DIFFERENTIAL EQUATIONS

(Math 103201)

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Foreword

When a scientist, engineer or economist tries to solve a practical problem, he or she must find equations which give a mathematical description of this problem, a process which is called *mathematical modeling*. Most of the quantities which one encounters in life are quantities of change, and therefore, more often than not, the equations in a mathematical model contain derivatives and thus turn out to be differential equations.

This course is an introduction into the broad field of differential equations, covering merely the basic and most important aspects of *ordinary differential equations*. It is meant as a foundation to help you solve and understand the many more advanced equations which you may encounter in your future studies. The focus lies on the techniques for solving such equations, and numerous exercises are included in this text to help you apply and practise these techniques. Because of time constraints only a limited number of practical examples can be presented, which nevertheless show the power and omnipresence of differential equations.

Here, a note on how to study is in order. Many students like to read a mathematics book like a novel and try to memorize the formulas and examples given in the text. This is not how you should study mathematics. The lectures and textbook only introduce you to the concepts, but it is when you try to solve a problem where most of the learning takes place because then you must think about the various aspects of the problem by yourself. Do not be discouraged if first you have no idea of how to attack a problem — think about it, compare it with the examples, discuss it with your friends, or ask the tutors or the lecturer for help. As all this takes time it is important that you study regularly from the beginning of the term on; be prepared to spend at least three hours of problem solving for every lecture hour. Experience has shown that students who actively solve all the assigned exercises will succeed in this course.

You will also notice that solving a differential equation quite often requires you to integrate, and are therefore advised to review the techniques of integration which you learnt in Calculus I, right now at the beginning of the course.

Eckart Schulz
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Chapter 1

First Order Differential Equations

1.1 Introduction

Definition An equation containing an unknown function and some of its derivatives is called a *differential equation.*

e.g.

$$\frac{dy}{dx} + 2xy = e^{-x^2} \tag{1.1}$$

$$\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0 \tag{1.2}$$

$$y''' + 2e^x y'' + yy' = 4 \tag{1.3}$$

$$y''' + xy' + (\cos^2 x)y = x^3 \tag{1.4}$$

In the above examples, y is an unknown function of x. Therefore, x is called the *independent variable* and y the *dependent variable.* All these equations are called *ordinary differential equations* (ODE) because they contain only one independent variable.

e.g.

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \tag{1.5}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0 \tag{1.6}$$

$$u_t = c^2 u_{xx} \tag{1.7}$$

These equations are called *partial differential equations* (PDE) because u, v and w are functions of several variables and the equations contain their partial derivatives.

Definition The *order* of a differential equation is the order of the highest order derivative which occurs in the equation.

e.g.
- (1.1) and (1.5) are of first order
- (1.2), (1.6) and (1.7) are of second order
- (1.3) and (1.4) are of third order

Definition An ordinary differential equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x) \qquad (a_n(x) \neq 0)$$

is called a *linear equation of order n*.

A linear equation can not contain products of y and its derivatives, such as y^2, yy', $y'y''$, etc.

e.g.
- (1.1) and (1.4) are linear.
- (1.2) and (1.3) are not linear (called *nonlinear*).

In this course, we will study the most common ordinary differential equations. Just as there are many different methods to integrate, there are many different methods to solve a differential equation. Our task will be to choose the best method which solves a given differential equation.

1.2 An Easy Equation

The simplest differential equation is of the form

$$\boxed{\frac{dy}{dx} = f(x)} \tag{1.8}$$

This equation can be solved by simple integration. Its general solution is

$$y = \int f(x)\, dx + c \tag{1.9}$$

e.g.
- If $\frac{dy}{dx} = \cos x$ then $y = \sin x + c$.
- If $\frac{dy}{dx} = e^{-x^2}$ then $y = \int e^{-x^2}\, dx + c$.
  (We can not evaluate this integral !)

1.3 Separable Equations

A differential equation which can be written as

$$\boxed{\frac{dy}{dx} = \frac{g(x)}{f(y)}} \tag{1.10}$$

is called *separable*. Here, f is a function of y and g is a function of x.

e.g. The equations

$$\frac{dy}{dx} = \frac{x^2}{y^2} \qquad \text{and} \qquad y\frac{dy}{dx} = \cos y \cdot \ln x$$

are separable.

To solve a separable equation, we move all terms which contain y to the left side and all the terms which contain x to the right side of the equation. This step is called *separating the variables*,

$$f(y)\,dy = g(x)\,dx$$

Then we integrate both sides,

$$\int f(y)\,dy = \int g(x)\,dx + c.$$

Note that we have combined the two integration constants to one integration constant, which is written on the right-hand side of the equation.

Example 1 Find the solution to

$$\frac{dy}{dx} = \frac{y^2}{x^2} \tag{1.11}$$

Solution. First we separate the variables,

$$\frac{1}{y^2}\,dy = \frac{1}{x^2}\,dx$$

Then we integrate,

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x^2}\,dx + c$$

$$-\frac{1}{y} = -\frac{1}{x} + c$$

Finally, we solve for y,

$$-y = \frac{1}{-\frac{1}{x} + c}$$

We simplify this fraction, and get the general solution

$$y = \frac{x}{1 - cx} \tag{1.12}$$

□

We see that there are infinitely many solutions, depending on the value of c. All these solutions together form the *general solution*. If we fix a value for c, then we obtain a *particular solution*.

For example, if we choose $c = 1$, then we obtain the particular solution

$$y = \frac{x}{1 - x}.$$

Remark We can always test whether we have found the correct solution by substituting it into the differential equation. If we take the derivative of (1.12), then we get

$$\frac{dy}{dx} = \frac{(1-cx) - x(-c)}{(1-cx)^2} = \frac{1}{(1-cx)^2}$$

On the other hand,

$$\frac{y^2}{x^2} = \left(\frac{x}{1-cx}\right)^2 \frac{1}{x^2} = \frac{1}{(1-cx)^2}$$

This shows that we have found the correct solution of (1.11).

Example 2 Find the general solution to

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

Solution. First separate the variables,

$$(1-y^2)\,dy = x^2\,dx$$

Then integrate,

$$\int (1-y^2)\,dy = \int x^2\,dx + c$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + c$$

Multiply by 3,

$$3y - y^3 - x^3 = c_1$$

where we have set $c_1 = 3c$. This solution is called an *implicit solution* because we can not solve for y directly. □

Example 3 Find the general solution to

$$\frac{dy}{dx} = ky$$

Solution. Separate the variables

$$\frac{dy}{y} = k\,dx$$

and integrate

$$\int \frac{dy}{y} = \int k\,dx$$

so that

$$\ln|y| = kx + C \tag{1.13}$$

Exponentiate,

$$|y| = e^C e^{kx}.$$

To eliminate the absolute value, we square

$$y^2 = e^{2C} e^{2kx}$$

and take positive or negative roots,

$$y = \pm e^C e^{kx}. \tag{1.14}$$

Setting $c = \pm e^C$ we have the general solution

$$y = ce^{kx}.$$

□

Remark Some books don't bother to write the absolute value when integrating, and obtain

$$\ln y = kx + C$$

in (1.13) or

$$y = e^C e^{kx}.$$

This is not the complete solution because the solutions $y = -e^C e^{kx}$ are missing ! However, when setting $c = e^C$, these solutions reappear because we can choose $c < 0$. So the omission of the absolute value in (1.14) can be justified, although it is not good practise.

Example 4 Find the solution to

$$\frac{dy}{dx} = y \tag{1.15}$$

which satisfies

$$y(0) = -1. \tag{1.16}$$

Solution. By example 3, equation (1.15) has general solution

$$y = ce^x \tag{1.17}$$

We have one other condition : If $x = 0$ then $y = -1$. If we substitute these numbers into (1.17) then we obtain

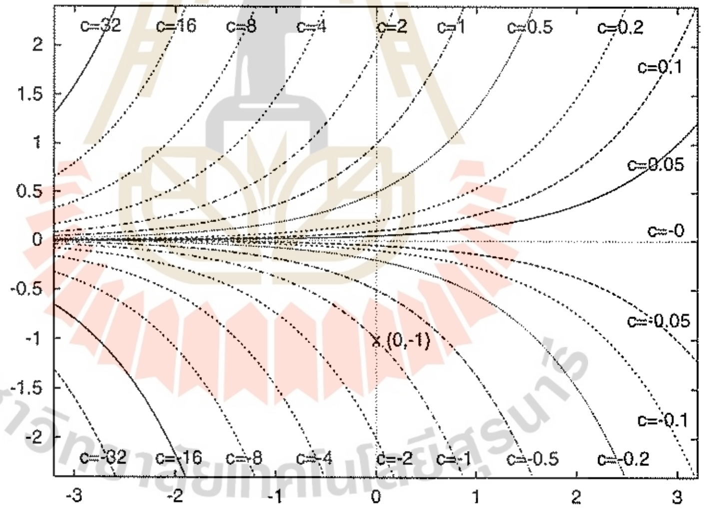
$$-1 = ce^0$$
$$c = -1$$

The particular solution which satisfies condition (1.16) is

$$y = -e^x.$$

□

If we graph all the solutions $y = ce^x$ of equation (1.15), we obtain a *one-parameter family* of curves, where the constant c is called a *parameter*. The extra condition (1.16) selects the one solution whose graph passes through the point (0,-1). Such a condition is called an *initial condition* or a *one-point boundary condition.*



The one-parameter family $y = ce^x$.

In general, a first order differential equation

$$F(x, y, y') = 0$$

together with an initial condition

$$y(x_o) = y_o$$

is called an *initial value problem* (IVP).

Example 5 Solve the initial value problem

$$\frac{dy}{dx} = \cos y \qquad y(0) = \pi.$$

Solution. It is WRONG to simply integrate and write

$$y = \int \cos y \, dx = \sin y + c$$

because on the right side we have the variable y and not x ! Instead, we separate the variables in differential equation (1.18)

$$\frac{1}{\cos y} = dx$$

Now we integrate both sides,

$$\int \sec y \, dy = \int dx + C$$

$$\ln|\sec y + \tan y| = x + C$$

and exponentiate

$$|\sec y + \tan y| = e^C e^x$$

Eliminate the absolute value,

$$\sec y + \tan y = ce^x$$

where $c = \pm e^C$. We have found the general solution of the differential equation in implicit form. Finally, the initial condition $y(0) = \pi$ gives us

$$-1 + 0 = ce^0$$

so that $c = -1$. The solution of this initial value problem is

$$\sec y + \tan y + e^x = 0.$$

□

Remark If we consider y the independent variable, and x the dependent variable, then we can rewrite the above problem as

$$\frac{dx}{dy} = \sec y \qquad x(\pi) = 0.$$

Here we can integrate both sides with respect to y and obtain

$$x = \ln|\sec y + \tan y| + C$$

Exponentiate as usual,

$$e^x = c(\sec y + \tan y)$$

where $c = \pm e^C$. Finally, the condition $x(\pi) = 0$ gives

$$e^0 = c(-1 + 0)$$

so that $c = -1$. Of course have the same solution as above,

$$e^x + \sec y + \tan y = 0.$$

We can also write a differential equation in *differential form*,

$$P(x, y)\, dx + Q(x, y)\, dy = 0$$

which is equivalent to writing

$$\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)}.$$

Example 6 Find the general solution to

$$x \sin y \, dx - (x^2+1)\cos y \, dy = 0.$$

Solution. To solve, we separate the variables:

$$\begin{aligned}(x^2+1)\cos y \, dy &= x \sin y \, dx \\ \frac{\cos y}{\sin y}\, dy &= \frac{x}{x^2+1}\, dx\end{aligned}$$

Now we integrate,

$$\begin{aligned}\int \frac{\cos y}{\sin y}\, dy &= \int \frac{x}{x^2+1}\, dx + C \\ \ln|\sin y| &= \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C\end{aligned}$$

Exponentiate,

$$|\sin y| = e^C\sqrt{x^2+1}$$

Eliminate the absolute value by squaring and taking roots,

$$\sin y = \pm e^C\sqrt{x^2+1},$$

and set $c = \pm e^C$ to obtain the general solution

$$\sin y = c\sqrt{x^2+1}. \tag{1.18}$$

□

Remark If we take the inverse sine function in (1.18) and write

$$y = \sin^{-1} c\sqrt{x^2+1}$$

then we loose some of the solutions because the sine function is not one-to-one. Therefore, we leave solution (1.18) in implicit form.

Exercises

1. Solve the following differential equations:
   - (a) $y' = e^{3x} - x$
   - (b) $xy' = 1$
   - (c) $y' + y\tan x = 0$
   - (d) $y' - y\tan x = 0$
   - (e) $y \ln y \, dx - x\, dy = 0$
   - (f) $(1+x^2)\, y' = \tan^{-1} x$
   - (g) $y' + 2xy = 0$
   - (h) $\frac{dy}{dx} = y \sin x$
   - (i) $y' + 2xy^2 = 0$
   - (j) $(1+x)\frac{dy}{dx} = 4y$
   - (k) $2\sqrt{x}\frac{dy}{dx} = \sqrt{1-y^2}$
   - (l) $y' = \sqrt[3]{64xy}$
   - (m) $xyy' = y - 1$
   - (n) $(1+x^2)\, dy + (1+y^2)\, dx = 0$
   - (o) $x^5 y' + y^5 = 0$
   - (p) $y' \sin y = x^2$
   - (q) $(y^2-1)x\, dx + (x+2)y\, dy = 0$
   - (r) $\tan\theta \, dr + 2r\, d\theta = 0$

2. Solve the following initial value problems.
   - (a) $y' = xe^x$ $\quad y(1) = 3$
   - (b) $y' = e^{3x-2y}$ $\quad y(0) = 0$
   - (c) $x(x^2-4)y' = 1$ $\quad y(1) = 0$
   - (d) $xyy' = (x+1)(y+1)$ $\quad y(1) = 0$
   - (e) $8\cos^2 y\, dx + \csc^2 x\, dy = 0$ $\quad y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

1.3.1 Homogeneous Equations

A differential equation of the form

$$\boxed{\frac{dy}{dx} = F\left(\frac{y}{x}\right)} \tag{1.19}$$

is called a *homogeneous equation.*

e.g. The equation

$$\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

is homogeneous.

A homogeneous equation can be solved by setting

$$\boxed{y = vx}$$

Taking derivatives and using the product rule,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting into (1.19) we obtain

$$v + x\frac{dv}{dx} = F(v)$$

which is a separable equation. To see this, let us separate the variables,

$$x\frac{dv}{dx} = F(v) - v$$

$$\frac{1}{F(v)-v}dv = \frac{1}{x}\,dx$$

This equation can now be solved by integration.

Example 1 Solve the equation

$$\frac{dy}{dx} = \frac{x+y}{x-y} \tag{1.20}$$

Solution. If we rewrite the equation as

$$\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} \tag{1.21}$$

we see that it is homogeneous. We therefore substitute

$$y = vx$$

Taking derivatives,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting into (1.21) we obtain

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

Now we separate the variables,

$$x\,\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}\,dv = \frac{1}{x}\,dx$$

To integrate, we split the first term,

$$\left(\frac{1}{1+v^2} - \frac{1}{2}\,\frac{2v}{1+v^2}\right) dv = \frac{1}{x}\,dx$$

and integrate,

$$\tan^{-1} v - \frac{1}{2}\ln\left(1+v^2\right) = \ln|x| + c$$

Now resubstitute $v = \frac{y}{x}$,

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln\sqrt{1+\left(\frac{y}{x}\right)^2} + \ln|x| + c$$

and simplify to obtain the general solution (in implicit form),

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln\sqrt{x^2+y^2} + c$$

□

Sometimes it is difficult to see whether a differential equation is homogeneous. For example, equation (1.20) is homogeneous, but we only notice this when the equation is written in form (1.21). We will now look for a criterion to determine whether an equation is homogeneous.

Definition A function $f(x,y)$ is *homogeneous of degree n* if

$$f(tx,ty) = t^n f(x,y) \tag{1.22}$$

e.g.
- $f(x,y) = x^2 + xy$. Since

  $$f(tx,ty) = (tx)^2 + (tx)(ty) = t^2(x^2+xy) = t^2 f(x,y)$$

  this function is homogeneous of degree 2.
- $g(x,y) = \sqrt{x^2+y^2}$. Since

  $$g(tx,ty) = \sqrt{(tx)^2+(ty)^2} = tg(x,y) \qquad (t \geq 0)$$

  this function is homogeneous of degree 1.
- $h(x,y) = x\sqrt{y} + y\sqrt{x}$. Since

  $$h(tx,ty) = (tx)\sqrt{ty} + (ty)\sqrt{tx} = t^{\frac{3}{2}}h(x,y) \qquad (t \geq 0)$$

  this function is homogeneous of degree $\frac{3}{2}$.

- $F(x,y) = x^2y + xy$ is *not* homogeneous. In fact,

  $$F(tx,ty) = (tx)^2(ty) + (tx)(ty) = t^2(tx^2y + y^2) \neq t^2F(x,y)$$

  This function is not homogeneous because the two terms have different exponents: We think of x^2y as a term with exponent $2+1=3$, while y^2 has exponent 2 only.

Remark Careful: We are using the word *homogeneous* in two situations:

1. A *homogeneous equation* is a differential equation of form (1.19).
2. A *homogeneous function* is a function that satisfies condition (1.22).

Theorem 1 *Consider the equation*

$$P(x,y)\,dx + Q(x,y)\,dy = 0. \tag{1.23}$$

If P and Q are homogeneous functions of the same degree, then (1.23) is a homogeneous equation.

Proof: Let us rewrite this equation as

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$$

Because P and Q are homogeneous of the same degree n, we can now write

$$\frac{dy}{dx} = -\frac{P(x, x\,\frac{y}{x})}{Q(x, x\,\frac{y}{x})} = -\frac{x^nP(1,\frac{y}{x})}{x^nQ(1,\frac{y}{x})} = -\frac{P(1,\frac{y}{x})}{Q(1,\frac{y}{x})}$$

Now we set

$$F(s) = -\frac{P(1,s)}{Q(1,s)}$$

and obtain

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

which is a homogeneous equation.

Example 2 Solve the initial value problem

$$(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0 \tag{1.24}$$

$$y(1) = 0.$$

Solution. Here,

$$P(x,y) = x^2 + 3xy + y^2 \qquad \text{and} \qquad Q(x,y) = -x^2$$

Both functions are homogeneous of degree two. Therefore, equation (1.24) is homogeneous. We can rewrite it as

$$\frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2} = 1 + 3\frac{y}{x} + \frac{y^2}{x^2} \tag{1.25}$$

Now set

$$y = vx$$

so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute into (1.25) and obtain

$$v + x\frac{dv}{dx} = 1 + 3v + v^2$$

Separate the variables

$$x\frac{dv}{dx} = (1+v)^2$$

$$\frac{1}{(1+v)^2}\,dv = \frac{1}{x}\,dx$$

and integrate

$$\frac{-1}{1+v} = \ln|x| + c$$

$$1 + v = \frac{-1}{\ln|x| + c}$$

Now solve for v,

$$v = \frac{-1}{\ln|x| + c} - 1$$

and resubstitute,

$$\frac{y}{x} = \frac{-1}{\ln|x| + c} - 1$$

Multiply by x to obtain the general solution

$$y = \frac{-x}{\ln|x| + c} - x$$

Finally, we use the initial condition: if $x = 1$ then $y = 0$, so that

$$0 = \frac{-1}{\ln 1 + c} - 1$$

$$c = -1$$

The particular solution of this initial value problem is

$$y = \frac{x}{1 - \ln x} - x.$$

□

Remark When solving an initial value problem we are only interested in a solution which is defined on an interval. The largest interval on which the above solution is defined and which contains $x = 1$ is $(0, \infty)$. This is why we have omitted the absolute value in $\ln x$.

Exercises

1. Solve the following equations

   (a) $(x+y)y' = x - y$

   (b) $xy' = y + 2(xy)^{1/2}$

   (c) $2xyy' = x^2 + 2y^2$

   (d) $xy' = x + y$

   (e) $xy' = 2x + 3y$

   (f) $(x^2 - 2y^2)\,dx + xy\,dy = 0$

   (g) $x^2y' - 3xy - 2y^2 = 0$

   (h) $x \sin\dfrac{y}{x}\, y' = y \sin\dfrac{y}{x} + x$

   (i) $xy' = y + 2xe^{-y/x}$

   (j) $xy' = \sqrt{x^2 + y^2}$

   (k) $(\sqrt{x+y} + \sqrt{x-y})\,dx + (\sqrt{x-y} - \sqrt{x+y})\,dy = 0$

   Supplementary exercises:

2. Show that a substitution $z = ax + by + c$ changes

$$y' = f(ax + by + c)$$

   into a separable equation. Use this idea to solve

   (a) $y' = (4x + y)^2$

   (b) $(x+y)y' = 1$

   (c) $y' = \sin^2(x - y + 1)$

   (d) $y' = (x + y + 1)^{1/2}$

3. Consider the equation

$$\frac{dy}{dx} = F\Big(\frac{ax + by + c}{dx + ey + f}\Big)$$

   (a) If $ae \neq bd$ then find constants h and k such that the substitution

$$x = z - h \qquad y = w - k$$

   changes $\frac{ax+by+c}{dx+ey+f}$ to an expression $G(\frac{w}{z})$. Use this substitution to obtain a homogeneous equation

$$\frac{dw}{dz} = F\Big(G\big(\frac{w}{z}\big)\Big)$$

   (b) If $ae = bd$ then show that the substitution

$$v = ax + by$$

   gives a separable equation

$$f(v)\,dv = g(x)\,dx$$

   (c) Use this to solve the following equations

   i. $\dfrac{dy}{dx} = \dfrac{x + y + 4}{x - y - 6}$

   ii. $\dfrac{dy}{dx} = \dfrac{x + y + 4}{x + y - 6}$

   iii. $\dfrac{dy}{dx} = \dfrac{x + y - 1}{x + 4y + 2}$

   iv. $(2x - 2y)\,dx + (y - 1)\,dy = 0$

   v. $(2x + 3y - 1)\,dx - (4x + 1)\,dy = 0$

1.4 Linear Equations

The general form of a *first order linear differential equation* is

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

where $a_1(x) \neq 0$. If we divide by $a_1(x)$ we obtain the equation in *standard form*,

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)} \tag{1.26}$$

where $p = \frac{a_0}{a_1}$ and $q = \frac{b}{a_1}$.

e.g.

$$\frac{dy}{dx} + (\tan x)\, y = \sin 2x$$

is a first order linear equation in standard form.

To find the solution of equation (1.26), we multiply it by

$$h(x) \;=\; e^{\int p(x)\, dx} \tag{1.27}$$

and obtain

$$h(x)\frac{dy}{dx} \;+\; h(x)p(x)\, y \;=\; q(x)h(x). \tag{1.28}$$

Because the derivative of $h(x)$ is

$$\frac{dh}{dx} \;=\; \frac{d}{dx}\left[e^{\int p(x)\, dx}\right] \;=\; e^{\int p(x)\, dx}\, p(x) \;=\; h(x)p(x),$$

(we have used the chain rule), equation (1.28) really is of the form

$$h(x)\frac{dy}{dx} \;+\; \frac{dh}{dx}\, y \;=\; q(x)h(x). \tag{1.29}$$

A good look shows that the left-hand side is the derivative of the product $h(x)y$:

$$\frac{d}{dx}\left[\, h(x)\, y\, \right] \;=\; q(x)h(x).$$

We can now integrate

$$h(x)\, y \;=\; \int q(x)h(x)\, dx + c$$

and solve for y,

$$y \;=\; \frac{1}{h(x)}\left[\int q(x)h(x)\, dx + c\,\right].$$

We resubstitute (1.27) and obtain the general solution to the linear equation (1.26):

$$\boxed{y = e^{-\int p(x)\, dx}\left[\int q(x)e^{\int p(x)\, dx}\, dx + c\,\right]} \tag{1.30}$$

Remark The function $h(x)$ is called an *integrating factor* because it allows us to integrate the left-hand side of equation (1.28). We usually do not use formula (1.30) because it looks complicated. Instead, we do the steps which have led us to the formula:

1. Write the equation in standard form.
2. Multiply each side by the integrating factor
$$h(x) = e^{\int p(x)\, dx}$$
3. Write the left-hand side as the derivative of a product:
$$\frac{d}{dx}\,[h(x)y] = q(x)h(x).$$
4. Integrate both sides.
5. Solve for y.

Example 1 Solve the linear differential equation

$$y' + 3y = 2xe^{-3x} \tag{1.31}$$

Solution. Here, $p(x) = 3$. To find an integrating factor, first integrate

$$\int p(x)\, dx = \int 3\, dx = 3x$$

(Because we only need *one* integrating factor, we could choose $c = 0$ as integrating constant) and then exponentiate,

$$h(x) = e^{\int p(x)\, dx} = e^{3x}.$$

Multiply equation (1.31) by the integrating factor $h(x)$,

$$e^{3x}y' + 3e^{3x}y = 2x.$$

We recognize the left-hand side as the derivative of $e^{3x}y$,

$$\frac{d}{dx}\left[e^{3x}y\right] = 2x.$$

Now integrate,

$$e^{3x}y = x^2 + c$$

and solve for y to obtain the general solution

$$y = x^2e^{-3x} + ce^{-3x}.$$

□

Example 2 Solve the differential equation

$$\frac{dy}{dx} + \frac{1}{x}y = 3x. \tag{1.32}$$

Solution. This is a linear equation with $p(x) = \frac{1}{x}$. To find the integrating factor we compute

$$\int p(x)\, dx = \int \frac{1}{x}\, dx = \ln|x|$$

Now exponentiate,

$$h(x) = e^{\ln|x|} = |x|$$

For the purpose of an integrating factor we may drop the absolute value, and simply multiply (1.32) by $h(x) = x$,

$$x\frac{dy}{dx} + y = 3x^2.$$

We recognize the left-hand side as the derivative of the product xy,

$$\frac{d}{dx}[xy] = 3x^2$$

Integrate,

$$xy = \int 3x^2\, dx + c = x^3 + c$$

The general solution is

$$y = x^2 + \frac{c}{x}.$$

□

Example 3 Solve the initial value problem

$$y' + y(\tan x) = \sin 2x \qquad y(0) = 1 \tag{1.33}$$

Solution. This is a linear equation with $p(x) = \tan x$. First compute the integrating factor,

$$\int p(x)\,dx = \int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \ln|\sec x|$$

$$h(x) = e^{\int p(x)\,dx} = e^{\ln|\sec x|} = |\sec x|$$

For the purposes of an integrating factor we may drop the absolute value. So we multiply the differential equation in (1.33) by $h(x) = \sec x$ and obtain

$$\begin{aligned}(\sec x)\,y' + (\sec x \tan x)\,y &= \sin 2x\,\sec x\\ (\sec x)\,y' + (\sec x)'y &= \frac{2\sin x\cos x}{\cos x}\\ \big[(\sec x)\,y\big]' &= 2\sin x\end{aligned}$$

We integrate,

$$(\sec x)\,y = -2\cos x + c$$

and multiply by $\cos x$ to obtain the general solution

$$y = -2\cos^2 x + c\cos x.$$

Finally, use the initial condition: if $x = 0$ the $y = 1$. Then,

$$\begin{aligned}1 &= -2\cos^2 0 + c\cos 0\\ c &= 3\end{aligned}$$

The solution of this initial value problem is

$$y = 3\cos x - 2\cos^2 x.$$

□

1.4.1 Bernoulli Equations

An equation

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)y^n} \tag{1.34}$$

where n is a real number, is called a *Bernoulli equation.* We exclude the case where $n = 0$ or $n = 1$, because then the equation is already linear.

A Bernoulli equation can always be changed into a linear equation as follows:

Multiply (1.34) by y^{-n},

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x) \tag{1.35}$$

Now change the independent variable and set

$$u = y^{1-n}.$$

By the chain rule,

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Substitute into (1.35) and get

$$\frac{1}{1-n}\frac{du}{dx} + p(x)u = q(x)$$

Multiply by $1-n$ and get

$$\boxed{\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)}$$

which is now a linear equation.

Example 4 Solve

$$\frac{dy}{dx} + y = xy^3$$

Solution. This is a Bernoulli equation. (n=3). So first multiply by y^{-3},

$$y^{-3}\frac{dy}{dx} + y^{-2} = x \tag{1.36}$$

Now set

$$u = y^{-2}$$

By the chain rule,

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$$

or

$$-\frac{1}{2}\frac{du}{dx} = y^{-3}\frac{dy}{dx}$$

Substituting into (1.36) we get

$$-\frac{1}{2}\frac{du}{dx} + u = x$$

Multiply by -2,

$$\frac{du}{dx} - 2u = -2x$$

This is now a linear equation. To solve it, we use the integrating factor

$$e^{\int -2\,dx} = e^{-2x}$$

and obtain

$$e^{-2x}\frac{du}{dx} - 2e^{-2x}u = -2xe^{-2x}$$

$$\frac{d}{dx}\left[e^{-2x}u\right] = -2xe^{-2x}$$

Integrate, using integration by parts,

$$\begin{aligned} e^{-2x}u &= \int -2xe^{-2x}\,dx + c \\ &= \frac{1}{2}(2x+1)e^{-2x} + c \end{aligned}$$

Now solve for u,

$$u = x + \frac{1}{2} + ce^{2x}$$

Finally, we resubstitute $u = y^{-2}$ and obtain the general solution,

$$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$$

or in explicit form,

$$y = \pm \frac{\sqrt{2}}{2x + 1 + c_1 e^{2x}} \qquad (c_1 = 2c)$$

□

Exercises

1. Solve the following linear equations:
   (a) $y' + y = 1$
   (b) $y' + y = e^{-x}$
   (c) $y' - 2y = e^{3x}$
   (d) $xy' + y = \cos x$
   (e) $x\dfrac{dy}{dx} - 3y = x^4$
   (f) $y' + y = \dfrac{1}{1 + e^{2x}}$
   (g) $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$
   (h) $y' + y = 2xe^{-x} + x^2$
   (i) $y' + y \cot x = 2x \csc x$
   (j) $(2y - x^3)\, dx = x\, dy$
   (k) $y - x + xy \cot x + xy' = 0$
   (l) $\dfrac{dy}{dx} - 2xy = 6xe^{x^2}$
   (m) $(x \ln x)y' + y = 3x^3$
   (n) $(y - 2xy - x^2)\, dx + x^2\, dy = 0$

2. Solve the following initial value problems:
   (a) $y' + 2y = 2, \quad y(0) = 1$
   (b) $xy' - y = x, \quad y(1) = 2$
   (c) $y' = (1 - y)\cos x, \quad y(\pi) = 0$
   (d) $xy' + 3y = 2x^5, \quad y(2) = 1$
   (e) $y' = 1 + x + y + xy, \quad y(0) = 0$
   (f) $(x^2 + 4)y' + 3xy = x, \quad y(0) = 1$

3. Solve the following equations:
   (a) $y^2y' + 2xy^3 = 6x$
   (b) $xy' + 6y = 3xy^{4/3}$
   (c) $3y^2y' + y^3 = e^{-x}$
   (d) $xy' + y = x^4y^3$
   (e) $xy^2y' + y^3 = x\cos x$
   (f) $x\, dy + y\, dx = xy^2\, dx$
   (g) $2xy' + y^3e^{-2x} = 2xy$

4. Solve the following equations using appropriate substitutions:
   (a) $xe^yy' = 2(e^y + x^3e^{2x})$
   (b) $(2x \sin y \cos y)y' = 4x^2 + 3\sin^2 y$
   (c) $(x + e^y)y' = xe^{-y} - 1$

5. Which of the equations in section 1.3, exercise 1 and section 1.3.1, exercise 1 are linear ? If an equation is linear, solve it by the method of this section.

   Supplementary exercises:

6. A differential equation of the form

$$y' = p(x)y^2 + q(x)y + r(x)$$

is called a *Riccati equation.*

(a) Show that if we know one solution y_1 of this equation, then the substitution

$$y = y_1 + \frac{1}{v}$$

will transform it into a linear equation

$$v' + [q(x) + 2p(x)y_1]v = -p(x).$$

(b) Show that $y_1 = x$ is a particular solution of each of the following equations, and then find the general solution by applying the above substitution.

i. $y' + y^2 = 1 + x^2$ ii. $y' + 2xy = 1 + x^2 + y^2$

7. One of the solutions of $y' \sin 2x = 2y + 2\cos x$ remains bounded as $x \to \pi/2$. Find it.

1.5 Exact Equations

Assume we have a family of curves,

$$f(x, y) = c \tag{1.37}$$

If we take the *total differential* on both sides we obtain the differential equation

$$df = dc$$

or

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0 \tag{1.38}$$

Therefore, the solution of the differential equation (1.38) is $f(x, y) = c$.

e.g. The family

$$x^2y^3 = c$$

is the solution to the differential equation

$$2xy^3\,dx + 3x^2y^2\,dy = 0$$

because $\frac{\partial}{\partial x}(x^2y^3) = 2xy^3$ and $\frac{\partial}{\partial y}(x^2y^3) = 3x^2y^2$.

Now suppose we are given a differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1.39}$$

If we can find a function $f(x, y)$ which satisfies

$$\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y) \tag{1.40}$$

then by the above comments, the solution must be of the form

$$f(x, y) = c.$$

If such a function exists, then we call equation (1.39) an *exact differential equation*, and

$$M(x, y)\,dx + N(x, y)\,dy$$

an *exact differential*.

Example 1 Is the equation

$$y\ dx + x\ dy \ = \ 0 \tag{1.41}$$

exact ? Try the function $f(xy) = xy$.

$$\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = x.$$

We see that equation (1.41) can be written as

$$\frac{\partial f}{\partial x}\, dx \ + \ \frac{\partial f}{\partial y}\, dy = 0$$

so that it is indeed an exact equation. Its general solution must be

$$f(x, y) = c$$

that is,

$$xy = c.$$

□

So to solve equation (1.39), we must

1. determine whether the equation is exact, and if it is exact,
2. find the function $f(x, y)$.

The next theorem answers the first question:

Theorem 2 *Suppose, M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on a rectangle in the xy-plane. Then the equation*

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Remark The functions we are usually using are all continuous as required. The words "if and only if" have the following meaning:

- **If** $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact, **then** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- **If** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ **then** $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact.

Once we have determined whether an equation is exact, we can integrate one of the two equations in (1.40) to find the function $f(x, y)$.

Example 2 Find the general solution of

$$(3x^2 + 4xy)\, dx + (2x^2 + 2y)\, dy \ = \ 0$$

Solution. First check whether this is an exact equation. Here,

$$M(x,y) = 3x^2 + 4xy \qquad \text{so that} \qquad \frac{\partial M}{\partial y} = 4x$$
$$N(x,y) = 2x^2 + 2y \qquad \text{so that} \qquad \frac{\partial N}{\partial x} = 4x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this is an exact equation.
The general solution of an exact equation is $f(x,y) = c$ where

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x,y) \tag{1.42}$$

To find f, let us integrate the function $M(x,y)$,

$$\begin{aligned} f(x,y) &= \int M(x,y)\,dx + g(y) \\ &= \int (3x^2 + 4xy)\,dx + g(y) \\ &= x^3 + 2x^2y + g(y) \end{aligned}$$

Where does the term $g(y)$ come from ? Because we are integrating with respect to x, the word "constant" means independent of x only; the integration "constant" may really be a function of y. To determine the value of this function $g(y)$, we make use of the second equation in (1.42). We take the derivative of $f(x,y)$ with respect to y,

$$\frac{\partial f}{\partial y} = 2x^2 + g'(y)$$

and compare it with N. We obtain

$$2x^2 + g'(y) = \frac{\partial f}{\partial y} = N(x,y) = 2x^2 + 2y$$

Thus,

$$g'(y) = 2y$$

and integrating,

$$g(y) = y^2 + c_1. \tag{1.43}$$

Therefore, $f(x,y) = x^3 + 2x^2y + y^2 + c_1$ and the general solution is

$$x^3 + 2x^2y + y^2 + c_1 = c$$

Combining the two constants into one, we get

$$x^3 + 2x^2y + y^2 = \tilde{c}.$$

□

Remark Because the integration constant c_1 can be combined with the constant c to a new constant $\tilde{c}$, we may omit to write the constant c_1 in step (1.43) .

Example 3 Solve the initial value problem

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0 \qquad y(0) = 2$$

Solution. First check whether the equation is exact.

$$M(x,y) = 2x\cos y + 3x^2y \qquad \frac{\partial M}{\partial y} = -2x\sin y + 3x^2$$

$$N(x,y) = x^3 - x^2\sin y - y \qquad \frac{\partial N}{\partial x} = 3x^2 - 2x\sin y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this is an exact equation.
The general solution of an exact equation is

$$f(x,y) = c$$

where

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x,y)$$

To find f, we integrate M (we could also integrate N!)

$$\begin{aligned} f(x,y) &= \int M(x,y)\,dx + g(y) \\ &= \int (2x\cos y + 3x^2y)\,dx + g(y) \\ &= x^2\cos y + x^3y + g(y) \end{aligned}$$

To determine g, take the derivative of f with respect to y,

$$\frac{\partial f}{\partial y} = -x^2\sin y + x^3 + g'(y)$$

and compare it to N,

$$\frac{\partial f}{\partial y} = N(x,y) = x^3 - x^2\sin y - y$$

We see that

$$g'(y) = -y$$

$$g(y) = -\frac{y^2}{2}$$

The general solution is

$$x^2\cos y + x^3y - \frac{y^2}{2} = c$$

Finally, we must find the particular solution which satisfies the initial condition $y(0) = 2$. Substituting these values into the solution we get:

$$0 + 0 - \frac{4}{2} = c$$

$$c = -2$$

The solution is

$$x^2\cos y + x^3y - \frac{y^2}{2} = -2.$$

□

Example 4 Find the general solution to

$$e^{x^2y}(1+2x^2y)\,dx + x^3e^{x^2y}\,dy \;=\; 0$$

Solution. First check whether the equation is exact.

$$M(x,y) = e^{x^2y}(1+2x^2y) \qquad \text{and} \qquad N(x,y) = x^3e^{x^2y}$$

Taking partial derivatives,

$$\begin{aligned}\frac{\partial M}{\partial y} &= e^{x^2y}x^2(1+2x^2y) + e^{x^2y}2x^2 &= e^{x^2y}(3x^2+2x^4y)\\ \frac{\partial N}{\partial x} &= 3x^2e^{x^2y} + x^3e^{x^2y}2xy &= e^{x^2y}(3x^2+2x^4y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this is an exact equation.

The general solution of an exact equation is

$$f(x,y) = c$$

where

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x,y).$$

To find f we could integrate M,

$$f(x,y) \;=\; \int M(x,y)\,dx + g(y) \;=\; \int e^{x^2y}(1+2x^2y)\,dx + g(y)$$

However, this looks quite difficult (although it is not impossible to do!). We better integrate the function N with respect to y,

$$f(x,y) \;=\; \int N(x,y)\,dy + h(x) \;=\; \int x^3e^{x^2y}\,dy + h(x) \;=\; xe^{x^2y} + h(x).$$

To find h, we must now compare the derivative $\frac{\partial f}{\partial x}$ with M,

$$\frac{\partial f}{\partial x} \;=\; e^{x^2y} + 2x^2ye^{x^2y} + h'(x) \;=\; e^{x^2y}(1+2x^2y) + h'(x).$$

The comparison with $M(x,y)$ shows that

$$h'(x) = 0 \quad \Rightarrow \quad h(x) = 0.$$

The general solution is

$$xe^{x^2y} \;=\; c.$$

□

Remark If we solve for y explicitly, then we obtain

$$y \;=\; \frac{1}{x^2}\ln\left(\frac{c}{x}\right)$$

If $x > 0$ then c must be > 0, and this can be written as

$$y \;=\; \frac{\tilde{c} - \ln x}{x^2} \qquad (\,\tilde{c} = \ln c\,)$$

If $x < 0$ then c must be < 0 and this can be written as

$$y \;=\; \frac{1}{x^2}\ln\left(\frac{-c}{-x}\right) \;=\; \frac{\tilde{c} - \ln(-x)}{x^2} \qquad (\,\tilde{c} = \ln(-c)\,)$$

Combining both cases we get

$$y \;=\; \frac{\tilde{c} - \ln|x|}{x^2}.$$

1.5.1 Integrating Factors

An equation which is not exact can sometimes be changed to an exact equation. Let us first explain this idea by an example.

Example 1 Consider the equation

$$y\,dx + (x^2y - x)\,dy \;=\; 0 \tag{1.44}$$

Is this equation exact ?

$$M(x,y) = y \qquad \text{so that} \qquad \frac{\partial M}{\partial y} = 1$$
$$N(x,y) = x^2y - x \qquad \text{so that} \qquad \frac{\partial N}{\partial x} = 2xy - 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this equation is not exact.
Now look what happens when we multiply this equation by $\frac{1}{x^2}$: We obtain

$$\frac{y}{x^2}\,dx + (y - \frac{1}{x})\,dy \;=\; 0. \tag{1.45}$$

Since

$$\frac{\partial}{\partial y}(\frac{y}{x^2}) = \frac{1}{x^2} \qquad \text{and} \qquad \frac{\partial}{\partial x}(y - \frac{1}{x}) = \frac{1}{x^2},$$

this is now an exact equation ! We know that its solution is $f(x,y) = c$ where

$$f(x,y) \;=\; \int \frac{y}{x^2}\,dx + g(y) \;=\; -\frac{y}{x} + g(y)$$

To find g, we compare $\frac{\partial f}{\partial y}$ and the dy-term in (1.45),

$$\frac{\partial f}{\partial y} \;=\; -\frac{1}{x} + g'(y) \;=\; y - \frac{1}{x}.$$

Therefore,

$$g'(y) \;=\; y$$
$$g(y) \;=\; \frac{y^2}{2}$$

The general solution to equation (1.44) is

$$-\frac{y}{x} + \frac{y^2}{2} \;=\; c.$$

□

In the above example, the given equation (1.44) is not exact, but can be changed to the exact equation (1.45) by multiplying with the function $\frac{1}{x^2}$. This function is therefore called an *integrating factor*.

Definition Consider a differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1.46}$$

A function $F(x,y)$ such that

$$FM\,dx + FN\,dy = 0 \tag{1.47}$$

is exact is called an *integrating factor.*

Most differential equations can be made exact in this way, however, it is usually difficult to find an integrating factor $F(x,y)$. In the following we will discuss two special cases where an integrating factor can easily be found.

Let us assume that $F(x,y)$ is an integrating factor for (1.46). Because (1.47) is exact, we know by theorem 2 that

$$(FM)_y = (FN)_x$$

By the product rule,

$$F_yM + FM_y = F_xN + FN_x \tag{1.48}$$

- Suppose, $F = F(x)$ is a function of x only. Then $F_y = 0$ and (1.48) becomes

$$F_xN = FM_y - FN_x$$

$$\frac{dF}{dx}\frac{1}{F} = \frac{M_y - N_x}{N} \tag{1.49}$$

Since $\frac{dF}{dx}\frac{1}{F}$ is a function of x alone we see that

$$\frac{M_y - N_x}{N} \tag{1.50}$$

is also a function of x alone.

Now we reverse the argument: Assume, (1.50) is a function of x alone. Let $F = F(x)$ be any solution of equation (1.49). Then (1.48) is true which shows that (1.47) is an exact equation. That is, F is an integrating factor for (1.46).

To find a solution of equation (1.49) we set

$$A(x) = \frac{M_y - N_x}{N}$$

and separate the variables:

$$\frac{1}{F}\,dF = A(x)\,dx$$

Integrating (we choose the integrating constant $c = 0$), we obtain

$$\begin{aligned} \ln F &= \int A(x)\,dx \\ F &= e^{\int A(x)\,dx} \end{aligned}$$

- If $F = F(y)$ is a function of y alone, we can obtain a similar formula, which is shown in the next theorem:

Theorem 3 *Consider an equation*

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

a) If

$$A(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

is a function of x only, then

$$F(x) = e^{\int A(x)\,dx}$$

is an integrating factor.

b) If

$$B(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M}$$

is a function of y only, then

$$F(y) = e^{\int B(y)\,dy}$$

is an integrating factor.

Example 2 Find the general solution to the equation

$$(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0 \tag{1.51}$$

Solution. First check whether this equation is exact.

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3xy + y^2) &= 3x + 2y \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + xy) &= 2x + y \end{aligned}$$

So the equation is not exact. We must look for an integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = (3x + 2y) - (2x + y) = x + y$$

Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x + y}{x^2 + xy} = \frac{1}{x}$$

is a function of x alone. By the theorem, we obtain an integrating factor

$$F(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$

(To be precise, we get $f(x) = e^{\ln|x|} = |x|$, but for the purpose of an integrating factor we may omit the absolute value.) Multiply equation (1.51) by x, and obtain the exact equation

$$(3x^2y + xy^2)\,dx + (x^3 + x^2y)\,dy = 0 \tag{1.52}$$

Its solution is $f(x,y) = c$ where

$$f(x,y) = \int (3x^2y + xy^2)\,dx + g(y) = x^3y + \frac{x^2y^2}{2} + g(y)$$

Differentiate, and compare with the dy-coefficient in (1.52),

$$\frac{\partial f}{\partial y} = x^3 + x^2 y = x^3 + x^2 y + g'(y).$$

Therefore,

$$\begin{aligned} g'(y) &= 0 \\ g(y) &= 0 \end{aligned}$$

The general solution to equation (1.51) is

$$x^3 y + \frac{x^2 y^2}{2} = c$$

□

Example 3 Solve the equation

$$2xy\,dx + (4y + 3x^2)\,dy = 0 \tag{1.53}$$

Solution. First we check whether the equation is exact.

$$\frac{\partial M}{\partial y} = 2x \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 6x$$

So this equation is not exact. Let us try to find an integrating factor. Now

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4x}{4y + 3x^2}$$

contains both variables, x and y, and we can not use it to get an integrating factor. However,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{-4x}{-2xy} = \frac{2}{y}$$

is a function of y only. Therefore, we can use the integrating factor

$$F(y) = e^{\int \frac{2}{y}\,dy} = e^{2\ln y} = y^2$$

We multiply equation (1.53) by this function and obtain

$$2xy^3\,dx + (4y^3 + 3x^2y^2)\,dy = 0 \tag{1.54}$$

This equation is now exact. Its solution is $f(x, y) = c$ where

$$f(x, y) = \int 2xy^3\,dx + g(y) = x^2y^3 + g(y)$$

Comparing $\frac{\partial f}{\partial y}$ with the dy-coefficient in (1.54),

$$\frac{\partial f}{\partial y} = 3x^2y^2 + g'(y) = 4y^3 + 3x^2y^2$$

we obtain

$$\begin{aligned} g'(y) &= 4y^3 \\ g(y) &= y^4 \end{aligned}$$

We have found that $f(x, y) = x^2y^3 + y^4$, so that the general solution is

$$x^2y^3 + y^4 = c$$

□

Exercises

1. Check whether the following equations are exact and solve them if they are exact.

   (a) $(2x + 3y)\,dx + (3x - 4)\,dy = 0$

   (b) $(3x^2 - 2y^2)\,dx + (6y^2 - 4xy)\,dy = 0$

   (c) $(2y^2 - 4x + 5)\,dx + (4 - 2y + 4xy)\,dy = 0$

   (d) $\cos x \cos^2 y\,dx + 2\sin x \sin y \cos y\,dy = 0$

   (e) $(\sin x \tan y + 1)\,dx - \cos x \sec^2 y\,dy = 0$

   (f) $(x^2 + y/x)\,dx + (y^2 + \ln x)\,dy = 0$

   (g) $(e^x \sin y + \tan y)\,dx + (e^x \cos y + x\sec^2 y)\,dy = 0$

   (h) $(\sin x \sin y - xe^y)\,dy = (e^y + \cos x \cos y)\,dx$

   (i) $2x(1 + \sqrt{x^2 - y})\,dx = \sqrt{x^2 - y}\,dy$

   (j) $(\theta^2 + 1)\cos r\,dr + 2\theta \sin r\,d\theta = 0$

2. Determine whether the following equations are exact. If not, find an integrating factor. Then solve the equations.

   (a) $2xy\,dx + (y^2 - x^2)\,dy = 0$

   (b) $(xy - 1)\,dx + (x^2 - xy)\,dy = 0$

   (c) $y\,dx + (2xy - e^{-2y})\,dy = 0$

   (d) $(x + 3y^2)\,dx + 2xy\,dy = 0$

   (e) $y\,dx + (2x - ye^y)\,dy = 0$

   (f) $\left(\dfrac{4x^3}{y^2} + \dfrac{3}{y}\right)dx + \left(\dfrac{3x}{y^2} + 4y\right)dy = 0$

   (g) $(4xy^2 + y)\,dx + (6y^3 - x)\,dy = 0$

   (h) $e^x\,dx + (e^x \cot y + 2y\csc y)\,dy = 0$

   (i) $(3x^2 - y^2)\,dy - 2xy\,dx = 0$

   (j) $(y + y\cos xy)\,dx + (x + x\cos xy)\,dy = 0$

   (k) $(y - x^3)\,dx + (x + y^3)\,dy = 0$

   (l) $3x^2(1 + \ln y)\,dx + \left(\dfrac{x^3}{y} - 2y\right)dy = 0$

   Supplementary exercises:

3. The equation

$$\frac{4y^2 - 2x^2}{4xy^2 - x^3}\,dx + \frac{8y^2 - x^2}{4y^3 - x^2 y}\,dy = 0$$

   is both, homogeneous and exact. Solve it as

   (a) a homogeneous equation.

   (b) an exact equation.

1.6 Reduction of Order

The general form of a second order differential equation is

$$F(x, y, y', y'') = 0$$

If some of the variables are missing then such an equation may be reduced to a first order equation. We have two cases to consider:

Case 1: The dependent variable y is missing

In this case, the differential equation is of the form

$$G(x, y', y'') = 0 \tag{1.55}$$

If we consider y' the dependent variable, then this is really a first order equation ! We therefore substitute

$$\boxed{v = y'}$$

Then differentiating with respect to x,

$$v' = y''$$

so that (1.55) changes to the first order equation

$$G(x, v, v') = 0$$

Example 1 Solve the equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2 \tag{1.56}$$

Solution. The variable y is missing. We therefore set

$$v = \frac{dy}{dx}$$

so equation (1.56) changes to

$$x\,\frac{dv}{dx} - v = 3x^2$$

This is a linear equation. Divide by x to bring it into standard form,

$$\frac{dv}{dx} - \frac{1}{x}v = 3x$$

We choose the integrating factor

$$e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}$$

and multiply by this factor to obtain

$$\frac{1}{x}\,\frac{dv}{dx} - \frac{1}{x^2}\,v = 3$$

$$\frac{d}{dx}\left[\frac{1}{x}\,v\right] = 3$$

Integrate,

$$\frac{1}{x}v = 3x + c$$

$$v = 3x^2 + cx$$

We must not forget to resubstitute,

$$y' = 3x^2 + cx$$

To find y, integrate once more

$$y = \int (3x^2 + cx)\,dx = x^3 + c_o x^2 + d$$

where we have set $c_o = c/2$. □

Remark You may notice that the general solution contains two constants, c_o and d. This is typical for second order equations.

Case 2: The independent variable x is missing

In this case we have a differential equation

$$H(y, y', y'') = 0 \tag{1.57}$$

Because x is missing, we may think of y as the independent variable, and of y' as the dependent variable. We choose the same substitution as in case 1,

$$\boxed{v = y'}$$

but consider v as a function of the variable y now ! What does y'' change to ? By the chain-rule,

$$y'' = \frac{d}{dx}(y') = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v$$

and (1.57) becomes the first order equation

$$H(y, v, \frac{dv}{dy}v) = 0$$

Example 2 Solve

$$(y+1)y'' - (y')^2 = 0$$

Solution. Clearly, the variable x is missing. We therefore make y the independent variable, and $v = y'$ the dependent variable. The equation changes to

$$(y+1)\frac{dv}{dy}v - v^2 = 0 \tag{1.58}$$

or dividing by v,

$$(y+1)\frac{dv}{dy} - v = 0$$

This equation is both separable and linear. Let us solve it by separation of variables,

$$\frac{1}{v}\,dv = \frac{1}{(y+1)}\,dy$$

Integrate,

$$\ln|v| = \ln|y+1| + C$$

and exponentiate,

$$|v| = e^C|y+1|$$

As usual, we set $k = \pm e^C$ depending on the effect of the absolute value, and get

$$v = k(y+1) \tag{1.59}$$

Now we must resubstitute $v = \frac{dy}{dx}$,

$$\frac{dy}{dx} = k(y+1) \tag{1.60}$$

This equation can be solved by separation of variables,

$$\frac{dy}{y+1} = k\,dx$$

$$\ln|y+1| = kx + D$$
$$y + 1 = ce^{kx}$$

where $c = \pm e^D$. The solution is thus

$$y = ce^{kx} - 1 \tag{1.61}$$

□

Remark Note that it would be wrong to simply integrate in (1.60) because on the right side there is the variable y ! In fact, at this stage we always have a separable equation because the variable x is missing.

Remark When we divided by v in (1.58) we lost the solution $v = 0$ which integrates to $y = const$. This solution appears again in (1.59) when we set $k = 0$.

Exercises

1. Solve the following equations:
   (a) $xy'' + 2y' = 6x$
   (b) $y'' + \tan xy' = \cos x$
   (c) $y'' - 2yy' = 0$
   (d) $y''y^2 = y'$
   (e) $2yy'' = 1 + (y')^2$
   (f) $xy'' + y' = 4x$
   (g) $yy'' + (y')^2 = 0$
   (h) $xy'' = y' + (y')^3$
   (i) $y'' - 4y = 0$
   (j) $x^2y'' = 2xy' + (y')^2$
2. Find the particular solutions of the following problems:
   (a) $(x^2 + 2y')y'' + 2xy' = 0 \quad y(0) = 1, \quad y'(0) = 0$
   (b) $yy'' = y^2y' + (y')^2 \quad y(0) = 1, \quad y'(0) = 1$
   (c) $y'' = y'e^y \quad y(0) = 0, \quad y'(0) = 1$
3. Solve each of the following equations using both methods of this section.
   (a) $y'' = 1 + (y')^2$
   (b) $y'' + (y')^2 = 1$

1.7 Applications

Let us now discuss some simple applications of differential equations to physical systems.

1.7.1 Orthogonal Trajectories

Suppose, two curves in the plane intersect at a point P. The *angle of intersection* is the angle at which their tangents at P intersect.

Let m_1 and m_2 denote the slopes of these tangents. The angle is a right angle ($= 90^o$) if and only if

$$m_2 = -\frac{1}{m_1}$$

In many interesting applications (*e.g.* electric field theory) we have the following problem: Given a one parameter family

$$F(x, y, c) = 0 \tag{1.62}$$

of curves in the plane, find another family

$$G(x, y, c) = 0 \tag{1.63}$$

of curves which intersect the curves of (1.62) at right angles. The curves of (1.63) are called *orthogonal trajectories.*

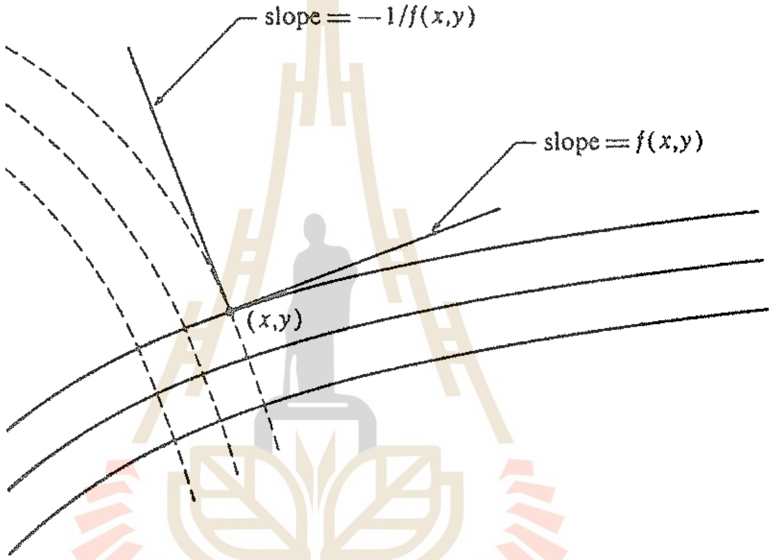
Suppose, the curve of the family (1.62) which passes through a point (x, y) has slope

$$\frac{dy}{dx} = m_1 = f(x, y)$$

Then its orthogonal trajectory must have slope

$$\frac{dy}{dx} = -\frac{1}{m_1} = -\frac{1}{f(x, y)}.$$

Solving this differential equation, we can find the orthogonal trajectory through (x, y).



orthogonal trajectories

Example 1 Find the orthogonal trajectories to the family of parabolas

$$y = cx^2 \tag{1.64}$$

Solution. First we must find the differential equation of this family. Take derivatives,

$$\frac{dy}{dx} = 2cx \tag{1.65}$$

We must eliminate the parameter c. To do this, solve (1.64) for c

$$c = \frac{y}{x^2}$$

and substitute into (1.65)

$$\frac{dy}{dx} = 2\left(\frac{y}{x^2}\right)x = \frac{2y}{x}$$

This is called the differential equation of the family of parabolas. It follows that an orthogonal trajectory must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

This is a separable equation. Solving,

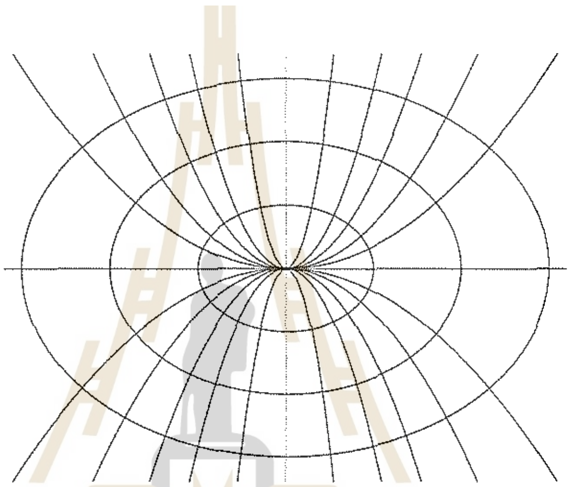
$$2y\,dy = -x\,dx$$

$$y^2 = -\frac{x^2}{2} + c.$$

The orthogonal trajectories are ellipses

$$y^2 + \frac{x^2}{2} = c.$$

□



Example 1: The orthogonal trajectories are ellipses.

Exercises

1. Find the orthogonal trajectories of

   (a) the family of hyperbolas $xy = c$,

   (b) the family of circles $(x-c)^2 + y^2 = c^2$,

   (c) the family of ellipses $x^2 - xy + y^2 = c^2$,

   (d) the family of parabolas $2cy + x^2 = c^2 \quad (c > 0)$.

2. In each exercise, sketch the family of curves. Find the orthogonal trajectories and sketch them as well

   (a) $y = ce^x$ (b) $y = x + ce^{-x}$ (c) $y^2 = ce^x + x + 1$

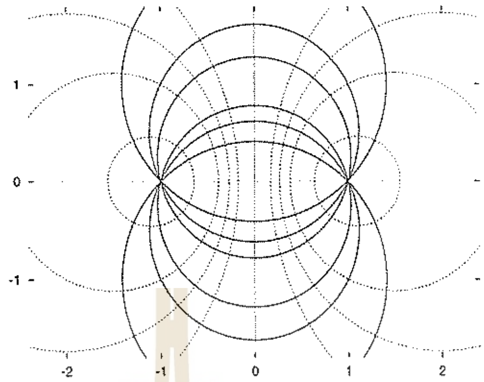
3. When two straight lines with slopes m_1 and m_2 intersect, then the angle θ between the lines is
$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$
   Find the family of curves which intersects the family

   (a) $x - 2y = c$ (b) $x^2 + y^2 = c$

at an angle of 45^o.

Supplementary exercises:

4. Experiments show that the *electric lines of force* of two opposite charges of the same strength at $(-1, 0)$ and $(1, 0)$ are circles through $(-1, 0)$ and $(1, 0)$. Show that these circles have equations $x^2 + (y - c)^2 = 1 + c^2$. Show that the *equipotential lines* (=orthogonal trajectories) are the circles $(x + d)^2 + y^2 = d^2 - 1$.



1.7.2 Growth Processes

Exponential Growth Let $x(t)$ be a quantity that changes with time. Assume that *the rate of change of this quantity is proportional to the amount present.* (Examples of this situation are radioactive decay, population growth, some chemical processes.) We get a differential equation

$$\frac{dx}{dt} = kx \tag{1.66}$$

If $k > 0$, the quantity increases with time – we have growth, and if $k < 0$, the quantity decreases with time – we have decay.

We already know that equation (1.66) has general solution

$$x = ce^{kt}$$

If x_o denotes the amount present at time $t = 0$, then

$$x_o = ce^0 = c$$

so the solution of (1.66) is

$$\boxed{x = x_o e^{kt}}$$

The quantity x changes exponentially; we therefore call (1.66) the equation of *exponential growth (decay)*. The constant k is sometimes called the *growth coefficient.*

Example 1 Radioactive Decay The radioactive element Thorium-234 decays at a quick rate. Experiments show that within 7 days an initial amount of 100 mg of Thorium-234 decays to only 82.04mg.

a) Find the amount of Thorium-234 present after t days

b) Find the *half-life* of Thorium-234.

Solution. Let x denote the amount of Thorium-234 present at time t (in days). We assume that the rate of change is proportional to the amount present. We have an initial value problem

$$\frac{dx}{dt} = -kx \qquad x(0) = 100$$

whose solution is

$$x(t) = 100e^{-kt}$$

Note the minus sign which indicates decay. Now let us use the condition $x(7) = 82.04$ to obtain

$$82.04 = 100e^{-7k}$$
$$k = \frac{\ln 0.8204}{-7} \approx 0.02828$$

So after t days,

$$x(t) = 100e^{-0.02828t}$$

milligrams of Thorium-234 are left. The *half-life* is the time when only 50% of the original amount is left. So we solve

$$50 = 100e^{-0.02828t}$$
$$t = \frac{\ln 0.5}{-0.02828} \approx 24.5$$

The half-life is 24.5 days. □

Example 2 Population Growth A small amount of bacteria is placed into a nutrient solution. After one hour, the bacteria population has tripled. When will the population have grown to 100 times its original size ?

Solution. Let $x(t)$ denote the population at time t (in hours), and x_o denote the population at $t = 0$. Assuming an unlimited supply of nutrients, we may assume that the population grows at a rate proportional to its present size,

$$\frac{dx}{dt} = kx$$

so that

$$x(t) = x_o e^{kt}$$

We know that $x(1) = 3x_o$. Substitute into the equation,

$$3x_o = x_o e^{k\cdot 1}$$
$$k = \ln 3$$

so that

$$x(t) = x_o e^{t\ln 3}$$

The population has increased to $100x_o$ when

$$100x_o = x_o e^{t\ln 3}$$
$$t = \frac{\ln 100}{\ln 3} \approx 4.18 \quad \text{hours}$$

□

Logistic Growth The equation of exponential growth used in the last example is realistic only for small populations, as it does not take conditions such as limited food supply, migration, etc. into consideration. Another commonly used equation is the *logistic equation*:

$$\frac{dx}{dt} = kx(m-x) \tag{1.67}$$

We can explain this equation as follows: The rate of change of the population is

- proportional to the present population x, and
- proportional to the difference between the present population x and the largest possible population m.

To solve the logistic equation, we separate the variables,

$$\frac{1}{x(m-x)}\,dx = k\,dt$$

and integrate,

$$\int \frac{1}{x(m-x)}\,dx = \int k\,dt + c$$

We solve this integral by partial fraction decomposition,

$$\int \frac{1}{m}\left[\frac{1}{x} + \frac{1}{m-x}\right] dx = \int k\,dt + c$$

so that after integration,

$$\frac{1}{m}\ln\left|\frac{x}{m-x}\right| = kt + c$$

Multiply by m and exponentiate,

$$\frac{x}{m-x} = Ce^{kmt} \tag{1.68}$$

where we have set $C = \pm e^{cm}$. Solve for x,

$$x = \frac{mCe^{kmt}}{1+Ce^{kmt}} \tag{1.69}$$

Now assume, we have an initial population $x(0) = x_o$. Then by (1.68),

$$\frac{x_o}{m-x_o} = C.$$

We substitute into (1.69) and obtain

$$x = \frac{m\frac{x_o}{m-x_o}e^{kmt}}{1+\frac{x_o}{m-x_o}e^{kmt}} = \frac{mx_oe^{kmt}}{(m-x_o)+x_oe^{kmt}}.$$

Finally, we multiply by e^{-kmt} and obtain the solution

$$\boxed{x(t) = \frac{mx_o}{x_o+(m-x_o)e^{-kmt}}.} \tag{1.70}$$

Note that $x(t) \to m$ as $t \to \infty$. This means that the population will approach the largest population possible.

Example 3 Bacteria are cultivated in a laboratory. An initial population of 1000 bacteria doubles within one day. Find the population of bacteria at time t if

1. there is unlimited food supply,
2. there is enough food for only 100,000 bacteria.

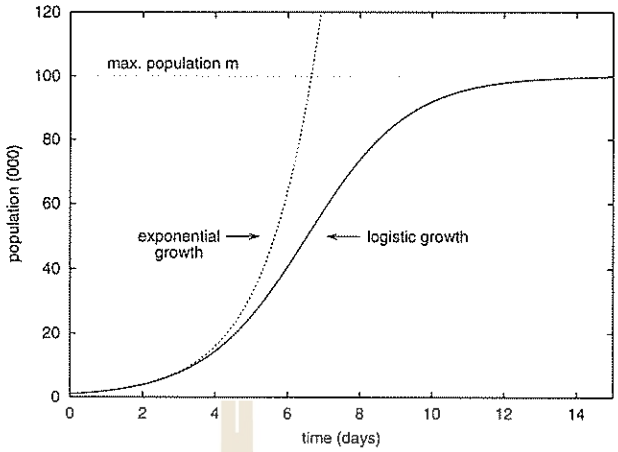
In each case, when will the population have grown to 80,000 bacteria ?

Solution. To make computations easier, let $x(t)$ denote the population of bacteria in *thousands*. The given data says that $x_o = x(0) = 1$ (the initial population is 1,000) and $x(1) = 2$ (after one day, the population is 2,000.

1. If there is unlimited food supply, then we choose the equation of exponential growth,
$$\frac{dx}{dt} = kx$$
whose solution is
$$x(t) = x_o e^{kt} = e^{kt}$$
Then $x(1) = 2$ gives
$$2 = e^k$$
so that the population at time t is
$$x(t) = \left(e^k\right)^t = 2^t$$
The population has reached 80,000 when
$$80 = 2^t$$
or
$$t = \log_2 80 \approx 6.32 \quad \text{(days)}$$

2. If there is food for only 100,000 bacteria, then we choose the equation of logistic growth with $m = 100$. By (1.70), the solution is
$$x(t) = \frac{100 \cdot 1}{1 + (100 - 1)e^{-100kt}}$$
The condition $x(1) = 2$ gives
$$2 = \frac{100}{1 + 99e^{-100k}}$$
so that
$$e^{-100k} = \frac{98}{198}.$$
Therefore,
$$-100k = \ln\frac{98}{198} \approx -0.7033.$$
Thus, the population at time t, in thousands, is
$$x(t) = \frac{100}{1 + 99e^{-0.7033t}}.$$
The population will have reached 80,000 when
$$80 = \frac{100}{1 + 99e^{-0.7033t}}$$
or
$$e^{-0.7033t} = \frac{20}{80 \cdot 99} = \frac{1}{396}$$
$$t = \frac{-\ln 396}{-0.7033} \approx 8.5 \text{ (days)}.$$

The picture below shows the graphs of both solutions. We notice that for small values of t, both solutions are nearly identical. After day 3, however, the population increases much faster in case of exponential growth.



Example 3: exponential growth compared with logistic growth.

□

Exercises

1. *Newton's Law of Cooling* says: A hot body cools at a rate which is proportional to the difference between the body's temperature and the surrounding temperature.

   A body is heated to 110°C and placed into air at 10°C. After 1 hour its temperature is 60°C. When will the body have cooled to a temperature of 30°C ?

2. A bottle of water is taken from the refrigerator at a temperature of 6°C, and placed into a room at temperature 22°C. After 10 minutes, the temperature of the water has risen to 14°C.

   (a) Find the temperature of the water after 20 minutes.

   (b) When will the water temperature have risen to 21°C ?

   (Note that Newton's law of cooling applies for warming as well)

3. Radium has a half-life of 1600 years. What percentage of the original amount will be left after 2400 years ? after 8000 years ?

4. Radioactive carbon C-14 has a half-life of 5568 years. Plants accumulate this isotope during their life spans, and the accumulated C-14 decays after their deaths. A fossile plant contains only 0.2% of its original amount of C-14. How old is this fossile plant ?

5. A city had a population of 100,000 in the year 1980, and of 120,000 in the year 1990. Assuming that the population increases exponentially, what population can we expect in the year 2020 ?

6. The population $x(t)$ of a certain country increases because of two factors:

   - the population growths naturally because of births, with growth coefficient k;
   - every year, I persons are immigrating to the country.

   Thus, we get the equation

$$\frac{dx}{dt} = kx + I$$

(a) If the country has a population of 50 million in the year 2000, a growth coefficient of $k = 4\%$ and an annual immigration of $500,000$ persons, what will the population be in the year 2005 ? (the year 2055 ?)

(b) repeat these computations, if the growth coefficient is 1%.

7. A highly contagious disease is spreading in a town of $15,000$ people. At time $t = 0$, there are $5,000$ people who have the disease, and the disease is increasing at a rate of 500 per day. How long will it take for another $5,000$ people to get the disease ? (Because the number of people to be infected is limited, we assume that the disease spreads according to the logistic equation.)

8. In a chemical reaction $A \to C$, substance A is converted to substance C at a rate which is proportional to the amount of substance A present. (This is called a *first order reaction.*) After 5 minutes, 10% of the original amount of chemical A has been converted.

   (a) How many percent of chemical A will have been converted after a total of 20 minutes ?

   (b) When will 60% of chemical A have been converted ?

9. In a first order chemical reaction $A \to C$, an unknown quantity of substance A is converted to substance C. After 1 hour, 50 g of substance A remain, and after 3 hours, only 25 g of substance A remain.

   (a) How many g of substance A were initially present ?

   (b) How many g of substance A will remain after 5 hours ?

   (c) After how many hours will only 2 g of substance A remain ?

10. A moon rock was found to contain equal numbers of potassium and argon atoms. Assume that all argon is the result of radioactive decay of potassium, and that one of every nine potassium atom disintegrations yields an argon atom. What is the age of the rock, assuming that originally the rock contained only potassium ? (The half-life of potassium is 1.28×10^9 years.)

1.7.3 Chemical Reactions

Consider a chemical reaction

$$A + B \to C$$

where α grams of substance A and β grams of substance B react to form 1 gram of a new substance C.

This is called a *second order reaction*, if substance C is formed at a rate which is proportional to the amounts of A and of B present,

$$\frac{dx}{dt} = k \cdot a(t) \cdot b(t). \tag{1.71}$$

Here, $a(t)$ and $b(t)$ denote the amounts of substances A and B present at time t, and $x(t)$ the amount of substance C which has been formed since time $t = 0$.

If a_o and b_o denote the initial amounts of substances A and B respectively, then the amount of A left at time t is

$$a(t) = \text{initial amount} - \text{amount converted at time } t = a_o - \alpha x(t),$$

and similarly, the amount of B left at time t is $b(t) = b_o - \beta x(t)$. Thus, (1.71) becomes

$$\boxed{\frac{dx}{dt} = k(a_o - \alpha x)(b_o - \beta x)}$$

This equation is separable, and has initial condition $x(0) = 0$.

Example 1 1g of substance A and 3g of substance B react to form 4g of substance C. Initially, 10g of substance A and 15g of substance B are present. After 15min, 5g of substance C have been formed. Find the amount of substance C formed at time t.

Solution. Here, $a_o = 10$, $b_o = 15$, $\alpha = 0.25$ and $\beta = 0.75$. We therefore must solve the equation

$$\frac{dx}{dt} = k(10 - 0.25x)(15 - 0.75x)$$

which can be simplified to

$$\frac{dx}{dt} = \tilde{k}(40 - x)(20 - x)$$

where we have set $\tilde{k} = \frac{3}{16}k$. To solve it, we separate the variables,

$$\frac{1}{(40 - x)(20 - x)}\, dx = \tilde{k}\, dt \tag{1.72}$$

Before we can integrate we do a partial fraction decomposition on the left side:

$$\frac{1}{(40 - x)(20 - x)} = \frac{A}{40 - x} + \frac{B}{20 - x}$$

$$1 = A(20 - x) + B(40 - x)$$

If $x = 40$ we obtain

$$1 = -20A \qquad \text{or} \qquad A = -\frac{1}{20}$$

If $x = 20$ we obtain

$$1 = 20B \qquad \text{or} \qquad B = \frac{1}{20}$$

Therefore, (1.72) becomes

$$\frac{1}{20}\left(\frac{1}{20 - x} - \frac{1}{40 - x}\right) dx = \tilde{k}\, dt$$

Multiply by 20 and integrate,

$$-\ln(20 - x) + \ln(40 - x) = 20\tilde{k}t + c$$

Exponentiate,

$$\frac{40 - x}{20 - x} = e^c e^{20\tilde{k}t}$$

Now let us use the initial condition $x(0) = 0$. It gives us

$$e^c = \frac{40 - 0}{20 - 0} = 2$$

so that

$$\frac{40 - x}{20 - x} = 2e^{20\tilde{k}t}$$

To find the value of $\tilde{k}$, use the second condition $x(15) = 5$,

$$\frac{40 - 5}{20 - 5} = 2e^{20\cdot 15\cdot \tilde{k}}$$

$$\frac{35}{15} = 2\left(e^{20\tilde{k}}\right)^{15}$$

$$e^{20\tilde{k}} = \left(\frac{7}{6}\right)^{\frac{1}{15}}$$

Therefore,

$$\frac{40-x}{20-x} = 2\left(\frac{7}{6}\right)^{\frac{t}{15}}$$

Now solve for x,

$$\begin{aligned} 40 - x &= 2(20-x)\left(\frac{7}{6}\right)^{\frac{t}{15}} \\ x\left(2\left(\frac{7}{6}\right)^{\frac{t}{15}} - 1\right) &= 40\left(\left(\frac{7}{6}\right)^{\frac{t}{15}} - 1\right) \end{aligned}$$

The solution is

$$\boxed{x(t) = 40\,\frac{\left(\frac{7}{6}\right)^{\frac{t}{15}} - 1}{2\left(\frac{7}{6}\right)^{\frac{t}{15}} - 1}}$$

□

Remark Note that as $t \to \infty$,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} 40\,\frac{\frac{1}{15}\cdot\left(\frac{7}{6}\right)^{\frac{t}{15}}\cdot\ln\left(\frac{7}{6}\right)}{2\cdot\frac{1}{15}\cdot\left(\frac{7}{6}\right)^{\frac{t}{15}}\cdot\ln\left(\frac{7}{6}\right)} = 20$$

by l'Hôpital's rule. This makes sense: Because there is initially only 15g substance B, at most 20 g of substance C can be formed.

