction results from the non-Abelian nature of QCD. In the language of group theory, quarks and gluons transform as fundamental and adjoint representations of $SU(3)_C$, respectively. However, quarks and gluons have never been observed in isolation in the normal phase of QCD; they can only be found in bound states, like mesons or baryons. This phenomenon is known as the color confinement.

Besides strong interaction, the SM also describes electroweak interaction which is stipulated by $SU(2)_L \times U(1)_Y$ gauge group. Under this gauge group, not only quark fields are charged but also lepton fields as well. The subscript L in $SU(2)$ specifies that weak gauge bosons interact with only left-handed matter fields. The up-type quark and down-type quark from the same generation and neutrino and its corresponding charged lepton field are grouped to form doublet representation of $SU(2)_L$ namely

$$Q_L^{i\alpha} = \begin{pmatrix} u \\ d \end{pmatrix}_L^{i\alpha}, \qquad L_L^i = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L^i, \tag{2.4}$$

where Greek and Latin indices denote color and generation indices, respectively. All right-handed fields are singlet under $SU(2)_L$. The hypercharge quantum number is related to electric charge by the famous Gell-Mann-Nishijima formula given as

$$Q = t^3 + Y, \tag{2.5}$$

where t^3 is the third generator of $SU(2)_L$ and Y is the $U(1)$ hypercharge. Finally, the only fundamental scalar particle which is crucial for paticle mass generation in the SM is the scalar Higgs boson. This Higgs boson also is a weak doublet of $SU(2)_L$ and represented by

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}, \tag{2.6}$$

where H^+ and H^0 are the positively charged and neutral components, respectively. In summary, the particle content of the SM and their corresponding gauge quantum numbers are shown in Table. 2.1

Table 2.1 SM particle content and their corresponding gauge quantum numbers

Particle content	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$Q_L^{i\alpha} = \binom{u}{d}_L^\alpha, \binom{c}{s}_L^\alpha, \binom{t}{b}_L^\alpha$	3	2	$+\frac{1}{6}$
$L_L^i = \binom{\nu_e}{e}_L, \binom{\nu_\mu}{\mu}_L, \binom{\nu_\tau}{\tau}_L$	1	2	$-\frac{1}{2}$
$U_R^i \equiv u_R, c_R, t_R$	3	1	$+\frac{2}{3}$
$D_R^i = d_R, s_R, b_R$	3	1	$-\frac{1}{3}$
$E_R^i = e_R, \mu_R, \tau_R$	1	1	-1
$H = \binom{H^+}{H^0}$	1	2	$+\frac{1}{2}$

2.1.2 The Lagrangian of SM

Interactions between fundamental particles are encoded in the Lagrangian. The SM Lagrangian comprises interactions of gauge boson (Yang-Mills), fermion, scalar (Higgs) and Yukawa sectors.

Gauge sector

The dynamics of gauge fields is given by the kinetic term containing the field strength of gauge bosons as

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}, \tag{2.7}$$

where $G^{a\mu\nu}$, $W^{a\mu\nu}$ and $B^{\mu\nu}$ are the fields strength tensors for $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$, respectively.

The $SU(3)_C$ field strength reads

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu, \tag{2.8}$$

where g_s is the strong coupling constant and f^{abc} is the fully-antisymmetric structure constant of Lie algebra of $SU(3)_C$ with a, b, c run from 1 to 8.

For $SU(2)_L$, the field strength is defined to be

$$W^a_{\mu\nu} \;=\; \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_2 f^{abc} W^b_\mu W^c_\nu, \tag{2.9}$$

where g_2 is the weak coupling constant and ϵ^{abc} is the fully-antisymmetric structure constant of Lie algebra of $SU(2)_L$ with a, b, c run from 1 to 3.

For $U(1)_Y$, the field strength is defined to be

$$B_{\mu\nu} \;=\; \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{2.10}$$

The presence of the third term in both $SU(3)_C$ and $SU(2)_L$ field strength tensors shows that the corresponding gauge fields will exhibit self-interactions which do not present in electromagnetic interaction. This is characteristic property of gauge field in non-Abelian theory. The strength of this self-interaction is contained in their corresponding coupling constant.

Under infinitesimal gauge transformations, the gauge fields of each gauge group transform as

$$B_\mu \;\rightarrow\; B_\mu + \frac{1}{g_1}\partial_\mu \theta_Y(x), \tag{2.11}$$

$$W^a_\mu \;\rightarrow\; W^a_\mu + \frac{1}{g_2}\partial_\mu \theta^a_L(x) + \epsilon^{abc} W^b_\mu \theta^c_L(x), \tag{2.12}$$

$$G^a_\mu \;\rightarrow\; G^a_\mu + \frac{1}{g_s}\partial_\mu \theta^a_C(x) + f^{abc} G^b_\mu \theta^c_C(x). \tag{2.13}$$

Fermion sector

The gauge interaction of quarks and leptons can be readily constructed by following their representation under the gauge group as prescribed in the previous subsection. The transformation of SM fermion fields under $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry read as

$$Q^{i\alpha}_L \;\rightarrow\; \left(e^{i\theta^a_C(x)T^a}\right)_{\alpha\beta} e^{i\theta^a_L(x)t^a} e^{i\theta_Y(x)Y_Q} Q^{i\beta}_L, \tag{2.14}$$

$$U^\alpha_R \;\rightarrow\; \left(e^{i\theta^a_C(x)T^a}\right)_{\alpha\beta} e^{i\theta_Y(x)Y_U} U^\beta_R, \tag{2.15}$$

$$D_R^{\alpha} \rightarrow \left(e^{i\theta_C^a(x)T^a}\right)_{\alpha\beta} e^{i\theta_Y(x)Y_D} D_R^{\beta}, \tag{2.16}$$

$$L_L \rightarrow e^{i\theta_L^a(x)t^a} e^{i\theta_Y(x)Y_L} L_L, \tag{2.17}$$

$$E_R \rightarrow e^{i\theta_Y(x)Y_e} L_L, \tag{2.18}$$

where T^a and t^a are generators of Lie algebra of $SU(3)$ and $SU(2)$, respectively. The gauge, Lorentz invariant fermionic Lagrangian is given by

$$\mathcal{L}_{\text{fermion}} = i\bar{Q}_L \not{D} Q_L + i\bar{U}_R \not{D} U_R + i\bar{D}_R \not{D} D_R + i\bar{L}_L \not{D} L_L + i\bar{E}_R \not{D} E_R, \tag{2.19}$$

where we have used the Feynman slashed notation defined by $\not{D} \equiv \gamma^{\mu} D_{\mu}$ and the covariant derivatives acting on each field are given by

$$D_{\mu} Q_L = \left(\partial_{\mu} - ig_s \frac{\lambda^a}{2} G_{\mu}^a - ig_2 \frac{\sigma^a}{2} W_{\mu}^a - ig_1 Y_Q B_{\mu}\right) Q_L, \tag{2.20}$$

$$D_{\mu} U_R = \left(\partial_{\mu} - ig_s \frac{\lambda^a}{2} G_{\mu}^a - ig_1 Y_U B_{\mu}\right) U_R, \tag{2.21}$$

$$D_{\mu} D_R = \left(\partial_{\mu} - ig_s \frac{\lambda^a}{2} G_{\mu}^a - ig_1 Y_U B_{\mu}\right) D_R, \tag{2.22}$$

$$D_{\mu} L_L = \left(\partial_{\mu} - ig_2 \frac{\sigma^a}{2} W_{\mu}^a - ig_1 Y_L B_{\mu}\right) L_L, \tag{2.23}$$

$$D_{\mu} E_R = \left(\partial_{\mu} - ig_1 Y_E B_{\mu}\right) E_R. \tag{2.24}$$

Here we have used $T^a = \lambda^a/2$ and $t^a = \sigma^a/2$ where λ^a and σ^a are the Gellmann and Pauli matrices, respectively. We have suppressed color and generation indices for clarity.

Scalar sector

The scalar sector contains kinetic term and potential of the Higgs field

$$\mathcal{L}_{\text{Higgs}} = (D_{\mu} H)^{\dagger} D^{\mu} H - V. \tag{2.25}$$

The covariant derivative acting on the scalar Higgs reads

$$D_{\mu} H = \left(\partial_{\mu} - ig_2 \frac{\sigma^a}{2} W_{\mu}^a - ig_1 Y_H B_{\mu}\right) H. \tag{2.26}$$

Under gauge transformation, the Higgs field transforms as

$$H \rightarrow e^{i\theta_L^a(x)\sigma^a/2} e^{i\theta_Y(x)Y_H} H. \tag{2.27}$$

The most generic renormalizable scalar potential takes the form

$$V = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2, \tag{2.28}$$

where μ^2 and λ are real parameters. In order to ensure that the potential is bounded from below at large field value, the λ parameter is required to be positive.

Yukawa sector

Lastly, the Yukawa part providing interactions between Higgs and fermions is given by

$$\mathcal{L}_{\text{Yuk}} = -Y_D^{ij} \bar{Q}_{Li} H D_{Rj} - Y_U^{ij} \bar{Q}_{Li} \tilde{H} U_{Rj} - Y_E^{ij} \bar{L}_{Li} H E_{Rj} + H.c., \tag{2.29}$$

where i, j indices of Yukawa coupling run over generations of fermion and the dual Higgs is defined as

$$\tilde{H} \equiv \sigma^2 H^* = \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix}. \tag{2.30}$$

In summary, the complete SM Lagrangian is given by

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yuk}}. \tag{2.31}$$

2.2 Electroweak Symmetry Breaking by Higgs Mechanism

So far, all gauge bosons and fermions are still massless in SM. For gauge bosons, one can verify that gauge symmetry are not respected by mass terms

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} m_W^2 W^\mu W_\mu + \frac{1}{2} m_B^2 B^\mu B_\mu. \tag{2.32}$$

Therefore, local gauge invariance of the SM Lagrangian prohibits non-zero bare mass of all gauge fields. In non-Abelian gauge theory, explicit mass terms will also damage the renormalizability of the theory. In the case of fermions, we know from Dirac Lagrangian

$$\mathcal{L}_{\text{Dirac}} = i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi \tag{2.33}$$

that the Dirac mass term couples right-handed and left-handed fields together. In SM, however, left-handed and right-handed fermion fileds are living in different irreducible representations of $SU(2)$. Their combination cannot yield $SU(2)$ singlet so that we can not directly have such term in the SM Lagrangian.

In Nature, weak gauge bosons and fermions are massive. They can acquire masses via interactions with Higgs field from the Higgs kinetic term and Yukawa interaction terms. This happens when temperature is low enough that the Higgs field develops its non-zero vacuum expectation value and the electroweak symmetry is said to be spontaneouly broken.

The spontaneous symmetry breaking (SSB) is of the following pattern

$$SU(2)_L \times U(1)_Y \xrightarrow{<H>} U(1)_{\text{EM}}, \tag{2.34}$$

where $U(1)_{\text{EM}}$ is the remaining unbroken gauge symmery associated to electromagnetic interaction. As a result, photon remains massless after SSB.

The ground state of theory can be obtained by minimizing the scalar potential, i.e.

$$\left.\frac{\partial V}{\partial H}\right|_{H,H^\dagger=\langle H\rangle,\langle H^\dagger\rangle} = \left\langle H^\dagger \right\rangle \left(-\mu^2 + 2\lambda \left\langle H^\dagger H \right\rangle\right) = 0, \tag{2.35}$$

$$\left.\frac{\partial V}{\partial H^\dagger}\right|_{H,H^\dagger=\langle H\rangle,\langle H^\dagger\rangle} = \left(-\mu^2 + 2\lambda \left\langle H^\dagger H \right\rangle\right) \langle H \rangle = 0. \tag{2.36}$$

If $\mu^2 < 0$, we only have a trivial minimum at $\langle H\rangle = \left\langle H^\dagger\right\rangle = 0$. However if $\mu^2 > 0$, these stationary equations have two solutions. The trivial solution corresponds to the maximum point of the potential. The non-trivial solution given by

$$\left\langle H^\dagger H \right\rangle = \frac{\mu^2}{2\lambda} \tag{2.37}$$

corresponds to the minimum of the potential. Due to the unbroken $U(1)_{\text{EM}}$ symmetry, only the neutral component of Higgs acquires non-zero VEV

$$\left\langle H^0 \right\rangle = \frac{v}{\sqrt{2}}. \tag{2.38}$$

We can expand the Higgs doublet around its VEV and rewrite it as four independent real fields

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} h_1 + ih_2 \\ v + h + ih_4 \end{pmatrix}. \tag{2.39}$$

Up to linear order in the field, it can be parameterized as

$$H = \frac{1}{\sqrt{2}}\exp\left(\frac{i\eta^a\sigma^a}{v}\right)\begin{pmatrix} 0 \\ v + h \end{pmatrix}. \tag{2.40}$$

Here we can identify $\eta^1 = h_2$, $\eta^2 = h_1$ and $h_4 = -\eta^3$, and h is the physical Higgs field interpreted as a fluctuation around the classical field value, v. We can use $SU(2)$ transformation to gauge away the complex matrix phase so that the Higgs field becomes

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h \end{pmatrix}. \tag{2.41}$$

This is called the unitary gauge where the physical spectrum of the theory is more transparent.

It can be easily verified that the vacuum state is invariant under $U(1)_{\text{EM}}$ transformation, i.e.

$$\langle H \rangle \longrightarrow e^{i\theta Q}\langle H \rangle = \exp\left\{ i\theta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \langle H \rangle = \langle H \rangle\,, \tag{2.42}$$

where $Q = \frac{\sigma^3}{2} + Y$ is the generator of $U(1)_{\text{EM}}$ symmetry.

Gauge boson masses

After EWSB, the gauge boson masses are generated from the kinetic part of the Higgs field,

$$\begin{aligned}\mathcal{L}_{\text{kin Higgs}} &\supset \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}\left(g_2\frac{\sigma^a}{2}W^a_\mu + \frac{1}{2}g_1 B_\mu\right)\left(g_2\frac{\sigma^a}{2}W^a_\mu + \frac{1}{2}g_1 B_\mu\right)\begin{pmatrix}0\\ v\end{pmatrix}\\ &= \frac{v^2}{8}\left[(W^1_\mu)^2 + (W^2_\mu)^2 + \begin{pmatrix}W^3_\mu & B_\mu\end{pmatrix}\begin{pmatrix}g_2^2 & -g_1g_2\\ -g_1g_2 & g_1^2\end{pmatrix}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix}\right].\end{aligned} \tag{2.43}$$

It can be seen that there is a mixing between W^3_μ and B_μ gauge bosons. The determinant of this mass matrix is zero implying that one mass eigenstate is massless. This state will be identified as a photon denoted by A_μ. The massive eigenstate corresponds to the neutral Z boson denoted by Z_μ.

Upon diagonalization, the mass eigenstates are related to weak eigenstates as

$$Z_\mu = \cos\theta_w W^3_\mu - \sin\theta_w B_\mu \qquad \text{with} \qquad m_Z = \frac{v\sqrt{g_1^2+g_2^2}}{2}, \tag{2.44}$$

$$A_\mu = \sin\theta_w W^3_\mu + \cos\theta_w B_\mu \qquad \text{with} \qquad m_A = 0. \tag{2.45}$$

The weak mixing angle is related to gauge couplings as

$$\sin\theta_w = \frac{g_1}{\sqrt{g_1^2+g_2^2}}, \qquad \cos\theta_w = \frac{g_2}{\sqrt{g_1^2+g_2^2}}. \tag{2.46}$$

For the remaining two gauge bosons of $SU(2)$, the mass eigenstates with definite electric charge are given by

$$W^\pm_\mu = \frac{W^1_\mu \mp W^2_\mu}{\sqrt{2}} \qquad \text{with} \qquad m_W = \frac{g_2 v}{2}. \tag{2.47}$$

Fermion masses

The fermions masses from Yukawa interactions read

$$\begin{aligned}\mathcal{L}_{Yuk} &= -\frac{Y_D^{ij}v}{\sqrt{2}}\bar{d}_{Li}D_{Rj} - \frac{Y_U^{ij}v}{\sqrt{2}}\bar{u}_{Li}U_{Rj} - \frac{Y_E^{ij}v}{\sqrt{2}}\bar{e}_{Li}E_{Rj} + H.c \\ &= -\bar{d}_{Li}\mathcal{M}_d D_{Rj} - \bar{u}_{Li}\mathcal{M}_u U_{Rj} - \bar{e}_{Li}\mathcal{M}_e E_{Rj} + H.c. \end{aligned} \quad (2.48)$$

Since the Yukawa couplings are not diagonal, the fields in this basis have no definite mass. By performing unitary transformations of the fields

$$u_L = U_L^u u_L', \;\; d_L = U_L^d d_L', \;\; e_L = U_L^e e_L' \;\; U_R = V_R^u U_R', \;\; D_R = V_R^d D_R', \; E_R = V_R^e E_R', \quad (2.49)$$

the complex Yukawa matrices are diagonalized by bi-unitary transformations as

$$U_L^{u\dagger}\mathcal{M}_u V_R^u = \mathrm{diag}(m_u, m_c, m_t), \quad (2.50)$$

$$U_L^{d\dagger}\mathcal{M}_d V_R^d = \mathrm{diag}(m_d, m_s, m_b), \quad (2.51)$$

$$U_L^{e\dagger}\mathcal{M}_e V_R^e = \mathrm{diag}(m_e, m_\mu, m_\tau). \quad (2.52)$$

The fermion mass terms become

$$\begin{aligned}\mathcal{L}_{\mathrm{mass}} &= -\begin{pmatrix}\bar{u}' & \bar{c}' & \bar{t}'\end{pmatrix}\begin{pmatrix}m_u & 0 & 0\\ 0 & m_c & 0\\ 0 & 0 & m_t\end{pmatrix}\begin{pmatrix}u'\\ c'\\ t'\end{pmatrix} - \begin{pmatrix}\bar{d}' & \bar{s}' & \bar{b}'\end{pmatrix}\begin{pmatrix}m_c & 0 & 0\\ 0 & m_s & 0\\ 0 & 0 & m_b\end{pmatrix}\begin{pmatrix}d'\\ s'\\ b'\end{pmatrix} \\ &\quad - \begin{pmatrix}\bar{e}' & \bar{\mu}' & \bar{\tau}'\end{pmatrix}\begin{pmatrix}m_e & 0 & 0\\ 0 & m_\mu & 0\\ 0 & 0 & m_\tau\end{pmatrix}\begin{pmatrix}e'\\ \mu'\\ \tau'\end{pmatrix}\end{aligned} \quad (2.53)$$

where the primed fields denote mass eigenstates. Note that neutrinos are massless because no right-handed neutrino exists in the SM.

Higgs mass

After SSB, the Higgs field obtains mass given by

$$m_h = \sqrt{2}\mu = \sqrt{2\lambda}v. \tag{2.54}$$

The value of VEV, $v \simeq 246$ GeV, is obtained from its relation with Fermi coupling constant G_F. The Higgs mass is therefore governed by the quartic self-coupling λ at the weak scale.

2.3 SM Gauge Interactions After Electroweak Symmetry Breaking

In terms of mass eigenstates, the covariant derivative takes the following form

$$\begin{aligned} D_\mu &= \partial_\mu - i\frac{g_2}{\sqrt{2}}(W_\mu^+ t^+ + W_\mu^- t^-) - i\frac{1}{\sqrt{g_1^2+g_2^2}}Z_\mu(g_2^2 t^3 - g_1^2 Y) \\ &\quad - i\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}}A_\mu(t^3+Y) \end{aligned} \tag{2.55}$$

with

$$t^\pm = \frac{\sigma^1 \pm i\sigma^2}{2}. \tag{2.56}$$

The last term corresponds to electromagnetic interaction. Thus, the electron charge e and the electric charge quantum number Q are dictated to be

$$e = \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}} \qquad \text{and} \qquad Q = t^3 + Y, \tag{2.57}$$

respectively. Furthermore, we may simplify Z_μ coupling by using the weak mixing angle relation from Eq. (2.46)

$$\frac{1}{\sqrt{g_1^2+g_2^2}}(g_2^2 t^3 - g_1^2 Y) = \frac{1}{\sqrt{g_1^2+g_2^2}}\left[(g_1^2+g_2^2)t^3 - g_1^2 Q\right],$$

$$
\begin{aligned}
&= \frac{g_1^2+g_2^2}{\sqrt{g_1^2+g_2^2}}\left[t^3-\sin^2\theta_w Q\right], \\
&= \frac{g_2}{\cos\theta_w}\left[t^3-\sin^2\theta_w Q\right]. \qquad (2.58)
\end{aligned}
$$

The covariant derivative in Eq.(2.55) becomes

$$
D_\mu = \partial_\mu - i\frac{g_2}{\sqrt{2}}(W_\mu^+ t^+ + W_\mu^- t^-) - i\frac{g_2}{\cos\theta_w}Z_\mu\left(t^3-\sin^2\theta_w Q\right) - ieA_\mu Q. \qquad (2.59)
$$

This covariant derivative classifies the interactions between fermions and gauge bosons into three classes: electromagnetic interaction, neutral current interaction,and charged current interaction. In mass basis, the form of electromagnetic and neutral current interactions are not affected by unitary transformations in Eq. (2.49) and hence are flavor diagonal

$$
\begin{aligned}
\mathcal{L}_{\mathrm{EM}} &= -e\left(Q_u(\bar{u}'_L\gamma^\mu u'_L + \bar{u}'_R\gamma^\mu u'_R) + Q_d(\bar{d}'_L\gamma^\mu d'_L + \bar{d}'_R\gamma^\mu d'_R)\right. \\
&\quad \left. + Q_e(\bar{e}'_L\gamma^\mu e'_L + \bar{e}'_R\gamma^\mu e'_R)\right) A_\mu, \qquad (2.60) \\
\mathcal{L}_{\mathrm{NC}} &= \frac{g_2}{\cos\theta_w}\left(\frac{1}{2}\bar{u}'_L\gamma^\mu u'_L - \frac{1}{2}\bar{d}'_L\gamma^\mu d'_L + \frac{1}{2}\bar{\nu}_{eL}\gamma^\mu \nu_{eL} - \frac{1}{2}\bar{e}'_L\gamma^\mu e'_L\right. \\
&\quad -\sin^2\theta_w\left[Q_u(\bar{u}'_L\gamma^\mu u'_L + \bar{u}'_R\gamma^\mu u'_R) + Q_d(\bar{d}'_L\gamma^\mu d'_L + \bar{d}'_R\gamma^\mu d'_R)\right. \\
&\quad \left.\left. + Q_e(\bar{e}'_L\gamma^\mu e'_L + \bar{e}'_R\gamma^\mu e'_R)\right]\right) Z_\mu. \qquad (2.61)
\end{aligned}
$$

The sum over generation is understood. The charged current part given by

$$
\begin{aligned}
\mathcal{L}_{CC} &= \frac{g_2}{\sqrt{2}}\left[\bar{u}'_L\gamma^\mu U_L^{u\dagger}U_L^d d'_L W_\mu^+ + \bar{d}'_L\gamma^\mu U_L^{d\dagger}U_L^u u'_L W_\mu^- + \bar{\nu}'_{eL}\gamma^\mu e'_L W_\mu^+\right. \\
&\quad \left. + \bar{e}'_L\gamma^\mu \nu'_{eL} W_\mu^-\right], \qquad (2.62)
\end{aligned}
$$

has mixing between quarks from different generations induced by the quantity $U_L^{u\dagger}U_L^d$. This is the well known CKM (Cabibbo-Kobayashi-Maskawa) matrix (Cabibbo, 1963; Kobayashi and Maskawa, 1973). In the last two terms, we have absorbed U_L^e into neutrino fields by $\nu_{eL} = U_L^e\nu'_{eL}$. This can be done because in SM flavor neutrino is also mass eigenstate. As a result, no lepton flavor mixing presents in the SM.