Exercises

1. Do example 1 again, but assume now that the amount of substance A is kept constant at 15g. (For example, the substances are dissolved in water and substance A is oversaturated.) Find the time when 19g of substance C have been formed.

2. Do example 1 again, but assume now that initially only 5g of substance A are present. Find the time when 19g of substance C have been formed.

3. The rate at which a certain substance dissolves in water is proportional to the amount undissolved, and proportional to the difference $c_1 - c_2$, where c_1 is the concentration of a saturated solution, and c_2 the concentration of the actual solution. In a saturated solution, 50 g of water dissolve 20 g of the substance.

   (a) If 10 g of the substance is placed in 50 g of water, and half of it dissolves within 30 minutes, how much will be dissolved after one hour ?

   (b) Answer the same question when 30 g of the substance is placed into 50 g of water.

1.7.4 Mixing

Let us simply give an example:

Example 1 Initially, a tank contains 25 l of water in which is dissolved 2 kg of salt. (Water in which salt is dissolved is called *brine*.) Saltwater which contains 0.2 kg of salt per liter flows into the tank at a rate of 5 l per minute. Every minute, 3 l of the saltwater flow out of the tank. How much salt does the tank contain at time t? (The mixture inside the tank is kept uniform by stirring.)

Solution. Let $x(t)$ denote the amount of salt contained in the tank at time t. We have

$$\frac{dx}{dt} = x_{\rm IN} - x_{\rm OUT}$$

where

$$x_{\rm IN} = \text{rate at which salt enters the tank}$$
$$x_{\rm OUT} = \text{rate at which salt leaves the tank}$$

Let use compute these quantities.
The amount of salt entering the tank is

$$x_{\rm IN} = \text{volume of water flowing in} \cdot \text{concentration of salt in inflowing water}$$

or

$$x_{\rm IN} = 5\,\text{l/min} \cdot 0.2\ \text{kg/l} = 1\,\text{kg/min}$$

Similarly, the amount of salt leaving the tank is

$$x_{\rm OUT} = \text{volume of water flowing out} \cdot \text{concentration of salt in outflowing water}$$

What is the concentration of the outflowing water ? This water comes from the inside of the tank, and so has concentration

$$\frac{\text{amount of salt in tank}}{\text{volume of brine in tank}} = \frac{x}{V(t)}$$

where $V(t)$ denotes the volume of brine in the tank. Now the initial volume is $V(0) = 25$, and every minute, 5 liters of brine flow in, and 3 liters flow out. Thus, $V(t) = 25 + (5-3)t$ and

$$x_{\rm OUT} = 3 \cdot \frac{x(t)}{25+2t}$$

We get the differential equation

$$\frac{dx}{dt} = x_{\rm IN} - x_{\rm OUT} = 1 - \frac{3x}{25+2t}$$

with initial condition $x(0) = 2$.
This equation is linear. We rewrite it as

$$\frac{dx}{dt} + \frac{3}{25+2t}\,x = 1 \tag{1.73}$$

Choose the integrating factor

$$e^{\int \frac{3}{25+2t}\,dt} = e^{\frac{3}{2}\ln(25+2t)} = (25+2t)^{\frac{3}{2}}$$

Multiplying by this factor, equation (1.73) becomes

$$\frac{d}{dt}\left((25+2t)^{\frac{3}{2}}x\right) = (25+2t)^{\frac{3}{2}}$$

Integrate,

$$(25+2t)^{\frac{3}{2}}x = \frac{2}{5}\,\frac{1}{2}\,(25+2t)^{\frac{5}{2}} + c$$

and solve for x,

$$x = 0.2(25+2t) + \frac{c}{(25+2t)^{\frac{3}{2}}}$$

Now we use the initial condition $x(0) = 2$. We get

$$2 \;=\; 0.2(25+0) + \frac{c}{(25+0)^{\frac{3}{2}}} \;=\; 5 + \frac{c}{125}$$

Then,

$$c = -375$$

The solution is

$$x(t) = (5 + 0.4t) - \frac{375}{(25+2t)^{\frac{3}{2}}}$$

□

Exercises

1. A tank of 200 l capacity is initially full of water in which there is dissolved 4 kg of a chemical. Water containing 50 g per liter of this chemical flows in at a rate of 5 l/min. Water flows out from the tank at a rate of 7 l/min. How much of the chemical is in the tank when the tank is half-full ? (The mixture in the tank is kept uniform by stirring)

2. A tank contains 100 l of water in which 5 kg of salt are dissolved. We want to reduce the concentration in the tank to 0.01 kg/l by pouring in pure water at the rate of 20 l/min and allowing the mixture to flow out at the same rate. How long will this take ? (Assume the mixture in the tank is kept uniform by stirring.)

3. A tank contains 40 liters of pure water. Saltwater which contains 3 kg of salt per liter flows in at a rate of 2 liter per minute. The stirred mixture flows out at a rate of 3 liters per minute. Find the amount of salt in the tank at time t. When is this amount largest ?

4. The air in a room 20m × 10m × 3m contains 0.2% carbon dioxide (CO_2). We want to reduce the concentration of carbon dioxide in the room by pumping outside air containing 0.05% of carbon dioxide into the room. At what (constant) rate must the outside air be pumped in so that after 30 minutes, the air in the room contains 0.1% of carbon dioxide ? (Hint: At what rate does air leave the room ? Think !)

1.7.5 Mechanics

Let us look at the *falling body problem* as an example of motion where *friction* is present.

Example 1 A man (called a sky diver) jumps from an airplane, opens a parachute and falls towards the earth. The following information is given:

- At some time (which we call $t = 0$), the man falls with velocity $v_o = 10$ m/s.
- The weight of man and equipment is $W = 712$ N.
- The air resistance R is proportional to the square of the velocity,

$$R = -bv^2 \qquad \text{where} \quad b = 30N\ \text{sec}^2/\text{m}^2$$

  (This is a reasonable assumption for high velocities. The minus sign indicates that the force of resistance has opposite direction to the velocity.)

Find the velocity $v(t)$ at time $t > 0$.

Solution. There are two opposing forces acting on the sky diver,

1. the weight $W = mg$, directed downward ↓, and
2. the frictional force R, directed upward ↑.

(g denotes the earth acceleration and m the mass of man and equipment.) The total force is

$$F = W + R = mg - bv^2$$

By Newton's second law, this force results in an acceleration given by $F = m\frac{dv}{dt}$. Thus,

$$m\,\frac{dv}{dt} \;=\; mg - bv^2 \tag{1.74}$$

Let us first simplify this equation,

$$\frac{dv}{dt} \;=\; g - \frac{b}{m}v^2$$

or

$$\frac{dv}{dt} \;=\; -\frac{b}{m}(v^2 - k^2) \qquad\qquad \left(\ k = \sqrt{\frac{gm}{b}}\ \right)$$

This is a separable equation, so separate the variables,

$$\frac{dv}{v^2 - k^2} \;=\; -\frac{b}{m}\,dt.$$

Use partial fraction decomposition on the left side,

$$\frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right)dv \;=\; -\frac{b}{m}\,dt,$$

and integrate,

$$\frac{1}{2k}\Big(\ln|v-k| - \ln|v+k|\Big) \;=\; -\frac{b}{m}\,t + c_o$$

$$\ln\left|\frac{v-k}{v+k}\right| \;=\; -\frac{2kb}{m}\,t + c_1 \qquad\qquad (\ c_1 = 2kc_o\ )$$

$$\left|\frac{v-k}{v+k}\right| \;=\; e^{c_1}e^{-pt} \qquad\qquad \left(\ p = \frac{2kb}{m}\ \right).$$

Set $c = \pm e^{c_1}$ to eliminate the absolute value,

$$\frac{v-k}{v+k} \;=\; ce^{-pt} \tag{1.75}$$

and solve for v,

$$\begin{aligned} v - k &= (v+k)ce^{-pt} \\ v(1 - ce^{-pt}) &= k(1 + ce^{-pt}) \end{aligned}$$

The general solution of equation (1.74) is thus

$$\boxed{v(t) \;=\; k\,\frac{1 + ce^{-pt}}{1 - ce^{-pt}}} \tag{1.76}$$

Now find the particular solution satisfying the initial value $v(0) = v_0$. It is best to start from equation (1.75). We obtain

$$\boxed{\frac{v_o - k}{v_o + k} = c} \tag{1.77}$$

Finally, let us substitute the given numbers. We get

$$k = \sqrt{\frac{W}{b}} = \sqrt{\frac{712}{30}} \approx 4.87 \text{ m/sec}$$

$$p = \frac{2kb}{W/g} = \frac{2 \cdot 4.87 \cdot 30}{72.7} \approx 4.02/\text{sec}$$

$$c = \frac{v_o - k}{v_o + k} = \frac{10 - 4.87}{10 + 4.87} \approx 0.345$$

$$v(t) = 4.87 \, \frac{1 + 0.345 e^{-4.02t}}{1 - 0.345 e^{-4.02t}} \text{ m/sec}$$

Note that in (1.76), $\lim_{t \to \infty} v(t) = k$. This limit represents the velocity where weight and force of friction have equal size and is called the *terminal velocity*. □

Exercises

1. A ball falls from rest towards the earth from a height of 1000 m. Assume the air resistance is proportional to the velocity v. If the terminal velocity is 245 m/sec, find the velocity $v(t)$.

2. A ball of mass 100 g is thrown vertically upward from a point 60 cm above the ground with an initial velocity of 150 cm/sec. It rises and then falls back towards the ground. The air resistance is $200v$ (in dynes).

   (a) Find the time when the ball starts falling back to the ground.

   (b) Find the velocity with which the ball falls back to the ground.

   (c) When does the ball hit the ground? (Find an approximation of the time only.)

3. A motor boat weighs 5000 N. The motor exerts a constant force of 200 N in the direction of motion. The water resistance is equal to 1.5 times the velocity, and the boat starts from rest.

   (a) Find the velocity of the boat after 20 seconds.

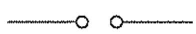
   (b) Find the velocity of the boat after one hour.

   (c) If the motor breaks down after 5 minutes (so that the boat is coasting), find the velocity of the boat after the breakdown.

1.7.6 Electric Circuits

Let us look at simple electric circuits. We will use the following components:

- a generator, or a battery, providing an *electromotive force* $E(t)$ (measured in *volt*).
  We will denote the current flowing through a component by $I(t)$ (measured in *ampere*)

  E(t)

- a *resistor*. The voltage drop across a resistor is proportional to the current flowing through it,

  $$E_R = RI$$

  R

  R is the *resistance* measured in *ohm*.

- an *inductor*. The voltage drop across an inductor is proportional to the rate of change of the current through it,

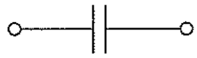
$$E_L = L\frac{dI}{dt}$$

L

L is the *inductance* measured in *henry*.

- a *capacitor*. The voltage drop across a capacitor is proportional to the charge Q which it is holding,

$$E_C = \frac{1}{C}\,Q$$



C

C is the *capacitance* measured in *farad*,
and the charge Q is measured in *coulomb*. Note that the current flowing through the capacitor is the rate of change of charge on it,

$$I = \frac{dQ}{dt}$$

Integrating I, we may also write the equation of a capacitor as

$$E_C = \frac{1}{C}\left[Q(t_o) + \int_{t_o}^{t} I(\tau)\,d\tau\right]$$

where $Q(t_o)$ is the charge on the capacitor at time t_o.

We will make use of *Kirchhoff's Voltage Law*: The sum of all voltage drops around a closed loop is zero.

Example 1 Consider the RL-circuit in the picture below. By Kirchhoff's law,

$$E(t) \;=\; E_R + E_L \;=\; RI + L\,\frac{dI}{dt}$$

or dividing by L,

$$\frac{dI}{dt} + \frac{R}{L}I \;=\; \frac{1}{L}E(t) \tag{1.78}$$

This is a linear equation, and its solution is given by (1.30),

$$I(t) \;=\; e^{-\frac{R}{L}t}\left[\int \frac{E}{L}\,e^{\frac{R}{L}t}\,dt + c\right] \tag{1.79}$$

Special case: Assume, the electromotive force is constant, $E(t) = E_o$. We have

$$\begin{aligned} I(t) &= e^{-\frac{R}{L}t}\left[\frac{E_o}{L}\int e^{\frac{R}{L}t}\,dt + c\right] \\ &= e^{-\frac{R}{L}t}\left[\frac{E_o}{L}\frac{L}{R}e^{\frac{R}{L}t} + c\right] \\ &= \frac{E_o}{R} + ce^{-\frac{R}{L}t} \end{aligned}$$

Now assume further that the initial current is given, $I(0) = I_o$. We get

$$I_o = \frac{E_o}{R} + ce^0$$

so that

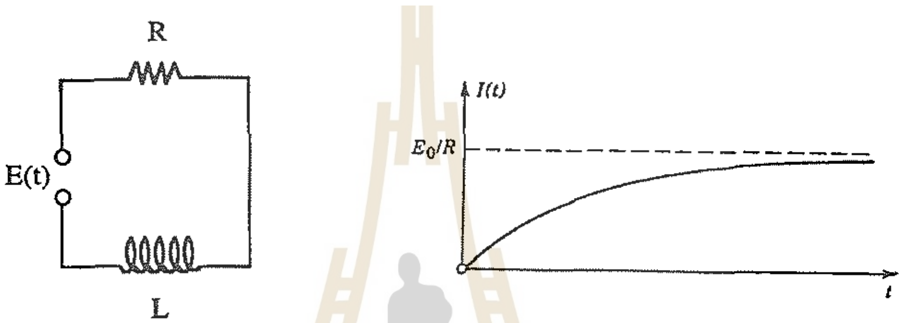
$$c = I_o - \frac{E_o}{R}$$

and thus

$$\boxed{I(t) = \frac{E_o}{R} + \left(I_o - \frac{E_o}{R}\right) e^{-\frac{R}{L}t}}$$

If $t \to \infty$ then $I(t) \to \frac{E_o}{R}$. This non-vanishing part is called the *steady state* current. The vanishing part $\left(I_o - \frac{E_o}{R}\right) e^{-\frac{R}{L}t}$ is called the *transient current*. We see that the steady state current is independent of the initial current.

□



RL-circuit with constant electromotive force.

Exercises

1. Solve example 1, but assume that the electromotive force is periodic, $E(t) = E_o \sin(\omega t)$, and the initial current is zero. Find the steady state current and the transient current.

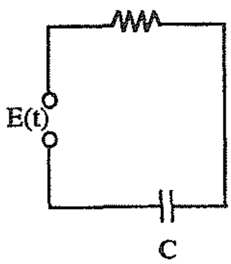
2. Find the current flowing through an RC-circuit

   (a) in general,

   (b) if the electromotive force is constant, $E(t) = E_o$, and the initial charge on the capacitor is zero, $Q(0) = 0$.

   (c) if the electromotive force is periodic, $E(t) = E_o \sin(\omega t)$, and the initial charge on the capacitor is zero, $Q(0) = 0$.

   Also, find the steady state and transient currents.



RC-circuit.

1.8 Existence of Solutions

In the previous sections we have seen some methods of solving first order ordinary differential equations. There exist many more types of first order differential equations; you have only seen the most important ones.

One question remains: Do there exist solutions other than those we have found ? The next theorem answers this question.

Theorem 4 *Consider an initial value problem*

$$\frac{dy}{dx} = F(x,y) \qquad y(x_o) = y_o \tag{1.80}$$

where F and $\frac{\partial F}{\partial y}$ are continuous on some rectangle in the xy-plane containing the point (x_o, y_o).

Then there exists a unique solution $y = f(x)$ to this initial-value problem. This solution is defined on some interval $(x_o - h, x_o + h)$ containing x_o.

Remark Note that the theorem says the following two things:

- *Existence.* Every initial value problem (1.80) has at least one solution, and
- *Uniqueness.* There can not exist two distinct solutions to an initial value problem (1.80).

For this reason, it is called an *existence and uniqueness theorem.* However, the theorem does not tell us how to find the solution.

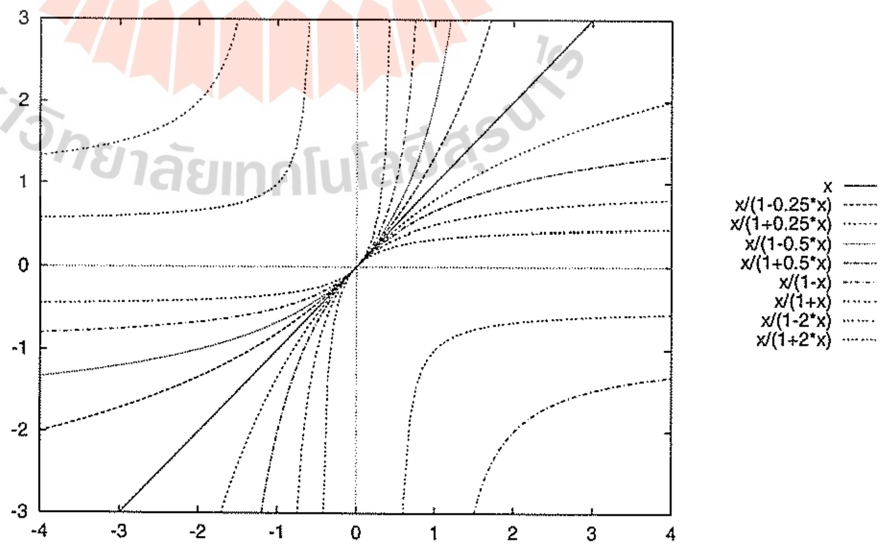
Graphically, the theorem says that for every first order initial value problem, there exists exactly one solution whose graph passes through a given point (x_o, y_o) in the xy-plane.

Example 1 We have seen in example 1 of section 1.3 that the equation

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

has general solution

$$y = \frac{x}{cx-1}$$



The solutions of example 1, section 1.3.

Why do all of the curves pass through the point $(0,0)$? Why does none of the curves pass through the points $(0,c)$ with $c \neq 0$? Does this contradict the theorem ?

Answer: The function

$$F(x,y) \;=\; \frac{y^2}{x^2}$$

is not continuous at $x=0$; therefore, the theorem does not apply at the points $(0,c)$. At all other points, the theorem applies: there exists exactly one solution curve through a point $(x_o, y_o) \neq (0,c)$. □

1.9 Review Exercises

The following exercises may help you review the concepts introduced in this section.

Exercises

1. Solve the following differential equations:
   (a) $\dfrac{dy}{dx} = \dfrac{e^{y-2x}}{y-1}$
   (b) $y' + y\cot x = 0$
   (c) $y' + y\cot x + \sin x = 0$
   (d) $(\sqrt{\frac{y}{x}} + \cos x)\,dx + (\sqrt{\frac{x}{y}} + \sin x)\,dy = 0$
   (e) $y'' = x - y'$
   (f) $xy' + 2y = x^3 y^{3/2}$
   (g) $\dfrac{dy}{dx} = \dfrac{y^2+y}{x^2+x}$
   (h) $(3x^2y + 2xy^{-2})dx + (3x^3 + 2y^{-1})dy = 0$
   (i) $\dfrac{dy}{dx} - \dfrac{y}{x} = x^2 e^{-x^2}$
   (j) $(e^y + e^{-x})dx + (e^y + 2ye^{-x})dy = 0$
   (k) $y'' + 2y(y')^3 = 0$

2. Solve the following initial value problems:
   (a) $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{xy}, \quad y(1) = -2$
   (b) $xyy' = 4y^2 + 3x^2, \quad y(1) = 2$
   (c) $y' = y^4 - y, \quad y(0) = 1$
   (d) $\dfrac{dy}{dx} = e^{-x}\sec y - \tan y, \quad y(0) = \dfrac{\pi}{2}$

3. Solve using an appropriate substitution:
   (a) $y' + xy^3 \sec(y^{-2}) = 0$
   (b) $y' = 1 + \cos^2(x-y)$

   The following are multiple choice questions. Choose the answer which is most correct.

4. The equation $(1+x)\,dy - y\,dx = 0$ is
   (a) homogeneous
   (b) linear and exact
   (c) separable and linear
   (d) exact
   (e) separable and exact

5. If $y = y(x)$ is the solution to the initial value problem

$$(1+x)\,dy - y\,dx = 0, \qquad y(0) = 1$$

   then $y(2)$ equals

   (a) 0 (b) $\ln 2$ (c) 1 (d) 3 (e) e

6. If $y = y(x)$ is the solution to the initial value problem

$$x^2y' + xy = 1, \qquad y(1) = 0$$

   then $y(e)$ equals

(a) 0 (b) $1/e$ (c) e (d) 3 (e) e^2

7. When solving the equation $(2\sqrt{xy}+y)\,dx - x\,dy = 0$ we
   - (a) multiply by the integrating factor $e^{\sqrt{xy}}$
   - (b) multiply by the integrating factor x^{-1}
   - (c) substitute $u = y^{-1}$
   - (d) substitute $v = yx^{-1}$
   - (e) substitute $p = \sqrt{y}$ or $u = yx^{-1}$

8. The equation $(2\sqrt{xy}+y)\,dx - x\,dy = 0$ has general solution
   - (a) $\sqrt{y} = \ln|x| + c$
   - (b) $y = c + 2x - 3y^{3/2}$
   - (c) $y = x(c + \ln|x|)^2$
   - (d) $2\sqrt{xy} - y = x^2 + c$
   - (e) none of the above

In the following 3 questions, consider the problem

$$(\cos x)y'' + (\sin x)y' = 1, \qquad y(0) = -1, \quad y'(0) = 1 \qquad (*)$$

9. When solving this equation, we substitute
   - (a) $v = \dfrac{dy}{dx}$ and $\dfrac{dv}{dy} = \dfrac{d^2y}{dx^2}$
   - (b) $v = \dfrac{dy}{dx}$ and $\dfrac{dv}{dx} = \dfrac{d^2y}{dx^2}$
   - (c) $v = \dfrac{dy}{dx}$ and $v\,\dfrac{dv}{dy} = \dfrac{d^2y}{dx^2}$
   - (d) $v = \dfrac{y}{x}$ and $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$
   - (e) $v = \dfrac{x}{y}$ and $\dfrac{dx}{dy} = v + y\,\dfrac{dv}{dy}$

10. After this substitution, and solving for v we get
    - (a) $v = \sin x + 1$
    - (b) $v = (1+x)\cos x$
    - (c) $v = \sin x + \cos x$
    - (d) $v = (1-x)\sin x + \cos x$
    - (e) $v = \dfrac{-2y}{1+y}$

11. If $y = y(x)$ is the solution to problem $(*)$, then $y(\pi)$ equals

    (a) 0 (b) 1 (c) -1 (d) 2 (e) -3

Chapter 2

Second Order Linear Differential Equations

2.1 Complex Numbers

You may remember that there exists no real number x which is solution of the equation

$$x^2 = -1$$

To obtain a solution of this and similar equations, we invent new numbers.

The first new number is a number which we call i. To make the usual arithmetic operations possible, we must form combinations of numbers of the form

$$z = x + iy$$

where x and y are arbitrary real numbers, and call them *complex numbers*.

- x is called the *real part* of the complex number z, written **Re** z.
- y is called the *imaginary part* of the complex number z, written **Im** z.

If **Im** $z = 0$, then z is a real number. If **Re** $z = 0$, then z is called a *purely imaginary number*.

e.g. For example,

$4 + i7$	is often written $4 + 7i$.
$4 - 2i$	is the usual way of writing $4 + i(-2)$
$7i$	is an abbreviation of $0 + i7$. This is a purely imaginary number.
5	can also be written $5 + i0$. This is a real number.

Algebraic Operations on Complex Numbers

We can now define algebraic operations such as addition, multiplication, etc.
Let $z = x + iy$ and $z_1 = x_1 + iy_1$ be complex numbers. We define

- **addition** by

$$z + z_1 = (x + iy) + (x_1 + iy_1) = (x + x_1) + i(y + y_1)$$

You can see that we add two complex numbers by adding their real parts and their imaginary parts.

- **subtraction** by

$$z - z_1 = (x + iy) - (x_1 + iy_1) = (x - x_1) + i(y - y_1)$$

  You can see that we subtract two complex numbers by subtracting their real parts and their imaginary parts.

- **multiplication** by

$$zz_1 = (x + iy)\,(x_1 + iy_1) = (xx_1 - yy_1) + i(xy_1 + yx_1)$$

  Notice that multiplication is done by expanding the brackets formally. Thinking that $i^2 = -1$ we then collect the real and imaginary parts.

  e.g.

$$\begin{aligned} (4+3i)+(7+2i) &= (4+7)+(3+2)i = 11+5i \\ (4+3i)-(2-i) &= (4-2)+(3-(-1))i = 2+4i \\ (4+3i)(2+i) &= (4\cdot 2 - 3\cdot 1)+(4\cdot 1+3\cdot 2)i = 5+10i \\ i^2 &= (0+1\cdot i)(0+1\cdot i) = -1+0i = -1 \end{aligned}$$

  Now the equation $x^2 = -1$ has the solution $x = i$! (And also the solution $x = -i$.) Before we can define division, we define the

- **complex conjugate** of $z = x + iy$ by

$$\overline{z} = \overline{x + iy} = x - iy$$

  That is, we simply change the sign of the imaginary part. Note that

$$z\overline{z} = (x+iy)(x-iy) = (x^2 - (-y)^2) + i(x(-y) + xy) = x^2 + y^2 \geq 0.$$

  So $z\overline{z}$ is always a real, nonnegative number. We can therefore define the

- **absolute value**, or **modulus** by

$$|z| = \sqrt{z\,\overline{z}} = \sqrt{x^2 + y^2}.$$

  e.g. If $z = 4 + 3i$, then

$$\overline{z} = 4 - 3i \qquad \text{and} \qquad |z| = \sqrt{4\cdot 4 + 3\cdot 3} = 5.$$

  Note that $z\overline{z} = |z|^2$. Finally, we can define

- **division** by

$$\frac{z_1}{z} = \frac{1}{|z|^2}\, z_1\overline{z}$$

  which we often write as

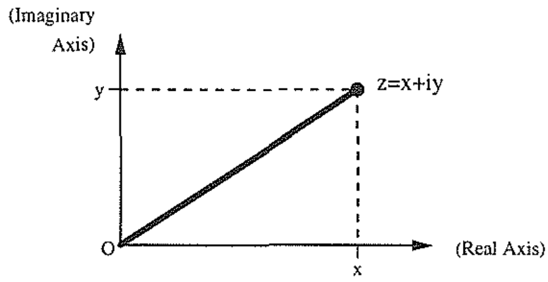
$$\frac{z_1}{z} = \frac{z_1\overline{z}}{z\overline{z}}.$$

  e.g. $\displaystyle \frac{2+i}{1-i} = \frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{(2-1)+i(2+1)}{1^2+1^2} = \frac{1}{2} + \frac{3}{2}i$

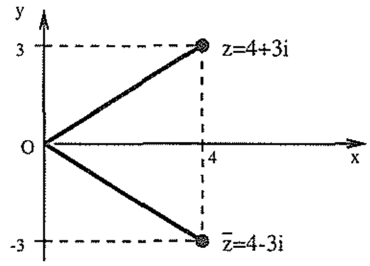
The usual rules of arithmetics hold. For example, the distributive law holds,

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

We can represent complex numbers as points or vectors in the plane. Forming the complex conjugate corresponds to reflection along the x-axis.



The point $z = x + iy$ in the plane.



$z = 4 + 3i$ and its conjugate $\overline{z} = 4 - 3i$.

The Polar and Exponential Representations of a Complex Number

Given a complex number z, let r denote the distance of the point $z = x + iy$ from the origin O, and let θ denote the angle which the line $\overline{Oz}$ forms with respect to the positive x-axis. By trigonometry,

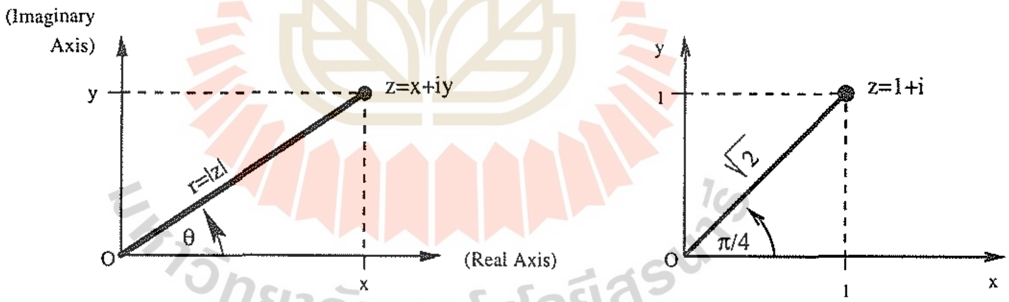
$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

where

$$r = \sqrt{x^2 + y^2} = |z| \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \qquad (\text{if } x \neq 0)$$

Then the *polar representation* of z is

$$z = x + iy = r(\cos\theta + i\sin\theta)$$



Polar representation $re^{i\theta}$ of $z = x + iy$ and $z = 1 + i$.

For convenience we set

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \tag{2.1}$$

so that the polar representation becomes

$$z = re^{i\theta}$$

Note that for any integer n,

$$re^{i(\theta+2n\pi)} = r\big(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)\big) = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

This shows that the polar representation is not unique.

e.g.

- If $z = \sqrt{3} + i$ then

$$r = |z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \qquad \text{and} \qquad \theta = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6},$$

  so that the polar representation is

$$z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$$

  or simply

$$z = 2e^{i\frac{\pi}{6}}.$$

- If $z = -1 + i$, then

$$r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \qquad \text{and} \qquad \tan\theta = \frac{1}{-1} = -1$$

  Because z lies in the second quadrant, we have $\theta = 3\pi/4$ (instead of $\theta = -\pi/4$). The polar representation is

$$z = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$$

  or simply

$$z = \sqrt{2}e^{i\frac{3\pi}{4}}.$$

In (2.1) we have defined what we mean by e^{iy}. We use this to define the *complex exponential function*: If $z = x + iy$ is a complex number, we set

$$\boxed{e^z = e^x e^{iy} = e^x(\cos y + i\sin y)}$$

e.g.

- If $z = 3 + 2i$ then

$$e^z = e^{3+2i} = e^3 e^{2i} = e^3\cos 2 + ie^3\sin 2$$

- If $z = i\pi$ then

$$e^z = e^{i\pi} = \cos\pi + i\sin\pi = -1$$

  so that $-1 = e^{i\pi}$!

Now let $z = re^{i\theta}$ be the polar representation of a complex number z. Since $r > 0$ we can write $r = e^u$ where $u = \ln r$. Thus, we may write

$$z = e^u e^{i\theta} = e^{u+i\theta}$$

which is called the *exponential representation* of z.

e.g. We have seen that $z = \sqrt{3} + i$ has the polar representation

$$\sqrt{3} + i = 2e^{i\frac{\pi}{6}}$$

The exponential representation is thus

$$\sqrt{3} + i = e^{\ln 2}e^{i\frac{\pi}{6}} = e^{\ln 2 + i\frac{\pi}{6}}$$

One can show that the usual properties of exponentials hold:

$$
\begin{aligned}
e^{z_1}e^{z_2} &= e^{z_1+z_2} \\
\frac{e^{z_1}}{e^{z_2}} &= e^{z_1-z_2} \\
(e^z)^n &= e^{nz} \qquad\qquad (n \text{ integer})
\end{aligned}
$$

We can define the derivative of the complex exponential function just like in the case of the usual exponential function. One can show that if $f(z) = e^z$, then the derivative is

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = e^z$$

Also, if $f(z) = e^{az}$ with a constant, then $f'(z) = ae^{az}$.

Complex Solutions of Polynomial Equations

One important property of the complex numbers is the following:

Theorem 5 (Fundamental Theorem of Algebra) *Every polynomial equation of degree n,*

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_o = 0$$

can be completely factored into n factors

$$a_n(x-z_1)(x-z_2)\cdots(x-z_n) = 0$$

for some complex numbers $z_1, \ldots, z_n$, and has therefore n solutions (possibly not distinct).

Remark The solutions of a polynomial equation are also called the *roots* of the equation.

In particular, every quadratic equation

$$ax^2 + bx + c = 0$$

factors as

$$a(x-z_1)(x-z_2) = 0$$

The complex numbers z_1 and z_2 can be found using the quadratic formula

$$z_1, z_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Note that

$$z_1 \text{ and } z_2 \text{ are } \begin{cases} \text{real and distinct, if} & b^2-4ac > 0 \\ \text{equal and real, if} & b^2-4ac = 0 \\ \text{complex, if} & b^2-4ac < 0 \end{cases}$$

Example 1 Factor the equation

$$2x^2 + 2x + 3 = 0$$

This equation has solutions

$$z_1, z_2 = \frac{-2 \pm \sqrt{2^2 - 4\cdot 2\cdot 3}}{2\cdot 2} = \frac{-2\pm\sqrt{-20}}{4} = -\frac{1}{2} \pm i\frac{\sqrt{5}}{2}$$

and thus factors as

$$2(x-z_1)(x-z_2) = 2(x+\frac{1}{2}-i\frac{\sqrt{5}}{2})(x+\frac{1}{2}+i\frac{\sqrt{5}}{2}) = 0$$

□

Example 2 Find the solution to

$$z^2 - (4+3i)z + (1+5i) = 0.$$

Solution. By the quadratic formula, the equation has exactly two solutions

$$\begin{aligned} z_1, z_2 &= \frac{4+3i \pm \sqrt{(4+3i)^2 - 4(1+5i)}}{2} \\ &= \frac{4+3i \pm \sqrt{3+4i}}{2}. \end{aligned} \tag{2.2}$$

Now what is $\sqrt{3+4i}$? We must find a number $x+iy$ such that

$$(x+iy)^2 = 3+4i$$
$$(x^2-y^2) + 2ixy = 3+4i$$

Comparing real and imaginary parts, we need that

$$x^2 - y^2 = 3 \qquad \text{and} \qquad 2xy = 4$$

By the second of these equations, $y = \frac{2}{x}$. Substitute into the first equation,

$$\begin{aligned} x^2 - \frac{4}{x^2} &= 3 \\ x^4 - 3x^2 - 4 &= 0 \\ (x^2-4)(x^2+1) &= 0 \\ x = \pm 2 \quad &\text{or} \quad x = \pm i. \end{aligned}$$

Since x must be a real number, we pick $x = 2$. Then, $y = \frac{2}{x} = 1$. Therefore,

$$\sqrt{3+4i} = 2+i,$$

and the two solutions of equation (2.2) are now

$$z_1, z_2 = \frac{4+3i \pm (2+i)}{2}$$

or

$$z_1 = 3+2i, \qquad z_2 = 1+i.$$

□

The simplest polynomial equations are of the form

$$z^n = z_o \tag{2.3}$$

where z_o is a fixed number. By the Fundamental Theorem of Algebra, this equation has n roots. To find these, we write z_o in polar representation,

$$z^n = r\, e^{i\theta}$$

One solution is easy to find: Set

$$z_1 = \sqrt[n]{r}\, e^{i\frac{\theta}{n}}$$

Then indeed,

$$(z_1)^n = \left(\sqrt[n]{r}\, e^{i\frac{\theta}{n}}\right)^n = (\sqrt[n]{r})^n \left(e^{i\frac{\theta}{n}}\right)^n = re^{i\frac{n\theta}{n}} = re^{i\theta} = z_o$$

To find the remaining roots, we add $\frac{2i\pi}{n}$ consecutively to the exponent,

$$z_1 = \sqrt[n]{r}\, e^{i\frac{\theta}{n}}, \qquad z_2 = \sqrt[n]{r}\, e^{i(\frac{\theta}{n}+\frac{2\pi}{n})}, \qquad z_3 = \sqrt[n]{r}\, e^{i(\frac{\theta}{n}+\frac{4\pi}{n})} \quad \ldots$$
$$z_k = \sqrt[n]{r}\, e^{i(\frac{\theta}{n}+\frac{2(k-1)\pi}{n})} \quad \ldots \quad z_n = \sqrt[n]{r}\, e^{i(\frac{\theta}{n}+\frac{2(n-1)\pi}{n})}$$

These numbers are all different, and we have for each k,

$$(z_k)^n = \left(\sqrt[n]{r}\, e^{i(\frac{\theta}{n}+\frac{2(k-1)\pi}{n})}\right)^n = \left(\sqrt[n]{r}\right)^n \left(e^{i\frac{\theta}{n}}\right)^n \left(e^{2i\pi}\right)^{\frac{n(k-1)}{n}} = re^{i\theta} = z_o$$

We have thus found all n solutions to equation (2.3).

Example 3 Find the solutions to

$$z^4 = -4.$$

Solution. We write the number -4 in polar form,

$$z^4 = 4e^{i\pi}$$

One solution is

$$z_1 = \sqrt[4]{4}\, e^{i\frac{\pi}{4}} = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

The remaining solutions can be obtained by consecutively adding $\frac{2i\pi}{4}$ to the exponent,

$$z_1 = \sqrt{2}\, e^{i\frac{\pi}{4}}, \quad z_2 = \sqrt{2}\, e^{i\frac{3\pi}{4}}, \quad z_3 = \sqrt{2}\, e^{i\frac{5\pi}{4}}, \quad z_4 = \sqrt{2}\, e^{i\frac{7\pi}{4}}$$

which can be written as

$$z_1 = 1+i, \quad z_2 = -1+i, \quad z_3 = -1-i, \quad z_4 = 1-i$$

□

Exercises

1. If $z_1 = 4-5i$ and $z_2 = 2+3i$, find
   (a) $z_1 + z_2$ (b) $z_1 z_2$ (c) $(z_1+z_2)^2$ (d) $\dfrac{1}{z_2}$ (e) $\dfrac{z_2}{z_1}$ (f) $3z_1 - 6z_2$ (g) $\dfrac{z_1}{(z_1+z_2)}$ (h) $0.2z_1^3$ (i) $\dfrac{338}{z_2^2}$
2. Show that
   (a) $i^2 = -1$ (b) $i^3 = -i$ (c) $i^4 = 1$ (d) $i^5 = i$ (e) $\dfrac{1}{i} = -i$ (f) $\dfrac{1}{i^2} = -1$ (g) $\dfrac{1}{i^3} = i$
3. If $z = x + iy$, find
   (a) $\mathbf{Re}\ \dfrac{1}{1+i}$ (b) $\mathbf{Im}\ \dfrac{3+4i}{7-i}$ (c) $(1+i)^8$ (d) $\mathbf{Im}\ \dfrac{z}{\bar{z}}$ (e) $\mathbf{Re}\ z^2$ (f) $\mathbf{Re}\ \dfrac{(2-3i)^2}{2+3i}$ (g) $(\mathbf{Re}\ z)^2$ (h) $\mathbf{Im}\ z^3$ (i) $(\mathbf{Im}\ z)^3$ (j) $(0.3+0.4i)^2$
4. Show that
   (a) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$ (b) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ (c) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ (d) $|z_1 z_2| = |z_1|\,|z_2|$ (e) $|z|^n = |z^n|$ (f) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(g) Verify these formulas for $z_1 = 31 - 34i$ and $z_2 = 2 - 5i$.

5. Find

(a) $|-0.2i|$ (b) $|1.5 + 2i|$ (c) $|z|^4$ (d) $|z^4|$ (e) $\left|\frac{\bar{z}}{z}\right|$ (f) $|\cos\theta + i\sin\theta|$ (g) $\left|\frac{5+7i}{7-5i}\right|$ (h) $\left|\frac{z+1}{z-1}\right|$ (i) $\left|\frac{(1+i)^6}{i^3(1+4i)^2}\right|$

6. Represent in polar form and exponential form, and sketch in the plane.

(a) $2i$ (b) $-2i$ (c) $1+i$ (d) -3 (e) $-1+\sqrt{3}i$ (f) $-1-i$ (g) $6+8i$ (h) $\sqrt{3}-i$ (i) $\frac{1+i}{1-i}$ (j) $\frac{i\sqrt{2}}{4+4i}$ (k) $\frac{3\sqrt{2}+2i}{-\sqrt{2}-2i/3}$ (l) $\frac{2+3i}{5+4i}$

7. Represent in the form $x + iy$

(a) $4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ (b) $2\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$ (c) $10(\cos 0.4 + i\sin 0.4)$ (d) $\cos(-1.8) + i\sin(-1.8)$

8. Solve the following equations, using the method of example 2,

(a) $z^2 + 4 = 0$ (b) $z^2 + z + 1 - i = 0$ (c) $z^4 = 1$ (d) $z^4 = 2$ (e) $z^2 - (5+i)z + 8 + i = 0$ (f) $z^4 - 3(1+2i)z^2 - 8 + 6i = 0$

9. Find all solutions to

(a) $z^3 = 1$ (b) $z^8 = 1$ (c) $z^2 = 1 - i$ (d) $z^2 = 1 + i\sqrt{3}$

10. If x is a real number and $\lambda = r + is$ is complex, find the values of

(a) $\frac{e^{\lambda x} + e^{\bar{\lambda}x}}{2}$ (b) $\frac{e^{\lambda x} - e^{\bar{\lambda}x}}{2i}$

2.2 The Homogeneous Equation - Theory

The general *second order linear* differential equation is of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

where $a(x) \neq 0$, $b(x)$, $c(x)$ and $f(x)$ are functions of x. If we divide by $a(x)$ we obtain this equation in *standard form*,

$$\boxed{y'' + p(x)y' + q(x)y = r(x)}$$

where we have set $p = \frac{b}{a}$, $q = \frac{c}{a}$ and $r = \frac{f}{a}$.

We say that this equation has *constant coefficients* if $p(x)$ and $q(x)$ are constant. We say that this equation is *homogeneous* if $r(x) = 0$, otherwise it is *nonhomogeneous*.

Remark The word "homogeneous" has a meaning here which is different from that of a "homogeneous equation" introduced in chapter 1.

e.g. • The equation

$$y'' - 4y' + 2y = x^2$$

is linear, nonhomogeneous and has constant coefficients.

- The equation
$$y'' - 4y' + 2y = 0$$
is linear, homogeneous and has constant coefficients.
- The equation
$$x^2y'' - 4xy' + 3y = 0$$
is linear and homogeneous. It does not have constant coefficients.

In this section, we will study the theory of the homogeneous equation,

$$\boxed{y'' + p(x)y' + q(x)y \;=\; 0} \tag{2.4}$$

Let us begin by looking at an example.

Example 1 The equation

$$y'' + y = 0. \tag{2.5}$$

You can easily verify that

$$y_1 = \cos x \qquad \text{and} \qquad y_2 = \sin x$$

are both solutions to this equation. (Just substitute these functions into the equation.) Now choose arbitrary numbers c_1 and c_2, and set

$$y \;=\; c_1 \cos x + c_2 \sin x$$

Then also,

$$y'' \;=\; -c_1 \cos x - c_2 \sin x$$

If we add the last two equations we obtain

$$y'' + y = 0.$$

This shows that $y = c_1 \cos x + c_2 \sin x$ is also a solution to equation (2.5). □

This observation can be generalized. But let us first make the following definition:

Definition Let y_1 and y_2 be two functions. A function

$$y \;=\; c_1y_1 + c_2y_2,$$

where c_1 and c_2 are arbitrary numbers, is called a *linear combination* of y_1 and y_2.

e.g.
- $y = 3\sin x + 4\cos x$ is a linear combination of $\sin x$ and $\cos x$.
- $y = 2e^x - 0.5e^{-3x}$ is a linear combination of e^x and e^{-3x} .

Theorem 6 (Superposition Principle) *If y_1 and y_2 are two solutions of the homogeneous linear equation*

$$y'' + p(x)y' + q(x)y \;=\; 0, \tag{2.6}$$

then every linear combination $c_1y_1 + c_2y_2$ is also a solution.

Proof: We can quickly prove this theorem by substituting the function

$$y = c_1y_1 + c_2y_2$$

into the equation. Since

$$y' = c_1y_1' + c_2y_2'$$
$$y'' = c_1y_1'' + c_2y_2''$$

we obtain that

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1y_1'' + c_2y_2'') + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2) \\ &= c_1\underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{=0 \text{ (by (2.6))}} + c_2\underbrace{(y_2'' + p(x)y_2' + q(x)y_2)}_{=0 \text{ (by (2.6))}} \\ &= c_1\,0 + c_2\,0 = 0. \end{aligned}$$

Thus, the linear combination $c_1y_1 + c_2y_2$ is also a solution to equation (2.6). □

Example 1 (continued) We have seen that every function

$$y = c_1 \cos x + c_2 \sin x \tag{2.7}$$

is a solution to the equation

$$y'' + y = 0$$

There are two parameters, c_1 and c_2. To determine particular values for c_1 and c_2 we now need *two initial conditions*. Suppose for example that we have the initial conditions

$$y(0) = 1 \qquad \text{and} \qquad y'(0) = \frac{1}{2}. \tag{2.8}$$

Then the first initial condition gives the equation

$$1 = c_1 \cos 0 + c_2 \sin 0$$
$$c_1 = 1.$$

To find c_2, take derivatives in (2.7),

$$y' = -c_1 \sin x + c_2 \cos x$$

and substitute the second initial condition $y'(0) = 1/2$,

$$1/2 = -c_1 \sin 0 + c_2 \cos 0$$

$$c_2 = 1/2.$$

We obtain the particular solution

$$y = \cos x + \frac{1}{2} \sin x.$$

□

Are there any other solutions to the initial value problem (2.7), (2.8) ? The next theorem says that not.

Theorem 7 (Existence and Uniqueness Theorem). *Consider a second order linear equation*

$$y'' + p(x)y' + q(x)y = r(x) \tag{2.9}$$

with two initial conditions $y(x_o) = y_o$ *and* $y'(x_o) = y'_o$. *(This is called an* initial value problem*). Assume that* $p(x)$, $q(x)$ *and* $r(x)$ *are continuous on an interval* I, *and that* x_o *is a point in* I. *Then there exists exactly one solution which satisfies the initial conditions. This solution exists over the whole interval* I.

Remark This theorem is the analogue to theorem 4, but now for second order equations, and it says two things:

- Every initial value problem (2.9) has a solution (existence).
- There can be at most one solution (uniqueness).

Definition Two functions $f(x)$ and $g(x)$ defined on an interval I are called *linearly dependent* if there exists a constant c such that

$$g = cf \tag{2.10}$$

Otherwise they are called *linearly independent.* Note that $c = 0$ is permitted.

Remark By $g = cf$ we mean that $g(x) = cf(x)$ for *every* x in the interval I.

e.g.
- $f(x) = 2x$ and $g(x) = 6x$ are linearly dependent because $g = 3f$.
- $f(x) = \sin x$ and $g(x) = \cos x$ are linearly independent because $\frac{\sin x}{\cos x} \neq$ *constant.*
- e^x and e^{2x} are linearly independent because $\frac{e^x}{e^{2x}} = e^{-x} \neq$ *constant.*

To test for linear independence, one can use the following tool:

Definition Given functions $f(x)$ and $g(x)$ we define the *Wronskian* to be the determinant

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - f'(x)g(x).$$

Note that $W(f,g)$ is a function of x !

e.g.
- $W(2x, 6x) = \begin{vmatrix} 2x & 6x \\ 2 & 6 \end{vmatrix} = 12x - 12x = 0$
- $W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$
- $W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$
- $W(e^{mx}, e^{nx}) = \begin{vmatrix} e^{mx} & e^{nx} \\ me^{mx} & ne^{nx} \end{vmatrix} = (n-m)e^{(m+n)x}$

Remark Note that the Wronskian $W(f,g)$ in the first example is zero, and in the other examples is nonzero. (By $W(f,g) = 0$ we mean that $W(f,g)$ is the zero function: $W(f,g)(x) = 0$ for *every* x in I.) The next theorem explains why:

Theorem 8 *Two functions $f(x)$ and $g(x)$ defined on an interval I are linearly dependent if and only if $W(f,g) = 0$.*

Proof: Assume first that $W(f,g) = 0$. This means that for all points x in I,

$$f(x)g'(x) - f'(x)g(x) = 0$$

We may assume that $f \neq 0$. (If $f = 0$ then $f = 0 \cdot g$ and the two functions are linearly dependent.) Divide by f^2,

$$\frac{fg' - f'g}{f^2} = 0$$

But the left is the derivative of the quotient g/f. Therefore,

$$\left(\frac{g}{f}\right)' = 0$$

Integrate both sides,

$$\frac{g}{f} = c$$

so that

$$g = cf$$

which shows that f and g are linearly dependent.

Now suppose that f and g are linearly dependent. Then there exists a constant c such that

$$g = cf$$

Take derivatives,

$$g' = cf'$$

Therefore,

$$W(f,g) = fg' - f'g = f(cf') - f'(cf) = 0$$

That is, the Wronskian is zero. □

e.g.
- Since $W(2x, 6x) = 0$, the functions $2x$ and $6x$ are linearly dependent.
- Since $W(x, x^2) = x^2 \neq 0$, the functions x and x^2 are linearly independent.
- Since $W(\cos x, \sin x) = 1 \neq 0$, the functions $\cos x$ and $\sin x$ are linearly independent.
- Since $W(e^{mx}, e^{nx}) = (n-m)e^{(m+n)x} \neq 0$, the functions e^{mx} and e^{nx} are linearly independent for $m \neq n$.

When the functions f and g are solutions of a differential equation, we can say even more:

Theorem 9 *Let y_1 and y_2 be two solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{2.11}$$

on some interval I. If for only one point x_o in I, we have $W(y_1,y_2)(x_o) = 0$, then $W(y_1,y_2)(x) = 0$ or all x in I.

Proof: Set $W = W(y_1, y_2)$. Then,

$$\begin{aligned} W' &= (y_1y_2' - y_1'y_2)' \\ &= (y_1y_2'' + y_1'y_2') - (y_1'y_2' + y_1''y_2) \\ &= y_1y_2'' - y_1''y_2 \end{aligned}$$

Since y_1 and y_2 are solutions to (2.11), we also have

$$y_2'' + p(x)y_2' + q(x)y_2 = 0 \quad (2.12)$$
$$y_1'' + p(x)y_1' + q(x)y_1 = 0 \quad (2.13)$$

Multiply equation (2.12) by y_1 and equation (2.13) by $-y_2$, and add the two equations,

$$(y_1y_2'' - y_1''y_2) + p(x)(y_1y_2' - y_1'y_2) + q(x)(y_1y_2 - y_2y_1) = 0$$

or simply,

$$W' + p(x)W = 0$$

This is a first order linear equation. It's general solution is by (1.30)

$$W(x) = ce^{-\int p(x)\,dx}$$

or solving for c,

$$c = W(x)e^{\int p(x)\,dx}$$

Now if $W(x_o) = 0$ at one point x_o, then c must be zero, so that $W(x) = 0$ for all x. This proves the theorem. □

Remark The theorem really says the following: Either

$$W(y_1, y_2)(x) = 0$$

for *all* x in I (which is the case when y_1 and y_2 are linearly dependent), or

$$W(y_1, y_2)(x) \neq 0$$

for *all* x in I (which is the case when y_1 and y_2 are linearly independent).

Finally, we can give a description of the general solution to the homogeneous equation:

Theorem 10 *Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \quad (2.14)$$

defined on some interval I. Then the general solution is of the form

$$y = c_1y_1 + c_2y_2$$

for constants c_1 and c_2.

Proof: Let $\tilde{y}$ be any solution of equation (2.14). Pick a point x_o in I and set

$$y_o = \tilde{y}(x_o) \qquad \text{and} \qquad y'_o = \tilde{y}'(x_o).$$

So $\tilde{y}$ is the unique solution to the initial value problem

$$y'' + p(x)y' + q(x)y = 0 \tag{2.15}$$
$$y(x_o) = y_o, \qquad y'(x_o) = y'_o \tag{2.16}$$

To prove the theorem, we must find a solution of this initial value problem which is of the form

$$y_c = c_1y_1 + c_2y_2.$$

It will then follow from the existence and uniqueness theorem that $\tilde{y} = y_c$, and we will be done.

By the superposition principle, every linear combination $c_1y_1 + c_2y_2$ is a solution to equation (2.15). We only need to find appropriate values for c_1 and c_2 so that the initial conditions (2.16) are also satisfied.

Taking derivatives, we have

$$y'_c = c_1y'_1 + c_2y'_2$$

To satisfy the initial conditions we need

$$\begin{aligned} y_o &= c_1y_1(x_o) + c_2y_2(x_o) \\ y'_o &= c_1y'_1(x_o) + c_2y'_2(x_o) \end{aligned}$$

We can write this system of equations in matrix form,

$$\begin{pmatrix} y_1(x_o) & y_2(x_o) \\ y'_1(x_o) & y'_2(x_o) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_o \\ y'_o \end{pmatrix}$$

Recall that this matrix equation can be solved for $\binom{c_1}{c_2}$ if the matrix is invertible, that is, if

$$\begin{vmatrix} y_1(x_o) & y_2(x_o) \\ y'_1(x_o) & y'_2(x_o) \end{vmatrix} \neq 0$$

But this is precisely the Wronskian $W(y_1, y_2)(x)$! Now because y_1 and y_2 are linearly independent solutions of the homogeneous equation, the Wronskian is never zero. (This is theorem 9.) Thus, the above system of equations can be solved for c_1 and c_2, that is, we can find a linear combination $y_c = c_1y_1 + c_2y_2$ which satisfies the initial conditions (2.16). This completes the proof. □

Remark This theorem tells us how to find the general solution of the second order homogeneous equation:

1. Find two linearly independent solutions y_1 and y_2.
2. Then the general solution is of the form $y = c_1y_1 + c_2y_2$.

Example 1 (continued) We have already seen that $y_1 = \cos x$ and $y_2 = \sin x$ are two solutions of the equation

$$y'' + y = 0.$$

Since these two functions are linearly independent, the general solution of this equation is

$$y = c_1 \cos x + c_2 \sin x.$$

□

Example 2 Consider the equation

$$y'' + 2y' - 3y = 0.$$

We can easily check that

$$y_1 = e^x \qquad \text{and} \qquad y_2 = e^{-3x}$$

are solutions. Since

$$\frac{y_1}{y_2} = e^{4x} \neq \text{constant}$$

these two solutions are linearly independent. The general solution is therefore

$$y = c_1 e^x + c_2 e^{-3x}.$$

□

Exercises

1. In each of the following cases, find the general solution to the given equation by "trial and error".
   (a) $y'' = 0$
   (b) $y'' - 2y' = 0$
   (c) $y'' - y = 0$
   (d) $(x-1)y'' - xy' + y = 0$
   (e) $y'' + 2y' = 0$

2. By eliminating the constants c_1 and c_2, find the differential equation of each of the following families of curves:
   (a) $y = c_1 x + c_2 x^2$
   (b) $y = c_1 e^{kx} + c_2 e^{-kx}$
   (c) $y = c_1 \sin kx + c_2 \cos kx$
   (d) $y = c_1 + c_2 e^{2x}$
   (e) $y = c_1 x + c_2 \sin x$
   (f) $y = c_1 e^x + c_2 x e^x$
   (g) $y = c_1 e^x + c_2 e^{-3x}$
   (h) $y = c_1 x + c_2 x^{-1}$

3. (a) Use the method of reduction of order to find the general solution of

$$y'' + (y')^2 = 0 \qquad (x > 0).$$

   (b) Verify that $y_1 = 1$ and $y_2 = \ln x$ are linearly independent solutions to this equation. Is $y = c_1 y_1 + c_2 y_2$ the general solution ? If not, does this contradict the results of this section ?

Supplementary exercises:

4. Show that $y = x^2 \sin x$ and $y = 0$ are both solutions to

$$x^2 y'' - 4xy' + (x^2 + 6)y = 0$$

and that both satisfy the conditions $y(0) = 0$ and $y'(0) = 0$. Does this contradict the existence and uniqueness theorem ? If not, why not?

5. If a solution to the equation
$$y'' + p(x)y' + q(x)y = 0$$
is tangent to the x-axis at any point of an interval I, then it must be zero on that interval. Can you explain why ?

6. Let y_1 and y_2 be two solutions of the linear second order homogeneous equation (2.4) on an open interval (a, b). Show that y_1 and y_2 are linearly dependent if either

   (a) $y_1(x_o) = 0 = y_2(x_o)$ at some point x_o in (a, b), or

   (b) y_1 and y_2 have their maximum or minimum at the same point of (a, b).

2.3 Using One Solution to Find Another

In the last section we have seen that in order to solve the second order linear homogeneous equation

$$y'' + p(x)y' + q(x)y \;=\; 0. \tag{2.17}$$

we need to find two linearly independent solutions y_1 and y_2 first.

Usually, it is very difficult to find even these two solutions. Sometimes, we may be able to find at least one solution by "trial and error". The method explained in the following can then be used to reduce the equation to one of first order, which is easier to solve.

Assume that y_1 is one solution to equation (2.17). We are looking for another solution y_2 which is not a constant multiple of y_1. That is, $v = \frac{y_2}{y_1}$ should not be constant. Solve for y_2,

$$\boxed{y_2 \;=\; vy_1}$$

Now take derivatives and use the product rule,

$$y_2' \;=\; v'y_1 + vy_1'$$

$$y_2'' \;=\; v''y_1 + 2v'y_1' + vy_1''$$

Substitute these derivatives into equation (2.17),

$$(\; v''y_1 + 2v'y_1' + vy_1'' \;) + p(x)(\; v'y_1 + vy_1' \;) + q(x)vy_1 = 0$$

Now rearrange,

$$v''y_1 + (\; 2y_1' + p(x)y_1 \;)v' + (\; \underbrace{y_1'' + p(x)y_1' + q(x)y_1}_{=0 \text{ (by (2.17))}} \;)v = 0.$$

We therefore have the equation

$$v''y_1 + (\; 2y_1' + p(x)y_1 \;)v' = 0 \tag{2.18}$$

which really is a first order equation because the variable v is missing. We separate the variables,

$$\frac{v''}{v'} \;=\; -\frac{2y_1' + p(x)y_1}{y_1} \;=\; -2\frac{y_1'}{y_1} - p(x)$$

integrate,

$$\ln v' = -2\ln y_1 - \int p(x)\,dx \tag{2.19}$$

and exponentiate.

$$v' = \frac{1}{y_1^2}\, e^{-\int p(x)\,dx}.$$

To find v we integrate once more,

$$v = \int \frac{1}{y_1^2}\, e^{-\int p(x)\,dx}\,dx.$$

Since $y_2 = vy_1$, we have obtained the second solution

$$\boxed{y_2 = y_1 \int \frac{1}{y_1^2}\, e^{-\int p(x)\,dx}\,dx} \tag{2.20}$$

Remark You may notice that we did not write the integration constant in the last steps. We don't need to because we are interested in only *one* extra solution. Similarly, we could drop the absolute value of $\ln|v'|$ in equation (2.19).

Example 1 Consider the equation

$$x^2y'' + xy' - y = 0 \tag{2.21}$$

Solution. Let us find one solution by "trial and error": If we substitute $y = 1, x, x^2, \ldots$ into the equation, we find that

$$y_1 = x$$

is one solution. To find a second solution, we set

$$y_2 = vx.$$

By the product rule,

$$y_2' = v'x + v \quad \text{and} \quad y_2'' = v''x + 2v'.$$

Substitute into the equation (2.21)

$$x^2(v''x + 2v') + x(v'x + v) - vx = 0$$

and rearrange,

$$x^3v'' + 3x^2v' = 0.$$

This is a separable equation in v'. Separate the variables,

$$\frac{v''}{v'} = \frac{-3x^2}{x^3} = \frac{-3}{x}$$

and integrate both sides,

$$\ln v' = -3\ln x.$$

Exponentiate,

$$v' = x^{-3}$$

and integrate again.

$$v = -\frac{1}{2}x^{-2} \tag{2.22}$$

We may drop the factor $-\frac{1}{2}$ and simply choose $v = x^{-2}$. Then we obtain

$$y_2 = vy_1 = x^{-2}x = x^{-1}.$$

The general solution is

$$y = c_1 x + \frac{c_2}{x}.$$

□

Remark We could omit the constant $-\frac{1}{2}$ in equation (2.22) because in a linear equation, by the superposition principle, every multiple of a solution is again a solution.

Example 2 Consider the equation

$$2x^2y'' + 3xy' - y = 0 \qquad (x > 0) \tag{2.23}$$

Solution. Let us first find one solution by "trial and error": We successively substitute $y = 1, x, x^{-1}, x^2, x^{-2}, \ldots, e^x, e^{2x}, \ldots$ into the equation, and find that

$$y_1 = \frac{1}{x}.$$

is one solution. To find a second solution, set

$$y_2 = vy_1 = vx^{-1}.$$

Then,

$$\begin{aligned} y_2' &= v'x^{-1} - vx^{-2} \\ y_2'' &= v''x^{-1} - 2v'x^{-2} + 2vx^{-3}. \end{aligned}$$

Substitute into the equation and rearrange,

$$\begin{aligned} 2x^2(v''x^{-1} - 2v'x^{-2} + 2vx^{-3}) + 3x(v'x^{-1} - vx^{-2}) - vx^{-1} &= 0 \\ 2xv'' - v' &= 0. \end{aligned}$$

Separate the variables and integrate,

$$\begin{aligned} \frac{v''}{v'} &= \frac{1}{2x} \\ \ln v' &= \frac{1}{2}\ln x. \end{aligned}$$

Exponentiate,

$$v' = e^{\frac{1}{2}\ln x} = x^{1/2}$$

and integrate.

$$v = \frac{2}{3}x^{3/2}$$

Simply choose $v = x^{3/2}$ and obtain

$$y_2 = vy_1 = x^{3/2}x^{-1} = x^{1/2}$$

The general solution is

$$y = \frac{c_1}{x} + c_2\sqrt{x}$$

□

Example 3 Consider the equation

$$(1-x^2)y'' - 2xy' + 2y = 0 \qquad (-1 < x < 1)$$

(This equation is called the *Legendre* equation.)

Solution. You can quickly see that

$$y_1 = x$$

is one solution. Therefore, we set

$$y_2 = vx$$

Then,

$$y_2' = v'x + v \qquad \text{and} \qquad y_2'' = v''x + 2v'$$

Substitute into the equation,

$$(1-x^2)(v''x + 2v') - 2x(v'x + v) + 2vx = 0$$

and rearrange,

$$(1-x^2)xv'' + (2-4x^2)v' = 0$$

Now separate the variables,

$$\frac{v''}{v'} = \frac{4x^2-2}{x(1-x^2)}.$$

Partial fraction decomposition gives

$$\frac{v''}{v'} = \frac{2x}{1-x^2} - \frac{2}{x}.$$

Integrate,

$$\ln v' = -\ln(1-x^2) - 2\ln|x| = \ln\frac{1}{x^2(1-x^2)}.$$

(We need not write $\ln|1-x^2|$ because we assume that $|x| < 1$.) Now exponentiate,

$$v' = \frac{1}{x^2(1-x^2)}.$$

Another partial fraction decomposition gives

$$v' = \frac{1}{x^2} + \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right).$$

Integrate again,

$$v = \frac{-1}{x} + \frac{1}{2}\ln\frac{1+x}{1-x}.$$

Therefore,

$$y_2 = vy_1 = -1 + x\ln\sqrt{\frac{1+x}{1-x}}.$$

The general solution is

$$y = c_1x + c_2\left(-1 + x\ln\sqrt{\frac{1+x}{1-x}}\right).$$

□

Exercises

1. In the following equations, find one solution by "trial and error". Then find the general solution.
   (a) $xy'' + 3y' = 0$ (b) $x^2y'' + xy' - 4y = 0$
2. Use the given solution y_1 to find the general solution using the method of this section.
   (a) $y'' + 4y = 0, \quad y_1 = \sin 2x$ (b) $y'' - y = 0, \quad y_1 = e^x$
   (c) $xy'' - (2x+1)y' + (x+1)y = 0, \quad y_1 = e^x$
3. Show that $y_1 = x$ is a solution of each of the following equations. Then find the general solution.
   (a) $(x-1)y'' - xy' + y = 0$ (b) $x^2y'' + 2xy' - 2y = 0$
   (c) $x^2y'' - x(x+2)y' + (x+2)y = 0$
4. Show that $y_1 = x^{-1/2}\sin x$ is one solution of the *Bessel equation of order 1/2*,

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0 \qquad (x > 0).$$

   Then find the general solution.

   Supplementary exercises:

5. Find the general solution of $y'' - xf(x)y' + f(x)y = 0$.
6. (a) If n is a positive integer, find one solution of

$$xy'' - (x+n)y' + ny = 0$$

   (b) Find the general solution of the equation in (a) for the cases $n = 1, 2, 3$.

2.4 The Linear Equation with Constant Coefficients

In this section, we will look at and solve the homogeneous second order linear equation with constant coefficients,

$$\boxed{y'' + by' + cy = 0} \tag{2.24}$$

(So b and c are constants.) By the discussion in section 2.2 we must first find two linearly independent solutions y_1 and y_2; the general solution will then be

$$y = c_1y_1 + c_2y_2.$$

Let us begin by trying solutions of the form

$$y = e^{\lambda x}.$$

Then,

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

When we substitute these into equation (2.24) we get

$$\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$

Divide by $e^{\lambda x}$,

$$\boxed{\lambda^2 + b\lambda + c = 0.} \tag{2.25}$$

We conclude: The function $y = e^{\lambda x}$ is a solution to equation (2.24) exactly when λ is a solution to the equation (2.25). The latter equation is called the *characteristic equation* or *auxiliary equation* of the differential equation (2.24). By the quadratic formula, its solutions are

$$\lambda_1, \lambda_2 \;=\; \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \tag{2.26}$$

These solutions are also called the *roots* of the characteristic equation. There are three possibilities to distinguish: Two real roots ($b^2 - 4c > 0$), one real root ($b^2 - 4c = 0$), or two complex roots ($b^2 - 4c < 0$).

Case I: The characteristic equation has two distinct real roots. — ($b^2 - 4c > 0$)

We then have found the two solutions

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

of the differential equation (2.24). Because

$$\frac{e^{\lambda_1 x}}{e^{\lambda_2 x}} \;=\; e^{(\lambda_1 - \lambda_2)x} \neq \text{constant},$$

these two functions are linearly independent. The general solution is therefore

$$\boxed{y \;=\; c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.}$$

Example 1 Consider the equation

$$y'' - 3y' + 2y \;=\; 0$$

Solution. The characteristic equation is

$$\lambda^2 - 3\lambda + 2 \;=\; 0$$

It factors as

$$(\lambda - 2)(\lambda - 1) = 0$$

and has real roots

$$\lambda = 1 \qquad \text{and} \qquad \lambda = 2$$

The general solution of the differential equation is therefore

$$y \;=\; c_1 e^{x} + c_2 e^{2x}.$$

□

Example 2 Consider the initial value problem

$$y'' + y' - 12y = 0 \qquad y(2) = 2, \quad y'(2) = 0$$

Solution. Its characteristic equation is

$$\lambda^2 + \lambda - 12 = 0,$$

factors as

$$(\lambda + 4)(\lambda - 3) = 0,$$

and has real roots

$$\lambda_1 = -4, \qquad \lambda_2 = 3.$$

Therefore, the general solution to the differential equation is

$$y = c_1 e^{-4x} + c_2 e^{3x}. \tag{2.27}$$

Let us find the particular solution which satisfies the two initial conditions. We differentiate,

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x}. \tag{2.28}$$

and substitute the initial conditions $y(2) = 2$ and $y'(2) = 0$ into (2.27) and (2.28),

$$\begin{aligned} 2 &= c_1 e^{-8} + c_2 e^6 \\ 0 &= -4c_1 e^{-8} + 3c_2 e^6 \end{aligned}$$

This is a system of two equations in two unknowns c_1 and c_2. If we solve it we obtain

$$c_2 = \frac{8}{7}e^{-6} \qquad \text{and} \qquad c_1 = \frac{6}{7}e^8$$

Therefore, the particular solution is

$$y = \frac{6}{7}e^8 e^{-4x} + \frac{8}{7}e^{-6}e^{3x}$$

which can be written as

$$y = \frac{6}{7}e^{8-4x} + \frac{8}{7}e^{3x-6}.$$

□

Case II: The characteristic equation has only one real root. — ($b^2 - 4c = 0$)

Note that this root equals $\lambda = -b/2$. We have found only one solution of the differential equation,

$$y_1 = e^{\lambda x}.$$

To find a second solution, we use the method of section 2.3, "Using one solution to find another". With $p(x) = b$ and $b = -2\lambda$, formula (2.20) gives

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, e^{-\int p(x)\,dx}\, dx \\ &= e^{\lambda x} \int e^{-2\lambda x}\, e^{-\int b\,dx}\, dx \\ &= e^{\lambda x} \int e^{bx} e^{-bx}\, dx \\ &= e^{\lambda x} \int 1\, dx \\ &= x e^{\lambda x} \end{aligned}$$

The general solution is therefore

$$\boxed{y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}}$$

Example 3 Solve the equation

$$2y'' + 12y' + 18y = 0.$$

Solution. The characteristic equation is

$$2\lambda^2 + 12\lambda + 18 = 0$$

$$2(\lambda + 3)^2 = 0$$

and has only one root

$$\lambda = -3.$$

The general solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x} = e^{-3x}(c_1 + x c_2).$$

□

Example 4 Solve the initial value problem

$$y'' - 4y' + 4y = 0 \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution. The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0.$$

It factors as

$$(\lambda - 2)^2 = 0$$

and has therefore a single real root

$$\lambda = 2.$$

The general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x}.$$

To find the particular solution, let us use the initial conditions. The condition $y(0) = 1$ gives

$$1 = c_1 e^0 + c_2 \cdot 0 e^0$$

so that $c_1 = 1$. Since

$$y' = 2c_1 e^{2x} + c_2(2x + 1)e^{2x},$$

the condition $y'(0) = 0$ gives

$$0 = 2c_1 e^0 + c_2 e^0 = 2 + c_2.$$

Thus, $c_2 = -2$. The particular solution is

$$y = e^{2x} - 2x e^{2x} = (1 - 2x)e^{2x}.$$

□

Case III: The characteristic equation has two complex roots. — $(b^2 - 4c < 0)$

By (2.26) these two roots are

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{-1}\sqrt{4c - b^2}}{2}.$$

So if we set

$$r = -\frac{b}{2} \qquad \text{and} \qquad s = \frac{\sqrt{4c - b^2}}{2}$$

then we can write the roots as

$$\lambda_1 = r + is \qquad \text{and} \qquad \lambda_2 = r - is$$

We thus have found two solutions to the differential equation, namely the functions

$$e^{\lambda_1 x} = e^{(r+is)x} \qquad \text{and} \qquad e^{\lambda_2 x} = e^{(r-is)x}.$$

These are functions with complex values! However, we are only interested in solutions which are real valued. Now remember: In a linear and homogeneous differential equation, any linear combination of two solutions is again a solution. Let us try to choose special numbers c_1 and c_2 making the linear combination

$$c_1 e^{(r+is)x} + c_2 e^{(r-is)x}$$

real valued. (If you have already done exercise 10 in section 2.1 then you know what values to choose.) Note that

$$\begin{aligned} c_1 e^{(r+is)x} + c_2 e^{(r-is)x} &= c_1 e^{rx}(\cos sx + i \sin sx) + c_2 e^{rx}(\cos sx - i \sin sx) \\ &= (c_1 + c_2)\, e^{rx} \cos sx + i(c_1 - c_2)\, e^{rx} \sin sx \end{aligned}$$

We first choose $c_1 = c_2 = \frac{1}{2}$, and get the function

$$y_1 = e^{rx} \cos sx$$

Then we choose $c_1 = \frac{1}{2i}$ and $c_2 = -\frac{1}{2i}$ and get the function

$$y_2 = e^{rx} \sin sx$$

Both, y_1 and y_2 are now real valued. Thus, the general solution is

$$y = c_1 e^{rx} \cos sx + c_2 e^{rx} \sin sx$$

or

$$\boxed{y = e^{rx}(c_1 \cos sx + c_2 \sin sx)}$$

Example 5 Consider the equation

$$y'' + 4y = 0$$

Solution. Its characteristic equation is

$$\lambda^2 + 4 = 0$$

and has roots

$$\lambda^2 = -4$$

$$\lambda_1, \lambda_2 = \pm 2\sqrt{-1} = \pm 2i.$$

These roots are purely imaginary, so that the general solution is

$$y = c_1 \sin(2x) + c_2 \cos(2x).$$

□

Example 6 Solve the initial value problem

$$16y'' - 8y' + 145y = 0 \qquad y(0) = -2, \quad y'(0) = 1$$

Solution. The characteristic equation is

$$16\lambda^2 - 8\lambda + 145 = 0.$$

Its solutions are

$$\lambda = \frac{8 \pm \sqrt{64 - 4 \cdot 16 \cdot 145}}{2 \cdot 16} = \frac{1 \pm \sqrt{1 - 145}}{4} = \frac{1}{4} \pm 3i.$$

Therefore, the general solution is

$$y = e^{x/4} \left(c_1 \cos 3x + c_2 \sin 3x \right).$$

Now find the particular solution which satisfies the initial conditions. First notice that by the product rule,

$$y' = e^{x/4} \left((\frac{1}{4}c_1 + 3c_2) \cos 3x + (\frac{1}{4}c_2 - 3c_1) \sin 3x \right)$$

Then use the initial conditions $y(0) = -2$ and $y'(0) = 1$. We get the system of equations

$$\begin{aligned} -2 &= e^0 \left(c_1 \cos 0 + c_2 \sin 0 \right) \\ 1 &= e^0 \left((\frac{1}{4}c_1 + 3c_2) \cos 0 + (\frac{1}{4}c_2 - 3c_1) \sin 0 \right) \end{aligned}$$

Solving this system we obtain

$$c_1 = -2 \qquad \text{and} \qquad c_2 = \frac{1}{2}$$

The particular solution is

$$y = e^{x/4} \left(\frac{1}{2} \sin 3x - 2 \cos 3x \right).$$

□

Remark In some books you will see a differential equation written in *operator notation.* If we denote the derivative $\frac{dy}{dx}$ by Dy and $\frac{d^2y}{dx^2}$ by D^2y, then equation (2.24) becomes

$$D^2y + bDy + cy = 0$$

which can be abbreviated by

$$(D^2 + bD + c)y = 0.$$

The expression in brackets looks just like the characteristic equation !

Exercises

1. Find the general solution of each equation:
   - (a) $y'' + y' - 6y = 0$
   - (b) $y'' + 2y' + y = 0$
   - (c) $y'' + 8y = 0$
   - (d) $2y'' - 4y' + 8y = 0$
   - (e) $y'' - 4y' + 4y = 0$
   - (f) $y'' - 9y' + 20y = 0$
   - (g) $2y'' + 2y' + 3y = 0$
   - (h) $4y'' - 12y' + 9y = 0$
   - (i) $y'' + y' = 0$
   - (j) $y'' - 6y' + 25y = 0$
   - (k) $4y'' + 20y' + 25y = 0$
   - (l) $y'' + 2y' + 3y = 0$
   - (m) $y'' = 4y$
   - (n) $4y'' - 8y' + 7y = 0$
   - (o) $2y'' + y' - y = 0$
   - (p) $16y'' - 8y' + y = 0$
   - (q) $y'' + 4y' + 5y = 0$
   - (r) $y'' + 4y' - 5y = 0$

2. Solve each of the following initial value problems:

   (a) $y'' - 5y' + 6y = 0, \quad y(1) = e^2, \quad y'(1) = 3e^2$
   (b) $y'' - 6y' + 5y = 0, \quad y(0) = 3, \quad y'(0) = 11$
   (c) $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 5$
   (d) $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$
   (e) $y'' + 4y' + 2y = 0, \quad y(0) = -1, \quad y'(0) = 2 + 3\sqrt{2}$
   (f) $y'' + 8y' - 9y = 0, \quad y(1) = 2, \quad y'(1) = 0$

Supplementary exercises:

3. Show that the general solution of (2.24) approaches 0 as $x \to \infty$ if and only if $b > 0$ and $c > 0$.

4. Show that the derivative of every solution of (2.24) is again a solution.

2.5 The Nonhomogeneous Equation

We now want to study the solution of the *nonhomogeneous* linear equation

$$\boxed{y'' + p(x)y' + q(x)y = r(x)} \tag{2.29}$$

where $r(x) \neq 0$. We will also need the equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2.30}$$

which is called the *related homogeneous equation*.

Assume that we have found one solution y_p to (2.29). That is,

$$y_p'' + p(x)y_p' + q(x)y_p = r(x) \tag{2.31}$$

Now suppose, y_2 is another solution. Then also,

$$y_2'' + p(x)y_2' + q(x)y_2 = r(x)$$

Subtracting the two equations, we get

$$(y_2 - y_p)'' + p(x)(y_2 - y_p)' + q(x)(y_2 - y_p) = 0$$

That is, the difference $y_h = y_2 - y_p$ is a solution to the related homogeneous equation. We can solve for y_2,

$$y_2 = y_h + y_p.$$

Conversely, assume that y_h is a solution to the related homogeneous equation. That is,

$$y_h'' + p(x)y_h' + q(x)y_h = 0 \tag{2.32}$$

Adding this equation and (2.31) we obtain

$$(y_h + y_p)'' + p(x)(y_h + y_p)' + q(x)(y_h + y_p) = 0 + r(x) = r(x)$$

That is, $y_h + y_p$ is also a solution of the nonhomogeneous equation. We have shown:

Theorem 11 *Assume, y_p is one particular solution of the nonhomogeneous equation (2.29). Then the general solution is of the form*

$$\boxed{y = y_h + y_p}$$

where y_h is the general solution to the related homogeneous equation (2.30).

Remark The theorem shows us a way to find the general solution of the nonhomogeneous equation:

1. First find the general solution y_h of the related *homogeneous* equation.
2. Next find one particular solution y_p of the *nonhomogeneous* equation.
3. The general solution is of the form $y = y_h + y_p$.

Example 1 Consider the equation

$$y'' + 3y' + 2y = e^x$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

Therefore, the general solution to the homogeneous equation is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}.$$

You can quickly check that one particular solution of the nonhomogeneous equation is

$$y_p = \frac{1}{6}e^x$$

Therefore, the general solution is

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6}e^x$$

□

Our goal is now to find one particular solution y_p of the homogeneous equation. There are two methods which we can use:

2.5.1 The Method of Undetermined Coefficients

This method can only be used in the case of an equation with *constant coefficients*,

$$y'' + by' + cy = r(x). \tag{2.33}$$

To get an idea of this method, let us do an example first.

Example 1 Consider the equation

$$y'' + 4y = 2e^{3x} \tag{2.34}$$

Solution. The characteristic equation

$$\lambda^2 + 4 = 0$$

has roots

$$\lambda = \pm 2i.$$

The solution to the related homogeneous equation is thus

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

Now we must find a particular solution y_p. The idea is as follows: We note that the function on the right-hand side of equation (2.34) is $r(x) = e^{3x}$. We try to find a particular solution of the same form,

$$y_p = Ae^{3x}$$

where just the constant A is different. Why can we expect that such a particular solution exists ? Notice that y_p and all its derivatives are just multiples of $r(x)$,

$$y_p' = 3Ae^{3x} \qquad \text{and} \qquad y_p'' = 9Ae^{3x}.$$

So if we substitute these functions into (2.34), then the left side becomes just a multiple of $r(x)$,

$$9Ae^{3x} + 4Ae^{3x} = 2e^{3x}.$$
$$13Ae^{3x} = 2e^{3x}.$$

You can see that the two sides can be made equal if we choose

$$A = 2/13.$$

Therefore,

$$y_p = \frac{2}{13}e^{3x}$$

and the general solution is

$$y = y_h + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{2}{13}e^{3x}.$$

□

Remark Because the main task was to find the value of the coefficient A belonging to y_p, this method is called the *method of undetermined coefficients*. Note that this rule works only if the derivatives of the function $r(x)$ are similar to $r(x)$ itself.

Rule 1: If $r(x)$ is one of the functions in the left column of the following table, then we choose a particular solution y_p as indicated on the right side of the table.

$r(x)$	choice for y_p
ae^{mx}	Ae^{mx}
$P_n(x)$	$Q_n(x)$
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$
$e^{mx}\, P_n(x)$	$e^{mx}\, Q_n(x)$
$e^{mx}(a\cos\omega x + b\sin\omega x)$	$e^{mx}(A\cos\omega x + B\sin\omega x)$
$P_n(x)\cos\omega x + R_n(x)\sin\omega x$	$Q_n(x)\cos\omega x + S_n(x)\sin\omega x$
$e^{mx}\,(P_n(x)\cos\omega x + R_n(x)\sin\omega x)$	$e^{mx}\,(Q_n(x)\cos\omega x + S_n(x)\sin\omega x)$

Undetermined coefficients - Choices for y_p.

In this table, P_n, R_n, Q_n and S_n denote *polynomials of degree* n, for example,

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_o$$

and

$$Q_n(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_o.$$

Just as in the last example, we can find the values of the undetermined coefficients A, $B \ldots$ by substituting our choice for y_p into equation (2.33).

Warning: This method can only be applied, if the function $r(x)$ is as in the table. That is, $r(x)$ must be

1) a polynomial, or
2) a simple exponential function, or
3) a linear combination of the sine and cosine functions, or
4) a sum or a product of functions in 1) – 3)

Example 2 Consider the problem

$$y'' - y' - 2y = 4x^2 \qquad y(0) = 0, \quad y(1) = \frac{3}{e} - 3 \tag{2.35}$$

Solution. The characteristic equation is

$$\begin{aligned} \lambda^2 - \lambda - 2 &= 0 \\ (\lambda - 2)(\lambda + 1) &= 0 \end{aligned}$$

Therefore, the general solution to the related homogeneous equation is

$$y_h = c_1 e^{-x} + c_2 e^{2x}$$

We think of $r(x) = 4x^2$ as a polynomial of degree two (in fact, $r(x) = 4x^2 + 0x + 0$) and therefore choose a particular solution which is also a polynomial of degree two,

$$y_p = Ax^2 + Bx + C.$$

Then,

$$y_p' = 2Ax + B \qquad \text{and} \qquad y_p'' = 2A.$$

Substitute into equation (2.35),

$$2A - (2Ax + B) - 2(Ax^2 + Bx + C) = 4x^2$$

Collect all like powers of x,

$$-2Ax^2 + (-2A - 2B)x + (2A - B - 2C) = 4x^2 + 0x + 0.$$

Now we compare coefficients,

$$\begin{aligned} -2A \qquad\qquad\qquad &= 4 \\ -2A - 2B \qquad\quad &= 0 \\ 2A - B - 2C &= 0 \end{aligned}$$

This is a system of three equations and three unknowns. If we solve it, we get

$$A = -2, \qquad B = 2, \qquad \text{and} \qquad C = -3.$$

Therefore,

$$y_p = -2x^2 + 2x - 3$$

so that the general solution to equation (2.35) is

$$y = c_1 e^{-x} + c_2 e^{2x} - 2x^2 + 2x - 3.$$

Now we must find the particular solution satisfying the two conditions $y(0) = 0$ and $y(1) = \frac{3}{e} - 3$. So we substitute these conditions into the general solution,

$$\begin{aligned} 0 &= c_1 + c_2 - 3 \\ \frac{3}{e} - 3 &= c_1 e^{-1} + c_2 e^2 - 3 \end{aligned}$$

Solving, we get

$$c_1 = 3 \qquad \text{and} \qquad c_2 = 0$$

The particular solution is

$$y = 3e^{-x} - 2x^2 + 2x - 3.$$

□

Remark This is not an initial value problem. The two conditions in (2.35) specify the values of y for different values of x. Such conditions are called *boundary conditions* and the problem is called a *boundary value problem.*

Example 3 Solve the equation

$$y'' + 6y' + 9y = 6\sin 2x + 4\cos 2x$$

Solution. The characteristic equation is

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0.$$

Therefore, the solution of the homogeneous equation is

$$y_h = c_1 e^{-3x} + c_2 x e^{-3x}.$$

We try a particular solution

$$y_p = A\cos 2x + B\sin 2x.$$

Taking derivatives,

$$\begin{aligned} y_p' &= -2A\sin 2x + 2B\cos 2x \\ y_p'' &= -4A\cos 2x - 4B\sin 2x. \end{aligned}$$

Now substitute all these into the differential equation,

$$\begin{aligned} (-4A\cos 2x - 4B\sin 2x) \;+\; 6(-2A\sin 2x + 2B\cos 2x) \;+\; 9(A\cos 2x + B\sin 2x) \\ = 6\sin 2x + 4\cos 2x. \end{aligned}$$

Collect like terms,

$$(5A + 12B)\cos 2x \;+\; (5B - 12A)\sin 2x = 6\sin 2x + 4\cos 2x.$$

Compare the coefficients,

$$\begin{aligned} 5A + 12B &= 4 \\ -12A + 5B &= 6 \end{aligned}$$

Solving this system in two unknowns, we get that

$$A = -\frac{4}{13} \qquad \text{and} \qquad B = \frac{6}{13}.$$

Therefore, the general solution is

$$y = y_h + y_p = c_1 e^{-3x} + c_2 x e^{-3x} - \frac{4}{13}\cos 2x + \frac{6}{13}\sin 2x.$$

□

Example 4 Solve the equation

$$y'' + 4y' + 4y = 2e^{-2x}. \tag{2.36}$$

Solution. The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.$$

It has one repeated root $\lambda = -2$. Therefore, the solution to the homogeneous equation is

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}. \tag{2.37}$$

Now as $r(x) = 2e^{-2x}$ we would think to try the homogeneous solution

$$y_p = Ae^{-2x}.$$

However, this function is already a solution of the homogeneous equation! (Just look at (2.37) with $c_2 = 0$.) This means that when we substitute it into the left side of equation (2.36) we get zero; we can never get $2e^{-2x}$. What to do now ? We multiply it by x to obtain a new choice for y_p,

$$y_p = Axe^{-2x}.$$

This is still a solution to the homogeneous equation. (Now look at (2.37) with $c_1 = 0$.) We multiply by x again, and try

$$y_p = Ax^2 e^{-2x}.$$

Then,

$$y_p' = (2x - 2x^2)Ae^{-2x}$$

$$y_p'' = (2 - 8x + 4x^2)Ae^{-2x}.$$

Substitute into the equation (2.36),

$$(2 - 8x + 4x^2)Ae^{-2x} + 4(2x - 2x^2)Ae^{-2x} + 4Ax^2e^{-2x} = 2e^{-2x}$$

Most terms on the left cancel,

$$2Ae^{-2x} = 2e^{-2x}$$

$$A = 1$$

The general solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 e^{-2x} = (c_1 + c_2 x + x^2)\, e^{-2x}.$$

□

Let us summarize this as a rule.

Rule 2: If our choice for y_p contains a term which is already a solution of the related homogeneous equation, then multiply this choice by x to get a new choice for y_p.

Example 5 Solve the initial value problem

$$y'' + 2y' = 2x + 5 \qquad y(0) = 0, \quad y'(0) = 0$$

Solution. The characteristic equation is

$$\lambda^2 + 2\lambda = \lambda(\lambda + 2) = 0$$

Therefore, the solution of the related homogeneous equation is

$$y_h = c_1 e^{0x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}.$$

Since $r(x) = 2x + 5$, we would try the particular solution

$$y_p = Ax + B$$

But note that the constant function $y = B$ is already a solution of the homogeneous equation. We must therefore multiply by x and obtain a new try,

$$y_p = Ax^2 + Bx$$

Then,

$$y_p' = 2Ax + B \qquad \text{and} \qquad y_p'' = 2A$$

Substituting into the equation,

$$\begin{aligned} 2A + 2(2Ax + B) &= 2x + 5 \\ 4Ax + (2A + 2B) &= 2x + 5 \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} 4A &= 2 \\ 2A + 2B &= 5 \end{aligned}$$

Therefore,

$$A = 1/2 \qquad \text{and} \qquad B = 2$$

The general solution is

$$y = y_h + y_p = c_1 + c_2 e^{-2x} + \frac{x^2}{2} + 2x$$

Now we must look for the particular solution which satisfies the initial conditions. We have

$$y' = -2c_2 e^{-2x} + x + 2$$

We substitute the initial conditions $y(0) = 0$ and $y'(0) = 0$ into the last two equations and obtain

$$\begin{aligned} 0 &= c_1 + c_2 \\ 0 &= -2c_2 + 2 \end{aligned}$$

Solving, we get $c_2 = 1$ and $c_1 = -1$. The particular solution is

$$y = -1 + e^{-2x} + \frac{x^2}{2} + 2x.$$

□

Remark The last two examples show that we must compute y_h *before* we can choose the correct y_p. Always check whether your choice of y_p is already part of y_h, because in this case you must adjust y_p !

If $r(x)$ can be written as a sum of functions listed in the table, then we can use the following rule to find a particular solution:

Theorem 12 (Superposition Principle). *Consider the equation*

$$y'' + p(x)y' + q(x)y = r_1(x) + r_2(x). \tag{2.38}$$

If y_{p_1} is a particular solution to the equation

$$y'' + p(x)y' + q(x)y = r_1(x)$$

and y_{p_2} a particular solution to

$$y'' + p(x)y' + q(x)y = r_2(x)$$

then the sum

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution to (2.38).

Proof: All we need to is substitute the function $y_p = y_{p_1} + y_{p_2}$ into the left side of equation (2.38). We then obtain

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) \\ &= (y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1}) + (y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2}) \\ &= r_1(x) + r_2(x) \end{aligned}$$

This proves the theorem. □

Example 6 Solve the equation

$$y'' + 2y' + 5y = 16e^x + 17\sin 2x \tag{2.39}$$

Solution. The characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

has solutions

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i.$$

Therefore,

$$y_h = e^{-x}(c_1\cos 2x + c_2 \sin 2x)$$

As $r_1(x) = 16e^x$ and $r_2(x) = 17\sin 2x$ we choose the particular solution

$$y_p = Ae^x + (B\cos 2x + C\sin 2x).$$

Then,

$$\begin{aligned} y_p' &= Ae^x - 2B\sin 2x + 2C\cos 2x \\ y_p'' &= Ae^x - 4B\cos 2x - 4C\sin 2x \end{aligned}$$

Substitute into equation (2.39) and collect terms,

$$(A+2A+5A)e^x + (-4B+4C+5B)\cos 2x + (-4C-4B+5C)\sin 2x = 16e^x + 17\sin 2x$$

Compare coefficients,

$$\begin{aligned} 8A &= 16 \\ B+4C &= 0 \\ -4B+C &= 17 \end{aligned}$$

Solving we get $A=2$, $B=-4$ and $C=1$. The general solution is

$$y = y_h + y_p = e^{-x}\,(\,c_1\cos 2x + c_2\sin 2x\,) + 2e^x - 4\cos 2x + \sin 2x.$$

□

Example 7 Solve the initial value problem

$$y'' + 9y = 18x + 5e^x + 12\cos 3x \qquad y(0) = 1/2, \quad y'(0) = 1 \tag{2.40}$$

Solution. The characteristic equation

$$\lambda^2 + 9 = 0$$

has two purely imaginary solutions

$$\lambda = \pm 3i.$$

Therefore, the solution to the homogeneous equation is

$$y_h = c_1\cos 3x + c_2\sin 3x.$$

We would try the particular solution

$$y_p = (Ax+B) + Ce^x + (D\cos 3x + E\sin 3x).$$

But note that $\cos 3x$ and $\sin 3x$ are already solutions to the homogeneous equation. So we multiply $\cos 3x$ and $\sin 3x$ by x, and try the new choice

$$y_p = (Ax+B) + Ce^x + (Dx\cos 3x + Ex\sin 3x).$$

Then,

$$\begin{aligned} y_p' &= A + Ce^x + (D+3Ex)\cos 3x + (E-3Dx)\sin 3x. \\ y_p'' &= Ce^x + (6E-9Dx)\cos 3x + (-6D-9Ex)\sin 3x. \end{aligned}$$

Substitute into the equation (2.40) and collect like terms,

$$(9Ax+9B) + 10Ce^x + 6E\cos 3x - 6D\sin 3x = 18x + 5e^x + 12\cos 3x.$$

Compare coefficients,

$$A=2, \quad B=0, \quad C=\frac{1}{2}, \quad D=0, \quad \text{and} \quad E=2$$

The general solution is

$$y = c_1\cos 3x + c_2\sin 3x + 2x + \frac{1}{2}e^x + 2x\sin 3x.$$

To find the particular solution satisfying the initial conditions, take the derivative,

$$y' = -3c_1 \sin 3x + 3c_2 \cos 3x + 2 + \frac{1}{2}e^x + 2\sin 3x + 6x\cos 3x.$$

and substitute the values $y(0) = \frac{1}{2}$ and $y'(0) = 1$ into the last two equations,

$$\begin{aligned} \frac{1}{2} &= c_1 + \frac{1}{2} \\ 1 &= 3c_2 + 2 + \frac{1}{2} \end{aligned}$$

Solving these equations we obtain

$$c_1 = 0 \qquad \text{and} \qquad c_2 = -\frac{1}{2}$$

The particular solution is therefore

$$y = 2x + \frac{1}{2}e^x + \left(2x - \frac{1}{2}\right)\sin 3x.$$

□

Example 8 Find the correct pattern for the particular solution y_p, if

a) $y'' + 4y' + 3y = x + e^x + \cos x$

b) $y'' + 4y = x^2e^x + \sin 2x - \cos 2x + 3x\sin 2x$

c) $y'' + y = \cos x + \sin 2x + e^x \sin x$

Do not evaluate the coefficients A, B, ...

Solution. Remember that we must always find y_h before we can choose the correct y_p.

a) In the first equation,

$$y_h = c_1e^{-x} + c_2e^{-3x}$$

We choose y_p of the form

$$y_p = (Ax + B) + Ce^x + (D\cos x + E\sin x)$$

b) In the second equation,

$$y_h = c_1\cos 2x + c_2\sin 2x$$

The function on the right must be thought of as

$$r(x) = (1\cdot x^2 + 0\cdot x + 0)e^x + [(3x+1)\sin 2x + (0\cdot x - 1)\cos 2x]$$

so that we choose

$$y_p = (Ax^2 + Bx + C)e^x + [(Dx + E)\sin 2x + (Fx + G)\cos 2x]$$

However, the terms $E\sin 2x$ and $G\cos 2x$ are already part of y_h, so we must multiply the second part by x,

$$y_p = (Ax^2 + Bx + C)e^x + \left[(Dx^2 + Ex)\sin 2x + (Fx^2 + Gx)\cos 2x\right]$$

c) In the last equation,

$$y_h = c_1 \cos x + c_2 \sin x$$

We first choose

$$y_p = (A\cos x + B\sin x) + (C\cos 2x + D\sin 2x) + e^x(E\cos x + F\sin x)$$

But $A\cos x$ and $B\sin x$ are already part of y_h, so we must multiply the first part of y_p by x, and choose

$$y_p = (Ax\cos x + Bx\sin x) + (C\cos 2x + D\sin 2x) + e^x(E\cos x + F\sin x)$$

□

Exercises

1. Find the general solution of the following equations:
   (a) $y'' - 2y' - 3y = 3e^{2x}$
   (b) $y'' - 2y' - 3y = -3xe^{-x}$
   (c) $y'' + 2y' + 5y = 3\sin 2x$
   (d) $y'' + 2y' = 3 + 4\sin 2x$
   (e) $y'' + 9y = x^2e^{3x} + 6$
   (f) $y'' + 2y' + y = 2e^{-x}$
   (g) $2y'' + 3y' + y = x^2 + 3\sin x$
   (h) $y'' + y = 3\sin 2x + x\cos 2x$
   (i) $y'' + \omega_o^2 y = \cos\omega x \qquad (\omega_o^2 \neq \omega^2)$
   (j) $y'' + \omega_o^2 y = \cos\omega_o x$
   (k) $y'' - y' + 4y = 2\sinh x$
   (l) $y'' - y' - 2y = \cosh 2x$
   (m) $y'' + 4y = 4\cos 2x + 6\cos x + 8x^2 - 4x$
   (n) $y'' + 9y = 2\sin 3x + 4\sin x - 26e^{-2x} + 27x^3$

2. Solve each initial value problem:
   (a) $y'' + y' - 2y = 2x, \quad y(0) = 0, \quad y'(0) = 1$
   (b) $y'' + 4y = x^2 + 3e^x, \quad y(0) = 0, \quad y'(0) = 2$
   (c) $y'' - 2y' + y = xe^x + 4, \quad y(0) = 1, \quad y'(0) = 1$
   (d) $y'' - 2y' - 3y = 3xe^{2x}, \quad y(0) = 1, \quad y'(0) = 0$
   (e) $y'' + 4y = 3\sin 2x, \quad y(0) = 2, \quad y'(0) = -1$
   (f) $y'' + 2y' + 5y = 4e^{-x}\cos 2x, \quad y(0) = 1, \quad y'(0) = 0$

3. In the following problems, find the correct choice for the particular solution y_p. Do not evaluate the constants $A, B, \ldots$.
   (a) $y'' + 3y' = 2x^4 + x^2e^{-3x} + \sin 3x$
   (b) $y'' + y = x(1 + \sin x)$
   (c) $y'' - 5y' + 6y = e^x\cos 2x + e^{2x}(3x + 4)\sin x$
   (d) $y'' + 2y' + 2y = 3e^{-x} + 2e^{-x}\cos x + 4e^{-x}x^2\sin x$
   (e) $y'' - 4y' + 4y = 2x^2 + 4xe^{2x} + x\sin 2x$
   (f) $y'' + 4y = x^2\sin 2x + (6x + 7)\cos 2x$
   (g) $y'' + 3y' + 2y = e^x(x^2 + 1)\sin 2x + 3e^{-x}\cos x + 4e^x$
   (h) $y'' + 2y' + 5y = 3xe^{-x}\cos 2x - 2xe^{-2x}\cos x$

4. In many physical systems, the nonhomogeneous term is specified by different formulas in different time periods. Solve the following equations and sketch their solutions.
   (a) $y'' + y = \begin{cases} t, & 0 \le t \le \pi, \\ \pi e^{\pi - t}, & t > \pi, \end{cases} \qquad y(0) = 0, \quad y'(0) = 1.$
   (b) $y'' + 2y' + 5y = \begin{cases} 1, & 0 \le t \le \pi/2, \\ 0, & t > \pi/2, \end{cases} \qquad y(0) = 0, \quad y'(0) = 0.$

2.5.2 Variation of Parameters

We now study a second method for finding a particular solution y_p. Consider a general nonhomogeneous equation,

$$y'' + p(x)y' + q(x)y = r(x) \tag{2.41}$$

and assume that we have already found the general solution y_h of the related homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2.42}$$

which always is a linear combination of the form

$$y_h = c_1y_1 + c_2y_2.$$

The idea is as follows: Since every particular solution y_p of (2.41) is different from y_h, there must exist nonconstant functions $u = u(x)$ and $v = v(x)$ such that

$$y_p = uy_1 + vy_2. \tag{2.43}$$

(That is, we *vary* the parameters c_1 and c_2.) To determine these functions u and v, we substitute this choice of y_p into equation (2.41). But first we must take derivatives,

$$y_p' = uy_1' + u'y_1 + v'y_2 + vy_2'.$$

If we differentiate once more, the second order derivatives u' and v' will also appear. In order to avoid this to happen, we impose one more condition and require that

$$u'y_1 + v'y_2 = 0 \tag{2.44}$$

Then simply,

$$y_p' = uy_1' + vy_2'$$

and

$$y_p'' = uy_1'' + u'y_1' + v'y_2' + vy_2''.$$

Now we can substitute y_p into (2.41),

$$(uy_1'' + u'y_1' + v'y_2' + vy_2'') + p(x)(uy_1' + vy_2') + q(x)(uy_1 + vy_2) = r(x).$$

Collect all terms of u and v,

$$u\,\underbrace{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}_{=0} + v\,\underbrace{\left(y_2'' + p(x)y_2' + q(x)y_2\right)}_{=0} + u'y_1' + v'y_2' = r(x)$$

Because y_1 and y_2 are solutions to the homogeneous equation (2.42), this reduces to

$$u'y_1' + v'y_2' = r(x). \tag{2.45}$$

Now together with condition (2.44) we have the system of two equations,

$$\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1' + v'y_2' &= r(x) \end{aligned} \tag{2.46}$$

In matrix form, this system can be written as

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \tag{2.47}$$

Note that the determinant of this matrix is the Wronskian $W = W(y_1, y_2)$! We know that $W \neq 0$ because y_1 and y_2 are linearly independent. Therefore, we can use Cramer's rule to solve this system and get

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 r}{W} \qquad \text{and} \qquad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 r}{W}$$

Integrate u' and v' and substitute into (2.43) to obtain the particular solution

$$\boxed{y_p = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx} \tag{2.48}$$

Remark This method is called *variation of parameters.* It can be used for EVERY linear equation and for EVERY function $r(x)$. However, we may end up with integrals which are difficult to evaluate.

Example 1 Find the solution to

$$y'' + y = \csc x \tag{2.49}$$

Solution. The characteristic equation is

$$\lambda^2 + 1 = 0$$

and has roots $\lambda = \pm i$. Therefore, the solution to the homogeneous equation is

$$y_h = c_1 \cos x + c_2 \sin x$$

To find a particular solution of the nonhomogeneous equation, set

$$y_p = u \cos x + v \sin x$$

where $u = u(x)$ and $v = v(x)$ are functions of x. Then,

$$y_p' = -u \sin x + u' \cos x + v' \sin x + v \cos x$$

Now impose the condition

$$u' \cos x + v' \sin x = 0 \tag{2.50}$$

so that simply

$$y_p' = -u \sin x + v \cos x$$

Take derivatives again,

$$y_p'' = -u \cos x - u' \sin x + v' \cos x - v \sin x$$

Now substitute y_p into the equation (2.49),

$$(-u \cos x - u' \sin x + v' \cos x - v \sin x) + u \cos x + v \sin x = \csc x$$

and simplify,

$$-u' \sin x + v' \cos x = \csc x.$$

Together with (2.50) we have the system of equations

$$\begin{aligned} u' \cos x + v' \sin x &= 0 \\ -u' \sin x + v' \cos x &= \csc x \end{aligned}$$

Solving this system for u' and v' we get

$$\begin{aligned} u'(\cos^2 x + \sin^2 x) &= -\sin x \csc x \\ v'(\sin^2 x + \cos^2 x) &= \cos x \csc x \end{aligned}$$

so that

$$u' = -1 \qquad \text{and} \qquad v' = \frac{\cos x}{\sin x}.$$

Integrate,

$$u(x) = -x \qquad \text{and} \qquad v(x) = \ln|\sin x|.$$

Therefore, the particular solution is

$$y_p = u \cos x + v \sin x = -x \cos x + \sin x \ln|\sin x|$$

Finally, the general solution to this differential equation is

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln|\sin x|$$

□

Remark Usually, we don't need to go through all these steps. After having found the homogeneous solution y_h we continue by solving the system of equations (2.46) or the matrix equation (2.47). This requires that the differential equation be in *standard form*, because (2.46) and (2.47) were derived from an equation of form (2.41).

Example 2 Solve the equation

$$x^2 y'' + xy' - y = x \ln x \qquad (\, x > 0\,) \tag{2.51}$$

Solution. We have seen in example (2.21) that the related homogeneous equation has solution

$$y_h = c_1 x + \frac{c_2}{x}.$$

(In the next section you will see another method for finding this solution.) We therefore try the particular solution

$$y_p = ux + vx^{-1}.$$

Let us start right away with the system of equations (2.46). But first we must bring equation (2.51) into standard form,

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = \frac{\ln x}{x}.$$

Since

$$y_1' = 1 \qquad \text{and} \qquad y_2' = -x^{-2},$$

system (2.46) is of the form

$$\begin{aligned} u'x + v'x^{-1} &= 0 \\ u' \quad - v'x^{-2} &= \frac{\ln x}{x} \end{aligned}$$

This system can be easily solved. For example, if we divide the first equation by x, we obtain

$$\begin{aligned} u' + v'x^{-2} &= 0 \\ u' \quad - v'x^{-2} &= \frac{\ln x}{x} \end{aligned}$$

Now adding both equations gives

$$2u' = \frac{\ln x}{x} \qquad \text{so that} \qquad u' = \frac{1}{2}\frac{\ln x}{x}$$

Subtracting both equations gives

$$2v'x^{-2} = -\frac{\ln x}{x} \qquad \text{so that} \qquad v' = -\frac{1}{2}x\ln x$$

Integrate,

$$u = \frac{1}{2}\int \frac{\ln x}{x}\,dx = \frac{1}{2}\int u\,du = \frac{1}{4}u^2 = \frac{1}{4}(\ln x)^2$$

and

$$\begin{aligned} v = -\frac{1}{2}\int x\ln x\,dx &= -\frac{1}{2}\Big(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,dx\Big) \\ &= -\frac{1}{2}\Big(\frac{x^2}{2}\ln x - \frac{x^2}{4}\Big) = \frac{x^2}{8}(1 - 2\ln x) \end{aligned}$$

Therefore, the particular solution is

$$y_p = ux + vx^{-1} = \frac{1}{4}x(\ln x)^2 + \frac{x}{8}(1 - 2\ln x)$$

and the general solution to equation (2.51) is

$$y = y_h + y_p = \tilde{c}_1 x + \frac{c_2}{x} + \frac{1}{4}x(\ln x)^2 - \frac{1}{4}x\ln x$$

where we have set $\tilde{c}_1 = c_1 + 1/8$. □

Remark In both examples we have omitted the integration constants because we require only *one* function u and *one* function v.

Exercises

1. Solve each of the following equations using both, the method of undetermined coefficients and variation of parameters.
   (a) $y'' - 5y' + 6y = 2e^x$
   (b) $y'' + 2y' + y = 3e^{-x}$
   (c) $y'' - y' - 2y = 2e^{-x}$
   (d) $4y'' - 4y' + y = 16e^{x/2}$

2. Find the general solution of the following equations:
   (a) $y'' + y = \tan x$
   (b) $y'' + 4y' + 4y = x^{-2}e^{-2x}$
   (c) $y'' + 9y = 9\sec^2 3x$
   (d) $y'' + 4y = 3\csc 2x$
   (e) $y'' + 4y = \tan 2x$
   (f) $4y'' + y = 2\sec(x/2)$
   (g) $y'' + 2y' + y = e^{-x}\ln x$
   (h) $y'' - 2y' + y = \frac{e^x}{1+x^2}$
   (i) $y'' - 2y' - 3y = 64xe^{-x}$
   (j) $y'' + 2y' + 5y = e^{-x}\sec 2x$
   (k) $2y'' + 3y' + y = e^{-3x}$
   (l) $y'' - 3y' + 2y = (1 + e^{-x})^{-1}$

3. In a previous exercise we have seen that the general solution of Bessel's equation

$$x^2y'' + xy' + (x^2 - 0.25)y = 0$$

is $y_h = c_1x^{-1/2}\sin x + c_2x^{-1/2}\cos x$. Find the general solution of

$$x^2y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x$$

4. Show that the equation $y'' + y = f(x)$ has a particular solution

$$y_p(x) = \int_{x_o}^{x} f(t)\sin(x - t)\,dt$$

Then find the solution to
   (a) $y'' + y = \sec x$
   (b) $y'' + y = \cot^2 x$
   (c) $y'' + y = \cot 2x$
   (d) $y'' + y = x\cos x$
   (e) $y'' + y = \tan x$
   (f) $y'' + y = \sec x \tan x$
   (g) $y'' + y = \sec x \csc x$

2.6 Cauchy-Euler Equations

An equation of the form

$$\boxed{x^2y'' + bxy' + cy = r(x)} \tag{2.52}$$

is called a second order *Cauchy-Euler equation*.

It can be transformed into an equation with constant coefficients by substituting

$$x = e^t$$

which is equivalent to $t = \ln x$. Then by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \tag{2.53}$$

and by the product rule,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \tag{2.54}$$

Now substitute into (2.52) and obtain

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + b\frac{dy}{dt} + cy = r(e^t)$$

or

$$\boxed{\frac{d^2y}{dt^2} + (b-1)\frac{dy}{dt} + cy = f(t)}$$

where we have set $f(t) = r(e^t)$. This is now a linear equation with constant coefficients.

Remark You may notice that this new equation looks similar to (2.52), except that b has been replaced by $b-1$, x has disappeared on the left, and x has been replaced by e^t on the right.

Remark Note that the substitution $x = e^t$ is only correct for $x > 0$. If $x < 0$, then we must substitute $x = -e^t$. For the sake of simplicity, we will assume that $x > 0$ unless stated otherwise.

Example 1 Solve the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3. \tag{2.55}$$

Solution. As this is an Euler equation, we substitute $x = e^t$. Then as shown above,

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

so that the equation becomes

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t} \tag{2.56}$$

The characteristic equation is

$$\lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0$$

and has roots $\lambda = 1, 2$. Therefore, the solution to the homogeneous equation is

$$y_h = c_1e^t + c_2e^{2t}.$$

To find a particular solution, it is easiest to use the method of undetermined coefficients. Set $y_p = Ae^{3t}$ and substitute into (2.56),

$$9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} = e^{3t}$$

so that

$$A = 1/2.$$

Therefore, the solution of (2.56) is

$$y = y_h + y_p = c_1e^t + c_2e^{2t} + \frac{1}{2}e^{3t}$$

We must not forget to resubstitute $x = e^t$, to obtain the general solution of (2.55),

$$y = c_1x + c_2x^2 + \frac{x^3}{2}.$$

□

Example 2 Solve the equation

$$x^2y'' - 4xy' + 6y = x^4 \sin x \tag{2.57}$$

Solution. This is an Euler equation. We set $x = e^t$, so that the equation is transformed into an equation with constant coefficients,

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = e^{4t} \sin e^t \tag{2.58}$$

The characteristic equation

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

has solutions $\lambda = 2, 3$. The general solution to the homogeneous equation (2.58) is

$$y_h = c_1e^{2t} + c_2e^{3t}.$$

Because of the term $\sin e^t$ we can not use the method of undetermined coefficients to find a particular solution y_p in (2.58). It is easier to resubstitute first,

$$y_h = c_1x^2 + c_2x^3$$

and use the method of variation of parameters with (2.57). So we set

$$y_p = ux^2 + vx^3.$$

Then,

$$y_p' = 2ux + u'x^2 + v'x^3 + 3vx^2 = 2ux + 3vx^2$$

because we have imposed the condition

$$u'x^2 + v'x^3 = 0 \tag{2.59}$$

Also

$$y_p'' = 2u + 2u'x + 3v'x^2 + 6vx.$$

Substituting y_p into (2.57) we get

$$x^2(2u + 2u'x + 3v'x^2 + 6vx) - 4x(2ux + 3vx^2) + 6(ux^2 + vx^3) = x^4 \sin x$$

which simplifies to

$$2u'x^3 + 3v'x^4 = x^4 \sin x. \tag{2.60}$$

Equations (2.59) and (2.60) give us a system of equations

$$\begin{array}{rcrcl} u' & + & v'x & = & 0 \\ 2u' & + & 3v'x & = & x \sin x. \end{array}$$

Solving this system, we get

$$u' = -x \sin x \qquad \text{and} \qquad v' = \sin x.$$

Integrate,

$$u = x \cos x - \sin x \qquad \text{and} \qquad v = -\cos x.$$

Therefore,

$$y_p = ux^2 + vx^3 = x^2(x \cos x - \sin x) + x^3(-\cos x) = -x^2 \sin x$$

The general solution is

$$y = c_1x^2 + c_2x^3 - x^2 \sin x.$$

□

Exercises

1. Find the general solution of each of the following equations. (Assume, $x > 0$.)
   (a) $x^2y'' - 3xy' + 3y = 0$
   (b) $4x^2y'' - 4xy' + 3y = 0$
   (c) $x^2y'' + xy' + 4y = 0$
   (d) $3x^2y'' - 4xy' + 2y = 0$
   (e) $x^2y'' + xy' + 9y = 0$
   (f) $9x^2y'' + 3xy' + y = 0$
   (g) $x^2y'' - 5xy' + 10y = 0$
   (h) $x^2y'' + xy' - 4y = 0$
   (i) $x^2y'' - 3xy' + 4y = 0$
   (j) $x^2y'' - 3xy' + 13y = 0$
   (k) $x^2y'' + xy' + 4y = 2x\ln x$
   (l) $x^2y'' + xy' + y = 4\sin(\ln x)$
   (m) $x^2y'' - 3xy' + 5y = 5x^2$
   (n) $x^2y'' - 2y = 3x^2 - 1$
   (o) $x^2y'' - 3xy' + 4y = x^2\ln x$

2. Solve each initial value problem:
   (a) $x^2y'' - 2xy' - 10y = 0, \quad y(1) = 5, \quad y'(1) = 4$
   (b) $x^2y'' - 4xy' + 6y = 0, \quad y(2) = 0, \quad y'(2) = 4$
   (c) $x^2y'' + 5xy' + 3y = 0, \quad y(1) = 1, \quad y'(1) = -5$
   (d) $x^2y'' - 2y = 4x - 8, \quad y(1) = 4, \quad y'(1) = -1$
   (e) $x^2y'' - 4xy' + 4y = 4x^2 - 6x^3, \quad y(2) = 4, \quad y'(2) = -1$
   (f) $x^2y'' + 2xy' - 6y = 10x^2, \quad y(1) = 1, \quad y'(1) = -6$
   (g) $x^2y'' - 5xy' + 8y = 2x^3, \quad y(2) = 0, \quad y'(2) = -8$
   (h) $x^2y'' - 6y = \ln x, \quad y(1) = \frac{1}{6}, \quad y'(1) = -\frac{1}{6}$

3. Find the general solution of
   (a) $(x+2)^2y'' - (x+2)y' - 3y = 0$
   (b) $(2x-3)^2y'' - 6(2x-3)y' + 12y = 0$

Supplementary exercises:

4. Let λ_1 and λ_2 denote the solutions to the equation
$$\lambda^2 + (b-1)\lambda + c = 0.$$
Show the that the homogeneous Cauchy-Euler equation
$$x^2y'' + bxy' + cy = 0$$
has general solution
   (a) $y = c_1x^{\lambda_1} + c_2x^{\lambda_2}$ if λ_1 and λ_2 are real.
   (b) $y = (c_1 + c_2\ln x)x^{\lambda}$ if $\lambda_1 = \lambda_2 = \lambda$.
   (c) $x^r(c_1\cos(s\ln x) + c_2\sin(s\ln x))$ if λ_1 and λ_2 are complex and $\lambda_1, \lambda_2 = r \pm is$.

5. Show that a substitution $y = uv$ allows us to transform the equation
$$y'' + p(x)y' + q(x)y = 0$$
into a linear second order equation in v where the term v' is missing. Find the function u which makes this work, and find and the new equation in terms of the functions $p(x)$ and $q(x)$. Then use this method to find the general solution of
$$y'' + 2xy' + (1 + x^2)y = 0.$$

6. Consider the *Riccati equation*

$$y' = p(x)y^2 + q(x)y + r(x)$$

(see the exercise in section 1.4.) Show that the substitution

$$y = -\frac{v'}{p(x)v}$$

transforms it into a second order linear equation

$$p(x)v'' - (\, p'(x) + p(x)q(x)\,)v' + p^2(x)r(x)v = 0.$$

Use this substitution to solve
(a) $x^2y' + x^2y^2 + xy - 4 = 0$ (b) $y' + 2xy = 1 + x^2 + y^2$.

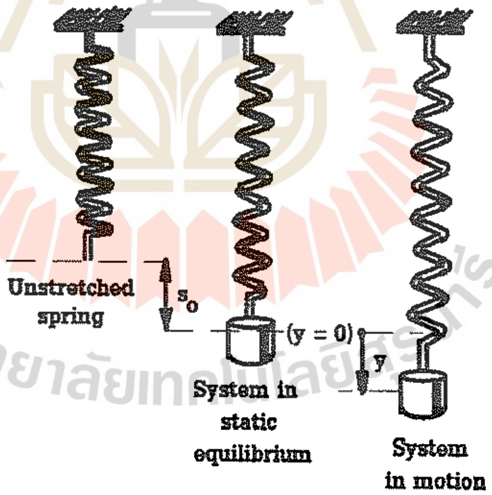
(You may need to use the result of the preceding exercise !)

2.7 Applications

In this section we will discuss oscillations of mechanical and electrical systems as a direct application of second order differential equations.

2.7.1 The Oscillating Spring

Let us consider a spring of length L whose upper end is kept fixed while a body of mass m is attached to its lower end. There are two opposing forces acting on the body:



A mass-spring system.

- The weight F_w of the body. By Newton's second law,

$$F_w = mg$$

(Note that we have chosen the positive direction to be downward.)

- The retracting force F_s of the spring. By Hooke's law,

$$F_s = -ks$$

where k denotes the spring constant and s the amount by which the spring has been stretched (or compressed). The minus sign indicates that the force is directed upward if the spring has been stretched, and downward if the spring has been compressed.

The total force on the body is therefore

$$F_w + F_s = mg - ks.$$

The position of the body where this resulting force is zero is called the *equilibrium position*. Let s_o denote the amount by which the spring is stretched when it is in this position. Then,

$$mg - ks_o = 0$$

If the body is not in equilibrium position, then the nonzero force $F_w + F_s$ will result in a motion of the body/spring system. Let $y = y(t)$ denote the displacement of the body from the equilibrium position at time t. So when the body is at position y, then the retracting force will be $F_s = -k(y + s_o)$.

- It is reasonable to assume that the motion is accompanied by friction, which results in a force F_r in direction opposite to the direction of movement. We assume that this force is proportional to the velocity of the body,

$$F_r = -c\frac{dy}{dt}$$

- Lastly, we may have an external force $F_{ext}(t)$ imposed in the body.

We have a total force acting on the body

$$F = F_w + F_s + F_r + F_{ext}$$

By Newton's second law, this force results in an acceleration $a = \frac{d^2y}{dt^2}$ with $F = ma$, so that

$$m\frac{d^2y}{dt^2} = mg - k(y + s_o) - c\frac{dy}{dt} + F_{ext}$$

Since $mg - ks_o = 0$ we get

$$\boxed{my'' + cy' + ky = F_{ext}.} \qquad (2.61)$$

This is a second order linear differential equation with constant coefficients. The nature of its solution depends on the values of m, k, c and F_{ext}.

Case I: Undamped Free Motion

Let us first assume that there is no damping and no external force ($c = 0$, $F_{ext} = 0$). Then we have an equation

$$my'' + ky = 0 \qquad (2.62)$$

The characteristic equation

$$\lambda^2 + \frac{k}{m} = 0$$

has roots $\lambda = i\omega_o$ where $\omega_o = \sqrt{\frac{k}{m}}$. Therefore the solution of equation (2.62) is

$$y = c_1 \cos \omega_o t + c_2 \sin \omega_o t \tag{2.63}$$

We want to write this solution in the form

$$\boxed{y = D \cos (\omega_o t - \phi)}. \tag{2.64}$$

By the difference of angles formula,

$$D \cos (\omega_o t - \phi) = D \cos \phi \cos \omega_o t + D \sin \phi \sin \omega_o t.$$

Comparing with (2.63) we see that we must choose ϕ and D such that

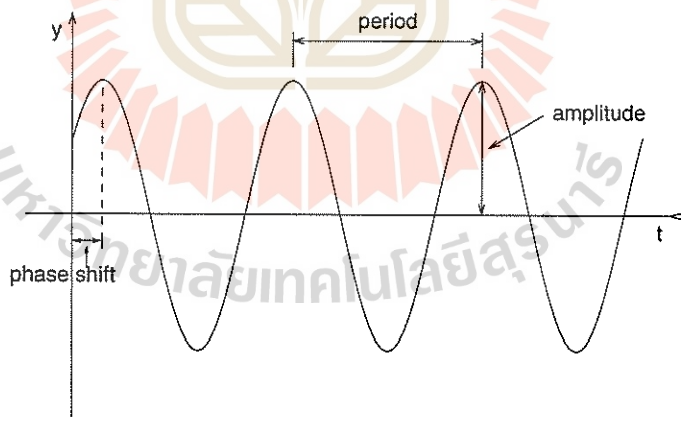
$$D \cos \phi = c_1 \qquad \text{and} \qquad D \sin \phi = c_2.$$

That is, we must set

$$D = \sqrt{c_1^2 + c_2^2} \qquad \text{and} \qquad \tan \phi = \frac{c_2}{c_1}$$

This is a periodic solution with period

$$T = \frac{2\pi}{\omega_o}$$



Simple harmonic motion: $y = D \cos (\omega_o t - \phi)$
(amplitude D, natural frequency ω_o, phase angle ϕ)

We call this *simple harmonic motion.* The number ω_o is called the *natural frequency of oscillation* (measured in hertz), D is called the *amplitude* and ϕ the *phase angle.* You may notice that the frequency depends only on the mass and the spring constant. The initial conditions determine the phase angle and the amplitude.

Example 1 A mass of 8 kg is placed at the lower end of a coil spring which is hanging from the ceiling. The weight comes to rest in its equilibrium position, stretching the spring 20 cm. The weight is then pulled down 4 cm below its equilibrium position and released at $t = 0$ with an initial velocity of 21 cm/sec upward. Neglecting resistance, find the amplitude, the period and the frequency of the resulting motion.

Solution. This is an example of free (no external force), undamped motion. We must first compute the value of k. In equilibrium position,

$$mg = ks_o$$

so that $k = mg/s_o = 8 \cdot 9.81/0.2 \approx 392.4$ N/m. We now have the differential equation

$$8y'' + 392.4y = 0$$

The characteristic equation has roots

$$\lambda = \pm i\sqrt{\frac{392.4}{8}} \approx \pm 7i$$

so that we have general solution

$$y = c_1 \cos 7t + c_2 \sin 7t,$$

and the natural frequency is 7 Hertz. Also,

$$y' = -7c_1 \sin 7t + 7c_2 \cos 7t.$$

So using the initial conditions $y(0) = 0.04$ and $y'(0) = -0.21$ we get

$$0.04 = c_1 \qquad \text{and} \qquad -0.21 = 7c_2$$

Therefore,

$$y = 0.04 \cos 7t - 0.03 \sin 7t.$$

Choosing D and ϕ so that

$$D\cos\phi = 0.04 \qquad \text{and} \qquad D\sin\phi = -0.03,$$

we have amplitude

$$D = \sqrt{0.04^2 + 0.03^2} = 0.05 \text{ (in meters)}$$

and phase angle

$$\tan\phi = -\frac{0.03}{0.04} = -0.75.$$

Since $\sin\phi < 0$ and $\cos\phi > 0$, the angle ϕ must be in the fourth quadrant. Therefore, the phase angle is

$$\phi = \tan^{-1}(-0.75) \approx -0.634$$

The equation of the motion is

$$y = 0.05 \cos(7t + 0.634).$$

□

Case II: Damped Free Motion

Now assume we have damping, but still no external force. We get the equation

$$my'' + cy' + ky = 0$$

The characteristic equation has roots

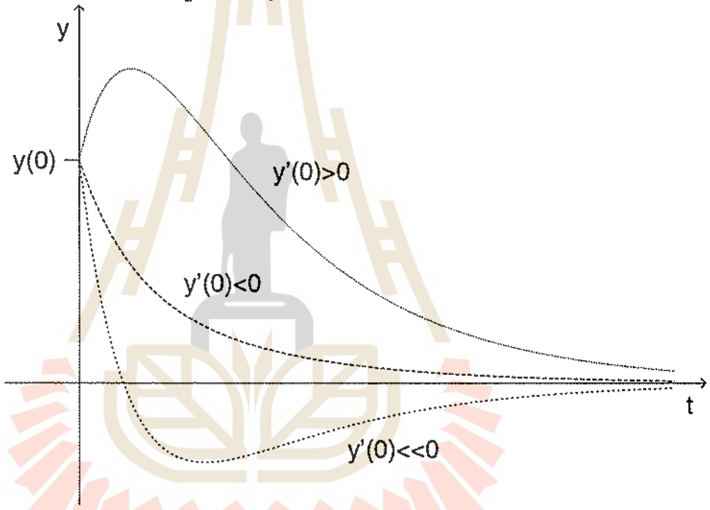
$$\lambda_1, \lambda_2 = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$$

There are three possibilities:

1. $c^2 - 4km > 0$. **(Overcritical Damping)**
   If we set $\alpha = \frac{c}{2m}$ and $\beta = \frac{\sqrt{c^2-4km}}{2m}$ then we have the general solution

   $$\boxed{y = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}} \tag{2.65}$$

   There is no periodic motion. Furthermore, as $\alpha - \beta > 0$, we see that $y(t) \to 0$ as $t \to \infty$. This is of course expected; the motion dies down after a while.

   Overcritical Damping: $y = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$
   at various initial velocities

2. $c^2 - 4km = 0$. **(Critical Damping)**
   If we set $\alpha = \frac{c}{2m}$ then we have the solution

   $$\boxed{y = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t}} \tag{2.66}$$

   We still have that $y(t) \to 0$ as $t \to \infty$. If damping is reduced further by only a little, then we have

3. $c^2 - 4km < 0$. **(Undercritical Damping or Damped Oscillations)**
   Setting

   $$\alpha = \frac{c}{2m} \qquad \text{and} \qquad \mu = \frac{\sqrt{4km - c^2}}{2m}$$

we have the general solution

$$y = e^{-\alpha t}(c_1 \cos \mu t + c_2 \sin \mu t)$$

or writing as in (2.64),

$$\boxed{y = De^{-\alpha t} \cos(\mu t - \phi)} \tag{2.67}$$

where

$$D = \sqrt{c_1^2 + c_2^2} \qquad \text{and} \qquad \tan \phi = \frac{c_2}{c_1}.$$

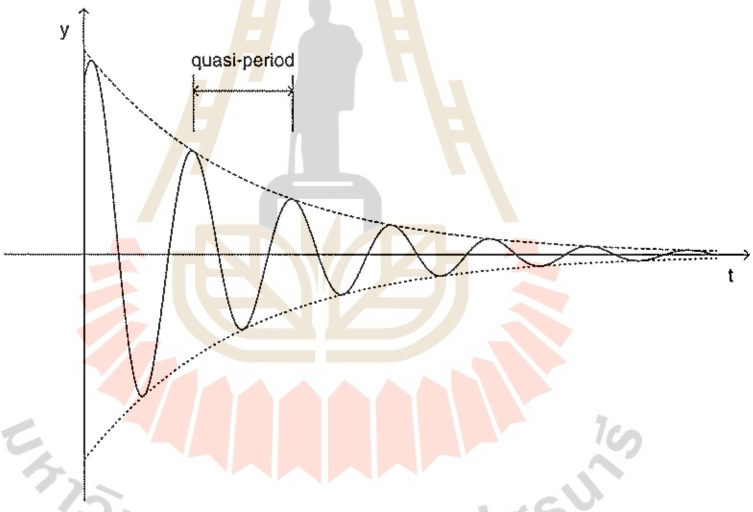
We have oscillations which decrease in amplitude and eventually vanish, because

$$\lim_{t \to \infty} e^{-\alpha t} = 0$$

Note that the frequency μ is smaller than the natural frequency ω_o of the undamped system. It is called the *quasi-frequency* of the system, and the period

$$T = \frac{2\pi}{\mu}$$

is called the *quasi-period* of the system.



Damped Oscillations: $y = De^{-\alpha t} \cos(\mu t - \phi)$

Example 2 How does the solution to example 1 change if there is damping with

1. $c = 50$ kg/sec.
2. $c = 112.057$ kg/sec.
3. $c = 192$ kg/sec.

Solution. 1. We have an initial value problem

$$8y'' + 50y + 392.4y = 0 \qquad y(0) = 0.04, \quad y'(0) = -0.21$$

Its characteristic equation has roots

$$\lambda = \frac{-50 \pm \sqrt{2500 - 4 \cdot 8 \cdot 392.4}}{2 \cdot 8} \approx 3.125 \pm 6.268i$$

The general solution is

$$y \;=\; e^{-3.125t}(c_1 \cos 6.268t + c_2 \sin 6.268t)$$

If we take derivatives, we get

$$y' \;=\; e^{-3.125t}\big(\,(-3.125c_1 + 6.268c_2)\cos 6.268t \;+\; (-3.125c_2 - 6.268c_1)\sin 6.268t\,\big)$$

Using the initial conditions, we get

$$0.04 = c_1$$

and

$$-0.21 = -3.125c_1 + 6.268c_2, \qquad \text{i.e.} \quad c_2 = -0.0136$$

Therefore,

$$y \;=\; e^{-3.125t}(0.04\cos 6.268t - 0.0136 \sin 6.268t)$$

or writing in form (2.67),

$$y \;=\; 0.0422e^{-3.125t}\cos(6.268t + 0.327)$$

This is undercritical damping.

2. We have an equation

$$8y'' + 112.057y + 392.4y = 0 \qquad\qquad y(0) = 0.04, \quad y'(0) = -0.21$$

Its characteristic equation has repeated roots

$$\lambda = \frac{-112.057 \pm \sqrt{112.057^2 - 4 \cdot 8 \cdot 392.4}}{2 \cdot 8} = \frac{-112.057 \pm 0}{16} \approx -7$$

The general solution is

$$y \;=\; c_1 e^{-7t} + c_2 t e^{-7t}$$

Using the initial conditions, we have

$$y' \;=\; -7c_1 e^{-7t} + c_2(1 - 7t)e^{-7t}$$

so that

$$0.04 = c_1 \quad \text{and} \quad -0.21 = -7c_1 + c_2 \quad \text{i.e.} \quad c_2 = 0.07$$

Therefore,

$$y = 0.04e^{-7t} + 0.07te^{-7t}.$$

This is critical damping.

3. We have an equation

$$8y'' + 192y + 392.4y = 0 \qquad\qquad y(0) = 0.04, \quad y'(0) = -0.21$$

Its characteristic equation has roots

$$\lambda = \frac{-192 \pm \sqrt{192^2 - 4 \cdot 8 \cdot 392.4}}{2 \cdot 8} \approx \frac{-192 \pm 155.91}{16} = -2.256, -21.74$$

The general solution is

$$y = c_1 e^{-2.256t} + c_2 e^{-21.74t}$$

Take the derivative

$$y' = -2.256 c_1 e^{-2.256t} - 21.74 c_2 e^{-21.74t}$$

and use the initial conditions,

$$0.04 = c_1 + c_2 \qquad \text{and} \qquad -0.21 = -2.256c_1 - 21.74c_2.$$

Solving, we obtain

$$y = 0.0339 e^{-2.256t} + 0.00615 e^{-21.74t}.$$

This is overcritical damping.

□

Case III: Forced Motion

Let us assume now that a periodic external force

$$F_{ext} = F_o \cos \omega t$$

is applied to our system. We then have the nonhomogeneous equation

$$my'' + cy' + ky = F_o \cos \omega t. \tag{2.68}$$

We have already seen that there are four possibilities for y_h, given by equations (2.64), (2.65), (2.66) and (2.67) above. For example, if there is no damping, then y_h is of the form (2.64),

$$y_h = D \cos(\omega_o t - \phi).$$

We use the method of undetermined coefficients to find a particular solution. Choose

$$y_p = A \cos \omega t + B \sin \omega t \tag{2.69}$$

and substitute into (2.68). Determining the values of the constants A and B we get

$$A = F_o \frac{m(\omega_o^2 - \omega^2)}{m^2(\omega_o^2 - \omega^2)^2 + \omega^2 c^2} \qquad \text{and} \qquad B = F_o \frac{\omega c}{m^2(\omega_o^2 - \omega^2)^2 + \omega^2 c^2} \tag{2.70}$$

where again $\omega_o = \sqrt{k/m}$. (Details are omitted.) Let us rewrite this particular solution as a single cosine function,

$$y_p = H \cos(\omega t - \theta) \tag{2.71}$$

where the amplitude is

$$H = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{m^2(\omega_o^2 - \omega^2)^2 + \omega^2 c^2}} \tag{2.72}$$

and the phase angle is given by

$$\tan \theta = \frac{B}{A} = \frac{\omega c}{m(\omega_o^2 - \omega^2)} \tag{2.73}$$

1. Undamped Forced Oscillations If there is no damping ($c = 0$) then $B = 0$, $H = A$ and $\theta = 0$. We have found the general solution

$$y \;=\; y_h + y_p \;=\; D\cos(\omega_o t - \phi) + \frac{F_o}{m(\omega_o^2 - \omega^2)}\cos\omega t$$

This is a sum of two periodic functions, one with the natural frequency of the system, and the other one with the frequency of the external force. For the sake of simplicity, let us assume that we have initial conditions $y(0) = y'(0) = 0$. Determining the values for D and ϕ (details are omitted) we obtain

$$y \;=\; \frac{F_o}{m(\omega_o^2 - \omega^2)}(\cos\omega t - \cos\omega_o t)$$

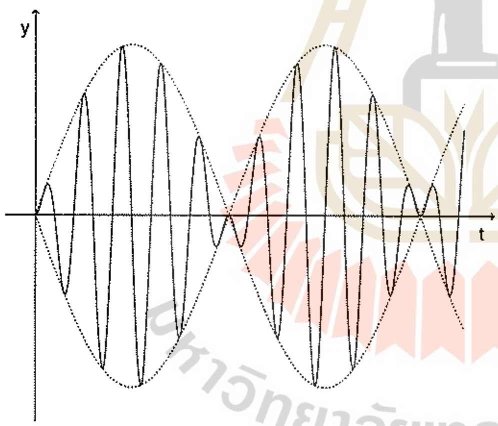
Using the formula

$$\cos a - \cos b = 2\,\sin\frac{b-a}{2}\,\sin\frac{b+a}{2}$$

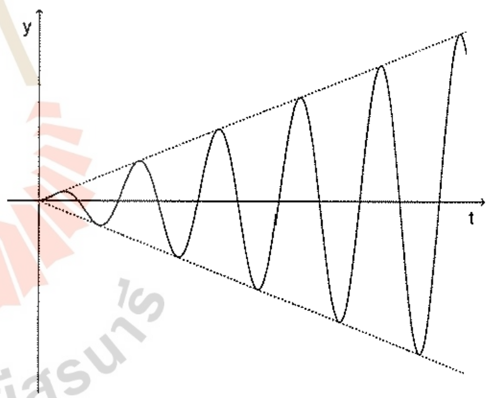
we can rewrite the solution as

$$\boxed{y \;=\; \underbrace{\frac{2F_o}{m(\omega_o^2 - \omega^2)}\sin\frac{(\omega_o - \omega)t}{2}}_{\text{slowly varying amplitude}}\;\underbrace{\sin\frac{(\omega_o + \omega)t}{2}}_{\text{fast oscillation}}} \tag{2.74}$$

If $|\omega_o - \omega|$ is small, then this is a rapidly oscillating motion (of frequency $\omega_o + \omega$) whose amplitude varies with slow frequency $|\omega_o - \omega|$. This slow variation of amplitude is called *beat*, and this is what a musician listens to when tuning his instrument.



Beat (equation (2.74))



Resonance (equation (2.75)).

Now as ω approaches the natural frequency ω_o, the amplitude of the slow motion increases, but its frequency decreases. But when $\omega = \omega_o$, the particular solution (2.71) is no longer valid because the function y_p is already a solution to the homogeneous equation. Instead, we will have a particular solution

$$y_p \;=\; At\cos\omega_o t + Bt\sin\omega_o t$$

When we substitute this y_p into (2.68) we see that

$$A = \frac{F_o}{2m\omega_o} \qquad \text{and} \qquad B = 0$$

so that we have a general solution of the form

$$y = y_h + y_p = D\cos(\omega_o t - \phi) + \frac{F_o}{2m\omega_o} t \sin\omega_o t. \tag{2.75}$$

Because of the factor t, the amplitude becomes unbounded as $t \to \infty$. This is called *resonance.*

2. Damped Forced Vibrations In practice, every system is under the influence of a damping force. Let us assume that we have undercritical damping ($0 < c < 2\sqrt{km}$), so the homogeneous equation has solution (2.67),

$$y_h = De^{-\alpha t}\cos(\mu t - \phi)$$

where $\alpha = c/2m$ and $\mu = \sqrt{4mk - c^2}/2m$. The particular solution is still as in (2.71),

$$y_p = H\cos(\omega t - \theta)$$

so that we have the general solution

$$\boxed{y = De^{-\alpha t}\cos(\mu t - \phi) + H\cos(\omega t - \theta)}. \tag{2.76}$$

You can see that y_h vanishes as $t \to \infty$. This is called the *transient solution.* On the other hand, y_p constitutes a periodic solution called the *steady state solution.* It is independent of the choice of the initial conditions.

Let us determine for which values of ω the amplitude of the steady state solution is largest. Taking derivatives in (2.72) we obtain

$$H'(\omega) = -\frac{F_o}{2}\frac{2m^2(\omega_o^2 - \omega^2)(-2\omega) + 2\omega c^2}{\sqrt{(m^2(\omega_o^2 - \omega^2)^2 + \omega^2 c^2)^3}}$$

Then $H'(\omega) = 0$ when

$$-4m^2\omega(\omega_o^2 - \omega^2) + 2\omega c^2 = 0$$

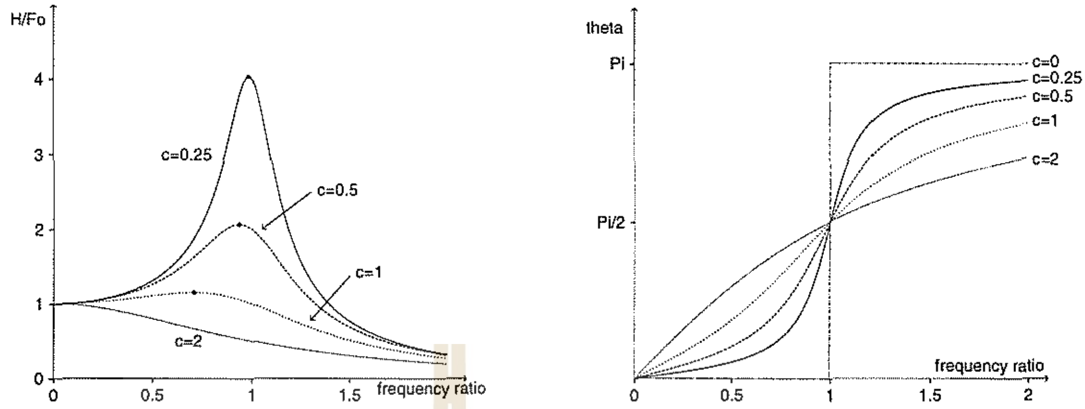
or

$$\omega = \omega_{max} = \sqrt{\omega_o^2 - \frac{c^2}{2m^2}}.$$

Only for $c^2 < 2m^2\omega_o^2 = 2mk$ is this solution real. Note that $\omega_{max} \to \omega_o$ as $c \to 0$. Computing the maximal amplitude from (2.72) one obtains

$$H(\omega_{max}) = \frac{2mF_o}{c\sqrt{4m^2\omega_o^2 - c^2}}$$

Remark If the damping factor c is small and the frequency ω of the external force is close to the natural frequency ω_o of the system, then the steady state solution has a very large amplitude H. This is the reason why a solid building may collapse in an earthquake, or a bridge may collapse under traffic.



Amplification H/F_o Phase angle θ

(external periodic force $F_{ext} = H\cos(\omega t - \theta)$, $m = 1$, x-axis: frequency ratio ω/ω_o.)

2.7.2 Electric Circuits

Let us look at simple electric circuits. We will use the following components:

- a generator, or a battery, providing an *electromotive force* $E(t)$ (measured in *volt*). We will denote the current flowing through a component by $I(t)$ (measured in *ampere*)

E(t)

- a *resistor*. The voltage drop across a resistor is proportional to the current flowing through it,

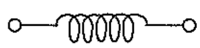
$$E_R = RI$$

R

R is the *resistance* measured in *ohm*.

- an *inductor*. The voltage drop across an inductor is proportional to the rate of change of the current through it,

$$E_L = L\frac{dI}{dt}$$

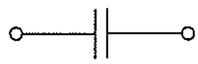


L

L is the *inductance* measured in *henry*.

- a *capacitor*. The voltage drop across a capacitor is proportional to the charge Q which it is holding,

$$E_C = \frac{1}{C}\,Q$$



C

C is the *capacitance* measured in *farad*, and the charge Q is measured in *coulomb*. Note that the current flowing through the capacitor is the rate of change of charge on it,

$$I = \frac{dQ}{dt}$$

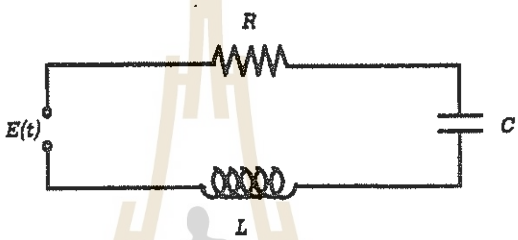
Integrating I, we may also write the equation of a capacitor as

$$E_C = \frac{1}{C}\left[Q(t_o) + \int_{t_o}^{t} I(\tau)\,d\tau\right]$$

where $Q(t_o)$ is the charge on the capacitor at time t_o.

We will make use of *Kirchhoff's Voltage Law*: The sum of all voltage drops around a closed loop is zero.

Let us now look at an RLC-circuit where resistor, capacitor and inductor are switched in series.



RLC-circuit.

By Kirchhoff's law,

$$E(t) = E_L + E_R + E_C$$

or

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$$

This equation contains both current I and charge Q as dependent variables. Since $I = \frac{dQ}{dt}$ we can express it in terms of the charge only,

$$\boxed{LQ'' + RQ' + \frac{1}{C}Q = E(t)} \tag{2.77}$$

Alternatively, we may differentiate the equation to obtain an equation in terms of the current only,

$$\boxed{LI'' + RI' + \frac{1}{C}I = E'(t)} \tag{2.78}$$

Example 1 A series RLC-circuit has a capacitor of 0.02 farad, a resistor of 12 ohms, and an inductor of 2 henry. The circuit is connected to a generator providing a voltage of $120\sin 5t$ volts. If the initial charge on the capacitor is 1 coulomb and the initial current is zero, find the current through the circuit at time t.

Solution. Since one of the initial conditions specifies the initial charge, we choose equation (2.77),

$$2Q'' + 12Q' + 50Q = 120\sin 5t$$

together with the initial conditions

$$Q(0) = 1, \qquad Q'(0) = 0$$

The characteristic equation is

$$2\lambda^2 + 12\lambda + 50 = 0$$

and thus has roots

$$\lambda = \frac{-12 \pm \sqrt{144 - 4\cdot 2\cdot 50}}{2\cdot 2} = -3 \pm 4i$$

Thus, the general solution of the related homogeneous equation is

$$Q_h = e^{-3t}\left[c_1\cos 4t + c_2 \sin 4t\right]$$

Now since the function on the right is $120\sin 5t$ we choose the particular solution

$$Q_p = A\cos 5t + B\sin 5t$$

Substituting into the differential equation, we get

$$2\left[-25A\cos 5t - 25B\sin 5t\right] + 12\left[-5A\sin 5t + 5B\cos 5t\right] + 50\left[A\cos 5t + B\sin 5t\right] = 120\sin 5t$$

Comparing coefficients we see that

$$-60A = 120 \qquad \text{and} \qquad 60B = 0$$

Thus, $A = -2$ and $B = 0$ so that

$$Q_p = -2\cos 5t$$

The general solution of the nonhomogeneous equation is

$$Q = Q_h + Q_p = e^{-3t}\left[c_1\cos 4t + c_2\sin 4t\right] - 2\cos 5t$$

Now differentiate using the product rule,

$$Q' = e^{-3t}\left[(-3c_1 + 4c_2)\cos 4t + (-4c_1 - 3c_2)\sin 4t\right] + 10\sin 5t$$

and use the initial conditions. $Q(0) = 1$ gives

$$c_1 - 2 = 1$$

so that $c_1 = 3$. Then $Q'(0) = 0$ gives

$$-3\cdot 3 + 4\cdot c_2 = 0$$

so that $c_2 = 2.25$. The current at time t is thus

$$I(t) = Q'(t) = -18.75e^{-3t}\sin 4t + 10\sin 5t$$

Note that the transient current is $-18.75e^{-3t}\sin 4t$ amperes, and the steady state current is $10\sin 5t$ amperes. □

You may notice that mathematically speaking, equations (2.77) or (2.78) of the RLC-circuit are the same as the equation of the oscillating spring (2.61), now with constants L, R and $\frac{1}{C}$ instead of m, c and k, and with current I instead of displacement y. We therefore have solutions of the same nature, and all the observations which we have made in the mechanical case apply here as well. For example, if $R \neq 0$, then there are three possibilities for the solutions Q_h or I_h of the homogeneous equation: overcritical damping (if $R^2 - 4L/C > 0$), critical damping (if $R^2 - 4L/C = 0$), and undercritical damping (if $R^2 - 4L/C < 0$).

Often, the voltage is a sinus function, $E(t) = E_o \sin \omega t$, and we are only interested in the steady state current. Since the initial conditions do not affect the steady state current, we may start with equation (2.78),

$$LI'' + RI' + \frac{1}{C} I = E_o \omega \cos \omega t$$

and search for the particular solution I_p of this equation. Just as in (2.68) (at least in case $R \neq 0$; this excludes the possibility of resonance) we get

$$I_p = A \cos \omega t + B \sin \omega t$$

where

$$A = E_o \omega \, \frac{L(w_o^2 - \omega^2)}{L^2(w_o^2 - \omega^2)^2 + \omega^2 R^2} \qquad \text{and} \qquad B = E_o \omega \, \frac{\omega R}{L^2(w_o^2 - \omega^2)^2 + \omega^2 R^2}$$

and now $\omega_o = \sqrt{\frac{1}{LC}}$.