2.4 Accidental Lepton and Baryon Number Symmetries in SM

The Standard Model has been constructed so far as a renormalized quantum field theory satisfying fundamental symmetries like Lorentz invariance and local gauge symmetry. With the given particle content, the SM lagrangian possesses another type of continuous symmetry that was not imposed as a requirement in its construction. It emerges accidentally once we have listed all terms that are consistent with the imposed symmetries so that it is called an accidental symmetry. Remarkably, despite being accidental, nature seems to preserve them. Nevertheless, the breaking of these symmetries is not considered sacrilege as they are not protected by the required fundamental symmetries.

The fermionic sector of the SM possesses a global gauge symmetry given by

$$U(1)_{L_e} \times U(1)_{L_\mu} \times U(1)_{L_\tau} \times U(1)_B, \tag{2.63}$$

associated with the first, second, third lepton generations, and baryon respectively. Noether's theorem then asserts that there will be conserved charges associated with these symmetries in all SM interactions. Let us take the first generation of lepton doublet as an example. We assign electronic lepton number $L_e = +1$ to electron and electron neutrino and $L_e = -1$ to their antiparticles, while all other particles have zero electronic lepton number. The total lepton number can be defined as $L = L_e + L_\mu + L_\tau$. These conserved charges are summarized in Table 2.2.

Historically, the lepton number was introduced to explain the absence of some physics processes. If ν_e and ν_μ were identical particles, it was expected that the following processes

$$\nu_\mu + n \longrightarrow e^- + p,$$

$$\bar{\nu}_\mu + p \longrightarrow e^+ + n$$

should happen at the same rate as the following ones

$$\nu_\mu + n \longrightarrow \mu^- + p,$$
$$\bar{\nu}_\mu + p \longrightarrow \mu^+ + n.$$

However, only the latter processes have been observed while not the former (Danby et al., 1962; Bienlein et al., 1964). This can be explained in terms of the separate lepton number conservation. Similarly, no positive signal of the proton decay has been reported hitherto. The explanation is that conservation of baryon number leads to the stability of the proton. For recent review about conservations of lepton and baryon number together with their experimental status, interested reader can consult Ref. (Pich and Musolf, 2019).

In fact, individual lepton number and baryon number are only conserved at the classical level. In SM, the non-perturbative effect from sphaleron process and quantum anomaly break L and B separately, but still preserve the combination $B - L$.* However, this process is only relevant in the early universe when the temperature of the surrounding plasma is extremely high. This is not the case as probed in any experimental setup. In addition, the exact masslessness of neutrino within the SM is due to the exact $B - L$ symmetry (Witten, 2001).

*sphaleron process and quantum anomaly also break $B + L$.

Table 2.2 Lepton and baryon number assignment. The corresponding antiparticles will have the same assigment but with opposite sign.

Particle	L_e	L_μ	L_τ	L	B
$\binom{\nu_e}{e}$	1	0	0	1	0
$\binom{\nu_\mu}{\mu}$	0	1	0	1	0
$\binom{\nu_\tau}{\tau}$	0	0	1	1	0
All quarks	0	0	0	0	1/3

CHAPTER III

BEYOND STANDARD MODEL

We have witnessed the astounding achievement of the SM in describing fundamental interactions of particles in TeV energy regime and its persistent predicitions which have been tested to high precision. However, several issues cannot be accomodated within the minimal SM, such as the massiveness of neutrino, the origin of electroweak symmetry breaking (EWSB), the stability of the Higgs mass from radiative correction, the presence of dark matter in our Universe. These indicate that the SM must be extended.

In this chapter, we focus on neutrino mass and radiative symmetry breaking. In the first section, two types of possible neutrino mass and variant neutrino mass models are presented. We pay particular attention to the so called type II seesaw model. Next, radiative symmetry breaking, which is an interesting solution to the origin of EWSB, is discussed. In the last section, the Gildener-Weinberg method for finding a non-trivial minimum of the effective potential is described for a scale invariant multiscalar potential.

3.1 Massive Neutrinos

Since 1998 (Fukuda et al., 1998), the neutrino oscillation experiments (Ahmad et al., 2001; Ahmad et al., 2002; Ahn et al., 2006; Eguchi et al., 2003) have provided us the solid evidence of massive neutrino. Unfortunately, they are sensitive only to the mass-squared difference of neutrinos, and the absolute mass scale of neutrino stays unknown. The upper bound of the sum of neutrino masses however has been found to be $\sum m_\nu < 0.12$ eV from cosmological observations of cosmic microwave background by Planck satellite (Aghanim et al., 2020). Moreover, the question regarding the Dirac or Majorana nature of neutrinos is still unanswered.

Dirac Neutrinos

If neutrinos are Dirac particles, the right-handed neutrino must be present. The Dirac mass term couples left-handed and right-handed fields as

$$\mathcal{L}_{Dirac} = -m_D(\bar{\nu}_R \nu_L + \bar{\nu}_L \nu_R). \tag{3.1}$$

The simplest extension of SM can be done by introducing right-handed singlet neutrinos. Similar to the quark and charged lepton masses, this neutrino mass can be generated via Yukawa interaction with the Higgs field.

Majorana Neutrinos

If neutrinos are their own antiparticles, the Majorana mass term can be constructed from only one left-handed neutrino field of the form

$$\mathcal{L}_{\text{Majorana},L} = -m_L(\bar{\nu}_L^C \nu_L + \bar{\nu}_L \nu_L^C). \tag{3.2}$$

However, neither this nor mass term generating from Yukawa interaction is allowed by $SU(2)_L \times U(1)_Y$ so that the SM must be non-trivially extended.

These neutrino problems constitute the active area of research in neutrino physics. The answers to these questions provide a key to the theoretical construction of a model beyond the SM. Typically, there are two approaches to extend the SM. The first approach is to consider the SM as an effective theory up to electroweak energy, and one could relax the renormalizablity by adding higher dimensional operator (Weinberg, 1979) which is suppressed by some higher energy scale. Another approach is to introduce a new kind of particle that interacts with neutrino at tree-level.

3.1.1 Canonical Seesaw Models

The minimality of SM particle content (i.e. without right-handed neutrino) does not legitimate us to form Dirac mass term for neutrino. Constructing a gauge invariant renormalizable Majorana mass term is also not permitted because we only have single scalar Higgs doublet in SM. As mentioned above, regarding the SM as an effective field theory gives us a hint to consider the Weinberg operator ($\Lambda^{-1}LLHH$) as a stepping stone to seriously reveal the UV-completion of this operator. Indeed, it had been proven in (Ma, 1998) that there are three realization of Weinberg operator at tree-level. They are well known as canonical seesaw models. The seesaw mechanism is one of the most appealing mechanisms to generate mass of neutrino. Its main principle is to introduce a new heavy particle into the SM and allow B-L violation. The tiny mass of neutrino is ascribed by the presence of heavy particle and lepton number violation. The followings are main scenarios that have been explored among theorists.

1. Type I Seesaw

   This is the simplest extension of SM in which heavy right-handed neutrinos ν_R are introduced (Minkowski, 1977; Gell-Mann et al., 1979; Yanagida, 1979; Mohapatra and Senjanović, 1980). They are singlet under SM gauge group so that they have no interaction with gauge bosons. The relevant parts of the Lagrangian read

$$-\mathcal{L} \supset Y_L^{ij}\overline{L}_L^i H E_R^j + Y_\nu^{ij}\overline{L}_L^i \tilde{H}\nu_R^j + \frac{1}{2}\overline{\nu_R^c}M_R\nu_R + H.c. \tag{3.3}$$

   where M_R is symmetric Majorana mass matrix. Once integrating out heavy right-handed neutrinos, we get an effective Weinberg operator

$$\mathcal{L}_{d=5} \;=\; M_\nu^{ij}\overline{L}_L^i \tilde{H}\tilde{H}^T L_L^{cj} + H.c. \tag{3.4}$$

where the light Majorana neutrino mass matrix (after substituting SM Higgs VEV) is given by

$$m_\nu \;=\; -\frac{v^2}{2} Y_\nu M_R^{-1} Y_\nu^T. \tag{3.5}$$

As the scale of M_R is not dictated by the electroweak scale, tiny neutrino mass can be obtained by having right-handed neutrino mass scale far above electroweak scale.

2. Type II Seesaw

In this scenario, the Higgs sector is extended by a Higgs triplet, Δ, which transforms as an adjoint representation of SU(2) with hypercharge +1 (Konetschny and Kummer, 1977; Schechter and Valle, 1980). With this additional triplet, the relevant terms generating neutrino mass are given by

$$-\mathcal{L} \supset Y_L^{ij}\overline{L}_L^i H E_R^j + L^T C Y_\nu i\sigma_2 \Delta L + \mu H^T i\sigma_2 \Delta^\dagger H + H.c. \tag{3.6}$$

Integration of heavy scalar triplet leads to the Weinberg operator of the form

$$\mathcal{L}_{d=5} \;=\; M_\nu^{ij} L_L^{Ti} C \tilde{H}^* \tilde{H}^\dagger L_L^j + H.c. \tag{3.7}$$

After spontaneous symmetry breaking, the VEV of the neutral multiplet of Higgs triplet gives the light neutrino mass matrix of the form

$$m_\nu \;=\; \frac{\mu v_h^2}{M_\Delta^2} Y_\nu. \tag{3.8}$$

In fact, the term $\mu v_h^2/M_\Delta^2$ is approximately equal to the VEV of scalar triplet Higgs, v_δ, which is bounded from above by few GeV by ρ parameter (Arhrib et al., 2011). Besides the suppression from the mass scale of triplet Higgs

M_Δ, tiny neutrino mass can be achieved either by small Yukawa coupling or by very small Higgs triplet VEV. Each scenario has its own distinctive feature which makes type II seesaw model has rich phenomenological aspects. Moreover, the presence of a charged scalar makes this model becoming one of the most active research models in scalar extension of SM. More details of this model is given in the next subsection.

3. Type III Seesaw

This is also known as fermionic seesaw. In this model, a triplet fermion F is introduced, and a new Yukawa interaction $\bar{L}\vec{\tau}H\vec{F}$ is allowed (Foot et al., 1989). The inclusion of heavy triplet fermion gives the following additional terms to the Lagrangian

$$-\mathcal{L} \supset Y_L^{ij}\overline{L}_L^i H E_R^j + \sqrt{2}Y_F^{ij}\overline{L}_L^i F^{cj}\tilde{H} + \frac{1}{2}\,\mathrm{Tr}\Big(\overline{F}M_F F^c\Big) + H.c, \tag{3.9}$$

where heavy fermion triplet is expressed in terms of 2×2 traceless Hermitian matrix as

$$F \equiv \frac{\vec{\sigma}}{\sqrt{2}}\cdot\vec{F} = \begin{pmatrix} F^0/\sqrt{2} & F^+ \\ F^- & -F^0/\sqrt{2} \end{pmatrix}. \tag{3.10}$$

Upon intergrating out heavy fermion triplet, the Weinberg operator reads

$$\mathcal{L}_{d=5} \;=\; M_\nu^{ij}\overline{L}_L^i\tilde{H}\tilde{H}^T L_L^{cj} + H.c, \tag{3.11}$$

where light neutrino mass matrix is given by

$$m_\nu \;=\; -\frac{v^2}{2}Y_F M_F^{-1}Y_F^T. \tag{3.12}$$

The scale of heavy fermion M_F allows us to get tiny netrino mass of the order eV.

3.1.2 Type II Seesaw Model

The extension of SM in scalar sector was pioneered by Kummer and Konetschny in 1977 (Konetschny and Kummer, 1977). In their study, the scalar boson can induce lepton number violating process. Later this idea was followed by Gelmini and Roncadelli in 1981 (Gelmini and Roncadelli, 1981). The confirmation from neutrinoless double-beta decay experiments about the Majorana nature of neutrinos has been being waited for long time. If positive results are obtained by the ongoing experiment (Andringa et al., 2016; Chen et al., 2017; Aalseth et al., 2018) and forthcoming experiments (Shirai, 2018; Abgrall et al., 2017; Wang et al., 2015), one should proceed to explore the mass generation beyond the SM. One of the most economical and natural extensions is the inclusion of triplet Higgs. Several aspects of this minimal version have been extensively investigated ranging from the property of neutrino mass (Garayoa and Schwetz, 2008; Kadastik et al., 2008; Akeroyd et al., 2008; Fileviez Pérez et al., 2008) to the decay rate of SM like-Higgs boson ($h \to \gamma\gamma, Z\gamma$) (Arhrib et al., 2012; Chen et al., 2013). The recent study of its collider phenomenology was given in Ref. (Du et al., 2019; Antusch et al., 2019; Primulando et al., 2019).

Besides providing a solution to small neutrino mass, another benefit of introducing the scalar triplet is that its existence can considerably alter the property of scalar Higgs potential in such a way that the vacuum becomes stable up to the Planck scale. A small range of the Higgs mass, $M_H \in (124, 126)$ GeV, was preferred by ATLAS (Aad et al., 2012b) and CMS (Chatrchyan et al., 2012b). This low Higgs mass will drive the λ_{SM} quartic coupling to be negative before the Planck scale is reached. This effect is primarily due to the radiative correction of top Yukawa coupling to the electroweak potential. The running of quartic coupling to negative value can be avoided by additional positive contribution due to interaction term between triplet and doublet Higgs (Chao et al., 2012; Bhupal Dev et al., 2013; Kobakhidze and Spencer-Smith, 2013). Since the triplet scalar also couples directly to the SM gauge boson (W, γ, Z), its VEV will change the value of ρ-parameter of SM at tree-level. Therefore, the extension of SM with scalar triplet in type II-seesaw provides much diverse phenomenology that can be tested at collider experiments.

The minimal type II seesaw requires the addition of a single scalar triplet $\vec{\Delta}$ with hypercharge $Y = +1$ to the SM particle content. This scalar triplet belongs to an adjoint representation of $SU(2)$

$$\vec{\Delta} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} \sim (\mathbf{1}, \mathbf{3}, 1) \tag{3.13}$$

and transforms as

$$\vec{\Delta} \to \vec{\Delta}' = U\vec{\Delta} \tag{3.14}$$

where $U \in SU(2)_{\text{adj}}$. A more convenient way is to express it in terms of 2×2 complex traceless matrix of the form

$$\Delta \equiv \frac{\vec{\sigma}}{\sqrt{2}} \cdot \vec{\Delta} = \begin{pmatrix} \delta^+/\sqrt{2} & \delta^{++} \\ \delta^0 & -\delta^+/\sqrt{2} \end{pmatrix} \tag{3.15}$$

which transforms as

$$\Delta \to U \Delta U^\dagger \tag{3.16}$$

where $U \in SU(2)_{\text{fund}}$. With the Higgs triplet, the Lagrangian of the scalar fields is given by

$$\mathcal{L}(H, \Delta) = (D^\mu H)^\dagger (D_\mu H) + \text{Tr}\Big((D^\mu \Delta)^\dagger (D_\mu \Delta)\Big) + \mathcal{L}_{\text{Yukawa}} - V(H, \Delta). \tag{3.17}$$

The Majorana neutrino mass is generated via the Yukawa interaction of Δ with the left-handed lepton doublet L,

$$-\mathcal{L}_{\text{Yukawa}} = L^T C Y_\nu i\sigma_2 \Delta L + H.c., \tag{3.18}$$

where Y_ν is the corresponding Yukawa coupling matrix. The Higgs scalar potential takes the following form

$$\begin{aligned} V(H,\Delta) &= -\mu_H^2 H^\dagger H + M_\Delta^2 \operatorname{Tr}\Delta^\dagger\Delta + \lambda_H (H^\dagger H)^2 + (\mu H^T i\sigma_2 \Delta^\dagger H + H.c.) \\ &\quad +\lambda_{H\Delta}(H^\dagger H)\operatorname{Tr}\Delta^\dagger\Delta + \lambda_\Delta(\operatorname{Tr}\Delta^\dagger\Delta)^2 + \lambda'_\Delta \operatorname{Tr}\Big(\Delta^\dagger\Delta\Big)^2 \\ &\quad +\lambda'_{H\Delta} H^\dagger\Delta\Delta^\dagger H, \end{aligned} \tag{3.19}$$

where all the couplings are taken to be real.

After symmetry breaking, the neutral components of doublet and triplet Higgs will acquire nonzero VEV

$$\langle H\rangle = \frac{v_h}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}, \quad \langle\Delta\rangle = \frac{v_\delta}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{3.20}$$

Therefore after symmetry breaking, neutrinos obtain masses approximately (assuming $M_\Delta^2 \gg m_\nu^2$) of the order

$$m_\nu = \frac{\mu v_h^2}{M_\Delta^2} Y_\nu. \tag{3.21}$$

The smallness of m_ν (i.e. $\sim 0.05\,\mathrm{eV}$) is set by having M_Δ at the TeV scale, $\mu \sim O(\mathrm{eV})$, and $Y_\nu \sim O(1)$. This mass generation can be illustrated by the Feynman diagram in Figure 3.1. Besides its practical purpose in yielding small mass for neutrino, one must consider the impact of the presence of Higgs triplet. The precise measurement of ρ parameter stringently bounds the value of VEV of Higgs triplet. One way to allow small deviation as small as possible is by giving smaller VEV to neutral Higgs triplet, few GeV, than those for the neutral doublet after electroweak symmetry breaking, viz. $|v_\Delta| \ll v$ (Gunion et al., 2000).

Assuming that the vacua given by Eq. (3.20) gives the lowest energy state, the stationary conditions for non-trivial solutions give us the following two relations

$$\mu_H^2 = \lambda_H v_h^2 + \frac{\lambda_{H\Delta} + \lambda'_{H\Delta}}{2} v_\delta^2 - \sqrt{2}\mu v_\delta, \tag{3.22}$$

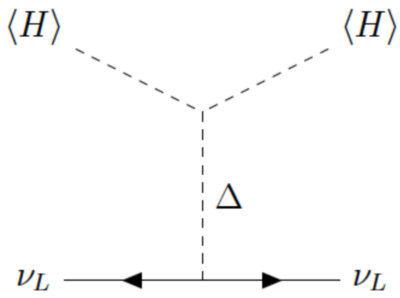

Figure 3.1 Feynman diagram of neutrino mass generation via coupling with Higgs triplet.

$$M_\Delta^2 \;=\; \frac{\mu v_h^2}{\sqrt{2}v_\delta} - \frac{\lambda_{H\Delta} + \lambda'_{H\Delta}}{2} v_h^2 - (\lambda_\Delta + \lambda'_\Delta) v_\delta^2. \tag{3.23}$$

By shifting the fields as

$$H = \begin{pmatrix} h^+ \\ \frac{v_h + h + iZ_h}{\sqrt{2}} \end{pmatrix}, \quad \Delta = \begin{pmatrix} \delta^+/\sqrt{2} & \delta^{++} \\ \frac{v_\delta + \delta^0 + iZ_\Delta}{\sqrt{2}} & -\delta^+/\sqrt{2} \end{pmatrix} \tag{3.24}$$

the mass spectra of scalar fields can be obtained by reading off the quadratic term in the fields. From this procedure, we will have doubly charged Higgs $\delta^{\pm\pm}$, singly charged Higgs $\phi^\pm$, two neutral CP-even h^0 and H^0 and neutral CP-odd A^0. The three remaining are the massless Goldstone bosons which become the longitudinal part of $W^\pm$ and Z bosons. These scalar mass eigenstates are presented below.

Mass of the doubly-charged Higgs

The mass of the doubly-charged fields is given by

$$m_{\delta^{\pm\pm}}^2 \;=\; \frac{\mu v_h^2}{\sqrt{2}v_\delta} - \frac{\lambda'_{H\Delta}}{2} v_h^2 - \lambda'_\Delta v_\delta^2. \tag{3.25}$$

Mass of the singly-charged Higgs

The mass-squared matrix of the singly-charged Higgs is

$$\mathcal{M}_\pm^2 \;=\; \left(\sqrt{2}\mu - \frac{\lambda'_{H\Delta}}{2} v_\delta\right) \begin{pmatrix} v_\delta & -v_h/\sqrt{2} \\ -v_h/\sqrt{2} & v_h^2/2v_\delta \end{pmatrix}. \tag{3.26}$$

The above mass matrix is diagonalized by orthogonal matrix $\mathcal{O}_\pm$ such that $\mathcal{O}_\pm \mathcal{M}^2_\pm \mathcal{O}^{\mathrm{T}}_\pm = \mathrm{diag}\left(0, m^2_{\phi^\pm}\right)$. The massless states $G^\pm$ correspond to the Goldstone bosons that will be absorbed by $W^\pm$ gauge boson. The physical massive singly-charged state has mass

$$m_{\phi^\pm} = (\sqrt{2}\mu/2v_\delta - \lambda_{H\Delta}/4)(v_h^2 + 2v_\delta^2). \tag{3.27}$$

The weak and mass eigenstates are related by

$$\begin{pmatrix} G^\pm \\ \phi^\pm \end{pmatrix} = \begin{pmatrix} c_\pm & s_\pm \\ -s_\pm & c_\pm \end{pmatrix} \begin{pmatrix} h^\pm \\ \delta^\pm \end{pmatrix}. \tag{3.28}$$

Mass of the neutral CP-even Higgs

The neutral scalar mass-squared matrix is

$$\mathcal{M}^2_{\text{CP-even}} = \begin{pmatrix} 2\lambda_H v_h^2 & -\sqrt{2}\mu v_h + (\lambda_{H\Delta} + \lambda'_{H\Delta})v_h v_\delta \\ -\sqrt{2}\mu v_h + (\lambda_{H\Delta} + \lambda'_{H\Delta})v_h v_\delta & \mu v_h^2/\sqrt{2}v_\delta + 2(\lambda_\Delta + \lambda'_\Delta)v_\delta^2 \end{pmatrix}. \tag{3.29}$$

The orthogonal transformation diagonalizes the mass matrix such that

$$\begin{pmatrix} m^2_{h^0} & 0 \\ 0 & m^2_{H^0} \end{pmatrix} = \begin{pmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} c_0 & -s_0 \\ s_0 & c_0 \end{pmatrix}. \tag{3.30}$$

The eigenvalues are given by

$$m^2_{h^0} = \frac{1}{2}\left[A + C - \sqrt{(A-C)^2 + 4B^2}\right], \tag{3.31}$$
$$m^2_{H^0} = \frac{1}{2}\left[A + C + \sqrt{(A-C)^2 + 4B^2}\right], \tag{3.32}$$

where

$$\begin{aligned} A &= 2\lambda_H v_h^2, \\ B &= -\sqrt{2}\mu v_h + (\lambda_{H\Delta} + \lambda'_{H\Delta})v_h v_\delta, \\ C &= \mu v_h^2/\sqrt{2}v_\delta + 2(\lambda_\Delta + \lambda'_\Delta)v_\delta^2. \end{aligned} \tag{3.33}$$

The mass eigenstates are related to the gauge eigenstates as

$$\begin{pmatrix} h^0 \\ H^0 \end{pmatrix} = \begin{pmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{pmatrix} \begin{pmatrix} h \\ \delta^0 \end{pmatrix}. \tag{3.34}$$

Mass of the neutral CP-odd Higgs

The neutral pseudoscalar mass-squared matrix is

$$\mathcal{M}^2_{\text{CP-odd}} = \sqrt{2}\mu \begin{pmatrix} v_\delta & -v_h \\ -v_h & v_h^2/2v_\delta \end{pmatrix}. \tag{3.35}$$