We may simplify these expressions by defining the *reactance* S and the *impedance* Z ,

$$S = \omega L - \frac{1}{\omega C} \qquad \text{and} \qquad Z = \sqrt{R^2 + S^2}.$$

Then

$$L(\omega_o^2 - \omega^2) = L\left(\sqrt{\frac{1}{LC}}\right)^2 - \omega^2 L = \frac{1}{C} - \omega^2 L = -\omega S$$

so that

$$A = E_o \omega \, \frac{-\omega S}{\omega^2 S^2 + \omega^2 R^2} = \frac{-E_o S}{Z^2} \qquad \text{and} \qquad B = E_o \omega \frac{\omega R}{\omega^2 S^2 + \omega^2 R^2} = \frac{E_o R}{Z^2}.$$

Now as the electromotive force is a sinus function, we would also like to express the current $I_p = A \cos \omega t + B \sin \omega t$ as a sinus function, of the form

$$I_p = I_o \sin(\omega t - \phi) \tag{2.79}$$

For this we use the trigonometric identity

$$I_o \sin(\omega t - \phi) = I_o(-\sin\phi \cos\omega t + \cos\phi \sin\omega t)$$

and choose I_o and ϕ so that

$$-I_o \sin\phi = A \qquad \text{and} \qquad I_o \cos\phi = B$$

That is,

$$I_o = \sqrt{A^2 + B^2} = \frac{E_o}{Z} \qquad \text{and} \qquad \tan\phi = -\frac{A}{B} = \frac{S}{R}$$

Exercises

1. In each of the following exercises, find ω_o, D, and ϕ so that you can write the given expression in the form $y = D\cos(\omega_o t - \phi)$.
   (a) $y = 3\cos 2t + 4\sin 2t$
   (b) $y = -\cos t + \sqrt{3}\sin t$
   (c) $y = 4\cos 3t - 2\sin 3t$
   (d) $y = -2\cos \pi t - 3\sin \pi t$

2. In each of the following exercises, write the given expression as a product of two periodic functions of different frequencies.
   (a) $y = \cos 9t - \cos 7t$
   (b) $y = \sin 7t - \sin 6t$
   (c) $y = \cos \pi t + \cos 2\pi t$
   (d) $y = \sin 3t + \sin 4t$

Some of the following questions will use different systems of units as indicated in the table.

system	length	mass	force	g
cgs	cm	g	dyne	981 cm/sec^2
SI (mkg)	m	kg	N	9.81 m/sec^2
British	ft	slug	lb	32ft/sec^2

In each of these exercises you should not use the formulas derived in the text, but begin by setting up a differential equation and solving it on your own.

3. How does the natural frequency ω_o change if we
   (a) double the mass,
   (b) take a stiffer spring (i.e. increase k),
   (c) change the initial conditions ?

4. A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/sec, and if there is no air resistance, determine the position of the mass at time t. When does the mass first return to its equilibrium position ?

5. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed 1 in upward from the equilibrium position, and set in motion with a downward velocity of 2 ft/sec, and if there is no air resistance, find the position of the mass at time t. Determine frequency, period, amplitude, and phase of the motion. (1 ft = 12 in.)

6. A mass of 20 g stretches a spring 5 cm. The mass is also attached to a viscous damper with a damping constant of 400 dyne-sec/cm. The mass is pulled down 2 cm below its equilibrium position and then released (no initial velocity).
   - Find the position of the mass at time t. Determine the ratio of the natural frequency to the quasi frequency of the system.
   - Express the position of the mass in the form $y = Re^{-at}\cos(\omega t - \phi)$. Determine the time τ when $|y| < R/50$ for all $t > \tau$.

7. A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is attached to the spring, and also a viscous damper that results in a force of 3 N when the velocity of the mass is 5 m/sec. The mass is pulled down 5 cm below the equilibrium position and given an initial downward velocity of 10 cm/sec. Find its position at time t. Determine the ratio of the natural frequency to the quasi frequency of the system.

8. Show that the period of motion of an undamped oscillation of a mass hanging from a spring is $2\pi\sqrt{s_o/g}$, where the weight of the mass stretches the spring by distance s_o.

9. Show that $c_1 \cos \omega_o t + c_2 \sin \omega_o t$ can be also written in the form

$$r \sin(\omega_o t - \theta).$$

If $R \cos(\omega_o t - \phi) = r \sin(\omega_o t - \theta)$, find the relationship between R, r, ϕ, and θ.

10. Assume that the system described by the equation $my'' + cy' + ky = 0$ is either critically damped or overdamped. Show that the mass can pass through the equilibrium position at most once, regardless of the initial conditions.

11. A mass of 5 kg stretches a spring 10 cm. The mass is acted upon by an external force of $10 \sin(t/2)$ N and moves in a liquid that gives a viscous force of 2 N when the speed of the mass is 4 cm/sec. The mass is set in motion from the equilibrium position with a downward velocity of 8 cm/sec.

    (a) Formulate the initial value problem describing the motion of the mass. (Use $g = 9.8$m/sec)
    (b) Find the steady state solution.
    (c) The external force is replaced by a force of $2 \cos \omega t$ N of variable frequency. Find the value of ω for which the steady state amplitude is largest.

12. A spring-mass system has spring constant $k = 3$ N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a fluid which gives a resistance equal (numerically) to the speed. The system is driven by an external force of $3 \cos 3t - 2 \sin 3t$. Find the steady state solution, and express it in the form $D \cos(\omega t - \phi)$.

13. A mass that weighs 8 lb stretches a spring 6 in. There is damping; the damping constant is 0.25 lb-sec/ft. The system is acted upon by an external force of $4 \cos 2t$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at time t. Determine the steady state response. How do we have to change the given mass so that the steady state solution has largest amplitude ?

14. Find the general solution of

$$my'' + cy' + ky = F_o \sin \omega t$$

where $c^2 < 4km$.

    (a) Find the solution which satisfies the initial conditions $y(0) = y_o, \quad y'(0) = 0$.
    (b) Find the solution which satisfies the initial conditions $y(0) = 0, \quad y'(0) = y'_o$.
    (c) Find the solution which satisfies the initial conditions $y(0) = y_o, \quad y'(0) = y'_o$.

15. Find the solution of the initial value problem

$$y'' + y = F(t), \quad y(0) = 0, \quad y'(0) = 0$$

where

$$F(t) = \begin{cases} F_o t & 0 \le t \le \pi, \\ F_o(2\pi - t), & \pi < t < 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

16. A series LC-circuit has a capacitor of 0.25×10^{-6} farad and an inductor of 1 henry. If the initial charge on the capacitor of 10^{-6} coulomb and and there is no initial current, find the charge on the capacitor at any time t.

17. A series RLC-circuit has a capacitor of 10^{-5} farad, a resistor of 300 ohms, and an inductor of 0.2 henry. If the initial charge on the capacitor of 10^{-6} coulomb and and there is no initial current, find the charge on the capacitor at time t.

18. A series RLC-circuit has a capacitor of 10^{-5} farad, and an inductor of 0.2 henry. Find the resistance R so that the circuit is critically damped.

19. A series RLC-circuit has a capacitor of 0.25×10^{-6} farad, a resistor of 5000 ohms, and an inductor of 1 henry. The initial charge on the capacitor is zero. A 12 volt battery is connected to the circuit and the circuit is closed at time $t = 0$. Find the charge $Q(t)$ on the capacitor at any time t. Find the limiting charge $\lim_{t\to\infty} Q(t)$.

20. When tuning the radio to a station we turn a knob which changes the capacitance C in an RLC-circuit so that the amplitude of the steady-state current becomes maximal. For what value of C (depending on the desired frequency ω, with R and L fixed) will this be the case ?

21. Find steady state and transient currents in an RLC circuit, assuming zero initial current and charge, when

    (a) $R = 80$ ohms, $L = 20$ henry, $C = 0.01$ farad, $E = 100$ volts.
    (b) $R = 160$ ohms, $L = 20$ henry, $C = 0.002$ farad, $E = 481 \sin 10t$ volts.
    (c) $R = 6$ ohms, $L = 1$ henry, $C = 0.04$ farad, $E = 24 \cos 5t$ volts.

22. Find the current in an LC circuit, assuming zero initial current and charge, when

    (a) $L = 0.4$ henry, $C = 0.1$ farad, $E = 110 \sin \omega t$ volts, $(\omega^2 \neq 25)$.
    (b) $L = 0.2$ henry, $C = 0.05$ farad, $E = 100$ volts.
    (c) $L = 2.5$ henry, $C = 0.001$ farad, $E = 10t^2$ volts.
    (d) $L = 10$ henry, $C = 0.004$ farad, $E = 250$ volts.
    (e) $L = 10$ henry, $C = \frac{1}{90}$ farad, $E = 10 \cos 2t$ volts.

23. Find the current in an LC circuit, assuming $L = 1$ henry, $C = 1$ farad, and zero initial current and initial charge, when

    (a) $E = 1$ when $0 < t < 1$ and $E = 0$ when $t > 1$.
    (b) $E = t$ when $0 < t < 1$ and $E = 1$ when $t > 1$.
    (c) $E = 1 - e^{-t}$ when $0 < t < \pi$ and $E = 0$ when $t > \pi$..

2.8 Higher Order Equations

In this section we will discuss the solution of the *n-th order linear equation*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0 y \;=\; f(x)$$

Dividing by the leading coefficient $a_n(x) \neq 0$ we obtain the equation in *standard form*

$$\boxed{y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y \;=\; r(x)} \tag{2.80}$$

The concepts and methods of solution for the second order equation extend to these higher order equations with only little modifications. We will therefore only summarize them without going into details.

An *initial value problem* will now consist of an equation of the form (2.80) together with n initial conditions,

$$y(x_o) = y_o, \quad y'(x_o) = y_o', \quad y''(x_o) = y_o'', \quad \ldots \quad , y^{(n-1)}(x_o) = y_o^{(n-1)}$$

Such an initial value problem always has a solution:

Theorem 13 (Existence and Uniqueness Theorem) *If $p_0(x)$, $p_1(x)$, ... , $p_{n-1}(x)$ and $r(x)$ are continuous on an interval I which contains the point x_o, then there exists exactly one solution to this initial value problem. This solution is valid over the whole interval I.*

2.8.1 The Homogeneous Equation - Theory

We must first study the theory of the related homogeneous equation,

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y \;=\; 0. \tag{2.81}$$

Definition Let $y_1,\ y_2,\ \ldots\ ,\ y_n$ be functions. A function

$$y \;=\; c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

where $c_1, c_2, \ldots, c_n$ are arbitrary numbers, is called a *linear combination* of $y_1,\ y_2,\ \ldots\ ,\ y_n$.

Theorem 14 (Superposition principle) *If $y_1,\ y_2,\ \ldots\ ,\ y_n$ are solutions of the homogeneous linear equation (2.81) then every linear combination $c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is also a solution.*

Definition The functions $f_1,\ f_2,\ \ldots\ ,\ f_n$ defined on an interval I are called *linearly dependent* if there exist constants $c_1,\ c_2,\ \ldots\ ,\ c_n$, not all zero, such that

$$c_1f_1 + c_2f_2 + \ldots + c_nf_n \;=\; 0.$$

If no such constants exist, then the functions are called *linearly independent.*

The Wronskian is now really an important tool to check for linear independence:

Definition Given functions $f_1,\ f_2,\ \ldots\ ,\ f_n$ we define the *Wronskian* to be the determinant

$$W(f_1, f_2, \ldots, f_n)(x) \;=\; \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Theorem 15 *The functions $f_1,\ f_2,\ \ldots\ ,\ f_n$ defined on an interval I are linearly dependent if and only if $W(f_1, f_2, \ldots, f_n) = 0$.*

When the functions involved are solutions to the homogeneous equation, we can say even more:

Theorem 16 *Let $y_1,\ y_2,\ \ldots\ ,\ y_n$ be n solutions of the homogeneous equation (2.81) on some interval I. Then either $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for all x in I (in which case these functions are linearly independent) or $W(y_1, y_2, \ldots, y_n)(x) = 0$ for all x in I (in which case these functions are linearly dependent).*

Theorem 17 *Let $y_1,\ y_2,\ \ldots\ ,\ y_n$ be n linearly independent solutions of the homogeneous equation (2.81) on some interval I. Then the general solution is of the form*

$$y_h \;=\; c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

for constants $c_1,\ c_2,\ \ldots\ ,\ c_n$.

Such a set $y_1,\ y_2,\ \ldots\ ,\ y_n$ of n linearly independent solutions of the homogeneous equation (2.81) is called *fundamental set of solutions.*

2.8.2 The Homogeneous Equation with Constant Coefficients

The standard form of the homogeneous equation with constant coefficients is

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y \;=\; 0. \tag{2.82}$$

Sometimes *operator notation* is used,

$$D^n y + a_{n-1}D^{n-1}y + \cdots + a_1 Dy + a_0 y \;=\; 0$$

where D^k denotes $\dfrac{d^k}{dx^k}$, differentiating k times. One can write this equation as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y \;=\; 0. \tag{2.83}$$

We form the *characteristic equation*,

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \;=\; 0$$

(Note the similarity between the characteristic equation and (2.83) !) and factor it completely,

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \;=\; 0$$

The solutions $\lambda_1, \ldots, \lambda_n$ are called the *roots* of the characteristic equation.

e.g. The equation

$$y^{(4)} - 6y''' + 14y'' - 14y' + 5y \;=\; 0. \tag{2.84}$$

Its characteristic equation is

$$\lambda^4 - 6\lambda^3 + 14\lambda^2 - 14\lambda + 5 \;=\; 0$$

and factors as

$$(\lambda - 1)^2(\lambda - (2+i))(\lambda - (2-i)) \;=\; 0$$

There are three distinct roots: The repeated real root $\lambda_1 = \lambda_2 = 1$, and the nonrepeated complex root $\lambda_3 = (2+i)$ together with its complex conjugate $\lambda_4 = \overline{\lambda_3} = (2-i)$.

Each root $\lambda = \lambda_k$ contributes one or several terms to the fundamental solution as follows:

Case I: λ is *a real, nonrepeated root.* In this case,

$$\boxed{e^{\lambda x}}$$

is part of the fundamental set of solutions.

Example 1 The equation

$$y''' - 2y'' - y' + 2y \;=\; 0$$

has characteristic equation

$$\lambda^3 - 2\lambda^2 - \lambda + 2 \;=\; (\lambda - 1)(\lambda + 1)(\lambda - 2) \;=\; 0$$

The roots $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 2$ are nonrepeated, so that the general solution is

$$y_h \;=\; c_1 e^x + c_2 e^{-x} + c_3 e^{2x}.$$

□

Case II: λ is a *real root, repeated m times.* In this case, the functions

$$\boxed{e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2e^{\lambda x}, \quad \dots \quad x^{m-1}e^{\lambda x}}$$

are part of the fundamental set of solutions.

Example 2 Consider the equation

$$y^{(5)} - 3y^{(4)} + 3y''' - y'' \;=\; 0$$

Its characteristic equation is

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 \;=\; \lambda^2(\lambda-1)^3 \;=\; 0$$

and has roots $\lambda = 0$ (repeated twice) and $\lambda = 1$ (repeated three times). Therefore, the general solution is

$$y_h \;=\; c_1e^{0x} + c_2xe^{0x} + c_3e^{1x} + c_4xe^{1x} + c_5x^2e^{1x}$$

or

$$y_h \;=\; c_1 + c_2x + e^x(c_3 + c_4x + c_5x^2).$$

□

Case III: $\lambda = r + is$ is a *complex, nonrepeated root.* One can show that the conjugate $\overline{\lambda} = r - is$ must also appear as a root because the characteristic equation has real coefficients. The pair λ, $\overline{\lambda}$ contributes the pair of functions

$$\boxed{e^{rx}\cos sx \quad \text{and} \quad e^{rx}\sin sx}$$

to the fundamental set of solutions.

Example 3 Consider the equation (2.84). The repeated real root $\lambda_1 = \lambda_2 = 1$ contributes the terms e^x and xe^x, while the complex roots $\lambda_3 = 2 + i$ and $\lambda_4 = 2 - i$ contribute the terms $e^{2x}\cos x$ and $e^{2x}\sin x$. The general solution is

$$y \;=\; c_1e^x + c_2xe^x + (c_3e^{2x}\cos x + c_4e^{2x}\sin x) \;=\; e^x(c_1 + c_2x) + e^{2x}(c_3\cos x + c_4\sin x).$$

□

Example 4 Consider the equation

$$y''' - 2y'' + 2y' \;=\; 0$$

Its characteristic equation is

$$\lambda^3 - 2\lambda^2 + 2\lambda \;=\; \lambda(\lambda^2 - 2\lambda + 2) \;=\; 0$$

and has nonrepeated roots

$$\lambda_1 = 0, \quad \lambda_2, \lambda_3 = \frac{2 \pm \sqrt{4 - 4\cdot 2}}{2} = 1 \pm i.$$

The general solutions is thus

$$y_h \;=\; c_1e^{0x} + c_2e^x\cos x + c_3e^x\sin x \;=\; c_1 + e^x(c_2\cos x + c_3\sin x)$$

□

Case IV: $\lambda = r + is$ is a *complex root, repeated m times.* One can show that its conjugate $\overline{\lambda} = r - is$ must also appear as a root m times. The pair λ, $\overline{\lambda}$ contributes the $2m$ functions

$$\boxed{e^{rx}\cos sx,\; e^{rx}\sin sx, \quad xe^{rx}\cos sx,\; xe^{rx}\sin sx, \quad \dots \quad x^{m-1}e^{rx}\cos sx,\; x^{m-1}e^{rx}\sin sx}$$

to the fundamental set of solutions.

Example 5 Consider the equation

$$y^{(7)} + 8y^{(5)} + 16y''' = 0.$$

Its characteristic equation is

$$\lambda^7 + 8\lambda^5 + 16\lambda^3 = \lambda^3(\lambda^2+4)^2 = \lambda^3(\lambda-2i)^2(\lambda+2i)^2 = 0$$

The roots are $\lambda = 0$ (repeated three times), $\lambda = 2i$ (repeated twice) and $\lambda = -2i$ (also repeated twice). The general solution is thus

$$y_h = c_1 + c_2x + c_3x^2 + c_4\cos 2x + c_5\sin 2x + c_6x\cos 2x + c_7x\sin 2x$$

□

2.8.3 The Nonhomogeneous Equation

To find the solution of the nonhomogeneous equation it is again enough to look for one particular solution:

Theorem 18 *Assume, y_p is one solution of the nonhomogeneous equation (2.80). Then the general solution is of the form*

$$\boxed{y = y_h + y_p}$$

where y_h is the general solution to the related homogeneous equation (2.81).

We have again two methods to find one particular solution to the nonhomogeneous equation.

The Method of Undetermined Coefficients. This method works exactly as described in the case of equations of order two, and can only be applied to equations with constant coefficients,

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = r(x) \tag{2.85}$$

and only with the choices for $r(x)$ as indicated in the table in section 2.5.1.

Example 6 Solve the equation

$$y''' - 4y' = x + 5\cos x + e^{-2x}$$

Solution. The characteristic equation is

$$\lambda^3 - 4\lambda = \lambda(\lambda-2)(\lambda+2) = 0$$

Thus, the solution to the homogeneous equation is

$$y_h = c_1 + c_2e^{2x} + c_3e^{-2x}$$

We would try the particular solution

$$y_p = (Ax+B) + (C\cos x + D\sin x) + Ee^{-2x}$$

But notice that the constant function and the function e^{-2x} are already solutions to the homogeneous equation. We must modify our choice for y_p,

$$y_p = (Ax^2+Bx) + (C\cos x + D\sin x) + Exe^{-2x}$$

Take derivatives

$$\begin{aligned} y_p' &= 2Ax + B - C\sin x + D\cos x + E(1-2x)e^{-2x} \\ y_p'' &= 2A - C\cos x - D\sin x + E(4x-4)e^{-2x} \\ y_p''' &= C\sin x - D\cos x + E(12-8x)e^{-2x} \end{aligned}$$

Then substitute into the equation and collect like terms,

$$-8Ax - 4B + 5C\sin x - 5D\cos x + 8Ee^{-2x} = x + 5\cos x + e^{-2x}$$

Comparing coefficients, we get

$$A = -1/8, \quad B = 0, \quad C = 0, \quad D = -1, \quad E = 1/8$$

The general solution is

$$y = c_1 + c_2e^{2x} + c_3e^{-2x} - \frac{1}{8}x^2 - \sin x + \frac{1}{8}xe^{-2x}$$

□

Example 7 Find the solution to

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4\ln x \tag{2.86}$$

Solution. This is a *Cauchy-Euler equation* of degree three. We again substitute

$$x = e^t$$

so that $t = \ln x$ and $\frac{dt}{dx} = \frac{1}{x}$. Then, as for second order equations (see (2.53) and (2.54)),

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

To replace the third order derivative, note that

$$\begin{aligned} \frac{d^3y}{dx^3} &= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right] \\ &= -\frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + \frac{1}{x^2}\left(\frac{d^3y}{dt^3}\frac{dt}{dx} - \frac{d^2y}{dt^2}\frac{dt}{dx}\right) \\ &= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) \end{aligned}$$

Then equation (2.86) becomes

$$\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) - 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 6\frac{dy}{dt} - 6y = te^{4t}$$

which simplifies to

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = te^{4t} \tag{2.87}$$

This is now a linear equation with constant coefficients. Its characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

so that the homogeneous equation has solution

$$y_h = c_1 e^t + c_2 e^{2t} + c_3 e^{3t}$$

Try the particular solution

$$y_p = (At + B)e^{4t}$$

The derivatives are

$$\begin{aligned} y_p' &= (4At + A + 4B)e^{4t} \\ y_p'' &= (16At + 8A + 16B)e^{4t} \\ y_p''' &= (64t + 48A + 64B)e^{4t} \end{aligned}$$

If we substitute into equation (2.87) we obtain

$$6Ate^{4t} + (11A + 6B)e^{4t} = te^{4t}$$

Therefore,

$$A = 1/6 \quad B = -11/36$$

The general solution is

$$y = c_1 e^t + c_2 e^{2t} + c_3 e^{3t} + (\frac{t}{6} - \frac{11}{36})e^{4t}$$

Resubstitute $x = e^t$,

$$y = c_1 x + c_2 x^2 + c_3 x^3 + \frac{x^4}{6}(\ln x - \frac{11}{6})$$

□

Variation of Parameters. This method can be applied to any linear equation.

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x). \tag{2.88}$$

Once we have found the general solution

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

of the related homogeneous equation, we vary the parameters and try the particular solution

$$y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n \tag{2.89}$$

where $u_1(x), u_2(x), \ldots u_n(x)$ are functions of x. Take derivatives,

$$y_p' = (u_1 y_1' + u_2 y_2' + \ldots + u_n y_n') + (u_1' y_1 + u_2' y_2 + \ldots + u_n' y_n)$$

In order not to get second order derivatives of the u_m we impose the condition

$$u_1' y_1 + u_2' y_2 + \ldots + u_n' y_n = 0 \tag{2.90}$$

so that

$$y_p' = u_1 y_1' + u_2 y_2' + \ldots + u_n y_n'$$

Note that this looks like (2.89) ! We continue differentiating this way, to get at each step

$$y_p^{(m)} = u_1 y_1^{(m)} + u_2 y_2^{(m)} + \ldots + u_n y_n^{(m)} \qquad (1 \le m \le n-1) \tag{2.91}$$

because we set

$$u_1' y_1^{(m-1)} + u_2' y_2^{(m-1)} + \ldots + u_n' y_n^{(m-1)} = 0 \qquad (1 \le m \le n-1) \tag{2.92}$$

Having taken derivatives n times, we are left with

$$y_p^{(n)} = (u_1 y_1^{(n)} + u_2 y_2^{(n)} + \ldots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \ldots + u_n' y_n^{(n-1)}) \tag{2.93}$$

Substitute all the derivatives (2.91) and (2.93) into (2.88). For each m, collect all terms containing u_m, and obtain

$$u_m \underbrace{(y_m^{(n)} + p_{n-1} y_m^{(n-1)} + \cdots p_1 y_m' + p_0 y_m)}_{=0} + u_m' y_m^{(n-1)}$$

which equals $u_m' y_m^{(n-1)}$ because every function y_m is a solution of the homogeneous equation. These terms add up to

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \ldots + u_n' y_n^{(n-1)} = r(x) \tag{2.94}$$

Now the equations (2.92) and (2.94) give us a system of equations

$$\begin{array}{ccccccccc} u_1' y_1 & + & u_2' y_2 & + & \ldots & + & u_n' y_n & = & 0 \\ u_1' y_1' & + & u_2' y_2' & + & \ldots & + & u_n' y_n' & = & 0 \\ u_1' y_1'' & + & u_2' y_2'' & + & \ldots & + & u_n' y_n'' & = & 0 \\ \vdots & & \vdots & & & & \vdots & = & \vdots \\ u_1' y_1^{(n-1)} & + & u_2' y_2^{(n-1)} & + & \ldots & + & u_n' y_n^{(n-1)} & = & r(x) \end{array}$$

In matrix form, this can be expressed as

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ r(x) \end{bmatrix} \tag{2.95}$$

This matrix is exactly the matrix occurring in the Wronskian $W = W(y_1, \ldots, y_n)$! Since $y_1, \ldots, y_n$ are linearly independent, this Wronskian is nonzero and we can use Cramer's rule to find that

$$u_m'(x) = \frac{r(x)\, W_m(x)}{W(x)}, \qquad m = 1, \ldots, n$$

where W_m denotes the determinant obtained from W by replacing the m-th column by the vector $[0, 0, 0, \ldots, r(x)]^t$. Now integrate each u_m' and obtain from (2.89),

$$\boxed{y_p = y_1 \int r(x) \frac{W_1(x)}{W(x)}\, dx + y_2 \int r(x) \frac{W_2(x)}{W(x)}\, dx + \cdots + y_n \int r(x) \frac{W_n(x)}{W(x)}\, dx}$$

Remark When using the method of variation of parameters it is best to start with the matrix equation (2.95) because it can easily be remembered.

Example 8 Find a particular solution to

$$y''' - y'' - y' + y = r(x) \tag{2.96}$$

(a) if $r(x) = 4e^x$,

(b) in general.

Solution. The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = (\lambda - 1)^2(\lambda + 1) = 0$$

Therefore, the homogeneous equation has solution

$$y_h = c_1 e^x + c_2 x e^x + c_3 e^{-x}$$

Using the method of variation of parameters, we set

$$y_p = u_1 e^x + u_2 x e^x + u_3 e^{-x}.$$

Then (2.95) gives us the system of equations,

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r(x) \end{bmatrix}$$

which becomes

$$\begin{bmatrix} e^x & xe^x & e^{-x} \\ e^x & (x+1)e^x & -e^{-x} \\ e^x & (x+2)e^x & e^{-x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r(x) \end{bmatrix}$$

To solve this system, we first compute the Wronskian using the rules of determinants,

$$\begin{aligned} W(x) &= W(e^x, xe^x, e^{-x}) = \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & (x+1)e^x & -e^{-x} \\ e^x & (x+2)e^x & e^{-x} \end{vmatrix} \\ &= e^x e^x e^{-x} \begin{vmatrix} 1 & x & 1 \\ 1 & (x+1) & -1 \\ 1 & (x+2) & 1 \end{vmatrix} = e^x \begin{vmatrix} 1 & x & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 0 \end{vmatrix} \\ &= e^x \begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} = 4e^x \end{aligned}$$

Therefore by Cramer's rule,

$$\begin{aligned} u_1'(x) &= \frac{\begin{vmatrix} 0 & xe^x & e^{-x} \\ 0 & (x+1)e^x & -e^{-x} \\ r(x) & (x+2)e^x & e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & (x+1)e^x & -e^{-x} \\ e^x & (x+2)e^x & e^{-x} \end{vmatrix}} = \frac{r(x) \begin{vmatrix} xe^x & e^{-x} \\ (x+1)e^x & -e^{-x} \end{vmatrix}}{W(x)} \\ &= \frac{(-2x-1)r(x)}{4e^x} \end{aligned}$$

$$u_2'(x) = \frac{\begin{vmatrix} e^x & 0 & e^{-x} \\ e^x & 0 & -e^{-x} \\ e^x & r(x) & e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & (x+1)e^x & -e^{-x} \\ e^x & (x+2)e^x & e^{-x} \end{vmatrix}} = \frac{-r(x)\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}}{W(x)} = \frac{2r(x)}{4e^x}$$

$$u_3'(x) = \frac{\begin{vmatrix} e^x & xe^x & 0 \\ e^x & (x+1)e^x & 0 \\ e^x & (x+2)e^x & r(x) \end{vmatrix}}{\begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & (x+1)e^x & -e^{-x} \\ e^x & (x+2)e^x & e^{-x} \end{vmatrix}} = \frac{r(x)\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}}{W(x)} = \frac{e^{2x}r(x)}{4e^x}$$

(a) Now if $r(x) = 4e^x$ we get

$$u_1' = -2x - 1, \qquad u_2' = 2, \qquad u_3' = e^{2x}$$

We integrate all three functions to obtain

$$u_1 = -x^2 - x, \qquad u_2 = 2x, \qquad u_3 = \frac{1}{2}e^{2x}$$

Thus, a particular solution is

$$\begin{aligned} y_p &= u_1e^x + u_2xe^x + u_3e^{-x} \\ &= (-x^2 - x)e^x + 2x^2e^x + \frac{1}{2}e^x \\ &= (x^2 - x + \frac{1}{2})e^x \end{aligned}$$

(b) In the general case, after integrating u_1', u_2', and u_3' we obtain the particular solution

$$\begin{aligned} y_p &= u_1e^x + u_2xe^x + u_3e^{-x} \\ &= e^x\int\frac{(-2x-1)r(x)}{4e^x}\,dx + xe^x\int\frac{2r(x)}{4e^x}\,dx + e^{-x}\int\frac{e^{2x}r(x)}{4e^x}\,dx \end{aligned}$$

This expression can be simplified if we express the antiderivatives as definite integrals. If $r(x)$ is defined on an interval I and a is a number in I, then by the fundamental theorem of calculus,

$$\begin{aligned} y_p &= e^x\int_a^x\frac{(-2t-1)r(t)}{4e^t}\,dt + xe^x\int_a^x\frac{2r(t)}{4e^t}\,dt + e^{-x}\int_a^x\frac{e^{2t}r(t)}{4e^t}\,dt \\ &= \int_a^x\frac{r(t)}{4e^t}\left(-e^x(2t+1) + 2xe^x + e^{-x}e^{2t}\right)dt. \end{aligned}$$

□

Exercises

1. Find the general solution of each equation:
   (a) $y''' - 3y'' + 2y' = 0$
   (b) $y''' - 3y'' + 4y' - 2y = 0$
   (c) $y''' - y = 0$
   (d) $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$
   (e) $y^{(4)} + 5y'' + 4y = 0$
   (f) $y^{(4)} - 2a^2y'' + a^4y = 0$
   (g) $y^{(4)} + 2a^2y'' + a^4y = 0$
   (h) $y^{(4)} + 2y''' + 2y'' + 2y' + y = 0$
   (i) $y^{(8)} + 8y^{(4)} + 16y = 0$
   (j) $y''' + y = 0$
   (k) $y''' + 3y'' + 3y' + y = 0$
   (l) $y^{(4)} + y = 0$
   (m) $y^{(4)} + 2y''' - 2y'' - 6y' + 5y = 0$
   (n) $y''' - 6y'' + 11y' - 6y = 0$
   (o) $y^{(4)} + y''' - 3y'' - 5y' - 2y = 0$
   (p) $y^{(6)} - y'' = 0$
   (q) $y^{(4)} - 8y' = 0$
   (r) $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$
   (s) $y^{(5)} - 6y^{(4)} - 8y''' + 48y'' + 16y' - 96y = 0$

2. Solve the following equations:
   (a) $y''' - y'' - y' + y = 2e^{-x} + 3$
   (b) $y^{(4)} - y = 3x + \cos x$
   (c) $y''' + y'' + y' + y = e^{-x} + 4x$
   (d) $y^{(4)} + y'' = \sin 2x$
   (e) $y''' + y' = \tan x$
   (f) $y^{(4)} - 4y'' = x^2 + e^x$
   (g) $y^{(4)} + 2y'' + y = 3 + 2\cos x$
   (h) $y^{(4)} + y''' = x$
   (i) $y''' - y' = x$
   (j) $y''' - 2y'' - y' + 2y = e^{4x}$
   (k) $y^{(4)} + 2y'' + y = 3x + 4, \quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1$
   (l) $y''' + 4y' = x, \quad y(0) = y'(0) = 0, \quad y''(0) = 1$
   (m) $y''' - 3y'' + 2y' = x + e^x, \quad y(0) = 1,\ y'(0) = -\frac{1}{4},\ y''(0) = -\frac{3}{2}$

3. Find a suitable choice for the particular solution y_p. Do not evaluate the constants.
   (a) $y''' - 2y'' + y' = x^3 + 2e^x$
   (b) $y''' - y' = xe^{-x} + 2\cos x$
   (c) $y^{(4)} - 2y'' + y = e^x + \sin x$
   (d) $y^{(4)} + 4y'' = \sin 2x + xe^x + 4$
   (e) $y^{(4)} - y''' - y'' + y' = x^2 + 4 + x\sin x$
   (f) $y^{(4)} + 2y''' + 2y'' = 3e^x + 2xe^{-x} + e^{-x}\sin x$

4. Solve the following equations
   (a) $x^3y''' + 3x^2y'' = 0$
   (b) $x^3y''' + 2x^2y'' + xy' - y = 0$
   (c) $x^3y''' + x^2y'' - 2xy' + 2y = 2x^4$

5. Show that the general solution of $y^{(4)} - y = 0$ can be written as

$$y = c_1 \cos x + c_2 \sin x + c_3 \cosh x + c_4 \sinh x$$

   Determine the solution satisfying the initial condition $y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$.

   Supplementary exercises:

6. Show that the definition of linear dependence given in (2.10) coincides with the definition of linear dependence given in this section.

Chapter 3

The Laplace Transform

3.1 Improper Integrals

You are already familiar with the integral $\int_a^b f(x)\,dx$ over a finite interval $[a,b]$. We now need to define the integral $\int_a^\infty f(x)\,dx$ over an *infinite interval* $[a,\infty)$. Such an integral is called an *improper integral*. It is defined by

$$\int_a^\infty f(x)\,dx \;=\; \lim_{b\to\infty}\int_a^b f(x)\,dx$$

provided that the limit on the right exists, in which case we say that the improper integral *converges*. If the limit on the right does not exist (or is infinity) then we say that the improper integral *diverges*.

Example 1 Evaluate

$$\int_1^\infty \frac{1}{x^2}\,dx, \qquad \int_1^\infty \frac{1}{\sqrt{x}}\,dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x}\,dx.$$

Solution.

- Evaluate the first integral,

$$\int_1^\infty \frac{1}{x^2}\,dx \;=\; \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx \;=\; \lim_{b\to\infty} -\frac{1}{x}\Big]_1^b \;=\; \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) \;=\; 1.$$

  This improper integral converges to 1.

- Now evaluate the second integral,

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx \;=\; \lim_{b\to\infty}\int_1^b x^{-1/2}\,dx \;=\; \lim_{b\to\infty} 2x^{1/2}\Big]_1^b \;=\; \lim_{b\to\infty}\left(2\sqrt{b}-2\right) \;=\; \infty.$$

  This improper integral diverges.

- Finally, we evaluate the third integral,

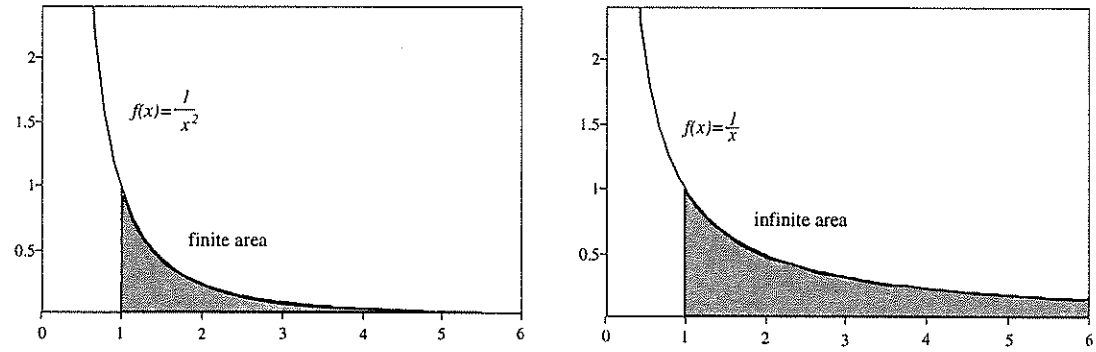
$$\int_1^\infty \frac{1}{x}\,dx \;=\; \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \;=\; \lim_{b\to\infty} \ln x\Big]_1^b \;=\; \lim_{b\to\infty}\left(\ln b - 0\right) \;=\; \infty.$$

  This improper integral diverges.

□

Remark One can show that in general,

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } 0 < p \le 1. \end{cases} \tag{3.1}$$



The graphs of the functions $\frac{1}{x^2}$ (left) and $\frac{1}{x}$ (right), see example 1.

Remark Because $\frac{1}{x^p} > 0$ for all $x \ge 1$, we can view

$$\int_1^\infty \frac{1}{x^p}\,dx$$

as the area of the infinite region which lies between the graph of $\frac{1}{x^p}$, the x-axis and the line $x = 1$. For $p > 1$ this area is finite, for $0 < p \le 1$ this area is infinite.

Example 2 Evaluate

$$\int_0^\infty e^{-x}\,dx \qquad \text{and} \qquad \int_0^\infty e^{2x}\,dx.$$

Solution.

- Evaluate the first integral,

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big]_0^b = \lim_{b\to\infty}(-e^{-b}+1) = 1.$$

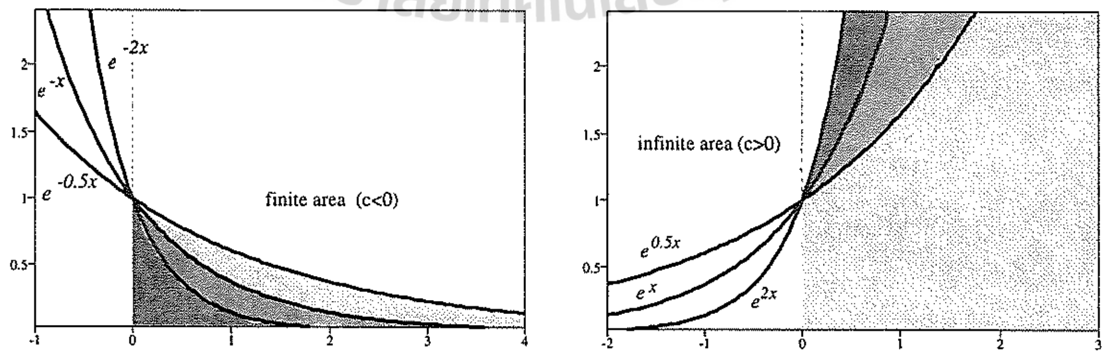
This improper integral converges to 1.

- Now evaluate the second integral,

$$\int_0^\infty e^{2x}\,dx = \lim_{b\to\infty}\int_0^b e^{2x}\,dx = \lim_{b\to\infty}\frac{1}{2}e^{2x}\Big]_0^b = \lim_{b\to\infty}\left(\frac{e^{2b}}{2}-\frac{1}{2}\right) = \infty.$$

This improper integral diverges.

□



The graphs of functions e^{cx} for $c < 0$ (left) and for $c > 0$ (right), see examples 2 and 3.

The last example can be generalized as follows:

Example 3 Determine for what values of the constant c the integral

$$\int_a^{\infty} e^{cx}\,dx$$

converges, and find its value.

Solution. Note that for $c \neq 0$,

$$\int_a^b e^{cx}\,dx \;=\; \frac{1}{c}\,e^{cx}\,\Big]_a^b \;=\; \frac{1}{c}\,e^{cb} - \frac{1}{c}\,e^{ca}$$

Now let $b \to \infty$. We must distinguish three cases:

- If $c < 0$, then
$$\lim_{b\to\infty}\left(\frac{1}{c}\,e^{cb} - \frac{1}{c}\,e^{ca}\right) \;=\; 0 - \frac{1}{c}e^{ca}.$$
- If $c > 0$, then
$$\lim_{b\to\infty} e^{cb} = \infty$$
and the integral diverges.
- If $c = 0$, then $e^{cx} = 1$, and we have an integral
$$\lim_{b\to\infty}\int_a^b 1\,dx \;=\; \lim_{b\to\infty} x\,\Big]_a^b \;=\; \lim_{b\to\infty}(b-a) \;=\; \infty,$$
which again diverges.

We can summarize this:

$$\int_a^{\infty} e^{cx}\,dx \;=\; \begin{cases} -\dfrac{1}{c}e^{ca} & \text{if } \; c < 0 \\ \text{diverges} & \text{if } \; c \geq 0 \end{cases} \tag{3.2}$$

□

Remark If $F(x)$ is an antiderivative of $f(x)$, then we write

$$\int_a^{\infty} f(x)\,dx \;=\; F(x)\,\Big]_a^{\infty}$$

where $F(x)\,\Big]_a^{\infty}$ means

$$F(x)\,\Big]_a^{\infty} \;=\; \lim_{b\to\infty} F(x)\,\Big]_a^b \;=\; \lim_{b\to\infty}\; F(b) - F(a)$$

For convenience we change b to x, and have

$$F(x)\,\Big]_a^{\infty} \;=\; \lim_{x\to\infty}\; F(x) - F(a)$$

Sometimes we can not compute an improper integral directly. By comparing the integrand with another function, we may still able to decide whether the integral converges:

Theorem 19 (Comparison Test). *Assume that $f(x)$ and $g(x)$ can be integrated over each finite subinterval of $[a,\infty)$.*

1. *If $|f(x)| \leq g(x)$ for all $x \geq a$, and if $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges also.*
2. *If $0 \leq g(x) \leq f(x)$ for all $x \geq a$, and if $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges also.*

Remark This theorem may tell us whether an improper integral converges or diverges, but does not tell us how to find the value of the integral !

Example 4 Determine whether the improper integral $\int_0^\infty \sin x^2\, e^{-x}\,dx$ converges.

Solution. We can not compute

$$\int \sin x^2\, e^{-x}\,dx.$$

However, we note that

$$|\sin x^2 e^{-x}| \;\leq\; e^{-x}.$$

From example 2 we know that

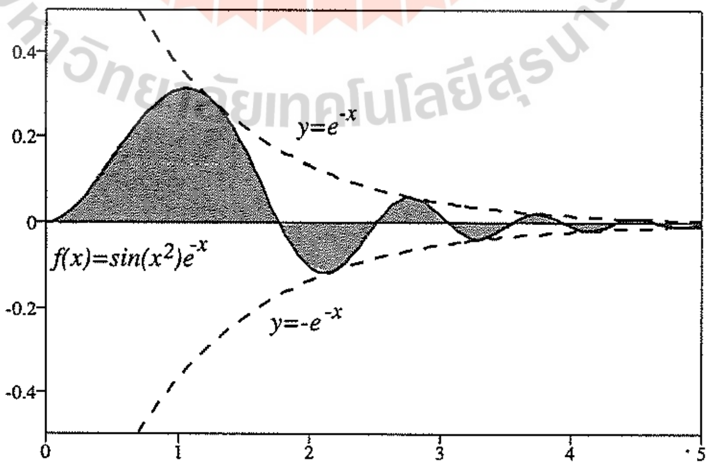
$$\int_0^\infty e^{-x}\,dx \;<\; \infty.$$

By the comparison test,

$$\int_0^\infty \sin x^2 e^{-x}\,dx \;<\; \infty.$$

That is, this improper integral converges. (However we don't know its value !)

□



Comparison test of example 4.

Example 5 Determine convergence of $\int_e^\infty x \ln x \, dx$.

Solution. Note that

$$0 \le x \le x \ln x$$

for all $x \ge e$. Therefore,

$$\int_e^\infty x \, dx \;\le\; \int_e^\infty x \ln x \, dx.$$

We can evaluate the integral on the left:

$$\int_e^\infty x \, dx \;=\; \frac{x^2}{2}\bigg]_e^\infty \;=\; \lim_{x\to\infty}\left(\frac{x^2}{2} - \frac{e^2}{2}\right) \;=\; \infty.$$

Thus we see that

$$\int_e^\infty x \ln x \, dx = \infty$$

also, that is, this improper integral diverges. □

Exercises

1. Find the following limits if they exist. (Throughout, $c > 0$ is constant).
   (a) $\lim_{b\to\infty} e^{-b}$
   (b) $\lim_{b\to\infty} e^{b}$
   (c) $\lim_{b\to\infty} e^{-cb}$
   (d) $\lim_{b\to\infty} e^{cb}$
   (e) $\lim_{b\to\infty} e^{cb} \quad (c = 0)$
   (f) $\lim_{b\to\infty} be^{-cb}$
   (g) $\lim_{b\to\infty} be^{cb}$
   (h) $\lim_{b\to\infty} e^{-cb}/b$
   (i) $\lim_{b\to\infty} e^{cb}/b$

2. Using the definition of the improper integral, determine whether the following improper integrals converge and find their values if they converge.
   (a) $\int_1^\infty \frac{1}{\sqrt{x^3}} \, dx$,
   (b) $\int_1^\infty \frac{1}{\sqrt{x}} \, dx$,
   (c) $\int_0^\infty xe^{-x^2} \, dx$,
   (d) $\int_1^\infty \ln x \, dx$,
   (e) $\int_0^\infty \frac{1}{x^2+1} \, dx$,
   (f) $\int_1^\infty xe^{-x} \, dx$.

3. Determine whether the following improper integrals converge or diverge. (You need not find the values of the integrals.)
   (a) $\int_0^\infty e^{-x^2} \, dx$,
   (b) $\int_0^\infty \sin xe^{-x} \, dx$,
   (c) $\int_0^\infty \sin x \, dx$,
   (d) $\int_1^\infty e^{x^2} \, dx$,
   (e) $\int_1^\infty x^{-2}e^{x} \, dx$.

4. Show that

$$\int_c^\infty e^{-(x-c)} \, dx \;=\; \int_0^\infty e^{-u} \, du$$

5. Prove equation (3.1).

   Supplementary exercises:

6. Let $f(x)$ be a positive, decreasing function such that $\int_0^\infty f(x) \, dx$ converges. Show that $\lim_{x\to\infty} f(x) = 0$.

3.2 Definition of the Laplace Transform

Definition Let $f(x)$ be a function defined on the interval $[0, \infty)$. The function

$$\boxed{F(s) \;=\; \int_0^\infty f(x)\, e^{-sx}\, dx} \tag{3.3}$$

is called the *Laplace transform of f*.

Remark Note that during integration the variable x disappears, and the number s becomes a variable. The Laplace transform is a function in the new variable s.

Let us find the Laplace transforms of some basic functions using the definition (3.2):

- The function $\boxed{f(x) = 1}$. Its Laplace transform is

$$F(s) \;=\; \int_0^\infty 1 \cdot e^{-sx}\, dx \;=\; -\frac{e^{-sx}}{s}\Big]_0^\infty \;=\; \begin{cases} \frac{1}{s} & \text{if } s > 0 \\ \infty & \text{if } s \le 0. \end{cases} \tag{3.4}$$

Therefore, the Laplace transform is

$$F(s) \;=\; \frac{1}{s}$$

and is defined for $s > 0$.

- The function $\boxed{f(x) = x}$. Its Laplace transform is

$$F(s) \;=\; \int_0^\infty x\, e^{-sx}\, dx.$$

Use integration by parts,

$$\begin{aligned} F(s) &= x\Big(-\frac{e^{-sx}}{s}\Big)\Big]_0^\infty - \int_0^\infty \Big(-\frac{e^{-sx}}{s}\Big)\, dx \\ &= \Big[-\frac{xe^{-sx}}{s} - \frac{e^{-sx}}{s^2}\Big]_0^\infty \\ &= \lim_{x\to\infty} -\frac{xe^{-sx}}{s} - \frac{e^{-sx}}{s^2} + 0 + \frac{1}{s^2} \end{aligned}$$

Now by l'Hôpital's rule,

$$\lim_{x\to\infty} xe^{-sx} \;=\; \lim_{x\to\infty} \frac{x}{e^{sx}} \;\overset{H}{=}\; \lim_{x\to\infty} \frac{1}{se^{sx}} \;=\; 0$$

for $s > 0$. Therefore, the Laplace transform is

$$F(s) \;=\; 0 - 0 + 0 + \frac{1}{s^2} \;=\; \frac{1}{s^2} \tag{3.5}$$

and is defined for $s > 0$.

- In a similar way, one can show that if $\boxed{f(x) = x^n}$ $(n = 1, 2, \ldots)$, then

$$F(s) \;=\; \int_0^\infty x^n\, e^{-sx}\, dx \;=\; \frac{n!}{s^{n+1}}. \tag{3.6}$$

- The function $\boxed{f(x) = e^{cx}}$. Its Laplace transform is

$$F(s) \;=\; \int_0^\infty e^{cx}\, e^{-sx}\, dx \;=\; \int_0^\infty e^{(c-s)x}\, dx.$$

But by (3.2), this integral converges to

$$F(s) \;=\; \frac{1}{s-c} \tag{3.7}$$

whenever $s > c$.

- The function $\boxed{f(x) = \sin ax}$. Its Laplace transform is

$$F(s) \;=\; \int_0^\infty \sin ax\; e^{-sx}\, dx.$$

Using integration by parts twice, one obtains the antiderivative

$$\int \sin ax\; e^{-sx}\, dx \;=\; \frac{-e^{-sx}}{s^2+a^2}\,(s\sin ax + a\cos ax) + C.$$

(check yourself !) Therefore,

$$\begin{aligned} F(s) \;&=\; \frac{-e^{-sx}}{s^2+a^2}\,(s\sin ax + a\cos ax)\Big]_0^\infty \\ &=\; \lim_{x\to\infty} \frac{-e^{-sx}}{s^2+a^2}\,(s\sin ax + a\cos ax) \;+\; \frac{a}{s^2+a^2}. \end{aligned}$$

To find this limit, we do a comparison: For all values of x,

$$0 \;\le\; \left|\frac{-e^{-sx}}{s^2+a^2}\,(s\sin ax + a\cos ax)\right| \;\le\; \frac{e^{-sx}}{s^2+a^2}\;(\,|s| + |a|\,)$$

because $|\sin ax\,| \le 1$ and $|\cos ax\,| \le 1$. Now when $s > 0$, we have

$$\lim_{x\to\infty} \frac{e^{-sx}}{s^2+a^2}\;(\,|s|+|a|\,) \;=\; \frac{0}{s^2+a^2}\;(\,|s|+|a|\,) \;=\; 0$$

so that by the Sandwich Theorem,

$$\lim_{x\to\infty} \frac{-e^{-sx}}{s^2+a^2}\,(s\sin ax + a\cos ax) \;=\; 0$$

also. The Laplace transform is thus

$$F(s) \;=\; \frac{a}{s^2+a^2}. \tag{3.8}$$

and is defined for $s > 0$.

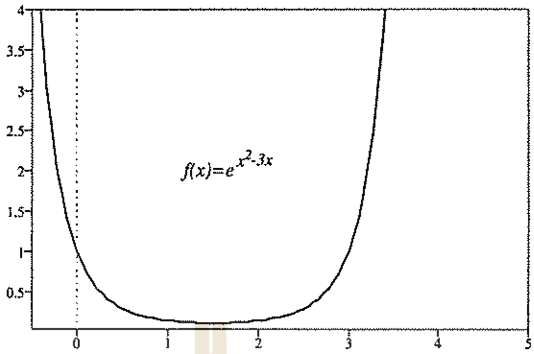
- In a similar way, one can show that $\boxed{f(x) = \cos ax}$ has the Laplace transform

$$F(s) = \frac{s}{s^2+a^2} \qquad (s > 0). \tag{3.9}$$

Not every function has a Laplace transform. For example, if $f(x) = e^{x^2}$ then the improper integral

$$F(s) = \int_0^\infty e^{x^2} e^{-sx}\, dx$$

diverges because $e^{x^2} e^{-sx}$ is increasing on $[\frac{s}{2}, \infty)$.



The improper integral $\int_0^\infty e^{x^2 - sx}\, dx$ diverges.

What functions f have a Laplace transform, that is when does the improper integral

$$\int_0^\infty f(x)\, e^{-sx}\, dx$$

converge ? As the above graph shows, the function f must not grow too fast. We make the following definition:

Definition A function $f(x)$ defined on an infinite interval $[a, \infty)$ is of *exponential order* if there exists a number $x_o \geq a$ and constants c and M such that

$$|f(x)| \leq Me^{cx}$$

for $x \geq x_o$. We write $f(x) = \mathcal{O}(e^{cx})$.

Remark One can show that if $\lim\limits_{x\to\infty} \dfrac{|f(x)|}{e^{cx}} < \infty$ then $f(x) = \mathcal{O}(e^{cx})$. Loosely speaking, "$f(x)$ is of exponential order" means that $|f(x)|$ grows not faster than the exponential function as $x \to \infty$.

e.g.

- If $f(x) = x^2$, then by L'Hôpitals rule,

  $$\lim_{x\to\infty} \frac{x^2}{e^{cx}} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{ce^{cx}} \overset{H}{=} \lim_{x\to\infty} \frac{2}{c^2 e^{cx}} = 0$$

  for every $c > 0$. This shows that for every $M > 0$ and $c > 0$ we can make

  $$x^2 \leq Me^{cx}$$

  provided that x is sufficiently large. Thus, $x^2 = \mathcal{O}(e^{cx})$ for every $c > 0$.
- The functions $\sin x$ and $\sin x\, e^{2x}$ are of exponential order.
- The function e^{x^2} is not of exponential order, as for every c.

$$\lim_{x\to\infty} \frac{e^{x^2}}{e^{cx}} = \lim_{x\to\infty} e^{x^2 - cx} = \infty.$$

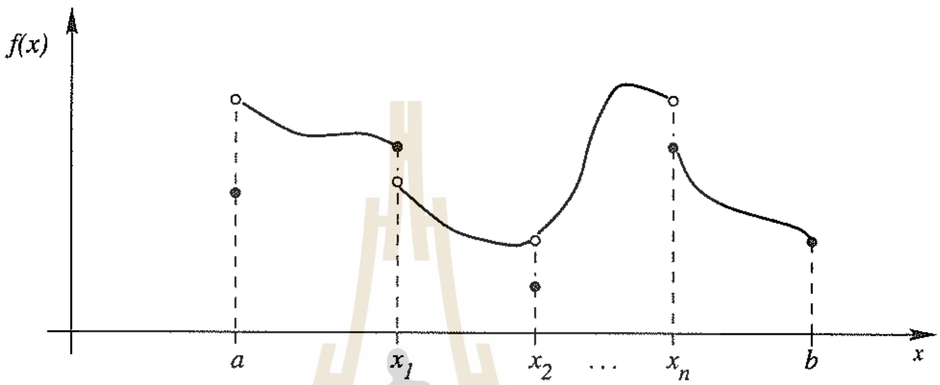
In practice we have to work with functions which are not continuous, so let us first make the following definition:

Definition A function f is *piecewise continuous* on an interval $[a, \infty)$ if

1. On every finite interval $[a, b]$, f is continuous except possibly at a finite number of points $x_1, x_2, \ldots, x_n$.
2. At each of these discontinuities x_i, the one-sided limits

$$\lim_{x \to x_i^+} f(x) \qquad \text{and} \qquad \lim_{x \to x_i^-} f(x)$$

exist.



A piecewise continuous function.

Remark A piecewise continuous function can still be integrated over the finite interval $[a, b]$. One simply integrates over each subinterval $[x_i, x_{i+1}]$ on which the function is continuous,

$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_n}^b f(x)\,dx.$$

We will now see that most functions have a Laplace transform:

Theorem 20 *If the function $f(x)$ is*

1. *piecewise continuous on $[0, \infty)$, and*
2. *of exponential order, $f = \mathcal{O}(e^{cx})$,*

then the Laplace transform $F(s)$ exists for $s > c$.

Proof. Let us split the integral which defines the Laplace transform,

$$\int_0^\infty f(x)\,e^{-sx}\,dx = \int_0^{x_o} f(x)e^{-sx}\,dx + \int_{x_o}^\infty f(x)e^{-sx}\,dx$$

where x_o, c and M are chosen so that $|f(x)| \leq Me^{cx}$ for $x \geq x_o$. Then

$$|f(x)e^{-sx}| \leq Me^{cx}e^{-sx} = Me^{(c-s)x}$$

for $x \geq x_o$, and since by (3.2), $\int_{x_o}^\infty e^{(c-s)x}\,dx$ converges for $s > c$, the comparison test shows that

$$\int_{x_o}^\infty f(x)e^{-sx}\,dx$$

also converges. Hence,

$$\int_0^\infty f(x)e^{-sx}\,dx$$

converges. □

Remark In the following, we will always assume that our functions are piecewise continuous and of exponential order, so that the Laplace transforms exist.

Exercises

1. Sketch the graph of each function f and determine whether f is continuous, piecewise continuous, or neither on the interval $[0,\infty)$.

   (a) $f(x) = \begin{cases} x^2, & 0 \le x \le 1, \\ 2+x, & 1 < x \le 2, \\ 6-x, & 2 < x \le 3 \\ 0, & x > 3 \end{cases}$ (c) $f(x) = \begin{cases} x^2, & 0 \le x \le 1, \\ 1, & 1 < x \le 2, \\ 3-x, & 2 < x \le 3 \\ 0, & x > 3 \end{cases}$

   (b) $f(x) = \begin{cases} 1/x^2, & 0 \le x \le 1, \\ 2+x, & 1 < x \le 2, \\ 0, & 2 < x \le 3 \\ 1, & x > 3 \end{cases}$ (d) $f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 3-x, & 1 < x \le 2, \\ 1, & 2 < x \le 3 \\ -1, & x > 3 \end{cases}$

2. Find the Laplace transform $F(s)$ using definition (3.3) of

   (a) $f(x) = xe^{ax}$, (b) $f(x) = x^2$ (c) $f(x) = (x-1)^2$ (d) $f(x) = x\cosh ax$

   Supplementary exercises:

3. Show that $f(x) = x^n$ is of exponential order.

4. Show that if the functions f and g are of exponential order, then $f+g$ is of exponential order.

5. Suppose, $f(x)$ is continuous on $[0,\infty)$ such that $f = \mathcal{O}(e^{-cx})$ for some $c > 0$. Show that the improper integral $\int_0^\infty f(x)\,dx$ converges.

6. Suppose, f and f' are continuous for $x \ge 0$, and of exponential order. Show that

   $$\lim_{s\to\infty} F(s) = 0.$$

   (Hint: use integration by parts)

3.3 Solutions of Initial Value Problems

One also uses the symbol $\mathfrak{L}\{f\}$ to denote the Laplace transform of a function $f(x)$. Thus,

$$F(s) \;=\; \mathfrak{L}\{f\} \;=\; \int_0^\infty f(x)\,e^{-sx}\,dx.$$

So far we have computed the following Laplace transforms:

$f(x)$	$F(s) = \mathfrak{L}\{f\}$
1	$\frac{1}{s}, \quad s > 0$
x	$\frac{1}{s^2}, \quad s > 0$
$x^n \quad (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}, \quad s > 0$
e^{cx}	$\frac{1}{s-c}, \quad s > c$
$\sin ax$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos ax$	$\frac{s}{s^2+a^2}, \quad s > 0$

Table 1: Basic Laplace Transforms

In this section we will learn to use the Laplace transform for solving initial value problems. The theory is based on the following two properties of the Laplace transform:

1. *Linearity:* If the Laplace transforms of f and g exist for $s > c$ then the Laplace transform of the linear combination $c_1 f + c_2 g$ exists for $s > c$, and equals

$$\mathfrak{L}\{c_1 f + c_2 g\} = c_1 \mathfrak{L}\{f\} + c_2 \mathfrak{L}\{g\}. \tag{3.10}$$

Proof. This is due to the corresponding property of the integral:

$$\begin{aligned} \mathfrak{L}\{c_1 f + c_2 g\} &= \int_0^\infty (\, c_1 f(x) + c_2 g(x)\,)\, e^{-sx}\, dx \\ &= c_1 \int_0^\infty f(x)\, e^{-sx}\, dx + c_2 \int_0^\infty g(x)\, e^{-sx}\, dx \\ &= c_1 \mathfrak{L}\{f\} + c_2 \mathfrak{L}\{g\}. \end{aligned}$$

□

Example 1 Find the Laplace transform of $f(x) = \cosh ax$.

Solution. Note that

$$\cosh ax = \frac{1}{2}e^{ax} + \frac{1}{2}e^{-ax}.$$

Now we know that

$$\mathfrak{L}\{\, e^{ax}\,\} = \frac{1}{s-a}$$

for $s > a$ and

$$\mathfrak{L}\{\, e^{-ax}\,\} \;=\; \frac{1}{s+a}$$

for $s > -a$. By linearity,

$$\mathfrak{L}\{\cosh ax\} \;=\; \frac{1}{2}\,\frac{1}{s-a} + \frac{1}{2}\,\frac{1}{s+a} \;=\; \frac{s}{s^2-a^2} \tag{3.11}$$

for $s > |a|$. □

Example 2 Find the Laplace transform of

$$f(x) \;=\; \sin 2x + e^{-4x} + 3x - 2.$$

Solution. Using the table and linearity, we get

$$\begin{aligned}\mathfrak{L}\{f\} &= \mathfrak{L}\{\sin 2x\} + \mathfrak{L}\{e^{-4x}\} + 3\mathfrak{L}\{x\} - 2\mathfrak{L}\{1\} \\ &= \frac{2}{s^2+4} + \frac{1}{s+4} + \frac{3}{s^2} - \frac{2}{s}.\end{aligned}$$

□

2. *Transform of a Derivative:* Suppose, f is continuous and f' is piecewise continuous on $[0,\infty)$, and that $f = \mathcal{O}(e^{cx})$. Then the Laplace transform of f' exists for $s > c$, and

$$\mathfrak{L}\{f'\} \;=\; s\mathfrak{L}\{f\} - f(0). \tag{3.12}$$

Loosely speaking, the Laplace transform changes "differentiation" to "multiplication by s".

Proof. To make the notation easier, we assume that f' is also continuous. First integrate over a finite interval using integration by parts,

$$\begin{aligned}\int_0^b f'(x)e^{-sx}\,dx &= f(x)e^{-sx}\Big]_0^b - \int_0^b f(x)(-s)e^{-sx}\,dx \\ &= f(b)e^{-sb} - f(0) + s\int_0^b f(x)e^{-sx}\,dx\end{aligned}$$

Now let $b \to \infty$. Since $|f(b)| \le Me^{cb}$ for large b, we have

$$|f(b)e^{-sb}| \le |Me^{cb}e^{-sb}| = Me^{(c-s)b} \quad\to\quad 0$$

whenever $s > c$. Therefore,

$$\begin{aligned}\mathfrak{L}\{f'\} &= \int_0^\infty f'(x)\,e^{-sx}\,dx \;=\; \lim_{b\to\infty}\left[f(b)e^{-sb} - f(0) + s\int_0^b f(x)e^{-sx}\right] \\ &= 0 - f(0) + s\int_0^\infty f(x)\,e^{-sx}\,dx \;=\; -f(0) + s\mathfrak{L}\{f\}.\end{aligned}$$

□

Example 3 Find the Laplace transform of $f(x) = x^2$.

Solution. We have $f'(x) = 2x$ and $f(0) = 0$. By the rule (3.12) for derivatives,

$$\begin{aligned} \mathfrak{L}\{f'(x)\} &= s\mathfrak{L}\{f(x)\} - f(0) \\ \mathfrak{L}\{2x\} &= s\mathfrak{L}\{x^2\} - 0 \\ 2\mathfrak{L}\{x\} &= s\mathfrak{L}\{x^2\}. \end{aligned}$$

Now divide by s,

$$\mathfrak{L}\{x^2\} = \frac{2}{s}\,\mathfrak{L}\{x\} = \frac{2}{s}\,\frac{1}{s^2} = \frac{2}{s^3}. \tag{3.13}$$

□

We can apply the rule for derivatives again to get a formula for the second order derivative. Replacing f by f' in (3.12) we get

$$\mathfrak{L}\{f''\} = s\mathfrak{L}\{f'\} - f'(0).$$

Now use (3.12) again for $\mathfrak{L}\{f'\}$,

$$\begin{aligned} \mathfrak{L}\{f''\} &= s\Big(\, s\mathfrak{L}\{f\} - f(0)\,\Big) - f'(0) \\ &= s^2\mathfrak{L}\{f\} - sf(0) - f'(0). \end{aligned} \tag{3.14}$$

Continuing this way, we obtain the Laplace transform of the n-th order derivative,

$$\mathfrak{L}\{f^{(n)}\} = s^n\mathfrak{L}\{f\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{3.15}$$

provided that $f, f', \ldots, f^{(n-1)}$ are continuous, $f^{(n)}$ is piecewise continuous and all functions are of exponential order.

The following two examples show how the Laplace transform can be used to solve initial value problems.

Example 4 Solve the initial value problem

$$y'' - y' - 2y = 0 \qquad y(0) = 1, \quad y'(0) = 0. \tag{3.16}$$

using the Laplace transform.

Solution. *Method 1 (Usual method):* The characteristic equation

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

has two solutions $\lambda = -1, 2$. The homogeneous equation (3.16) therefore has general solution

$$y = c_1e^{-x} + c_2e^{2x}.$$

The initial conditions will give the particular solution

$$y = \frac{2}{3}e^{-x} + \frac{1}{3}e^{2x}.$$

Method 2 (Laplace transform): Take the Laplace transform on both sides of equation (3.16). We obtain

$$\mathfrak{L}\{y'' - y' - 2y\} \;=\; \mathfrak{L}\{0\}.$$

By linearity,

$$\mathfrak{L}\{y''\} - \mathfrak{L}\{y'\} - 2\mathfrak{L}\{y\} \;=\; 0.$$

Now use the rules (3.12) and (3.14) for derivatives,

$$\Big(s^2\mathfrak{L}\{y\} - sy(0) - y'(0)\Big) - \Big(s\mathfrak{L}\{y\} - y(0)\Big) - 2\mathfrak{L}\{y\} \;=\; 0.$$

To simplify notation, set $Y(s) = \mathfrak{L}\{y\}$ and use the initial conditions $y(0) = 1$ and $y'(0) = 0$. We obtain

$$\Big(\, s^2Y(s) - s \,\Big) - \Big(sY(s) - 1\,\Big) - 2Y(s) \;=\; 0.$$

This is now a simple equation in one unknown variable $Y(s)$. Solve for $Y(s)$,

$$Y(s)(s^2 - s - 2) - s + 1 \;=\; 0$$

$$Y(s) \;=\; \frac{s-1}{s^2 - s - 2}.$$

We have now found the Laplace transform of the solution y. To find y itself, we use partial fraction decomposition for $Y(s)$.

$$\frac{s-1}{s^2-s-2} \;=\; \frac{s-1}{(s-2)(s+1)} \;=\; \frac{A}{s+1} + \frac{B}{s-2}$$

Multiply by the common denominator $(s-2)(s+1)$,

$$s - 1 \;=\; A(s-2) + B(s+1)$$

Now if $s = -1$, this becomes $-2 = -3A$, so that $A = 2/3$. If $s = 2$, then this becomes $1 = 3B$, so that $B = 1/3$.

Thus,

$$Y(s) \;=\; \frac{2}{3}\,\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{s-2}.$$

Now look at the table of Laplace transforms. We see that

$$\frac{1}{s+1} \quad \text{is the Laplace transform of} \quad e^{-x}$$

$$\frac{1}{s-2} \quad \text{is the Laplace transform of} \quad e^{2x}$$

Therefore,

$$y(x) \;=\; \frac{2}{3}\,e^{-x} + \frac{1}{3}\,e^{2x}$$

which is of course the same solution as we got using the first method. □

Remark If $F(s)$ is the Laplace transform of a function $f(x)$, then we call $f(x)$ the *inverse Laplace transform* of $F(s)$ and write $f(x) = \mathfrak{L}^{-1}\{\, F(s) \,\}$. This definition makes sense as one can show the following: If $f(x)$ and $g(x)$ have the same Laplace transform $F(s)$, then $f(x) = g(x)$ for all $x > 0$ except for the points of discontinuity.

Remark Because finding the Laplace transform is linear, the process of finding the inverse Laplace transform is also linear,

$$\mathfrak{L}^{-1}\Big\{\, c_1F(s) + c_2G(s) \,\Big\} \;=\; c_1\mathfrak{L}^{-1}\Big\{\, F(s) \,\Big\} + c_2\mathfrak{L}^{-1}\Big\{\, G(s) \,\Big\}.$$

Example 5 Solve the initial value problem

$$y'' + 4y = 4x \qquad y(0) = 1, \quad y'(0) = 5.$$

by using the Laplace transform.

Solution. Take the Laplace transform on both sides,

$$\mathfrak{L}\{y''\} + 4\mathfrak{L}\{y\} = \mathfrak{L}\{4x\}.$$

Now if $Y(s) = \mathfrak{L}\{y\}$ is the Laplace transform of y, then by (3.14), this equation becomes

$$\Big(s^2Y(s) - sy(0) - y'(0) \Big) + 4Y(s) = \frac{4}{s^2}$$

Now substitute the initial conditions,

$$s^2Y(s) - s - 5 + 4Y(s) = \frac{4}{s^2}$$

and solve for Y,

$$Y(s)(s^2+4) = s + 5 + \frac{4}{s^2}$$

$$Y(s) = \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{4}{s^2(s^2+4)}.$$

As the last step, we must find the inverse Laplace transform of $Y(s)$. We use partial fraction decomposition in order to write $Y(s)$ as a sum of simple fractions which can be found in table 1:

Because s^2 is a repeated factor, we must set

$$\frac{4}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

Multiply by the common denominator $s^2(s^2+4)$,

$$4 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2.$$

Collect terms with like powers of s,

$$4 = s^3(A+C) + s^2(B+D) + s(4A) + 4B$$

Now compare coefficients,

$$\begin{aligned} A+C &= 0 \\ B+D &= 0 \\ 4A &= 0 \\ 4B &= 4 \end{aligned}$$

Solving, we obtain $A = C = 0, B = 1$ and $D = -1$.

So we have

$$\begin{aligned} Y(s) &= \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{1}{s^2} - \frac{1}{s^2+4} \\ &= \frac{s}{s^2+4} + 2\,\frac{2}{s^2+4} + \frac{1}{s^2} \end{aligned}$$

To find the inverse Laplace transform, we look at the table and obtain

$$y(x) = \cos 2x + 2\sin 2x + x.$$

□

Remark In general, the method of the Laplace transform can be used with every linear equation with constant coefficients,

$$y^{(n)} + a_1 y^{(n-1)} \cdots + a_{n-1} y' + a_o y \;=\; r(x)$$

provided that the solution $y(x)$ satisfies the conditions in (3.15). The most difficult part is to find the inverse Laplace transform of the function $Y(s)$, and we will in the following investigate more properties of the Laplace transform which can help find the inverse transform.

Exercises

1. Using the transforms in (3.5), (3.7) and (3.8) and the rules (3.10) – (3.15) , show that

   (a) $\mathfrak{L}\{\cos ax\} = \dfrac{s}{s^2+a^2}, \quad s > 0,$

   (b) $\mathfrak{L}\{\sinh ax\} = \dfrac{a}{s^2-a^2}, \quad s > |a|,$

   (c) $\mathfrak{L}\{x^3\} = \dfrac{6}{s^4}, \quad s > 0,$

   (d) $\mathfrak{L}\{x^n\} = \dfrac{n!}{s^{n+1}}, \quad s > 0.$

2. Without integrating, but using formulas (3.5) – (3.8) and rule (3.10), find $\mathfrak{L}\{\sin^2 ax\}$ and $\mathfrak{L}\{\cos^2 ax\}$. How are the two transforms related to another ?

3. Find the Laplace transform of

   (a) $f(x) = 10,$

   (b) $f(x) = x^5 + \cos 2x,$

   (c) $f(x) = 2e^{3x} - \sin 5x,$

   (d) $f(x) = 4\sin x\cos x + 2e^{-x}.$

4. Find the inverse Laplace transform of $F(s) =$

   (a) $\dfrac{30}{s^4},$

   (b) $\dfrac{2}{s+3},$

   (c) $\dfrac{4}{s^3} + \dfrac{6}{s^2+4},$

   (d) $\dfrac{1}{s^2+s},$

   (e) $\dfrac{1}{s^4+s^2},$

   (f) $\dfrac{1}{s(s-3)},$

   (g) $\dfrac{3}{s(s+5)},$

   (h) $\dfrac{1}{s(s^2+4)},$

   (i) $\dfrac{2s+1}{s(s^2+9)},$

   (j) $\dfrac{1}{s^2(s^2+1)},$

   (k) $\dfrac{1}{s(s^2-9)},$

   (l) $\dfrac{1}{s^2(s^2-a^2)},$

   (m) $\dfrac{1}{s(s+1)(s+2)}.$

5. Using the rules discussed in this section, show that

   (a) $\mathfrak{L}\{xe^{ax}\} = \dfrac{1}{(s-a)^2} \quad s > a,$

   (b) $\mathfrak{L}\{x\cos ax\} = \dfrac{s^2-a^2}{(s^2+a^2)^2} \quad s > a,$

   (c) $\mathfrak{L}\{x\sinh ax\} = \dfrac{2as}{(s^2-a^2)^2} \quad s > |a|.$

6. Use the Laplace transform to solve the initial value problem:

   (a) $y'' - y' - 6y = 0 \quad y(0) = 1, \quad y'(0) = -1$

   (b) $y'' + 3y' + 2y = 0 \quad y(0) = 1, \quad y'(0) = 0$

   (c) $y'' + y' - 2y = 2e^{-x} \quad y(0) = 1, \quad y'(0) = 0$

   (d) $y'' + 4y = x^2 \quad y(0) = 0, \quad y'(0) = 2$

   (e) $y'' - 2y' - 3y = \sin x \quad y(0) = \frac{1}{2}, \quad y'(0) = \frac{1}{5}$

   (f) $y^{(4)} - y = 0 \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad , y'''(0) = 0$

   (g) $y^{(4)} - 4y = 0 \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$

   (h) $y'' + \omega^2 y = \cos 2x \quad y(0) = 1, \quad y'(0) = 0, \quad \omega^2 \neq 4$

3.4 Properties of the Laplace Transform

3.4.1 General Properties

It is usually quite complicated to compute the Laplace transform by using its definition. In this section we will study properties of the Laplace transform which can be used to derive the transform of a function from that of another function. Let $F(s) = \mathfrak{L}\{f\}$ and $G(s) = \mathfrak{L}\{g\}$ be the Laplace transform of functions f and g. We have already studied two properties:

- *Linearity:*

$$\boxed{\mathfrak{L}\{c_1 f + c_2 g\} \;=\; c_1 F(s) + c_2 G(s).} \tag{3.17}$$

- *Transform of a Derivative:*

$$\mathfrak{L}\{f'\} \;=\; sF(s) - f(0) \tag{3.18}$$
$$\mathfrak{L}\{f''\} \;=\; s^2F(s) - sf(0) - f'(0) \tag{3.19}$$
$$\vdots \qquad \vdots$$
$$\boxed{\mathfrak{L}\{f^{(n)}\} \;=\; s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

There are further properties:

- *Transform of an Integral:*

$$\boxed{\mathfrak{L}\left\{ \int_0^x f(u)\, du \right\} \;=\; \frac{F(s)}{s}} \tag{3.20}$$

We say that "integration" corresponds to 'division by s".

Proof. Set $g(x) = \displaystyle\int_0^x f(u)\, du$. Then by the fundamental theorem of calculus,

$$g(0) = 0 \qquad \text{and} \qquad g'(x) = f(x).$$

First we must verify whether $g(x)$ is of exponential order to make sure that its Laplace transform exists. Note that

$$\lim_{x\to\infty} \frac{|g(x)|}{e^{cx}} = \lim_{x\to\infty} \frac{\left|\int_0^x f(u)\, du\right|}{e^{cx}} \leq \lim_{x\to\infty} \frac{\int_0^x |f(u)|\, du}{e^{cx}}.$$

– Now if $\lim_{x\to\infty} \int_0^x |f(u)|\, du = M$ exists and is finite, then

$$\lim_{x\to\infty} \frac{|g(x)|}{e^{cx}} \leq \lim_{x\to\infty} \frac{M}{e^{cx}} \;=\; 0$$

whenever $c > 0$.

– On the other hand, if $\lim_{x\to\infty} \int_0^x |f(u)|\, du = \infty$, then by l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{|g(x)|}{e^{cx}} \leq \lim_{x\to\infty} \frac{\frac{d}{dx}\int_0^x |f(u)|\, du}{\frac{d}{dx}\, e^{cx}} \overset{H}{=} \lim_{x\to\infty} \frac{|f(x)|}{ce^{cx}} < \infty$$

because $f(x)$ is of exponential order.

Both cases show that $g(x)$ is of exponential order and therefore has a Laplace transform. Then using (3.18), we get

$$\mathfrak{L}\{g'\} = s\mathfrak{L}\{g\} - g(0) = s\mathfrak{L}\{g\}$$

Divide by s,

$$\mathfrak{L}\{g\} = \frac{\mathfrak{L}\{g'\}}{s} = \frac{\mathfrak{L}\{f\}}{s} = \frac{F(s)}{s}.$$

□

Example 1 Find the inverse Laplace transform of $\frac{1}{s(s+1)}$.

Solution. *Method 1*: We can use partial fraction decomposition.
Method 2: We use the above rule. Note that if we set

$$F(s) = \frac{1}{s+1}$$

then by the table, its inverse Laplace transform is

$$f(x) = \mathfrak{L}^{-1}\left\{ \frac{1}{s+1} \right\} = e^{-x}.$$

So if

$$G(s) = \frac{1}{s(s+1)} = \frac{\frac{1}{s+1}}{s}$$

then by rule (3.20) the inverse Laplace transform is,

$$g(x) = \mathfrak{L}^{-1}\left\{ \frac{F(s)}{s} \right\} = \int_0^x e^{-u}\, du = -e^{-u}\Big]_0^x = 1 - e^{-x}.$$

□

- *Translation on the s-axis:*

$$\boxed{\mathfrak{L}\{\, e^{cx} f(x) \,\} = F(s-c)} \qquad (3.21)$$

Proof. Simply compute the Laplace transform.

$$\mathfrak{L}\{\, e^{cx} f(x) \,\} = \int_0^\infty e^{cx} f(x)\, e^{-sx}\, dx = \int_0^\infty f(x) e^{-(s-c)x}\, dx = F(s-c).$$

□

Example 2 Find the Laplace transform of $f(x) = x^n e^{cx}$.

Solution. Since

$$\mathfrak{L}\{x^n\} = \frac{n!}{s^{n+1}}$$

we get

$$\mathfrak{L}\{x^n e^{cx}\} = \frac{n!}{(s-c)^{n+1}}. \qquad (3.22)$$

□

Example 3 Find the Laplace transform of $f(x) = e^{3x}\cos 2x$.

Solution. Since

$$\mathfrak{L}\{\cos 2x\} = \frac{s}{s^2+4}$$

we get that

$$\mathfrak{L}\{\ e^{3x}\cos 2x\ \} = \frac{s-3}{(s-3)^2+4} = \frac{s-3}{s^2-6s+13}.$$

□

Example 4 Find $\mathfrak{L}^{-1}\{\ G(s)\ \}$ where

$$G(s) = \frac{1}{s^2-4s+5}$$

Solution. Because the equation $s^2 - 4s + 5 = 0$ has no real solution, the denominator can not be factored. Instead, we complete the square and write

$$G(s) = \frac{1}{(s-2)^2+1}$$

Remember that $\mathfrak{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin x$. The translation rule (3.21) now shows that

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\} = e^{2x}\sin x.$$

□

- *Derivative of the Transform:*

$$\mathfrak{L}\{xf(x)\} = -F'(s),$$

and repeating,

$$\boxed{\mathfrak{L}\{x^n f(x)\} = (-1)^n F^{(n)}(s)} \tag{3.23}$$

We say that "multiplication by x" corresponds to "differentiating the negative Laplace transform".

Proof. Suppose that

$$F(s) = \int_0^\infty f(x)\,e^{-sx}\,dx$$

is the Laplace transform of $f(x)$ for $s > c$. Take the derivative with respect to s,

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty f(x)\,e^{-sx}\,dx = \int_0^\infty \frac{d}{ds}\Big(f(x)e^{-sx}\Big)\,dx \\ &= \int_0^\infty -xf(x)\,e^{-sx}\,dx = -xF(s). \end{aligned}$$

(In a course on advanced calculus you will see why we are allowed to move the derivative d/ds under the integral sign.) Finally, multiply by -1. □

Example 5 Find the Laplace transform of $f(x) = x \sin ax$ $(a > 0)$.

Solution. Recall from (3.8) that

$$\mathfrak{L}\{\sin ax\} = \frac{a}{s^2+a^2}.$$

Therefore by rule (3.23),

$$\mathfrak{L}\{x \sin ax\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = -\frac{0-2sa}{(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}$$

□

Remark If we divide divide by $2a$, we get the formula

$$\boxed{\mathfrak{L}\left\{\frac{x}{2a}\sin ax\right\} = \frac{s}{(s^2+a^2)^2}} \tag{3.24}$$

which is useful for finding inverse Laplace transforms.

- *Integral of the transform:*

$$\boxed{\mathfrak{L}\left\{\frac{f(x)}{x}\right\} = \int_s^\infty F(u)\, du} \tag{3.25}$$

provided that $\lim_{x\to 0^+} \frac{f(x)}{x}$ exists. We say that "division by x" corresponds to "integrating the Laplace transform".

Proof. Suppose, the Laplace transform $F(s)$ of $f(x)$ exists for $s > c$. Then,

$$\int_s^\infty F(u)\, du = \int_s^\infty \left[\int_0^\infty f(x)e^{-ux}\, dx\right] du$$

Now interchange the order of integration. (In an advanced calculus course you will see why this is permitted for this improper integral), and get for $s > c$,

$$\int_0^\infty F(u)\, du = \int_0^\infty \left[\int_s^\infty f(x)e^{-ux} du\right] dx$$

$$= \int_0^\infty f(x)\left[\int_s^\infty e^{-ux} du\right] dx.$$

Now by (3.2), this simplifies to

$$\int_0^\infty F(u)\, du = \int_0^\infty f(x)\frac{e^{-sx}}{x}\, dx = \mathfrak{L}\left\{\frac{f(x)}{x}\right\}.$$

□

Example 6 Find the Laplace transform of $f(x) = \dfrac{\sin x}{x}$.

Solution. Since $\mathfrak{L}\{\sin x\} = \dfrac{1}{s^2+1}$ for $s > 0$, rule (3.25) shows that

$$\mathfrak{L}\left\{\frac{\sin x}{x}\right\} = \int_s^\infty \frac{1}{u^2+1}\, du = \tan^{-1} u\,\Big]_s^\infty$$

$$= \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1} s\right) = \frac{\pi}{2} - \tan^{-1} s,$$

and is defined for $s > 0$. □

Remark We can use this result to find $\int_0^\infty \frac{\sin x}{x}\,dx$. In fact,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \lim_{s\to 0} \frac{\sin x}{x} e^{sx}\,dx = \lim_{s\to 0} \int_0^\infty \frac{\sin x}{x} e^{sx}\,dx$$

$$= \lim_{s\to 0} \mathfrak{L}\Big\{ \frac{\sin x}{x} \Big\}(s) = \lim_{s\to 0} \Big(\frac{\pi}{2} - \tan^{-1} s\Big) = \frac{\pi}{2}.$$

Exercises

1. Using the transforms for x^n, $\cos ax$ and $\sin ax$, and the rules for Laplace transforms, find the Laplace transform of

   (a) xe^{3x} (b) $x^4 e^{\pi x}$ (c) $x^2 e^{-x}$ (d) $x^2 \sin ax$ (e) $e^{cx} \sin ax$ (f) $e^{cx}\cos ax$ (g) $xe^{cx}\sin ax$ (h) $xe^{cx}\cos ax$

2. Find the inverse Laplace transform of $F(s) =$

   (a) $\dfrac{3}{s^2+4}$ (b) $\dfrac{4}{(s-1)^3}$ (c) $\dfrac{2}{s^2+3s-4}$ (d) $\dfrac{3s}{s^2-s-6}$ (e) $\dfrac{2s+2}{s^2+2s+2}$ (f) $\dfrac{2s-3}{s^2-4}$ (g) $\dfrac{2s+1}{s^2-2s+2}$ (h) $\dfrac{8s^2-4s+12}{s(s^2+4)}$ (i) $\dfrac{1-2s}{s^2+4s+5}$ (j) $\dfrac{2s-3}{s^2+2s+10}$

3. Find the inverse Laplace transform of $F(s) =$

   (a) $\dfrac{s-1}{(s+1)^3}$ (b) $\dfrac{2s-3}{9s^2-12s+20}$ (c) $\dfrac{3s+5}{s^2-6s+25}$ (d) $F(s) = \dfrac{1}{s^3-5s^2}$ (e) $F(s) = \dfrac{1}{(s^2+s-6)^2}$ (f) $F(s) = \dfrac{1}{s^4-16}$ (g) $F(s) = \dfrac{s^3}{(s-4)^4}$ (h) $F(s) = \dfrac{s^2-2s}{s^4+5s^2+4}$ (i) $F(s) = \dfrac{1}{s^4-8s^2+16}$

4. Use the Laplace transform to solve the following initial value problems:

   (a) $y'' - 2y' + 2y = 0 \quad y(0) = 0, \quad y'(0) = 1$
   (b) $y'' - 4y' + 4y = 0 \quad y(0) = 1, \quad y'(0) = 1$
   (c) $y'' - 2y' - 2y = 0 \quad y(0) = 2, \quad y'(0) = 0$
   (d) $y'' + 2y' + 5y = 0 \quad y(0) = 2, \quad y'(0) = -1$
   (e) $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0 \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1$
   (f) $y'' - 2y' + 2y = \cos x \quad y(0) = 1, \quad y'(0) = 0$
   (g) $y'' - 2y' + 2y = e^{-x} \quad y(0) = 0, \quad y'(0) = 1$
   (h) $y'' + 2y' + y = 4e^{-x} \quad y(0) = 2, \quad y'(0) = -1$
   (i) $y'' - 6y' + 8y = 2 \quad y(0) = 0, \quad y'(0) = 0$
   (j) $y'' - 4y = 3x \quad y(0) = 0, \quad y'(0) = 0$
   (k) $y'' + 4y' + 13y = xe^{-x} \quad y(0) = 0, \quad y'(0) = 2$
   (l) $y^{(4)} + 2y'' + y = e^{2x} \quad y(0) = y'(0) = y''(0) = y'''(0) = 0$

5. Show that if $F(s) = \mathfrak{L}\{f(x)\}$, then

$$\mathfrak{L}\{f(cx)\} = \frac{1}{c}\,F\Big(\frac{s}{c}\Big). \qquad (c > 0)$$

6. Find the inverse Laplace transform of $F(s) =$

   (a) $\ln \dfrac{s-2}{s+2}$ (c) $\ln \dfrac{s^2+1}{(s+2)(s-3)}$ (e) $\ln\left(1+\dfrac{1}{s^2}\right)$

   (b) $\ln \dfrac{s^2+1}{s^2+4}$ (d) $\tan^{-1} \dfrac{3}{s+2}$ (f) $\dfrac{s}{(s^2+1)^3}$

7. Transform each differential equation to find one nonzero solution such that $y(0) = 0$.

   (a) $xy'' + (x-2)y' + y = 0$ (d) $xy'' + 2(x-1)y' - 2y = 0$

   (b) $xy'' + (3x-1)y' + 3y = 0$ (e) $xy'' - 2y' + xy = 0$

   (c) $xy'' - (4x+1)y' + 2(2x+1)y = 0$ (f) $xy'' + (4x-2)y' + (13x-4)y = 0$

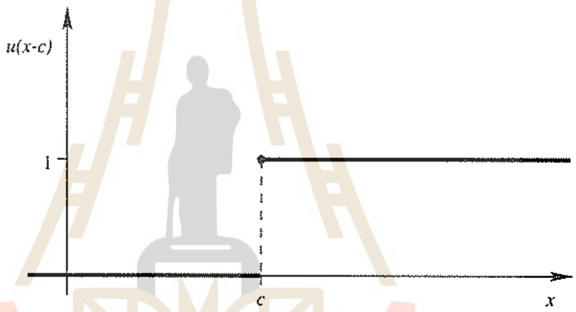
8. Solve exercises 21 and 22 in section 2.7 on page 111 using the Laplace transform.

3.4.2 The Step Function and Translation

The function

$$u_c(x) \;=\; u(x-c) \;=\; \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } c \le x \end{cases}$$

is called the *unit step function* at c. It is not continuous, but piecewise continuous. This function often appears in applications, for example, it may represent an electric switch which is turned on at time c.



The unit step function $u_c(x)$.

Now let us find the Laplace transform of the unit step function. By definition,

$$\mathfrak{L}\{u_c(x)\} \;=\; \int_0^\infty u_c(x)\, e^{-sx}\, dx.$$

Whenever we have an integral containing the step function u_c, it is best to split the integral at the point c where the function "jumps". We write

$$\begin{aligned} \mathfrak{L}\{u_c(x)\} &= \int_0^c u_c(x)\, e^{-sx}\, dx + \int_c^\infty u_c(x)\, e^{-sx}\, dx \\ &= \int_0^c 0 \cdot e^{-sx}\, dx + \int_c^\infty 1 \cdot e^{-sx}\, dx \\ &= 0 + \int_c^\infty e^{-sx}\, dx \;=\; \frac{1}{s}\, e^{-cs}, \end{aligned}$$

provided that $s > 0$, where in the last line we have used (3.2). That is,

$$\boxed{\mathfrak{L}\{u_c(x)\} \;=\; \frac{e^{-cs}}{s}} \tag{3.26}$$

Example 1 Find the values of the function $g(x) = 1 - u_c(x)$. (It is called the *step-down function* at c.) Then find its Laplace transform.

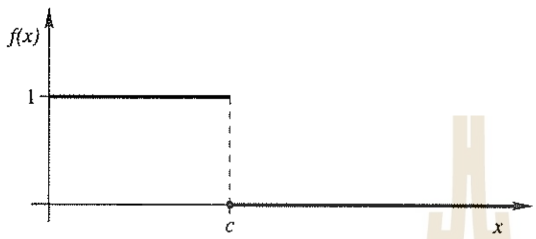
Solution. We simply subtract the unit step from the function $f(x) = 1$ and obtain

$$g(x) = \begin{cases} 1 & \text{if } x < c \\ 0 & \text{if } x \geq c \end{cases}$$

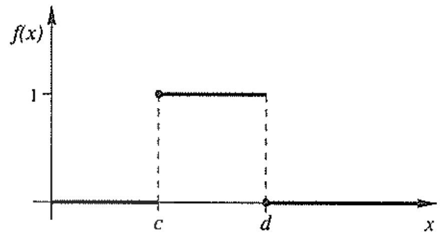
By linearity,

$$\mathfrak{L}\{g(x)\} = \mathfrak{L}\{1 - u_c(x)\} = \mathfrak{L}\{1\} - \mathfrak{L}\{u_c(x)\} = \frac{1}{s} - \frac{e^{-cs}}{s} = \frac{1 - e^{-cs}}{s}.$$

□



The step-down function $1 - u_c(x)$.



The impulse from c to d, $u_c(x) - u_d(x)$.

Example 2 Find the values of the function $h(x) = u_c(x) - u_d(x)$. (It is called the *impulse* from c to d). Then find its Laplace transform.

Solution. The definitions of u_c and u_d change at $x = c$ and $x = d$. We therefore must look at the intervals $[0, c)$, $[c, d)$ and $[d, \infty)$ separately.

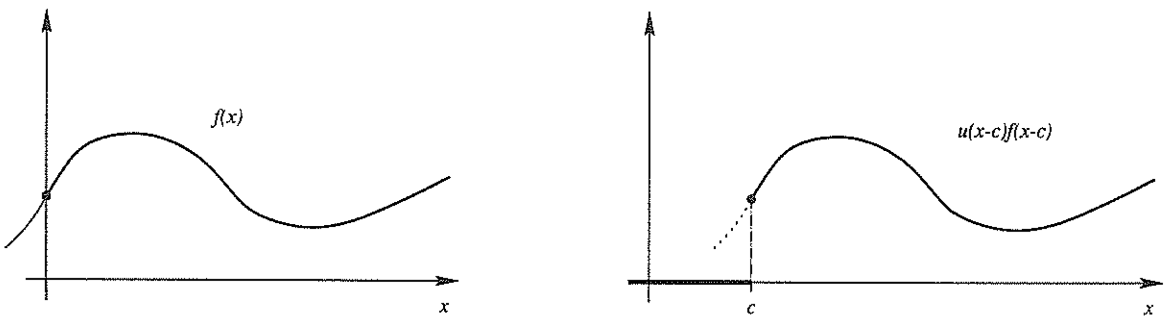
$$h(x) = u_c(x) - u_d(x) = \begin{cases} 0 - 0 = 0 & \text{if } x < c \\ 1 - 0 = 1 & \text{if } c \leq x < d \\ 1 - 1 = 0 & \text{if } d \leq x \end{cases}.$$

The Laplace transform is

$$\mathfrak{L}\{h(x)\} = \mathfrak{L}\{u_c(x)\} - \mathfrak{L}\{u_d(x)\} = \frac{e^{-cs}}{s} - \frac{e^{-ds}}{s} = \frac{e^{-cs} - e^{-ds}}{s}.$$

□

We now define the *translate* of a function $f(x)$ as follows: First we shift its graph c units to the right, which corresponds to the function $f(x - c)$. Then we cut the graph to the left of the point $x = c$. This cutting is achieved by multiplication with the step function $u_c(x)$, and we obtain the function $u_c(x)f(x - c)$.



The translate of $f(x)$ is $u_c(x)f(x - c)$.

The Laplace transforms of $f(x)$ and its translate $u_c(x)f(x-c)$ are related as follows:

$$\boxed{\mathfrak{L}\{u_c(x)f(x-c)\} \;=\; e^{-cs}\mathfrak{L}\{f(x)\}} \tag{3.27}$$

Proof. By the definition of the Laplace transform,

$$\mathfrak{L}\{u_c(x)f(x-c)\} \;=\; \int_0^\infty u_c(x)f(x-c)\,e^{-sx}\,dx \;=\; \int_c^\infty f(x-c)e^{-sx}\,dx$$

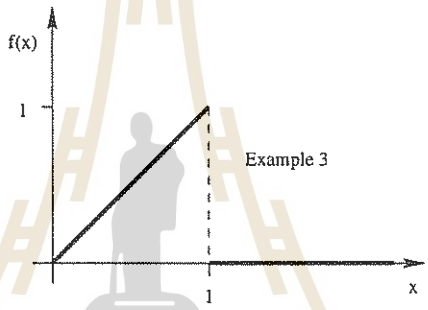
because $u_c(x)=0$ when $x<c$ and $u_c(x)=1$ when $x>c$. Now substitute $u=x-c$. Then, $x=u+c$ and $du=dx$, and we obtain

$$\mathfrak{L}\{u_c(x)f(x-c)\} \;=\; \int_0^\infty f(u)e^{-s(u+c)}du \;=\; e^{-cs}\int_0^\infty f(u)e^{-su}du \;=\; e^{-cs}\mathfrak{L}\{f(x)\}.$$

□

Example 3 Find the Laplace transform of $f(x)=\begin{cases} x & \text{if } 0\le x<1 \\ 0 & \text{if } x\ge 1 \end{cases}$

Solution. Let us first sketch the graph.



$f(x)=x-u_1(x)\,x.$

This looks like the graph of $y=x$ cut off to the right of $x=1$. This cutting is done multiplying by the "step-down" function $1-u_1(x)$, so we can write

$$f(x) \;=\; x\big(1-u_1(x)\big) \;=\; x-u_1(x)\,x.$$

To be able to use formula (3.27) we must express the factor next to $u_1(x)$ as a function in $x-1$. This is simple here, as $x=(x-1)+1$. Then,

$$f(x) \;=\; x-u_1(x)\,(x-1+1) \;=\; x-u_1(x)\,(x-1)-u_1(x),$$

and the Laplace transform is

$$F(s) \;=\; \frac{1}{s^2}-\frac{e^{-s}}{s^2}-\frac{e^{-s}}{s}.$$

□

Remark There is a second way to find this Laplace transform, using formula (3.23) for the product $xf(x)$. Then,

$$\begin{aligned}\mathfrak{L}\{x-u_1(x)\cdot x\} \;&=\; \mathfrak{L}\{x\}-\left(-\frac{d}{ds}\mathfrak{L}\{u_1(x)\}\right) \;=\; \frac{1}{s^2}+\frac{d}{ds}\frac{e^{-s}}{s} \\ &=\; \frac{1}{s^2}+\frac{-se^{-s}-e^{-s}}{s^2}\end{aligned}$$

which is the same as above.

Example 4 Find the inverse Laplace transform of $F(s) = \dfrac{1 - e^{-2s}}{s^2}$.

Solution. First we rewrite

$$F(s) = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2}$$

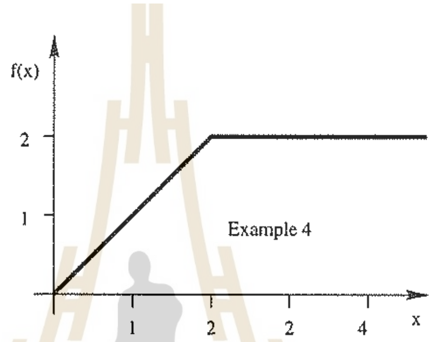
As $\mathfrak{L}^{-1}\left\{ \dfrac{1}{s^2} \right\} = x$, the translation rule gives

$$\mathfrak{L}^{-1}\left\{ e^{-2s}\frac{1}{s^2} \right\} = u_2(x)(x-2).$$

Therefore,

$$f(x) = x - u_2(x)(x-2) = \begin{cases} x - 0 = x & \text{if } 0 \le x < 2 \\ x - (x-2) = 2 & \text{if } 2 \le x \end{cases}$$

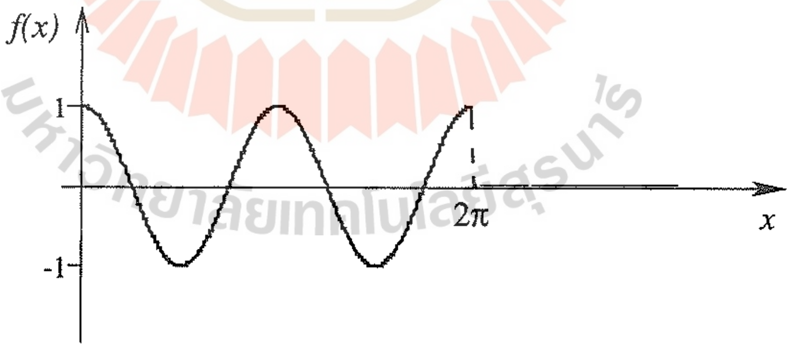
□



$f(x) = x - u_2(x)(x-2).$

Example 5 Find the Laplace transform of $f(x) = \begin{cases} \cos 2x & \text{if } 0 \le x < 2\pi \\ 0 & \text{if } x \ge 2\pi \end{cases}$

Solution. We first must find a simple expression for this function.



Observe that its graph can be obtained from the graph of $\cos 2x$ by cutting to the right of the point 2π, which is the same as multiplying by $1 - u_{2\pi}$. Therefore, we can write

$$f(x) = \left(1 - u_{2\pi}(x) \right) \cos 2x = \cos 2x - u_{2\pi}(x) \cos 2x.$$

Because $\cos 2x = \cos 2(x - 2\pi)$, this can be rewritten

$$f(x) = \cos 2x - u_{2\pi}(x) \cos 2(x - 2\pi).$$

Now we can use the translation rule (3.27), and get

$$\begin{aligned}\mathfrak{L}\{f\} &= \mathfrak{L}\{\cos 2x\} - \mathfrak{L}\{u_{2\pi}(x)\cos 2(x-2\pi)\}\\ &= \frac{s}{s^2+4} - e^{-2\pi s}\,\frac{s}{s^2+4} = \frac{s(1-e^{-2\pi s})}{s^2+4}.\end{aligned}$$

□

Example 6 A mass of 1 kg is attached to a long spring with spring constant $k = 4$ N/m. At time $t = 0$, an external force of $f(t) = \cos 2t$ is applied to the mass, and turned off at $t = 2\pi$. Initially, the mass is at rest in the equilibrium position. There is no damping. Find the position $y(t)$ of the mass at time t.

Solution. We use equation (2.61) with $m = 1$, $c = 0$ and $k = 4$, We have an initial value problem

$$y'' + 4y = \begin{cases} \cos 2t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}, \qquad y(0) = y'(0) = 0.$$

Note that the independent variable is called t. Take the Laplace transform, and use the result of the previous example,

$$\begin{aligned} s^2Y(s) + 4Y(s) &= \frac{s}{s^2+4} - e^{-2\pi s}\,\frac{s}{s^2+4}\\ Y(s) &= \frac{s}{(s^2+4)^2} - e^{-2\pi s}\frac{s}{(s^2+4)^2}.\end{aligned}$$

Now by (3.24), the inverse Laplace transform of the first fraction is

$$\mathfrak{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}\,t\sin 2t.$$

Using the translation rule (3.27), the inverse Laplace transform of the second fraction is

$$\mathfrak{L}^{-1}\left\{e^{-2\pi s}\,\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}\,u_{2\pi}(t)(t-2\pi)\sin 2(t-2\pi).$$

Therefore,

$$y(t) = \frac{1}{4}\,t\sin 2t - \frac{1}{4}\,u_{2\pi}(t)(t-2\pi)\sin 2(t-2\pi).$$

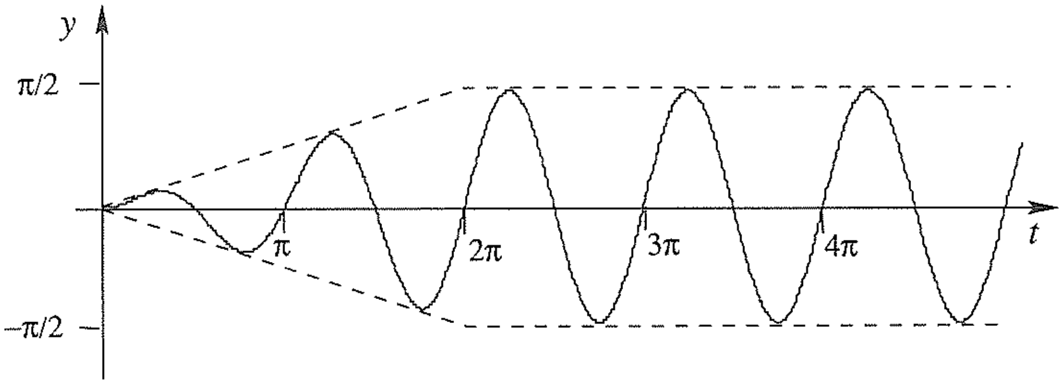
Because $\sin 2t$ is periodic, this simplifies to

$$y(t) = \frac{1}{4}t\sin 2t - \frac{1}{4}u_{2\pi}(t)(t-2\pi)\sin 2t.$$

Using the definition of the step function we can write two separate formulas,

$$y(t) = \begin{cases} \frac{1}{4}t\sin 2t - 0 = \frac{1}{4}t\sin 2t & \text{if } 0 \le t < 2\pi \\ \frac{1}{4}t\sin 2t - \frac{1}{4}(t-2\pi)\sin 2t = \frac{\pi}{2}\sin 2t & \text{if } t \ge 2\pi \end{cases}$$

□



The solution to example 6.

Exercises

1. Sketch the graph of

   (a) $h(x) = u_1(x) + 2u_3(x) - 6u_4(x)$.

   (b) $h(x) = (x-3)u_2(x) - (x-2)u_3(x)$.

   (c) $h(x) = u_\pi(x)f(x-\pi)$ where $f(x) = x^2$.

   (d) $h(t) = u_3(t)f(t-3)$ where $f(t) = \sin t$.

   (e) $h(x) = u_2(x)f(x-1)$ where $f(x) = 2x$.

   (f) $h(x) = u_1(x)(x-1) - 2u_2(x)(x-2) + u_3(x)(x-3)$.

2. Find the Laplace transform of

   (a) $f(x) = \begin{cases} 2 & \text{if } 0 \le x < 3 \\ 0 & \text{if } x \ge 3 \end{cases}$

   (b) $f(x) = \begin{cases} 3 & \text{if } 1 \le x < 4 \\ 0 & \text{else} \end{cases}$

   (c) $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < 2\pi \\ 0 & \text{if } x \ge 2\pi \end{cases}$

   (d) $f(x) = \begin{cases} \cos \pi x & \text{if } 0 \le x < 2 \\ 0 & \text{if } x \ge 2 \end{cases}$

   (e) $f(t) = \begin{cases} \sin t & \text{if } 0 \le t < 3\pi \\ 0 & \text{if } t \ge 3\pi \end{cases}$

   (f) $f(x) = \begin{cases} 0 & \text{if } 0 \le x < \pi \\ \sin 2x & \text{if } \pi \le x < 2\pi \\ 0 & x \ge 2\pi \end{cases}$

   (g) $f(x) = \begin{cases} \sin \pi x & \text{if } 2 \le x < 3 \\ 0 & \text{else} \end{cases}$

   (h) $f(t) = \begin{cases} \cos \frac{\pi t}{2} & \text{if } 3 \le t < 5 \\ 0 & \text{else} \end{cases}$

   (i) $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ x & \text{if } x \ge 1 \end{cases}$

   (j) $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$

   (k) $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2-x & \text{if } 1 \le x < 2 \\ 0 & \text{if } x \ge 2 \end{cases}$

   (l) $f(x) = \begin{cases} x^3 & \text{if } 1 \le x < 2 \\ 0 & \text{else} \end{cases}$

3. Find the inverse Laplace transform and sketch the graphs of

   (a) $F(s) = \dfrac{3!}{(s-2)^4}$

   (b) $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$

   (c) $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$

   (d) $F(s) = \dfrac{2e^{-2s}}{s^2-4}$

   (e) $F(s) = \dfrac{(s-2)e^{-s}}{s^2-4s+3}$

   (f) $F(s) = \dfrac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}$

   (g) $F(s) = \dfrac{e^{-3s}}{s^2}$

   (h) $F(s) = \dfrac{e^{-s}}{s+2}$

(i) $F(s) = \dfrac{e^{-\pi s}}{s^2+1}$

(j) $F(s) = \dfrac{1-e^{-2\pi s}}{s^2+1}$

(k) $F(s) = \dfrac{s(1+e^{-3s})}{s^2+\pi^2}$

(l) $F(s) = \dfrac{2s(e^{-\pi s}-e^{-2\pi s})}{s^2+4}$

4. If we have an RLC-circuit with initially zero current and zero charge on the capacitor, we obtain an equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}q(t) = e(t) \qquad i(0) = 0, \quad q(0) = 0,$$

where $q(t)$ is the charge on the capacitor, $i(t)$ the current and $e(t)$ the electromotive force. Now as $i(t) = \frac{dq}{dt}$ and $q(0) = 0$, we can write $q(t) = \int_0^t i(\tau)d\tau$. We obtain the equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau = e(t) \qquad i(0) = 0.$$

Find the solution of this equation if

(a) $L = 0, \quad R = 100, \quad C = 10^{-3}, \quad e(t) = \begin{cases} 100 & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}.$

(b) $L = 1, \quad R = 0, \quad C = 10^{-4}, \quad e(t) = \begin{cases} 100 & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}.$

(c) $L = 1, \quad R = 0, \quad C = 10^{-4}, \quad e(t) = \begin{cases} 100 \sin 10t & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases}.$

(d) $L = 1, \quad R = 150, \quad C = 2 \cdot 10^{-4}, \quad e(t) = \begin{cases} 100t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}.$

(e) $L = 1, \quad R = 100, \quad C = 4 \cdot 10^{-4}, \quad e(t) = \begin{cases} 50t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}.$

The units are in henry, ohm, farad and volt.

5. A mass m is attached to a spring with spring constant k. Initially, the mass is at rest in equilibrium. An external force $f(t)$ is applied. There is damping; the damping factor is c. So we get the initial value problem

$$my'' + cy' + ky = f(t) \qquad y(0) = y'(0) = 0.$$

Solve this equation if

(a) $m = 1, \quad k = 4, \quad c = 0, \quad f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases}.$

(b) $m = 1, \quad k = 4, \quad c = 5, \quad f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ 0 & \text{if } t \ge 2 \end{cases}.$

(c) $m = 1, \quad k = 9, \quad c = 0, \quad f(t) = \begin{cases} \sin t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}.$

(d) $m = 1, \quad k = 1, \quad c = 0, \quad f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}.$

(e) $m = 1, \quad k = 4, \quad c = 4, \quad f(t) = \begin{cases} t & \text{if } 0 \le t < 2 \\ 0 & \text{if } t > 2 \end{cases}.$

Units are in kg, Nsec/m, N/m and N.

6. Solve the following initial value problems:

   (a) $y'' + y = f(x), \quad y(0) = 0, \quad y'(0) = 1, \quad f(x) = \begin{cases} 1, & 0 \le x < \pi/2 \\ 0 & x \ge \pi/2 \end{cases}$

   (b) $y'' + 2y' + 2y = h(x), \quad y(0) = 0, \quad y'(0) = 1, \quad h(x) = \begin{cases} 1, & \pi \le x < 2\pi \\ 0 & \text{else} \end{cases}$

   (c) $y'' + 4y = \sin x - u_{2\pi}(x)\sin x, \quad y(0) = 0, \quad y'(0) = 0$

   (d) $y'' + 4y = \sin x + u_{\pi}(x)\sin(x - \pi), \quad y(0) = 0, \quad y'(0) = 0$

   (e) $y'' + 2y' + y = f(x), \quad y(0) = 1, \quad y'(0) = 0, \quad f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0 & x > 1 \end{cases}$

   (f) $y'' + 3y' + 2y = u_2(x), \quad y(0) = 0, \quad y'(0) = 1$

   (g) $y'' + y = u_{\pi}(x), \quad y(0) = 1, \quad y'(0) = 0$

   (h) $y'' + y' + \frac{5}{4}y = x - u_{\pi/2}(x)(x - \pi/2), \quad y(0) = 0, \quad y'(0) = 0$

   (i) $y'' + y = g(x), \quad y(0) = 0, \quad y'(0) = 1, \quad g(x) = \begin{cases} x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$

   (j) $y'' + y' + \frac{5}{4}y = g(x), \quad y(0) = 0, \quad y'(0) = 0, \quad g(x) = \begin{cases} \sin x & 0 \le x < \pi \\ 0 & x \ge \pi \end{cases}$

   (k) $y'' + 4y = u_{\pi}(x) - u_{2\pi}(x), \quad y(0) = 1, \quad y'(0) = 0$

   (l) $y^{(4)} - y = u_1(x) - u_2(x), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0$

   (m) $y^{(4)} + 5y'' + 4y = 1 - u_{\pi}(x), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0$

7. Solve exercise 23 in section 2.7 on page 111 using the Laplace transform.

3.4.3 The Delta Function

The function

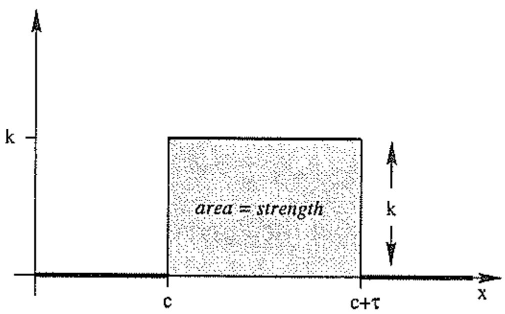
$$g(x) = \begin{cases} k & \text{if } c \le x < c + \tau \\ 0 & \text{else} \end{cases} \tag{3.28}$$

($c \ge 0$) is called an *impulse at c of length τ and height k*. The *strength* I of this impulse is measured by the area under its graph,

$$I = k\,\tau$$

which equals the integral

$$I = \int_0^{\infty} g(x)\,dx.$$



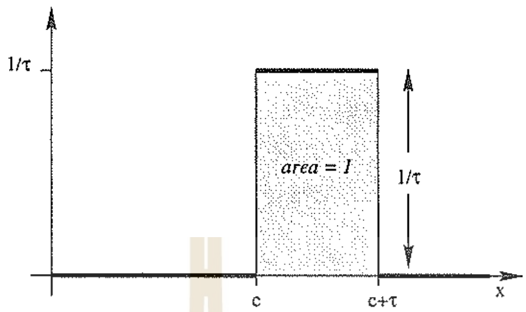
An impulse of length τ and height k.

Now consider impulses of *unit* strength,

$$\delta_{c,\tau}(x) \;=\; \begin{cases} 1/\tau & \text{if } c \le x < c+\tau \\ 0 & \text{else,} \end{cases}$$

so that

$$\int_0^\infty \delta_{c,\tau}(x)\,dx \;=\; 1.$$



A unit impulse of length τ.

If the length τ of the unit impulse is very small, then the amplitude $1/\tau$ must be very large in order that the impulse strength equal one. Now if x is any point different from c, then when the length τ is sufficiently small, x will lie outside of the interval $[c, c+\tau]$, and thus

$$\delta_{c,\tau}(x) \;=\; 0$$

while still

$$\delta_{c,\tau}(c) \;=\; 1/\tau.$$

Now let $\tau \to 0$. As the limit, we obtain a new function $\delta_c(x)$ given by

$$\boxed{\delta_c(x) \;=\; \begin{cases} \infty & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}} \tag{3.29}$$

and whose integral we define by

$$\boxed{\int_0^\infty \delta_c(x)\,dx \;=\; 1.} \tag{3.30}$$

This function is called the *delta function*. We think of it as an impulse of zero length and of strength one. Note that $\delta_c(x)$ is not a function in the usual sense; it is called a *generalized function*, and it can not be graphed.

In order to define its Laplace transform, we must first say what we mean by the integral $\int_0^\infty \delta_c(x) f(x)\,dx$. Note that for every impulse $\delta_{c,\tau}(x)$ of nonzero length and every continuous function $f(x)$,

$$\int_0^\infty \delta_{c,\tau}(x) f(x)\,dx \;=\; \int_c^{c+\tau} \frac{1}{\tau} f(x)\,dx \;=\; \frac{\int_c^{c+\tau} f(x)\,dx}{\tau}.$$

Now let $\tau \to 0$. Using l'Hôpital's rule and the Fundamental Theorem of Calculus we obtain

$$\lim_{\tau\to 0} \int_0^\infty \delta_{c,\tau}(x) f(x)\,dx \;=\; \lim_{\tau\to 0} \frac{\frac{d}{d\tau}\int_c^{c+\tau} f(x)\,dx}{\frac{d}{d\tau}\tau} \;=\; \lim_{\tau\to 0} \frac{f(c+\tau)}{1} \;=\; f(c).$$

It therefore makes sense to define the integral

$$\int_0^\infty \delta_c(x) f(x) dx \;=\; f(c).$$

Now we can define the Laplace transform of the delta function in the usual way,

$$\boxed{\mathfrak{L}\{\ \delta_c(x)\ \} \;=\; \int_0^\infty \delta_c(x)\, e^{-sx}\, dx \;=\; e^{-cs}.} \qquad (3.31)$$

When $c = 0$, we simply write $\delta(x) = \delta_0(x)$, so that

$$\mathfrak{L}\{\ \delta(x)\ \} \;=\; e^0 \;=\; 1.$$

Remark Although the delta function is not a function in the usual sense, we can work with its Laplace transform in the usual way.

Example 1 A mass of 1 kg is attached to a long spring with spring constant 4 N/m. The mass is released from rest at position $y(0) = 3$ m. At time $t = 2\pi$, the mass is hit with a hammer, transferring a momentum of 8 Nsec. Find the position of the mass at time t. (There is no damping.)

Solution. We have an initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 3, \quad y'(0) = 0 \qquad (3.32)$$

where $f(t)$ is the external force acting on the mass. How can we find $f(t)$?

Suppose, a force $f(t)$ acts on a mass m from time $t = a$ to time $t = b$. It transfers a momentum Δp onto the mass, which results in a change of velocity Δv with

$$\Delta p = m \cdot \Delta v.$$

By Newton's second law,

$$\Delta p \;=\; m \int_a^b v'(t)\, dt \;=\; \int_a^b m v'(t)\, dt \;=\; \int_a^b f(t)\, dt.$$

Since $f(t) = 0$ for $t < a$ and $t > b$, we can write this as an improper integral,

$$\Delta p \;=\; \int_0^\infty f(t)\, dt.$$

In our example, we have $a = b = 2\pi$, so we must use the force

$$f(t) \;=\; c\delta_{2\pi}(t).$$

Then

$$8 \;=\; \Delta p \;=\; \int_0^\infty f(t)\, dt \;=\; \int_0^\infty c\delta_{2\pi}(t)\, dt \;=\; c$$

so that $c = 8$ and $f(t) = 8\delta_{2\pi}(t)$.

Thus, equation (3.32) becomes

$$y'' + 4y = 8\delta_{2\pi}(t), \qquad y(0) = 3, \quad y'(0) = 0 \tag{3.33}$$

Apply the Laplace transform to both sides,

$$\left(s^2 Y(s) - 3s\right) + 4Y(s) = 8e^{-2\pi s}.$$

Solve for $Y(s)$,

$$Y(s) = \frac{3s}{s^2+4} + \frac{8e^{-2\pi s}}{s^2+4}$$

Using the table, we have

$$\mathfrak{L}^{-1}\left\{ \frac{3s}{s^2+4} \right\} = 3\cos 2t \qquad \text{and} \qquad \mathfrak{L}^{-1}\left\{ \frac{8}{s^2+4} \right\} = 4\sin 2t.$$

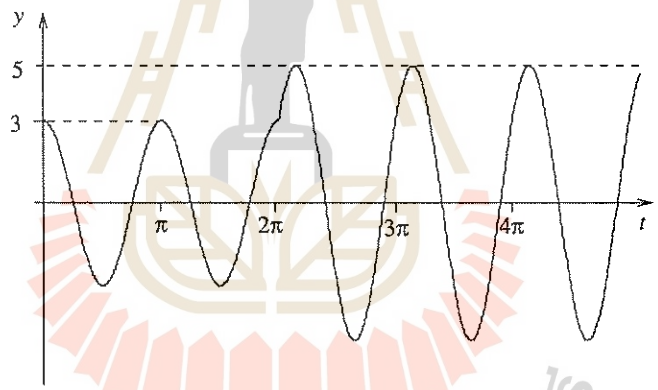
Using the translation rule,

$$\begin{aligned} \mathfrak{L}^{-1}\left\{ \frac{3s}{s^2+4} + e^{-2\pi s}\frac{8}{s^2+4} \right\} &= 3\cos 2t + 4u_{2\pi}(t)\sin 2(t-2\pi) \\ &= 3\cos 2t + 4u_{2\pi}(t)\sin 2t \end{aligned}$$

In the last line have used the periodicity of $\sin t$ .Thus,

$$y(t) = \begin{cases} 3\cos 2t & \text{if } \ 0 \le t < 2\pi \\ 3\cos 2t + 4\sin 2(t) & \text{if } \ t \ge 2\pi \end{cases}$$

□



The solution to example 1.

Remark We often think of the delta function $\delta_c(x)$ as the derivative of the step function $u_c(x)$. This makes sense because

$$\int_0^x \delta_c(u)\,du = \left\{ \begin{array}{ll} 0 & \text{if } \ x < c \\ 1 & \text{if } \ x > c \end{array} \right\} = u_c(x).$$

Exercises

1. Solve each initial value problem:

   (a) $y'' + 4y = \delta(x), \qquad y(0) = y'(0) = 0$

   (b) $y'' + 4y = \delta(x) + \delta_\pi(x), \qquad y(0) = y'(0) = 0$

(c) $y'' + 4y' + 4y = 1 + \delta_2(x), \quad y(0) = y'(0) = 0$
(d) $y'' + 2y' + y = x + \delta(x), \quad y(0) = 0, \quad y'(0) = 1$
(e) $y'' + 2y' + 2y = 2\delta_\pi(x), \quad y(0) = y'(0) = 0$
(f) $y'' + 9y = \delta_{3\pi}(x) + \cos 3x, \quad y(0) = y'(0) = 0$
(g) $y'' + 4y' + 5y = \delta_\pi(x) + \delta_{2\pi}(x), \quad y(0) = 0, \quad y'(0) = 2$
(h) $y'' + 2y' + y = \delta(x) - \delta_2(x), \quad y(0) = y'(0) = 2$

Supplementary exercises:

2. A mass m on a spring with spring constant k (no damping) is at rest in equilibrium position and receives an impulse $p_0 = mv_0$ at time $t = 0$. So we have an initial value problem

$$my'' + ky = 0 \qquad y(0) = 0, \quad y'(0) = v_0.$$

Show that the initial value problem

$$my'' + ky = p_0\delta(t), \qquad y(0) = y'(0) = 0$$

has the same solution. So the effect of $p_0\delta(t)$ is to give the mass an initial momentum p_0.

3.4.4 Periodic Functions

In many applications we encounter functions which are periodic. A nonconstant function $f(x)$ defined on $[0, \infty)$ is said to be *periodic* if there exists a number $p > 0$ such that

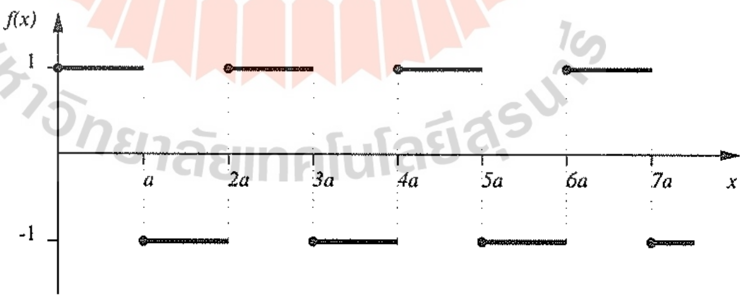
$$\boxed{f(x+p) = f(x)}$$

for all $x \geq 0$. The smallest such p is called the *period* of f.

e.g.
- The functions $\sin x$ and $\cos x$ are periodic with period 2π.
- The *square wave function* $f_s(x)$ is defined by

$$f_s(x) = \begin{cases} 1 & \text{if } 2na \leq x < (2n+1)a \\ -1 & \text{if } (2n+1)a \leq x < (2n+2)a \end{cases} \tag{3.34}$$

where n is integer. It is periodic with period $2a$.

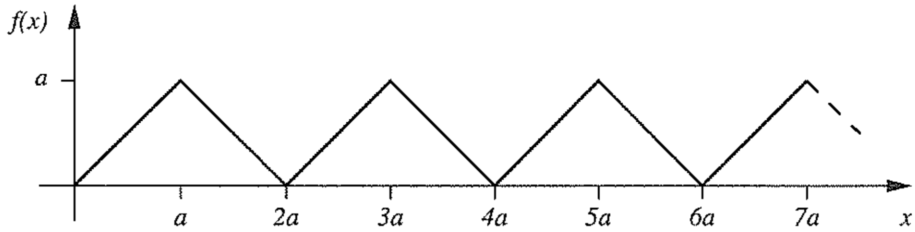


The square wave function $f_s(x)$.

- The *triangular wave function* is defined by

$$f_t(x) = \begin{cases} x - 2na & \text{if } 2na \leq x < (2n+1)a \\ (2n+2)a - x & \text{if } (2n+1)a \leq x < (2n+2)a \end{cases}$$

where n is integer. It is periodic with period $2a$. As you can see from the graph, $f_t'(x) = f_s(x)$ except for the points of discontinuity $0, a, 2a, \cdots$.



The triangular wave function.

The Laplace transform of a periodic function can be computed by integrating over a *finite* interval:

Theorem 21 *Suppose, $f(x)$ is periodic with period p and piecewise continuous on the interval $[0,\infty)$. Then, the Laplace transform of f exists for $s>0$, and*

$$\boxed{\mathfrak{L}\{f\} \;=\; \frac{1}{1-e^{-ps}}\int_0^p f(x)e^{-sx}\,dx.} \tag{3.35}$$

Proof. By the definition of the Laplace transform,

$$\begin{aligned}\mathfrak{L}\{f\} &= \int_0^\infty f(x)\,e^{-sx}\,dx \qquad (3.36)\\ &= \int_0^p f(x)e^{-sx}\,dx + \int_p^{2p} f(x)e^{-sx}\,dx + \int_{2p}^{3p} f(x)e^{-sx}\,dx\\ &\quad + \cdots + \int_{np}^{(n+1)p} f(x)e^{-sx}\,dx + \cdots\end{aligned}$$

Now use the substitution $u = x - np$ on the integral over $[np,(n+1)p]$, so that $x = u + np$ and $dx = du$, and obtain

$$\int_{np}^{(n+1)p} f(x)e^{-sx}\,dx \;=\; \int_0^p f(u+np)e^{-s(u+np)}\,dx.$$

Because $f(u+np) = f(u)$ this equals

$$\int_{np}^{(n+1)p} f(x)e^{-sx}\,dx \;=\; e^{-snp}\int_0^p f(u)e^{-su}\,dx$$

so that (3.36) becomes

$$\mathfrak{L}\{f\} \;=\; \int_0^p f(x)e^{-sx}\,dx\Big(1 + e^{-sp} + e^{-2sp} + \cdots + e^{-nsp} + \cdots\Big).$$

Now recall the geometric series: For $|v|<1$,

$$1 + v + v^2 + v^3 + \cdots + v^n + \cdots \;=\; \frac{1}{1-v}.$$

Applying this to $v = e^{-sp}$ we see that

$$\mathfrak{L}\{f\} \;=\; \int_0^p f(x)e^{-sx}\,dx\,\frac{1}{1-e^{-ps}}.$$

□

Example 1 The Laplace transform of the square wave function is

$$\begin{aligned}
\mathfrak{L}\{f_s(x)\} &= \frac{1}{1-e^{-2sa}} \int_0^{2a} f_s(x)e^{-sx}\,dx \\
&= \frac{1}{1-e^{-2sa}} \left(\int_0^a e^{-sx}\,dx \;-\; \int_a^{2a} e^{-sx}\,dx \right) \\
&= \frac{1}{1-e^{-2sa}} \left(-\frac{1}{s}e^{-sx}\Big]_0^a + \frac{1}{s}e^{-sx}\Big]_a^{2a} \right) \\
&= \frac{1}{s(1-e^{-2sa})} \left(-e^{-sa} + e^0 + e^{-2as} - e^{-sa} \right) \\
&= \frac{(1-e^{-sa})^2}{s(1-e^{-2sa})} \\
&= \frac{(1-e^{-sa})^2}{s(1-e^{-sa})(1+e^{-sa})} \;=\; \frac{(1-e^{-as})}{s(1+e^{-as})}.
\end{aligned}$$

Now multiply the numerator and denominator by $e^{as/2}$,

$$\mathfrak{L}\{f_s(x)\} = \frac{(e^{as/2} - e^{-as/2})}{s(e^{as/2} + e^{-as/2})} = \frac{1}{s}\tanh\frac{as}{2}.$$

□

Example 2 The Laplace transform of the triangular wave function $f_t(x)$. Since

$$f_s(x) = f_t'(x)$$

and $f_t(0) = 0$, the rule for derivatives gives

$$\mathfrak{L}\{f_s(x)\} = s\mathfrak{L}\{f_t(x)\} - f_t(0).$$

$$\mathfrak{L}\{f_t(x)\} = \frac{\mathfrak{L}\{f_s(x)\}}{s} = \frac{1}{s^2}\tanh\frac{as}{2}.$$

□

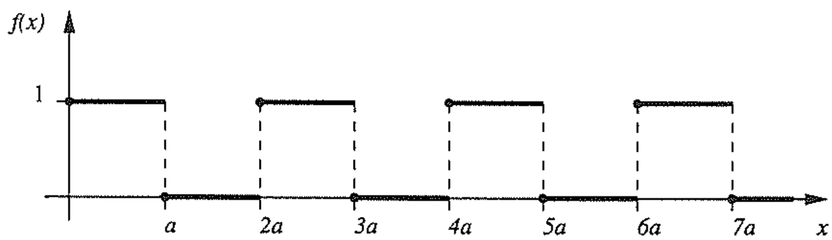
Exercises

1. Sketch the graph and find the Laplace transform of a different *square wave function.* This function is defined on the interval $[0, 2a)$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < a \\ 0 & \text{if } a \le x < 2a, \end{cases}$$

and in general, by

$$f(x + 2na) = f(x), \qquad (n = 0, 1, 2, \ldots, \quad 0 \le x < 2a).$$



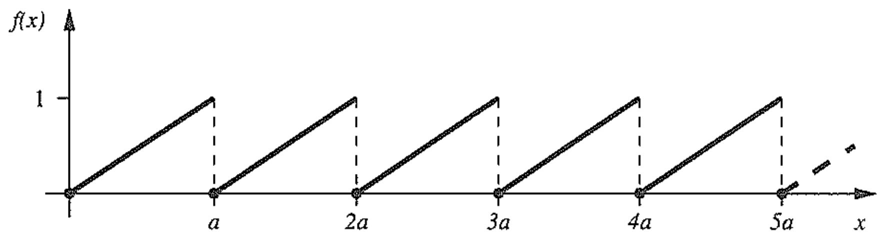
The square wave function of exercise 1.

2. Sketch the graph and find the Laplace transform of the *sawtooth function.* This function is defined on the interval $[0, a)$ by

$$g(x) \;=\; \frac{x}{a},$$

and in general, by

$$g(x + na) = g(x), \qquad (n = 0, 1, 2, \ldots, \quad 0 \le x < a).$$

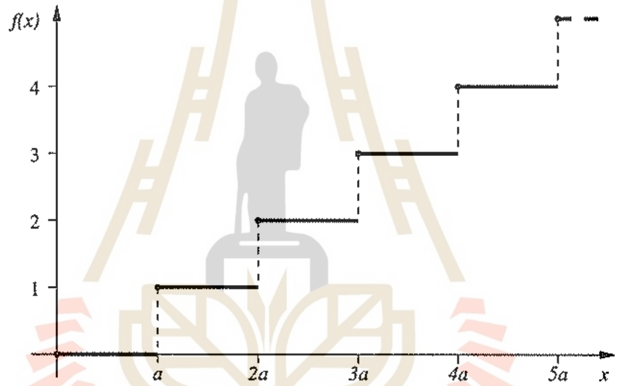


The sawtooth function.

3. Sketch the graph and find the Laplace transform of the *staircase function*

$$f(x) \;=\; \frac{x}{a} - g(x)$$

where $g(x)$ is the sawtooth function.



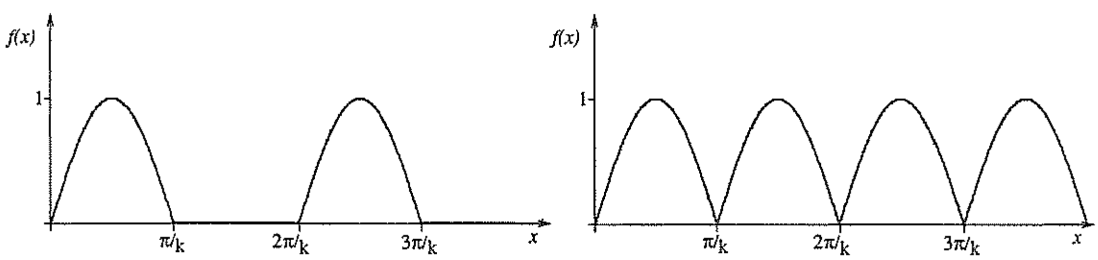
The staircase function.

4. Sketch the graph and find the Laplace transform of the *half-wave rectification* of $\sin kx$. This function is defined on the interval $[0, 2\pi/k)$ by

$$f(x) \;=\; \begin{cases} \sin kx & \text{if } \; 0 \le x < \pi/k \\ 0 & \text{if } \; \pi/k \le x < 2\pi/k \end{cases}$$

and in general, by

$$f(x + 2n\pi/k) = f(x), \qquad (n = 0, 1, 2, \ldots, \quad 0 \le x < 2\pi/k).$$



The half-wave and full-wave rectifications of $\sin kx$.

5. Sketch the graph and find the Laplace transform of the *full-wave rectification* of $\sin kx$,

$$g(x) = f(x) + u_{\pi/k} f(x - \pi/k) = |\sin ak|$$

where $f(x)$ is the half-wave rectification of $\sin kx$.

6. Consider an RC-circuit (no charge on the capacitor) with a battery supplying e_0 volts.

(a) If the switch to the battery is closed at time $t = a$ and opened at $t = b > a$, show that the current $i(t)$ satisfies the initial value problem

$$Ri' + \frac{1}{C}i = e_0\big(\delta_a(t) - \delta_b(t)\big), \qquad i(0) = 0.$$

(b) Solve this problem if $R = 100$ ohm, $C = 10^{-4}$ farad, $e_0 = 100$ volt, $a = 1$ sec and $b = 2$ sec. Show that $i(t) > 0$ if $1 < t < 2$ and $i(t) < 0$ if $t > 2$.

7. Consider an initially passive LC-circuit (no charge on the capacitor) with a battery supplying e_0 volts.

(a) If the switch to the battery is closed at time $t = 0$ and opened at $t = a$, show that the current $i(t)$ satisfies the initial value problem

$$Li'' + \frac{1}{C}i = e_0\big(\delta(t) - \delta_a(t)\big), \qquad i(0) = i'(0) = 0.$$

(b) Solve this problem if $L = 1$ henry, $C = 10^{-2}$ farad, $e_0 = 10$ volt, and $a = \pi$ sec. Show that $i(t) = \begin{cases} \sin 10t & \text{if } t < \pi \\ 0 & \text{if } t > \pi \end{cases}$. Graph the solution.

(c) Now assume, the switch is alternately closed and opened at $t = 0, 0.1\pi, 0.2\pi, \ldots$. Show that $i(t)$ satisfies the initial value problem

$$\begin{aligned} i'' + 100i &= 10\,[\delta(t) - \delta_{\pi/10}(t) + \delta_{2\pi/10}(t) - \delta_{3\pi/10}(t) + \delta_{4\pi/10}(t)\ldots] \\ &= 10\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi/10}, \qquad i(0) = i'(0) = 0. \end{aligned}$$

Solve this problem to show that

$$i(t) = (n+1)\sin 10t, \qquad n\pi/10 < t < (n+1)\pi/10.$$

We have some kind of resonance. Graph the solution.

(d) Now assume, the switch is alternately closed and opened at $t = 0, 0.2\pi, 0.4\pi, \ldots$. Show that $i(t)$ satisfies the initial value problem

$$\begin{aligned} i'' + 100i &= 10\,[\delta(t) - \delta_{\pi/5}(t) + \delta_{2\pi/5}(t) - \delta_{3\pi/5}(t) + \delta_{4\pi/5}(t)\ldots] \\ &= 10\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi/5}, \qquad i(0) = i'(0) = 0. \end{aligned}$$

Solve this problem to show that

$$i(t) = \begin{cases} \sin 10t & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Sketch the graph.

3.4.5 Convolution

There is no simple formula for the Laplace transform of a product. In general, the Laplace transform does not preserve products,

$$\mathfrak{L}\{fg\} \neq \mathfrak{L}\{f\}\mathfrak{L}\{g\} \text{ !}$$

However, we can give a formula for the inverse Laplace transform of a product.

Definition Let f and g be piecewise continuous functions defined on $[0, \infty)$. The *convolution* $f * g$ is defined by

$$\boxed{(f * g)(x) \;=\; \int_0^x f(u)g(x-u)\,du.} \tag{3.37}$$

Example 1 The convolution of the two functions $f(x) = \cos x$ and $g(x) = \sin x$ is

$$\cos x * \sin x \;=\; \int_o^x \cos u \sin(x-u)\,du.$$

Convert the product into a difference using the identity

$$\cos A \sin B \;=\; \frac{1}{2}\big[\sin(A+B) - \sin(A-B)\big].$$

We obtain the convolution

$$\begin{aligned} \cos x * \sin x &= \frac{1}{2}\int_0^x \Big(\sin x - \sin(2u-x)\Big)\,du \\ &= \frac{1}{2}\Big[\,u\sin x + \frac{\cos(2u-x)}{2}\,\Big]_0^x \\ &= \frac{1}{2}\Big(x\sin x + \frac{1}{2}\cos x - \frac{1}{2}\cos(-x)\Big) \;=\; \frac{1}{2}x\sin x \end{aligned}$$

□

Remark Substituting $s = x - u$ in (3.37) we get $u = x - s$ and $ds = -du$ so that also

$$(f * g)(x) \;=\; \int_x^0 f(x-s)g(s)(-ds) \;=\; \int_0^x g(s)f(x-s)ds.$$

This shows that convolution is *commutative*,

$$(f * g)(x) \;=\; (g * f)(x).$$

In general, convolution behaves like a product of functions. One can show that

1. $(f * g) * h \;=\; f * (g * h)$ (associative law),
2. $(f + g) * h \;=\; f * h + g * h$ (distributive law), and
3. $0 * f \;=\; 0$.

Now we show that the Laplace transform converts convolution to an ordinary product:

Theorem 22 (Convolution Theorem) *Suppose, $f(x)$ and $g(x)$ have Laplace transforms $F(s)$ and $G(s)$ for $s > c$. Then the Laplace transform of $f * g$ exists for $s > c$, and*

$$\boxed{\mathfrak{L}\{f * g\} \;=\; F(s)\,G(s).} \tag{3.38}$$

Proof. By the definition, the Laplace transforms of $f(x)$ and $g(x)$ are

$$F(s) \;=\; \int_0^\infty f(u)e^{-su}\,du \qquad \text{and} \qquad G(s) \;=\; \int_0^\infty g(v)e^{-sv}\,dv.$$

Therefore,

$$F(s)\,G(s) \;=\; \left(\int_0^\infty f(u)e^{-su}\,du\right)\left(\int_0^\infty g(v)e^{-sv}\,dv\right).$$

Now write as an iterated integral,

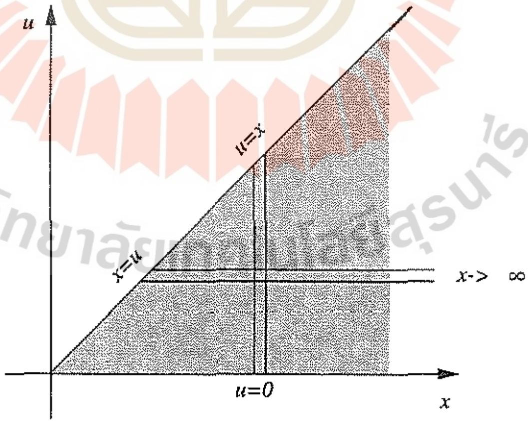
$$F(s)\,G(s) \;=\; \int_0^\infty \int_0^\infty f(u)g(v)e^{-s(u+v)}\,dv\,du.$$

Substitute $x = u + v$. Then, $v = x - u$ and $dx = dv$. If $v = 0$, then $x = u + 0 = u$ so that x ranges from u to ∞. We get

$$F(s)\,G(s) \;=\; \int_0^\infty \int_u^\infty f(u)g(x-u)e^{-sx}\,dx\,du.$$

Now interchange the order of integration. As the region of integration is bounded below by the x-axis and above by the line $u = x$ (see sketch), the limits of integration change to

$$\begin{aligned} F(s)\,G(s) \;&=\; \int_0^\infty \int_0^x f(u)g(x-u)e^{-sx}\,du\,dx \\ &=\; \int_0^\infty \left(\int_0^x f(u)g(x-u)\,du\right) e^{-sx}\,dx \;=\; \mathfrak{L}\{f * g\}. \end{aligned}$$



Change of order of integration.

□

Remark In this proof we have assumed that the order of integration can be exchanged even for improper integrals. Books on advanced calculus explain under what conditions this is allowed.

Example 2 Find the inverse Laplace transform of $H(s) = \dfrac{1}{(s^2+a^2)^2}$.

Solution. We can write

$$H(s) = F(s)F(s)$$

where

$$F(s) = \frac{1}{s^2+a^2}.$$

Now by the table,

$$f(x) = \mathfrak{L}^{-1}\left\{ \frac{1}{s^2+a^2} \right\} = \frac{1}{a}\sin ax.$$

By the convolution theorem,

$$h(x) = (f*f)(x) = \int_0^x \frac{1}{a^2} \sin au \sin a(x-u) \ du.$$

Using the trigonometric formula

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

we obtain

$$\begin{aligned} h(x) &= \frac{1}{2a^2}\int_0^x \Big(\cos a(2u-x) - \cos ax \Big)\, du \\ &= \frac{1}{2a^2}\left[\frac{\sin a(2u-x)}{2a} - u\cos ax \right]_0^x \\ &= \frac{1}{2a^2}\left(\frac{\sin ax}{2a} - \frac{\sin(-ax)}{2a} - x\cos ax - 0 \right) = \frac{1}{2a^3}\left(\sin ax - ax\cos ax\right). \end{aligned}$$

We have found that

$$\mathfrak{L}\left\{ \frac{1}{2a^3}\left(\sin ax - ax\cos ax\right) \right\} = \frac{1}{(s^2+a^2)^2} \tag{3.39}$$

□

Example 3 Find the inverse Laplace transform of $H(s) = \dfrac{2}{(s-1)(s^2+4)}$.

Solution. *Method 1*: Use partial fraction decomposition and obtain

$$H(s) = \frac{2}{5}\frac{1}{s-1} - \frac{2}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{2}{s^2+4}.$$

Using the table we get

$$h(x) = \mathfrak{L}^{-1}\{ H(s) \} = \frac{2}{5}e^x - \frac{2}{5}\cos 2x - \frac{1}{5}\sin 2x.$$

Method 2: Notice that $H(s) = F(s)\,G(s)$ where

$$F(s) = \frac{2}{s^2+4} \qquad \text{and} \qquad G(s) = \frac{1}{s-1}.$$

By the table, the inverse Laplace transform are

$$f(x) = \mathfrak{L}^{-1}\{ F(s) \} = \sin 2x \qquad \text{and} \qquad g(x) = \mathfrak{L}^{-1}\{ G(s) \} = e^x.$$

By the convolution theorem,

$$h(x) = (f*g)(x) = \int_0^x \sin(2u)e^{x-u}\,du = e^x\int_0^x \sin(2u)\,e^{-u}\,du.$$

Integration by parts applied twice gives

$$\begin{aligned} h(x) &= e^x\left[\frac{e^{-u}}{5}\left(-\sin 2u - 2\cos 2u\right)\right]_0^x \\ &= e^x\left(\frac{e^{-x}}{5}\left(-\sin 2x - 2\cos 2x\right) + 0 + \frac{2}{5}\right) \\ &= \frac{2}{5}e^x - \frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x. \end{aligned}$$

□

Example 4 Solve the general second order initial value problem

$$y'' + by' + cy = r(t), \qquad y(0) = y'(0) = 0.$$

Solution. Take the Laplace transform on both sides,

$$s^2Y(s) + bsY(s) + cY(s) = R(s).$$

Now solve for $Y(s)$,

$$(s^2 + bs + c)Y(s) = R(s)$$

$$Y(s) = H(s)R(s) \tag{3.40}$$

where we have set

$$H(s) = \frac{1}{s^2 + bs + c}.$$

In applications, the function often $r(t)$ represents the *input* of a system (e.g. external force, voltage, etc.), and the solution $y(t)$ the *output* of a system. The function $H(s)$ is called the *transfer function.* You can see that the transfer function $H(s)$ depends on the system under consideration (here the values of the constants b and c), while the function $R(s)$ represents the input. Applying the convolution theorem to (3.40), we have

$$y(t) = (h*r)(t) = \int_0^t h(u)r(t-u)\,du \tag{3.41}$$

where $h(t) = \mathfrak{L}^{-1}\left\{ H(s) \right\}$.

If we choose $R(s) = 1$, then $y(t) = \mathfrak{L}^{-1}\left\{ H(s) \right\} = h(t)$, and this is the solution to the initial value problem

$$y'' + by' + cy = \delta(t), \qquad y(0) = y'(0) = 0.$$

For this reason, $h(t)$ is called the *impulse response* of the system. Often, the system will be described by this impulse response, and the output can be obtained by forming the convolution of the input with the impulse response as in (3.41). □

Exercises

1. Find the convolution $(f * g)(x)$ if

   (a) $f(x) = x, \quad g(x) = 1$
   (b) $f(x) = x, \quad g(x) = e^{ax}$
   (c) $f(x) = \sin x, \quad g(x) = \sin x$
   (d) $f(x) = x^2, \quad g(x) = \cos x$
   (e) $f(x) = e^{ax}, \quad g(x) = e^{ax}$
   (f) $f(x)e^{ax}x, \quad g(x) = e^{bx}, b \neq a$

2. Using the convolution theorem, find the inverse Laplace transform of $F(s) =$

   (a) $\dfrac{1}{s^4(s^2+1)}$ (b) $\dfrac{s}{(s^2+4)(s+1)}$ (c) $\dfrac{1}{(s+1)^2(s^2+4)}$ (d) $\dfrac{G(s)}{s^2+1}$

3. Using the table, find the inverse Laplace transform of

   (a) $\dfrac{s^2+3}{(s^2+2s+2)^2}$ (b) $\dfrac{2s^3-s^2}{(4s^2-4s+5)^2}$

4. In each of the following initial value problems, express the solution as a convolution integral:

   (a) $y'' + \omega^2 y = g(t), \quad y(0) = 0, \quad y'(0) = 1$
   (b) $y'' + 2y' + 2y = \sin \alpha x, \quad y(0) = 0, \quad y'(0) = 0$
   (c) $4y'' + 4y' + 17y = g(x), \quad y(0) = 0, \quad y'(0) = 0$
   (d) $y'' + y' + \frac{5}{4}y = 1 - u_\pi(x), \quad y(0) = 1, \quad y'(0) = -1$
   (e) $y'' + 4y' + 4y = g(t), \quad y(0) = 2, \quad y'(0) = -3$
   (f) $y'' + 3y' + 2y = \cos \alpha x, \quad y(0) = 1, \quad y'(0) = 0$
   (g) $y^{(4)} - y = g(x), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0$
   (h) $y^{(4)} + 5y'' + 4y = g(t), \quad y(0) = 1, \quad y'(0) = y''(0) = y'''(0) = 0$

5. An equation

   $$y(x) = r(x) + \int_0^x y(u)f(x-u)\,du$$

   with y an unknown function, is called an *integral equation*. It can be solved taking the Laplace transform on both sides. Find the solution to

   (a) $y(x) = x + \displaystyle\int_0^x y(u)\sin(x-u)\,du$

   (b) $y(x) = \sin 2x - \displaystyle\int_0^x y(u)(x-u)\,du$

6. A mass $m = 1$ kg is attached to a spring with $k = 4$ N/m. Initially, the mass is at rest in equilibrium position. There is no damping. An external force of $f(t)$ is applied, where $f(t)$ is the square wave function (3.34) with amplitude 2 and period 2π. Find the position $y(t)$ of the mass at time t, and identify the steady state and transient solutions.