The determinant of the above CP-odd mass matrix is zero. The massless state G^0 corresponds to the Goldstone boson that will be absorbeb by Z gauge boson. This mass matrix can be diagonalized by orthogonal transformation such that $\mathcal{O}_0 \mathcal{M}^2_{\text{CP-odd}} \mathcal{O}_0^{\text{T}} = \text{diag}\,(0, m_A^2)$, and the mass eigenstates and interaction eigenstates are related by

$$\begin{pmatrix} G^0 \\ A^0 \end{pmatrix} = \begin{pmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{pmatrix} \begin{pmatrix} Z_h \\ Z_\delta \end{pmatrix}. \tag{3.36}$$

In order to ensure that the vacuum is stable, the potential must be bounded from below (BFB) in the limit where the scalar field value is large in any direction in the field space. This theoretical constraint is called the vacuum stability or BFB condition. For type II seesaw model, it has been derived in Ref. (Arhrib et al., 2011; Bonilla et al., 2015) that the quartic scalar couplings need to satisfy the following conditions

$$\begin{gathered} \lambda_H > 0, \quad \lambda_\Delta + \frac{\lambda'_\Delta}{2} > 0, \quad \lambda_\Delta + \lambda'_\Delta > 0, \\ 2\sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta)} + (\lambda_{H\Delta} + \lambda'_{H\Delta}) > 0,\ 2\sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta)} + \lambda_{H\Delta} > 0, \\ \Bigg(2\lambda'_\Delta \sqrt{\lambda_H} \le |\lambda'_{H\Delta}| \sqrt{\lambda_\Delta + \lambda'_\Delta} \ \text{ OR} \\ \lambda_{H\Delta} + \frac{\lambda'_{H\Delta}}{2} + \frac{1}{2}\sqrt{\left(\frac{2\lambda_\Delta}{\lambda'_\Delta} + 1\right)(8\lambda'_\Delta \lambda_H - \lambda'^2_{H\Delta}) > 0} \Bigg). \end{gathered} \tag{3.37}$$

Another set of theoretical constraints is coming from the requirement that unitarity must be preserved in various scattering processes, such as scalar-scalar scattering, gauge boson-gauge boson scattering and scalar-gauge boson scattering (Cornwall et al., 1974; Lee et al., 1977). By considering only scalar-scalar scattering processes dominated by quartic interactions, unitarity constraints on the scalar masses for type II seesaw are given in (Arhrib et al., 2011):

$$
\begin{aligned}
|\lambda_H| &\leq 8\pi, \\
|\lambda_\Delta| &\leq 8\pi, \\
|\lambda_\Delta + \lambda'_\Delta| &\leq 8\pi, \\
|2\lambda_\Delta - \lambda'_\Delta| &\leq 16\pi, \\
|\lambda_{H\Delta}| &\leq 16\pi, \\
|\lambda_{H\Delta} + \lambda'_{H\Delta}| &\leq 16\pi, \\
|2\lambda_{H\Delta} + 3\lambda'_{H\Delta}| &\leq 32\pi, \\
|2\lambda_{H\Delta} - \lambda'_{H\Delta}| &\leq 32\pi, \\
\left|4\lambda_H + 4\lambda_\Delta + 8\lambda'_\Delta \pm \sqrt{(4\lambda_H - 4\lambda_\Delta - 8\lambda'_\Delta)^2 + 16\lambda'^2_{H\Delta}}\right| &\leq 64\pi, \\
\left|12\lambda_H + 16\lambda_\Delta + 12\lambda'_\Delta \pm \sqrt{(12\lambda_H - 16\lambda_\Delta - 12\lambda'_\Delta)^2 + 24(\lambda_{H\Delta} + \lambda_{H\Delta})'^2}\right| &\leq 64\pi.
\end{aligned}
\tag{3.38}
$$

3.2 Radiative Symmetry Breaking

As shown in the previous chapter, the sign of μ^2 parameter plays an essential role in determining whether the SM symmetry is broken or not. This dimensionful parameter is put by hand in order to realize successful symmetry breaking. It turns out to be the only dimensionful parameter existing in the model. In fact, this negative mass squared term sets the electroweak scale and hence the source of hierarchy problem (the huge separation between electroweak scale $\sim$ 100 GeV and Planck scale, $M_{Pl} \sim 10^{19}$ GeV). In the limit $\mu \to 0$, the SM Lagrangian enjoys a scale symmetry. Generally, if the underlying symmetry of nature is classical scale invariance, all kinds of dimensionful parameters will be forbidden. The scale invariance is advocated thereby omitting this dimensionful parameter and the generation of mass scale can be succesfully realized dynamically by dimensional transmutation.

A seminal paper by Coleman and Weinberg (Coleman and Weinberg, 1973) elucidated the possibility of symmetry breaking in the absence of dimensionful parameter. Instead of breaking the symmetry classically at tree-level, the CW mechanism is shown to successfully break the symmetry by taking quantum corrections into account in the case of scalar electrodynamics. In the next subsections, we firstly review the evaluation of the effective potential from the effective action using the path integral formalism (Jackiw, 1974). It is a necessary tool for the determination of vacua of the theory. Subsequently, the dynamical symmetry breaking in scalar electrodynamics is demonstrated. Later, we shortly revisit the SM to verify its inability to be a classically scale invariant theory given the masses of its bosonic content. In the last subsection, a short comment on the scale invariant type II seesaw is given.

3.2.1 Effective Action and Effective Potential

The effective action is an essential tool to study the symmetry breaking of a given quantum field theory. The vacuum structure of quantum theory can be scrutinized by using the effective action. For an interacting real scalar field theory with action $S[\phi] = \int d^4x\, \mathcal{L}[\phi]$, the vacuum-to-vacuum amplitude in the presence of the source $J(x)$ is given by

$$Z[J] \;=\; \int \mathcal{D}\phi \,\exp\Big[\frac{i}{\hbar}\int d^4x(\mathcal{L}[\phi]+J(x)\phi(x))\Big]. \tag{3.39}$$

$Z[J]$ is also known as generating functional of Green's function which includes both connected and disconneted Green's functions. The generating fuctional of all connected Green's function $W[J]$ can be defined by

$$Z[J]=\exp\Big(\frac{i}{\hbar}W[J]\Big). \tag{3.40}$$

The vacuum expectation value of quantum field operator in the presence of J is given by

$$\phi_{cl}(x) \;=\; \frac{\delta W[J]}{\delta J(x)} \equiv \frac{\langle 0|\,\phi\,|0\rangle}{\langle 0|0\rangle}. \tag{3.41}$$

An important type of connected Feynman's diagrams is known as the one-particle-irreducible (1PI) diagram, which cannot be split into two by cutting any internal line. The generating functional for the 1PI diagrams is called an effective action $\Gamma[\phi]$. It is defined by the Legendre transformation of $W[J]$ as follows

$$\Gamma[\phi_{cl}] \;=\; W[J]-\int d^4x\; J(x)\phi_{cl}(x). \tag{3.42}$$

In order to calculate $\Gamma[\phi_{cl}]$, one has to solve Eq.(3.41) and express J in terms of ϕ_{cl}. Starting from Eq. (3.40),

$$Z[J]=\exp\Big(\frac{i}{\hbar}W[J]\Big)=\int \mathcal{D}\phi \,\exp\Big(\frac{i}{\hbar}S[\phi,J]\Big), \tag{3.43}$$

where

$$\begin{aligned} S[\phi,J] \;&=\; \int d^4x(\mathcal{L}[\phi]+J\phi), \\ &=\; S[\phi]+\int d^4x\,J\phi. \end{aligned} \tag{3.44}$$

The equation of motion is given by

$$\frac{\delta S[\phi,J]}{\delta\phi}\Big|_{\phi=\varphi} \;=\; 0,$$

$$\frac{\delta S[\phi]}{\delta \phi}\Big|_{\phi=\varphi} = -J. \tag{3.45}$$

In order to evaluate the integral in Eq. (3.43), we may use saddle-point approximation by expanding the argument of exponential around the solution φ as follows

$$\begin{aligned} S[\phi+\varphi] &= S[\varphi] + \int d^4x \frac{\delta S}{\delta \phi(x)}\Big|_{\phi=\varphi} \phi(x) \\ &\quad + \frac{1}{2!}\int d^4x\, d^4y \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\varphi} \phi(x)\phi(y) + \dots\ . \end{aligned} \tag{3.46}$$

The generating functional $Z[J]$ becomes

$$\begin{aligned} Z[J] &\simeq \int \mathcal{D}\phi\ \exp\left[\frac{i}{\hbar}\left(S[\varphi] + \int d^4x \frac{\delta S}{\delta\phi(x)}\Big|_{\phi=\varphi}\phi(x)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\int d^4x\, d^4y \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\varphi}\phi(x)\phi(y) + \int d^4x\ J(\phi+\varphi)\right)\right] \\ &= \int \mathcal{D}\phi\ \exp\left[\frac{i}{\hbar}\left(S[\varphi, J] + \frac{1}{2}\int d^4x\, d^4y \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\varphi}\phi(x)\phi(y)\right)\right] \\ &= \exp\Big(\frac{i}{\hbar}S[\varphi, J]\Big)\int \mathcal{D}\phi\ \exp\left[i\left(\frac{1}{2}\int d^4x\, d^4y \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\varphi}\phi(x)\phi(y)\right)\right] \\ &= \exp\Big(\frac{i}{\hbar}S[\varphi, J]\Big)\ \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\varphi}\right)^{-1/2}. \end{aligned} \tag{3.47}$$

The generating functional $W[J]$ is then

$$\begin{aligned} \frac{i}{\hbar}W[J] &= \frac{i}{\hbar}S[\varphi] + \frac{i}{\hbar}\int d^4x\ J\varphi - \frac{1}{2}\ln \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\varphi}\right), \\ W[J] &= S[\varphi] + \int d^4x\ J\varphi + \frac{i\hbar}{2}\ln \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\varphi}\right). \end{aligned} \tag{3.48}$$

We need to find the relation between φ and ϕ_{cl}. If $\hbar$ were zero, we would find

$$\frac{\delta W[J]}{\delta J} = \frac{\delta S[\varphi]}{\delta \phi}\frac{\delta \phi}{\delta J} + \varphi + J\frac{\delta \phi}{\delta J} = \varphi. \tag{3.49}$$

Thus, at order $\hbar$, they are related as

$$\phi_{cl} = \varphi + \varphi^1, \tag{3.50}$$

where φ^1 is a functional of ϕ_{cl} and it is of the order of $\hbar$. Substituting back into Eq.(3.42), we have

$$\Gamma[\phi_{cl}] = S[\phi_{cl} - \varphi^1] + \int d^4x\ J[\phi_{cl} - \varphi^1](\phi_{cl} - \varphi^1)$$

$$
\begin{aligned}
&+\frac{i\hbar}{2}\ln \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi_{cl}-\varphi^1}\right)-\int d^4x\; J[\phi_{cl}-\varphi^1]\phi_{cl} \\
= \quad & S[\phi_{cl}]-\left(\frac{\delta S[\phi_{cl}]}{\delta\phi_{cl}}+J\right)\varphi^1+\frac{1}{2}\varphi^1\frac{\delta^2 S[\phi_{cl}]}{\delta\phi_{cl}^2}\varphi^1+\varphi^1\frac{\delta J[\phi_{cl}]}{\delta\phi_{cl}}\varphi^1 \\
&+\frac{i\hbar}{2}\ln \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi_{cl}}\right)-\varphi^1\frac{i\hbar}{2}\frac{\delta}{\delta\phi_{cl}}\ln \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi_{cl}}\right) \\
&+\mathcal{O}\Big(\hbar^3\Big). \qquad (3.51)
\end{aligned}
$$

Taking into account only the one-loop correction, we find

$$
\Gamma[\phi_{cl}] \quad = \quad S[\phi_{cl}]+\frac{i\hbar}{2}\ln \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi_{cl}}\right). \qquad (3.52)
$$

For translationally invariant theory, the effective action and effective potential are related by

$$
\Gamma[\phi_{cl}] \quad = \quad -V_{\text{eff}}[\phi_{cl}]\int d^4x. \qquad (3.53)
$$

The effective potential then reads

$$
V_{\text{eff}}[\phi_{cl}]\int d^4x \quad = \quad V_0[\phi_{cl}]\int d^4x-\frac{i\hbar}{2}\ln \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi_{cl}}\right). \qquad (3.54)
$$

3.2.2 Dynamical Symmetry Breaking in Scalar Electrodynamics

In this subsection, we are going to show how the hierarchy of the couplings is an active ingredient for dynamical symmetry breaking by quantum correction. The simplest theory that exhibits such feature is scalar electrodymanics. The action of the massless scalar electrodynamics is

$$
S=\int d^4x\left(-\frac{1}{4}F_{\mu\nu}^2-\frac{1}{2\xi}(\partial_\mu A^\mu)^2+|D_\mu\phi|^2-\frac{\lambda}{3!}|\phi|^4\right) \qquad (3.55)
$$

where ξ is a gauge-fixing parameter and the covariant derivative is $D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$. In order to calculate the effective potential Eq. (3.54), it is more convenient to express the action such that the bilenarity in the fields is obviously shown. Therefore,

$$\begin{aligned} S &= \int d^4x \left(\frac{1}{2} A^\mu \left[g_{\mu\nu} \partial_\mu \partial^\mu - \partial_\mu \partial_\nu (1 - \frac{1}{\xi}) \right] A^\nu - \frac{1}{2} \phi_a \partial_\mu \partial^\mu \phi_a + e A^\mu \epsilon_{ab} \phi_a \partial_\mu \phi_b \right. \\ & \left. + \frac{1}{2} e^2 A_\mu^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right) \end{aligned} \tag{3.56}$$

where $\epsilon = i\sigma^2$ and ϕ_a is the real component of the complex field ϕ $(a, b = 1, 2)$. The second functional derivative of the action is taken with respect to the fields A^μ and ϕ. A straightforward computation gives

$$\left. \frac{\delta^2 S}{\delta A^\mu \delta A^\nu} \right|_{\phi_{cl}} \equiv S''_{AA} = \left[g_{\mu\nu} \partial_\mu \partial^\mu - \partial_\mu \partial_\nu (1 - \frac{1}{\xi}) + e^2 g_{\mu\nu} \phi_{\rm cl}^2 \right] \delta^4(x - y), \tag{3.57}$$

$$\left. \frac{\delta^2 S}{\delta A^\mu \delta \phi_a} \right|_{\phi_{cl}} \equiv S''_{A\phi_a} = -e \epsilon_{ab} \phi_{b,cl} \partial_\mu \delta^4(x - y), \tag{3.58}$$

$$\left. \frac{\delta^2 S}{\delta \phi_b \delta A^\nu} \right|_{\phi_{cl}} \equiv S''_{\phi_b A} = e \epsilon_{bd} \phi_{d,cl} \partial_\nu \delta^4(x - y), \tag{3.59}$$

$$\left. \frac{\delta^2 S}{\delta \phi_1 \delta \phi_1} \right|_{\phi_{cl}} \equiv S''_{\phi_1 \phi_1} = \left[-\partial_\mu \partial^\mu - \frac{\lambda}{4!} \left(12 \phi_{1cl}^2 + 4 \phi_{2cl}^2 \right) \right] \delta^4(x - y), \tag{3.60}$$

$$\left. \frac{\delta^2 S}{\delta \phi_1 \delta \phi_2} \right|_{\phi_{cl}} \equiv S''_{\phi_1 \phi_2} = -\frac{\lambda}{4!} 8 \phi_{1cl} \phi_{2cl} \delta^4(x - y), \tag{3.61}$$

$$\left. \frac{\delta^2 S}{\delta \phi_2 \delta \phi_2} \right|_{\phi_{cl}} \equiv S''_{\phi_2 \phi_2} = \left[-\partial_\mu \partial^\mu - \frac{\lambda}{4!} \left(12 \phi_{2cl}^2 + 4 \phi_{1cl}^2 \right) \right] \delta^4(x - y). \tag{3.62}$$

By Fourier transforming into momentum space, we have

$$S''_{AA} = -g_{\mu\nu} p^2 + p_\mu p_\nu (1 - \frac{1}{\xi}) + e^2 g_{\mu\nu} \phi_{\rm cl}^2, \tag{3.63}$$

$$S''_{A\phi_a} = -ie \epsilon_{ab} \phi_{b,cl} p_\mu, \tag{3.64}$$

$$S''_{\phi_b A} = ie \epsilon_{bd} \phi_{d,cl} p_\nu, \tag{3.65}$$

$$S''_{\phi_1 \phi_1} = p^2 - \frac{\lambda}{4!} \left(12 \phi_{1cl}^2 + 4 \phi_{2cl}^2 \right), \tag{3.66}$$

$$S''_{\phi_1 \phi_2} = -\frac{\lambda}{3} \phi_{1cl} \phi_{2cl}, \tag{3.67}$$

$$S''_{\phi_2 \phi_2} = p^2 - \frac{\lambda}{4!} \left(12 \phi_{2cl}^2 + 4 \phi_{1cl}^2 \right). \tag{3.68}$$

In matrix form, this reads

$$S'' = \begin{pmatrix} -g_{\mu\nu}p^2 + p_\mu p_\nu(1-\frac{1}{\xi}) + e^2 g_{\mu\nu}\phi_{\rm cl}^2 & -ie\epsilon_{ab}\phi_{b,cl}p_\mu \\ ie\epsilon_{bd}\phi_{d,cl}p_\nu & \left(p^2 - \frac{\lambda}{6}\phi_{cl}^2\right)\delta_{ab} - \frac{\lambda}{3}\phi_{a,cl}\phi_{b,cl} \end{pmatrix}. \quad (3.69)$$

The determinant of a matrix

$$M = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \quad (3.70)$$

where A, B and C are themselves matrices is

$$\det(M) \;=\; \det(A)\det\Big(B - C^T A^{-1} C\Big). \quad (3.71)$$

Let us denote the entries of matrix S'' as follows

$$\Delta_{\mu\nu} \;=\; -g_{\mu\nu}\left(p^2 - e^2\phi_{\rm cl}^2\right) + p_\mu p_\nu(1 - \frac{1}{\xi}), \quad (3.72)$$

$$D_{ab} \;=\; \left(p^2 - \frac{\lambda}{6}\phi_{cl}^2\right)\delta_{ab} - \frac{\lambda}{3}\phi_{a,cl}\phi_{b,cl}. \quad (3.73)$$

The determinant of $\Delta_{\mu\nu}$ is

$$\det(\Delta_{\mu\nu}) \;=\; -\left(p^2 - e^2\phi_{\rm cl}^2\right)^3 \left(p^2 - (1-\frac{1}{\xi})p^2 - e^2\phi_{\rm cl}^2\right), \quad (3.74)$$

and its inverse is

$$\Delta_{\mu\nu}^{-1} \;=\; \frac{-g_{\mu\nu}\left(p^2 - (1-\frac{1}{\xi})p^2 - e^2\phi_{\rm cl}^2\right) - (1-\frac{1}{\xi})p_\mu p_\nu}{\left(p^2 - (1-\frac{1}{\xi})p^2 - e^2\phi_{\rm cl}^2\right)(p^2 - e^2\phi_{\rm cl}^2)}. \quad (3.75)$$

Finally, the last determinant becomes

$$\begin{aligned}\det\Big(D_{ab} - C^T\Delta_{\mu\nu}^{-1}C\Big) &= \det\left(\left[p^2 - \frac{\lambda}{6}\phi_{cl}^2\right]\delta_{ab} - \frac{\lambda}{3}\phi_{a,cl}\phi_{b,cl} + \frac{p^2\xi e^2\phi_{cl}^2\sigma_{ac}\sigma_{bd}\phi_{c,\rm cl}\phi_{d,cl}}{(p^2 - \xi e^2\phi_{\rm cl}^2)}\right) \\ &= \frac{p^2 - \frac{\lambda}{2}\phi_{cl}^2}{p^2 - \xi e^2\phi_{cl}^2}\left[\left(p^2 - \frac{\lambda}{6}\phi_{cl}^2\right)\left(p^2 - \xi e^2\phi_{cl}^2\right) + \xi e^2\phi_{cl}^2 p^2\right].\end{aligned} \quad (3.76)$$

Therefore, the determinant of S'' is obatained to be

$$\begin{aligned}\det(S'') &= -\frac{1}{\xi}\left(p^2 - e^2\phi_{\rm cl}^2\right)^3\left(p^2 - \frac{\lambda}{2}\phi_{cl}^2\right)\left[\left(p^2 - \frac{\lambda}{6}\phi_{cl}^2\right)\left(p^2 - \xi e^2\phi_{cl}^2\right) + \xi e^2\phi_{cl}^2 p^2\right] \\ &= -\frac{1}{\xi}\left(p^2 - e^2\phi_{\rm cl}^2\right)^3\left(p^2 - \frac{\lambda}{2}\phi_{cl}^2\right)\left[p^4 - \frac{\lambda}{6}\phi_{cl}^2 p^2 + \frac{\lambda}{6}\xi e^2\phi_{cl}^4\right].\end{aligned} \quad (3.77)$$

Inserting Eq. (3.77) into Eq. (3.54), we need to calculate the integral of the form (Di Luzio and Mihaila, 2014)

$$-\frac{i}{2}\mu^{2\epsilon}\int\frac{d^4p}{(2\pi)^d}\ln\Big(-p^2+A\Big) \;=\; \frac{1}{64\pi^2}A^2\left(\ln\frac{A}{\mu^2}-\frac{3}{2}-\Delta_\epsilon\right), \quad (3.78)$$

$$-(d-1)\frac{i}{2}\mu^{2\epsilon}\int\frac{d^4p}{(2\pi)^d}\ln\Big(-p^2+A\Big) \;=\; \frac{1}{64\pi^2}3A^2\left(\ln\frac{A}{\mu^2}-\frac{5}{6}-\Delta_\epsilon\right). \quad (3.79)$$

where the modified minimal subtraction term, Δ_ϵ, is defined to be

$$\Delta_\epsilon \;=\; \frac{1}{\epsilon}+\log 4\pi-\gamma_E. \quad (3.80)$$

The effective potential reads

$$\begin{aligned} V_{\text{eff}}[\phi_{cl}] \;=\; & \frac{\lambda+\delta\lambda}{4!}\phi_{cl}^4+\frac{1}{64\pi^2}\left[3e^4\phi_{\text{cl}}^4\left(\ln\frac{e^2\phi_{\text{cl}}^2}{\mu^2}-\frac{5}{6}\right)+\frac{\lambda^2}{4}\phi_{cl}^4\left(\ln\frac{\frac{\lambda}{2}\phi_{cl}^2}{\mu^2}-\frac{3}{2}\right)\right. \\ & \left.+m_+^4\left(\frac{m_+^2}{\mu^2}-\frac{3}{2}\right)+m_-^4\left(\frac{m_-^2}{\mu^2}-\frac{3}{2}\right)\right] \\ & -\frac{\Delta_\epsilon}{64\pi^2}\left[3e^4\phi_{\text{cl}}^4+\frac{\lambda^2}{4}\phi_{cl}^4+m_+^4+m_-^4\right], \end{aligned} \quad (3.81)$$

where we denote

$$m_\pm^2 \;=\; \frac{\phi_{cl}^2}{2}\left(\frac{\lambda}{6}\pm\sqrt{\frac{\lambda^2}{36}-\frac{2\lambda}{3}\xi e^2}\right). \quad (3.82)$$

In order to reproduce the Coleman-Weinberg result, we rewrite the effective potential as follows

$$\begin{aligned} V_{\text{eff}}[\phi_{cl}] \;=\; & \frac{\lambda+\delta\lambda}{4!}\phi_{cl}^4+\frac{\phi_{cl}^4}{64\pi^2}\left[\frac{5}{18}\lambda^2+3e^4-\frac{\lambda}{3}\xi e^2\right]\ln\frac{\phi_{cl}^2}{\mu^2}+\zeta(e,\xi.\lambda)\phi_{cl}^4 \\ & -\frac{\Delta_\epsilon}{64\pi^2}\left[3e^4\phi_{\text{cl}}^4+\frac{\lambda^2}{4}\phi_{cl}^4+m_+^4+m_-^4\right] \end{aligned} \quad (3.83)$$

where the remaining terms in Eq. (3.81) are contained in the ζ function. The renormalization conditions are imposed as

$$V_{\text{eff}}''[0]=0, \quad V_{\text{eff}}''''[M]=\lambda, \quad (3.84)$$

where M is an arbitrary renormalization scale. We find that

$$V_{\text{eff}}''''[M] \;=\; \lambda+\delta\lambda+\frac{3}{8\pi^2}\left[\frac{5}{18}\lambda^2+3e^4-\frac{\lambda}{3}\xi e^2\right]\ln\frac{M^2}{\mu^2}+24\zeta(e,\xi.\lambda)$$

$$+\frac{25}{16\pi^2}\left[\frac{5}{18}\lambda^2+3e^4-\frac{\lambda}{3}\xi e^2\right]-\frac{3\Delta_\epsilon}{8\pi^2}\left[\frac{5}{18}\lambda^2+3e^4-\frac{\lambda}{3}\xi e^2\right] \quad (3.85)$$

Hence the counterterm is

$$\begin{aligned}\delta\lambda &= \frac{3\Delta_\epsilon}{8\pi^2}\left[\frac{5}{18}\lambda^2+3e^4-\frac{\lambda}{3}\xi e^2\right]-\frac{3}{8\pi^2}\left[\frac{5}{18}\lambda^2+3e^4-\frac{\lambda}{3}\xi e^2\right]\ln\frac{M^2}{\mu^2}\\ &\quad -24\zeta(e,\xi.\lambda)-\frac{25}{16\pi^2}\left[\frac{5}{18}\lambda^2+3e^4-\frac{\lambda}{3}\xi e^2\right]. \end{aligned} \quad (3.86)$$

Plugging this into Eq. (3.83), finally we obtain

$$V_{\text{eff}}[\phi_{cl}] = \frac{\lambda}{4!}\phi_{cl}^4+\frac{\phi_{cl}^4}{64\pi^2}\left[\frac{5}{18}\lambda^2+3e^4-\frac{\lambda}{3}\xi e^2\right]\left(\ln\frac{\phi_{cl}^2}{M^2}-\frac{25}{6}\right). \quad (3.87)$$

Once fixing the gauge $\xi=0$, the result is consistent with Coleman-Weinberg's computation (Coleman and Weinberg, 1973)

$$V_{\text{eff}}[\phi_{cl}] = \phi_{cl}^4\left[\frac{\lambda}{4!}+\frac{1}{64\pi^2}\left(\frac{5}{18}\lambda^2+3e^4\right)\left(\ln\frac{\phi_{cl}^2}{M^2}-\frac{25}{6}\right)\right]. \quad (3.88)$$

We now investigate the minimum of potential. Let us firstly set $e=0$ at which the potential reduces to that of the ϕ^4 toy model as

$$V_{\text{eff}}[\phi_{cl}] = \phi_{cl}^4\left[\frac{\lambda}{4!}+\frac{5}{1152\pi^2}\lambda^2\left(\ln\frac{\phi_{cl}^2}{M^2}-\frac{25}{6}\right)\right]. \quad (3.89)$$

The shape of the potential will be altered if one-loop contributions can compete with the tree-level one. The minimum of potential can occur at nonzero value of the field determined by

$$V'_{\text{eff}}[\langle\phi_{cl}\rangle] = \langle\phi_{cl}\rangle^3\left[\frac{\lambda}{3!}+\frac{5}{288\pi^2}\lambda^2\left(\ln\frac{\langle\phi_{cl}\rangle^2}{M^2}-\frac{11}{3}\right)\right]=0. \quad (3.90)$$

This requires that either $|\lambda|$ or $|\lambda\ln\frac{\phi_{cl}^2}{M^2}|$ is large, and the perturbative method is expected to be invalid. In the presence of the couping e, the minimum of Eq. (3.88) can be simplified if we choose the renormalization scale at the minimum point $M=\langle\phi_{cl}\rangle$. At the minimum, we have

$$V'_{\text{eff}}[\langle\phi_{cl}\rangle] = \langle\phi_{cl}\rangle^3\left[\frac{\lambda}{3!}-\frac{11}{48\pi^2}\left(\frac{5}{18}\lambda^2+3e^4\right)\right]=0. \quad (3.91)$$

Since perturbative calculation demands that λ is small, this allows us to omit λ^2 term. The non-trivial minimum is achieved if the following relation is satisfied

$$\lambda = \frac{33}{8\pi^2} e^4. \tag{3.92}$$

The effecitve potential finally becomes

$$V_{\text{eff}}[\phi_{cl}] \quad = \quad \frac{3e^4}{64\pi^2} \phi_{cl}^4 \left(\ln \frac{\phi_{cl}^2}{\langle \phi_{cl} \rangle^2} - \frac{1}{2} \right). \tag{3.93}$$

In conclusion, we have shown that the essential ingredient for the radiative symmetry breaking to occur is the presence of hierarchy between couplings such that the one-loop correction can balance the tree-level potential without producing large logarithm. For ϕ^4 theory, it fails to meet this requirement as we only have one coupling at our disposal, λ.

3.2.3 The Classically Scale Invariant Standard Model

In this subsection, we briefly show that the dynamical symmetry breaking cannot be successful in the classically scale invariance SM. The reason is due to the negative contribution from heavy top quark mass. The effective potential for scale invariance SM is given by

$$V_{\text{eff}} \quad = \quad \frac{\lambda}{4} h^4 + B h^4 \left(\ln \left(\frac{h^2}{v_h^2} \right) - \frac{25}{6} \right) \tag{3.94}$$

where the renormalization scale $M = v_h$ and the coefficient B for SM is given by

$$B = \frac{3}{64\pi^2} \left(2 \frac{m_W^4}{v_h^4} + \frac{m_Z^4}{v_h^4} - 4 \frac{m_t^4}{v_h^2} \right), \tag{3.95}$$

where other contributions have been omitted including the Higgs contribution. Performing minimization of this potential, we get

$$\left. \frac{\partial V_{\text{eff}}}{\partial h} \right|_{h=v_h} = (\lambda - \frac{44}{3} B) v_h^3 = 0. \tag{3.96}$$

For a non-trvial minimum, the solution is

$$\lambda = \frac{44}{3}B. \tag{3.97}$$

It can be seen that B is negative due to negative contribution from top quark. This results in an unstable potential, i.e. $\lambda < 0$. Hence, the CW symmetry breaking mechanism can not be directly applied to the SM potential. However, this mechanism is utilized in various models beyond SM in which contributions from additional bosonic degrees of freedom dominate over the top quark one.

3.2.4 A Short Comment about Scale Invariant Type II Seesaw Model

In this subsection, we give a short comment on promoting the type-II seesaw being a scale-invariant theory. There are three dimensionful parameters in the scalar potential of the type II seesaw: μ_H^2, M_Δ, and μ. The first two terms are dimension 2 as they are the normal mass terms for scalar doublet and triplet, respectively. The last term having dimension one is the source of lepton number violating term. Explicitly including this term in the scalar potential will avoid us from having Goldstone particle in the spectra as easily seen from the mass of the CP-odd scalar Eq. (3.35) which is directly proportional to μ. Thus omitting this μ term from the scalar potential will result in the spontaneously broken global lepton number after the scalar triplet develops non-zero VEV, and by Goldstone theorem there will be Goldstone boson associated with this breaking. Because the majoron here has $SU(2)_L$ and $U(1)_Y$ gauge interactions, this will conflict with the invisible decay of Z boson. Therefore, the scale-invariant of type II seesaw has been already ruled out experimentally (Gonzalez-Garcia and Nir, 1989).

3.3 Gildener-Weinberg Scheme

As we have demonstrated in the previous section, in order to realize viable scale invariant scalar potential, one has to supply a bosonic field to balance the top quark contribution to the effective potential. To introduce additional bosonic degrees of freedom to the theory, it is typically simpler to extend the scalar sector than the gauge sector. Nevertheless, analyzing the minimum of effective potential is a non-trivial task to do even in the single scalar setup. The presence of several scalar fields in the model considerably makes it more complicated. Fortunately, a framework to deal with multiple scalar potential analytically was introduced by Gildener and Weinberg (Gildener and Weinberg, 1976). The central idea is the appearance of the so-called flat direction of field space. Along this ray, the tree-level potential has zero value and perturbative analysis can be performed.* In the generic direction of field space with the scalar coupling satisfying the power counting relation $\lambda \sim g^2 << 1$, the tree-level potential dominates over the loop corrections. If the one-loop corrections of order $g^4\phi^4 \ln\frac{\phi^2}{\mu^2}$ could shift the minimum point, the logarithmic term must be of order $1/g^2$. This large logarithm would invalidate perturbative calculations. However, before doing perturbation the renormalization scale μ_{GW} can be chosen in such a way that the tree-level potential has a minimum equal to zero along a specific direction. On this ray, even small loop corrections can be significant. They could change the shape of potential by developing a small curvature in the radial direction, and hence single out $\langle\phi\rangle$ of ϕ. In this sense, CW symmetry breaking occurs in the flat direction.

Let us consider the most general renormalizable scale invariant potential

*The loop correction can significantly change the tree-level potential if the tree-level potential vanishes. In the multi-dimensional field space, it is sufficient if the tree-level potential vanishes along a certain direction at a particular renormalization scale.

of weakly coupled scalar fields ϕ_i $(i = 1, 2, .., n)$

$$V_0(\vec{\Phi}) \;=\; \frac{1}{4!}\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \tag{3.98}$$

with λ_{ijkl} being totally symmetric quartic couplings. The field can be parameterized as $\phi_i = \varphi N_i$, where φ is the radial coordinate and N_i has unit norm. At some particular scale $\mu = \mu_{GW}$ and in the flat direction defined by $N_i = n_i$, the tree-level potential has degenerate non-trivial minima and satisfy

$$\min_{n_i n_i = 1}\left(\lambda_{ijkl}(\mu_{GW})n_i n_j n_k n_l\right) \;=\; 0. \tag{3.99}$$

The condition of the flat direction being a stationary line is given by

$$\frac{1}{\varphi^4}\frac{\partial V_0}{\partial N_i}\bigg|_{N_i=n_i} \;=\; \lambda_{ijkl}(\mu_{GW})n_j n_k n_l = 0. \tag{3.100}$$

Lastly, the Hessian matrix given by

$$P_{ij} \;=\; \frac{1}{\varphi^4}\frac{\partial^2 V_0}{\partial N_i \partial N_j}\bigg|_{N_i=n_i} = \frac{1}{2}\lambda_{ijkl}(\mu_{GW})n_k n_l \tag{3.101}$$

must be positive-semidefinite in order to guarantee that the flat direction is a local minimum. This implies that the eigenvalues of the Hessian matrix P will be either zero or positive.

At this stage, we only consider the classical potential. Unlike spontaneously broken compact continuous symmetry which will pick up a single VEV (or perhaps a discrete sets of VEVs), spontaneously breaking scale invariance which is a non-compact continuous symmetry leads to the non-trivial vacua on a ray in the field space. In other words, the non-trivial vacua constitute a continuum value of VEV along the flat direction. The value of the classical potential has a flat minimum in this direction.

Turning on the one-loop correction will reshape the tree-level potential and single out the unique VEV $\langle\varphi\rangle$ along the radial direction. A small deviation of

ϕ_i from the direction n_i by an amount $\delta\phi_i$ will also be induced. The stationary equation becomes

$$\frac{\partial}{\partial\phi_i}\left(V_0(\vec{\Phi}) + \delta V(\vec{\Phi})\right)\Big|_{\vec{n}\langle\varphi\rangle+\delta\vec{\phi}} = 0. \tag{3.102}$$

Expanding up to first order, we have

$$\begin{aligned}\frac{\partial V_0(\vec{\Phi})}{\partial\phi_i}\Big|_{\vec{n}\langle\varphi\rangle} + \delta\phi_j\frac{\partial^2 V_0(\vec{\Phi})}{\partial\phi_i\partial\phi_j}\Big|_{\vec{n}\langle\varphi\rangle} + \frac{\partial\delta V(\vec{\Phi})}{\partial\phi_i}\Big|_{\vec{n}\langle\varphi\rangle} &= 0,\\ \frac{1}{\langle\varphi\rangle}\frac{\partial V_0(\vec{\Phi})}{\partial N_i}\Big|_{\vec{n}\langle\varphi\rangle} + \frac{1}{\langle\varphi\rangle^2}\delta\phi_j\frac{\partial^2 V_0(\vec{\Phi})}{\partial N_i\partial N_j}\Big|_{\vec{n}\langle\varphi\rangle} + \frac{\partial\delta V(\vec{\Phi})}{\partial\phi_i}\Big|_{\vec{n}\langle\varphi\rangle} &= 0. \end{aligned} \tag{3.103}$$

The first term is zero by Eq. (3.100). Contracting with n_i and using Eq. (3.100) and (3.101), we arrive at

$$n_i\frac{\partial\delta V(\vec{\Phi})}{\partial\phi_i}\Big|_{\vec{n}\langle\varphi\rangle} = \frac{\partial\delta V(\vec{\Phi})}{\partial\varphi}\Big|_{\langle\varphi\rangle} = 0. \tag{3.104}$$

Along the flat direction the one-loop effective potential can be recast as

$$\delta V_{\text{1-loop}}(\varphi) = A(\vec{n})\varphi^4 + B(\vec{n})\varphi^4\log\frac{\varphi^2}{\mu_{GW}^2}. \tag{3.105}$$

In $\overline{MS}$ scheme, the dimensionless parameters $A(\vec{n})$ and $B(\vec{n})$ read

$$\begin{aligned} A(\vec{n}) &= \frac{1}{64\pi^2\langle\varphi\rangle^4}\left[6M_W^4\left(\log\frac{M_W^2}{\langle\varphi\rangle^2}-\frac{5}{6}\right)+3M_Z^4\left(\log\frac{M_Z^2}{\langle\varphi\rangle^2}-\frac{5}{6}\right)\right.\\ &\quad\left.+\sum_i n_i M_{H_i}^4\left(\log\frac{M_{H_i}^2}{\langle\varphi\rangle^2}-\frac{3}{2}\right)-12M_t^4\left(\log\frac{M_t^2}{\langle\varphi\rangle^2}-\frac{3}{2}\right)\right],\\ B(\vec{n}) &= \frac{1}{64\pi^2\langle\varphi\rangle^4}\left[6M_W^4+3M_Z^4+\sum_i n_i M_{H_i}^4-12M_t^4\right], \end{aligned} \tag{3.106}$$

where the sum runs over the number of scalar mass eigenstates with $n_i = 2$ for charged scalar and $n_i = 1$ for neutral scalar. Computing Eq. (3.104), we obtain

$$\left(2A(\vec{n}) + 2B(\vec{n})\log\frac{\langle\varphi\rangle^2}{\mu_{GW}^2} + B(\vec{n})\right)2\langle\varphi\rangle^3 = 0. \tag{3.107}$$

This leads to a non-trivial solution satisfying the relation

$$\log\frac{\langle\varphi\rangle^2}{\mu_{GW}^2} = -\frac{1}{2} - \frac{A(\vec{n})}{B(\vec{n})}. \tag{3.108}$$

We can eliminate μ_{GW} in Eq. (3.105) in favor of $\langle\varphi\rangle$ by using Eq. (3.108),

$$\delta V_{\text{1-loop}}(\varphi) \;=\; B(\vec{n})\varphi^4\left(\log\frac{\varphi^2}{\langle\varphi\rangle^2}-\frac{1}{2}\right). \tag{3.109}$$

The tree-level mass spectra of scalar bosons can be obtained from the eigenvalues of the Hessian matrix

$$(M_0^2)_{ij} \;=\; \frac{\partial^2 V_0}{\partial\phi_i\partial\phi_j}\bigg|_{\vec{n}\langle\varphi\rangle} = \frac{1}{2}\lambda_{ijkl}(\mu_{GW})n_k n_l\langle\varphi\rangle^2 = P_{ij}\langle\varphi\rangle^2. \tag{3.110}$$

As mentioned above, the Hessian matrix P can have zero eigenvalue corresponding to massless scalon due to the spontaneous breaking of scale invariance. The corresponding eigenvector is $\vec{n}$ satisfying the relation

$$P_{ij}n_j \;=\; 0, \tag{3.111}$$

where we have used Eq. (3.100). If the theory also possess a continuous symmetry under which the infinitesimal change in the field reads

$$\delta\phi_i \;=\; \epsilon\Theta_{ij}n_j \tag{3.112}$$

and some generators do not annihilate the ground state (the broken generators), i.e., $\Theta n \neq 0$, then by Goldstone theorem we have

$$P_{ij}(\Theta n)_j \;=\; 0. \tag{3.113}$$

Finally the massive scalar bosons associated with the direction perpendicular to the flat direction are given by

$$M_0^2 \;=\; n_\perp^i\frac{\partial^2 V_0}{\partial\phi_i\partial\phi_j}\bigg|_{\vec{\phi}}n_\perp^j = \vec{n}_\perp\cdot P\cdot\vec{n}_\perp\langle\varphi\rangle^2 \tag{3.114}$$

where $\vec{n}\cdot\vec{n}_\perp = 0$. Therefore, in the lowest order the Hessian matrix contains a set of massless Goldstone bosons associated with the broken continuous symmetry, one massless scalar (scalon) associated with the broken scale invariance, and a set of massive scalar Higgs bosons.

Upon including the 1-loop corrections, the mass matrix will shift to

$$
\begin{aligned}
(M_0^2 + \delta M^2)_{ij} &= \frac{\partial^2 \left(V_0(\vec{\phi}) + \delta V(\vec{\phi})\right)}{\partial\phi_i \partial\phi_j}\Big|_{\langle\vec{\phi}\rangle + \delta\vec{\phi}} \\
&= \frac{\partial^2 V_0(\vec{\phi})}{\partial\phi_i \partial\phi_j}\Big|_{\langle\vec{\phi}\rangle} + \frac{\partial^2 \delta V(\vec{\phi})}{\partial\phi_i \partial\phi_j}\Big|_{\langle\vec{\phi}\rangle} \\
&\quad + \langle\varphi\rangle \lambda_{ijkl}(\mu_{GW}) n_k \delta\phi_l \qquad (3.115)
\end{aligned}
$$

where we have kept up to first order in small VEV correction. Thus we identify the loop corrections to mass matrix as

$$
(\delta M^2)_{ij} = \frac{\partial^2 \delta V(\vec{\phi})}{\partial\phi_i \partial\phi_j}\Big|_{\langle\vec{\phi}\rangle} + \langle\varphi\rangle \lambda_{ijkl}(\mu_{GW}) n_k \delta\phi_l. \qquad (3.116)
$$

We will prove that eigenvalues of the mass matrix in Eq. (3.115) are positive-definite with an exception for the Goldstone direction Θn. Here we have assumed that the eigenvectors Θn of M_0^2 are also eigenvectors of $(M_0^2 + \delta M^2)$ with zero eigenvalue, provided that δV also shares the same continuous symmetry as V_0.

Contracting Eq. (3.115) with $n_i n_j$, we find that the scalon becomes massive and its mass is given by

$$
\begin{aligned}
M_s^2 &= n_i n_j (\delta M^2)_{ij} \\
&= n_i n_j \frac{\partial^2 \delta V(\vec{\phi})}{\partial\phi_i \partial\phi_j}\Big|_{\langle\vec{\phi}\rangle} + \langle\varphi\rangle \lambda_{ijkl}(\mu_{GW}) n_i n_j n_k \delta\phi_l \\
&= \frac{d^2 \delta V(\vec{\phi})}{d\varphi^2}\Big|_{\langle\varphi\rangle} = 8B(\vec{n}) \langle\varphi\rangle^2 \qquad (3.117)
\end{aligned}
$$

where Eq. (3.100) has been used. In order for the vacuum to be stable, we then require that $B > 0$. Under this condition, the mass eigenvalues of the mass matrix in Eq. (3.115) is still positive-definite upon the inclusion of radiative correction. This, in turn, makes the potential at the stationary point to be the local minimum, i.e.

$$
V(\vec{n}\varphi) = -\frac{1}{2} B(\vec{n}) \langle\varphi\rangle^4 < 0. \qquad (3.118)
$$

Indeed, the tree-level mass eigenvalues get corrected by δM. As long as this correction is small compared to the tree-level one, we may use the tree-level mass matrix as a good approximation. We find that the mass of scalon reads

$$M_s^2 = \frac{1}{8\pi^2 \langle\varphi\rangle^2}\left[6M_W^4 + 3M_Z^4 + \sum_i n_i M_{H_i}^4 - 12M_t^4\right]. \qquad (3.119)$$

CHAPTER IV

SCALE INVARIANCE OF SINGLET EXTENSION OF TYPE II SEESAW MODEL

An argument by Bardeen (Bardeen, 1995) has always been an inspiring guidance in constructing model with classical scale invariance which aims to decipher the hierarchy problem in the last several years. It is stated that in the absence of dimensionful parameter in theory and the breaking is only allowed by quantum correction as restated in Ref. (Iso et al., 2009), there will be no quadratic divergence. Moreover, as the cutoff regularization procedure maximally breaks scale symmetry, dimensional regularization must be applied which does the least breaking, i.e. the renormalization scale introduced in dimensional regularization only appears in logarithmic fashion. Besides these two conditions (Meissner and Nicolai, 2008), no intermediate scale between the electroweak scale and the Planck scale as well as the absence of instability or Landau pole before the Planck scale must be fulfilled. A plethora of proposals had been appeared putting forward a scale invariance as a possible solution to the hierarchy problem such as (Meissner and Nicolai, 2008) and several of them apply Gildener-Weinberg framework in studying the radiative breaking of potential (Foot et al., 2007; Alexander-Nunneley and Pilaftsis, 2010; Farzinnia et al., 2013; Ghorbani, 2018; Karam and Tamvakis, 2015). Nevertheless, other frameworks to address the hierarchy problem also exist in the literature such as supersymmetry which impose a beautiful symmetry between fermion and scalar particles such that a nice cancellations occur at the loop level yielding no quadratic sensitivity to scalar mass and little Higgs model featuring a shift symmetry.

As we have seen in the previous chapter, promoting the type II seesaw model to be scale invariant is not favored phenomenologically due to the appearance of massless majoron which has full $SU(2)_L \times U_Y(1)$ gauge interaction. It is thus inevitable to add one more scalar field which is singlet under SM gauge group. In this case, the majoron is still in the particle spectrum but it is mainly singlet. Such model extension had been initially proposed in Ref (Schechter and Valle, 1982) and its collider phenomenology was done in (Díaz et al., 1998; Bonilla et al., 2016).

In this chapter, we aim to study the possibility of scale invariance of singlet extension type II seesaw being as viable model. We derive a necessary and sufficient conditions for the scalar potential to be bounded from below (BFB). In order to accomplish this job, firstly we derive the orbit space of this model using both conventional method and P-matrix approach. Next, we employ Gildener-Weinberg scheme and obtain the flat direction. We then collect the points that satisfy BFB condition as well as the points that reproduce SM-like Higgs boson, $m_h = 125.09$ GeV, as one of the eigenvalues of the CP-even scalar mass matrix. Finally, we radiatively break the tree-level potential by taking into account the role of one-loop effective potential along the flat direction thereby picking up the unique radial VEV.