   Supplementary exercises:

7. Show that the convolution has the following properties:

   (a) $(f+g)*h = f*h + g*h,$
   (b) $0*f = 0,$
   (c) $(f*g)*h = f*(g*h),$

8. Find functions f and g such that

   (a) $1*f \neq f,$
   (b) $f*f \not\geq 0.$

9. If the function $f(x)$ is continuous on $(0, \infty)$ but $\lim_{x\to 0^+} f(x) = \pm\infty$ we can define the improper integral $\int_0^\infty f(x)\,dx$ by

$$\int_0^\infty f(x)\,dx = \lim_{v\to 0^+}\int_v^1 f(x)\,dx + \lim_{b\to\infty}\int_1^b f(x)\,dx$$

provided the two limits exist.

(a) Find the Laplace transform of $f(x) = x^{-1/2}$.

(b) Apply the convolution theorem to show that $\mathfrak{L}^{-1}\left\{\dfrac{1}{(s-1)\sqrt{s}}\right\} = \dfrac{2e^x}{\sqrt{\pi}}\displaystyle\int_0^{\sqrt{x}} e^{-u^2}\,du.$

3.5 Review of Partial Fractions Decomposition

When looking for the inverse Laplace transform we often have to express a fraction

$$F(s) = \frac{P(s)}{Q(s)}$$

as a sum of simpler fractions. This procedure is called *partial fraction decomposition*, and you have already used it with integration. Let us give a quick review.

Step 1: Factor the denominator $Q(s)$ completely. You will obtain factors of the form

$$s - c \qquad \text{or} \qquad s^2 + es + f.$$

Step 2: Find the terms included in the partial fraction decomposition.

- If the *linear* factor $s - c$ appears n times (that is, you have an expression $(s-c)^n$), then you must include the terms

$$\frac{A_1}{s-c}, \quad \frac{A_2}{(s-c)^2}, \quad \frac{A_3}{(s-c)^3}, \quad \dots \quad \frac{A_n}{(s-c)^n}$$

  in the partial fraction decomposition.

- If the *quadratic* factor $s^2 + es + f$ appears n times (that is, you have an expression $(s^2+es+f)^n$), then you must include the terms

$$\frac{B_1 s + C_1}{s^2+es+f}, \quad \frac{B_2 s + C_2}{(s^2+es+f)^2}, \quad \frac{B_3 s + C_3}{(s^2+es+f)^3}, \quad \dots \quad \frac{B_n s + C_n}{(s^2+es+f)^n}$$

  in the partial fraction decomposition.

Step 3: Find the inverse Laplace transform of each term. In the case of a quadratic denominator, you must first complete the square, and write

$$\frac{Bs + C}{(\,s^2+es+f\,)^n} = \frac{Bs+C}{(\,(s-c)^2+a^2\,)^n}$$

where $-2c = e$ and $c^2 + a^2 = f$. Then you separate into two fractions,

$$\frac{Bs+C}{(\,(s-c)^2+a^2\,)^n} = \frac{B(s-c)}{(\,(s-c)^2+a^2\,)^n} + \frac{Bc+C}{(\,(s-c)^2+a^2\,)^n}$$

and use the table to find the inverse Laplace transform of each of the two fractions.

Example 1 Find the inverse Laplace transform of $F(s) = \dfrac{s^2+1}{s^3-2s^2-8s}$.

Solution. The denominator factors as

$$Q(s) \;=\; s(s+2)(s-4).$$

We have three linear, nonrepeated factors. We therefore must use the partial fraction decomposition

$$\frac{s^2+1}{s(s+2)(s-4)} \;=\; \frac{A}{s}+\frac{B}{s+2}+\frac{C}{x-4}.$$

To find the values of A, B, and C, multiply by the common denominator $s(s+2)(s-4)$,

$$s^2+1 \;=\; A(s+2)(s-4)+Bs(s-4)+Cs(s+2).$$

Substitute $s=0$, $s=-2$, and $s=4$ to obtain

$$\begin{aligned} -8A &= 1 \qquad & A &= -1/8 \\ 12B &= 5 \qquad & B &= 5/12 \\ 24C &= 17 \qquad & C &= 17/24 \end{aligned}$$

Therefore,

$$F(s) \;=\; -\frac{1}{8}\frac{1}{s} \;+\; \frac{5}{12}\,\frac{1}{s+2} \;+\; \frac{17}{24}\,\frac{1}{s-4},$$

and

$$f(x) \;=\; -\frac{1}{8} \;+\; \frac{5}{12}\,e^{-2x} \;+\; \frac{17}{24}\,e^{4x} \;=\; \frac{1}{24}\Big(-3+10e^{-2x}+17e^{4x}\Big).$$

□

Example 2 Find the inverse Laplace transform of $F(s) = \dfrac{2}{s^3(s+2)^2}$

Solution. The denominator contains the linear factor s, repeated $n=3$ times, and the linear factor $s+2$, repeated $n=2$ times. We therefore must use the partial fraction decomposition

$$\frac{2}{s^3(s+2)^2} \;=\; \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+2}+\frac{E}{(s+2)^2}.$$

Multiply by the common denominator $s^3(s+2)^2$.

$$\begin{aligned} 2 &= As^2(s+2)^2+Bs(s+2)^2+C(s+2)^2+Ds^3(s+2)+Es^3 \\ &= A(s^4+4s^3+4s^2)+B(s^3+4s^2+4s)+C(s^2+4s+4)+D(s^4+2s^3)+Es^3 \end{aligned}$$

We can not use the method of the last example to find all of the coefficients because we have repeated factors. Instead, we expand and collect like powers of s,

$$2 \;=\; s^4(A+D)+s^3(4A+B+2D+E)+s^2(4A+4B+C)+s(4B+4C)+4C,$$

and then compare coefficients:

$$\begin{array}{llllllllllll} s^4: & A & & & & & + & D & & & = & 0 \\ s^3: & 4A & + & B & & & + & 2D & + & E & = & 0 \\ s^2: & 4A & + & 4B & + & C & & & & & = & 0 \\ s: & & & 4B & + & 4C & & & & & = & 0 \\ 1: & & & & & 4C & & & & & = & 2 \end{array}$$

If we solve this system of equations, we get

$$C = 1/2, \qquad B = -1/2, \qquad A = 3/8, \qquad D = -3/8, \qquad E = -1/4.$$

Therefore,

$$F(s) = \frac{3}{8}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{1}{2}\frac{1}{s^3} - \frac{3}{8}\frac{1}{s+2} - \frac{1}{4}\frac{1}{(x+2)^2}.$$

and

$$f(x) = \frac{3}{8} - \frac{x}{2} + \frac{x^2}{4} - \frac{3}{8}e^{-2x} - \frac{1}{4}xe^{-2x}.$$

□

Example 3 Find the inverse Laplace transform of $F(s) = \dfrac{5s^2 - s - 2}{(s+2)(s^2+1)}$.

Solution. There is one linear factor and one quadratic factor. We use the partial fraction decomposition

$$\frac{5s^2 - s - 2}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}.$$

Multiply by the common denominator,

$$5s^2 - s - 2 = A(s^2+1) + (Bs+C)(s+2) = s^2(A+B) + s(2B+C) + (A+2C).$$

Now compare coefficients,

$$\begin{aligned} s^2: &\quad A + B &&= 5 \\ s: &\quad 2B + C &&= -1 \\ 1: &\quad A + 2C &&= -2 \end{aligned}$$

Solving, we obtain

$$A = 4, \qquad B = 1, \qquad C = -3.$$

Therefore,

$$F(s) = \frac{4}{s+2} + \frac{s-3}{s^2+1} = \frac{4}{s+2} + \frac{s}{s^2+1} - 3\frac{1}{s^2+1}.$$

The inverse Laplace transform is

$$f(x) = 4e^{-2x} + \cos x - 3\sin x.$$

□

Example 4 Find the inverse Laplace transform of $F(s) = \dfrac{60}{(s^2+4)(s^2+6s+34)}$.

Solution. When completing the square we obtain

$$s^2 + 6s + 34 = (s+3)^2 + 25.$$

This shows that we can not factor further. We have two quadratic factors, and must use the partial fraction decomposition

$$\frac{60}{(s^2+4)(s^2+6x+34)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+6s+34}.$$

Multiply by the common denominator,

$$60 = (As+B)(s^2+6s+34)+(Cs+D)(s^2+4),$$

and expand and collect like powers of s.

$$60 = s^3(A+C)+s^2(6A+B+D)+s(34A+6B+4C)+(34B+4D).$$

Now compare coefficients,

$$\begin{array}{lccccccccc} s^3: & A & & & + & C & & & = & 0 \\ s^2: & 6A & + & B & & & + & D & = & 0 \\ s: & 34A & + & 6B & + & 4C & & & = & 0 \\ 1: & & & 34B & & & + & 4D & = & 60 \end{array}$$

Solving this system of equations, we get

$$A = -\frac{10}{29}, \qquad B = \frac{50}{29}, \qquad C = D = \frac{10}{29}.$$

Therefore,

$$\begin{aligned} F(s) &= \frac{10}{29}\left(\frac{-s+5}{s^2+4}+\frac{s+1}{(s+3)^2+25}\right) \\ &= \frac{10}{29}\left(-\frac{s}{s^2+4}+\frac{5}{s^2+4}+\frac{s+3}{(s+3)^2+25}-\frac{2}{(s+3)^2+25}\right) \end{aligned}$$

Using the table, the inverse transform is

$$f(s) = \frac{10}{29}\left(-\cos 2x+\frac{5}{2}\sin 2x+e^{-3x}\cos 5x-\frac{2}{5}e^{-3x}\sin 5x\right).$$

□

Example 5 Find the inverse Laplace transform of $F(s) = \dfrac{s^4+11s^2-2s+15}{(s-1)(s^2+4)^2}$.

Solution. We have one linear factor and one repeated quadratic factor. Therefore, we use the partial fraction decomposition

$$\frac{s^4+11s^2-2s+15}{(s-1)(s^2+4)^2} = \frac{A}{s-1}+\frac{Bs+C}{s^2+4}+\frac{Ds+E}{(s^2+4)^2}.$$

Multiply by the common denominator

$$s^4+11s^2-2s+15 = A(s^2+4)^2+(Bs+C)(s-1)(s^2+4)+(Ds+E)(s-1)$$

and collect like powers of s,

$$\begin{aligned} s^4+11s^2-2s+15 &= (A+B)s^4+(-B+C)s^3+(8A+4B-C+D)s^2 \\ &+ (-4B+4C-D+E)s+(16A-4C-E). \end{aligned}$$

Compare coefficients,

$$\begin{array}{lccccccccccc} s^4: & A & + & B & & & & & & & = & 1 \\ s^3: & & - & B & + & C & & & & & = & 0 \\ s^2: & 8A & + & 4B & - & C & + & D & & & = & 11 \\ s: & & - & 4B & + & 4C & - & D & + & E & = & -2 \\ 1: & 16A & & & - & 4C & & & - & E & = & 15 \end{array}$$

Solving, we obtain

$$A = 1, \qquad B = C = 0, \qquad D = 3, \qquad E = 1.$$

Therefore,

$$F(s) = \frac{1}{s-1} + \frac{3s+1}{(s^2+4)^2} = \frac{1}{s-1} + \frac{3s}{(s^2+4)^2} + \frac{1}{(s^2+4)^2}.$$

Using the table, we obtain an inverse Laplace transform

$$f(x) = e^x + 3\frac{x}{2\cdot 2}\sin 2x + \frac{1}{2\cdot 8}(\sin 2x - 2x\cos 2x) = e^x + \frac{12x+1}{16}\sin 2x - \frac{x}{8}\cos 2x.$$

□

3.6 Tables of Laplace Transforms

$f(x)$	$F(s)$	occurs
1	$\frac{1}{s}, \quad s>0$	(3.4)
x	$\frac{1}{s^2}, \quad s>0$	(3.5)
$x^n \quad (n=1,2,\dots)$	$\frac{n!}{s^{n+1}}, \quad s>0$	$n=1:$ (3.13) $n\geq 2:$ exercise
e^{cx}	$\frac{1}{s-c}, \quad s>c$	(3.7)
$\sin ax$	$\frac{a}{s^2+a^2}, \quad s>0$	(3.8)
$\cos ax$	$\frac{s}{s^2+a^2}, \quad s>0$	exercise
$\sinh ax$	$\frac{a}{s^2-a^2}, \quad s>\lvert a\rvert$	exercise
$\cosh ax$	$\frac{s}{s^2-a^2}, \quad s>\lvert a\rvert$	(3.11)
$e^{cx}\sin ax$	$\frac{a}{(s-c)^2+a^2}, \quad s>c$	exercise
$e^{cx}\cos ax$	$\frac{s-c}{(s-c)^2+a^2}, \quad s>c$	exercise
$x^n e^{cx} \quad (n=1,2,\dots)$	$\frac{n!}{(s-c)^{n+1}}, \quad s>c$	(3.22)
$\frac{1}{2a^3}(\sin ax - ax\cos ax)$	$\frac{1}{(s^2+a^2)^2}, \quad s>0$	(3.39)
$\frac{x}{2a}\sin ax$	$\frac{s}{(s^2+a^2)^2}, \quad s>0$	(3.24)
$u_c(x)$	$\frac{e^{-cs}}{s}, \quad s>0$	(3.26)
$\delta_c(x)$	$e^{-cs}, \quad c\geq 0$	(3.31)

Elementary Laplace Transforms

$f(x)$	$F(s)$	occurs
$c_1 f_1(x) + c_2 f_2(x)$	$c_1 F_1(s) + c_2 F_2(s)$	(3.10)
$f'(x)$	$sF(s) - f(0)$	(3.12)
$f''(x)$	$s^2 F(s) - sf(0) - f'(0)$	(3.14)
$f^{(n)}(x)$	$s^n F(s) - s^{n-1} f(0) - \cdots$ $\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$	(3.15)
$\int_0^x f(u)\, du$	$\frac{F(s)}{s}$	(3.20)
$e^{cx} f(x)$	$F(s-c)$	(3.21)
$f(cx)$	$\frac{1}{c} F\left(\frac{s}{c}\right), \quad c > 0$	exercise
$(f * g)(x) = \int_0^x f(u) g(x-u)\, du$	$F(s)\, G(s)$	(3.38)
$x^n f(x)$	$(-1)^n F^{(n)}(s)$	(3.23)
$\frac{f(x)}{x}$	$\int_s^\infty F(u)\, du$	(3.25)
$u_c(x) f(x-c)$	$e^{-cs} F(s), \quad c > 0$	(3.27)

Properties of the Laplace Transform

Chapter 4

Power Series Solutions

In this chapter we will study solutions of differential equations by power series. We must first discuss power series in detail.

4.1 Power Series

4.1.1 Introduction

An infinite series in powers of the variable x,

$$\sum_{n=0}^{\infty} a_n x^n \;=\; a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a *power series centered at zero*. In general, a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_o)^n \;=\; a_0 + a_1(x - x_o) + a_2(x - x_o)^2 + a_3(x - x_o)^3 + \cdots + a_n(x - x_o)^n + \cdots \tag{4.1}$$

is called a *power series centered at* x_o. The numbers a_n are called the *coefficients* of the power series, and the number x_o is called the *center* of the power series.

Remark Every power series of the form (4.1) can be made into a power series centered at zero by substituting $z = x - x_o$. Thus, it is enough to study power series centered at zero.

Note that a power series is a sum of *infinitely* many terms. The next example explain what we mean by the sum of all these terms.

Example 1 Discuss the power series

$$\sum_{n=0}^{\infty} x^n \;=\; 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

(This is called the the *geometric series.*)

Solution. Let us first look at the sum of the first $N + 1$ terms,

$$S_N \;=\; \sum_{n=0}^{N} x^n \;=\; 1 + x + x^2 + x^3 + \cdots + x^N.$$

To simplify this expression, we multiply both sides by $1-x$,

$$\begin{aligned} S_N(1-x) &= (1+x+x^2+x^3+\cdots+x^N)(1-x) \\ &= (1+x+x^2+x^3+\cdots+x^N) - (x+x^2+x^3+x^4+\cdots+x^{N+1}) \\ &= 1-x^{N+1}. \end{aligned} \tag{4.2}$$

We can divide by $1-x$ if $x \neq 1$,

$$S_N = \frac{1-x^{N+1}}{1-x}$$

Now we add more and more of the terms, that is, we let $N \to \infty$. We must distinguish three cases.

- If $|x| < 1$ then

  $$\lim_{N\to\infty} x^{N+1} = 0$$

  and thus we obtain

  $$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x} = \frac{1}{1-x}.$$

- If $|x| > 1$ or $x = -1$, then

  $$\lim_{N\to\infty} x^{N+1}$$

  does not exists, so

  $$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x}$$

  does not exist either.

- If $x = 1$ then we can not divide by $1-x$ in equation (4.2). However, we can compute S_N directly,

  $$S_N = \underbrace{1+1+1^2+1^3\cdots+1^N}_{N+1 \text{ terms}} = N+1$$

  so that

  $$\lim_{N\to\infty} S_N = \lim_{N\to\infty} N+1 = \infty.$$

We have shown: If $|x| < 1$, then

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} x^n = \frac{1}{1-x},$$

otherwise this limit does not exist. We say that the geometric series converges for $|x| < 1$, and define its sum for such x by

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}} \tag{4.3}$$

□

Let us define the concept of convergence properly:

Definition Given a power series

$$\sum_{n=0}^{\infty} a_n(x - x_o)^n,$$

the finite sum

$$S_N = \sum_{n=0}^{N} a_n(x - x_o)^n$$

is called the *N-th partial sum*. We say that the series *converges* at a point x to the number $f(x)$, if the limit of partial sums

$$f(x) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} a_n(x - x_o)^n.$$

exists. Otherwise, we say that the series *diverges* at x.

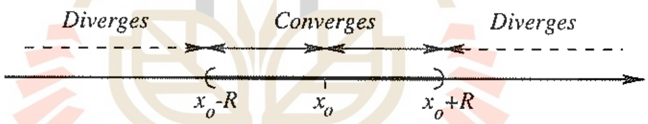
The situation of example 1 is typical for a power series, as the following theorem shows:

Theorem 23 *Consider a power series* $\sum_{n=0}^{\infty} a_n(x - x_o)^n$ *and suppose,*

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists. ($R = \infty$ is permitted.) Then

- *the power series converges for $|x - x_o| < R$, and*
- *the power series diverges for $|x - x_o| > R$.*

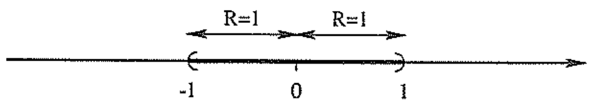


Radius and Interval of Convergence

Remark The number R is called the *radius of convergence*, and the interval $(x_o - R, x_o + R)$ is called the *interval of convergence* of the series.

Remark The theorem does not tell us what is happening at the endpoints $x_o - R$ and $x_o + R$ of the interval of convergence. At these endpoints, some series converge while others diverge; you can try to check the following examples for convergence at the endpoints by yourself.

Example 2 From example 1 we see that the geometric series $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$ and interval of convergence $(-1, 1)$. On this interval, the geometric series equals the function $f(x) = \frac{1}{1-x}$. □

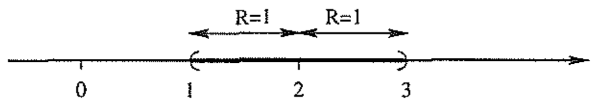


Example 3 Find the interval of convergence of the series $\sum_{n=0}^{\infty}(-1)^{n+1}n(x-2)^n$.

Solution. This is a power series with center $x_o = 2$ and coefficients $a_n = (-1)^{n+1}n$. Its radius of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

Therefore, this power series converges for $|x-2| < 1$, that is whenever $1 < x < 3$, and its interval of convergence is $(1, 3)$. □

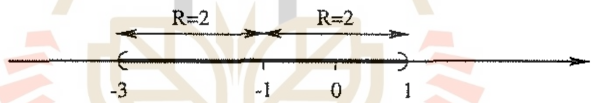


Example 4 Find the interval of convergence of the series $\sum_{n=1}^{\infty}\frac{(x+1)^n}{n\,2^n}$.

Solution. This is a power series with center $x_o = -1$ and coefficients $a_n = \frac{1}{n2^n}$. Its radius of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{1}{n2^n}}{\frac{1}{(n+1)2^{n+1}}} = \lim_{n\to\infty}\frac{2(n+1)}{n} = 2.$$

Thus, the interval of convergence is $(-3, 1)$. □

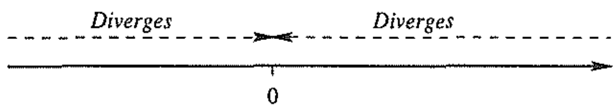


Example 5 Find the interval of convergence of the series $\sum_{n=1}^{\infty} n!\,x^n$.

Solution. This is a power series with center $x_o = 0$ and coefficients $a_n = n!$. Its radius of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Thus, the series converges only at its center $x = 0$. (Note that every power series converges at its center because every term of the series is zero at $x = x_o$.) □

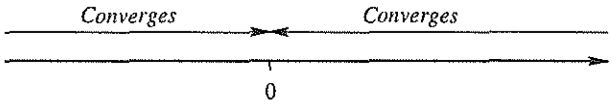


Example 6 Find the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. This is a power series with center $x_o = 0$ and coefficients $a_n = \frac{1}{n!}$. Its radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

Hence, the interval of convergence is $(-\infty, \infty)$. (You may recognize it as the power series for e^x, see (4.6) below.) □



Example 7 Find the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{3n} = 1 - \frac{x^3}{2} + \frac{x^6}{4} - \frac{x^9}{8} + \frac{x^{12}}{16} - \cdots$$

Solution. We can not use the theorem directly because

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

does not exists. (The coefficients $a_1, a_2, a_4, a_5, \cdots, a_{3k+1}, a_{3k+2}, \cdots$ are zero.) Instead, we can think of this as a power series in $z = x^3$, and write it in the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x^3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n. \tag{4.4}$$

The formula for the radius of convergence gives us

$$\lim_{n\to\infty} \left|\frac{\frac{(-1)^n}{2^n}}{\frac{(-1)^{n+1}}{2^{n+1}}}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} = \lim_{n\to\infty} 2 = 2$$

So the series (4.4) converges for

$$|z| < 2,$$

or

$$|x| < \sqrt[3]{2}.$$

The radius of convergence is $\sqrt[3]{2}$, and the interval of convergence is $(-\sqrt[3]{2}, \sqrt[3]{2})$. □

Given a function $f(x)$, we often need to find a power series $\sum_{n=0}^{\infty} a_n(x - x_o)^n$ so that

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_o)^n.$$

In this case, we say that the power series *represents* (or is a *series expansion* of) the function $f(x)$. One commonly used way to find such a series is to construct the *Taylor series at a point x_o*:

If $f(x)$ is infinitely differentiable at x_o, then its Taylor series at x_o is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n = f(x_o) + f'(x_o)(x-x_o) + \frac{f''(x_o)}{2}(x-x_o)^2 + \cdots$$

We say that $f(x)$ is *analytic* at x_o, if it can be represented by its Taylor series, that is if

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n. \tag{4.5}$$

in an interval containing x_o.

The functions which we are commonly using are analytic at every x_o in their domains. These include all functions composed of rational functions, trigonometric functions and exponential functions.

e.g. You may already know the Taylor series expansion of the following functions, valid for all x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{4.6}$$

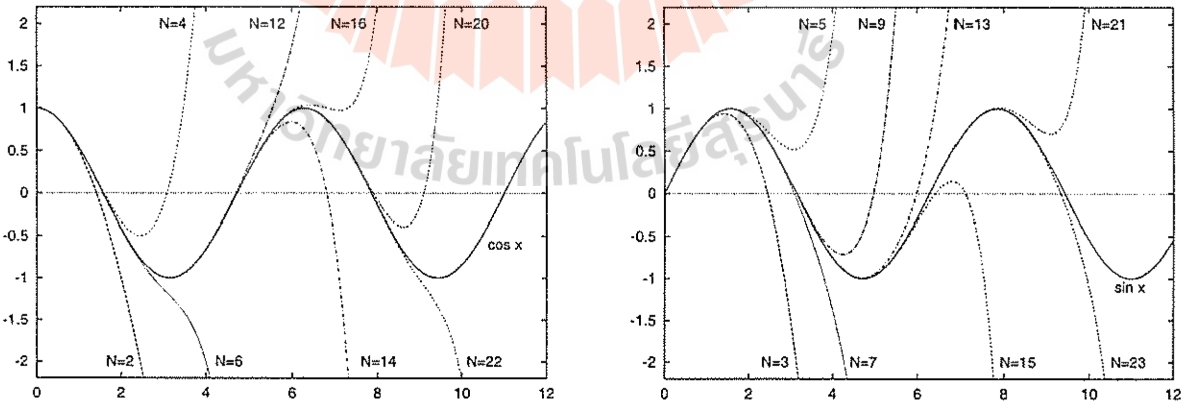
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{4.7}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{4.8}$$

In many practical applications, we can only compute finite partial sums

$$S_N = \sum_{n=0}^{N} a_n (x-x_o)^n$$

of a power series. (In the case of a Taylor series, this is called a Taylor polynomial). The following two sketches show the graphs of the Taylor polynomials of $\cos x$ and $\sin x$ for various values of N.



Approximations of $\cos x$ and $\sin x$ by Taylor polynomials $S_N = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n$.

As we move away from the center $x_o = 0$, we need to add more and more terms of the series to get a good approximation for $\cos x$ and $\sin x$.

Exercises

1. Find the Taylor series centered at $x_o = 0$ (also called the Maclaurin series) by using the known Taylor series in (4.6) – (4.8).
   (a) $f(x) = e^{-x}$
   (b) $f(x) = e^{2x}$
   (c) $f(x) = e^{-3x}$
   (d) $f(x) = e^{x^3}$
   (e) $f(x) = \cos 2x$
   (f) $f(x) = \sin(x/2)$
   (g) $f(x) = \sin x^2$
   (h) $f(x) = \cos\sqrt{x}$

2. Find the Taylor series centered at the point x_o:
   (a) $f(x) = \ln(1+x), \quad x_o = 0$
   (b) $f(x) = \ln x, \quad x_o = 1$
   (c) $f(x) = \cos x + \sin x, \quad x_o = \pi/2$
   (d) $f(x) = \sinh x, \quad x_o = 0$
   (e) $f(x) = \dfrac{1}{x}, \quad x_o = 1$
   (f) $f(x) = \sin x, \quad x_o = \pi/4$
   (g) $f(x) = x\tan^{-1} x, \quad x_o = 0$

3. Find the radius of convergence R and the interval of convergence of
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^2+1}$
   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2 x^n$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n-1}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n x^n}{5^n}$
   (f) $\displaystyle\sum_{n=0}^{\infty} (5x-3)^n$
   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^4+16}$
   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n^2}$
   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)! x^n}{n!}$
   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n+1)x^n}{n!}$
   (k) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 (x+1)^n}{3^n}$
   (l) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n^2}$
   (m) $\displaystyle\sum_{n=1}^{\infty} \frac{(3-x)^n}{n^3}$
   (n) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{n!}(x-10)^n$
   (o) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}(x-5)^n$
   (p) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\cdot 10^n}(x-2)^n$
   (q) $\displaystyle\sum_{n=0}^{\infty} x^{2^n}$
   (r) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x^2+1}{5}\right)^n$

4.1.2 Properties of Power Series

In the following, we will list some useful properties of power series without proof. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

be two power series which converge for $|x| < R$.

- *Sum of Power Series.* We can add and subtract the two series term by term,

$$\boxed{\left(\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n + b_n) x^n}$$

  whenever $|x| < R$.

- *Multiplication by a constant.* For every constant c,

$$\boxed{c\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} c\, a_n\, x^n}$$

  whenever $|x| < R$.

e.g. To find a power series for $\cosh x$, we use the series (4.6) for e^x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{4.9}$$

and replace x by $-x$ to obtain a series for e^{-x},

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}. \tag{4.10}$$

Now add the two series,

$$\begin{aligned} e^x + e^{-x} &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \\ &= 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + 2\frac{x^6}{6!} + \cdots = 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \end{aligned}$$

Finally divide by 2, and obtain

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

This series converges for all x because the series (4.9) and (4.10) have radius of convergence $R = \infty$.

- *Product of Power Series.* To multiply two series, we multiply each term of the first series with each term of the second series (the same way we multiply two expressions in brackets),

$$\boxed{f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k},$$

and this series converges for $|x| < R$.

e.g.

$$\begin{aligned} \sin x \cos x &= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right) \\ &= 1 \cdot 1 \cdot x + \left(-\frac{1}{6} \cdot 1 + 1 \cdot \left(-\frac{1}{2}\right)\right) x^3 \\ &\quad + \left(1 \cdot \frac{1}{24} + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{120} \cdot 1\right) x^5 + \cdots \\ &= x - \frac{4}{6}x^3 + \frac{16}{120}x^5 - \cdots \\ &= \frac{1}{2}\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots\right) = \frac{1}{2} \sin 2x \end{aligned}$$

where in the last line we have used (4.8).

- *Identity of power series.* If

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all $|x| < R$, then

$$\boxed{a_n = b_n}$$

for all n.

e.g. Suppose,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = 0.$$

The right hand side can be written as a power series,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 0 \cdot x^n.$$

Comparing all coefficients we see that $a_n = 0$ for all n.

- *Shift of Summation Index.*

$$\boxed{\sum_{n=a}^{\infty} a_n\, x^{n-r} = \sum_{n=a-r}^{\infty} a_{n+r}\, x^n}$$

Proof. We start from $\sum_{n=a}^{\infty} a_n\, x^{n-r}$ and simply do the substitution

$$k = n - r.$$

Then, $n = k + r$. When $n = a$ then we have $k = a - r$, so the sum starts with this index. Also, when $n \to \infty$ then $k \to \infty$. We obtain

$$\sum_{n=a}^{\infty} a_n\, x^{n-r} = \sum_{k=a-r}^{\infty} a_{k+r}\, x^k.$$

Finally, we rename the summation index, by setting $n = k$ on the right. □

e.g. – Given the series $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}$, we set $k = n - 1$. Then, $n = k + 1$. When $n = 2$ then we have $k = 2 - 1 = 1$. Thus,

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-1} = \sum_{k=1}^{\infty} (k+1)k\, a_{k+1}\, x^k.$$

We can rename our index of summation, and set $n = k$ to obtain

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-1} = \sum_{n=1}^{\infty} (n+1)n\, a_{n+1}\, x^n.$$

– Given the series $\sum_{n=0}^{\infty}(n+1)(n+2)\,a_n\,x^{n+2}$, we set $k = n+2$. Then $n = k-2$, and the sum starts with $k = 0+2 = 2$. We obtain

$$\sum_{n=0}^{\infty}(n+1)(n+2)\,a_n\,x^{n+2} = \sum_{k=2}^{\infty}(k-1)k\,a_{k-2}\,x^k.$$

Finally, rename $n = k$ on the right,

$$\sum_{n=0}^{\infty}(n+1)(n+2)\,a_n\,x^{n+2} = \sum_{n=2}^{\infty}(n-1)n\,a_{n-2}\,x^n.$$

But note that

$$(n-1)n = 0$$

when $n = 0$ or $n = 1$. Therefore, we can start with summation index $n = 0$ on the right, and write

$$\sum_{n=2}^{\infty}(n-1)na_{n-2}x^n = \sum_{n=0}^{\infty}(n-1)na_{n-2}x^n.$$

We can differentiate and integrate a power series: Let

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n \tag{4.11}$$

be a power series with radius of convergence $R > 0$.

- *Derivative of power series.* The function $f(x)$ has derivatives of all orders for $|x| < R$. To find the derivative, we simply differentiate each term of the series (4.11),

$$\boxed{f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_n\,x^{n-1}.}$$

  Furthermore, this power series also has radius of convergence R.

- *Integral of powers series.* The function $f(x)$ can be integrated on the interval of convergence. To find the indefinite integral, we simply integrate each term of the series (4.11),

$$\boxed{\int f(x)\,dx = C + a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4} + \cdots = C + \sum_{n=0}^{\infty}\frac{a_n}{n+1}\,x^{n+1}.}$$

  Often we want the antiderivative whose initial value is zero,

$$\boxed{\int_0^x f(t)\,dt = a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty}\frac{a_n}{n+1}\,x^{n+1}.}$$

  These power series also have radius of convergence R.

e.g.

- Begin with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \tag{4.12}$$

and take its derivative,

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

for $|x| < 1$. After reindexing ($k = n - 1$), we obtain the power series representation

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

- Replace x by $-x$ in the geometric series (4.12) and obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Now integrate from 0 to x,

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \end{aligned}$$

for $|x| < 1$. After reindexing, we have a power series representation

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n}.$$

- Replace x by $-x^2$ in the geometric series (4.12), and obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots + (-1)^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Integrate from 0 to x,

$$\begin{aligned} \tan^{-1} x &= \int_0^x \frac{1}{1+t^2}\,dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

for $|x| < 1$.

Remark The differentiation rule says that the powers series

$$f(x) = \sum_{n=0}^{\infty} a_n\, x^n$$

has first order derivative

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}$$

and second order derivative

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}.$$

Note that the term $n\, a_n\, x^{n-1}$ is zero when $n = 0$, and the term $n(n-1)\, a_n x^{n-2}$ is zero when $n = 0$ or $n = 1$. Therefore, we can let both series start at $n = 0$ and write

$$f'(x) = \sum_{n=0}^{\infty} n\, a_n\, x^{n-1}$$

and

$$f''(x) = \sum_{n=0}^{\infty} n(n-1)\, a_n\, x^{n-2}.$$

Here is a table of some power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \qquad (\,|x| < 1\,)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{(n-1)} \frac{x^n}{n} + \cdots \qquad (\,|x| < 1\,)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \qquad (\,|x| < 1\,)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

Exercises

1. Use known series to find a power series representation of the following functions. Also, determine the radius and interval of convergence.

   (a) $f(x) = x^2 e^{-3x}$

   (b) $f(x) = \dfrac{1}{10+x}$

   (c) $f(x) = \sin x^2$

   (d) $f(x) = \cos^2 x = \frac{1}{2}(1+\cos 2x)$

   (e) $f(x) = \dfrac{1}{(1+x)^3}$

   (f) $f(x) = \dfrac{\ln(1+x)}{x}$

   (g) $f(x) = \dfrac{x - \arctan x}{x^3}$

2. Find a power series representation for

   (a) $f(x) = \displaystyle\int_0^x \sin t^3 \, dt$

   (b) $f(x) = \displaystyle\int_0^x \frac{\sin t}{t} \, dt$

   (c) $f(x) = \displaystyle\int_0^x e^{-t^3} \, dt$

   (d) $f(x) = \displaystyle\int_0^x \frac{\tan^{-1} t}{t} \, dt$

   (e) $f(x) = \displaystyle\int_0^x \frac{1-e^{-t^2}}{t^2} \, dt$

   (f) $\tanh^{-1} x = \displaystyle\int_0^x \frac{1}{1-t^2} \, dt$

3. Use the series for $\tan^{-1} x$ to show that

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{3}\right)^n .$$

4. If $y = \displaystyle\sum_{n=0}^{\infty} n x^n$, compute y' and y''.

5. Reindex to write in the form $\sum a_{n+r}(x-x_o)^n$:

   (a) $\displaystyle\sum_{n=0}^{\infty} a_n (x-1)^{n+1}$ (b) $\displaystyle\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ (c) $\displaystyle\sum_{n=0}^{\infty} a_n x^{n+2}$

6. Determine the coefficients a_n so that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

   What function is represented by the series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$?

   Supplementary exercises:

7. Suppose, $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n (x-x_o)^n$ converges for $|x-x_o| < R$, and let $\displaystyle\sum_{n=0}^{\infty} c_n (x-x_o)^n$ be the Taylor series of f. Show that $a_n = c_n$ for all n. (Hint: Take the derivatives of all orders.)

4.2 The Series Method - First Order Equations

We have seen in the preceding two chapters how to solve linear equations with constant coefficients. However, we have no method of solution in the case of equations with non-constant coefficients. In many cases, it is possible to find a power series solution. To get an idea of this method, we look at a few simple first order equations first.

Example 1 Solve the equation

$$y' - y = 0. \tag{4.13}$$

Solution. We assume that there exists a solution y which is can be expressed as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiate,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and substitute into the differential equation,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

In order to add the two series, we must reindex the first series to obtain powers x^n,

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now we can add the two series and obtain

$$\sum_{n=0}^{\infty} \Big[(n+1) a_{n+1} - a_n \Big] x^n = 0.$$

By the identity property all coefficients must be zero,

$$(n+1) a_{n+1} - a_n = 0$$

for all n. That is,

$$a_{n+1} = \frac{a_n}{n+1}.$$

This is called a *recurrence relation*. Once we know a_0, we can compute all the coefficients a_n in terms of a_0. For example, if $n = 0$, then the recurrence relation gives

$$a_1 = \frac{a_0}{1} = a_0.$$

If $n = 1$, the recurrence relation gives

$$a_2 = \frac{a_1}{2} = \frac{a_0}{2 \cdot 1}$$

If $n = 2$, the recurrence relation gives

$$a_3 = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2 \cdot 1}.$$

You can see the pattern. To compute the coefficient a_n from a_{n-1} we divide by n. After n steps we have

$$a_n = \frac{1}{n!} a_0.$$

The solution is therefore,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

We recognize this as the power series of the function $y = e^x$, so that

$$y = a_0 e^x.$$

This is the solution which we have expected. In fact, equation (4.13) is a first order linear equation, and using the methods of chapter we can find the same solution. □

Example 2 Find the solution to $(x-3)y' + 2y = 0$.

Solution. Substitute

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

We obtain

$$(x-3)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Multiply $(x-3)$ into the first series, to obtain

$$\left[\sum_{n=1}^{\infty} n a_n x^n - 3\sum_{n=1}^{\infty} n a_n x^{n-1}\right] + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Before we can add these series, we must shift the index in the middle series (setting $k = n-1$). Also, we can let the first series should start at zero, because then $n a_n x^n = 0$,

$$\sum_{n=0}^{\infty} n a_n x^n - 3\sum_{n=0}^{\infty} (n+1)a_{(n+1)} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Add all the series,

$$\sum_{n=0}^{\infty} \Big[(n+2)a_n - 3(n+1)a_{n+1}\Big] x^n = 0$$

By the identity property,

$$(n+2)a_n - 3(n+1)a_{n+1} = 0$$

for all n. Solving for a_{n+1} we obtain the recurrence relation

$$a_{n+1} = \frac{n+2}{3(n+1)}\, a_n.$$

Now let $n = 0, 1, 2, 3, \ldots$. We obtain

$$\begin{array}{llllll}
n=0: & a_1 & = & \frac{2}{3}a_0 & & \\
n=1: & a_2 & = & \frac{3}{6}a_1 & = & \frac{3}{9}a_0 \\
n=2: & a_3 & = & \frac{4}{9}a_2 & = & \frac{4}{27}a_0 \\
n=3: & a_4 & = & \frac{5}{12}a_3 & = & \frac{5}{81}a_0. \\
\vdots & \vdots & & \vdots & & \vdots
\end{array}$$

You can see the pattern. The n-th coefficient is

$$a_n = \frac{n+1}{3^n}\, a_0.$$

We therefore have the formal power series solution

$$y = a_0 \sum_{n=0}^{\infty} (n+1)\left(\frac{x}{3}\right)^n. \tag{4.14}$$

□

Remark The power series which we have found can only be a solution if it converges. So we should determine its interval of convergence as well. Now

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{(n+1)3^{n+1}}{(n+2)3^n} = 3.$$

So the power series converges for $|x| < 3$ and is a valid solution on the interval $(-3, 3)$.

Remark What function does this series represent ? Let us substitute $z = x/3$ in (4.14) to obtain

$$y = a_0 \sum_{n=0}^{\infty} (n+1)z^n.$$

But note that $(n+1)z^n$ is the derivative of z^{n+1} ! Thus,

$$y = a_0 \frac{d}{dz} \left(\sum_{n=0}^{\infty} z^{n+1} \right) = a_0 \frac{d}{dz} \left(\sum_{n=1}^{\infty} z^n \right).$$

where we have re-indexed on the right. This is the geometric series, without its first term. Therefore,

$$y = a_0 \frac{d}{dz} \left(\sum_{n=0}^{\infty} z^n - 1 \right) = a_0 \frac{d}{dz} \left(\frac{1}{1-z} - 1 \right)$$

and taking the derivative on the right,

$$y = a_0 \frac{1}{(1-z)^2}.$$

Resubstitute,

$$y = a_0 \frac{1}{(1-\frac{x}{3})^2} = \frac{c}{(3-x)^2}$$

where we have set $c = 9a_0$.

Remark We can easily check that the function $\dfrac{1}{(3-x)^2}$ is a solution for all $x \neq 3$. On the other hand, our power series solution only converges on the interval $(-3, 3)$. So the power series method does not give us the best possible solution. If we want a series solution valid outside this interval, for example valid at $x = 5$, then we should try a series

$$\sum_{n=0}^{\infty} a_n (x-5)^n$$

centered at 5 to make sure that it converges there.

Exercises

1. Find a power series solution of each equation. Determine the radius of convergence of your solution, and identify it as an elementary function.
   (a) $y' = y$
   (b) $y' = 4y$
   (c) $2y' + 3y = 0$
   (d) $y' + 2xy = 0$
   (e) $y' = x^2 y$
   (f) $(x-2)y' + y = 0$
   (g) $(2x-1)y' + 2y = 0$
   (h) $2(x+1)y' = y$
   (i) $(x-1)y' + 2y = 0$
   (j) $2(x-1)y' = 3y$

2. Show that the power series method does not work to give you a power series solution centered at $x_0 = 0$ if
   (a) $xy' + y = 0$
   (b) $2xy' = y$
   (c) $x^2 y' + y = 0$
   (d) $x^3 y' = 2y$

3. Find a power series solution for example 2 which is valid at $x = 5$. What is its radius of convergence ?

4.3 The Series Method - Ordinary Points

Let us discuss the solution of second order linear differential equations of the form

$$a(x)y'' + b(x)y' + c(x)y = 0$$

by the power series method. Dividing by $a(x)$ we obtain the equation in standard form,

$$\boxed{y'' + p(x)y' + q(x)y = 0} \tag{4.15}$$

where we have set $p(x) = \frac{b(x)}{a(x)}$ and $q(x) = \frac{c(x)}{a(x)}$. If both functions, $p(x)$ and $q(x)$ are analytic at x_o so that we have a power series expansion valid at x_o, then we call x_o an *ordinary point*. Otherwise, x_o is called a *singular point.*

e.g.

- Consider the equation $(1-x^2)y' - 2xy' + y = 0$. Dividing by $1-x^2$,

  $$y' - \frac{2x}{1-x^2}y' + \frac{1}{1-x^2}y = 0.$$

  The points $x_o = 1$ and $x_o = -1$ are singular points because $p(x) = \dfrac{2x}{1-x^2}$ is not analytic at these points. (In general, a rational function is analytic wherever it is defined, and $p(x)$ is undefined at $x = \pm 1$.)
- Consider the equation $xy'' + (\sin x)y' + x^2 y = 0$. Dividing by x,

  $$y'' + \frac{\sin x}{x}y' + xy = 0.$$

  Note that

  $$\begin{aligned} \frac{\sin x}{x} &= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots\right) \\ &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots \end{aligned}$$

  so that $x_o = 0$ is an ordinary point !
- Consider the equation $y'' + \dfrac{2x}{(2x-1)(x+2)}y' + \dfrac{\cos x}{x^2}y = 0$. The points $x = 0$, $x = 1/2$ and $x = -2$ are singular points.

We can always find a power series solution centered at an ordinary point:

Theorem 24 *Consider a second order linear equation*

$$y'' + p(x)y' + q(x)y = 0. \tag{4.16}$$

Suppose p and q are analytic at $x = x_o$ and have a power series representation on an interval $|x - x_o| < R$. Then every solution of equation (4.16) is analytic at x_o and can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n (x - x_o)^n. \tag{4.17}$$

valid on the same interval.

Example 1 Find a power series solution of the equation

$$y'' + y = 0 \tag{4.18}$$

Solution. Here, $p(x) = 0$ and $q(x) = 1$, so we expect a series solution which converges and is a valid for $-\infty < x < \infty$.

We substitute the series

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Its derivatives are

$$y' = \sum_{n=1}^{\infty} a_n\, n\, x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} a_n\, n(n-1)\, x^{n-2}.$$

If we substitute these series into (4.18) we get

$$\sum_{n=2}^{\infty} a_n\, n(n-1)\, x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

We shift the summation index in the first series,

$$\sum_{n=0}^{\infty} a_{n+2}\, (n+2)(n+1)\, x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

and add the two series,

$$\sum_{n=0}^{\infty} \Big[a_{n+2}\, (n+2)(n+1) + a_n \Big] x^n = 0.$$

By the identity property,

$$a_{n+2}\, (n+2)(n+1) + a_n = 0$$

for all n. Solving for a_{n+2} we get the recurrence relation

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}.$$

So if we choose $n = 0, 2, 4 \ldots$, we obtain

$$\begin{array}{lllllll}
n=0: & a_2 = -\dfrac{a_0}{1\cdot 2} & = -\dfrac{a_0}{2!} & & \\
n=2: & a_4 = -\dfrac{a_2}{3\cdot 4} & = \dfrac{a_0}{1\cdot 2\cdot 3\cdot 4} & = \dfrac{a_0}{4!} & \\
n=4: & a_6 = -\dfrac{a_4}{5\cdot 6} & = -\dfrac{a_0}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} & = -\dfrac{a_0}{6!} \\
\vdots & \vdots \quad \vdots & \vdots & \vdots
\end{array}$$

We can write the even indices as $n = 2k$, and then we have in general,

$$a_{2k} = (-1)^k \frac{a_0}{(2k)!}.$$

Now if we choose $n = 1, 3, 5, \cdots$, we obtain

$$\begin{array}{lllllll}
n=1: & a_3 = -\dfrac{a_1}{2\cdot 3} & = -\dfrac{a_1}{3!} & & \\
n=3: & a_5 = -\dfrac{a_3}{4\cdot 5} & = \dfrac{a_1}{2\cdot 3\cdot 4\cdot 5} & = \dfrac{a_1}{5!} & \\
n=5: & a_7 = -\dfrac{a_5}{6\cdot 7} & = -\dfrac{a_1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} & = -\dfrac{a_1}{7!} \\
\vdots & \vdots \quad \vdots & \vdots & \vdots
\end{array}$$

We can write the odd indices as $n = 2k+1$, and then we have in general

$$a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}.$$

The general solution is

$$\begin{aligned} y &= a_0 + a_1 x - \frac{a_0}{2!}\,x^2 - \frac{a_1}{3!}\,x^3 + \frac{a_0}{4!}\,x^4 + \frac{a_1}{5!}x^5 - \cdots \\ &= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= a_0 \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

□

Remark We recognize the first series as the Taylor series for $\cos x$ and the second series as the Taylor series of $\sin x$. This is not surprising; if we use the methods of chapter 2, we obtain the general solution

$$y = a_0 \cos x + a_1 \sin x.$$

Example 2 Find a power series solution of the equation

$$y'' - x^2 y' - 2xy = 0. \tag{4.19}$$

Solution. We substitute the series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and its derivatives

$$y' = \sum_{n=1}^{\infty} a_n\, n\, x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} a_n\, n(n-1)\, x^{n-2}$$

into (4.19),

$$\sum_{n=2}^{\infty} a_n\, n(n-1)\, x^{n-2} - x^2 \sum_{n=1}^{\infty} a_n\, n\, x^{n-1} - 2x \sum_{n=0}^{\infty} a_n x^n = 0.$$

We would like to add all three series. But before, we must make some adjustments. First we move all x inside the series,

$$\sum_{n=2}^{\infty} a_n\, n(n-1)\, x^{n-2} - \sum_{n=1}^{\infty} a_n\, n\, x^{n+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Then we reindex all three series in order to have terms x^n,

$$\sum_{n=0}^{\infty} a_{n+2}\,(n+2)(n+1)\, x^n - \sum_{n=2}^{\infty} a_{n-1}\,(n-1)\, x^n - 2 \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

Finally, we want all three series to start with the same initial value of the index n. The second series can start with $n = 1$, because when $n = 1$, then $a_{n-1}(n-1)x^n = 0$,

$$\sum_{n=0}^{\infty} a_{n+2}\,(n+2)(n+1)\, x^n - \sum_{n=1}^{\infty} a_{n-1}\,(n-1)\, x^n - 2 \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

As the first series starts with $n = 0$, we write its first term separately,

$$2a_2 \ + \ \sum_{n=1}^{\infty} a_{n+2}\,(n+2)(n+1)\,x^n \ - \ \sum_{n=1}^{\infty} a_{n-1}\,(n-1)\,x^n \ - \ 2\sum_{n=1}^{\infty} a_{n-1}x^n \ = \ 0.$$

Now we can add the three series,

$$2a_2 \ + \ \sum_{n=1}^{\infty} \Big[a_{n+2}(n+2)(n+1) - a_{n-1}(n-1) - 2a_{n-1}\Big]\,x^n \ = \ 0.$$

By the identity property,

$$\begin{cases} 2a_2 \ = \ 0 & \\ a_{n+2}(n+2)(n+1) - a_{n-1}(n-1) - 2a_{n-1} \ = \ 0 & (n = 1, 2, 3...) \end{cases}$$

These two equations give use

$$a_2 = 0$$

and the recurrence relation

$$a_{n+2} \ = \ \frac{a_{n-1}(n+1)}{(n+1)(n+2)} \ = \ \frac{a_{n-1}}{n+2}. \qquad (n = 1, 2, 3...)$$

The recurrence relation jumps three steps. If we choose $n = 1, 4, 7\ldots$, we obtain

$$\begin{array}{lllll} n=1: & a_3 & = & \dfrac{a_0}{3} & \\ n=4: & a_6 & = & \dfrac{a_3}{6} & = \ \dfrac{a_0}{3\cdot 6} \\ n=7: & a_9 & = & \dfrac{a_6}{9} & = \ \dfrac{a_0}{3\cdot 6\cdot 9} \\ \vdots & & \vdots & & \vdots \end{array}$$

If we choose $n = 2, 5, 8, \ldots$, we obtain

$$\begin{array}{lllll} n=2: & a_4 & = & \dfrac{a_1}{4} & \\ n=5: & a_7 & = & \dfrac{a_4}{7} & = \ \dfrac{a_1}{4\cdot 7} \\ n=8: & a_{10} & = & \dfrac{a_7}{10} & = \ \dfrac{a_1}{4\cdot 7\cdot 10} \\ \vdots & & \vdots & & \vdots \end{array}$$

Finally, if $n = 3, 6, 9, \cdots$ we obtain

$$\begin{array}{llll} n=3: & a_5 & = \ \dfrac{a_2}{5} & = \ 0 \\ n=6: & a_8 & = \ \dfrac{a_5}{8} & = \ 0 \\ \vdots & \vdots & & \\ n=3k: & a_{3k+2} & = \ 0. & \end{array}$$

The solution is therefore

$$y \ = \ a_0\left(1 + \frac{x^3}{3} + \frac{x^6}{3\cdot 6} + \frac{x^9}{3\cdot 6\cdot 9} + \cdots\right) \ + a_1\left(x + \frac{x^4}{4} + \frac{x^7}{4\cdot 7} + \frac{x^{10}}{4\cdot 7\cdot 10} + \cdots\right).$$

□

Remark Since the functions $p(x) = -x^2$ and $q(x) = -2x$ are polynomials, the series solution converges and is valid for all x by theorem 24. But what functions do the two series represent ? The first series can be written

$$\begin{aligned} y_1 &= 1 + \frac{x^3}{3} + \frac{x^6}{3 \cdot 6} + \frac{x^9}{3 \cdot 6 \cdot 9} + \cdots \\ &= 1 + \frac{x^3}{3}\,\frac{1}{1} + \left(\frac{x^3}{3}\right)^2 \frac{1}{1 \cdot 2} + \left(\frac{x^3}{3}\right)^3 \frac{1}{1 \cdot 2 \cdot 3} + \left(\frac{x^3}{3}\right)^4 \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots \\ &= e^{x^3/3}. \end{aligned}$$

The second series has no such simple representation. We therefore write the solution as

$$y = a_0 e^{x^3/3} + a_1 \left(x + \frac{x^4}{4} + \frac{x^7}{4 \cdot 7} + \frac{x^{10}}{4 \cdot 7 \cdot 10} + \cdots \right).$$

Remark Since we have found one solution $y_1 = e^{x^3/3}$ in closed form, we might be tempted to use the method "Using one Solution to find another" to obtain the second solution in closed form. If we try this and substitute

$$y = v e^{x^3/3},$$

then equation (4.19) changes to the separable equation

$$v'' + x^2 v' = 0$$

whose solution is

$$v' = c\, e^{-x^3/3}.$$

Then

$$v = c \int e^{-x^3/3}\, dx + d,$$

and we can not evaluate this integral! Nevertheless, the second solution is

$$y_2 = v y_1 = e^{x^3/3} \left[c \int e^{-x^3/3}\, dx + d \right].$$

By the Fundamental Theorem of Calculus, we can rewrite the indefinite integral as

$$y_2 = e^{x^3/3} \left[c \int_0^x e^{-t^3/3}\, dt + d \right].$$

To determine the values of c and d, we take derivatives (and use the product rule),

$$y_2' = e^{x^3/3} x^2 \left[c \int_0^x e^{-t^3/3}\, dt + d \right] + c.$$

Now the second power series satisfies $y(0) = 0$ and $y'(0) = 1$. If we substitute these conditions into the last two equations, we get $d = 0$ and $c = 1$. We have found a power series for the integral:

$$y_2 = e^{x^3/3} \int_0^x e^{-t^3/3}\, dt = x + \frac{x^4}{4} + \frac{x^7}{4 \cdot 7} + \frac{x^{10}}{4 \cdot 7 \cdot 10} + \cdots.$$

Example 3 Find the power series solution to the initial value problem

$$(x^2 - 2x) y'' + (x - 1) y' - 4y = 0 \qquad y(1) = 1,\ y'(1) = 2. \tag{4.20}$$

Solution. Note that the points $x_o = 0$ and $x_o = 2$ are singular points of the equation. Therefore, we can not expect a power series solution centered at 0. Because the initial conditions are specified at $x_o = 1$, we instead look for a solution

$$\sum_{n=0}^{\infty} a_n(x-1)^n$$

centered at $x = 1$. By theorem 24, such a power series solution will be valid in the interval $(0, 2)$. To simplify computations, we substitute

$$z = x - 1$$

and look for a solution

$$y = \sum_{n=0}^{\infty} a_n z^n.$$

Then equation (4.20) becomes

$$\left(\,(z+1)^2 - 2(z+1)\,\right) y'' + zy' - 4y = 0$$

so that we have an initial value problem

$$(z^2-1)y'' + zy' - 4z = 0 \qquad y(0) = 1;\ y'(0) = 2. \tag{4.21}$$

To solve it, we substitute

$$y = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \tag{4.22}$$

$$y' = \sum_{n=1}^{\infty} a_n n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots \tag{4.23}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) z^{n-2} = 2a_2 + 6a_3 z + 12a_4 z^2 \cdots$$

and obtain

$$(z^2-1)\sum_{n=2}^{\infty} a_n n(n-1)z^{n-2} + z\sum_{n=1}^{\infty} a_n n z^{n-1} - 4\sum_{n=0}^{\infty} a_n z^n = 0.$$

Multiply into the series,

$$\left(\sum_{n=2}^{\infty} a_n n(n-1)z^n - \sum_{n=2}^{\infty} a_n n(n-1)z^{n-2}\right) + \sum_{n=1}^{\infty} a_n n z^n - \sum_{n=0}^{\infty} 4a_n z^n = 0$$

Now reindex the second series, and let the first and third series start at $n = 0$,

$$\sum_{n=0}^{\infty} a_n n(n-1)z^n - \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)z^n + \sum_{n=0}^{\infty} a_n n z^n - \sum_{n=0}^{\infty} 4a_n z^n = 0$$

and add,

$$\sum_{n=0}^{\infty}\left[\,a_n n(n-1) - a_{n+2}(n+2)(n+1) + a_n n - 4a_n\,\right] z^n = 0$$

$$\sum_{n=0}^{\infty}\left[\,a_n(n^2-4) - a_{n+2}(n+2)(n+1)\,\right] z^n = 0$$

By the identity property,

$$a_n(n^2-4) - a_{n+2}(n+2)(n+1) \;=\; 0$$

for all n. We obtain the recurrence relation

$$a_{n+2} \;=\; \frac{a_n(n^2-4)}{(n+2)(n+1)}$$

or

$$a_{n+2} \;=\; \frac{a_n(n-2)}{n+1}.$$

Because we are given initial conditions, we can right away determine the values of a_0 and a_1. Substitute $y(0) = 1$ into (4.22) to get

$$a_0 \;=\; y(0) \;=\; 1$$

and substitute $y'(0) = 2$ into (4.23) to get

$$a_1 \;=\; y'(0) \;=\; 2.$$

Now choose $n = 0, 2, 4, \cdots$,

$$\begin{aligned}
n=0: \quad & a_2 \;=\; -2a_0 \;=\; -2 \\
n=2: \quad & a_4 \;=\; \frac{0}{3}\,a_2 \;=\; 0 \\
n=4: \quad & a_6 \;=\; \frac{2}{5}\,a_4 \;=\; 0 \\
\vdots \quad & \qquad \vdots
\end{aligned}$$

In general, $a_n = 0$ when $n > 2$ is even. Now choose $n = 1, 3, 5, \cdots$,

$$\begin{aligned}
n=1: \quad & a_3 \;=\; \frac{-1}{2}a_1 \;=\; -2\cdot\left(\frac{1}{2}\right) \;=\; -1 \\
n=3: \quad & a_5 \;=\; \frac{1}{4}a_3 \;=\; -2\cdot\left(\frac{1}{2\cdot 4}\right) \\
n=5: \quad & a_7 \;=\; \frac{3}{6}a_5 \;=\; -2\cdot\left(\frac{3\cdot 1}{6\cdot 4\cdot 2}\right) \\
n=7: \quad & a_9 \;=\; \frac{5}{8}a_7 \;=\; -2\cdot\left(\frac{5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\right) \\
\vdots \quad & \qquad \vdots \qquad\qquad \vdots
\end{aligned}$$

In general, for $n \geq 5$ odd we get

$$a_n \;=\; -2\,\frac{(n-4)\cdot(n-6)\cdot\ldots\cdot 5\cdot 3\cdot 1}{(n-1)\cdot(n-3)\cdot\ldots\cdot 4\cdot 2}.$$

We can write the odd integers as $n = 2k+1$ and obtain

$$a_{2k+1} \;=\; -2\,\frac{(2k-3)\cdot(2k-5)\cdot\ldots\cdot 5\cdot 3\cdot 1}{2k\cdot(2k-2)\cdot\ldots\cdot 4\cdot 2}$$

for $k \geq 2$. The power series solution is therefore

$$\begin{aligned}
y \;&=\; 1 + 2z - 2z^2 - z^3 - 2\Big(\frac{1}{2\cdot 4}\,z^5 + \frac{3\cdot 1}{6\cdot 4\cdot 2}\,z^7 + \frac{5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\,z^9 + \cdots\Big) \\
&=\; 1 + 2z - 2z^2 - z^3 - 2\sum_{k=2}^{\infty}\frac{(2k-3)(2k-5)\cdot\ldots\cdot 5\cdot 3\cdot 1}{2k\cdot(2k-2)\cdot\ldots\cdot 4\cdot 2}\,z^{2k+1}.
\end{aligned}$$

Finally, we resubstitute $z = x - 1$ and obtain

$$y = 1 + 2(x-1) - 2(x-1)^2 - (x-1)^3 - 2\sum_{k=2}^{\infty} \frac{(2k-3)(2k-5)\cdot\ldots\cdot 5\cdot 3\cdot 1}{2k\cdot(2k-2)\cdot\ldots\cdot 4\cdot 2}(x-1)^{2k+1}.$$

□

Exercises

1. Use the series method to find two linearly independent solutions of each equation. Determine radius of convergence, and identify the solutions as elementary functions.
   (a) $y'' = y$
   (b) $y'' = 4y$
   (c) $y'' + 9y = 0$
   (d) $y'' + y = x$

2. Solve the following initial value problems using the power series method. Identify the solutions as elementary functions.
   (a) $y'' + 4y = 0;\quad y(0) = 0,\quad y'(0) = 3$
   (b) $y'' - 4y = 0;\quad y(0) = 2,\quad y'(0) = 0$
   (c) $y'' - 2y' + y = 0;\quad y(0) = 0,\quad y'(0) = 1$
   (d) $y'' + y' - 2y = 0;\quad y(0) = 1,\quad y'(0) = -2$

3. Use the power series method to solve each of the following equations, and find the radius of convergence of each series which is guaranteed by theorem 24.
   (a) $(x^2-1)y'' + 4xy' + 2y = 0$
   (b) $(x^2+2)y'' + 4xy' + 2y = 0$
   (c) $y'' + xy' + y = 0$
   (d) $(x^2+1)y'' + 6xy' + 4y = 0$
   (e) $(x^2-3)y'' + 2xy' = 0$
   (f) $(x^2-1)y'' - 6xy' + 12y = 0$
   (g) $(x^2+3)y'' - 7xy' + 16y = 0$
   (h) $(2-x^2)y'' - xy' + 16y = 0$
   (i) $(x^2-1)y'' + 8xy' + 12y = 0$
   (j) $3y'' + xy' - 4y = 0$
   (k) $5y'' - 2xy' + 10y = 0$
   (l) $y'' - x^2y' - 3xy = 0$
   (m) $y'' + x^2y' + 2xy = 0$
   (n) $y'' + xy = 0$
   (o) $y'' + x^2y = 0$

4. Use the power series method to solve each initial value problem.
   (a) $(1+x^2)y'' + 2xy' - 2y = 0;\quad y(0) = 0,\quad y'(0) = 1$
   (b) $y'' + xy' - 2y = 0;\quad y(0) = 1,\quad y'(0) = 0$

5. Find a power series solution centered at the point specified by the initial conditions.
   (a) $y'' + (x-1)y' + y = 0;\quad y(1) = 2,\quad y'(1) = 1$
   (b) $(2x-x^2)y'' - 6(x-1)y' - 4y = 0;\quad y(1) = 0,\quad y'(1) = 1$
   (c) $(x^2-6x+10)y'' - 4(x-3)y' + 6y = 1;\quad y(3) = 2,\quad y'(3) = 0$
   (d) $(4x^2+16x+17)y'' = 8y;\quad y(-2) = 1,\quad y'(-2) = 0$
   (e) $(x^2+6x)y'' + (3x+9)y' - 3y = 0;\quad y(-3) = 0,\quad y'(-3) = 2$

6. Find the recurrence relation for the power series solution, and determine the first three nonzero terms of two linearly independent solutions.
   (a) $y'' + (1+x)y = 0$
   (b) $(x^2-1)y'' + 2xy' + 2xy = 0$
   (c) $y'' + x^2y' + x^2y = 0$
   (d) $(1+x^3)y'' + x^4y = 0$

7. Using the power series method, solve

$$y'' + xy = e^x.$$

8. Solve example 3 of chapter 2, section 2.3 using the power series method.

4.4 The Series Method - Singular Points

Now let us look at a second order linear differential equation

$$\boxed{y'' + p(x)y' + q(x)y = 0} \tag{4.24}$$

where $x = x_o$ is a singular point. That is, at least one of $p(x)$ or $q(x)$ is not analytic at x_o. The point x_o is called a *regular singular point*, if the functions

$$(x - x_o)p(x) \qquad \text{and} \qquad (x - x_o)^2 q(x)$$

are analytic at x_o. Otherwise it is called an *irregular singular point*.

e.g.

- Consider the equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

(This is called *Bessel's equation of order n*.) If we divide by x^2 we obtain

$$y'' + \frac{1}{x}\, y' + (1 - \frac{n^2}{x^2})\, y = 0.$$

Here,

$$p(x) = \frac{1}{x} \qquad \text{and} \qquad q(x) = 1 - \frac{n^2}{x^2}$$

so that $x = 0$ is a singular point. However, the functions

$$xp(x) = 1 \qquad \text{and} \qquad x^2q(x) = x^2 - n^2.$$

are analytic at $x = 0$; therefore the point $x = 0$ is a regular singular point.

- Consider the equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

(This is called the *Legendre equation of order n*.) Divide by $1 - x^2$,

$$y'' - \frac{2x}{1 - x^2}\, y' + \frac{n(n+1)}{1 - x^2}\, y = 0$$

to see that the points $x = -1$ and $x = 1$ are singular points. Let us look at $x = 1$ first. We have

$$(x - 1)p(x) = \frac{-2x}{1 + x} \qquad \text{and} \qquad (x - 1)^2 q(x) = \frac{n(n+1)(1 - x)}{1 + x}.$$

Both functions are analytic at $x = 1$; therefore the point $x = 1$ is a regular singular point. In a similar way, the point $x = -1$ is also a regular singular point.

- Consider the equation

$$2(x - 2)^2xy'' + 3xy' + (x - 2)y = 0,$$

which we first rewrite as

$$y'' + \frac{3}{2(x - 2)^2}\, y' + \frac{1}{2(x - 2)x}\, y = 0.$$

There are two singular points: $x = 0$ and $x = 2$. Now

$$xp(x) = \frac{3x}{2(x-2)^2} \qquad \text{and} \qquad xq(x) = \frac{1}{2(x-2)}.$$

Since both these functions are analytic at $x = 0$, this is a regular singular point. Now if $x = 2$, then

$$(x-2)p(x) = \frac{3}{2(x-2)}$$

is not analytic at $x = 2$. Therefore, $x = 2$ is an irregular singular point.

- Finally. consider the equation

$$x^2y'' - 3(\sin x)y' + (1+x^2)y = 0$$

Dividing by x^2 we obtain

$$y'' - \frac{3\sin x}{x^2}y' + \frac{1+x^2}{x^2}y = 0.$$

Thus, $x = 0$ is a singular point. Note that

$$xp(x) = 3\,\frac{\sin x}{x} \qquad \text{and} \qquad x^2q(x) = 1 + x^2.$$

Is $\frac{\sin x}{x}$ an analytic function ? Take the power series for $\sin x$, and note that

$$\frac{1}{x}\sin x = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots .$$

This is an analytic function, and therefore $x = 0$ is a regular singular point.

If x_o is a singular point, then there may not exist a power series solution

$$y = \sum_{n=0}^{\infty} a_n(x-x_o)^n$$

centered at x_o. However, if x_o is a *regular singular point*, then there always exists at least one series solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n(x-x_o)^n$$

where r can be *any* number ! A series of this form might look as follows:

$$y = a_0x^{-1/2} + a_1x^{1/2} + a_2x^{3/2} + a_3x^{5/2} + \cdots ,$$

and is called a *Frobenius series.* Let us explain this method of finding a series solution, called the *Frobenius Method*, by an example.

Example 1 Find a series solution of the equation

$$2x^2y'' - xy' + (1+x)y = 0 \tag{4.25}$$

centered at zero.

Solution. First check whether $x=0$ is a singular point. Divide by $2x^2$ to obtain the equation in standard form,

$$y'' - \frac{1}{2x}y' + \frac{1+x}{x^2}y \; = \; 0.$$

We now see that $x=0$ is a regular singular point, and we try a series solution

$$y \; = \; x^r \sum_{n=0}^{\infty} a_n x^n \; = \; \sum_{n=0}^{\infty} a_n x^{n+r}.$$

The derivatives are

$$y' \; = \; \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \qquad \text{and} \qquad y'' \; = \; \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

Now substitute all these series into (4.25),

$$2x^2 \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} - x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + (1+x)\sum_{n=0}^{\infty} a_n x^{n+r} \; = \; 0,$$

and multiply the three terms $2x^2$, x and $(1+x)$ into the series,

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} \; = \; 0.$$

Before we can add we must re-index the last series,

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r} \; = \; 0.$$

Because the last series starts with the index $n=1$, we add the terms belonging to $n=0$ separately,

$$\Big[2a_0r(r-1) - a_0r + a_0\Big]x^r + \sum_{n=1}^{\infty}\Big[2a_n(n+r)(n+r-1) - a_n(n+r) + a_n + a_{n-1}\Big]x^{n+r} \; = \; 0.$$

Because the right side is zero, every term on the left must be zero. The term belonging to $n=0$ gives

$$2a_0r(r-1) - a_0r + a_0 \; = \; 0$$

and the terms belonging to $n \geq 1$ give

$$a_n\Big[2(n+r)(n+r-1) - (n+r) + 1\Big] + a_{n-1} \; = \; 0 \qquad (n \geq 1)$$

Let us divide the first equation by $a_0 \neq 0$, to obtain

$$2r^2 - 3r + 1 \; = \; 0$$

which factors as

$$(2r-1)(r-1) \; = \; 0. \tag{4.26}$$

This equation is called the *indicial equation*, and its two solutions

$$r_1 = 1 \qquad \text{and} \qquad r_2 = 1/2.$$

are called the *exponents* of the differential equation. The second equation can be written

$$a_n\Big(\,[n+r-1]\;[2(n+r)-1]\,\Big) \; = \; -a_{n-1}$$

so that

$$a_n = -\frac{a_{n-1}}{[n+r-1]\,[2(n+r)-1]}. \qquad (4.27)$$

Let us first consider the larger of the two exponents $r = 1$. We get the recurrence relation

$$a_n = -\frac{a_{n-1}}{n\,(2n+1)}.$$

Thus,

$$\begin{aligned} a_1 &= -\frac{a_0}{3} = -\frac{a_0}{3\cdot 1} \\ a_2 &= -\frac{a_1}{2\cdot 5} = \frac{a_0}{5\cdot 3\cdot 2\cdot 1} \\ a_3 &= -\frac{a_2}{3\cdot 7} = -\frac{a_0}{(7\cdot 5\cdot 3\cdot 1)\,(3\cdot 2\cdot 1)} \end{aligned}$$

and in general,

$$a_n = (-1)^n \frac{a_0}{[\,(2n+1)(2n-1)\cdots 3\cdot 1\,]\;\cdot\; n!}.$$

Choosing $a_0 = 1$ we have found one series solution

$$y_1 = x\sum_{n=0}^{\infty} a_n x^n = x\left(\sum_{n=0}^{\infty} \frac{(-1)^n\, x^n}{[\,(2n+1)(2n-1)\cdots 3\,]\; n!}\right).$$

What is the radius of convergence of this series ?

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+3)(2n+1)\cdot 5\cdots 3\,(n+1)!}{(2n+1)\cdot 5\cdots 3\; n!}\right| = \infty.$$

Thus the series converges for all x.

Now let us take the other exponent $r = 1/2$. The recurrence relation (4.27) becomes

$$a_n = -\frac{a_{n-1}}{(n-1/2)\cdot(2n)} = -\frac{a_{n-1}}{(2n-1)n}$$

so that

$$\begin{aligned} a_1 &= -\frac{a_0}{1} \\ a_2 &= -\frac{a_1}{2\cdot 3} = \frac{a_0}{3\cdot 3} \\ a_3 &= -\frac{a_2}{3\cdot 5} = -\frac{a_0}{(3\cdot 5)\,(2\cdot 3)} \\ a_4 &= -\frac{a_3}{4\cdot 7} = \frac{a_0}{(3\cdot 5\cdot 7)\,(2\cdot 3\cdot 4)} \end{aligned}$$

and in general,

$$a_n = (-1)^n \frac{a_0}{[\,3\cdot 5\cdot 7\cdots(2n-1)\,]\; n!}.$$

Setting $a_0 = 1$ we have found a second series solution

$$y_2 = x^{1/2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, x^n}{3\cdot 5\cdot 7\cdots(2n-1)\, n!}\right). \qquad (x > 0)$$

This series also converges for all x. Because of the factor $x^{1/2}$, however, the solution y_2 is valid for $x \geq 0$ only. Since y_1 and y_2 are linearly independent, we have found the general solution to equation (4.25),

$$y = c_1 y_1 + c_2 y_2.$$

□

Now let us discuss the general theory. To simplify the notation, we suppose that $x = 0$ is a regular singular point of a differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

If we multiply this equation by x^2, we obtain

$$x^2y'' + x\,[\,xp(x)\,]\,y' + [\,x^2q(x)\,]\,y = 0 \tag{4.28}$$

where now $xp(x)$ and $x^2q(x)$ are analytic at zero, and thus can be expressed by power series

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \text{and} \qquad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$$

in some interval $(-R, R)$. When substituting these two series and the expected solution

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

$(a_0 \neq 0, x > 0)$ into equation (4.28), one always obtains an expression of the form

$$a_0\,[r(r-1) + p_o r + q_o]\,x^r + \sum_{n=1}^{\infty} [\cdots]x^{n+r} = 0,$$

just as in the last example. The equation

$$r(r-1) + p_o r + q_o = 0$$

is called the *indicial equation* and my have two real solutions. Let us call these solutions r_1 and r_2, labeled so that $r_1 \geq r_2$.