4.1 Scalar potential

The most general renormalizable scale invariant scalar potential for the extended type II seesaw with singlet scalar takes the form of

$$\begin{aligned}V &= \lambda_H(H^\dagger H)^2 + \lambda_S(S^\dagger S)^2 + \lambda_\Delta \operatorname{Tr}\left(\Delta^\dagger\Delta\right)^2 + \lambda'_\Delta \operatorname{Tr}\left(\Delta^\dagger\Delta\Delta^\dagger\Delta\right) \\ &+ \lambda_{H\Delta} H^\dagger H \operatorname{Tr}\left(\Delta^\dagger\Delta\right) + \lambda'_{H\Delta} H^\dagger\Delta\Delta^\dagger H + \lambda_{HS} H^\dagger H S^\dagger S + \lambda_{S\Delta} S^\dagger S \operatorname{Tr}\left(\Delta^\dagger\Delta\right) \\ &+ \frac{1}{2}(\lambda_{SH\Delta} S H^T \varepsilon \Delta^\dagger H + \text{h.c.}).\end{aligned} \tag{4.1}$$

The last term is essential due to the appearance of majoron, Goldstone boson associated to spontaneously breaking global lepton number, which otherwise will have electroweak gauge interaction in the type II seesaw model. With the assignment lepton number L to fields S and H by $+2$ and 0, respectively, the last term will still be invariant under global lepton number symmetry. After Δ and S develop VEV, the lepton number symmetry will be spontaneously broken resulting in the emergence of massless Goldstone boson, majoron.

4.1.1 Boundary of the Orbit Space

The requirement for vacuum stability of the scalar potential is one of the constraints that will impose the acceptable value of quartic couplings. As the fields go to infinity, the potential must be bounded from below (BFB) in any generic direction in the field space. It is non-trivial job to find the set of BFB conditions in the presence of multiple scalar fields in the potential. Fortunately, for the class of homogenous potential in the fields the bounded from below depends only on the ratios of the norms for multiple fields. The useful tool that one usually utilizes in deriving BFB condition is so-called orbit parameter (El Kaffas et al., 2007; Arhrib et al., 2011; Bonilla et al., 2015; Kannike, 2016). For short intoduction about orbit parameter, we will follow closely Ref. (Kim, 1982; Kim, 1984).

The orbit of ϕ_a is defined to be the set of states ϕ^a that are reached by symmetry transformation given by $\phi^a = T(\theta)\phi_a$ with $T(\theta)$ an element of symmetry group G. The subgroup of G that leaves state ϕ_a invariant, $\phi_a = T(\xi_a)\phi_a$ for $T(\xi_a) \in G'_a \subset G$, is called a little group of ϕ_a. Suppose that G'_b is a little group of ϕ_b. If ϕ_b and ϕ_a are on the same orbit, then G'_a and G'_a are conjugate of each other. This can be proved as follows. Because ϕ_b and ϕ_a are on the same orbit then they are related by $\phi_b = T(\theta_b)\phi_a$. Let $T(\xi_b) \in G'_b$ then $\phi_b = T(\theta_b)T(\xi_a)T(\theta_b)^{-1}T(\theta_b)\phi_a$. This is $\phi_b = T(\theta_b)T(\xi_a)T(\theta_b)^{-1}\phi_b$ meaning that $T(\theta_b)T(\xi_a)T(\theta_b)^{-1} \in G'_b$ for $\forall T(\theta_b) \in G$. The set of all orbits that are invariant under the same little group up to conjugation form a stratum of little group. For unitary group, all the state ϕ^a will have the same norm $\phi^*_a\phi_a$.

Higgs scalar potential is a gauge invariant function. It takes a constant value on an orbit. Furthermore, it contains invariant polynomials which we call basic invariant polynomials. Thus, given an orbit, each basic invariant polynomial will be constant on that orbit meaning that Higgs potential is actually a function of orbits. For any pair of distinct orbits, there exist at least one basic invariant polynomial that takes different value on each orbit. A number of basic invariant polynomials, q, depends on the representation. In this q-dimensional space of basic invariant polynomial, an orbit is just a point. In conclusion, the basic invariant polynomials separate the orbits.

In deriving bounded from below conditions, the following dimensionless ratio of invariant quantities is a useful tool

$$\alpha = \frac{f_{ijkl}\phi^*_i\phi_j\phi^*_k\phi_l}{(\phi^*_i\phi_i)^2}, \tag{4.2}$$

where f_{ijkl} denotes gauge contraction. This quantity is known as orbit parameter. Given the definition above, it certainly shows us that orbit parameter will be bounded from below and above, $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$. The physical region of orbit parameters is called orbit space. Once we re-express our potential in terms of these orbit parameters, we can derive the bounded from below conditions. In conventional method, the orbit space is derived by taking particular fields to zero, and the set of equations that give us the boundary of the orbit space is obtained. Inside the orbit space, the potential must be positive. This procedure is more obvious when the orbit parameter space is three dimensions or lower.

Another powerful method in deriving the boundary of orbit space is through the so called P-matrix approach (Talamini, 2006; Talamini, 2006). The P-matrix is an $q \times q$ symmetric and positive semi-definite matrix defined by

$$P_{ij} = \frac{\partial p_i}{\partial \phi_a^\dagger} \frac{\partial p_j}{\partial \phi_a}, \tag{4.3}$$

where p_i are gauge invariant polynomials running from p_1 to p_q, and ϕ_a runs over the fields component and their complex conjugates. The boundaries of orbit space are obtained by solving $\det P = 0$. The P-matrix approach is much more efficient when the orbit parameter space is beyond three dimensions.

Below, we are going to demonstrate these two methods to obtain the boundaries of orbit spaces for our potential.

Conventional Approach

We define the orbit parameters $r, k, \zeta, \xi, \eta, \alpha$ as the followings

$$H^\dagger H \equiv r \ \mathrm{Tr}\left(\Delta^\dagger \Delta\right), \tag{4.4}$$

$$S^\dagger S \equiv k \ \mathrm{Tr}\left(\Delta^\dagger \Delta\right), \tag{4.5}$$

$$\mathrm{Tr}\left(\Delta^\dagger \Delta\right)^2 \equiv \zeta \ (\mathrm{Tr}\, \Delta^\dagger \Delta)^2, \tag{4.6}$$

$$H^\dagger \Delta \Delta^\dagger H \equiv \xi \ (H^\dagger H) \, \mathrm{Tr}\left(\Delta^\dagger \Delta\right), \tag{4.7}$$

$$S H^T \epsilon \Delta^\dagger H \equiv \eta e^{i\alpha} H^\dagger H \sqrt{S^\dagger S} \sqrt{\mathrm{Tr}(\Delta^\dagger \Delta)}. \tag{4.8}$$

The range of these orbit parameters are given in the following interval

$$0 \leq r, \tag{4.9}$$

$$0 \leq k, \tag{4.10}$$

$$1/2 \leq \zeta \leq 1, \tag{4.11}$$

$$0 \leq \xi \leq 1, \tag{4.12}$$

$$0 \leq \alpha \leq 2\pi, \tag{4.13}$$

$$0 \leq \eta \leq 1. \tag{4.14}$$

In term of orbit parameters, the potential reads

$$\begin{aligned} \frac{V(S,H,\Delta)}{(\mathrm{Tr}\,\Delta^\dagger\Delta)^2} &= \lambda_H r^2 + \lambda_{H\Delta} r + \lambda_\Delta + \lambda'_\Delta \zeta + \lambda'_{H\Delta}\xi r + \lambda_S k^2 + \lambda_{HS} r k \\ &\quad + \lambda_{S\Delta} k + \lambda_{SH\Delta}\eta r \sqrt{k}\cos\alpha. \end{aligned} \tag{4.15}$$

From Eqs.(4.6) and (4.7), ξ and ζ are not independent. They are related as

$$2\xi^2 - 2\xi + 1 = \zeta + (1-2\zeta)\gamma, \tag{4.16}$$

$$0 \leq \gamma \leq 0.5. \tag{4.17}$$

The well-known shape of $\zeta - \xi$ allowed region in type II seesaw model is shown in Fig. 4.1. For a given value of γ, Eq.(4.16) is a parabola inside the allowed region. When γ parameter varies, the curve spans over the shaded region. Extending the scalar content by one complex singlet scalar introduces additional orbit parameters, and the allowed orbit space will form a non-trivial shape.

In order to gain an understanding of the shape of the $\xi - \eta$ and the $\eta - \zeta$ projected planes. We expand Eqs. (4.6)-(4.8) in terms of the field components as

$$\xi = S_\Delta^2 C_h^2 + C_\Delta^2 S_h^2 + 2\sqrt{2} Re\left[\left(\delta^+\delta^{0*} - \delta^{++}\delta^+\right) h^0 h^{+*}\right], \tag{4.18}$$

$$\begin{aligned} \eta^2 &= S_\Delta^2 C_h^4 + C_\Delta^2 S_h^4 - \frac{|\delta^+|^2}{2}\left(C_h^4 + S_h^4 - 4C_h^2 S_h^2\right) - 2Re(\delta^{++*}\delta^0 h^{0*2} h^{+2}) \\ &\quad + 2\sqrt{2}\left[S_h^2\, Re(\delta^{+*}\delta^0 h^{0*} h^+) - C_h^2\, Re(\delta^{++*}\delta^+ h^{0*} h^+)\right], \end{aligned} \tag{4.19}$$

$$\zeta = S_\Delta^4 + C_\Delta^4 + |\delta^+|^2\left(1 - |\delta^+|^2\right) - 2Re(\delta^{+2}\delta^{0*}\delta^{++*}), \tag{4.20}$$

with

$$S_\Delta^2 = \frac{|\delta^+|^2}{2} + |\delta^{++}|^2, \tag{4.21}$$

$$C_\Delta^2 = \frac{|\delta^+|^2}{2} + |\delta^0|^2, \tag{4.22}$$

$$S_h^2 = |h^0|^2, \tag{4.23}$$

$$C_h^2 = |h^+|^2. \tag{4.24}$$

Note that the fields appearing on these equations are the normalized fields (i.e., $S_\Delta^2 + C_\Delta^2 = 1$ and $S_h^2 + C_h^2 = 1$). It can be seen that no singlet field appears on these expressions. The boundary solutions can be obtained by picking up specific field directions in which some fields take zero value.

Firstly, consider the direction where $\delta^+ = 0$. In this limit, Eqs. (4.18)-(4.20) are simplified as

$$\lim_{\delta^+\to 0} \xi = S_\Delta^2 C_h^2 + C_\Delta^2 S_h^2, \tag{4.25}$$

$$\lim_{\delta^+\to 0} \eta^2 = S_\Delta^2 C_h^4 + C_\Delta^2 S_h^4 - 2Re(\delta^{++*}\delta^0 h^{0*2} h^{+2}), \tag{4.26}$$

$$\lim_{\delta^+\to 0} \zeta = S_\Delta^4 + C_\Delta^4. \tag{4.27}$$

From Eqs. (4.25) and (4.27), we obtain Eq. (4.16). The $\xi - \eta$ relation is given by

$$\lim_{\delta^+\to 0} \eta = \sqrt{\xi - S_h^2 C_h^2 - 2Re(\delta^{++*}\delta^0 h^{0*2} h^{+2})}. \tag{4.28}$$

It is instructive to set $h^+ = 0$. In this limit $C_h^2 = 0$ and $S_h^2 = 1$. We get

$$\lim_{\delta^+, h^+\to 0} \eta = \sqrt{\xi}, \tag{4.29}$$

$$\lim_{\delta^+, h^+\to 0} \zeta = 2\eta^4 - 2\eta^2 + 1. \tag{4.30}$$

Eq. (4.30) corresponds to Eq. (4.16) with $\gamma = 0$. We denote the vector of three orbit space parameter as $\vec{\rho} = (\xi, \zeta, \eta)$. The first boundary solution is then expressed as

$$\vec{\rho}_I = (\xi, 2\xi^2 - 2\xi + 1, \sqrt{\xi}), \qquad 0 \leq \xi \leq 1. \tag{4.31}$$

From Eqs. (4.25) -(4.27) if setting $\delta^{++} = 0$ so that $S_\Delta^2 = 0$ and $C_\Delta^2 = 1$, we get

$$\lim_{\delta^+, \delta^{++} \to 0} \eta = \xi, \tag{4.32}$$

$$\lim_{\delta^+, \delta^{++} \to 0} \zeta = 1. \tag{4.33}$$

This corresponds to the second boundary solution

$$\vec{\rho}_{II} = (\xi, 1, \xi), \qquad 0 \leq \xi \leq 1. \tag{4.34}$$

Next, we explore another direction where $\delta^0 = \delta^{++} = 0$. In this limit $S_\Delta^2 = C_\Delta^2 = 1/2$, and we have

$$\lim_{\delta^0, \delta^{++} \to 0} \xi = \frac{1}{2}, \tag{4.35}$$

$$\lim_{\delta^0, \delta^{++} \to 0} \eta = \sqrt{2 S_h^2 C_h^2}, \tag{4.36}$$

$$\lim_{\delta^0, \delta^{++} \to 0} \zeta = \frac{1}{2}. \tag{4.37}$$

This corresponds to the solution

$$\vec{\rho}_{III} = (1/2, 1/2, \eta), \qquad 0 \leq \eta \leq 1/\sqrt{2}. \tag{4.38}$$

Lastly, the fourth solution is obtained when we take $h^0 = \delta^{++} = 0$. In this limit $S_h^2 = 0$ and $C_h^2 = 1$. We get

$$\lim_{h^0, \delta^{++} \to 0} \xi = S_\Delta^2, \tag{4.39}$$

$$\lim_{h^0, \delta^{++} \to 0} \eta^2 = S_\Delta^2 - \frac{|\delta^+|^2}{2} = 0, \tag{4.40}$$

$$\lim_{h^0, \delta^{++} \to 0} \zeta = S_\Delta^4 + C_\Delta^4 + |\delta^+|^2 \left(1 - |\delta^+|^2\right). \tag{4.41}$$

Eq. (4.41) can be rewritten in terms of ξ as

$$\lim_{h^0, \delta^{++} \to 0} \zeta = 1 - 2\xi^2, \tag{4.42}$$

so that the fourth solution is

$$\vec{\rho}_{IV} = (\xi, 1-2\xi^2, 0), \qquad 0 \leq \xi \leq 1/2. \tag{4.43}$$

P-matrix Approach

We define gauge invariant polynomials p_1 to p_6 as the following

$$p_1 = S^\dagger S \equiv s^2, \tag{4.44}$$
$$p_2 = H^\dagger H \equiv h^2, \tag{4.45}$$
$$p_3 = \mathrm{tr}\Big(\Delta^\dagger \Delta\Big) \equiv \delta^2, \tag{4.46}$$
$$p_4 = H^\dagger \Delta \Delta^\dagger H \equiv \xi h^2 \delta^2, \tag{4.47}$$
$$p_5 = \mathrm{tr}\Big(\Delta^\dagger \Delta \Delta^\dagger \Delta\Big) \equiv \zeta \delta^4, \tag{4.48}$$
$$p_{6R} + ip_{6I} = SH^T \epsilon \Delta^\dagger H \equiv \eta e^{i\alpha} s \delta h^2. \tag{4.49}$$

Here, δ, h and s are the norms of the fields, and the orbit space variables ξ, ζ, η and α are the same as ones in Eqs. (4.6)-(4.8). In terms of these variables, the potential is given by

$$\begin{aligned} V &= \lambda_H h^4 + \lambda_S s^4 + (\lambda_\Delta + \lambda'_\Delta \zeta)\delta^4 + (\lambda_{H\Delta} + \lambda'_{H\Delta}\xi)h^2\delta^2 + \lambda_{HS} h^2 s^2 + \lambda_{S\Delta} s^2 \delta^2 \\ &\quad + |\lambda_{SH\Delta}| \eta s \delta h^2 \cos(\alpha + \phi_{\lambda_{SH\Delta}}), \end{aligned} \tag{4.50}$$

where we parameterize $\lambda_{SH\Delta} = |\lambda_{SH\Delta}| e^{i\phi_{\lambda_{SH\Delta}}}$. Since the minimum of the last term

$$\min |\lambda_{SH\Delta}| \eta s \delta h^2 \cos(\alpha + \phi_{\lambda_{SH\Delta}}) = -|\lambda_{SH\Delta}| \eta s \delta h^2, \tag{4.51}$$

therefore we can consider the absolute value of p_6

$$|p_6|^2 = p_{6R}^2 + p_{6I}^2 = \left| SH^T \varepsilon \Delta^\dagger H \right|^2 = \eta^2 s^2 \delta^2 h^4 \tag{4.52}$$

instead of separate p_{6R} and p_{6I}. We calculate the elements of the P-matrix defined in Eq. (4.3) where p_i are given by p_1 to p_5 and $|p_6|^2$. In general, the P-matrix elements are gauge invariant quantities, and can be expressed in terms of the gauge invariant polynomials. For the present model, unfortunately, that would necessitate introducing higher order invariants which would complicate things considerably. However, we can find an equation for the boundary of the orbit space directly in terms of field components.

We have already seen in Chapter 3 that the $SU(2)$ triplet can be expressed as a complex traceless matrix of the form

$$\Delta = \frac{\vec{\sigma}}{\sqrt{2}} \cdot \vec{\Delta}. \tag{4.53}$$

We can use an $SU(2)$ gauge rotation to get rid of three real components of the triplet. We parameterize the remaining components as

$$\delta_1 = x, \quad \delta_2 = iy, \quad \delta_3 = z, \tag{4.54}$$

so that the norm is given by

$$\delta^2 = x^2 + y^2 + z^2. \tag{4.55}$$

It is easy to show that the orbit space parameters can in principle only depend on the difference of the phases of the components h_1 and h_2 of the Higgs doublet. Real solutions for real components of the fields, however, are only obtained when the phase difference is zero. For that reason, we take h_1 and h_2 to be real on the orbit space boundary without loss of generality.

The equation $\det P = 0$ for the boundary of the orbit space is then given by

$$\begin{aligned} y(x^2 - y^2 + z^2)(4x^2 + 4z^2 + h_1^2 + h_2^2)(2xh_1h_2 + z(h_1^2 - h_2^2)) \\ \times((x + y)h_1^2 - 2zh_1h_2 + (y - x)h_2^2) = 0. \end{aligned} \tag{4.56}$$

The boundary equation (4.56) has total 10 solutions. Two of them are imaginary and thus spurious, and some give the same result; in the end, four distinct edges are given in the parametric form as

$$\vec{\rho}_{\mathrm{I}} = (\xi, 1 - 2\xi + 2\xi^2, \sqrt{\xi}), \qquad 0 \le \xi \le 1, \tag{4.57}$$

$$\vec{\rho}_{\mathrm{II}} = (\xi, 1, \xi), \qquad 0 \le \xi \le 1, \tag{4.58}$$

$$\vec{\rho}_{\mathrm{III}} = (\tfrac{1}{2}, \tfrac{1}{2}, \eta), \qquad 0 \le \eta \le 1/\sqrt{2}, \tag{4.59}$$

$$\vec{\rho}_{\mathrm{IV}} = (\xi, 1 - 2\xi^2, 0), \qquad 0 \le \xi \le 1/2, \tag{4.60}$$

which are consistent with the results obtained from conventional method. In details, edge I is given e.g. by $h_2 = 0, z = 0$, edge II is given by $z^2 = x^2 + y^2$, edge III is given by $y = 0$, and edge IV is given by $h_1 = 0, z = 0$ or $h_2 = 0, y = -x$.

The orbit space has three vertices at the ends of the edges:

$$\vec{\rho}_{\mathrm{A}} = (0, 1, 0), \quad \vec{\rho}_{\mathrm{B}} = (\tfrac{1}{2}, \tfrac{1}{2}, 0), \quad \vec{\rho}_{\mathrm{C}} = (1, 1, 1). \tag{4.61}$$

Only the neutral components of the Higgs douplet and triplet should obtain VEVs. Inserting these VEVs into the orbit space parameters, we find we must require that the global minimum be in the vertex $\vec{\rho}_{\mathrm{C}} = (1, 1, 1)$ of the orbit space. The extrema on other vertices and edges must have greater potential energy. Moreover, because the edges II and III are straight line segments, it is not necessary to consider them separately in the minimization of the potential. They are automatically included in the convex hull of the orbit space.

Lastly, the numerical scan of field components are also performed. The projection of allowed points on the $\zeta - \xi$, $\eta - \xi$ and $\zeta - \eta$ planes are shown in Figure. 4.1, Figure. 4.2 and Figure. 4.3, respectively. The 3D shape for this orbit space is shown in Figure. 4.4. In these plots, the boundary solutions are also shown.