1. **Case I:** If r_1 and r_2 don't differ by an integer, $r_1 \neq r_2 + N$ for some integer N, then equation (4.28) has two linearly independent series solutions

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n.$$

2. **Case II:** If $r_1 = r_2 + N$ for some positive integer N, then equation (4.28) has two linearly independent solutions

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n + c\, y_1(x) \ln x$$

for some constant c (which can be zero.)

3. **Case III:** If $r_1 = r_2$, then equation (4.28) has two linearly independent solutions

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = x^{r_1+1} \sum_{n=0}^{\infty} b_n x^n + y_1(x) \ln x.$$

All these solutions are valid in the interval $(0, R)$.

Example 2 Find the general solution of the equation

$$x^2y'' + (6x + x^2)y' + xy = 0. \tag{4.29}$$

Solution. If we divide by x^2, we obtain the equation

$$y'' + \frac{6+x}{x}y' + \frac{1}{x}y = 0$$

and see that $x = 0$ is a regular singular point. We therefore try the series solution

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

We substitute this series into (4.29) and obtain

$$x^2 \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + (6x+x^2) \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Now multiply the factors x^2, $6x + x^2$ and x into the series,

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 6a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

and change the index of summation in the last two series,

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 6a_n(n+r)x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}(n+r-1)x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0.$$

Now add all these sums,

$$a_0\,[r(r-1)+6r]\,x^r + \sum_{n=1}^{\infty} \Big\{ a_n\big[\,(n+r)(n+r-1)+6(n+r)\,\big] + a_{n-1}\big[\,(n+r-1)+1\,\big]\Big\} x^{n+r} = 0.$$

As all the terms on the left side must be zero, we have the indicial equation

$$\begin{aligned} r(r-1)+6r &= 0 \\ r(r+5) &= 0 \end{aligned} \tag{4.30}$$

and the recurrence relation

$$a_n\big[\,(n+r)(n+r-1)+6(n+r)\,\big] + a_{n-1}(n+r) = 0$$

which can be rewritten as

$$a_n(n+r)(n+r+5) = -a_{n-1}(n+r). \tag{4.31}$$

The indicial equation (4.30) has the two solutions $r_1 = 0$ and $r_2 = -5$. So this is case II.

Let us first choose the larger exponent $r = 0$. The recurrence relation (4.31) becomes

$$a_n = -\frac{a_{n-1}}{n+5} \tag{4.32}$$

and we get

$$\begin{aligned} a_1 &= -\frac{a_0}{6} \\ a_2 &= -\frac{a_1}{7} = \frac{a_0}{7\cdot 6} \\ a_3 &= -\frac{a_2}{8} = -\frac{a_0}{8\cdot 7\cdot 6} \\ a_4 &= -\frac{a_3}{9} = \frac{a_0}{9\cdot 8\cdot 7\cdot 6} \\ &\vdots \end{aligned}$$

In general,

$$a_n = (-1)^n \frac{a_0}{(n+5)(n+4)\ldots 7\cdot 6} = (-1)^n \frac{a_0\cdot 5!}{(n+5)!}.$$

Choosing $a_0 = \frac{1}{5!}$ we have found one series solution

$$y_1 = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n+5)!} = \frac{1}{5!} - \frac{x}{6!} + \frac{x^2}{7!} - \frac{x^3}{8!} + \cdots.$$

Now take $r = -5$. The recurrence relation (4.31) becomes

$$a_n(n-5)n = -a_{n-1}(n-5) \tag{4.33}$$

which for $n \neq 5$ equals

$$a_n = -\frac{a_{n-1}}{n}. \tag{4.34}$$

So if we choose $a_0 = 1$, then we get

$$\begin{aligned} a_1 &= -a_0 = -1 \\ a_2 &= -\frac{a_1}{2} = \frac{1}{2} \\ a_3 &= -\frac{a_2}{3} = -\frac{1}{3\cdot 2} \\ a_4 &= -\frac{a_3}{4} = \frac{1}{4\cdot 3\cdot 2} \end{aligned}$$

Note that for $n = 5$, the recurrence relation (4.34) is not valid, because it was obtained from (4.33) by dividing by $n - 5$ which is zero now. We must look at (4.33) instead. This equation is satisfied for any choice of a_5; the easiest choice is $a_5 = 0$. Then,

$$a_n = 0$$

for all $n \geq 5$. We have found a second series solution

$$y_2 = x^{-5}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right) = \frac{1}{x^5} - \frac{1}{x^4} + \frac{1}{2x^3} - \frac{1}{6x^2} + \frac{1}{24x}.$$

The general solution is

$$y = c_1y_1 + c_2y_2.$$

Remark Note that the recurrence relation (4.34) is identical to the relation (4.32) for $n > 5$, just shifted by 5 indices. So if we choose $a_5 \neq 0$ for y_2, then we obtain

$$\begin{aligned} a_6 &= -\frac{a_5}{6} \\ a_7 &= -\frac{a_6}{7} = \frac{a_5}{7 \cdot 6} \\ a_8 &= -\frac{a_7}{8} = -\frac{a_5}{8 \cdot 7 \cdot 6} \\ a_9 &= -\frac{a_8}{9} = \frac{a_5}{9 \cdot 8 \cdot 7 \cdot 6} \\ &\vdots \qquad \vdots \qquad \vdots \end{aligned}$$

so that

$$y_2 = x^{-5}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right) + (5!)\, a_5\, y_1.$$

Remark It is quite easy to represent the solution in closed form. We can write y_1 as

$$\begin{aligned} y_1 &= -x^{-5}\left(-\frac{x^5}{5!} + \frac{x^6}{6} - \frac{x^7}{7!} + \frac{x^8}{8!} - \cdots\right) = -x^{-5}\sum_{n=5}^{\infty}(-1)^n\frac{x^n}{n!} \\ &= x^{-5}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}\right) \\ &= y_2 - x^{-5}e^{-x}. \end{aligned}$$

It follows that

$$y_3 = x^{-5}e^{-x}$$

is also a solution to the equation. The general solution can now be written as the linear combination

$$y = c_1 y_3 + c_2 y_2 = x^{-5}\left(c_1\, e^{-x} + c_2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right)\right).$$

Example 3 Find the general solution of

$$x^2 y'' + (x^2 - 3x)y' + 3y = 0. \tag{4.35}$$

Solution. Since we can write this equation as

$$y'' + (1 - \frac{3}{x})\, y' + \frac{3}{x^2}\, y = 0$$

we see that $x = 0$ is a regular singular point. We substitute

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into (4.35), where $a_0 \neq 0$. We obtain

$$x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2}$$
$$+ (x^2 - 3x)\sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1} + 3\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Multiplying the factors x^2, x^2-3x and 3 into the series and re-index the last series,

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - 3\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}(n+r-1)x^{n+r} + 3\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Now we add,

$$a_0\Big[\, r(r-1) - 3r + 3 \,\Big]x^r + \sum_{n=1}^{\infty} \Big\{ a_n\Big[(n+r)(n+r-1) - 3(n+r) + 3\Big] + a_{n-1}\Big[\, n+r-1 \,\Big] \Big\}\, x^{n+r} = 0.$$

We have the indicial equation

$$\begin{aligned} r(r-1) - 3r + 3 &= 0 \\ (r-3)(r-1) &= 0 \end{aligned} \tag{4.36}$$

and the recurrence relation

$$a_n\Big[\,(n+r)(n+r-1) - 3(n+r) + 3\,\Big] + a_{n-1}\Big[\, n+r-1 \,\Big] = 0. \tag{4.37}$$

The indicial equation has the two solutions $r_1 = 3$ and $r_2 = 1$. So this is again case II. Let us look at the exponent $r = 3$ first. The recurrence relations (4.37) gives

$$a_n\Big[\,(n+3)(n-1) + 3\,\Big] + a_{n-1}\Big[\, n+2 \,\Big] = 0$$

or

$$a_n(n^2+2n) = -a_{n-1}(n+2)$$

so that

$$a_n = -\frac{a_{n-1}}{n}.$$

We now have

$$\begin{aligned} a_1 &= -a_0 \\ a_2 &= -\frac{a_1}{2} = \frac{a_0}{2} \\ a_3 &= -\frac{a_2}{3} = -\frac{a_0}{3\cdot 2} \\ a_4 &= -\frac{a_3}{4} = \frac{a_0}{4\cdot 3\cdot 2} \\ &\vdots \qquad \vdots \qquad \vdots \end{aligned}$$

or in general,

$$a_n = (-1)^n \frac{a_0}{n!}.$$

Choosing $a_0 = 1$ we have found one series solution

$$y_1 = x^3 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = x^3 - x^4 + \frac{x^5}{2} - \frac{x^6}{6} + \cdots \tag{4.38}$$

We recognize the series as the power series for e^{-x}. Therefore,

$$y_1 = x^3 e^{-x}.$$

Now let us pick the exponent $r = 1$. The recurrence relation (4.37) becomes

$$a_n\Big[\,(n+1)n - 3(n+1) + 3\,\Big] + a_{n-1}\,n \;=\; 0$$

or

$$a_n\Big[\,n^2 + 2n\,\Big] \;=\; -a_{n-1}n$$

so that

$$a_n(n-2) \;=\; -a_{n-1} \tag{4.39}$$

If we choose $a_0 \neq 0$, then

$$a_1 \;=\; -\frac{a_0}{-1} \;=\; a_0$$

But note that for $n = 2$, equation (4.39) becomes

$$0 = -a_1$$

which is never true. What has happened ? There simply does not exist a second series solution of the form

$$y_2(x) \;=\; x^{r_2}\sum_{n=0}^{\infty} b_n x^n.$$

Remember that this is case II, and the second solution should be of the form

$$y_2(x) \;=\; x^{r_2}\sum_{n=0}^{\infty} b_n x^n \;+\; c\,y_1(x)\,\ln x \tag{4.40}$$

In this case, the constant c must be different from 0. So how can we find this second solution y_2 ? Since we have the first solution $y_1 = x^3e^{-x}$ in closed form, we can use the method "Using one Solution to find another". We therefore set

$$y \;=\; x^3e^{-x}v$$

for some function v to be determined. We differentiate twice and we get

$$y' \;=\; x^3e^{-x}v' + (3x^2 - x^3)e^{-x}v$$

and

$$y'' \;=\; x^3e^{-x}v'' + (6x^2 - 2x^3)e^{-x}v' + (6x - 6x^2 + x^3)e^{-x}v.$$

Substituting into (4.35) and simplifying, we get

$$xv'' + (3-x)v' \;=\; 0.$$

This is a separable equation in v', and

$$\frac{v''}{v'} \;=\; \frac{x-3}{x} \;=\; 1 - \frac{3}{x}.$$

Integrating, we obtain

$$\ln v' = x - 3\ln x$$

and exponentiate

$$v' \;=\; x^{-3}e^{x}.$$

Hence,

$$v \;=\; \int x^{-3}e^{x}\,dx.$$

We can not evaluate this integral directly. Instead, we use the powers series again,

$$\begin{aligned} x^{-3}e^{x} &= x^{-3}\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{4}\cdots\right) \\ &= x^{-3}+x^{-2}+\frac{1}{2}x^{-1}+\frac{1}{6}+\frac{1}{24}x+\cdots \end{aligned}$$

and integrate this series,

$$v = \int x^{-3}e^{x}\,dx = -\frac{1}{2x^2}-\frac{1}{x}+\frac{1}{2}\ln x+\frac{1}{6}x+\frac{1}{48}x^2+\cdots.$$

We have found a second solution

$$y_2 = x^3e^{-x}v = x^3e^{-x}\left(-\frac{1}{2x^2}-\frac{1}{x}+\frac{1}{2}\ln x+\frac{1}{6}x+\frac{1}{48}x^2+\cdots\right).$$

The general solution is now of the form

$$y = c_1y_1+c_2y_2.$$

□

Remark Let us bring this solution into the form (4.40). First separate the term containing $\ln x$, and then express e^{-x} as a power series,

$$\begin{aligned} y_2 &= e^{-x}\left(-\frac{1}{2}x-x^2+\frac{1}{6}x^4+\frac{1}{48}x^5+\cdots\right)+\frac{1}{2}x^3e^{-x}\ln x \\ &= \left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\cdots\right)\left(-\frac{1}{2}x-x^2+\frac{1}{6}x^4+\frac{1}{48}x^5+\cdots\right) \\ &= \left(-\frac{1}{2}x-\frac{1}{2}x^2+\frac{3}{4}x^3-\frac{1}{6}x^4+\cdots\right)+\frac{1}{2}x^3e^{-x}\ln x. \end{aligned}$$

Example 4 Solve the differential equation

$$x^2y'' + xy' + x^2y = 0. \tag{4.41}$$

(This is called the *Bessel equation of order zero*)

Solution. Dividing by x^2 we obtain

$$y'' + \frac{1}{x}y' + y = 0$$

and see that $x=0$ is a regular singular point. Therefore, we can expect one series solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

We substitute this series into (4.41) and obtain

$$x^2\sum_{n=0}^{\infty} a_n\,(n+r)(n+r-1)\,x^{n+r-2}+x\sum_{n=0}^{\infty} a_n\,(n+r)\,x^{n+r-1}+x^2\sum_{n=0}^{\infty} a_nx^{n+r} = 0.$$

We multiply the factors x^2 and x into the series, and obtain

$$\sum_{n=0}^{\infty} a_n\,(n+r)(n+r-1)\,x^{n+r} + \sum_{n=0}^{\infty} a_n\,(n+r)\,x^{n+r} + \sum_{n=0}^{\infty} a_n\,x^{n+r+2} = 0.$$

Then we add the first two series, and shift the index of summation in the last series,

$$\sum_{n=0}^{\infty} a_n\,(n+r)^2\,x^{n+r} \;+\; \sum_{n=2}^{\infty} a_{n-2}\,x^{n+r} \;=\; 0.$$

To add again, we must separate the cases $n = 0$ and $n = 1$,

$$a_0 r^2 x^r \;+ a_1(1+r)^2 x^{1+r} \;+\; \sum_{n=2}^{\infty} a_n\,(n+r)^2 \;+ a_{n-2}\,x^{n+r} \;=\; 0.$$

When $n = 0$ we obtain the indicial equation

$$a_0\,r^2 \;=\; 0$$

which has a repeated, single solution

$$r \;=\; r_1 \;=\; r_2 \;=\; 0.$$

Now when $n = 1$ we obtain (using the fact that $r = 0$),

$$a_1\,(1+0)^2 \;=\; 0$$

which is only possible if $a_1 = 0$. When $n \geq 2$ we obtain (again using $r = 0$)

$$a_n\,n^2 + a_{n-2} \;=\; 0$$

which gives the recurrence relation

$$a_n \;=\; -\frac{a_{n-2}}{n^2}.$$

Because $a_1 = 0$ we see that $a_n = 0$ for all *odd* indices n. For *even* indices n we obtain

$$\begin{aligned}
n = 2: &\quad a_2 \;=\; -\frac{a_0}{2^2} \\
n = 4: &\quad a_4 \;=\; -\frac{a_2}{4^2} \;=\; \frac{a_0}{4^2\cdot 2^2} \\
n = 6: &\quad a_6 \;=\; -\frac{a_4}{6^2} \;=\; -\frac{a_0}{6^2\cdot 4^2\cdot 2^2} \\
&\qquad\quad \vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

In general, we can write even integers as $n = 2k$, and then

$$a_{2k} \;=\; (-1)^k\,\frac{a_0}{(2k)^2\cdot(2k-2)^2\ldots 4^2\cdot 2^2}$$

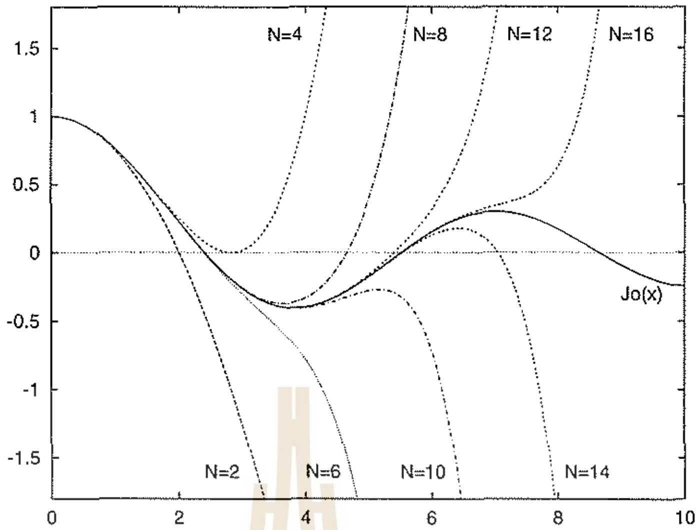
which we can rewrite as

$$a_{2k} \;=\; (-1)^k\,\frac{a_0}{2^2k^2\cdot 2^2(k-1)^2\ldots 2^2 2^2\cdot 2^2 1^2} \;=\; \frac{(-1)^k a_0}{2^{2k}\,(k!)^2}$$

Setting $a_0 = 1$ we have found one power series solution

$$\begin{aligned}
y_1 \;&=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\,(k!)^2}\,x^{2k} \\
&=\; 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147,456} - \cdots
\end{aligned}$$

This is a fast converging series with radius of convergence $R = \infty$. (Check!) The function defined by this series is called the *Bessel function of order zero of the first kind*, and often denoted by $J_o(x)$.



Partial sums of $J_o(x)$, $S_N = a_0 + a_2x^2 + a_4x^4 + \ldots + a_Nx^N$.

Now as the indicial equation has repeated roots $r_1 = r_2 = 0$, the second solution must be of the form

$$y_2 \;=\; x^1 \sum_{n=0}^{\infty} a_n x^n \;+\; y_1 \ln x.$$

If we multiply the factor x into the series, and change the names of the coefficients, we obtain

$$y_2 \;=\; \sum_{n=1}^{\infty} b_n x^n \;+\; y_1 \ln x.$$

To determine the values of the coefficients b_n we substitute this function into equation (4.41),

$$\left[\, x^2 y_1'' \;+\; x y_1' \;+\; x^2 y_1 \,\right] \ln x + 2x y_1' - y_1 + \sum_{n=2}^{\infty} b_n\, n(n-1)\, x^n$$
$$+\; y_1 \;+\; \sum_{n=1}^{\infty} b_n\, n\, x^n \;+\; x^2 \sum_{n=1}^{\infty} b_n\, x^n \;=\; 0.$$

The sum in the brackets is zero, because y_1 is already a solution of the differential equation. So the above can be written as

$$2x \sum_{k=0}^{\infty} \frac{(-1)^k\, 2k}{2^{2k}\, (k!)^2}\, x^{2k-1} \;+\; b_1 x \;+ (2b_2 + 2b_2) x^2$$
$$+\; \sum_{n=3}^{\infty} \Big[\, b_n \left[\, n(n-1) + n \,\right] + b_{n-2} \,\Big]\, x^n \;=\; 0$$

or

$$\sum_{k=0}^{\infty} \frac{(-1)^k\, 4k}{2^{2k}\, (k!)^2}\, x^{2k} \;+\; b_1 x \;+ 4b_2 x^2 + \sum_{n=3}^{\infty} \Big[\, n^2 b_n + b_{n-2} \,\Big]\, x^n \;=\; 0$$

Now we compare coefficients:

- x^1 : $b_1 = 0$.
- x^2 : $-1 + 4b_2 = 0$ so that $b_2 = 1/4$.
- x^n, $n > 1$ odd: The first series contains no odd powers of x, so that

$$n^2 b_n + b_{n-2} = 0$$

and solving for b_n,

$$b_n = -\frac{b_{n-2}}{n^2}.$$

But since $b_1 = 0$, we see that $b_n = 0$ for all odd n.

- x^n, $n > 2$ even: Setting $n = 2k$ and using the recurrence relation, we get

$$\frac{(-1)^k\, 4k}{2^{2k}(k!)^2} + (2k)^2 b_{2k} + b_{2k-2} = 0.$$

Instead of solving this equation for b_{2k} it is better to substitute

$$b_{2k} = \frac{(-1)^{k+1}\, c_{2k}}{2^{2k}(k!)^2}. \tag{4.42}$$

Then we obtain

$$\frac{(-1)^k 4k}{2^{2k}(k!)^2} + (2k)^2 \frac{(-1)^{k+1}\, c_{2k}}{2^{2k}\,(k!)^2} + \frac{(-1)^k\, c_{2k-2}}{2^{2k-2}\,(\,(k-1)!\,)^2} = 0.$$

Now multiply by $(-1)^k 2^{2k} k!$,

$$4k - (2k)^2 c_{2k} + 4k^2 c_{2k-2} = 0$$

and solve for c_{2k},

$$c_{2k} = c_{2k-2} + \frac{1}{k}.$$

By (4.42) the first coefficient is

$$c_2 = 4b_2 = 1$$

and then the remaining coefficients are

$$\begin{aligned} c_4 &= c_2 + \frac{1}{2} = 1 + \frac{1}{2} \\ c_6 &= c_4 + \frac{1}{3} = 1 + \frac{1}{2} + \frac{1}{3} \\ &\vdots \quad \vdots \\ c_{2k} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{k} =: H_k \end{aligned}$$

We have found a second solution

$$\begin{aligned} y_2(x) &= y_1(x)\ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} H_k}{2^{2k}(k!)^2}\, x^{2k} \\ &= y_1(x)\ln x + \frac{x^2}{4} - \frac{3x^4}{128} + \frac{11x^6}{13,824} - \frac{25x^8}{1,769,472} + \cdots \end{aligned}$$

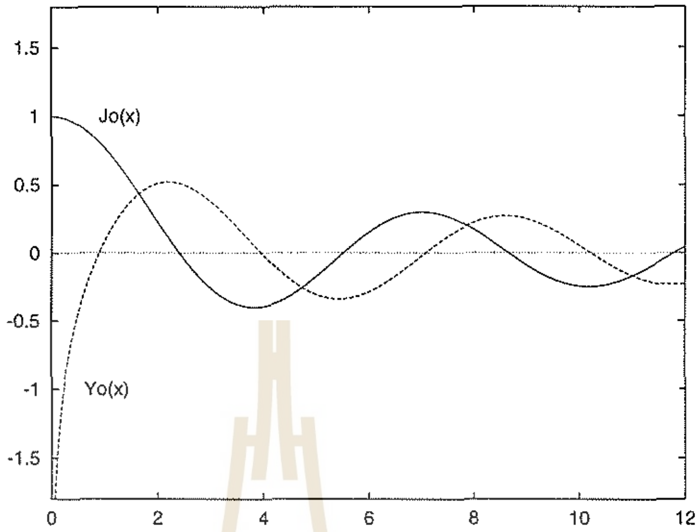
Instead of y_2, one often uses the linear combination

$$Y_o(x) = \frac{2}{\pi}\Big[\,(\gamma - \ln 2)y_1 + y_2\,\Big]$$

as the second solution, where

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) \approx 0.57722.$$

This second function is called the *Bessel function of order zero of the second kind.*



The Bessel functions of order zero.

□

Exercises

1. Find all regular singular points and the root of the indicial equation if
   (a) $xy'' + 2xy' + 6e^x y = 0$
   (b) $x(x-1)y'' + 6x^2y' + 3y = 0$
   (c) $x^2y'' + 3(\sin x)y' - 2y = 0$
   (d) $2x(x+2)y'' + y' - xy = 0$
   (e) $x^2(1-x)y'' - (1+x)y' + 2xy = 0$
   (f) $(4-x^2)y'' + 2xy' + 3y = 0$

2. Use the series method to solve the following Cauchy-Euler equations
   (a) $x^2y'' + 4xy' + 2y = 0$
   (b) $x^2y'' - 3xy' + 4y = 0$
   (c) $x^2y'' - xy' + y = 0$
   (d) $(x-1)^2y'' + 8(x-1)y' + 12y = 0$

   Compare your solution with the solution obtained using the substitution $x = e^t$.

3. Find two linearly independent series solutions:
   (a) $4xy'' + 2y' + y = 0$
   (b) $2xy'' + 3y' - y = 0$
   (c) $2xy'' - y' - y = 0$
   (d) $3xy'' + 2y' + 2y = 0$
   (e) $2x^2y'' + xy' - (1+2x^2)y = 0$
   (f) $2x^2y'' + xy' - (3-2x^2)y = 0$
   (g) $6x^2y'' + 7xy' - (x^2+2)y = 0$
   (h) $3x^2y'' + 2xy' + x^2y = 0$
   (i) $2xy'' + (1+x)y' + y = 0$
   (j) $2xy'' + (1-2x^2)y' - 4xy = 0$
   (k) $xy'' + 2y' + 9xy = 0$
   (l) $xy'' + 2y' - 4xy = 0$
   (m) $4xy'' + 8y' + xy = 0$
   (n) $xy'' - y' + 4x^2y = 0$

4. Find two series solution of
   (a) $xy'' + (3-x)y' - y = 0$
   (b) $xy'' + (5-x)y' - y = 0$
   (c) $xy'' + (5+3x)y' + 3y = 0$
   (d) $xy'' - (4+x)y' + 3y = 0$
   (e) $x^2y'' + (2x+3x^2)y' - 2y = 0$
   (f) $x(1-x)y'' - 3y' + 2y = 0$

5. In each of the following equations, find one series solution y_1. Then use the method "using one solution to find another" to find a second linearly independent solution y_2. (Compute only the first three terms in the series part of y_2.)
   (a) $x^2y'' + xy' + (x-1)y = 0$
   (b) $x^2y'' - xy' + 8(x^2-1)y = 0$
   (c) $xy'' + y' + 2y = 0$
   (d) $x^2y'' - xy' + (x^2+1)y = 0$
   (e) $x^2y'' + (x^2-3x)y' + 4y = 0$
   (f) $x^2y'' + x^2y' - 2y = 0$

6. Use the series method to show that $y_1 = x$ is one solution of

$$x^3y'' - xy' + y = 0.$$

   Then use the method "using one solution to find another" to find the general solution of this equation.

7. Use the power series method to solve the differential equations in chapter II, section 2.3, exercises 3 and 4.

Supplementary exercises:

8. Show that

$$x^2y'' + (\sin x)y' - (\cos x)y = 0$$

   has a regular singular point at $x = 0$, and that the indicial equation has solutions ± 1. Find two linearly independent solutions. (Determine only the first three terms in the series involved).

Appendix A

Mathematics Dictionary

absolute	สัมบูรณ์
acceleration	ความเร่ง
add	บวก
addition	การบวก
altitude	ระดับความสูง
amplitude	แอมพลิจูด
angle	มุม
angular	เชิงมุม
antiderivative	ปฏิยานุพันธ์
applied	ประยุกต์
approximate	ใกล้เคียง, ประมาณ
approximation	ค่าใกล้เคียง, การประมาณ
arbitrary	ไม่เจาะจง
arbitrary constant	ค่าคงตัวไม่เจาะจง
arc	ส่วนโค้ง, อาร์ก
area	พื้นที่
argument	1. ข้อโต้แย้ง 2. อาร์กิวเมนต์
assumption	การสมมุติ, ข้อสมมุติ
asymptote	เส้นกำกับ
at infinity	ณ อนันต์
axis	แกน
- real axis	- แกนจริง
Bernoulli	แบร์นูลลี (นักคณิตศาสตร์ชาวสวิสต์)
bound	ค่าขอบเขต
boundary	ขอบ

boundary value problem	ปัญหาเงื่อนไขค่าขอบเขต
boundary condition	เงื่อนไขขอบเขต
bounded	มีขอบเขต
capacity	ความจุ
centre, center	ศูนย์กลาง
check	การตรวจสอบ
constraint	เงื่อนไขบังคับ
continuity	ความต่อเนื่อง
chain rule	กฎลูกโซ่
characteristic	ลักษณะเฉพาะ
characteristic equation	สมการลักษณะเฉพาะ
circle	วงกลม
circumference	เส้นรอบวง
circumscribe	เขียนล้อม
clockwise	ตามเข็มนาฬิกา, เวียนขวา
closed	ปิด
coefficient	สัมประสิทธิ์
commutative law	กฎการสลับที่
compatibility	ความไม่ขัดแย้งกัน, ความเข้ากันได้
complementary angle	มุมประกอบมุมฉาก
complex	เชิงซ้อน
complex plane	ระนาบเชิงซ้อน
component	ส่วนประกอบ
composition	1. การประกอบ 2. ผลประกอบ

concave	เว้า
concept	แนวคิด
conclusion	ข้อยุติ
condition	ภาวะ, เงื่อนไข
cone	กรวย
conjugate	สังยุค
conjugates	คู่สังยุค
consequence	ผลสืบเนื่อง
constant (n)	ค่าคงตัว
constant function	ฟังก์ชันคงตัว
continuous	ต่อเนื่อง
contradiction	1. ความขัดแย้งกัน, การตั้งข้อขัดแย้ง 2. ความเท็จโดยรูปแบบ
convention	สัญนิยม
convergent	ลู่เข้า
converse	บทกลับ
convolution	ผลการประสาร
coordinate	พิกัด
coordinate axes	แกนพิกัด
corollary	อนุนัย, บทแทรก
correspond	สมนัย
correspondence	การสมนัย
counter-clockwise	ทวนเข็มนาฬิกา, เวียนซ้าย
counter example	ตัวอย่างค้าน
criterion	เกณฑ์
critical	วิกฤต
critical point	จุดวิกฤต
cross	ไขว้
cross product	ผลคูณเชิงเวกเตอร์ (ดู vector product)
cube (n)	ลูกบาศก์
cube root	รากที่สาม
cubic root	รากที่สาม
curvature	ความโค้ง
curve	เส้นโค้ง
cylinder	ทรงกระบอก
cylindrical coordinates	พิกัดทรงกระบอก
damped oscillation	การแกว่งกวัดแบบหน่วง
damping factor	ตัวคูณหน่วง
decay	การลดลง
decimal place	ตำแหน่งทศนิยม
decreasing function	ฟังก์ชันลด
define	นิยาม
definite	กำหนดแน่
definite integral	อินทิกรัลจำกัดเขต
definition	บทนิยาม
degenerate	สภาพเสื่อมคลาย
degree	1. องศา 2. ระดับขั้น
denominator	ตัวส่วน
dependent	ไม่อิสระ
dependent variable	ตัวแปรตาม
derivative	อนุพันธ์
derived	อนุพัทธ์
descriptive	พรรณนา
determinant	ตัวกำหนด
diagonal	แนวทแยงมุม เส้นทแยงมุม
diagram	แผนภาพ
diameter	เส้นผ่านศูนย์กลาง,
differentiable function	ฟังก์ชันที่หาอนุพันธ์ได้
differential (a)	เชิงอนุพันธ์
differential (n)	ผลต่างอนุพัทธ์
differential equation	สมการเชิงอนุพันธ์
differentiation	การหาอนุพันธ์
digit	เลขโดด
dimension	มิติ
distance	ระยะทาง
divide	หาร, แบ่ง
divisible	หารลงตัว
division	การหาร, การแบ่ง

domain	โดเมน
domain (of a function)	เซตฝ่ายให้ (ของฟังก์ชัน) ชุดฝ่ายให้ (ของฟังก์ชัน) โดเมน (ของฟังก์ชัน)
dot product	ผลคูณเชิงสเกลาร์ (ดู scalar product)
double	ทวีคูณ
duration	ช่วงเวลา
dynamic	พลวัต
echelon matrix	เมทริกซ์ลดรูปเป็นขั้น
echelon-reduced matrix	เมทริกซ์ลดรูปเป็นขั้น
elementary row-operation	การดำเนินการเปลี่ยนแถวเชิงธาตุมูล
eliminate	กำจัด
ellipse	วงรี
ellipsoid	ทรงรี
elliptic	เชิงวงรี
endpoint	จุดปลาย
equality	สมภาพ
equation	สมการ
equation of degree n	สมการระดับขั้น n
equidistant	ระยะห่างเท่ากัน
equilateral	ด้านเท่า
equilibrium	สมดุล
equivalent	สมมูล
evaluation	การประเมินค่า
even	คู่
even number	จำนวนคู่
exact	แม่นตรง
exact differential	ผลต่างอนุพันธ์แม่นตรง
example	ตัวอย่าง
excess (n)	ส่วนเกิน
expansion	การขยาย
explicit	โดยชัดแจ้ง
exponent	เลขชี้กำลัง
exponential	ชี้กำลัง
expression	นิพจน์
extension	1. การยืดขยาย 2. ภาคยืดขยาย
extreme	สุดขีด
extreme point	จุดสุดขีด
factor	ตัวประกอบ
factorial	แฟกทอเรียล
family of curves	วงศ์เส้นโค้ง
figure	1. รูป 2. ตัวเลข
finite	จำกัด
fixed	ตรึง
fraction	เศษส่วน
- proper fraction	- เศษส่วนแท้
general solution	คำตอบทั่วไป
generalize	วางนัยทั่วไป
generate (v)	ก่อกำเนิด
geometric; geometrical	เรขาคณิต
geometry	เรขาคณิต
grade	1. ลำดับขั้น 2. เกรด (หน่วยวัดมุม)
gradient	เกรเดียนต์ (มีความหมายเหมือนกับ grad)
gradient of curve	ความชันของเส้นโค้ง
graph	กราฟ
graphical	เชิงกราฟ
growth	การเพิ่มขึ้น
harmonic	ฮาร์มอนิก
helix	เกลียว, ฮีลิกซ์
hemisphere	กึ่งทรงกลม
homogeneous equation	สมการเอกพันธ์
homogeneous function of degree n	ฟังก์ชันเอกพันธ์ดีกรี n
horizontal	แนวนอน, แนวราบ, แนวระดับ
hyperbola	ไฮเพอร์โบลา
hyperbolic	ไฮเพอร์โบลิก

hyperboloid	ไฮเพอร์โบลอยด์
one-sheet hyperboloid	ไฮเพอร์โบลอยด์ เชื่อมโยงกัน
identity	เอกลักษณ์
identity matrix	เมทริกซ์เอกลักษณ์ (มีความหมายเหมือนกับ unit matrix)
image	ภาพ
imaginary	จินตภาพ
imaginary axis	แกนจินตภาพ
implicit	โดยปริยาย
improper fraction	เศษเกิน
improper integral	อินทิกรัลไม่ตรงแบบ
inclination	ความเอียง
inclination, angle of	มุมเท, มุมเอียง
inclusion	การเป็นเซตย่อย, เป็นชุดย่อย
increasing function	ฟังก์ชันเพิ่ม
increment	ส่วนที่เปลี่ยน
indefinite integral	อินทิกรัลไม่จำกัดเขต
independent	อิสระ
independent variable	ตัวแปรต้น
indeterminate form	รูปแบบยังไม่กำหนด
index	ดรรชนี
inequality	อสมการ, การไม่เท่ากัน
infinite	อนันต์
initial	เริ่มต้น
initial condition	เงื่อนไขเริ่มต้น
initial value problem	ปัญหาเงื่อนไขค่าเริ่มต้น
inner product	ผลคูณภายใน
inscribe	เขียนแนบใน, บรรจุภายใน
integer	จำนวนเต็ม (มีความหมายเหมือนกับ integral number)
integrable	อินทิเกรดได้
integral	อินทิกรัล
integrand	ตัวถูกอินทิเกรต
integrate	อินทิเกรต
integration constant	ค่าคงที่ของการอินทิเกรต

integrating factor	ตัวประกอบเพื่ออินทิเกรต
integration by parts	การอินทิเกรตทีละส่วน
interchange	สับเปลี่ยน
interior	ข้างใน
intersect	ตัดกัน,ตัด
intersection	ผลตัด
interval	ช่วง
intuition	สหัชญาณ
inversion	การผกผัน
invertible	หาตัวผกผันได้
isosceles	หน้าจั่ว
iterated; iterative	เกิดซ้ำ,ทำซ้ำ
jump (n)	ค่ากระโดด
known	ตัวรู้ค่า
Laplace	ลาปลาซ (นักคณิตศาสตร์ชาวฝรั่งเศส)
leading coefficient	สัมประสิทธิ์นำ
lemma	ทฤษฎีบทประกอบ
lemniscate	เส้นโค้งเลมนิสเคต
level line	เส้นระดับ
limacon	เส้นโค้งลีมาซอง
limit	ขีดจำกัด, ลิมิต
limit point	จุดลิมิต
line	เส้นตรง (ดู straight line)
line segment	เส้นจำกัด
linear	เชิงเส้น
linear combination (of vectors)	ผลบวกเชิงเส้น (ของเวกเตอร์)
linear equation	สมการเชิงเส้น
linearly independent	อิสระต่อกันในตัวเอง
local	เฉพาะที่, เฉพาะท้องถิ่น
local properties	สมบัติเฉพาะแห่ง (ดู global properties ประกอบ)
locus	โลกัส
log	ล็อก
logarithm	ลอการิทึม

logarithmic	ลอการิทึม
long division	การหารยาว
lower boundary	ขอบล่าง
magnitude	ขนาด
major (a)	ใหญ่
major (n)	อันใหญ่,ส่วนใหญ่
major axis	แกนเอก
map (n)	การส่ง
match	จับคู่
mathematical model	ตัวแบบเชิงคณิตศาสตร์
mathematics	คณิตศาสตร์
matrix	เมตริกซ์
maximal	สูงสุดเฉพาะกลุ่ม ใหญ่สุดเฉพาะกลุ่ม
maximum (a)	สูงสุด,ใหญ่สุด
maximum (n)	ค่าสูงสุด
mean	มัชฌิม, ตัวกลาง, ค่าเฉลี่ย
method	ระเบียบวิธี
metric	เมตริก
metrical coordinates	พิกัดอิงระยะทาง
mid point	จุดกึ่งกลาง
minimal	ต่ำสุดเฉพาะกลุ่ม เล็กสุดเฉพาะกลุ่ม
minimax point	จุดมินิแมกซ์ (มีความหมายเหมือนกับ saddle point)
minimum (a)	ต่ำสุด, เล็กสุด
minimum (a)	ค่าต่ำสุด
minor (a)	น้อย
minor (n)	อันน้อย, ส่วนน้อย
minor axis	แกนโท
minute	1. ลิปดา 2. นาที
mirror image	ภาพกระจกเงา (ดู reflected image)
mistake	ความผิด
modal	ฐานนิยม
mode	ฐานนิยม
model	ตัวแบบ
model, mathematical	ตัวแบบเชิงคณิตศาสตร์
moment	โมเมนต์
moment of inertia	โมเมนต์ความเฉื่อย
monotone function	ฟังก์ชันทางเดียว
monotonic function	ฟังก์ชันทางเดียว
multiple	พหุคูณ
multiplicand	ตัวตั้งคูณ
multiplication	การคูณ
multiplicity	ภาระรากค่าซ้ำ
natural logarithm	ลอการิทึมธรรมชาติ
natural number	จำนวนธรรมชาติ
negative	นิเสธ,ลบ
negative integer	จำนวนเต็มลบ
negligible	ละเลยได้
n-factorial	แฟกทอเรียล n
non-negative	ไม่เป็นลบ
norm	ค่าประจำ
normal (a)	1. แนวฉาก 2. ปรกติ 3. นอร์แมล
normal (n)	เส้นแนวฉาก (มีความหมายเหมือนกับ normal line)
normal curve	เส้นโค้งปรกติ
normal line	เส้นแนวฉาก
normal plane	ระนาบแนวฉาก
n-tuple n	สิ่งที่เป็นอันดับ
number	จำนวน
numeral	ตัวเลข
numerator	ตัวเศษ
oblique	เฉียง
obtuse angle	มุมป้าน
octant	อัฐภาค
odd function	ฟังก์ชันคี่
odd number	จำนวนคี่

operation	การดำเนินการ
operator	ตัวดำเนินการ
order	อันดับ
order (of magnitude)	อันดับขนาด
order of differential equation	อันดับของสมการเชิงอนุพันธ์
ordered pair	คู่อันดับ
ordinary differential equation	สมการเชิงอนุพันธ์สามัญ
origin	จุดกำเนิด
oscillate	แกว่งกวัด
osculating plane	ระนาบสัมผัสประชิด
pairwise	ทีละคู่
parabola	พาราโบลา
paraboloid	พาราโบลอยด์
parallel (a)	ขนาน
parallel (n)	เส้นขนาน
parallelepiped	ทรงสี่เหลี่ยมหน้าขนาน
parallelogram	รูปสี่เหลี่ยมด้านขนาน
parameter	พารามิเตอร์,ตัวแปรเสริม
one parameter family of curves	วงศ์เส้นโค้งพารามิเตอร์หนึ่งตัว
parametric equation	สมการอิงตัวแปรเสริม
parenthesis	วงเล็บ
parity, same	ภาวะเดียวกัน
partial	ย่อย, บางส่วน
partial derivative	อนุพันธ์ย่อย
partial differential equation	สมการเชิงอนุพันธ์ย่อย
partial sum	ผลบวกย่อย
particular solution	ผลเฉลยเฉพาะราย, คำตอบเฉพาะ
partition (v)	แบ่งกั้น
partition (n)	ผลแบ่งกั้น
path	วิถี
pattern	แบบอย่าง
percent	ร้อยละ, เปอร์เซ็นต์
percentage	อัตราร้อยละ
percentage error	ค่าผิดพลาดร้อยละ
perfect	สมบูรณ์
perimeter	เส้นรอบรูป, ความยาวรอบรูป
period	คาบ
periodic	เป็นคาบ
perpendicular (a)	ตั้งฉาก
perpendicular (n)	เส้นตั้งฉาก
piecewise continuous	ต่อเนื่องเป็นช่วง ๆ (มีความหมายเหมือนกับ sectionally continous)
place, value	ค่าประจำหลัก
plane (a)	ราบ
plane (n)	ระนาบ
point	จุด
isolated point	จุดเอกเทศ
ordinary point	จุดสามัญ
singular point	จุดเอกฐาน
polar (a)	เชิงขั้ว
polar (n)	เส้นเชิงขั้ว
polar coordinates	พิกัดเชิงขั้ว
pole	1. ขั้ว 2. โพล (ใช้กับฟังก์ชันตัวแปรเชิงซ้อน)
population	ประชากร
position	ตำแหน่ง
position vector	เวกเตอร์บอกตำแหน่ง
positive	บวก
positive definite	บวกแน่นอน
power series	อนุกรมกำลัง
principal (a)	มุขสำคัญ
principal (n)	เงินต้น
principal axis	แกนมุขสำคัญ
problem	ปัญหา
product	ผลคูณ
proof	ข้อพิสูจน์

proper	แท้
proportion	สัดส่วน,ปฏิภาค
proportional	เป็นสัดส่วน
proposition	ประพจน์
quadrant	จตุภาค
quadric (a)	กำลังสอง
quadric (n)	พื้นผิวกำลังสอง (มีความหมายเหมือนกับ quadric surface)
quantity	ปริมาณ
quarter	หนึ่งในสี่,เสี้ยว
quotient (a)	ปันส่วน
quotient (n)	ผลหาร
radius	รัศมี
radix	ฐาน (ดู base)
range	พิสัย
rank	ค่าลำดับชั้น
ratio	อัตราส่วน
rational	ตรรกยะ
real axis	แกนจริง
real line	เส้นจำนวนจริง (มีความหมายเหมือนกับ real number line)
real number	จำนวนจริง
reciprocal	ส่วนกลับ
reduce	ทดทอน
reflected image	ภาพสะท้อน (มีความหมายเหมือนกับ mirror image)
reflection	การสะท้อน
region	บริเวณ
relative	สัมพัทธ์
removable discontinuity	ความไม่ต่อเนื่องที่ขจัดได้
right angle	มุมฉาก
right circular cone	กรวยกลมตรง
right triangle	รูปสามเหลี่ยมมุมฉาก
root	ราก
saddle point	จุดอานม้า
satisfy	สอดคล้องกับ
scalar	สเกลาร์
scalar product	ผลคูณเชิงสเกลาร์ (มีความหมายเหมือนกับ dot product)
scale	มาตราส่วน, มาตรา, สเกล
score	1. คะแนน 2. ยี่สิบ
sec	เซก (ดู secant)
secant	เซแคนต์
second	1. ที่สอง 2. วินาที 3. ฟิลิปดา
section	1. ภาคตัด 2. ตอน
sectionally continuous	ต่อเนื่องเป็นช่วง ๆ (ดู piecewise continuous)
semi	กึ่ง , ครึ่ง
semicircle	รูปครึ่งวงกลม
semisphere	ครึ่งวงกลม
separable	แยกกันได้
separable variables	ตัวแปรแยกกันได้
separable equation	สมการแบบแยกตัวแปรได้
sequence	ลำดับ
series	อนุกรม
set	เซต, ชุด
sheet	ผิว
shift	เลื่อน
similar	คล้าย
similar triangles	รูปสามเหลี่ยมคล้าย
simple	เชิงเดียว
slope	ความชัน
solid (a)	สามมิติ
solid (n)	ทรงสามมิติ
solution	ผลเฉลย
solve (v)	หาผลเฉลย , แก้
sound (v)	หยั่ง

space	ปริภูมิ
- 3-space	ปริภูมิสามมิติ
sphere	ทรงกลม
spheroid	ทรงคล้ายทรงกลม
spiral (a)	เวียนก้นหอย
spiral (n)	เส้นเวียนก้นหอย
square (a)	1. จัตุรัส 2. กำลังสอง
square (n)	1. รูปสี่เหลี่ยมจัตุรัส 2. กำลังสอง 3. ตาราง
square matrix	เมทริกซ์จัตุรัส
square root	รากที่สอง
stable	เสถียร
statement	ถ้อยแถลง, ข้อความ
straight line	เส้นตรง (มีความหมายเหมือนกับ line)
strategy	ยุทธศาสตร์
strictly increasing function	ฟังก์ชันเพิ่มโดยแท้
strictly positive	บวกโดยแท้
subscript	ดรรชนีล่าง
subtract	ลบ
summand	ส่วนของผลบวก
summation sign; $\sum$ (capital sigma)	เครื่องหมายรวมยอด
symmetric; symmetrical	สมมาตร
symmetry	สมมาตร
system of equations	ระบบสมการ (มีความหมายเหมือนกับ simultaneous equations)
table	ตาราง
tangent (v)	สัมผัส
tangent (n)	1. เส้นสัมผัส 2. แทนเจนต์
tangent line	เส้นสัมผัส (ดู tangent (n))
term	พจน์
theorem	ทฤษฎี
theory	ทฤษฎี
torsion	การบิด, ความบิด
total differential	ผลต่างอนุพัทธ์รวม
totally	โดยสิ้นเชิง
trajectory	แนววิถี
transform (v)	แปลง
transform (n)	ผลการแปลง
transformation	การแปลง
trigonometry	ตรีโกณมิติ
triple integration	การอินทิเกรตสามชั้น
undetermined coefficients	ระเบียบวิธีเทียบสัมประสิทธิ์
uniform	เอกรูป, สม่ำเสมอ
unique	เป็นได้อย่างเดียว
unit (a)	หนึ่งหน่วย
unit (n)	หน่วย
unity	หนึ่ง
unknown	ตัวไม่รู้ค่า
unsigned	ไม่ระบุเครื่องหมาย
value	ค่า
variable (a)	แปรผันได้
variable (n)	ตัวแปร
variable, random	ตัวแปรสุ่ม
vector	เวกเตอร์
vector product	ผลคูณเชิงเวกเตอร์ (มีความหมายเหมือนกับ cross product)
verification	ทวนสอบ
vertical	แนวยืน , แนวตั้ง, แนวดิ่ง
volume	ปริมาตร
well-defined	แจ่มชัด
zero	ศูนย์
zero vector	เวกเตอร์ศูนย์ (มีความหมายเหมือนกับ null vector)

Appendix B

Translation of Word Problems

The following is a translation into the Thai language of some of the applied word problems in chapter 1.

Section 1.7.2

1. กฎของนิวตันเกี่ยวกับการเย็นลงกล่าวว่า วัตถุร้อนจะเย็นลงในอัตราซึ่งเป็นสัดส่วนกับผลต่างระหว่าง อุณหภูมิของวัตถุนั้นและอุณหภูมิรอบข้าง

   วัตถุหนึ่งถูกทำให้ร้อนถึง 110^oC และได้ถูกนำไปวางในที่ซึ่งอุณหภูมิของอากาศเป็น 10^oC หลังจากนั้น 1 ชั่วโมง อุณหภูมิของวัตถุเป็น 60^oC เมื่อใดที่วัตถุจะเย็นตัวลงไปที่อุณหภูมิ 30^oC

2. ขวดน้ำขวดหนึ่งถูกนำออกจากตู้เย็นขณะที่อุณหภูมิ 6^oC และถูกนำไปวางในห้องซึ่งมีอุณหภูมิ 22^oC หลังจากนั้น 10 นาที อุณหภูมิของน้ำสูงขึ้นเป็น 14^oC

   ก) จงหาอุณหภูมิของน้ำหลังจากเวลาผ่านไป 20 นาที
   ข) เมื่อใดอุณหภูมิของน้ำจะสูงขึ้นเป็น 21^oC

3. เรเดียมมี half-life เป็นเวลา 1600 ปี จงคำนวณว่าปริมาณเรเดียมจะยังคงหลงเหลืออยู่กี่เปอร์เซนต์ของปริมาณเริ่มต้นหลังจากเวลาผ่านไป 2400 ปี และหลังจากเวลาผ่านไป 8000 ปี

4. สารกัมมันตภาพรังสีคาร์บอน C-14 มี half-life เป็นเวลา 5568 ปี พืชต่าง ๆ จะสะสมกัมมันตภาพรังสีนี้ในช่วงที่มีชีวิตอยู่ และ C-14 ที่สะสมไว้จะสลายตัวหลังจากพืชเหล่านั้นตาย ซากพืชฟอสซิลชิ้นหนึ่งมี C-14 เพียง 0.2% ของปริมาณเริ่มต้น อยากทราบว่าซากพืชฟอสซิลนี้มีอายุเท่าใด

5. ในปี ค.ศ. 1980 เมือง ๆ หนึ่งมีประชากร 100,000 คน และมี 120,000 คน ในปี ค.ศ. 1990 สมมุติว่า ประชากรเพิ่มแบบชี้กำลัง (exponential) จงคาดคะเนประชากรในปี ค.ศ. 2020 ว่าเป็นเท่าใด

6. ประชากรของประเทศหนึ่งมีอัตราการเจริญเติบโตเป็นค่าคงตัว k และทุกปีมีจำนวนคนย้ายถิ่นฐาน หรืออพยพมา I คน ดังนั้นสมการที่กำหนดสถานการณ์นี้คือ

$$\frac{dx}{dt} = kx + I$$

   ก) ถ้าในปี ค.ศ. 2000 ประเทศนี้มีประชากร 50 ล้านคน อัตราการเจริญเติบโตหรืออัตราการเพิ่ม $k = 4\%$ และจำนวนผู้อพยพในรอบปีเป็น 500,000 คน จะมีประชากรเท่าใดในปี ค.ศ. 2005 และ ค.ศ. 2055

   ข) สมมุติว่าอัตราการเพิ่มเป็น 1% จงคำนวณหาจำนวนประชากรดังคำถามในข้อ ก)

7. โรคติดต่อร้ายแรงได้แพร่ระบาดในเมืองซึ่งมีพลเมือง 15,000 คน เมื่อเวลา $t = 0$ มีประชากร 5,000 คน เป็นโรคนี้ และอัตราการเพิ่มของโรคระบาดนี้เป็น 500 คนต่อวัน จะใช้เวลานานเท่าใดที่พลเมืองอีก 5,000 คน จะเป็นโรคนี้ (เนื่องจากว่าจำนวนพลเมืองที่จะติดเชื้อได้มีจำกัด ดังนั้นเราให้การแพร่กระจายของเชื้อโรคโดยใช้สมการแบบลอจิสติก)

8. ในปฏิกิริยาเคมี $A \to C$ สาร A ถูกทำให้เปลี่ยนเป็นสาร C ด้วยอัตราซึ่งเป็นสัดส่วนกับปริมาณสาร A ที่ปรากฏอยู่ (เรียกว่าปฏิกิริยาอันดับหนึ่ง) หลังจากเวลาผ่านไป 5 นาที ปริมาณสารเคมี A ถูกทำให้เปลี่ยนไป 10% จากปริมาณเริ่มต้น

   ก) หลังจากเวลาผ่านไปแล้ว 20 นาที สารเคมี A จะถูกทำให้เปลี่ยนไปกี่เปอร์เซนต์ของปริมาณเริ่มต้น

   ข) เมื่อใดสารเคมี A จะถูกทำให้เปลี่ยนไป 60 เปอร์เซนต์ของปริมาณเริ่มต้น

9. ในปฏิกิริยาเคมีอันดับหนึ่ง $A \to C$ สาร A ซึ่งไม่ทราบว่ามีปริมาณเริ่มต้นเป็นเท่าใดถูกทำให้เปลี่ยนไปเป็นสาร C หลังจากนั้น 1 ชั่วโมง สาร A เหลืออยู่ 50 กรัม และหลังจากเวลาผ่านไป 3 ชั่วโมง เหลือสาร A เพียง 25 กรัม

   ก) สาร A มีปริมาณเริ่มต้นกี่กรัม

   ข) สาร A จะเหลืออยู่กี่กรัมเมื่อเวลาผ่านไป 5 ชั่วโมง

   ค) หลังจากเวลาผ่านไปเท่าใดสาร A จึงจะเหลือ 2 กรัมเท่านั้น

10. ก้อนหินของดวงจันทร์ถูกค้นพบว่า มีจำนวนอะตอมของโปแตสเซียมและอาร์กอนเท่ากัน สมมุติว่า อาร์กอนทั้งหมดเป็นผลจากปฏิกิริยาการสลายตัวของโปแตสเซียม และพบว่าทุก ๆ หนึ่งในเก้าอะตอมของโปแตสเซียมสลายตัวเป็นอาร์กอนหนึ่งอะตอม จงคำนวณว่าหินนี้มีอายุเท่าใด โดยสมมุติว่า หินก้อนนี้เริ่มต้นมีแต่โปแตสเซียมเท่านั้น (half-life ของโปแตสเซียมเป็น 1.28×10^9 ปี)

Section 1.7.3

1. จงแก้ปัญหาในตัวอย่างที่ 1 โดยสมมุติว่าปริมาณของสาร B มีปริมาณคงที่เท่ากับ 15 กรัม (ตัวอย่างเช่น สาร A เป็นสารที่ละลายในน้ำและอยู่ในภาวะอิ่มตัว) จงหาว่าเวลาใดที่ได้สาร C มีปริมาณ 19 กรัม

2. จงแก้ปัญหาในตัวอย่างที่ 1 โดยสมมุติว่าปริมาณเริ่มต้นของสาร A มี 5 กรัม จงหาว่าเวลาใดที่ได้สาร C มีปริมาณ 19 กรัม

3. อัตราส่วนของสารชนิดหนึ่งที่ละลายในน้ำเป็นสัดส่วนกันกับปริมาณของสารที่ยังไม่ละลาย และเป็นสัดส่วนกันกับผลต่างของ $c_1 - c_2$ เมื่อ c_1 เป็นความเข้มข้นของสารละลายอิ่มตัว และ c_2 เป็นความเข้มข้นที่แท้จริงของสารละลาย ในภาวะอิ่มตัวน้ำ 50 กรัม สามารถละลายสารได้ 20 กรัม

   ก) ถ้าใส่สาร 10 กรัม ลงในน้ำ 50 กรัม แล้วภายใน 30 นาที สารนั้นจะละลายไปครึ่งหนึ่ง จงคำนวณว่าสารนั้นละลายไปเท่าไรเมื่อเวลาผ่านไป 1 ชั่วโมง

   ข) จงตอบคำถามเหมือนข้อ ก) เมื่อใส่สาร 3 กรัม ลงในน้ำ 50 กรัม

Section 1.7.4

1. ถังขนาด 200 ลิตร ใบหนึ่งบรรจุสารละลายชนิดหนึ่งอยู่เต็ม และในสารละลายนี้มีสารเคมีละลายอยู่ 4 kg เติมสารละลายที่มีสารเคมีชนิดเดียวกันนี้ละลายอยู่ 50 g/l ลงไปในถังด้วยอัตรา 2 l/min และ ปล่อยส่วนผสมที่คนให้เข้าเป็นเนื้อเดียวกันแล้วออกจากถังด้วยอัตรา 7 l/min จงหาปริมาณสารเคมีในถังเมื่อมีสารละลายเหลืออยู่ครึ่งถัง

2. ถังใบหนึ่งบรรจุน้ำ 100 l ซึ่งมีเกลือละลายอยู่ 5 kg ถ้าเราต้องการลดความเข้มข้นของน้ำในถังลงไปเป็น 0.01 kg/l โดยการเติมน้ำจืดเข้าถังด้วยอัตรา 20 l/min ในขณะเดียวกันก็ปล่อยส่วนผสมที่คนให้เข้าเป็นเนื้อเดียวกันแล้วออกจากถังด้วยอัตราเท่ากันกับเติมเข้า จงคำนวณว่าจะใช้เวลานานเท่าใด

3. ถังใบหนึ่งบรรจุน้ำบริสุทธิ์ 40 ลิตร เติมน้ำเกลือซึ่งมีเกลือละลายอยู่ 3kg/l เข้าไปในถังด้วยอัตรา 2 l/min คนให้เข้ากันแล้วปล่อยออกจากถังด้วยอัตรา 3 l/min จงหาปริมาณของเกลือในถัง ณ เวลา t ใด ๆ และเมื่อใดที่มีปริมาณของเกลือมากที่สุด

4. อากาศในห้องขนาด 20m x 10m x 3m มีก๊าซ carbondioxide (CO_2) ผสมอยู่ 0.2% ให้เวลาเริ่มต้นเป็น $t = 0$ ทำการปั๊มอากาศนอกห้องซึ่งมีก๊าซ CO_2 ผสมอยู่ 0.05% เข้าไปในห้อง จงหาว่าจะปั๊มอากาศเข้าไปในห้องด้วยอัตราเท่าใด (อัตราคงตัว) เพื่อให้ได้อากาศในห้องที่มีก๊าซ (CO_2) ผสมอยู่ 0.1% เมื่อเวลาผ่านไป 30 นาที

Section 1.7.5

1. ลูกบอลลูกหนึ่งตกลงบนพื้นโลกจากความสูง 1000 เมตร ด้วยความเร็วเริ่มต้น $v_0 = 0$ สมมุติว่าแรงต้านทานของอากาศเป็นสัดส่วนกับความเร็ว v ถ้าความเร็วปลาย (terminal velocity) เป็น 245 เมตรต่อวินาที จงหาความเร็ว ณ เวลา t ใด ๆ

2. ลูกบอลลูกหนึ่งมีมวล 100 กรัม ถูกขว้างขึ้นไปในแนวดิ่งจากจุดที่อยู่สูงจากพื้นดิน 60 เซนติเมตรด้วยความเร็วเริ่มต้น 150 เซนติเมตรต่อวินาที ลูกบอลลอยขึ้นและแล้วก็ตกลงมาสู่พื้นดิน โดยแรงต้านทานของอากาศเป็น $200v$ (หน่วยเป็น dynes)

   ก) จงหาเวลาเมื่อลูกบอลเริ่มตกลงสู่พื้นดิน

   ข) จงหาความเร็วที่ลูกบอลตกลงสู่พื้นดิน

   ค) เมื่อใดที่ลูกบอลตกถึงพื้นดิน (หาค่าประมาณของเวลา)

3. เรือยนต์ลำหนึ่งมีน้ำหนัก 5000 N เครื่องยนต์มีพลังแรงคงที่ 200 N ในทิศทางของการเคลื่อนที่ นั้น ๆ แรงต้านทานของน้ำเท่ากับ 1.5 เท่าของความเร็ว ถ้าเรือเริ่มต้นจากที่จอดพัก

   ก) จงหาความเร็วของเรือหลังจากเวลาผ่านไป 20 วินาที

   ข) จงหาความเร็วของเรือหลังจากเวลาผ่านไป 1 ชั่วโมง

   ค) และหลังจากเวลาผ่านไป 5 นาที เครื่องยนต์พัง (เรือจึงแล่นเรื่อย ๆ โดยไม่ใช้เครื่องยนต์) จงหาความเร็วของเรือหลังจากที่เครื่องยนต์พัง

Appendix C

Solutions to the Exercises

Solutions for Chapter 1

Section 1.3

1. The solutions are
   (a) $y = \frac{1}{3}e^{3x} - \frac{1}{2}x^2 + c$
   (b) $y = \ln|x| + c$
   (c) $y = c\cos x$
   (d) $y = c\sec x$
   (e) $y = e^{cx}$
   (f) $y = \frac{1}{2}(\tan^{-1} x)^2 + c$
   (g) $y = ce^{-x^2}$
   (h) $y = ce^{-\cos x}$
   (i) $y = 1/(x^2 - c)$
   (j) $y = c(1+x)^4$
   (k) $y = \sin(c + \sqrt{x})$
   (l) $y = (2x^{4/3} + c)^{3/2}$
   (m) $(y-1)e^y = cx$
   (n) $y = (c - x)/(1 + cx)$
   (o) $y^4 = x^4/(cx^4 - 1)$
   (p) $3\cos y = c - x^3$
   (q) $1 - y^2 = ce^{-2x}(x+2)^4$
   (r) $r = c/\sin^2\theta$

2. The solutions are
   (a) $y = e^x(x-1) + 3$
   (b) $y = \frac{1}{2}\ln\left[\frac{2}{3}e^{3x} + \frac{1}{3}\right]$
   (c) $y = \frac{1}{8}\ln\frac{|x^2-4|}{3x^2}$
   (d) $\dfrac{e^{y+1}}{y+1} = xe^x$
   (e) $\tan y = 2\sin 2x - 4x + \frac{\pi}{3}$

Section 1.3.1

1. The solutions are
   (a) $x^2 - 2xy - y^2 = c$
   (b) $y = x(c + \ln|x|)^2$
   (c) $y^2 = x^2(\ln|x| + c)$
   (d) $y = x(\ln|x| + c)$
   (e) $y = cx^3 - x$
   (f) $y^2 = x^2 + cx^4$
   (g) $y = x^3/(c - x^2)$
   (h) $y = x\cos^{-1}(c - \ln|x|)$
   (i) $y = x\ln(2\ln|x| + c)$
   (j) $ue^{yu/x^2} = cx^3$ where $u = y + \sqrt{x^2+y^2}$

Section 1.4

1. The solutions are
   (a) $y = 1 + ce^{-x}$
   (b) $y = xe^{-x} + ce^{-x}$
   (c) $y = e^{3x} + ce^{2x}$
   (d) $y = c/x + (\sin x)/x$
   (e) $y = x^4 + cx^3$
   (f) $y = e^{-x}[\tan^{-1}(e^x) + c]$
   (g) $y = \frac{1}{1+x^2}[\ln|\sin x| + c]$
   (h) $y = x^2e^{-x} + x^2 - 2x + 2 + ce^{-x}$
   (i) $y = (x^2 + c)\csc x$
   (j) $y = cx^2 - x^3$
   (k) $y = 1/x - \cot x + (c\csc x)/x$
   (l) $y = (3x^2 + c)e^{x^2}$
   (m) $y = (x^3 + c)/\ln|x|$
   (n) $y = x^2[1 + ce^{1/x}]$

2. The solutions are
   (a) $y = 1$
   (b) $y = 2x + x\ln x$
   (c) $y = 1 - e^{-\sin x}$
   (d) $y = x^5/4 - 56/x^3$
   (e) $y = e^{x+x^2/2} - 1$
   (f) $y = \frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}$

3. These are all Bernoulli equations.
   (a) $y^3 = 3 + ce^{-3x^2}$
   (b) $y = 1/(x+cx^2)^3$
   (c) $y^3 = xe^{-x} + ce^{-x}$
   (d) $y^2 = 1/(cx^2 - x^4)$
   (e) $y^3 = 3\sin x + 9x^{-1}\cos x - 18x^{-2}\sin x - 18x^{-3}\cos x + cx^{-3}$
   (f) $y = 1/(cx - x\ln|x|)$
   (g) $y^2 = e^{2x}/(c + \ln|x|)$

4. We substitute
   (a) $u = e^y, \quad y = \ln|x^2(e^{2x}+c)|$
   (b) $u = \sin^2 y, \quad \sin^2 y = cx^3 - 4x^2$
   (c) $u = (x+e^y), \quad (x+e^y)^2 = 2x^2 + c$

5. Section 1.3, exercise 1, equations (c), (d), (g), (h), (j) and section 1.3.1, exercise 1, equation (e) are linear.

Section 1.5

1. The solutions are
   (a) $y = (c - x^2)/(3x-4)$
   (b) $x^3 - 2y^2x + 2y^3 = c$
   (c) $2y^2x - 2x^2 + 5x + 4y - y^2 = c$
   (d) not exact.
   (e) $\tan y \cos x = x - c.$
   (f) $x^3 + 3y\ln x + y^3 = c$
   (g) $e^x \sin y + x\tan y = c$
   (h) $xe^y + \sin x \cos y = c$
   (i) $y = x^2 - (c - \frac{3}{2}x^2)^{2/3}$
   (j) $\theta^2 = c\csc r - 1$

2. An integrating factor and the solution are
   (a) $y^{-2}, \quad x^2 = cy - y^2$
   (b) $x^{-1}, \quad y^2 + 2\ln|x| - 2xy = c$
   (c) $e^{2y}/y, \quad xe^{2y} = c + \ln|y|$
   (d) $x^2, \quad x^4/4 + x^3y^2 = c$
   (e) $y, \quad y^2x - e^y(y^2 - 2y + 2) = c$
   (f) $y^2, \quad x^4 + 3xy + y^4 = c$
   (g) $y^{-2}, \quad 2yx^2 + x + 3y^3 - cy = 0$
   (h) $\sin y, \quad e^x\sin y + y^2 = c$
   (i) $y^{-4}, \quad y^2 - x^2 = cy^3$
   (j) exact, $\quad xy + \sin(xy) = c$
   (k) exact, $\quad x^4 - 4xy - y^4 = c$
   (l) exact, $\quad x^3(1+\ln y) - y^2 = c$

Section 1.6

1. The solutions are
   (a) $y = x^2 - c/x + d$
   (b) $y = x\sin x + c\sin x + \cos x + d$
   (c) $y = c\tan(cx + d)$
   (d) $cy + \ln|cy - 1| = c^2x + d$
   (e) $y = 1/(4c) + c(x+d)^2$
   (f) $y = x^2 + c\ln|x| + d$
   (g) $y^2 = cx + d$
   (h) $y = d \pm \sqrt{c^2 - x^2}$
   (i) $2y + \sqrt{4y^2 \pm c^2} = de^{\pm 2x}$ (Compare to the method of chapter 2 !)
   (j) $y = -\frac{1}{2}x^2 + cx - c^2\ln|x+c| + d$

2. The solutions are
   (a) $y = 1;\ y = 1 - x^3/3$
   (b) $y = 1/(1-x)$
   (c) $y = -\ln(1-x)$

3. The solutions are
   (a) $y = \ln|\sec(x+c)| + d$
   (b) $y = \ln(1 + ce^{2x}) - x + d$

Section 1.7.1

1. The solutions are
   (a) $y^2 - x^2 = c$
   (b) $x^2 + (y-c)^2 = c^2$
   (c) $x - y = c(x+y)^3$
   (d) $x^2 - 2cy = c^2 \quad (c > 0)$
2. The solutions are
   (a) $y^2 + 2x = c$ (b) $(x + 2 - y) = ce^{-y}$ (c) $3x + y^2 = c\sqrt{y}$
3. The solutions are
   (a) $y - 3x = c$
   (b) $\ln\sqrt{x^2+y^2} + \tan^{-1}(y/x) = c$

Section 1.7.2

1. Temperature $T = 10 + 100/2^t$, time $t \approx 2.32$ hours.
2. (a) 18°C, (b) 40 min.
3. 35.36%.
4. 49,921 years.
5. 207,360 people.
6. (a) 63,837,672 (551,563,344), (b) 55,127,110 (123,325,302)
7. $\frac{20}{3}\ln 4 \approx 9.242$ days.
8. (a) 34.39%, (b) 43.49 min.
9. $A(t) = 50(\sqrt{2})^{1-t}$, $A(0) = 50\sqrt{2}$, $A(5) = 12.5$, $t \approx 10.29$ hours
10. $x(t) = x(0)\, e^{-5.415\cdot 10^{-10}\cdot t}$, $T = 4.252 \cdot 10^9$ years.

Section 1.7.3

1. $x(t) = 20(1 - e^{-0.0192t})$, 156.02 min.
2. $x = 20 - 1800/(2t + 90)$, 855 min.
3. Let $x(t)$ denote the amount of substance dissolved at time t.
   (a) $x(t) = 20\,\frac{1-e^{-kt}}{2-e^{-kt}}$ where $k = \frac{\ln 3 - \ln 2}{30} \approx 0.01352$, $x(60) \approx 7.14$g
   (b) $x(t) = 60\,\frac{1-e^{-kt}}{3-2e^{-kt}}$ where $k = \frac{\ln 10 - \ln 9}{30} \approx 0.10536$, $x(60) \approx 8.26$g

Section 1.7.4

1. $x(t) = 10{,}000 - 100t - 0.0006\sqrt{(100-t)^7}$, $x(50) = 4470$
2. $5\ln 5 \approx 8.047$ min.
3. $x(t) = 120 - 3t - \frac{3}{1600}(40-t)^3$, $t \approx 16.9$ min
4. 22 m^3/min.

Section 1.7.5

1. $v(t) = 245(1 - e^{-0.04t})$
2. (a) $t = 0.1334$ sec, (b) $v(t) = 490.5 - 640.5e^{-2t}$ ($t > 0.1334$ sec), (c) 0.563 sec.
3. (a) 7.765 m/sec (b) 133.331 m/sec, $v(t) = 133.\overline{3}(1 - e^{-0.003t})$m/sec
   (c) $v(t) = 79.12e^{-0.003t}$m/sec (choose $t = 0$ at breakdown)

Section 1.7.6

1. Steady state: $\frac{E_o}{R^2+\omega^2L^2}[R\sin\omega t - \omega L\cos\omega t]$ Transient: $\frac{E_o\omega L}{R^2+\omega^2L^2}e^{-(R/L)t}$

2. The solutions are
   (a) $I = e^{-t/RC}\left[\int \frac{dE}{dt}\frac{1}{R}e^{t/RC}\,dt + c\right]$
   (b) $I = E_o/Re^{-t/RC}$
   (c) transient: $E_o\left[\frac{1}{R} - \frac{\omega C}{1+\omega^2C^2R^2}\right]e^{-t/RC}$
   steady state: $\frac{E_o\omega C}{1+\omega^2C^2R^2}[\cos\omega t + RC\omega\sin\omega t]$.

Section 1.9

1. The solutions are
   (a) $2ye^{-y} = e^{-2x} + c$
   (b) $y = c\csc x$
   (c) $y = (c - \frac{x}{2})\csc x + \frac{1}{2}\cos x$
   (d) $2\sqrt{xy} + \sin x - \cos y = c$
   (e) $y = \frac{1}{2}x^2 - x + ce^{-x} + d$
   (f) $y = 16/(cx - x^3)^2$
   (g) $y = \dfrac{cx}{1+(1-c)x}$
   (h) $x^3y^3 + x^2 + y^2 = c$
   (i) $y = \frac{1}{2}x(c - e^{-x^2})$
   (j) $e^{x+y} + x + y^2 = c$
   (k) $y^3 + cy = 3x + d$
2. The solutions are
   (a) $y = -\sqrt{1+3/x^2}$
   (b) $y = \sqrt{5x^8 - x^2}$
   (c) $y = 1$ (best solved as Bernoulli equation)
   (d) $\sin y = (1-x)e^{-x}$
3. The substitutions and solutions are
   (a) $u = y^{-2}$, $\sin(y^{-2}) = x^2 + c$
   (b) $u = x - y$, $\tan(x-y) = c - x$

4. (c) 5. (d) 6. (b) 7. (e) 8. (c) 9. (b) 10. (c) 11. (b)

Solutions for Chapter 2

Section 2.1

1. The solutions are
   (a) $6 - 2i$
   (b) $23 + 2i$
   (c) $32 - 4i$
   (d) $\frac{2}{13} - \frac{3}{13}i$
   (e) $-\frac{7}{41} + \frac{22}{41}i$.
   (f) $-33i$.
   (g) $\frac{17}{20} - \frac{11}{20}i$.
   (h) $-47.2 - 23i$.
   (i) $-10 - 24i$.
3. Where no numbers are given we assume that $z = x + iy$.
   (a) $1/2$
   (b) $31/50$
   (c) 16
   (d) $\frac{2xy}{x^2+y^2}$
   (e) $x^2 - y^2$
   (f) $-46/13$
   (g) x^2
   (h) $i(3x^2y - y^3)$
   (i) y^3
   (j) $-0.007 + 0.24i$
5. Where no numbers are given we assume that $z = x + iy$.
   (a) 0.2
   (b) 2.5
   (c) $(x^2 + y^2)^2$
   (d) $(x^2 + y^2)^2$
   (e) 1
   (f) 1
   (g) 1
   (h) $\sqrt{1 + \frac{4x}{(x-1)^2+y^2}}$
   (i) $8/17$
6. The solutions are
   (a) $2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2e^{i\pi/2}$
   (b) $2(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}) = 2e^{-i\pi/2}$
   (c) $\sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}e^{i\pi/4}$
   (d) $3(\cos\pi + i\sin\pi) = 3e^{i\pi}$
   (e) $2(\cos 2\pi/3 + i\sin 2\pi/3) = 2e^{i2\pi/3}$
   (f) $\sqrt{2}(\cos 5\pi/4 + i\sin 5\pi/4) = \sqrt{2}e^{i5\pi/4}$
   (g) $10(\cos 0.927 + i\sin 0.927) = 10e^{0.927i}$
   (h) $2(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}) = 2e^{-i\pi/6}$
   (i) $(\cos\pi/2 + i\sin\pi/2) = e^{i\pi/2}$
   (j) $\frac{1}{4}(\cos\pi/4 + i\sin\pi/4) = \frac{1}{4}e^{i\pi/4}$
   (k) $3(\cos\pi + i\sin\pi) = 3e^{i\pi}$
   (l) $.563(\cos .308 + i\sin .308) = .5631e^{.308i}$

7. The solutions are
   (a) $2+2\sqrt{3}i$.
   (b) $-2+2i$.
   (c) $9.21+3.89i$.
   (d) $-0.227-0.974i$.

8. The solutions are
   (a) $\pm 2i$.
   (b) $-1-i, i$
   (c) $\pm 1, \pm i$
   (d) $\pm\sqrt[4]{2}, \pm\sqrt[4]{2}i$
   (e) $3+2i, 2-i$.
   (f) $\pm(1+i), \pm(2+i)$.