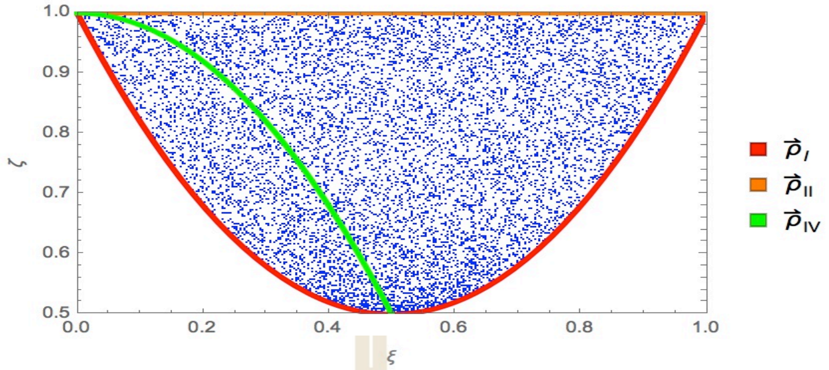

Figure 4.1 The projected $\zeta - \xi$ plane

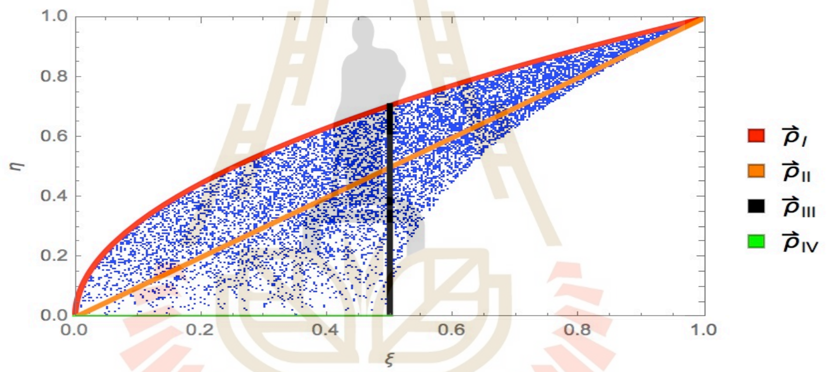

Figure 4.2 The projected $\eta - \xi$ plane

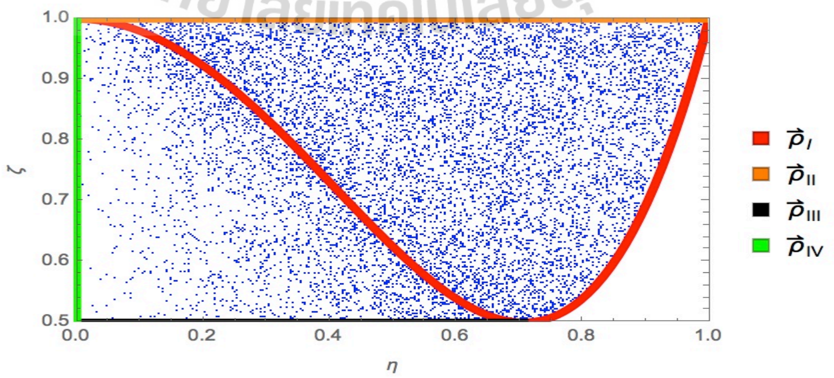

Figure 4.3 The projected $\zeta - \eta$ plane

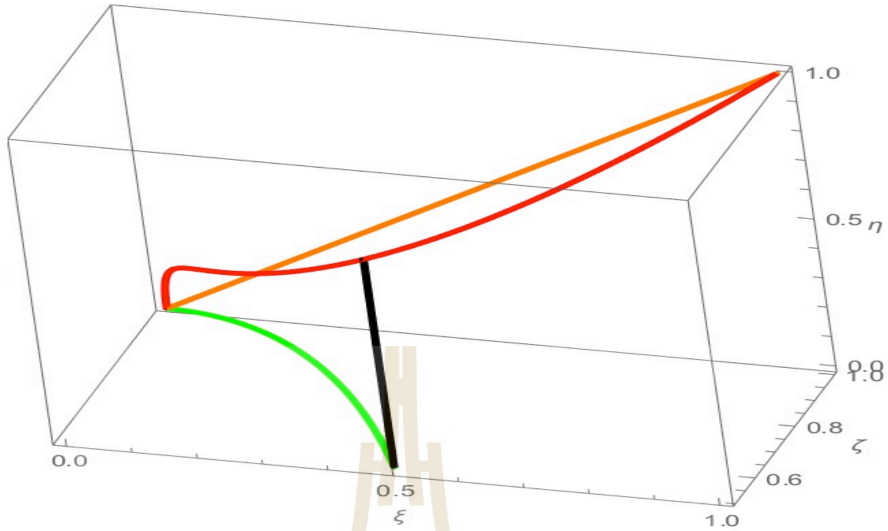

Figure 4.4 The boundary of orbit space of $\xi - \eta - \zeta$

4.1.2 Theoretical Constraints on the Quartic Couplings

For a well-defined theory, the vacuum must be stable. This criteria can be fulfilled if the scalar potential is bounded from below at large field value. The BFB condition imposes non-trivial relations among the quartic couplings. In addition, the quartic couplings are also required to be perturbative all the way to the Planck scale with the condition $|\lambda_i| < 4\pi$. Below we present the sufficient and necassary conditions for the potential to be bounded from below. For detail derivation, one can consult Appendix A.

$$\lambda_H > 0,\ \lambda_\Delta + \frac{\lambda'_\Delta}{2} > 0,\ \lambda_\Delta + \lambda'_\Delta > 0,\ \lambda_S > 0, -|\lambda_{SH\Delta}| + 2\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} > 0,$$

$$\lambda_{S\Delta} + 2\min\left[\sqrt{\lambda_S(\lambda_\Delta + \lambda'_\Delta)}, \sqrt{\lambda_S(\lambda_\Delta + \frac{\lambda'_\Delta}{2})}\right] > 0,$$

$$\left[-|\lambda_{SH\Delta}|\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} + 2\lambda_{HS}\lambda'_{H\Delta} \leq 0 \ \vee \ 4\lambda_{HS}\lambda'_{H\Delta} - |\lambda_{SH\Delta}|^2 > 0\right],$$

$$\Big([\lambda_{HS} > 0,\ \lambda_{H\Delta} > 0,\ \lambda_{H\Delta} + \lambda'_{H\Delta} > 0] \ \vee \ \big[\lambda_{HS} \leq 0,\ \lambda_{H\Delta} \leq 0,\ \lambda_{H\Delta} + \lambda'_{H\Delta} \leq 0,$$

$$
\begin{gathered}
2\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta)}+(\lambda_{H\Delta}+\lambda'_{H\Delta})>0,\;\; 2\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta)}+\lambda_{H\Delta}>0,\\
\lambda_{H\Delta}+\lambda'_{H\Delta}/2+\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta/2)}>0,\;\; \left(2\lambda'_\Delta\sqrt{\lambda_H}\le|\lambda'_{H\Delta}|\sqrt{\lambda_\Delta+\lambda'_\Delta}\ \text{ OR }\ F_1\right),\\
\left(2\lambda'_\Delta\sqrt{\lambda_H}\le\lambda'_{H\Delta}\sqrt{\lambda_\Delta+\lambda'_\Delta/2}\ \text{ OR }\ \lambda'_{H\Delta}\le 0\ \text{ OR }\ F_1\right),\\
\left(2\lambda'_\Delta\sqrt{\lambda_H}\ge\lambda'_{H\Delta}\sqrt{\lambda_\Delta+\lambda'_\Delta/2}\ \text{ OR }\ \lambda'_{H\Delta}\ge 0\ \text{ OR }\ F_2\right),\\
2\sqrt{\lambda_H\lambda_S}+\lambda_{HS}>0,\; D_{\min}>0\wedge(Q_{\min}>0\ \vee\ R_{\min}>0)\Big]\Big), \qquad (4.62)
\end{gathered}
$$

where

$$
F_1 = \lambda_{H\Delta}+\frac{\lambda'_{H\Delta}}{2}+\frac{1}{2}\sqrt{\left(\frac{2\lambda_\Delta}{\lambda'_\Delta}+1\right)(8\lambda'_\Delta\lambda_H-\lambda'^2_{H\Delta})}>0, \qquad (4.63)
$$

$$
F_2 = \lambda_{H\Delta}+\left(\frac{8\lambda_H|\lambda'_\Delta|-\lambda'^2_{H\Delta}}{2|\lambda'_\Delta|}\right)\sqrt{\frac{2\lambda'_\Delta(\lambda_\Delta+\lambda'_\Delta)}{8\lambda'_\Delta\lambda_H+\lambda'^2_{H\Delta}}}>0, \qquad (4.64)
$$

while $D_{\min}$, $Q_{\min}$, and $R_{\min}$ are defined in Appendix A.

4.1.3 Minimization of Potential

In this subsection, we minimize the scalar potential by employing the Gildener-Weinberg scheme. The tree-level potential dominates over entire region of the field space except along the flat direction. Therefore, the minimization is performed at the scale where the flat direction is developed. In the symmetry breaking vertex, the tree-level potential reads

$$\begin{aligned} 4V(h,\delta,s) &= \lambda_H h^4 + \left[(\lambda_{H\Delta}+\lambda'_{H\Delta})\delta^2 + \lambda_{HS}s^2\right]h^2 + \lambda_S s^4 + \lambda_{S\Delta}s^2\delta^2 \\ &\quad +(\lambda_\Delta+\lambda'_\Delta)\delta^4 - \lambda_{SH\Delta}s\delta h^2. \end{aligned} \tag{4.65}$$

In the light of Gildener-Weinberg framework, we parameterize the fields as

$$\begin{aligned} h &= \varphi\, N_h, \\ s &= \varphi\, N_s, \\ \delta &= \varphi\, N_\delta. \end{aligned} \tag{4.66}$$

Taking the first derivative with respect to each field variables and minimizing on the unit sphere, we obtain

$$0 = \lambda_H n_h^3 + \frac{1}{2}\left[(\lambda_{H\Delta}+\lambda'_{H\Delta})n_\delta^2 + \lambda_{HS}n_s^2\right]n_h - \frac{\lambda_{SH\Delta}}{2}n_s n_\delta n_h, \tag{4.67}$$

$$0 = (\lambda_\Delta+\lambda'_\Delta)n_\delta^3 + \frac{1}{2}\left[\lambda_{S\Delta}n_s^2 + (\lambda_{H\Delta}+\lambda'_{H\Delta})n_h^2\right]n_\delta^2 - \frac{\lambda_{SH\Delta}}{4}n_s n_h^2, \tag{4.68}$$

$$0 = \lambda_S n_s^3 + \frac{1}{2}\left[\lambda_{S\Delta}n_\delta^2 + \lambda_{HS}n_h^2\right]n_s - \frac{\lambda_{SH\Delta}}{4}n_\delta n_h^2, \tag{4.69}$$

$$1 = n_h^2 + n_\delta^2 + n_s^2. \tag{4.70}$$

For nonzero VEV, one may solve n_h^2,

$$n_h^2 = \frac{\lambda_{SH\Delta}n_s n_\delta - (\lambda_{H\Delta}+\lambda'_{H\Delta})n_\delta^2 - \lambda_{HS}n_s^2}{2\lambda_H}. \tag{4.71}$$

Inserting this expression into Eq. (4.68) and (4.69), one finds

$$0 = (8\lambda_H[\lambda_\Delta+\lambda'_\Delta] - 2[\lambda_{H\Delta}+\lambda'_{H\Delta}]^2)n_\delta^3 + 3\lambda_{SH\Delta}(\lambda_{H\Delta}+\lambda'_{H\Delta})n_\delta^2 n_s$$

$$+(4\lambda_H\lambda_{S\Delta} - 2\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta}) - \lambda^2_{SH\Delta})n_\delta n_s^2 + \lambda_{HS}\lambda_{SH\Delta}n_s^3, \tag{4.72}$$

$$\begin{aligned} 0 &= \lambda_{SH\Delta}(\lambda_{H\Delta} + \lambda'_{H\Delta})n_\delta^3 + (4\lambda_H\lambda_{S\Delta} - 2\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta}) - \lambda^2_{SH\Delta})n_\delta^2 n_s \\ &\quad +3\lambda_{HS}\lambda_{SH\Delta}n_\delta n_s^2 + (8\lambda_H\lambda_S - 2\lambda^2_{HS})n_s^3. \end{aligned} \tag{4.73}$$

Dividing both equations by n_s^3, we have

$$\begin{aligned} 0 &= (8\lambda_H[\lambda_\Delta + \lambda'_\Delta] - 2[\lambda_{H\Delta} + \lambda'_{H\Delta}]^2)u^3 + 3\lambda_{SH\Delta}(\lambda_{H\Delta} + \lambda'_{H\Delta})u^2 \\ &\quad +(4\lambda_H\lambda_{S\Delta} - 2\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta}) - \lambda^2_{SH\Delta})u + \lambda_{HS}\lambda_{SH\Delta}, \end{aligned} \tag{4.74}$$

$$\begin{aligned} 0 &= \lambda_{SH\Delta}(\lambda_{H\Delta} + \lambda'_{H\Delta})u^3 + (4\lambda_H\lambda_{S\Delta} - 2\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta}) - \lambda^2_{SH\Delta})u^2 \\ &\quad +3\lambda_{HS}\lambda_{SH\Delta}u + (8\lambda_H\lambda_S - 2\lambda^2_{HS}), \end{aligned} \tag{4.75}$$

where

$$u \equiv \frac{n_\delta}{n_s}. \tag{4.76}$$

4.2 Mass Spectra of Scalar Bosons

At tree-level the squared masses of scalar particles can be obtained by evaluating the eigenvalues of the Hessian matrix of tree-level scalar potential

$$\mathcal{M}_{ij} \;=\; \frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\Big|_{\phi=(v_h,v_s,v_\delta)},\ \phi_i=(h_0,S_R,\delta_0),\ (h^\pm,\delta^\pm,\delta^{\pm\pm}),\ \text{or}\,(Z_1,S_I,Z_2) \tag{4.77}$$

with

$$\begin{aligned} v_h &= \varphi\, n_h, \\ v_s &= \varphi\, n_s, \\ v_\delta &= \varphi\, n_\delta. \end{aligned} \tag{4.78}$$

Mass of the neutral CP-even Higgs

The mass squared matrix of the neutral CP-even Higgs in the weak basis (s,h,δ) is given by

$$\mathcal{M}^2_{\text{CP-even}} = \varphi^2 \begin{pmatrix} 2\lambda_S n_s^2+\frac{\lambda_{SH\Delta}}{4}n_h^2\frac{n_\delta}{n_s} & a & b \\ a & 2\lambda_H n_h^2 & c \\ b & c & 2(\lambda_\Delta+\lambda'_\Delta)n_\delta^2+\frac{\lambda_{SH\Delta}}{4}n_h^2\frac{n_s}{n_\delta} \end{pmatrix}, \tag{4.79}$$

where $a = \lambda_{HS}n_hn_s - \frac{\lambda_{SH\Delta}}{2}n_hn_\delta$, $b = \lambda_{S\Delta}n_sn_\delta - \frac{\lambda_{SH\Delta}}{4}n_h^2$, and $c = (\lambda_{H\Delta} + \lambda'_{H\Delta})n_hn_\delta - \frac{\lambda_{SH\Delta}}{2}n_hn_\delta$.

Mass of the neutral CP-odd Higgs

The mass squared matrix of the neutral CP-odd Higgs is

$$\mathcal{M}^2_{\text{CP-odd}} = \frac{\lambda_{SH\Delta}}{2}\varphi^2 \begin{pmatrix} \frac{n_h^2}{2}\frac{n_\delta}{n_s} & n_hn_\delta & -\frac{n_h^2}{2} \\ n_hn_\delta & 2n_sn_\delta & -n_hn_s \\ -\frac{n_h^2}{2} & -n_hn_s & \frac{n_h^2}{2}\frac{n_s}{n_\delta} \end{pmatrix}. \tag{4.80}$$

The null space of this matrix is two dimensions. All the three columns are not independent of each other, i.e., Rank($\mathcal{M}^2_{\text{CP-odd}}$)=1. Hence, there are two massless fields. One is the unphysical Goldstone boson G which will become the longitudinal component of Z boson, while another one is the physical majoron J. The eigensystem is given below

$$\begin{pmatrix} m_J=0 & m_G=0 & m_A=\frac{\lambda_{SH\Delta}}{2}\varphi^2\frac{(n_h^2n_\delta^2+n_h^2n_s^2+4n_s^2n_\delta^2)}{2n_sn_\delta} \\ \left(n_s(n_h^2+4n_\delta^2),-2n_\delta^2n_h,n_\delta n_h^2\right)^{\mathrm{T}} & (0,n_h,2n_\delta)^{\mathrm{T}} & \left(-\frac{n_\delta}{n_s},-\frac{2n_\delta}{n_h},1\right)^{\mathrm{T}} \end{pmatrix} \tag{4.81}$$

where the first row corresponds to mass eigenvalues and the second row is the associated eigenvectors. The matrix $\mathcal{M}^2_{\text{CP-odd}}$ can be diagonalized by

$$\mathcal{O}_I\,\mathcal{M}^2_{\text{CP-odd}}\,\mathcal{O}_I^{\mathrm{T}} = \operatorname{diag}\left(0,0,m_A^2\right), \tag{4.82}$$

where

$$\mathcal{O}_I^{\mathrm{T}} = \begin{pmatrix} n_s(n_h^2+4n_\delta^2)a & 0 & -n_hn_\delta b \\ -2n_\delta^2n_ha & n_hV & -2n_sn_\delta b \\ n_\delta n_h^2a & 2n_\delta V & n_hn_sb \end{pmatrix}, \tag{4.83}$$

with

$$\begin{aligned} V &= \frac{1}{\sqrt{n_h^2+4n_\delta^2}}, \\ a &= \frac{1}{\sqrt{\frac{n_s^2}{V^2}+4n_h^2n_\delta^4+n_\delta^2n_h^4}}, \\ b &= \frac{1}{\sqrt{\frac{n_h^2}{V^2}+4n_s^2n_\delta^2}}. \end{aligned} \tag{4.84}$$

The mass and weak eigenstates are related by

$$\begin{pmatrix} J \\ G \\ A \end{pmatrix} = \mathcal{O}_I \begin{pmatrix} S_I \\ Z_1 \\ Z_2 \end{pmatrix}. \tag{4.85}$$

Mass of the singly-charged Higgs

The mass squared matrix of the singly-charged Higgs is

$$\begin{aligned}\mathcal{M}^2_{\pm} &= \frac{\varphi^2}{2}\begin{pmatrix} \lambda_{SH\Delta}n_s n_\delta - \lambda'_{H\Delta}n_\delta^2 & \frac{\lambda'_{H\Delta}}{\sqrt{2}}n_h n_\delta - \frac{\lambda_{SH\Delta}}{\sqrt{2}}n_h n_s \\ \\ \frac{\lambda'_{H\Delta}}{\sqrt{2}}n_h n_\delta - \frac{\lambda_{SH\Delta}}{\sqrt{2}}n_h n_s & \frac{\lambda_{SH\Delta}}{2}\frac{n_s}{n_\delta}n_h^2 - \frac{\lambda'_{H\Delta}}{2}n_h^2 \end{pmatrix} \\ &= \frac{\varphi^2}{4}\left(\lambda_{SH\Delta}n_h n_s - \lambda'_{H\Delta}n_h n_\delta\right)\begin{pmatrix} 2\frac{n_\delta}{n_h} & -\sqrt{2} \\ \\ -\sqrt{2} & \frac{n_h}{n_\delta}\end{pmatrix} \end{aligned} \tag{4.86}$$

in the weak basis $(h^\pm, \delta^\pm)$. One eigenvalue is zero which corresponds to the charged Goldstone bosons that will be absorbed by $W^\pm$. This mass matrix can be diagonalized by the orthogonal matrix $\mathcal{O}_\pm$ such that $\mathcal{O}_\pm \mathcal{M}^2_\pm \mathcal{O}^{\mathrm{T}}_\pm = \mathrm{diag}(m_{H^\pm}, 0)$, where

$$\mathcal{O}^{\mathrm{T}}_\pm = \begin{pmatrix} c_\pm & -s_\pm \\ s_\pm & c_\pm \end{pmatrix} = \frac{1}{\sqrt{n_h^2 + 2n_\delta^2}}\begin{pmatrix} \sqrt{2}n_\delta & n_h \\ -n_h & \sqrt{2}n_\delta \end{pmatrix}, \tag{4.87}$$

and the physical Higgs mass is

$$m^2_{H^\pm} = \frac{\varphi^2}{4}\left(\lambda_{SH\Delta}n_h n_s - \lambda'_{H\Delta}n_h n_\delta\right)\frac{n_h^2 + 2n_\delta^2}{n_h n_\delta}. \tag{4.88}$$

Mass of doubly charged field

The mass squared of the doubly-charged Higgs is given by

$$m^2_{H^{\pm\pm}} = \varphi^2\left(\lambda_\Delta n_\delta^2 + \frac{\lambda_{H\Delta}}{2}n_h^2 + \frac{\lambda_{S\Delta}}{2}n_s^2\right). \tag{4.89}$$

Applying the tadpole condition, the mass squared takes the form

$$m^2_{H^{\pm\pm}} = \varphi^2\left(\frac{\lambda_{SH\Delta}}{4}\frac{n_s}{n_\delta}n_h^2 - \lambda'_\Delta n_\delta^2 - \frac{\lambda'_{H\Delta}}{2}n_h^2\right). \tag{4.90}$$

4.3 RGEs of quartic coupling

Below we present the analytic formulae for the beta functions of all scalar quartic couplings at 1-loop level (we have ignored all Yukawa couplings except the top Yukawa).

$$
\begin{aligned}
\frac{\mathrm{d}\lambda_H}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[24\lambda_H^2 + \frac{1}{2}\lambda_{SH\Delta}^2 + 3\lambda_{H\Delta}^2 + \lambda_{HS}^2 + 3\lambda_{H\Delta}\lambda'_{H\Delta} + \frac{5}{4}\lambda'^2_{H\Delta} \\
&\quad + \frac{3}{8}g_1^4 + \frac{9}{8}g_2^4 + \frac{3}{4}g_1^2g_2^2 - (3g_1^2 + 9g_2^2)\lambda_H - 6y_t^4 + 12\lambda_H y_t^2\Big], \qquad (4.91) \\
\frac{\mathrm{d}\lambda_\Delta}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[28\lambda_\Delta^2 + 24\lambda_\Delta\lambda'_\Delta + 6\lambda'^2_\Delta + 2\lambda_{H\Delta}^2 + 2\lambda_{H\Delta}\lambda'_{H\Delta} + \lambda_{S\Delta}^2 \\
&\quad + 6g_1^4 + 15g_2^4 - 12g_1^2g_2^2 - \left(12g_1^2 + 24g_2^2\right)\lambda_\Delta\Big], \qquad (4.92) \\
\frac{\mathrm{d}\lambda'_\Delta}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[18\lambda'^2_\Delta + 24\lambda_\Delta\lambda'_\Delta + \lambda'^2_{H\Delta} - 6g_2^4 + 24g_1^2g_2^2 \\
&\quad - \left(12g_1^2 + 24g_2^2\right)\lambda'_\Delta\Big], \qquad (4.93) \\
\frac{\mathrm{d}\lambda_S}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[20\lambda_S + 2\lambda_{HS}^2 + 3\lambda_{S\Delta}^2\Big], \qquad (4.94) \\
\frac{\mathrm{d}\lambda_{H\Delta}}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[3g_1^4 + 6g_2^4 - 6g_1^2g_2^2 - \left(\frac{15}{2}g_1^2 + \frac{33}{2}g_2^2\right)\lambda_{H\Delta} + 12\lambda_H\lambda_{H\Delta} \\
&\quad + 4\lambda_H\lambda'_{H\Delta} + 4\lambda_{H\Delta}^2 + 16\lambda_\Delta\lambda_{H\Delta} + 12\lambda'_\Delta\lambda_{H\Delta} + \lambda'^2_{H\Delta} + 6\lambda_\Delta\lambda'_{H\Delta} \\
&\quad + 2\lambda'_\Delta\lambda'_{H\Delta} + 2\lambda_{HS}\lambda_{S\Delta} + 6\lambda_{H\Delta}y_t^2\Big], \qquad (4.95) \\
\frac{\mathrm{d}\lambda'_{H\Delta}}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[12g_1^2g_2^2 - \left(\frac{15}{2}g_1^2 + \frac{33}{2}g_2^2\right)\lambda'_{H\Delta} + 4\lambda_H\lambda'_{H\Delta} + 8\lambda_{H\Delta}\lambda'_{H\Delta} \\
&\quad + 4\lambda'^2_{H\Delta} + 4\lambda_\Delta\lambda'_{H\Delta} + 8\lambda'_\Delta\lambda'_{H\Delta} + 2\lambda_{SH\Delta}^2 + 6\lambda'_{H\Delta}y_t^2\Big], \qquad (4.96) \\
\frac{\mathrm{d}\lambda_{HS}}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[4\lambda_{HS}^2 + 8\lambda_{HS}\lambda_S + 12\lambda_H\lambda_{HS} + 6\lambda_{S\Delta}\lambda_{H\Delta} + 3\lambda_{S\Delta}\lambda'_{H\Delta} \\
&\quad + 3\lambda_{SH\Delta}^2 - \left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2\right)\lambda_{HS} + 6\lambda_{HS}y_t^2\Big], \qquad (4.97) \\
\frac{\mathrm{d}\lambda_{S\Delta}}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[4\lambda_{S\Delta}^2 + \lambda_{HS}(4\lambda_{H\Delta} + 2\lambda'_{H\Delta}) + \lambda_{S\Delta}(16\lambda_\Delta + 12\lambda'_\Delta + 8\lambda_S) \\
&\quad + \lambda_{SH\Delta}^2 - (6g_1^2 + 12g_2^2)\lambda_{S\Delta}\Big], \qquad (4.98) \\
\frac{\mathrm{d}\lambda_{SH\Delta}}{\mathrm{d}t} &= \frac{1}{16\pi^2}\Big[4\lambda_H + 4\lambda_{H\Delta} + 6\lambda'_{H\Delta} + 4\lambda_{HS} + 2\lambda_{S\Delta} + 6y_t^2 \\
&\quad - \frac{9}{2}g_1^2 - \frac{21}{2}g_2^2\Big]\lambda_{SH\Delta}, \qquad (4.99)
\end{aligned}
$$

where g_1, g_2, g_3 are the gauge coupling of $U(1)_Y$, $SU(2)_L$, and $SU(3)_c$, respectively.

We found that there are a few mismatches between our results and those in ref. (Okada et al., 2015). One term for each RGE of λ_Δ and λ'_Δ. Another mismatch is in the RGE of $\lambda_{H\Delta}$ involving the term $\lambda_{HS}\lambda_{S\Delta}$.

4.4 Numerical Study on Radiative Symmetry Breaking

In this section, a numerical study on radiative symmetry breaking will be presented. The scalar potential contains total nine quartic couplings. They are subjected to the following theoretical constraints:

- Three stationary equations,
- Bounded from below conditions,
- Perturbative conditions,

In addition, we also demand that the SM-like Higgs mass is consistent with the experimental observation $M_H = 125.09$ GeV.

In our numerical study, we first consider three stationary equations and the Higgs mass constraint. We therefore have five free parameters left. The following quartic couplings are chosen as our free parameters

$$\lambda_\Delta, \lambda'_\Delta, \lambda_{H\Delta}, \lambda'_{H\Delta}, \lambda_{S\Delta}.$$

We randomly generate these free parameters and numerically solve the stationary equations by requiring that the VEVs of SM-Higgs doublet and Higgs triplet satisfy $v \equiv \sqrt{v_h^2 + 2v_t^2} = 246$ GeV as well as the SM-Higgs mass. We identify the SM-Higgs boson with one of the non-zero eigenvalues of CP-even mass matrix. We then proceed to feed these parameters into the bounded from below conditions. Furthermore, these couplings are required to be perturbative as they evolve from weak scale to Planck scale through their corresponding RGEs. We have used the following initial values of gauge couplings and top Yukawa coupling in solving the RGEs (Buttazzo et al., 2013)

$$g_1(\mu = M_t) \quad = \quad 0.35745,$$

$$
\begin{aligned}
g_2(\mu = M_t) &= 0.64779, \\
g_3(\mu = M_t) &= 1.1666, \qquad (4.100) \\
y_t(\mu = M_t) &= 0.93690.
\end{aligned}
$$

We also check which set of quartic couplings can make the evolution of λ_H remain positive all the way to the Planck scale.

The set of quartic couplings for three benchmark points A,B,C that satisfy all the above constraints are listed in Table 4.1.

Table 4.1 The set of quartic coupling satisfying the perturbativity and vacuum stability criteria. The BP A gives $M_s = 3.9$ GeV, the BP B gives $M_s = 2.38$ GeV, and the BP C gives $M_s = 1.9$ GeV.

BP	λ_H	λ_Δ	λ'_Δ	λ_S	λ_{HS}	$\lambda_{H\Delta}$	$\lambda'_{H\Delta}$	$\lambda_{S\Delta}$	$\lambda_{SH\Delta}$
A	0.122	0.03	0.25	0.0005	-0.01565	-0.05	-0.327	0.103	0.00261
B	0.122	0.04	0.2	0.0005	-0.01565	-0.02	-0.3	0.1	0.00254
C	0.122	0.055	0.2186	0.0005	-0.01565	-0.168	-0.179	0.1053	0.0026

Figure. 4.5 shows the tree level and 1-loop level effective potential in the flat direction for the BP A. The inclusion of radiative potential along the flat direction successfully realizes non-zero VEV at the scale $\varphi = 1$ TeV. The three corresponding VEVs are $v_h = 245.99$ GeV, $v_\delta = 1$ GeV, and $v_s = 969.27$ GeV.* Note that the triplet VEV has been chosen to have such a small value due to the constraint from the ρ-parameter.

We also give a comment about the fate of running of quartic SM-Higgs doublet from the weak scale to the Planck scale. As is well known that RGE running of quartic coupling in SM crosses zero somewhere at high energy $\simeq 10^8$ GeV due to

*With this value of the triplet VEV, the neutrino mass given by $m_\nu = Y_\nu v_\delta$ of the order 0.01 eV can be produced by having small Yukawa coupling, i.e., $Y_\nu \simeq 10^{-11}$.

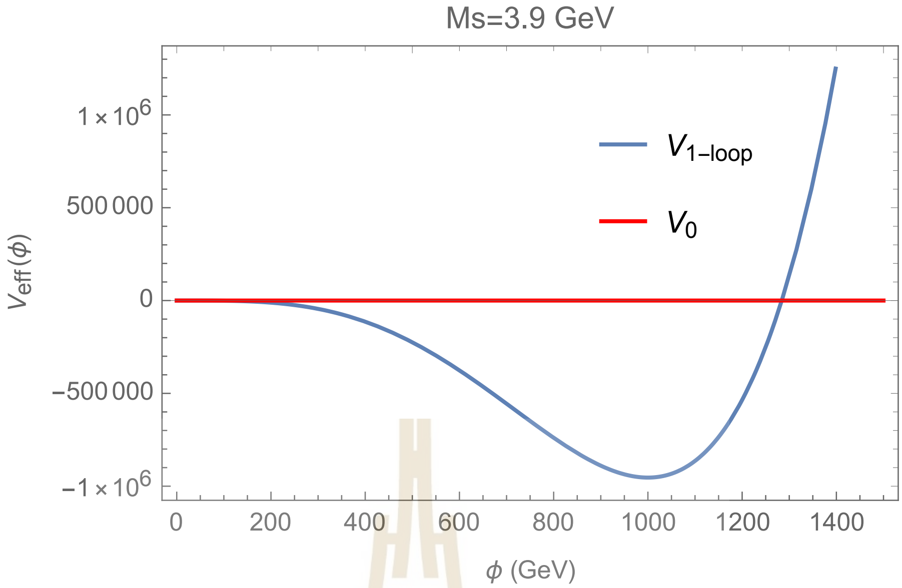

Figure 4.5 Effective potential in the flat direction for the BP A.

the strong negative contribution from the top Yukawa term. The situation can be dramatically changed when positive contribution from additional bosonic particles can equally compete. In the case of singlet extension of type II seesaw, there are three new additional contributions coming from $\lambda_{H\Delta}$, $\lambda'_{H\Delta}$, and λ_{HS}. It can be seen in Figures. 4.6-4.8 that the λ_H can remain to be positive up to the Planck scale signaling that the vacuum of the electroweak theory can be accomplished to be stable.

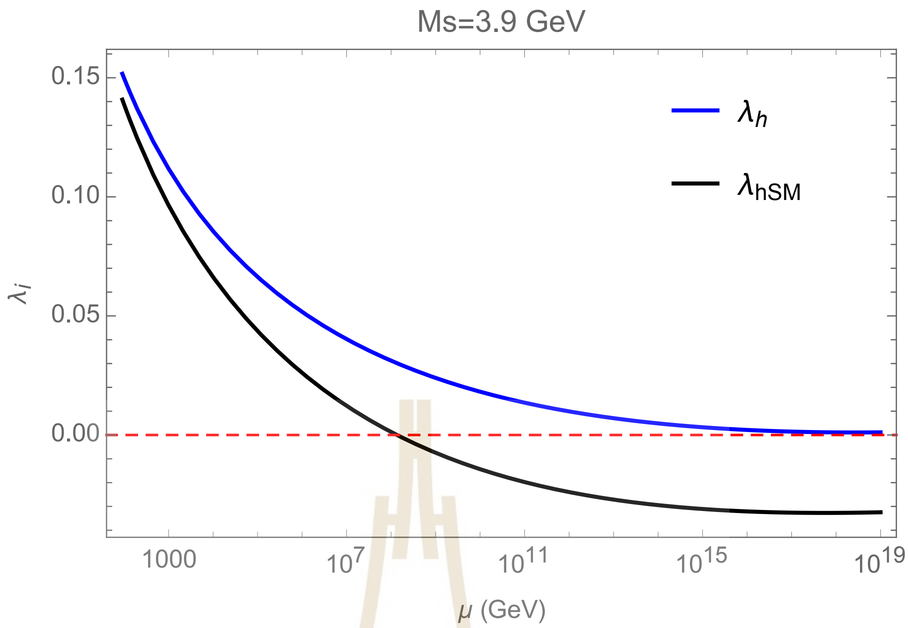

Figure 4.6 RGE running of quartic coupling of SM-Higgs doublet from weak scale to Planck scale for the BP A.

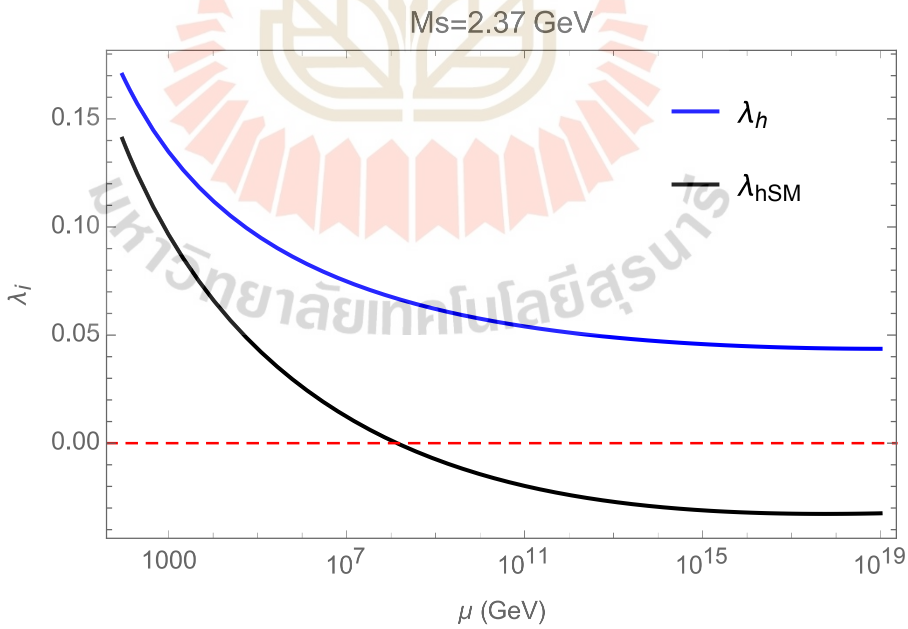

Figure 4.7 RGE running of quartic coupling of SM-Higgs doublet from weak scale to Planck scale for the BP B.

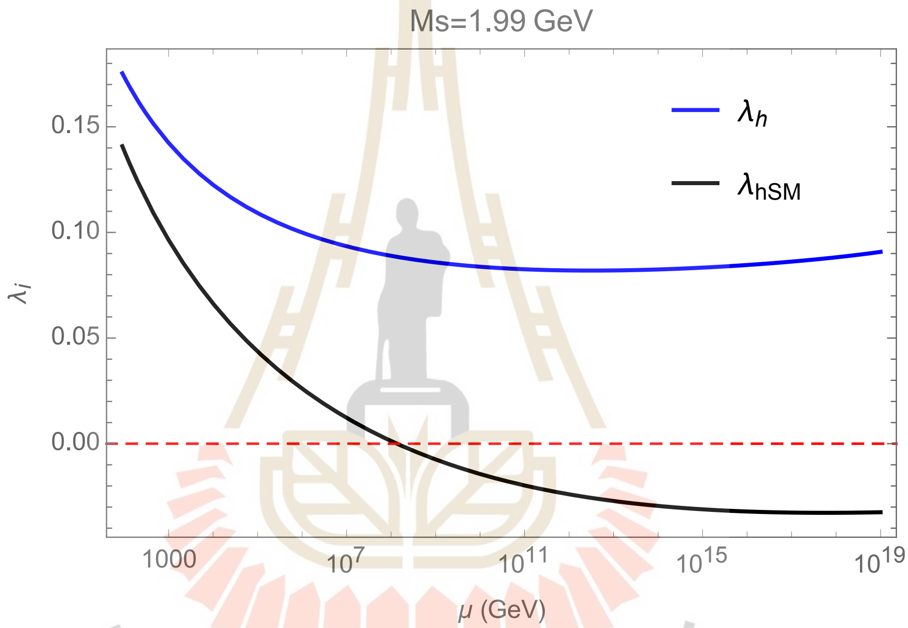

Figure 4.8 RGE running of quartic coupling of SM-Higgs doublet from weak scale to Planck scale for the BP C.

CHAPTER V

CONCLUSIONS AND DISCUSSIONS

In this thesis, we have considered the singlet extension of type II seesaw possessing a classical scale-invariance. A new singlet scalar has been introduced to prohibit the Goldstone boson of global lepton number symmetry from having fully $SU(2)_L$ gauge interaction. In this case, the majoron constitutes mostly singlet component of the gauge group of the SM. This framework is particularly interesting in three aspects. Firstly, the feature of type II seesaw can address the neutrino mass problem. Secondly, a classical scale-invariant theory paves the way to the origin of the electroweak potential which also allows us to cure the theoretical shortcoming of the hierarchy problem. Lastly, the incorporation of a new bosonic degree of freedom can save the vacuum of the theory from being unstable.

We derived for the first time a full set of sufficient and necessary conditions for the scalar Higgs potential being bounded from below without making approximation on parameters. The RGEs of quartic couplings are also calculated at the 1-loop level. Due to the complexity of multiscalar potential, we resort to the novel Gildener-Weinberg approach in minimizing the potential. An important role of the radiative corrections of the tree-level potential can be at play along the flat direction. At the scale $\varphi = 1$ TeV, the effective potential develops a non-zero VEV signaling that the symmetry has been spontaneously broken. In this regard, the scale has been generated dynamically by quantum corrections unlike the prior given scale in the spontaneously breaking of electroweak symmetry in the SM.

We showed that the stability of electroweak vacuum can be maintained all the way up to the Planck scale with the new contributions coming from the singlet and triplet scalars. Particularly, the evolution of λ_H with the energy scale can be prevented from crossing zero value at high energy due to sizeable contributions from $\lambda_{H\Delta}$ and $\lambda'_{H\Delta}$. It can be observed that this vacuum stability can be realized if the mass of the scalon is only a few GeV. For the purpose of numerical illustration, three benchmark points were presented. The evolution of the λ_H with scalon mass 3.9 GeV is lower than the evolution of the λ_H (almost cross zero value at the Planck scale) with scalon mass 2.37 GeV and 1.99 GeV.

In conclusion, we have shown in this work that the radiative symmetry breaking can be realized in the scale-invariant singlet extension of type II seesaw model.

For future works, a full scan of viable parameter space should be performed. The model has rich phenomenology since it contains new scalar fields. Constraints from collider experiments are deserved to be investigated. Another aspect that can also be studied is the possibility for this model to exhibit a first-order electroweak phase transition, which is one of the requirements for successful baryogenesis.

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APPENDICES

APPENDIX A

DERIVATION OF BOUNDED-FROM-BELOW CONDITIONS

The scalar potential in terms of orbit parameters is given by

$$\begin{aligned}\frac{V_{tree}(S,H,\Delta)}{(Tr\Delta^\dagger\Delta)^2} &= \lambda_H r^2 + \left(\lambda_{H\Delta} + \lambda'_{H\Delta}\xi + \lambda_{HS}k + |\lambda_{SH\Delta}|\eta\sqrt{k}\cos(\alpha+\phi_{\lambda_{SH\Delta}})\right) r \\ &\quad +\lambda_\Delta + \lambda'_\Delta\zeta + \lambda_S k^2 + \lambda_{S\Delta}k. \end{aligned} \tag{A.1}$$

Let us define the following functions:

$$f_I(k,\zeta) \equiv \lambda_\Delta + \lambda'_\Delta\zeta + \lambda_S k^2 + \lambda_{S\Delta}k, \tag{A.2}$$

$$f_{II}(k,\xi,\eta,\cos\beta) \equiv \lambda_{H\Delta} + \lambda'_{H\Delta}\xi + \lambda_{HS}k + |\lambda_{SH\Delta}|\eta\sqrt{k}\cos\beta. \tag{A.3}$$

with $\beta = \alpha + \phi_{\lambda_{SH\Delta}}$.

The potential is BFB if the following conditions are satisfied

$$\begin{aligned}\lambda_H > 0, \quad \lambda_\Delta + \lambda'_\Delta\zeta > 0, \quad \lambda_S > 0, \quad \lambda_{S\Delta} + 2\sqrt{\lambda_S(\lambda_\Delta+\lambda'_\Delta\zeta)} &> 0, \\ f_{II}(k,\xi,\eta,\cos\beta) + 2\sqrt{\lambda_H f_I(k,\zeta)} &> 0. \end{aligned} \tag{A.4}$$

For the last condition, we consider two possible scenarios:

- $f_{II}(k,\xi,\eta,\cos\beta) > 0$

$$\lambda_{HS} > 0, \quad \lambda_{H\Delta} + \lambda'_{H\Delta}\xi > 0, \quad |\lambda_{SH\Delta}|\eta\cos\beta + 2\sqrt{\lambda_{HS}(\lambda_{H\Delta}+\lambda'_{H\Delta}\xi)} > 0. \tag{A.5}$$

- $f_{II}(k,\xi,\eta,\cos\beta) \leq 0$ and $4\,\lambda_H f_I > f_{II}^2$

$$\lambda_{HS} \leq 0, \quad \lambda_{H\Delta} + \lambda'_{H\Delta}\xi \leq 0, \quad -|\lambda_{SH\Delta}|\eta\cos\beta + 2\sqrt{\lambda_{HS}(\lambda_{H\Delta}+\lambda'_{H\Delta}\xi)} > 0,$$

$$
\begin{aligned}
&(4\lambda_H\lambda_S-\lambda_{HS}^2)k^2-2\lambda_{HS}|\lambda_{SH\Delta}|\eta\cos\beta\, k\sqrt{k}\\
&-\left(|\lambda_{SH\Delta}|^2\eta^2\cos^2\beta+2\lambda_{HS}(\lambda_{H\Delta}+\lambda'_{H\Delta}\xi)-4\lambda_H\lambda_{S\Delta}\right)k\\
&-2|\lambda_{SH\Delta}|(\lambda_{H\Delta}+\lambda'_{H\Delta}\xi)\eta\cos\beta\,\sqrt{k}\\
&+4\lambda_H(\lambda_\Delta+\lambda'_\Delta\zeta)-(\lambda_{H\Delta}+\lambda'_{H\Delta}\xi)^2>0. \qquad (A.6)
\end{aligned}
$$

Let us define

$$
g_I(\eta,\xi,\cos\beta) \;\equiv\; |\lambda_{SH\Delta}|\eta\cos\beta+2\sqrt{\lambda_{HS}(\lambda_{H\Delta}+\lambda'_{H\Delta}\xi)}. \qquad (A.7)
$$

Since ξ and η are not independent quantities, we have to study $g_I(\eta,\xi,\cos\beta)$ at region bounded by each solution.

<u>Scenario I</u>

1. Region bounded by $\vec{\rho}_I=(\eta^2,2\eta^4-2\eta^2+1,\eta)$

$$
\begin{aligned}
g_I(\eta) &= -|\lambda_{SH\Delta}|\eta+2\sqrt{\lambda_{HS}(\lambda_{H\Delta}+\lambda'_{H\Delta}\eta^2)}, && (A.8)\\
g'_I(\eta) &\equiv \frac{dg_I}{d\eta}=-|\lambda_{SH\Delta}|+\frac{2\lambda_{HS}\lambda'_{H\Delta}\eta}{\sqrt{\lambda_{HS}(\lambda_{H\Delta}+\lambda'_{H\Delta}\eta^2)}}, && (A.9)\\
g''_I(\eta) &\equiv \frac{d^2g_I}{d\eta^2}=\frac{2\lambda_{HS}^2\lambda_{H\Delta}\lambda'_{H\Delta}}{\sqrt{[\lambda_{HS}(\lambda_{H\Delta}+\lambda'_{H\Delta}\eta^2)]^3}}. && (A.10)
\end{aligned}
$$

In minimum, we have $g_I(\eta,\eta^2,-1)$. That explains the minus sign in Eq. (A.8). We observe that the sign of $g''_I=sgn(\lambda_{H\Delta}\lambda'_{H\Delta})$ is constant implying that $g'_I(\eta)$ is monotonic function. Thus we can find the value of η_0 such that $g'_I(\eta_0)=0$. Since $g_I(\eta_0)$ is minimum if $g''_I(\eta_0)>0$, $\lambda_{H\Delta}$ and $\lambda'_{H\Delta}$ must have the same sign. Taking into account the second condition from Eq. (A.5), we have

$$
\lambda_{HS}>0,\qquad \lambda_{H\Delta}>0,\qquad \lambda'_{H\Delta}>0. \qquad (A.11)
$$

From Eqs. (A.8) and (A.9), the value of η_0 and $g_I(\eta_0)$ are

$$
\eta_0^2 \;=\; \frac{|\lambda_{SH\Delta}|^2\lambda_{H\Delta}}{4\lambda_{HS}\lambda'^2_{H\Delta}-|\lambda_{SH\Delta}|^2\lambda'_{H\Delta}}, \qquad (A.12)
$$

$$g_I(\eta_0) = \left(4|\lambda_{HS}||\lambda'_{H\Delta}| - |\lambda_{SH\Delta}|^2\right)\sqrt{\frac{\lambda_{H\Delta}}{\lambda'_{H\Delta}(4\lambda_{HS}\lambda'_{H\Delta} - |\lambda_{SH\Delta}|^2)}}. \tag{A.13}$$

From these expressions, we require

$$4\lambda_{HS}\lambda'_{H\Delta} - |\lambda_{SH\Delta}|^2 > 0. \tag{A.14}$$

Requiring that η_0 is confined within $0 \le \eta_0 \le 1$ is equvalent to $g'_I(0) < 0$ and $g'_I(1) > 0$:

$$-|\lambda_{SH\Delta}| < 0, \quad -|\lambda_{SH\Delta}|\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} + 2\lambda_{HS}\lambda'_{H\Delta} > 0. \tag{A.15}$$

The next possible minima of $g_I(\xi, \eta)$ are at $g_I(0,0)$, $g_I(1,1)$, and $g_I(1/2, 0)$. Hence we require

$$\begin{aligned} &\lambda_{HS} > 0,\ \lambda_{H\Delta} > 0,\ g_I(1,1) = -|\lambda_{SH\Delta}| + 2\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} > 0, \\ &\lambda_{H\Delta} + \lambda'_{H\Delta}/2 > 0. \end{aligned} \tag{A.16}$$

If $\lambda'_{H\Delta} < 0$, the function $g_I(\eta)$ will have no stationary point. The minimum of $g_I(\eta)$ will be at $\eta = 1$. Therefore, for solution I we have the following conditions

$$\begin{aligned} &\lambda_{HS},\ \lambda_{H\Delta},\ \lambda_{H\Delta} + \lambda'_{H\Delta},\ -|\lambda_{SH\Delta}| + 2\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} > 0, \\ &\Big[-|\lambda_{SH\Delta}|\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} + 2\lambda_{HS}\lambda'_{H\Delta} \le 0 \ \vee \\ &\qquad 4\lambda_{HS}\lambda'_{H\Delta} - |\lambda_{SH\Delta}|^2 > 0\Big]. \end{aligned} \tag{A.17}$$

2. Region bounded by $\vec{\rho}_{IV} = (\xi, 1 - 2\xi^2, 0)$ with $0 \le \xi \le \frac{1}{2}$

$$\lambda_{HS} > 0, \quad \lambda_{H\Delta} > 0, \quad \lambda_{H\Delta} + \frac{\lambda'_{H\Delta}}{2} > 0. \tag{A.18}$$

Scenario II

The analysis of g_I is the same as in the scenario I but with $\cos\beta = +1$. In this scenario we have conditions

$$\lambda_{HS},\ \lambda_{H\Delta},\ \lambda_{H\Delta} + \lambda'_{H\Delta} \leq 0,\ -|\lambda_{SH\Delta}| + 2\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} > 0,$$

$$\left[-|\lambda_{SH\Delta}|\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} + 2\lambda_{HS}\lambda'_{H\Delta} \leq 0 \ \ \vee \ \ 4\lambda_{HS}\lambda'_{H\Delta} - |\lambda_{SH\Delta}|^2 > 0\right]. \tag{A.19}$$