9. The solutions are
   (a) $e^{2i\pi/3}, e^{4i\pi/3}, 1$.
   (b) $\pm 1, \pm i, \frac{1}{\sqrt{2}}(\pm 1 \pm i)$.
   (c) $\sqrt{2}e^{-i\pi/8}, \quad \sqrt{2}e^{7i\pi/8}$.
   (d) $\sqrt{2}e^{i\pi/6}, \quad \sqrt{2}e^{7i\pi/6}$.

10. These are the functions $e^{rx}\cos sx$ and $e^{rx}\sin sx$.

Section 2.2

1. The solutions are
   (a) $y = c_1x + c_2$.
   (b) $y = c_1 + c_2e^{2x}$.
   (c) $y = c_1e^x + c_2e^{-x}$.
   (d) $y = c_1x + c_2e^x$.
   (e) $y = c_1 + c_2e^{-x}$

2. the solutions are
   (a) $x^2y'' - 2xy' + 2y = 0$.
   (b) $y'' - k^2y = 0$.
   (c) $y'' + k^2y = 0$.
   (d) $y'' - 2y' = 0$.
   (e) $(1 - x\cot x)y'' - xy' + y = 0$.
   (f) $y'' - 2y' + y = 0$.
   (g) $y'' + 2y' - 3y = 0$.
   (h) $x^2y'' + xy' - y = 0$.

3. The solutions are
   (a) $y = \ln|x+c| + d$.
   (b) No. Not a linear equation.

Section 2.3

1. The solutions are
   (a) $y = c_1 + c_2x^{-2}$.
   (b) $y = c_1x^2 + c_2x^{-2}$.

2. The solutions are
   (a) $y = c_1\sin 2x + c_2\cos 2x$.
   (b) $y = c_1e^x + c_2e^{-x}$.
   (c) $y = c_1e^x + c_2x^2e^x$.

3. The solutions are
   (a) $y = c_1x + c_2e^x$.
   (b) $y = c_1x + c_2x^{-2}$.
   (c) $y = c_1x + c_2xe^x$.

4. $y = c_1x^{-1/2}\sin x + c_2x^{-1/2}\cos x$.

Section 2.4

1. The solutions are
   (a) $y = c_1e^{2x} + c_2e^{-3x}$.
   (b) $y = c_1e^{-x} + c_2xe^{-x}$.
   (c) $y = c_1\cos 2\sqrt{2}x + c_2\sin 2\sqrt{2}x$.
   (d) $y = e^x(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x)$.
   (e) $y = c_1e^{2x} + c_2xe^{2x}$.
   (f) $y = c_1e^{5x} + c_2e^{4x}$.
   (g) $y = e^{-x/2}\left(c_1\cos\frac{\sqrt{5}}{2}x + c_2\sin\frac{\sqrt{5}}{2}x\right)$.
   (h) $y = c_1e^{3x/2} + c_2xe^{3x/2}$.
   (i) $y = c_1 + c_2e^{-x}$.
   (j) $y = e^{3x}(c_1\cos 4x + c_2\sin 4x)$.
   (k) $y = c_1e^{-5x/2} + c_2exe^{-5x/2}$.
   (l) $e^{-x}(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x)$.
   (m) $y = c_1e^{2x} + c_2e^{-2x}$.
   (n) $y = e^x(c_1\cos\frac{\sqrt{3}x}{2} + c_2\sin\frac{\sqrt{3}x}{2})$.
   (o) $y = c_1e^{x/2} + c_2e^{-x}$.
   (p) $y = c_1e^{x/4} + c_2xe^{x/4}$.
   (q) $y = e^{-2x}(c_1\cos x + c_2\sin x)$.
   (r) $y = c_1e^x + c_2e^{-5x}$

2. The solutions are
   (a) $y = e^{3x-1}$.
   (b) $y = e^x + 2e^{5x}$.
   (c) $y = 5xe^{3x}$.
   (d) $y = e^{-2x}(\cos x + 2\sin x)$.
   (e) $y = e^{(-2+\sqrt{2})x} - 2e^{(-2-\sqrt{2})x}$.
   (f) $y = \frac{9}{5}e^{x-1} + \frac{1}{5}e^{-9(x-1)}$.

Section 2.5.1

1. The solutions are
   (a) $y = c_1e^{3x} + c_2e^{-x} - e^{2x}$.
   (b) $y = c_1e^{3x} + c_2e^{-x} + \frac{3}{16}xe^{-x} + \frac{3}{8}x^2e^{-x}$.
   (c) $y = c_1e^{-x}\cos 2x + c_2e^{-x}\sin 2x + \frac{3}{17}\sin 2x - \frac{12}{17}\cos 2x$.
   (d) $y = c_1 + c_2e^{-2x} + \frac{3}{2}x - \frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x$.
   (e) $y = c_1\cos 3x + c_2\sin 3x + \frac{1}{162}(9x^2 - 6x + 1)e^{3x} + \frac{2}{3}$.
   (f) $y = c_1e^{-x} + c_2xe^{-x} + x^2e^{-x}$.
   (g) $y = c_1e^{-x} + c_2e^{-x/2} + x^2 - 6x + 14 - \frac{3}{10}\sin x - \frac{9}{10}\cos x$.
   (h) $y = c_1\cos x + c_2\sin x - \frac{1}{3}x\cos 2x - \frac{5}{9}\sin 2x$.
   (i) $y = c_1\cos\omega_o x + c_2\sin\omega_o x + (\omega_o^2 - \omega^2)^{-1}\cos\omega x$.
   (j) $y = c_1\cos\omega_o x + c_1\sin\omega_o x + \frac{1}{2\omega_o}x\sin\omega_o x$.
   (k) $y = c_1e^{x/2}\cos(\sqrt{15}x/2) + c_2e^{x/2}\sin(\sqrt{15}x/2) + \frac{1}{4}e^x - \frac{1}{6}e^{-x}$.
   (l) $y = c_1e^{-x} + c_2e^{2x} + \frac{1}{6}xe^{2x} + \frac{1}{8}e^{-2x}$.
   (m) $y = c_1\sin 2x + c_2\cos 2x + x\sin 2x + 2\cos x - 1 - x + 2x^2$.
   (n) $y = c_1\sin 3x + c_2\cos 3x - \frac{1}{3}x\cos 3x + \frac{1}{2}\sin x - 2e^{-2x} + 3x^3 - 2x$.

2. (a) $y = e^x - \frac{1}{2}e^{-2x} - x - \frac{1}{2}$.
   (b) $y = \frac{7}{10}\sin 2x - \frac{19}{40}\cos 2x + \frac{1}{4}x^2 - \frac{1}{8} + \frac{3}{5}e^x$.
   (c) $y = 4xe^x - 3e^x + \frac{1}{6}x^3e^x + 4$.
   (d) $y = e^{3x} + \frac{2}{3}e^{-x} - \frac{2}{3}e^{2x} - xe^{2x}$.
   (e) $y = 2\cos 2x - \frac{1}{8}\sin 2x - \frac{3}{4}x\cos 2x$.
   (f) $y = e^{-x}\cos 2x + \frac{1}{2}e^{-x}\sin 2x + xe^{-x}\sin 2x$.

3. (a) $y_p = x(Ax^4 + Bx^3 + Cx^2 + Dx + E) + x(Fx^2 + Gx + H)e^{-3x} + K\sin 3x + L\cos 3x$.
   (b) $y_p = Ax + B + x(Cx + D)\sin x + x(Ex + F)\cos x$.
   (c) $y_p = e^x(A\cos 2x + B\sin 2x) + (Cx + D)e^{2x}\cos x + (Ex + F)e^{2x}\sin x$.
   (d) $yp = Ae^{-x} + x(Bx^2 + Cx + D)e^{-x}\cos x + x(Ex^2 + Fx + G)e^{-x}\sin x$.
   (e) $y_p = Ax^2 + Bx + C + x^2(Dx + E)e^{2x} + (Fx + G)\cos 2x + (Hx + I)\sin 2x$.
   (f) $y_p = x(Ax^2 + Bx + C)\sin 2x + x(Dx^2 + Ex + F)\cos 2x$.
   (g) $y_p = (Ax^2+Bx+C)e^x\sin 2x + (Dx^2+Ex+F)e^x\cos 2x + e^{-x}(G\cos x + H\sin x) + Ke^x$.
   (h) $y_p = x(Ax + B)e^{-x}\cos 2x + x(Cx + D)e^{-x}\sin 2x + (Ex + F)e^{-2x}\cos x + (Gx + H)e^{-2x}\sin x$.

4. (a) $y = \begin{cases} t, & 0 \le t \le \pi \\ -(1 + \frac{\pi}{2})\sin t - \frac{\pi}{2}\cos t + \frac{\pi}{2}e^{\pi - t}, & t > \pi \end{cases}$
   (b) $y = \begin{cases} \frac{1}{5} - \frac{1}{10}e^{-t}\sin 2t - \frac{1}{5}e^{-t}\cos 2t & 0 \le t \le \pi/2 \\ -\frac{1}{5}(1 + e^{\pi/2})e^{-t}\cos 2t - \frac{1}{10}(1 + e^{\pi/2})e^{-t}\sin 2t & t > \pi/2 \end{cases}$

Section 2.5.2

1. The following are the solutions y_p which one gets using the method of undetermined coefficients. Using the method of variation of coefficients, one may obtain different y_p. Why is this not wrong ?
   (a) $y_p = e^x$.
   (b) $y_p = \frac{3}{2}x^2e^{-x}$.
   (c) $y_p = -\frac{2}{3}xe^{-x}$.
   (d) $y_p = 2x^2e^{x/2}$.

2. (a) $y = c_1\cos x + c_2\sin x - (\cos x)\ln(\tan x + \sec x)$.
   (b) $y = c_1e^{-2x} + c_2xe^{-2x} - e^{-2x}\ln x$.
   (c) $y = c_1\cos 3x + c_2\sin 3x + (\sin 3x)\ln(\tan 3x + \sec 3x) - 1$.
   (d) $y = c_1\cos 2x + c_2\sin 2x + \frac{3}{4}(\sin 2x)\ln\sin 2x - \frac{3}{2}x\cos 2x$.
   (e) $y_p = -\frac{1}{4}\cos 2x\ln(\sec 2x + \tan 2x)$.
   (f) $y = c_1\cos(x/2) + c_2\sin(x/2) + x\sin(x/2) + 2[\ln\cos(x/2)]\cos(x/2)$.
   (g) $y_p = \frac{1}{2}x^2e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$.
   (h) $y = c_1e^x + c_2xe^x - \frac{1}{2}e^x\ln(1 + x^2) + xe^x\arctan x$.
   (i) $y_p = -e^{-x}(8x^2 + 4x + 1)$.
   (j) $y_p = \frac{1}{2}xe^{-x}\sin 2x + \frac{1}{4}e^{-x}\cos 2x\log(\cos 2x)$.
   (k) $y_p = \frac{1}{10}e^{-3x}$.
   (l) $y_p = e^x\ln(1 + e^{-x}) - e^x + e^{2x}\ln(1 + e^{-x})$.

3. $y_p = -\frac{3}{2}x^{1/2}\cos x$.

4. (a) $y_p = x\sin x + \cos x\ln(\cos x)$.
   (b) $y_p = \cos x\ln(\csc x + \cot x) - 2$.
   (c) $y_p = \frac{1}{2}\cos x\ln(\sec x + \tan x) - \frac{1}{2}\sin x\ln(\csc x + \cot x)$.
   (d) $y_p = \frac{1}{4}(x^2\sin x + x\cos x - \sin x)$.
   (e) $y_p = -\cos x\ln(\sec x + \tan x)$.
   (f) $y_p = x\cos x - \sin x - \sin x\ln(\cos x)$.
   (g) $y_p = -\sin x\ln(\csc x + \cot x) - \cos x\ln(\sec x + \tan x)$.

Section 2.6

1. The solutions are
   (a) $y = c_1x + c_2x^3$.
   (b) $y = c_1x^{1/2} + c_2x^{3/2}$.
   (c) $y = c_1\sin(\ln x^2) + c_2\cos(\ln x^2)$.
   (d) $y = c_1x^2 + c_2x^{1/3}$.
   (e) $y = c_1\cos(\ln x^3) + c_2\sin(\ln x^3)$
   (f) $y = (c_1 + c_2\ln x)x^{1/3}$.
   (g) $y = x^3(c_1\cos(\ln x) + c_2\sin(\ln x))$
   (h) $y = c_1x^2 + c_2x^{-2}$
   (i) $y = c_1x^2 + c_2x^2\ln x$
   (j) $y = x^2(c_1\cos(\ln x^3) + c_2\sin(\ln x^3))$
   (k) $y = c_1\cos(\ln x^2) + c_2\sin(\ln x^2)$
   $+ x(\frac{2}{5}\ln x - \frac{4}{25})$
   (l) $y = c_1\sin(\ln x) + c_2\cos(\ln x)$
   $- 2\ln x\cos(\ln x)$.
   (m) $y = x^2(c_1\cos(\ln x) + c_2\sin(\ln x) + 5)$
   (n) $y_p = c_1x^2 + \frac{c_2}{x} + \frac{1}{2} + x^2\ln x$.
   (o) $y = c_1x^2 + c_2x^2\ln x + \frac{1}{6}x^2(\ln x)^3$.

2. The solutions are
   (a) $y = \frac{3}{x^2} + 2x^5$.
   (b) $y = x^3 - 2x^2$
   (c) $y = -\frac{1}{x} + \frac{2}{x^3}$.
   (d) $y = x^2 - 2x + 4 + 1/x$
   (e) $y = \frac{5x}{3} - 2x^2 + 3x^3 - \frac{23x^4}{24}$.
   (f) $y = \frac{2}{x^3} - x^2 + 2x^2\ln x$
   (g) $y = 4x^2 - 2x^3$
   (h) $y = \frac{1}{18}x^3 + \frac{1}{12x^2} - \frac{1}{6}\ln x + \frac{1}{36}$

3. (a) $y = c_1(x+2)^3 + \frac{c_2}{x+2}$.

Section 2.7

1. (a) $y = 5\cos(2t - \phi), \quad \phi = \arctan\frac{4}{3} \approx 0.9273$.
   (b) $y = 2\cos(t - \frac{2\pi}{3})$.
   (c) $y = 2\sqrt{5}\cos(3t - \phi), \quad \phi = -\arctan\frac{1}{2} \approx -0.4636$.
   (d) $y = \sqrt{13}\cos(\pi t - \phi), \quad \phi = \pi + \arctan\frac{3}{2} \approx 4.1244$.

2. The solutions are
   (a) $y = -2\sin 8t \sin t$.
   (b) $y = 2\sin\frac{t}{2}\cos\frac{13t}{2}$.
   (c) $y = 2\cos\frac{3\pi}{2}t\cos\frac{\pi}{2}t$.
   (d) $y = 2\sin\frac{7}{2}t\cos\frac{1}{2}t$.

3. Lower by $\sqrt{2}$, higher, unchanged.

4. $y = \frac{5}{7}\sin 14t$ cm, $\quad t = \frac{\pi}{14}$ sec.

5. $y = \frac{1}{4\sqrt{2}}\sin(8\sqrt{2}t) - \frac{1}{12}\cos(8\sqrt{2}t)$ ft, $\omega = 8\sqrt{2}$ rad/sec, $\quad T = \frac{\pi}{4\sqrt{2}}$ sec, $\quad D = \sqrt{\frac{11}{288}}$ ft, $\phi = \pi - \arctan\frac{3}{\sqrt{2}} \approx 2.0113$.

6. $y = e^{-10t}(2\cos 4\sqrt{6}t + \frac{5}{\sqrt{6}}\sin 4\sqrt{6}t)$ cm, $\quad \mu = 4\sqrt{6}$ rad/sec, $\quad T_d = \frac{\pi}{2\sqrt{6}}$ sec, $T_d/T = \frac{7}{2\sqrt{6}} \approx 1.4289$,
   $y = \frac{7}{\sqrt{6}}e^{-10t}\cos(4\sqrt{6}t - \phi)$ cm, $\quad \phi = \arctan\frac{5}{2\sqrt{6}} \approx 0.7956$ sec, $\quad \tau \geq 0.3912$ sec.

7. $y \approx 0.057198e^{-0.15t}\cos(3.87008t - 0.50709)$ m,
   $\mu = 3.87008$ rad/sec, $\mu/\omega_o = 3.87008/\sqrt{15} \approx 0.99925$.

9. $r = \sqrt{A^2 + B^2}, \quad r\cos\theta = B, \quad r\sin\theta = -A, \quad R = r, \quad \phi = \theta + (4n+1)\frac{\pi}{2}, \ (n = 0, 1, 2, \dots)$.

11. (a) $y'' + 10y' + 98y = 2\sin(t/2), \quad y(0) = 0, \quad y'(0) = 0.08$,
    (b) $y = \frac{160}{153281}[-\cos(t/2) + \frac{391}{20}\sin(t/2)]$, $\quad$ (c) $\omega = 4\sqrt{3}$ rad/sec.

12. $y = \frac{\sqrt{2}}{6}\cos(3t - 3\pi/4)$ m.

13. Steady state: $\frac{8}{901}(30\cos 2t + \sin 2t)$ ft. $\quad$ Replace by $m = 4$ slugs.

14. $y = e^{-ct/2m}(c_1\cos\mu t + c_2\sin\mu t) + \dfrac{F_o}{\Delta^2}[m(\omega_o^2 - \omega^2)\sin\omega t - c\omega\cos\omega t]$,
    $\mu = (4km - c^2)^{1/2}/2m, \quad \omega_o^2 = k/m, \quad \Delta^2 = m^2(\omega_o^2 - \omega^2)^2 + c^2\omega^2$,

    (a) $c_1 = y_o + \dfrac{c\omega F_o}{\Delta^2}, \qquad c_2 = \dfrac{cy_o}{2m\mu} + \dfrac{F_o\omega}{\mu\Delta^2}\left[\dfrac{c^2}{2m} - m(\omega_o^2 - \omega^2)\right]$,

    (b) $c_1 = \dfrac{c\omega F_o}{\Delta^2}, \qquad c_2 = \dfrac{u_o'}{\mu} + \dfrac{F_o\omega}{\mu\Delta^2}\left[\dfrac{c^2}{2m} - m(\omega_o^2 - \omega^2)\right]$,

    (c) $c_1 = y_o + \dfrac{c\omega F_o}{\Delta^2}, \qquad c_2 = \dfrac{u_o'}{\mu} + \dfrac{cy_o}{2m\mu} + \dfrac{F_o\omega}{\mu\Delta^2}\left[\dfrac{c^2}{2m} - m(\omega_o^2 - \omega^2)\right]$.

15. $$y = \begin{cases} F_o/m(t - \sin t), & 0 \leq t \leq \pi \\ F_o/m[(2\pi - t) - 3\sin t], & \pi < t <\leq 2\pi \\ -4F_o/m\sin t, & 2\pi < t < \infty \end{cases}$$

16. $Q = 10^{-6}\cos 2000t$ coulombs.

17. $Q = 10^{-6}(2e^{-500t} - e^{-1000t})$ coulombs.

18. $R = 200\sqrt{2}$ ohms.

19. $Q(t) = 10^{-6}(e^{-4000t} - 4e^{-1000t} + 3)$ coulombs, $\qquad Q(t) \to 3 \cdot 10^{-6}$ as $t \to \infty$.

20. If $E(t) = E_o \sin \omega t$, then by equation (2.79), $I_p = I_o \sin(\omega t - \phi)$ with $I_o = E_o/Z$, $Z = \sqrt{R^2 + S^2}$ and $S = \omega L - \frac{1}{\omega C}$. So I_o is largest when $S = 0$, that is, $C = \frac{1}{\omega^2 L}$.

21. (a) $I = 5e^{-2t} \sin t$.
    (b) $I = e^{-4t}(1.5 \cos 3t - (10/3) \sin 3t) - 1.5 \cos 10t + 1.6 \sin 10t$
    (c) $I = e^{-3t}(3 \sin 4t - 4 \cos 4t) + 4 \cos 5t$

22. Solve for Q first,
    (a) $I = \dfrac{275\omega}{25 - \omega^2}(\cos \omega t - \cos 5t)$
    (b) $I = 50 \sin 10t$
    (c) $I = 0.001 \sin 20t + 0.02t$.
    (d) $I = 5 \sin 5t$
    (e) $I = \frac{3}{5} \sin 3t - \frac{2}{5} \sin 2t$

23. (a) $I = \begin{cases} \sin t & 0 < t < 1 \\ \sin t - \sin(t-1) & t > 1 \end{cases}$

    (b) $I = \begin{cases} 1 - \cos t & 0 < t < 1 \\ (\cos 1 - 1) \cdot \cos t + \sin 1 \cdot \sin t & t > 1 \end{cases}$

    (c) $I = \begin{cases} \frac{1}{2}(e^{-t} - \cos t + \sin t) & 0 < t < \pi \\ -\frac{1}{2}(1 + e^{-\pi}) \cos t + \frac{1}{2}(3 - e^{-\pi}) \sin t & t > \pi \end{cases}$

Section 2.8

1. The solutions are
   (a) $y = c_1 + c_2 e^x + c_3 e^{2x}$.
   (b) $y = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x)$.
   (c) $y = c_1 e^x + e^{-x/2}(c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2})$.
   (d) $y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^{-x}$.
   (e) $y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$.
   (f) $y = (c_1 + c_2 x)e^{ax} + (c_3 + c_4 x)e^{-ax}$.
   (g) $y = (c_1 + c_2 x) \cos ax + (c_3 + c_4 x) \sin ax$.
   (h) $y = (c_1 + c_2 x)e^{-x} + c_3 \cos x + c_4 \sin x$.
   (i) $y = e^x[(c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x] + e^{-x}[(c_5 + c_6 x) \cos x + (c_7 + c_8 x) \sin x]$.
   (j) $y = c_1 e^{-x} + e^{x/2}(c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2})$.
   (k) $y = (c_1 + c_2 x + c_3 x^2)e^{-x}$.
   (l) $y = e^{x/\sqrt{2}}(c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}) + e^{-x/\sqrt{2}}(c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}})$.
   (m) $y = (c_1 + c_2 x)e^x + e^{-2x}(c_3 \cos x + c_4 \sin x)$.
   (n) $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$.
   (o) $y = c_1 e^{2x} + (c_2 + c_3 x + c_4 x^2)e^{-x}$.
   (p) $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} + c_5 \cos x + c_6 \sin x$.
   (q) $y = c_1 + c_2 e^{2x} + e^{-x}(c_3 \cos \sqrt{3}x + c_4 \sin \sqrt{3}x)$.
   (r) $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 e^{-x} + c_5 x e^{-x} + c_6 x^2 e^{-x}$.
   (s) $y = (c_1 + c_2 x)e^{2x} + (c_3 + c_4 x)e^{-2x} + c_5 e^{6x}$.

2. In some exercises, only a particular solution is given.
   (a) $y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + \frac{1}{2} x e^{-x} + 3$.
   (b) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - 3x - \frac{1}{4} x \sin x$.
   (c) $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{2} x e^{-x} + 4(x - 1)$.
   (d) $y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x + \frac{1}{12} \sin 2x$.
   (e) $y_p = -\ln \cos x - (\sin x) \ln(\sec x + \tan x)$.

(f) $y = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^{2x} - \frac{1}{3}e^x - \frac{1}{48}x^4 - \frac{1}{16}x^2$.

(g) $y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x + 3 - \frac{1}{4}x^2 \cos x$.

(h) $y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + \frac{x^4}{24} - \frac{x^3}{6}$.

(i) $y_p = -x^2/2$.

(j) $y_p = e^{4x}/30$.

(k) $y = (x-4)\cos x - (\frac{3}{2}x + 4)\sin x + 3x + 4$.

(l) $y = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2$.

(m) $y = 1 + \frac{1}{4}(x^2 + 3x) - xe^x$.

3. The particular solutions are

   (a) $y_p = x(Ax^3 + Bx^2 + Cx + D) + Ex^2 e^x$.

   (b) $y_p = x(Ax + B)e^{-x} + C\cos x + D\sin x$.

   (c) $y_p = Ax^2 e^x + B\cos x + C\sin x$.

   (d) $y_p = Ax^2 + (Bx + C)e^x + x(D\cos 2x + E\sin 2x)$.

   (e) $y_p = x(Ax^2 + Bx + C) + (Dx + E)\cos x + (Fx + G)\sin x$.

   (f) $y_p = Ae^x + (Bx + C)e^{-x} + xe^{-x}(D\cos x + E\sin x)$.

4. These are Euler equations.

   (a) $y = c_1 + c_2 x + c_3 x^{-1}$.

   (b) $y = c_1 x + c_2 \cos(\ln x) + c_3 \sin(\ln x)$.

   (c) $y_p = x^4/15$.

5. $y = \frac{1}{2}(\cosh x - \cos x) + \frac{1}{2}(\sinh x - \sin x)$.

Solutions for Chapter 3

Section 3.1

1. The limits a), c), f), h) converge are zero. The limit in e) equals 1. All other limits don't exist (and equal ∞.)

2. The solutions are as follows:

   (a) 2,

   (b) diverges,

   (c) 1/2,

   (d) diverges,

   (e) $\pi/2$,

   (f) $2/e$.

3. The integrals (a) and (b) converge; (c), (d) and (e) diverge.

Section 3.2

1. (c) is continuous, (a) and (d) are piecewise continuous, (b) is not piecewise continuous.

2. The Laplace transforms are

   (a) $\dfrac{1}{(s-a)^2}$ (b) $\dfrac{2}{s^3}$ (c) $\dfrac{s^2 - 2s + 2}{s^3}$ (d) $\dfrac{s^2 + a^2}{(s^2 - a^2)^2}$

Section 3.3

1. Use the fact that $\cos x = \frac{d}{dx}(\sin x)$, $\sinh x = \frac{d}{dx}(\cosh x)$, $\frac{d^n}{dx^n}(x^n) = n!x^{n-1}$ etc.

2. Use the trigonometric formulas $\cos^2 ax = \frac{1}{2}(1 + \cos 2ax)$ and $\sin^2 ax = \frac{1}{2}(1 - \cos 2ax)$. You will get

$$\mathfrak{L}\{\sin^2 ax\} = \frac{1}{2}\Big(\frac{1}{s} - \frac{s}{s^2 + 4a^2}\Big) \qquad \text{and} \qquad \mathfrak{L}\{\cos^2 ax\} = \frac{1}{2}\Big(\frac{1}{s} + \frac{s}{s^2 + 4a^2}\Big).$$

   Since $\sin^2 ax + \cos^2 ax = 1$, you also get $\mathfrak{L}\{\sin^2 ax\} + \mathfrak{L}\{\cos^2 ax\} = \mathfrak{L}\{1\} = 1/s$.

3. The solutions are
   (a) $\dfrac{10}{s}$,
   (b) $\dfrac{120}{s^6} + \dfrac{s}{s^2+4}$,
   (c) $\dfrac{2}{s-3} - \dfrac{5}{s^2+25}$,
   (d) $\dfrac{4}{s^2+4} + \dfrac{2}{s+1}$.

4. In some of these exercises, you have to use partial fractions decomposition.
   (a) $5x^3$,
   (b) $2e^{-3x}$,
   (c) $2x^2 + 3\sin 2x$,
   (d) $1 - e^{-x}$,
   (e) $x - \sin x$,
   (f) $\frac{1}{3}(e^{3x} - 1)$
   (g) $\frac{3}{5}(1 - e^{-5x})$
   (h) $\frac{1}{4}(1 - \cos 2x) = \frac{1}{2}\sin^2 x$
   (i) $\frac{1}{9}(6\sin 3x - \cos 3x + 1)$
   (j) $x - \sin x$
   (k) $\frac{1}{9}(\cosh 3x - 1)$
   (l) $\frac{1}{a^3}(\sinh ax - ax)$
   (m) $\frac{1}{2}(e^{-2x} - 2e^{-x} + 1)$

5. Use the rule for the derivative of the Laplace transform.

6. The solutions are
   (a) $y = \frac{1}{5}(e^{3x} + 4e^{-2x})$.
   (b) $y = 2e^{-x} - e^{-2x}$.
   (c) $y = e^{-2x} - e^{-x} + e^{x}$
   (d) $y = \frac{1}{8}[\cos 2x + 4\sin 2x + 2x^2 - 1]$
   (e) $y = \frac{1}{10}[2e^{3x} + 2e^{-x} - 2\sin x + \cos x]$
   (f) $y = \cosh x$.
   (g) $y = \cos\sqrt{2}x$.
   (h) $y = (\omega^2-4)^{-1}[(\omega^2-5)\cos\omega x + \cos 2x]$.

Section 3.4.1

1. The transforms are
   (a) $\dfrac{1}{(s-3)^2}$
   (b) $\dfrac{24}{(s-\pi)^5}$
   (c) $\dfrac{2}{(s+1)^3}$
   (d) $\dfrac{2a(3s^2-a^2)}{(s^2+a^2)^3}$
   (e) $\dfrac{a}{(s-c)^2+a^2}$
   (f) $\dfrac{s-c}{(s-c)^2+a^2}$
   (g) $\dfrac{2a(s-c)}{[(s-c)^2+a^2]^2}$
   (h) $\dfrac{(s-c)^2-a^2}{[(s-c)^2+a^2]^2}$

2. The solutions are
   (a) $\frac{3}{2}\sin 2x$
   (b) $2x^2e^x$
   (c) $\frac{2}{5}e^x - \frac{2}{5}e^{-4x}$
   (d) $\frac{9}{5}e^{3x} + \frac{6}{5}e^{-2x}$
   (e) $2e^{-x}\cos x$
   (f) $2\cosh 2x - \frac{3}{2}\sinh 2x$
   (g) $2e^x\cos x + 3e^x\sin x$
   (h) $3 - 2\sin 2x + 5\cos 2x$
   (i) $-2e^{-2x}\cos x + 5e^{-2x}\sin x$
   (j) $2e^{-x}\cos 3x - \frac{5}{3}e^{-x}\sin 3x$

3. Use partial fraction decomposition.
   (a) $e^{-x}(x - x^2)$
   (b) $\frac{1}{36}e^{2x/3}(8\cos\frac{4}{3}x - 5\sin\frac{4}{3}x)$
   (c) $e^{3x}(3\cos 4x + \frac{7}{2}\sin 4x)$
   (d) $\frac{1}{25}(e^{5x} - 1 - 5x)$
   (e) $\frac{1}{125}\left(e^{2x}(5x-2) + e^{-3x}(2+5x)\right)$
   (f) $\frac{1}{16}(\sinh 2x - \sin 2x)$
   (g) $e^{4x}(1 + 12x + 24x^2 + \frac{32}{3}x^3)$
   (h) $\frac{1}{3}(2\cos 2x + 2\sin 2x - 2\cos x - \sin x)$.
   (i) $\frac{1}{32}\left(e^{2x}(2x-1) + e^{-2x}(2x+1)\right)$

4. The solutions are
   (a) $y = e^x\sin x$.
   (b) $y = e^{2x} - xe^{2x}$.
   (c) $y = 2e^x\cosh\sqrt{3}x - (2/\sqrt{3})e^x\sinh\sqrt{3}x$.
   (d) $y = 2e^{-x}\cos 2x + \frac{1}{2}e^{-x}\sin 2x$.
   (e) $y = xe^x - x^2e^x + \frac{2}{3}x^3e^x$.
   (f) $y = \frac{1}{5}(\cos x - 2\sin x + 4e^x\cos x - 2e^x\sin x)$.
   (g) $y = \frac{1}{5}(e^{-x} - e^x\cos x + 7e^x\sin x)$.
   (h) $y = 2e^{-x} + xe^{-x} + 2x^2e^{-x}$.
   (i) $y = \frac{1}{4}(1 - 2e^{2x} + e^{4x})$
   (j) $y = \frac{1}{8}(3\sinh 2x - 6x)$
   (k) $y = \frac{1}{50}\left(e^{-x}(5x-1) + e^{-2x}(\cos 3x + 32\sin 3x)\right)$
   (l) $y = \frac{1}{50}(2e^{2x} + (10x-2)\cos x - (5x+14)\sin x)$

5. Evaluate the integral defining the Laplace transform, and substitute $u = cx$.

6. Take the derivative of $F(s)$ or integrate $F(s)$ (in the last question).
   (a) $-2x^{-1}\sinh 2x$ (c) $x^{-1}(e^{-2x}+e^{3x}-2\cos x)$ (e) $2x^{-1}(1-\cos x)$
   (b) $2x^{-1}(\cos 2x-\cos x)$ (d) $x^{-1}e^{-2x}\sin 3x$ (f) $\frac{1}{8}(x\sin x - x^2\cos x)$

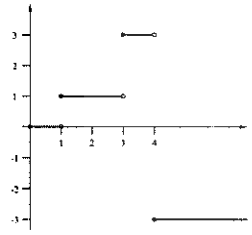
7. If you apply the Laplace transform, you will get first order equations in $Y(s)$. The transforms of the equations and the solutions are

   (a) $(s+1)Y'(s)+4Y(s)=0, \quad y=Cx^3e^{-x}, \quad C\neq 0$
   (b) $(s^2+3s)Y'(s)+3sY(s)=0, \quad y=Cx^2e^{-3x}, \quad C\neq 0$
   (c) $(s-2)Y'(s)+3Y(s)=0, \quad y=Cx^2e^{2x}, \quad C\neq 0$
   (d) $(s^2+2s)Y'(s)+(4s+4)Y(s)=0, \quad y=C(1-x-e^{-2x}-xe^{-2x}), \quad C\neq 0$
   (e) $(s^2+1)Y'(s)+4sY(s)=0, \quad y=C(\sin x - x\cos x), \quad C\neq 0$
   (f) $(s^2+4s+13)Y'(s)+(4s+8)Y(s)=0, \quad y=Ce^{-2x}(\sin 3x-3x\cos 3x), \quad C\neq 0$

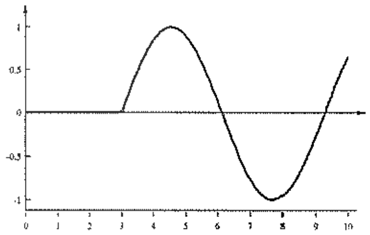
   Recall: Once you have found one particular solution, you can find the general solution by reducing the order of the equation.

Section 3.4.2

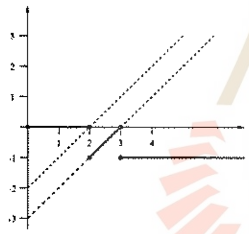
1. The graphs are



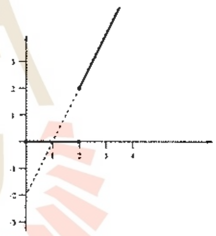
(a)



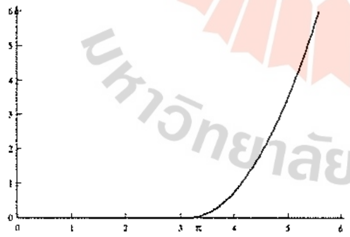
(d)



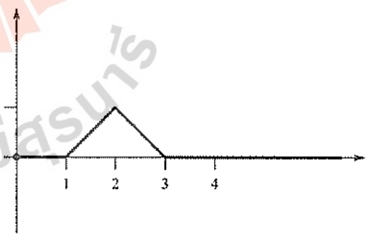
(b)



(e)



(c)



(f)

2. The solutions are
   (a) $F(s)=2(1-e^{-3s})/s$
   (b) $F(s)=3(e^{-s}-e^{-4s})/s$
   (c) $F(s)=(1-e^{-2\pi s})/(s^2+1)$
   (d) $F(s)=s(1-e^{-2s})/(s^2+\pi^2)$
   (e) $F(s)=(1+e^{-3\pi s})/(s^2+1)$
   (f) $F(s)=2(e^{-\pi s}-e^{-2\pi s})/(s^2+4)$
   (g) $F(s)=\pi(e^{-2s}+e^{-3s})/(s^2+\pi^2)$
   (h) $F(s)=2\pi(e^{-3s}+e^{-5s})(4s^2+\pi^2)$
   (i) $F(s)=e^{-s}(s^{-1}+s^{-2})$
   (j) $F(s)=(1-e^{-s})/s^2$
   (k) $F(s)=(1-2e^{-s}+e^{-2s})/s^2$
   (l) $F(s)=\dfrac{1-e^{-as}-ase^{-as}}{s^2(1-e^{-2as})}$

3. The solutions are
   (a) $f(x) = x^3 e^{2x}$
   (b) $f(x) = \frac{1}{3}u_2(x)[e^{x-2} - e^{-2(x-2)}]$
   (c) $f(x) = 2u_2(x)e^{x-2}\cos(x-2)$
   (d) $f(x) = u_2(x)\sinh 2(x-2)$
   (e) $f(x) = u_1(x)e^{2(x-1)}\cosh(x-1)$
   (f) $f(x) = u_1(x) + u_2(x) - u_3(x) - u_4(x)$
   (g) $f(x) = u_3(x)(x-3)$
   (h) $f(x) = u_1(x)e^{-2(x-1)}$
   (i) $f(x) = -u_\pi(x)\sin x$
   (j) $f(x) = (1 - u_{2\pi}(x))\sin x$
   (k) $f(x) = (1 - u_3(x))\cos \pi x$
   (l) $f(x) = 2\cos 2x[u_\pi(x) - u_{2\pi}(x)]$

4. The solutions are
   (a) $i(t) = e^{-10t} - u_1(t)e^{-10(t-1)}$
   (b) $i(t) = (1 - u_{2\pi}(t))\sin 100t$
   (c) $i(t) = \frac{10}{99}(\cos 10t - \cos 100t)$ if $t < \pi$; $i(t) = 0$ if $t > \pi$
   (d) $i(t) = \frac{1}{50}(1 - e^{-50t})^2 - \frac{1}{50}u_1(t)[1 + 98e^{-50(t-1)} - 99e^{-100(t-1)}]$
   (e) $i(t) = \frac{1}{50}(1 - e^{-50t}) - te^{-50t}$ if $0 \le t < 1$; $i(t) = \frac{1}{50}(-e^{-50t} + e^{-50(t-1)}) - te^{50t} - 49(t-1)e^{-50(t-1)}$ if $t \ge 1$.

5. The solutions are
   (a) $y(t) = \frac{1}{2}(1 - u_\pi(t))\sin^2 t$
   (b) $y(t) = \frac{1}{12}(3 + e^{-4t} - 4e^{-t})$ if $t < 2$; $y(t) = \frac{1}{12}(e^{-4t} - 4e^{-t} - e^{-4(t-2)} + 4e^{-(t-2)})$ if $t \ge 2$
   (c) $y(t) = \frac{1}{8}(1 - u_{2\pi}(t))(\sin t - \frac{1}{3}\sin 3t)$
   (d) $y(t) = t - \sin t$ if $t < 1$; $y(t) = -\sin t + \sin(t-1) + \cos(t-1)$ if $t > 1$
   (e) $y(t) = \frac{1}{4}[t - 1 + (t+1)e^{-2t} + u_2(t)(1 - t + (3t-5)e^{-2(t-2)})]$

6. The solutions are
   (a) $y = 1 - \cos x + \sin x - u_{\pi/2}(x)(1 - \sin x)$
   (b) $y = e^{-x}\sin x + \frac{1}{2}u_\pi(x)(1 + e^{-(x-\pi)}\cos x + e^{-(x-\pi)}\sin x) - \frac{1}{2}u_{2\pi}(x)(1 - e^{-(x-2\pi)}\cos x - e^{-(x-2\pi)}\sin x)$
   (c) $y = \frac{1}{6}[1 - u_{2\pi}(x)](2\sin x - \sin 2x)$
   (d) $y = \frac{1}{6}(2\sin x - \sin 2x) - \frac{1}{6}u_\pi(x)(2\sin x + \sin 2x)$
   (e) $y = 1 - u_1(x)[1 - e^{-(x-1)} - (x-1)e^{-(x-1)}]$
   (f) $y = e^{-x} - e^{-2x} + u_2(x)[\frac{1}{2} - e^{-(x-2)} + \frac{1}{2}e^{-2(x-2)}]$
   (g) $y = \cos x + u_\pi(x)(1 + \cos x)$
   (h) $y = h(x) + u_{\pi/2}(x)h(x - \pi/2)$, $h(x) = \frac{4}{25}(-4 + 5x + 4e^{-x/2}\cos x - 3e^{-x/2}\sin x)$
   (i) $y = x - u_1(x)[x - 1 - \sin(x-1)]$
   (j) $y = h(x) + u_\pi(x)h(x-\pi)$, $h(x) = \frac{4}{17}(-4\cos x + \sin x + 4e^{-x/2}\cos x + e^{-x/2}\sin x)$
   (k) $y = \cos 2x + u_\pi(x)h(x-\pi) - u_{2\pi}(x)h(x-2\pi)$, $h(x) = (1 - \cos 2x)/4$
   (l) $y = u_1(x)h(x-1) - u_2(x)h(x-2)$, $h(x) = -1 + (\cos x + \cosh x)/2$
   (m) $y = h(x) - u_\pi(x)h(x-\pi)$, $h(x) = (3 - 4\cos x + \cos 2x)/12$

Section 3.4.3

1. The solutions are
   (a) $y = \frac{1}{2}\sin 2x$
   (b) $y = \frac{1}{2}\sin 2x$ if $x < \pi$ and $y = \sin 2x$ if $x \ge \pi$.
   (c) $y = \frac{1}{4}(1 - e^{-2x}) - \frac{1}{2}xe^{-2x} + u_2(x)(x-2)e^{-2(x-2)}$

(d) $y = x - 2 + 2e^{-x} + 3xe^{-x}$

(e) $y = 0$ if $0 \le x < \pi$ and $y = -2e^{-(x-\pi)} \sin x$ if $x \ge \pi$

(f) $y = -\frac{1}{3}u_{3\pi}(x) \sin 3x + \frac{1}{6}x \sin x$

(g) $y = (2 - e^{2\pi}u_\pi(x) + e^{4\pi}u_{2\pi}(x))e^{-2x} \sin x$

(h) $y = (5x + 2)e^{-x} - u_2(x)(x - 2)e^{-(x-2)}$

Section 3.4.4

1. $F(s) = \dfrac{1}{s(1 + e^{-as})}$
2. $G(s) = \dfrac{1}{as^2} - \dfrac{e^{-as}}{s(1 - e^{-as})}$
3. $F(s) = \dfrac{e^{-as}}{s(1 - e^{-as})}$
4. $F(s) = \dfrac{k}{(s^2 + k^2)(1 - e^{-\pi s/k})}$
5. $G(s) = \dfrac{k}{s^2 + k^2} \coth \dfrac{\pi s}{2k}$
6. $i(t) = u_1(t)e^{-100(t-1)} - u_2(t)e^{-100(t-2)}$

Section 3.4.5

1. The solutions are

(a) $f(x) = \frac{1}{2}x^2$

(b) $f(x) = (e^{ax} - ax - 1)/a^2$

(c) $f(x) = \frac{1}{2}(\sin x - x \cos x)$

(d) $f(x) = 2(x - \sin x)$

(e) $f(x) = xe^{ax}$

(f) $f(x) = (e^{ax} - e^{bx})/(a - b)$

2. The solutions are

(a) $f(x) = \frac{1}{6}\int_0^x (x - u)^3 \sin u \, du$

(b) $f(x) = \int_0^x e^{-(x-u)} \cos 2u \, du$

(c) $f(x) = \frac{1}{2}\int_0^x (x - u)e^{-(x-u)} \sin 2u \, du$

(d) $f(x) = \int_0^x \sin (x - u)g(u) \, du$

3. The solutions are

(a) $f(x) = \frac{1}{2}e^{-x}(5 \sin x - 3x \cos x - 2x \sin x)$

(b) $f(x) = \frac{1}{64}e^{x/2}((4x + 8) \cos x + (4 - 3x) \sin x)$

4. The solutions are

(a) $y = \frac{1}{\omega} \sin \omega t + \frac{1}{\omega}\int_0^t \sin \omega(t - u)g(u) \, du$

(b) $y = \int_0^x e^{-(x-u)} \sin (x - u) \sin \alpha u \, du$

(c) $y = \frac{1}{8}\int_0^x e^{-(x-u)/2} \sin 2(x - u)g(u) \, du$

(d) $y = e^{-x/2} \cos x - \frac{1}{2}e^{-x/2} \sin x + \int_0^x e^{-(x-u)/2} \sin (x - u)[1 - u_\pi(u)] \, du.$

(e) $y = 2e^{-2t} + te^{-2t} + \int_0^t (t - u)e^{-2(t-u)}g(u) \, du.$

(f) $y = 2e^{-x} - e^{-2x} + \int_0^x [e^{-(x-u)} - e^{-2(x-u)}] \cos \alpha u \, du.$

(g) $y = \frac{1}{2}\int_o^x [\sinh (x - u) - \sin (x - u)]g(u) \, du.$

(h) $y = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t + \frac{1}{6}\int_0^t [2 \sin (t - u) - \sin 2(t - u)]g(u) \, du.$

5. The solutions are

(a) $y = x + \frac{1}{6}x^3$

(b) $y = \frac{1}{3}(4 \sin 2x - 2 \sin x)$

6. $Y(s) = \dfrac{4}{(s^2 + 4)s} \tanh \dfrac{as}{2}$. Then, $y(t) = 2 \sin 2t * f_s(t)$ where f_s is the square wave function. If you compute this integral (Separate the cases $0 < t < \pi$, $\pi < t < 2\pi$) and use periodicity, you get $y = |\sin t| \cdot \sin t$.

Solutions for Chapter 4

Section 4.1.1

1. The solutions are

(a) $e^{-x} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^n}{n!}$

(b) $e^{2x} = 1 + 2x + \dfrac{4x^2}{2!} + \dfrac{8x^3}{3!} + \dfrac{16x^4}{4!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{2^n x^n}{n!}$

(c) $e^{-3x} = 1 - 3x + \dfrac{9x^2}{2!} - \dfrac{27x^3}{3!} + \dfrac{64x^4}{4!} - \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n 3^n x^n}{n!}$

(d) $e^{x^3} = 1 + x^3 + \dfrac{x^6}{2!} + \dfrac{x^9}{3!} + \dfrac{x^{12}}{4!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{3n}}{n!}$

(e) $\cos 2x = 1 - \dfrac{4x^2}{2!} + \dfrac{16x^4}{4!} - \dfrac{64x^6}{6!} + \dfrac{256x^8}{8!} - \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{4^n x^{2n}}{(2n)!}$

(f) $\sin \dfrac{x}{2} = \dfrac{x}{2} - \dfrac{x^3}{8 \cdot 3!} + \dfrac{x^5}{32 \cdot 5!} - \dfrac{x^7}{128 \cdot 7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2^{2n+1} \cdot (2n+1)!}$

(g) $\sin x^2 = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \dfrac{x^{18}}{9!} - \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n+2}}{(2n+1)!}$

(h) $\cos \sqrt{x} = 1 - \dfrac{x}{2!} + \dfrac{x^2}{4!} - \dfrac{x^3}{6!} + \dfrac{x^4}{8!} - \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^n}{(2n)!}$

2. The solutions are

(a) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots = \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^n}{n}$

(b) $(x-1) - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{3}(x-1)^3 - \dfrac{1}{4}(x-1)^4 + \cdots = \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}(x-1)^n}{n}$

(c) $1 - \left(x - \dfrac{\pi}{2}\right) - \dfrac{1}{2!}\left(x - \dfrac{\pi}{2}\right)^2 + \dfrac{1}{3!}\left(x - \dfrac{\pi}{2}\right)^3 + \dfrac{1}{4!}\left(x - \dfrac{\pi}{2}\right)^4 - \dfrac{1}{5!}\left(x - \dfrac{\pi}{2}\right)^5 - \ldots$

(d) $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$

(e) $1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

(f) $\dfrac{\sqrt{2}}{2}\left[1 + \left(x - \dfrac{\pi}{4}\right) - \dfrac{(x-\pi/4)^2}{2!} - \dfrac{(x-\pi/4)^3}{3!} + \dfrac{(x-\pi/4)^4}{4!} + \cdots\right]$

(g) $x^2 - \dfrac{x^4}{3} + \dfrac{x^6}{5} + \cdots + (-1)^n \dfrac{x^{2n+2}}{2n+1} + \cdots \qquad (\,|x| < 1\,)$

3. We have not studied what happens at the endpoints of the interval of convergence, but we don't worry about this.

(a) $R = 1, \quad (-1, 1)$	(g) $R = 0.5, \quad (0, 1)$	(m) $R = 1, \quad (2, 4)$
(b) $R = 1, \quad (-1, 1)$	(h) $R = 0.5, \quad (2.5, 3.5)$	(n) $R = \infty, \quad (-\infty, \infty)$
(c) $R = 1, \quad (-1, 1)$	(i) $R = 0$	(o) $R = 0$
(d) $R = 1, \quad (-1, 1)$	(j) $R = 0.5, \quad (-0.5, 0.5)$	(p) $R = 10, \quad (-8, 12)$
(e) $R = 5, \quad (-5, 5)$	(k) $R = 3, \quad (-4, 2)$	(q) $R = 1, \quad (-1, 1)$
(f) $R = 0.2, \quad (0.4, 0.8)$	(l) $R = 1, \quad (1, 3)$	(r) $R = 2, \quad (-2, 2)$

Section 4.1.2

1. The solutions are

(a) $f(x) = x^2 - 3x^3 + \dfrac{9x^4}{2!} - \dfrac{27x^5}{3!} + \dfrac{81x^6}{4!} - \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3^n x^{n+2}}{n!}, \qquad R = \infty$

(b) $f(x) = \dfrac{1}{10}\left[1 - \dfrac{x}{10} + \left(\dfrac{x}{10}\right)^2 - \left(\dfrac{x}{10}\right)^3 + \left(\dfrac{x}{10}\right)^4 - \cdots\right] = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{10^{n+1}}, \qquad R = 10$

(c) $f(x) = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \dfrac{x^{18}}{9!} - \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}, \qquad R = \infty$

(d) $f(x) = 1 - \dfrac{2x^2}{2!} + \dfrac{8x^4}{4!} - \dfrac{32x^6}{6!} + \cdots = 1 + \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!}, \qquad R = \infty$

(e) $f(x) = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots = \dfrac{3}{2}\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n(n+1)x^n, \qquad R = 1$

(f) $f(x) = 1 - \dfrac{x}{2} + \dfrac{x^2}{3} - \dfrac{x^3}{4} + \dfrac{x^4}{5} - \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}, \qquad R = 1$

(g) $f(x) = \dfrac{1}{3} - \dfrac{x^2}{5} + \dfrac{x^4}{7} - \dfrac{x^6}{9} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+3}, \qquad R = 1$

2. The solutions are

(a) $f(x) = \dfrac{x^4}{4} - \dfrac{x^{10}}{3!10} + \dfrac{x^{16}}{5!16} - \dfrac{x^{22}}{7!22} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(6n+4)} x^{6n+4}$

(b) $f(x) = x - \dfrac{x^3}{3 \cdot 3!} + \dfrac{x^5}{5 \cdot 5!} - \dfrac{x^7}{7 \cdot 7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot (2n+1)!}, \qquad R = \infty$

(c) $f(x) = x - \dfrac{x^4}{4} + \dfrac{x^7}{2!7} - \dfrac{x^{10}}{3!10} + \dfrac{x^{13}}{4!13} - \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(3n+1)} x^{3n+1}$

(d) $f(x) = x - \dfrac{x^3}{3^2} + \dfrac{x^5}{5^2} - \dfrac{x^7}{7^2} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} x^{2n+1}$

(e) $f(x) = x - \dfrac{x^3}{2!3} + \dfrac{x^5}{3!5} - \dfrac{x^7}{4!7} + \dfrac{x^9}{6!9} - \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(n+1)!(2n+1)}$

(f) $f(x) = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \qquad R = 1$

3. Use the series for $\tan^{-1} x$ with $x = 1/\sqrt{3}$. The first six terms show that $3.14130878 < \pi < 3.1416744$.

4. $y' = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)^2 x^n$,
$y'' = 2^2 + 3^2 \cdot 2x + 4^2 \cdot 3x^2 + 5^2 \cdot 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+2)^2 (n+1) x^n$

5. The solutions are

(a) $\displaystyle\sum_{n=1}^{\infty} a_{n-1}(x-1)^n$ (b) $\displaystyle\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$ (c) $\displaystyle\sum_{n=2}^{\infty} a_{n-2}x^n$

6. $a_n = (-2)^n a_0/n! \quad (n = 1, 2, \ldots), \qquad y = \displaystyle\sum_{n=0}^{\infty} a_n x^n = a_0 e^{-2x}$

Section 4.2

1. The solutions are

(a) $y = a_0(1 + x + x^2/2! + x^3/3! + \cdots) = a_0 e^x, \qquad R = \infty$

(b) $y = a_0(1 + 4x + \frac{4^2x^2}{2!} + \frac{4^3x^3}{3!} + \cdots) = a_0 e^{4x}, \qquad R = \infty$

(c) $y = a_0(a - \frac{3x}{2} + \frac{3^2x^2}{2!2^2} - \frac{3^3x^3}{3!2^3} + \frac{3^4x^4}{4!2^4} - \cdots) = a_0 e^{-(3/2)x}, \qquad R = \infty$

(d) $y = a_0(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots) = a_0 e^{-x^2}, \qquad R = \infty$

(e) $y = a_0(1 + \frac{x^3}{3} + \frac{x^6}{3^2 \cdot 2!} + \frac{x^9}{3^3 \cdot 3!} + \cdots) = a_0 e^{x^3/3}, \qquad R = \infty$

(f) $a_{n+1} = \dfrac{a_n}{2}, \qquad y = \dfrac{2a_0}{2 - x}, \qquad R = 2$

(g) $y = a_0(1 + 2x + 4x^2 + 8x^3 + \cdots) = \dfrac{a_0}{1 - 2x}, \qquad R = 1/2.$

(h) $y = a_{n+1} = \frac{a_n(1-2n)}{2(n+1)}, \qquad R = 1, \qquad y = a_0(1 + x)^{1/2}$

(i) $y = a_0(1 + 2x + 3x^2 + 4x^3 + \cdots) = a_0/(1 - x)^2, \qquad R = 1$

(j) $a_{n+1} = \frac{a_n(2n-3)}{2(n+1)} \qquad R = 1, \qquad y = a_0(1 - x)^{3/2}$

2. The recurrence formulas give you

(a) $(n + 1)a_n = 0$ for all n, so that $a_n = 0$.

(b) $2na_n = a_n$ for all n, so that $a_n = 0$.

(c) $a_0 = a_1 = 0$ and $a_{n+1} = -na_n$ for $n \geq 1$, so that $a_n = 0$ for all n.

(d) $a_n = 0$ for all n.

Section 4.3

1. The solutions are

(a) $y = a_0(1 + x^2/2! + x^4/4! + x^6/6! + \cdots) + a_1(x + x^3/3! + x^5/5! + x^7/7! + \cdots)$
$= a_0 \cosh x + a_1 \sinh x, \qquad R = \infty$

(b) $a_{n+2} = \frac{4a_n}{(n+1)(n+2)}, \qquad R = \infty, \qquad y = a_0 \cosh 2x + \frac{a_1}{2} \sinh 2x$

(c) $y = a_0(1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} - \frac{3^6x^6}{6!} + \cdots) + \frac{a_1}{3}(3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \frac{3^7x^7}{7!} + \cdots)$
$= a_0 \cos 3x + \frac{a_1}{3} \sin 3x, \qquad R = \infty$

(d) $y = x + a_0 \cos x + (a_1 - 1) \sin x, \qquad R = \infty$

2. The solutions are

(a) $a_0 = 0, \quad a_1 = 3, \quad a_n = -\frac{4a_{n-2}}{n(n-1)}, \qquad y = \frac{3}{2} \sin 2x$

(b) $y = 2 \cosh 2x$

(c) $a_0 = 0, \quad a_1 = 1, \quad a_{n+1} = \frac{2na_n - a_{n-1}}{n(n+1)}, \qquad y = xe^x$

(d) $a_{n+2} = \frac{2a_n - (n+1)a_{n+1}}{(n+1)(n+2)}, \qquad y = e^{-2x}$

3. Using the power series method, one obtains

(a) $a_{n+2} = a_n, \qquad y = a_0 \sum_{n=0}^{\infty} x^{2n} + a_1 \sum_{n=0}^{\infty} x^{2n+1} = \dfrac{a_0 + a_1 x}{1 - x^2}, \qquad R = 1$

(b) $a_{n+2} = -\dfrac{1}{2} a_n, \quad y = a_0 \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{2^n} + a_1 \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2^n} = \dfrac{2(a_0 + a_1 x)}{x^2 + 2}, \quad R = \sqrt{2}$

(c) $a_{n+2} = \dfrac{-a_n}{n+2}$, $\quad y = a_0 e^{-x^2/2} + a_1 \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{1\cdot 3\cdot 5\cdots(2n+1)}$

can also be written as $y = a_0 e^{-x^2/2} + a_1 \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n n! x^{2n+1}}{(2n+1)!}$, $\quad R = \infty$

(d) $a_{n+2} = -\dfrac{a_n(n+4)}{n+2}$, $\quad R = 1$,

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n} + \frac{1}{3} a_1 \sum_{n=0}^{\infty} (-1)^n (2n+3) x^{2n+1} = \frac{3a_0 + a_1(x^3+3x)}{3(1+x^2)^2}$$

(e) $a_{n+2} = \dfrac{n a_n}{3(n+2)}$, $\quad y = a_0 + a_1 \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)3^n} = a_0 + \dfrac{a_1\sqrt{3}}{2} \ln\left[\dfrac{\sqrt{3}+x}{\sqrt{3}-x}\right]$,

$R = \sqrt{3}$ (can also use the method of reduction of order)

(f) $a_{n+2} = \dfrac{a_n(n-3)(n-4)}{(n+1)(n+2)}$, $\quad y = a_0(1+6x^2+x^4) + a_1(x+x^3)$, $\quad R = 1$

(g) $a_{n+2} = \dfrac{-a_n(n-4)^2}{3(n+1)(n+2)}$, $\quad R = \sqrt{3}$, $\quad y = a_0(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4)$

$$+ a_1\left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5 + \sum_{n=3}^{\infty} \frac{(-1)^n \cdot 1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-5)^2 \cdot x^{2n+1}}{(2n+1)!3^{n-2}}\right)$$

(h) $a_{n+2} = \dfrac{a_n(n-4)(n+4)}{2(n+1)(n+2)}$, $\quad R = \sqrt{2}$,

$$y = a_0(1 - 4x^2 + 2x^4) + a_1\left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5 + \sum_{n=3}^{\infty} \frac{(2n-5)!(2n+3)!x^{2n+1}}{n!(n-3)!2^{3n-3}}\right)$$

(i) $a_{n+2} = \dfrac{(n+3)(n+4)}{(n+1)(n+2)} a_n$, $\quad R = 1$,

$$y = a_0 \sum_{n=0}^{\infty} (n+1)(2n+1)x^{2n} + \frac{1}{3}a_1 \sum_{n=0}^{\infty} (n+1)(2n+3)x^{2n+1}$$

(j) $a_{n+2} = -\dfrac{(n-4)a_n}{3(n+1)(n+2)}$, $\quad R = \infty$,

$$y = a_0(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5 + \sum_{n=3}^{\infty} \frac{(-1)^n (2n-5)! x^{2n+1}}{2^{n-3}3^{n-1}(n-3)!(2n+1)!}\right)$$

(k) $a_{n+2} = \dfrac{2(n-5)a_n}{5(n+1)(n+2)}$, $\quad R = \infty$, $\quad y = a_1(x - \frac{4}{15}x^3 + \frac{4}{375x^5})$

$$+ a_0\left(1 - x^2 + \frac{1}{10}x^4 + \frac{1}{750}x^6 - 15 \sum_{n=4}^{\infty} \frac{1\cdot 3\cdot 5\cdot(2n-9)\cdot(2n-7)\cdot 2^n \cdot x^{2n}}{(2n)!5^n}\right)$$

$$= a_1(x - \frac{4}{15}x^3 + \frac{4}{375x^5}) + a_0\left(1 - x^2 + \frac{1}{10}x^4 + \frac{1}{750}x^6 - 240 \sum_{n=4}^{\infty} \frac{(2n-7)!x^{2n}}{(n-4)!(2n)!5^n}\right)$$

(l) $a_2 = 0$, $\quad a_{n+3} = \dfrac{a_n}{n+2}$, $\quad R = \infty$, $\quad y = a_0\left(1 + \sum_{n=1}^{\infty} \dfrac{x^{3n}}{2\cdot 5\cdots(3n-1)}\right) + a_1 \sum_{n=0}^{\infty} \dfrac{x^{3n+1}}{n!3^n}$

(m) $a_2 = 0$, $\quad a_{n+3} = -\dfrac{a_n}{n+3}$, $\quad R = \infty$, $\quad y = a_0 \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n}}{n!3^n} + a_1 \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n+1}}{1\cdot 4\cdots(3n+1)}$

(n) $a_2 = 0$, $\quad a_{n+3} = -\dfrac{a_n}{(n+2)(n+3)}$, $\quad R = \infty$,

$$y = a_0\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{3^n \cdot n! \cdot 2\cdot 5\cdots(3n-1)}\right) + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3^n \cdot n! \cdot 1\cdot 4\cdots(3n+1)}$$

(o) $a_2 = a_3 = 0, \quad a_{n+4} = -\dfrac{a_n}{(n+3)(n+4)}, \qquad R = \infty,$

$$y = a_0\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{4^n \cdot n! \cdot 3 \cdot 7 \cdots (4n-1)}\right) + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4^n \cdot n! \cdot 5 \cdot 9 \cdots (4n+1)}$$

4. the solutions are

(a) $y = x$ (b) $y = 1 + x^2$

5. The power series method gives

(a) $a_{n+2} = -\dfrac{a_n}{n+2}, \qquad y = 2e^{-(x-1)^2/2} + \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!(x-1)^{2n+1}}{(2n+1)!}$

(b) $y = \dfrac{1}{3}\displaystyle\sum_{n=0}^{\infty} (2n+3)(x-1)^{2n+1},$ converges if $0 < x < 2$.

(c) $y = 2 - \frac{11}{2}(x-3)^2,$ converges for all x.

(d) $y = 1 + 4(x+2)^2$

(e) $a_0 = 0, \quad a_1 = 2, \qquad a_{n+2} = \dfrac{a_n(n+3)(n-1)}{9(n+2)(n+1)}, \qquad y = 2(x+3)$

6. The solutions are

(a) $2a_2 + a_0 = 0, \quad (n+1)(n+2)a_{n+2} + a_n + a_{n-1} = 0, \quad (n \geq 1),$
$y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots, \qquad y_2 = x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots$

(b) $a_2 = 0, \quad (n+1)na_n + 2a_{n-1} - (n+2)(n+1)a_{n+2} = 0,$
$y_1 = 1 + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{45}x^6 + \cdots, \qquad y_2 = x + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{5}x^5 + \frac{2}{15}x^6 + \cdots$

(c) $a_2 = a_3 = 0, \quad (n+3)(n+4)a_{n+4} + (n+1)a_{n+1} + a_n = 0,$
$y_1 = 1 - \frac{1}{12}x^4 + \frac{1}{126}x^7 + \cdots, \qquad y_2 = x - \frac{1}{12}x^4 - \frac{1}{20}x^5 \cdots$

(d) $y = a_0(1 - \frac{1}{30}x^6 + \frac{1}{72}x^9 + \cdots) + a_1(x - \frac{1}{42}x^7 + \frac{1}{90}x^{10} + \cdots)$

7. There are two methods which you can use. Either find a power series solution for

$$y'' + xy = 0.$$

You will get the recurrence relation $a_2 = 0, \quad a_{n+2} = -\frac{a_{n-1}}{(n+1)(n+2)}$ which gives the solution

$$a_0\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)x^{3n}}{(3n)!}\right) + a_1\left(x + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2 \cdot 5 \cdots (3n-1)x^{3n+1}}{(3n)!}\right)$$

Then use the method of variation of parameters. (Note that the method of undetermined coefficients can not be used here !) The second method is to substitute the power series for e^x from the beginning. You get the recurrence relation $a_2 = 0.5, \quad a_{n+2} = \dfrac{\frac{1}{n!} - a_{n-1}}{(n+1)(n+2)}$.

Section 4.4

1. (a) $x = 0, \quad r(r-1) = 0, \quad r_1 = 1,\ r_2 = 0$

(b) $x = 0, \quad r(r-1) = 0, \quad r_1 = 1,\ r_2 = 0; \qquad x = 1, \quad r(r+5) = 0, \quad r_1 = 0,\ r_2 = -5$

(c) $x = 0, \quad r^2 + 2r - 2 = 0, \quad r_1, r_2 = -1 \pm \sqrt{3}$

(d) $x = 0, \quad r(r - \frac{3}{4}) = 0, \quad r_1 = \frac{3}{4},\ r_2 = 0$

(e) $x = -2, \quad r(r - \frac{5}{4}) = 0, \quad r_1 = \frac{5}{4},\ r_2 = 0$

(f) $x = 1, \quad r(r+1) = 0; \quad r_1 = 0,\ r_2 = -1$

(g) $x = 2, \quad r(r-2) = 0; \quad r_1 = 2,\ r_2 = 0; \qquad x = -2, \quad r(r-2) = 0, \quad r_1 = 2,\ r_2 = 0$

2. The solutions are
   (a) $y = c_1x^{-1} + c_2x^{-2}$
   (b) $y = c_1x^2 + c_2x^2 \ln|x|$
   (c) $y = c_1x + c_2x \ln|x|$
   (d) $y = c_1(x-1)^{-3} + c_2(x-1)^{-4}$

3. (a) $y_1 = \cos\sqrt{x}, \quad y_2 = \sin\sqrt{x}$

   (b) $y_1 = \sum_{n=0}^{\infty} \frac{x^n}{n!\,(2n+1)!}, \qquad y_2 = x^{-1/2}\sum_{n=0}^{\infty} \frac{x^n}{n!\,(2n-1)!}$

   (c) $y_1 = x^{3/2}\left(1 + 3\sum_{n=1}^{\infty} \frac{x^n}{n!(2n+3)!}\right), \qquad y_2 = 1 - x - \sum_{n=2}^{\infty} \frac{x^n}{(2n-3)!}$

   (d) $y_1 = x^{1/3}\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!\cdot 4\cdot 7\cdots(3n+1)}, \qquad y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!\cdot 2\cdot 5\cdots(3n-1)}$

   (e) $y_1 = x\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\cdot 7\cdot 11\cdots(4n+3)}\right), \qquad y_2 = x^{-1/2}\sum_{n=0}^{\infty} \frac{x^{2n}}{n!\cdot 1\cdot 5\cdots(4n+1)}$

   (f) $y_1 = x^{3/2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n!\cdot 9\cdot 13\cdots(4n+5)}\right), \; y_2 = x^{-1}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n!\cdot 3\cdot 7\cdots(4n-1)}\right)$

   (g) $y_1 = x^{1/2}\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n\cdot n!\cdot 19\cdot 31\cdots(12n+7)}\right),$

   $y_2 = x^{-2/3}\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n\cdot n!\cdot 5\cdot 17\cdots(12n-7)}\right)$

   (h) $y_1 = x^{1/3}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n\cdot n!\cdot 7\cdot 13\cdots(6n+1)}\right), \; y_2 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n\cdot n!\cdot 5\cdot 11\cdots(6n-1)}$

   (i) $y_1 = x^{1/2}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!\,2^n} = x^{1/2}e^{-x/2}, \qquad y_2 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n-1)!}$

   (j) $y_1 = x^{1/2}\sum_{n=0}^{\infty} \frac{x^{2n}}{n!\,2^n} = x^{1/2}e^{x^2/2}, \qquad y_2 = 1 + \sum_{n=1}^{\infty} \frac{2^n x^{2n}}{3\cdot 7\cdots(4n-1)}$

   (k) $y_1 = (\cos 3x)/x, \quad y_2 = (\sin 3x)/x$
   (l) $y_1 = (\cosh 2x)/x, \quad y_2 = (\sinh 2x)/x$
   (m) $y_1 = (\cos\frac{x}{2})/x, \quad y_2 = (\sin\frac{x}{2})/x$
   (n) $y_1 = \cos x^2, \quad y_2 = \sin x^2$

4. (a) $y_1 = x^{-2} + x^{-1}, \qquad y_2 = 1 + 2\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$

   (b) $y_1 = \frac{1}{x^4} + \frac{1}{x^3} + \frac{1}{2x^2} + \frac{1}{6x^3}, \qquad y_2 = 1 + 24\sum_{n=1}^{\infty} \frac{x^n}{(n+4)!}$

   (c) $y_1 = x^{-4}(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3), \qquad y_2 = 1 + 24\sum_{n=1}^{\infty} \frac{(-1)^n 3^n x^n}{(n+4)!}$

   (d) $y_1 = 1 + \frac{3}{4}x + \frac{1}{4}x^2 + \frac{1}{24}x^3, \qquad y_2 = x^5\left(1 + 120\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+5)!}\right)$

   (e) $y_1 = x^{-2}(2 - 6x + 9x^2), \qquad y_2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n x^n}{(n+2)!}$

   (f) $y_1 = 3 + 2x + x^2, \qquad y_2 = x^4/(1-x)^2$

5. (a) $y_1 = x\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(n+2)!}\right), \qquad y_2 = x^{-1}\left(-\frac{1}{2} - \frac{x}{2} + \frac{29x^2}{144} + \cdots\right) + \frac{1}{4}y_1 \ln x$

(b) $y_1 = x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10}\cdots\right)$, $\quad y_2 = x^{-2}\left(-\frac{1}{6} - \frac{x^2}{6} - \frac{x^4}{6} + \cdots\right) + \frac{2}{9}y_1 \ln x$

(c) $y_1 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{(n!)^2}$, $\quad y_2 = 4x - 3x^2 + \frac{22x^3}{7} + \cdots + y_1 \ln x$

(d) $y_1 = x\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right)$, $\quad y_2 = \frac{x^3}{4} - \frac{3x^5}{128} + \frac{11x^7}{13824} + \cdots + y_1 \ln x$

(e) $y_1 = x^2(1 - 2x + \frac{3}{2}x^2 - \frac{2}{3}x^3\cdots)$, $\quad y_2 = y_1(\ln x + 3x + \frac{11}{4}x^2 + \frac{49}{18}x^3 + \cdots)$

(f) $y_1 = x^2(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \cdots)$, $\quad y_2 = y_1\left(\frac{-1}{3x^3} + \frac{1}{20x} + \frac{3x}{100} + \cdots\right)$

Index

$f(x)$	$F(s)$
$c_1 f_1(x) + c_2 f_2(x)$	$c_1 F_1(s) + c_2 F_2(s)$
$f'(x)$	$sF(s) - f(0)$
$f''(x)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{(n)}(x)$	$s^n F(s) - s^{n-1} f(0) - \cdots$ $\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
$\int_0^x f(u)\,du$	$\frac{F(s)}{s}$
$e^{cx} f(x)$	$F(s-c)$
$f(cx)$	$\frac{1}{c} F\left(\frac{s}{c}\right), \quad c > 0$
$(f * g)(x) = \int_0^x f(u)g(x-u)\,du$	$F(s)G(s)$
$x^n f(x)$	$(-1)^n F^{(n)}(s)$
$\frac{f(x)}{x}$	$\int_s^\infty F(u)\,du$
$u_c(x) f(x-c)$	$e^{-cs} F(s), \quad c > 0$

Properties of the Laplace Transform

$f(x)$	$F(s)$
1	$\frac{1}{s}, \quad s > 0$
x	$\frac{1}{s^2}, \quad s > 0$
$x^n \quad (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}, \quad s > 0$
e^{cx}	$\frac{1}{s-c}, \quad s > c$
$\sin ax$	$\frac{a}{s^2 + a^2}, \quad s > 0$
$\cos ax$	$\frac{s}{s^2 + a^2}, \quad s > 0$
$\sinh ax$	$\frac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$
$\cosh ax$	$\frac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$
$e^{cx} \sin ax$	$\frac{a}{(s-c)^2 + a^2}, \quad s > c$
$e^{cx} \cos ax$	$\frac{s-c}{(s-c)^2 + a^2}, \quad s > c$
$x^n e^{cx} \quad (n = 1, 2, \ldots)$	$\frac{n!}{(s-c)^{n+1}}, \quad s > c$
$\frac{1}{2a^3}(\sin ax - ax \cos ax)$	$\frac{1}{(s^2 + a^2)^2}, \quad s > 0$
$\frac{x}{2a} \sin ax$	$\frac{s}{(s^2 + a^2)^2}, \quad s > 0$
$u_c(x)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$\delta_c(x)$	$e^{-cs}, \quad c \geq 0$

Elementary Laplace Transforms