Finally come to the last condition of scenario II. It is a quartic polynomial in $\sqrt{k}$:

$$a_4 k^2 + a_3 k\sqrt{k} + a_2 k + a_1\sqrt{k} + a_0, \tag{A.20}$$

with

$$\begin{aligned}
a_4 &= 4\lambda_H\lambda_S - \lambda_{HS}^2, \\
a_3 &= -2\lambda_{HS}|\lambda_{SH\Delta}|\eta\cos\beta, \\
a_2 &= 4\lambda_H\lambda_{S\Delta} - 2\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta}\xi) - |\lambda_{SH\Delta}|^2\eta^2\cos^2\beta, \\
a_1 &= -2|\lambda_{SH\Delta}|(\lambda_{H\Delta} + \lambda'_{H\Delta}\xi)\eta\cos\beta, \\
a_0 &= 4\lambda_H(\lambda_\Delta + \lambda'_\Delta\zeta) - (\lambda_{H\Delta} + \lambda'_{H\Delta}\xi)^2.
\end{aligned} \tag{A.21}$$

The condition that this quartic polynomial is positive is given by

$$\begin{aligned}
&4\lambda_H\lambda_S - \lambda_{HS}^2,\ 4\lambda_H(\lambda_\Delta + \lambda'_\Delta\zeta) - (\lambda_{H\Delta} + \lambda'_{H\Delta}\xi)^2 > 0, \\
&[(D \leq 0,\ a_3\sqrt{a_0} + a_1\sqrt{a_4} > 0) \vee (-2\sqrt{a_0 a_4} < a_2 < 6\sqrt{a_0 a_4},\ D \geq 0,\ \Lambda_1 \leq 0) \\
&\vee (6\sqrt{a_0 a_4} < a_2,\ [(a_1 > 0,\ a_3 > 0) \vee (D \geq 0,\ \Lambda_2 \leq 0)])],
\end{aligned} \tag{A.22}$$

where

$$D(\xi,\zeta,\eta) = 256a_0^3a_4^3 - 4a_1^3a_3^3 - 27a_0^2a_3^4 + 16a_0a_2^4a_4 - 6a_0a_1^2a_3^2a_4 - 27a_1^4a_4^2$$

$$
\begin{aligned}
& -192a_0^2a_1a_3a_4^2 - 4a_2^3(a_0a_3^2 + a_1^2a_4) + 18a_2(a_1a_3 + 8a_0a_4) \\
& \times(a_0a_3^2 + a_1^2a_4) + a_2^2(a_1^2a_3^2 - 80a_0a_1a_3a_4 - 128a_0^2a_4^2), \qquad \text{(A.23)} \\
\Lambda_1(\xi,\zeta,\eta) \;=\; & (a_3\sqrt{a_0} - a_1\sqrt{a_4})^2 - 32(a_0a_4)^{\frac{3}{2}} - 16\left(a_0a_2a_4 + a_0^{\frac{5}{4}}a_3a_4^{\frac{3}{4}} + a_0^{\frac{3}{4}}a_1a_4^{\frac{5}{4}}\right), \\
\Lambda_2(\xi,\zeta,\eta) \;=\; & (a_3\sqrt{a_0} - a_1\sqrt{a_4})^2 \\
& -\frac{4\sqrt{a_0a_4}(a_2 + 2\sqrt{a_0a_4})(\sqrt{a_0}a_3 + a_1\sqrt{a_4} + 4\sqrt{a_0a_4}\sqrt{a_2 - 2\sqrt{a_0a_4}})}{\sqrt{a_2 - 2\sqrt{a_0a_4}}}.
\end{aligned}
$$

(A.24)

These three expressions are intricate functions of three variables (ξ, ζ, η) needed to be minimised inside the allowed orbit space as well as on the boundary.

One may also use the conditions in which the variables are not restricted to only positive value. They are more convenient to handle compared with the conditions in Eq. (A.22). These conditions are given by

$$D(\xi,\zeta,\eta) > 0 \wedge (Q(\xi,\zeta,\eta) > 0 \;\vee\; R(\xi,\zeta,\eta) > 0) \qquad \text{(A.25)}$$

with

$$
\begin{aligned}
Q(\xi,\zeta,\eta) &= 8a_2a_4 - 3a_3^2, \\
R(\xi,\zeta,\eta) &= 64a_0a_4^3 + 16a_2a_3^2a_4 - 16a_4^2(a_2^2 + a_1a_3) - 3a_3^4. \qquad \text{(A.26)}
\end{aligned}
$$

It is easily resort to numerical analysis to find the minimum of these functions in the allowed region of orbit space.

The second condition in Eq. (A.22) is investigated within the solution I boundary. We require

$$\hat{p}_3(\xi,\zeta) \equiv 2\sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta\zeta)} + (\lambda_{H\Delta} + \lambda'_{H\Delta}\xi) > 0. \qquad \text{(A.27)}$$

Since $\hat{p}_3(\xi,\zeta)$ is monotonic in both ξ and ζ, therefore the minimum will be on the boundary solution. Let us define

$$\bar{p}_3(\xi) \equiv \hat{p}_3(\xi, 2\xi^2 - 2\xi + 1). \qquad \text{(A.28)}$$

We study the behaviour of this function by the taking the first and the second derivatives :

$$\frac{d\bar{p}_3}{d\xi} = \lambda'_{H\Delta} + \frac{2(2\xi-1)\lambda'_\Delta\lambda_H}{\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta\zeta)}}, \tag{A.29}$$

$$\frac{d^2\bar{p}_3}{d\xi^2} = \frac{2\lambda'_\Delta\lambda_H^2(2\lambda_\Delta+\lambda'_\Delta)}{\sqrt{[\lambda_H(\lambda_\Delta+\lambda'_\Delta\zeta)]^3}}. \tag{A.30}$$

In order for $\bar{p}_3$ to exhibit a minimum, it is required that $\lambda'_\Delta > 0$. Solving for ξ from $\frac{d\bar{p}_3}{d\xi} = 0$ and insert these solutions into $\bar{p}_3$, we get

$$\begin{aligned} \xi_\pm &= \frac{1}{2}\left[1 \pm \frac{1}{8\lambda'_\Delta\lambda_H - \lambda'^2_{H\Delta}}\sqrt{\lambda'^2_{H\Delta}\left(\frac{2\lambda_\Delta}{\lambda'_\Delta}+1\right)(8\lambda'_\Delta\lambda_H - \lambda'^2_{H\Delta})}\right], \\ &= \frac{1}{2}\left[1 \pm \frac{\mathrm{sgn}\,(\lambda'_{H\Delta})\lambda'_{H\Delta}}{8\lambda'_\Delta\lambda_H - \lambda'^2_{H\Delta}}\sqrt{\left(\frac{2\lambda_\Delta}{\lambda'_\Delta}+1\right)(8\lambda'_\Delta\lambda_H - \lambda'^2_{H\Delta})}\right], \end{aligned} \tag{A.31}$$

$$\begin{aligned} \bar{p}_{3\pm} &= \lambda_{H\Delta} + \frac{\lambda'_{H\Delta}}{2} + \frac{1}{2}\left(\frac{8\lambda'_\Delta\lambda_H \pm \lambda'^2_{H\Delta}\,\mathrm{sgn}\,\lambda'_{H\Delta}}{8\lambda'_\Delta\lambda_H - \lambda'^2_{H\Delta}}\right) \\ &\quad \times\sqrt{\left(\frac{2\lambda_\Delta}{\lambda'_\Delta}+1\right)(8\lambda'_\Delta\lambda_H - \lambda'^2_{H\Delta})}. \end{aligned} \tag{A.32}$$

From the second derivative we should have only one solution. To pick up the right solution, the first derivative shows us that $\lambda'_{H\Delta}$ and $2\xi-1$ should have different signs in order for the stationary solution to exist. Moreover, the solution ξ should be confined within the inverval $0 \le \xi \le 1$ which is equivalent to $\bar{p}'_3(0) < 0$ and $\bar{p}'_3(1) > 0$. That is

- $\hat{p}'_3(0) < 0$

$$\begin{aligned} \lambda'_{H\Delta} - \frac{2\lambda'_\Delta\lambda_H}{\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta)}} &< 0, \\ \frac{\lambda'_{H\Delta}\sqrt{\lambda_\Delta+\lambda'_\Delta} - 2\lambda'_\Delta\sqrt{\lambda_H}}{\sqrt{\lambda_\Delta+\lambda'_\Delta}} &< 0, \end{aligned} \tag{A.33}$$

- $\bar{p}'_3(1) > 0$

$$\lambda'_{H\Delta} + \frac{2\lambda'_\Delta\lambda_H}{\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta)}} > 0,$$

$$\frac{\lambda'_{H\Delta}\sqrt{\lambda_\Delta+\lambda'_\Delta}+2\lambda'_\Delta\sqrt{\lambda_H}}{\sqrt{\lambda_\Delta+\lambda'_\Delta}}>0. \tag{A.34}$$

Therefore from these two conditions, we have

$$2\lambda'_\Delta\sqrt{\lambda_H}>|\lambda'_{H\Delta}|\sqrt{\lambda_\Delta+\lambda'_\Delta}. \tag{A.35}$$

The next possible minima are either at $\xi=0$ or $\xi=1$. These give us

$$2\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta)}+\lambda_{H\Delta}>0,\quad 2\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta)}+(\lambda_{H\Delta}+\lambda'_{H\Delta})>0. \tag{A.36}$$

The remaining analysis is for the region bounded by solution I and solution IV.

1. Along the line $\zeta=2\xi^2-2\xi+1$, the minimum is ensured to lie in the interval $0<\xi<1/2$ by

$$\lambda'_{H\Delta}\sqrt{\lambda_\Delta+\lambda'_\Delta}<2\lambda'_\Delta\sqrt{\lambda_H},\ \lambda'_{H\Delta}>0. \tag{A.37}$$

The minimum value of $\bar{p}_3$ is given by

$$\min\bar{p}_{3\pm}=\bar{p}_->0. \tag{A.38}$$

2. Along the line $\zeta=1-2\xi^2$:

$$\frac{d\bar{p}_3}{d\xi} = \lambda'_{H\Delta}-\frac{4\xi\lambda'_\Delta\lambda_H}{\sqrt{\lambda_H(\lambda_\Delta+\lambda'_\Delta\zeta)}}, \tag{A.39}$$

$$\frac{d^2\bar{p}_3}{d\xi^2} = -\frac{4\lambda'_\Delta\lambda_H^2(\lambda_\Delta+\lambda'_\Delta)}{\sqrt{[\lambda_H(\lambda_\Delta+\lambda'_\Delta\zeta)]^3}}. \tag{A.40}$$

$\bar{p}_3$ may exhibit a minimum if $\lambda'_\Delta<0$. Solving for ξ from $\frac{d\bar{p}_3}{d\xi}=0$ and insert these solutions into $\bar{p}_3$, we get

$$\xi_\pm=\pm\sqrt{\frac{\lambda'^2_{H\Delta}}{16\lambda'_\Delta\lambda_H+2\lambda'^2_{H\Delta}}\left(\frac{\lambda_\Delta}{\lambda'_\Delta}+1\right)}. \tag{A.41}$$

The only physical solution in this scenario II is given by ξ_+, and

$$\bar{p}_{3\pm} = \lambda_{H\Delta}+\left(4\lambda_H\pm\frac{\lambda'^2_{H\Delta}\,\mathrm{sgn}(\lambda'_{H\Delta})}{2|\lambda'_\Delta|}\right)\sqrt{\frac{2\lambda'_\Delta(\lambda_\Delta+\lambda'_\Delta)}{8\lambda'_\Delta\lambda_H+\lambda'^2_{H\Delta}}}$$

$$= \lambda_{H\Delta} + \left(\frac{8\lambda_H|\lambda'_\Delta| \pm \lambda'^2_{H\Delta}\,\mathrm{sgn}(\lambda'_{H\Delta})}{2|\lambda'_\Delta|} \right) \sqrt{\frac{2\lambda'_\Delta(\lambda_\Delta + \lambda'_\Delta)}{8\lambda'_\Delta\lambda_H + \lambda'^2_{H\Delta}}}. \quad \text{(A.42)}$$

Moreover, the solution ξ should be confined within the inverval $0 < \xi < 1/2$ which is equivalent to $\bar{p}'_3(0) < 0$ and $\bar{p}'_3(1/2) > 0$. That is

- $\bar{p}'_3(0) < 0$

$$\lambda'_{H\Delta} < 0, \quad \text{(A.43)}$$

- $\bar{p}'_3(1/2) > 0$

$$\begin{aligned} \lambda'_{H\Delta} - \frac{2\lambda'_\Delta\lambda_H}{\sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta/2)}} &> 0, \\ \frac{\lambda'_{H\Delta}\sqrt{\lambda_\Delta + \lambda'_\Delta/2} - 2\lambda'_\Delta\sqrt{\lambda_H}}{\sqrt{\lambda_\Delta + \lambda'_\Delta/2}} &> 0. \end{aligned} \quad \text{(A.44)}$$

Therefore from these two conditions, we have

$$\begin{aligned} \lambda'_{H\Delta} &< 0, \\ \lambda'_{H\Delta}\sqrt{\lambda_\Delta + \lambda'_\Delta/2} &> 2\lambda'_\Delta\sqrt{\lambda_H}. \end{aligned} \quad \text{(A.45)}$$

These are sufficient and necessary conditions for the minimum inside the given interval. In order to take into account other possibilities several comments are in order:

(a) If the first condition is satisfied, i.e. $\lambda'_{H\Delta} < 0$, then strictly $\lambda'_\Delta < 0$.

(b) The opposite conditions, namely

$$\begin{aligned} \lambda'_{H\Delta} &\geq 0, \\ \lambda'_{H\Delta}\sqrt{\lambda_\Delta + \lambda'_\Delta/2} &\leq 2\lambda'_\Delta\sqrt{\lambda_H}, \end{aligned}$$

strictly allow $\lambda'_\Delta > 0$. That is the stationary point lies inside the interval and it is a maximum.

(c) For only the condition

$$\lambda'_{H\Delta}\sqrt{\lambda_\Delta + \lambda'_\Delta/2} \leq 2\lambda'_\Delta\sqrt{\lambda_H},$$

if $\lambda'_\Delta > 0$, $\lambda'_{H\Delta}$ can be either positive or negative. If $\lambda'_{H\Delta} < 0$, there will be a maximum point somewhere on the negative ξ-axis. If $\lambda'_{H\Delta} > 0$, the stationary point will be inside the given interval. On the other hand, if $\lambda'_\Delta < 0$, $\lambda'_{H\Delta}$ can only be negative. There will be a minimum point somewhere on the positive ξ-axis outside the interval.

Therefore all the possibilities can be taken into account by the following condition

$$\begin{aligned} &\lambda'_{H\Delta}\sqrt{\lambda_\Delta + \lambda'_\Delta/2} \leq 2\lambda'_\Delta\sqrt{\lambda_H} \quad \vee \quad \lambda'_{H\Delta} \geq 0 \\ &\vee \quad \bar{p}_{3+} = \lambda_{H\Delta} + \left(4\lambda_H - \frac{\lambda'^2_{H\Delta}}{2|\lambda'_\Delta|}\right)\sqrt{\frac{2\lambda'_\Delta(\lambda_\Delta + \lambda'_\Delta)}{8\lambda'_\Delta\lambda_H + \lambda'^2_{H\Delta}}} > 0. \end{aligned}$$

The other possible minima are either at $(\xi, \zeta) = (1/2, 1/2)$ or $(\xi, \zeta) = (0, 1)$ giving conditions

$$\lambda_{H\Delta} + \lambda'_{H\Delta}/2 + \sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta/2)} > 0, \qquad \lambda_{H\Delta} + \sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta)} > 0. \quad \text{(A.46)}$$

In summary, the sufficient and necessary conditions from this full analysis are given below

$$\lambda_H > 0,\ \lambda_\Delta + \frac{\lambda'_\Delta}{2} > 0,\ \lambda_\Delta + \lambda'_\Delta > 0,\ \lambda_S > 0, -|\lambda_{SH\Delta}| + 2\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} > 0,$$

$$\lambda_{S\Delta} + 2\min\left[\sqrt{\lambda_S(\lambda_\Delta + \lambda'_\Delta)}, \sqrt{\lambda_S(\lambda_\Delta + \frac{\lambda'_\Delta}{2})}\right] > 0,$$

$$\left[-|\lambda_{SH\Delta}|\sqrt{\lambda_{HS}(\lambda_{H\Delta} + \lambda'_{H\Delta})} + 2\lambda_{HS}\lambda'_{H\Delta} \leq 0 \ \vee \ 4\lambda_{HS}\lambda'_{H\Delta} - |\lambda_{SH\Delta}|^2 > 0\right],$$

$$\Big(\ [\lambda_{HS} > 0,\ \lambda_{H\Delta} > 0,\ \lambda_{H\Delta} + \lambda'_{H\Delta} > 0] \ \vee \ \big[\lambda_{HS} \leq 0, \lambda_{H\Delta} \leq 0, \lambda_{H\Delta} + \lambda'_{H\Delta} \leq 0,$$

$$2\sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta)} + (\lambda_{H\Delta} + \lambda'_{H\Delta}) > 0,\ 2\sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta)} + \lambda_{H\Delta} > 0,$$

$$
\begin{aligned}
&\lambda_{H\Delta} + \lambda'_{H\Delta}/2 + \sqrt{\lambda_H(\lambda_\Delta + \lambda'_\Delta/2)} > 0, \ \left(2\lambda'_\Delta\sqrt{\lambda_H} \leq |\lambda'_{H\Delta}|\sqrt{\lambda_\Delta + \lambda'_\Delta} \ \text{OR} \ F_1\right), \\
&\left(2\lambda'_\Delta\sqrt{\lambda_H} \leq \lambda'_{H\Delta}\sqrt{\lambda_\Delta + \lambda'_\Delta/2} \ \text{OR} \ \lambda'_{H\Delta} \leq 0 \ \text{OR} \ F_1\right), \\
&\left(2\lambda'_\Delta\sqrt{\lambda_H} \geq \lambda'_{H\Delta}\sqrt{\lambda_\Delta + \lambda'_\Delta/2} \ \text{OR} \ \lambda'_{H\Delta} \geq 0 \ \text{OR} \ F_2\right), \\
&2\sqrt{\lambda_H\lambda_S} + \lambda_{HS} > 0, \ D_{\min} > 0 \wedge (Q_{\min} > 0 \ \vee \ R_{\min} > 0)\Big]\Big)
\end{aligned} \tag{A.47}
$$

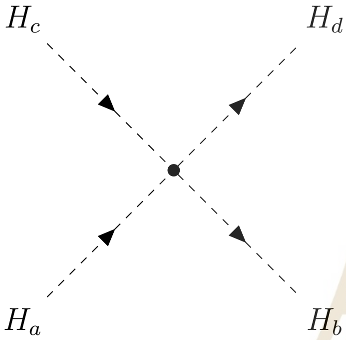
where

$$
F_1 = \lambda_{H\Delta} + \frac{\lambda'_{H\Delta}}{2} + \frac{1}{2}\sqrt{\left(\frac{2\lambda_\Delta}{\lambda'_\Delta} + 1\right)(8\lambda'_\Delta\lambda_H - \lambda'^2_{H\Delta})} > 0, \tag{A.48}
$$

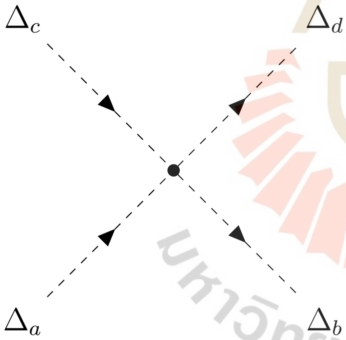
$$
F_2 = \lambda_{H\Delta} + \left(\frac{8\lambda_H|\lambda'_\Delta| - \lambda'^2_{H\Delta}}{2|\lambda'_\Delta|}\right)\sqrt{\frac{2\lambda'_\Delta(\lambda_\Delta + \lambda'_\Delta)}{8\lambda'_\Delta\lambda_H + \lambda'^2_{H\Delta}}} > 0. \tag{A.49}
$$

APPENDIX B

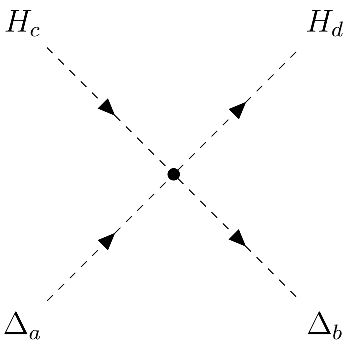
FEYNMAN RULES FOR SCALAR-SCALAR INTERACTIONS



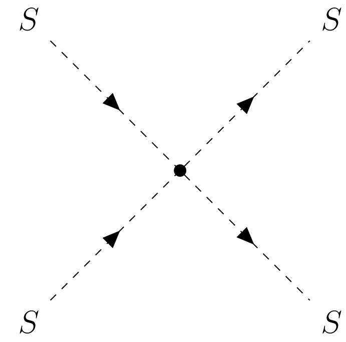
$$: \quad -i\mu^{\epsilon}2\lambda_H(\delta_{ab}\delta_{cd}+\delta_{ad}\delta_{bc}) \tag{B.1}$$



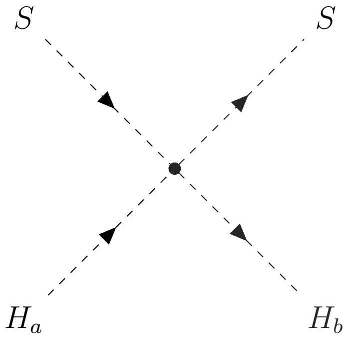
$$: \quad -i\mu^{\epsilon}2\left[(\lambda_{\Delta}+\lambda'_{\Delta})\left(\delta_{ab}\delta_{cd}+\delta_{ad}\delta_{bc}\right)-\lambda'_{H\Delta}\,\delta_{ac}\delta_{bd}\right] \tag{B.2}$$



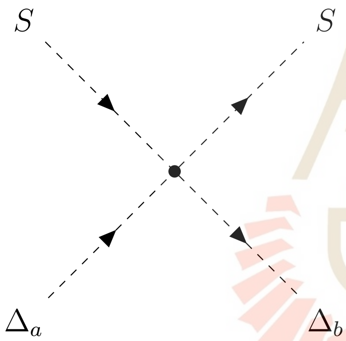
$$: \quad -i\mu^{\epsilon}\left[\left(\lambda_{H\Delta}+\frac{\lambda'_{H\Delta}}{2}\right)\delta_{ab}\delta_{cd}-\frac{\lambda'_{H\Delta}}{2}\,i\epsilon_{bam}\sigma^{m}_{dc}\right] \tag{B.3}$$



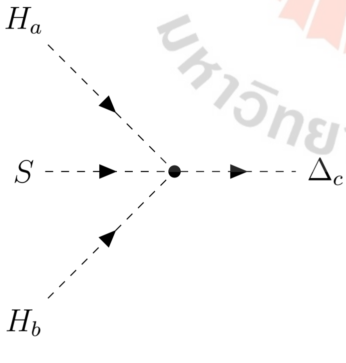
$$: \quad -i\mu^{\epsilon} 4\lambda_S \tag{B.4}$$



$$: \quad -i\mu^{\epsilon} \lambda_{HS} \delta_{ab} \tag{B.5}$$



$$: \quad -i\mu^{\epsilon} \lambda_{S\Delta} \delta_{ab} \tag{B.6}$$



$$: \quad -i\mu^{\epsilon} \frac{\sqrt{2}}{2} \lambda_{SH\Delta} (\epsilon\sigma^c)_{ab} \tag{B.7}$$

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